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A NUMERICAL SOLUTION OF A COUPLING SYSTEM OF CONFORMABLE TIME-DERIVATIVE TWO-DIMENSIONAL BURGERS' EQUATIONS

ILHEM MOUS[1] AND ABDELHAMID LAOUAR[2*]

Abstract. In this paper, we deal with a numerical solution of a coupling system of fractional conformable time-derivative two-dimensional (2D) Burgers' equations. The presence of both the fractional time derivative and the nonlinear terms in this system of equations makes solving it more difficult. Firstly, we use the Cole-Hopf transformation in order to reduce the coupling system of equations to a conformable time-derivative 2D heat equation for which the numerical solution is calculated by the explicit and implicit schemes. Secondly, we calculate the numerical solution of the proposed system by using both the obtained solution of the conformable time-derivative heat equation and the inverse Cole-Hopf transformation. This approach shows its efficiency to deal with this class of fractional nonlinear problems. Some numerical experiments are displayed to consolidate our approach.

1. Introduction

In the last two decades, the fractional derivatives regained an important interest, and have been widely used in various fields, such as modelling viscoelastic problems, signal processing, control theory, finance, etc. Thus, many classical mathematical models have been reformulated into new models with fractional-order derivatives for their important numerous applications (see [7, 8, 10, 13, 16]). As a result, the scientists introduced different fractional derivative definitions (see [4, 5, 10]): Caputo

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fractional derivative, Riemann-Liouville fractional derivative, Grünwald–Letnikov fractional derivative and others. We give for example two popular definitions below. For $\alpha \in [n-1, n)$, the α-derivatives of the function f are given as

(i) Riemann-Liouville definition

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx;$$

(ii) Caputo definition

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

All definitions including (i) and (ii) satisfy the property that the fractional derivative is linear. We cite for example some research works linked with the subject. Tarasov [16] investigated some properties of the chain rule and Leibniz rule for fractional derivatives. Khalil et al. [9] introduced a new definition of a fractional derivative and fractional integral (called also fractional conformable derivative and fractional integral) for which there are large number of numerous works done (see [1, 4, 5, 9, 17, 18]). Anderson et al. [4] introduced more precise definition of the conformable derivative motiving by a proportional-derivative controller. Ortigueira et al. [14] analyzed the definitions of the Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives. For instance, Abdeljawad [1] gave conformable versions of the chain rule, integration by parts, Taylor power series expansions and Laplace transform. In [2], the authors introduced the fractional conformable semi-group of operators whose generator will be the fractional derivative of the semigroup at $t = 0$. In [3] the authors studied the fractional logistic models in the frame of fractional operator generated by conformable derivatives. Yavuz et al. [19] introduced the conformable derivative operator in modelling neuronal dynamics.

In this work, we are interested in studying a coupling system of the fractional conformable derivative 2D Burgers' equations which incorporate the interaction between the nonlinear convection processes and the diffusive viscous processes. Many works studied the one/two viscous Burgers' equation (with integer-order derivative) using the Cole-Hopf transformation [11, 15]. It is known that the Burgers' equation has been used as a mathematical model in various areas such as number theory, gas dynamics, heat conduction, elasticity theory, etc. It has a lot of similarity to the famous Navier-Stokes equations [6, 12] and has often been used as a simple model equation for comparing the accuracy of different computational algorithms. However the inviscid Burgers' equation lacks one most important property attributed to turbulence since the solutions do no exhibit chaotic features like sensitivity with respect to initial conditions. The purpose of the current study focuses in the use of the Cole-Hopf transformation for this class of the fractional nonlinear problems. So, we transform with the help of Cole-Hopf transformation the coupling system of the conformable time-derivative 2D Burgers' equations into conformable time-derivative heat equation. The numerical solution of the latter is obtained by the explicit and implicit schemes. Therefore, we can easily calculate the solution of the system of the conformable time-derivative 2D Burgers' equations via the inverse Cole-Hopf transformation. For illustration, some numerical experiments are displayed to show the efficiency of this approach.

The paper is organized as follows. Section 2 gives some useful materiel and position of the problem. Section 3 uses the 2D Cole-Hopf transformation. Section 4 proposes the calculation of numerical solutions to heat equation by the explicit and implicit schemes and gives the required solutions for the coupling of 2D Burgers' equations. The last section displays the numerical results.

2. Preliminaries and Position of the Problem

Let us recall below a definition and a theorem which summarizes some important properties.

Definition 2.1 ([5, 9])**.** Given a function $f : [0, \infty) \to \mathbb{R}$, then the conformable fractional derivative of f with order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in (0.1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

Theorem 2.1 ([5, 9])**.** *Let $0 < \alpha \leq 1$ and f, g be α-differentiable at a point $t > 0$. Then*

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$*;*
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$*;*
3. $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$*;*
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$*;*
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$*;*
6. *in addition, if f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$*.*

2.1. Coupling system of the conformable derivatives 2D Burgers' equations.

Let us consider the following coupling system of 2D Burgers' equations

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right), \end{cases} \tag{2.3}$$

where $\alpha \in (0, 1)$, $r > 0$ the diffusion coefficient, $(x, y) \in \Omega$ (a rectangular domain), $t > 0$ and $\partial^\alpha u/\partial t^\alpha$, $\partial^\alpha v/\partial t^\alpha$ mean conformable derivatives respectively of the functions $u(x, y, t)$ and $v(x, y, t)$.

Subject to the initial conditions

$$\begin{cases} u(x,y,0) = u_0(x,y), & \text{for any } (x,y) \in \Omega, \\ v(x,y,0) = v_0(x,y), & \text{for any } (x,y) \in \Omega, \end{cases} \tag{2.4}$$

and the boundary conditions

$$\begin{cases} u(x,y,t) = f(x,y,t), & \text{for any } (x,y) \in \partial\Omega,\ t > 0, \\ v(x,y,t) = g(x,y,t), & \text{for any } (x,y,t) \in \partial\Omega,\ t > 0, \end{cases} \tag{2.5}$$

where f, g are two given functions.

We need later to use the following potential symmetry condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2.6}$$

3. Linearizing System (2.3) by the Cole-Hopf Transformation

Using the property 6 of Theorem 2.1, we can rewrite system (2.3) as follows

$$\begin{cases} t^{(1-\alpha)}\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ t^{(1-\alpha)}\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right). \end{cases} \tag{3.1}$$

The Cole-Hopf transformation is performed in two steps.

First step. Suppose that $u = \psi_x$ and $v = \psi_y$, thus the system (3.1) becomes

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \psi_x\psi_{xx} + \psi_y\psi_{xy} = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ t^{(1-\alpha)}\psi_{yt} + \psi_x\psi_{yx} + \psi_y\psi_{yy} = r\left(\psi_{yxx} + \psi_{yyy}\right), \end{cases} \tag{3.2}$$

which can be rewritten as

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ t^{(1-\alpha)}\psi_{yt} + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{yxx} + \psi_{yyy}\right). \end{cases} \tag{3.3}$$

Integrating respectively the first equation of system (3.3) with respect to x and the second with respect to y, we obtain

$$\begin{cases} t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_1(y,t), \\ t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_2(x,t), \end{cases} \tag{3.4}$$

where $\eta_1(y,t)$ and $\eta_2(x,t)$ are arbitrary functions depending respectively of y and x. Using the condition (2.6), we can combine two equations of system (3.4) and conclude that ψ satisfies the following equation (see [11])

$$t^{(1-\alpha)}\psi_t + \left(\frac{1}{2}\psi_x^2\right) + \left(\frac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta(t). \tag{3.5}$$

Second step. Introducing the transformation as $\psi = -2r \ln \phi$, we have

$$u = -2r\frac{\phi_x}{\phi} \quad \text{and} \quad v = -2r\frac{\phi_y}{\phi}. \tag{3.6}$$

Both the derivatives of function ψ are

$$\psi_t = -2r\frac{\phi_t}{\phi}, \quad \psi_x = -2r\frac{\phi_x}{\phi}, \quad \psi_y = -2r\frac{\phi_y}{\phi}, \tag{3.7}$$

$$\psi_{xx} = -2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}, \quad \psi_{yy} = -2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}. \tag{3.8}$$

Inserting the derivatives ψ_t, ψ_x and ψ_y in the left side of (3.5) and the derivatives ψ_{xx} and ψ_{yy} in the right side, we obtain

$$\begin{aligned} &-2rt^{(1-\alpha)}\frac{\phi_t}{\phi} + \frac{1}{2}\left(-2r\frac{\phi_x}{\phi}\right)^2 + \frac{1}{2}\left(-2r\frac{\phi_y}{\phi}\right)^2 \\ =&r\left(-2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2} - 2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}\right) + \eta(t). \end{aligned} \tag{3.9}$$

Equation (3.9) can be reduced to

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}) + \zeta(t)\phi, \quad \text{where } \zeta(t) = \frac{-\eta(t)}{2r}. \tag{3.10}$$

We now state the following theorem in order to show that the calculus of the functions $u(x, y, t)$ and $v(x, y, t)$ is independent of the function $\zeta(t)$.

Theorem 3.1. *Let $\phi(x, y, t)$ be the solution of (3.10), $u(x, y, t)$ and $v(x, y, t)$ are defined in (3.6), then the solution u and v are independent of $\zeta(t)$.*

Proof. Let

$$\beta(t) = \int \frac{1}{t^{1-\alpha}}\zeta(t)dt,$$

then

$$\beta'(t) = \frac{1}{t^{1-\alpha}}\zeta(t).$$

Multiply by $e^{-\beta(t)}$ the two sides of (3.10), yields

$$\frac{\partial^\alpha \phi}{\partial t^\alpha}e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)} + \zeta(t)\phi e^{-\beta(t)}. \tag{3.11}$$

By using the property 6 of Theorem 2.1, (3.11) becomes

$$t^{1-\alpha}\frac{\partial \phi}{\partial t}e^{-\beta(t)} - \zeta(t)\phi e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)}. \tag{3.12}$$

Then

$$t^{1-\alpha}\frac{\partial}{\partial t}\left(e^{-\beta(t)}\phi\right) = r((e^{-\beta(t)}\phi)_{xx} + (e^{-\beta(t)}\phi)_{yy}). \tag{3.13}$$

Now, let $\psi(x,y,t) = e^{-\beta(t)}\phi(x,y,t)$. Then $\psi(x,y,t)$ satisfies the following 2D heat equation

$$t^{1-\alpha}\frac{\partial \psi}{\partial t} = r(\psi_{xx} + \psi_{yy}), \tag{3.14}$$

which rewrites in other form

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = r(\psi_{xx} + \psi_{yy}). \tag{3.15}$$

Note that the difference between the solution of (3.10) and (3.15) is the factor $e^{-\beta(t)}$. Therefore,

$$u(x,y,t) = \frac{\phi_x}{\phi} = \frac{e^{-\beta(t)}\phi_x}{e^{-\beta(t)}\phi} = \frac{\psi_x}{\psi}, \tag{3.16}$$

$$v(x,y,t) = \frac{\phi_y}{\phi} = \frac{e^{-\beta(t)}\phi_y}{e^{-\beta(t)}\phi} = \frac{\psi_y}{\psi}. \tag{3.17}$$

It is clear that the solutions u and v are independent of the function $\zeta(t)$. □

For simplicity of the present study, we can take for example $\zeta(t) \equiv 0$. Then we get the diffusion equation

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}). \tag{3.18}$$

3.1. **Initial and boundary conditions.** We now try to determine a new derivation of the initial and boundary conditions which correspond to (3.18). For the sake of simplicity, let us take

$$\Omega = [a,b] \times [a,b], \quad \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

with

$$\Gamma_1 = \{a \le x \le b,\, y = a\}, \quad \Gamma_2 = \{a \le x \le b,\, y = b\},$$
$$\Gamma_3 = \{x = a,\; a \le y \le b\} \quad \text{and} \quad \Gamma_4 = \{x = b,\, a \le y \le b\}.$$

Initial condition (IC). From (3.6), we can rewrite

$$\frac{\phi_x}{\phi} = \frac{u(x,y,t)}{-2r}. \tag{3.19}$$

Integrating the left and right sides of (3.19) with respect to x, we obtain

$$\ln(\phi) = \frac{-1}{2r}\int_a^x u(s,y,t)ds + \ln(\phi(a,y,t)).$$

Then we get

$$\phi(x,y,t) = \phi(a,y,t)\exp\left(\frac{-1}{2r}\int_a^x u(s,y,t)ds\right). \tag{3.20}$$

On the other hand, we rearrange the second term of (3.6) as follows

$$\frac{\phi_y}{\phi} = \frac{v(x,y,t)}{-2r}. \tag{3.21}$$

Integration of the above equation with respect to y, then we obtain

$$\ln(\phi) = \frac{-1}{2r}\int_a^y u(x,s,t)ds + \ln(\phi(x,a,t)),$$

which yields

$$\phi(x,y,t) = \phi(x,a,t)\exp\left(\frac{-1}{2r}\int_a^y v(x,s,t)ds\right). \tag{3.22}$$

At $x = a$, (3.22) gives

$$\phi(a,y,t) = \phi(a,a,t)\exp\left(\frac{-1}{2r}\int_a^y v(a,s,t)ds\right). \tag{3.23}$$

Inserting (3.23) into (3.20), yields

$$\phi(x,y,t) = \phi(a,a,t)\exp\left(-\frac{1}{2r}\int_a^y v(a,s,t)ds - \frac{1}{2r}\int_a^x u(s,y,t)ds\right), \tag{3.24}$$

and at $t = 0$ in (3.24), then the initial condition is written as

$$\phi(x,y,0) = \phi(a,a,0)\exp\left(-\frac{1}{2r}\int_a^y v(a,s,0)ds - \frac{1}{2r}\int_a^x u(s,y,0)ds\right). \tag{3.25}$$

From (3.6), it is clear that $\phi(a,a,0)$ has no effect on the final solution of system (2.3). In our case, we can consider for example $\phi(a,a,0) = 1$. It yields

$$\phi_0(x,y) = \exp\left(-\frac{1}{2r}\int_a^y v(a,s,0)ds - \frac{1}{2r}\int_a^x u(s,y,0)ds\right). \tag{3.26}$$

Boundary conditions (BC). Using (3.6), the boundary conditions are reduced to

$$\begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), & (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), & (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{3.27}$$

Therefore, the time-conformable diffusion equation with the initial and Neumann boundary conditions is given by

$$\begin{cases} \text{Eq.}: & \dfrac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}), \\ \text{IC}: & \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\int_a^y v(a,s,0)ds - \dfrac{1}{2r}\int_a^x u(s,y,0)ds\right), \\ \text{BC}: & \begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \end{cases} \tag{3.28}$$

Reformulating problem (3.28) by using the property 6 of Theorem 2.1, it yields

$$\tag{3.29} \begin{cases} \text{Eq. :} \quad t^{(1-\alpha)}\dfrac{\partial \phi}{\partial t} = r(\phi_{xx} + \phi_{yy}), \\ \text{IC :} \quad \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\int_a^y v(a,s,0)ds - \dfrac{1}{2r}\int_a^x u(s,y,0)ds\right), \\ \text{BC:} \quad \begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \end{cases}$$

The solution of the problem (3.29) can be found in [13]. Finally, once the solution of the problem (3.29) is known, we can easily obtain the solution of the coupled problem (2.3)-(2.5) via the formula (3.6).

4. Numerical Study of the Problem (3.29)

We discretize the domain Ω by the finite difference method (FDM) into nx each of length $\Delta x = (b-a)/nx$ and into ny each of length $\Delta y = (b-a)/ny$ along, respectively the x-axis and y-axis. We define then the discrete mesh points (x_i, y_j, t_n) by $(a + i\Delta x, a + j\Delta y, n\Delta t)$, where $i = 0, \dots, nx$, $j = 0, \dots, ny$, $n = 0, \dots, T$.

4.1. **An explicit scheme.** By using a simple forward in time and centered in space discretization at point (x_i, y_j, t_n), the explicit scheme of (3.29) is given by

$$t_n^{(1-\alpha)}\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = r\left(\frac{\phi_{i+1,j}^n - 2\phi_{i,j}^n + \phi_{i-1,j}^n}{\Delta x^2} + \frac{\phi_{i,j+1}^n - 2\phi_{i,j}^n + \phi_{i,j-1}^n}{\Delta y^2}\right).$$

For every interior point (x_i, y_j, t_n), with $i = 1, \dots, nx-1$, $j = 1, \dots, ny-1$, we have

$$\tag{4.1} \phi_{i,j}^{n+1} = A\phi_{i,j}^n + B(\phi_{i+1,j}^n + \phi_{i-1,j}^n) + C(\phi_{i,j+1}^n - \phi_{i,j-1}^n),$$

where

$$\begin{aligned} A =& 1 - \frac{2r\Delta t}{\Delta x^2\, t_n^{(1-\alpha)}} - \frac{2r\Delta t}{\Delta y^2 t_n^{(1-\alpha)}}, \\ B =& \frac{r\Delta t}{\Delta x^2\, t_n^{(1-\alpha)}}, \quad C = \frac{r\Delta t}{\Delta y^2\, t_n^{(1-\alpha)}}. \end{aligned}$$

Now, let us consider the so-called BC described as

$$\tag{4.2} \begin{cases} \phi_x(x_i, y_j, t_n) \simeq \dfrac{\phi_{i+1,j}^n - \phi_{i-1,j}^n}{2\Delta x} = -\dfrac{1}{2r}u_{i,j}^n\phi_{i,j}^n, \\ \phi_y(x_i, y_j, t_n) \simeq \dfrac{\phi_{i,j+1}^n - \phi_{i,j-1}^n}{2\Delta y} = -\dfrac{1}{2r}v_{i,j}^n\phi_{i,j}^n, \end{cases}$$

which can be rewritten as

$$\tag{4.3} \begin{cases} \phi_{i+1,j}^n = \phi_{i-1,j}^n - \dfrac{\Delta x}{r}u_{i,j}^n\phi_{i,j}^n, \\ \phi_{i,j+1}^n = \phi_{i,j-1}^n - \dfrac{\Delta y}{r}v_{i,j}^n\phi_{i,j}^n. \end{cases}$$

Thus, we give the details for each discrete side as follows.

On the side Γ_1, let $j=0$ in (4.3), then we have

$$\begin{cases} \phi^n_{i+1,0} = \phi^n_{i-1,0} - \dfrac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}, \\ \phi^n_{i,1} = \phi^n_{i,-1} - \dfrac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}. \end{cases} \tag{4.4}$$

Substituting this constraint into (4.1) at the boundary points, for $i = 1, \dots, nx$, we obtain

$$\begin{aligned} \phi^{n+1}_{i,0} =& A\phi^n_{i,0} + B(\phi^n_{i+1,0} + \phi^n_{i-1,0}) + C(\phi^n_{i,1} + \phi^n_{i,-1}) \\ =& A\phi^n_{i,0} + B\left(2\phi^n_{i-1,0} - \frac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}\right) + C\left(2\phi^n_{i,1} - \frac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}\right). \end{aligned}$$

In same way as previously, we can calculate respectively the expressions both of the side Γ_2 for $j = ny, \Gamma_3$ for $i = 0$ and Γ_4 for $i = nx$, for all $i = 1, \dots, nx$,

$$\phi^{n+1}_{i,ny} = A\phi^n_{i,ny} + B\left(2\phi^n_{i-1,ny} - \frac{\Delta x}{r} u^n_{i,ny}\phi^n_{i,ny}\right) + C\left(2\phi^n_{i,ny-1} - \frac{\Delta y}{r} v^n_{i,ny}\phi^n_{i,ny}\right). \tag{4.5}$$

And for $j = 1, \dots, ny$,

$$\begin{aligned} \phi^{n+1}_{0,j} =& A\phi^n_{0,j} + B\left(2\phi^n_{1,j} + \frac{\Delta x}{r} u^n_{0,j}\phi^n_{0,j}\right) + C\left(2\phi^n_{0,j-1} - \frac{\Delta y}{r} v^n_{0,j}\phi^n_{0,j}\right), \\ \phi^{n+1}_{nx,j} =& A\phi^n_{nx,j} + B\left(2\phi^n_{nx-1,j} - \frac{\Delta x}{r} u^n_{nx,j}\phi^n_{nx,j}\right) + C\left(2\phi^n_{nx,j-1} - \frac{\Delta y}{r} v^n_{nx,j}\phi^n_{nx,j}\right). \end{aligned}$$

Adding the left-lower corner point (x_0, y_0), we obtain

$$\phi^{n+1}_{0,0} = A\phi^n_{0,0} + B\left(2\phi^n_{1,0} + \frac{\Delta x}{r} u^n_{0,0}\phi^n_{0,0}\right) + C\left(2\phi^n_{0,1} + \frac{\Delta y}{r} v^n_{0,0}\phi^n_{0,0}\right).$$

4.2. **An implicit scheme.** By using a simple forward in time and centered in space (FTCS) discretization at point (x_i, y_j, t_n), the implicit scheme for (3.29) is given by

$$t_n^{1-\alpha}\frac{\phi^{n+1}_{i,j} - \phi^n_{i,j}}{\Delta t} = r\left(\frac{\phi^{n+1}_{i+1,j} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i-1,j}}{\Delta x^2} + \frac{\phi^{n+1}_{i,j+1} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i,j-1}}{\Delta y^2}\right),$$

which can rewrite as

$$-\alpha(\phi^{n+1}_{i+1,j} + \phi^{n+1}_{i-1,j}) + \gamma\phi^{n+1}_{i,j} - \beta(\phi^{n+1}_{i,j+1} + \phi^{n+1}_{i,j-1}) = \phi^n_{i,j}, \tag{4.6}$$

where

$$\alpha = \frac{r\Delta t}{\Delta x^2 t_n^{1-\alpha}}, \quad \beta = \frac{r\Delta t}{\Delta y^2 t_n^{1-\alpha}}, \quad \gamma = 1 + 2\alpha + 2\beta,$$

or in matrix form

$$\mathcal{A}.\mathcal{X} = \mathcal{B},$$

where

$$\mathcal{A} = \begin{pmatrix} A & B & 0 & \cdots & 0 \\ C & D & K & 0 \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & C & D & K \\ 0 & \cdots & 0 & L & M \end{pmatrix}_{(nx\times ny, nx\times ny)},$$

$\mathcal{X}^t = (\phi^{n+1}_{0,0}, \phi^{n+1}_{0,1}, ..., \phi^{n+1}_{nx,ny})$ and $\mathcal{B} = (\phi^{n}_{0,0}, \phi^{n}_{0,1}, ..., \phi^{n}_{nx,ny})$, A, B, C, D, K, L and M are the submatrices with dimension (nx, ny) and are defined respectively by

$$A = \begin{pmatrix} a & -2\beta & 0 & \cdots & 0 \\ -2\beta & \gamma & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & \gamma \end{pmatrix}, \quad B = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & b_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & b_n \end{pmatrix},$$

$$C = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & -2\beta & 0 & \cdots & 0 \\ -\beta & \gamma & -\beta & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma & -\beta \\ 0 & \cdots & 0 & -2\beta & d_{ny} \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad L = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix},$$

$$M = \begin{pmatrix} m' & -2\beta & 0 & \cdots & 0 \\ -2\beta & m_1 & 0 & \cdots & \vdots \\ 0 & \ddots & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & m_{ny} \end{pmatrix},$$

with

$$\begin{aligned} a =& -\frac{\alpha\Delta x}{r} u^n_{0,0} + \gamma - \frac{\beta\Delta y}{r} v^n_{0,0}, \\ b_j =& -2\alpha - \frac{\alpha\Delta x}{r} u^n_{0,j} + \frac{\beta\Delta y}{r} v^n_{0,j}, \quad j = 1, \ldots, ny, \end{aligned}$$

$$
\begin{aligned}
d_1 &= -\frac{\alpha\Delta x}{r}u^n_{1,0} + \gamma - \frac{\beta\Delta y}{r}v^n_{1,0},\\
d_{ny} &= -\frac{\alpha\Delta x}{r}u^n_{nx,ny} + \gamma + \frac{\beta\Delta y}{r}v^n_{nx,ny},\\
m' &= d_1, \quad m_i = -\frac{\alpha\Delta x}{r}u^n_{i,ny} + \gamma + \frac{\beta\Delta y}{r}v^n_{i,ny}, \quad \text{for } i = 1, \dots, nx.
\end{aligned}
$$

4.3. **Calculating the required solution.** The calculation of solution to system (2.3) can be obtained by the inverse Cole-Hopf transformation.

Let $D_x\phi^n_{i,j}$ and $D_y\phi^n_{i,j}$ denote the derivative of ϕ, respectively at point (x_i, y_j, t_n) with respect to x and y. The $D_x\phi^n_{i,j}$ and $D_y\phi^n_{i,j}$ can be calculated from the first order centered difference formula, for $i = 1, \dots, nx - 1$, $j = 1, \dots, ny - 1$,

$$
\begin{aligned}
D_x\phi^n_{i,j} &= \frac{\partial\phi}{\partial x} \simeq \frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x},\\
D_y\phi^n_{i,j} &= \frac{\partial\phi}{\partial y} \simeq \frac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y}.
\end{aligned}
$$

Note that the derivatives $D_y\phi^n_{0,j}, D_y\phi^n_{nx,j}, D_x\phi^n_{i,0}$ and $D_x\phi^n_{i,ny}$ at the end points are known. Once the approximated values of ϕ, ϕ_x and ϕ_y are known at any discrete point (x_i, y_j, t_n), then the approximated values of u and v at discrete points can be calculated from the following discrete version, for $i = 1, \dots, nx$, $j = 1, \dots, ny$,

$$
\begin{cases}
u^n_{i,j} = -2r\dfrac{D_x\phi^n_{i,j}}{\phi^n_{i,j}},\\
v^n_{i,j} = -2r\dfrac{D_y\phi^n_{i,j}}{\phi^n_{i,j}}.
\end{cases} \tag{4.7}
$$

5. Numerical Experiments

For illustration of the proposed method, we will report the accuracy of the method based on relative error L_1-norm and L_∞-norm which are defined by:

$$
||Erreuru||_{L_1} = \frac{||u_a - u_n||_1}{||u_a||_1}, \quad ||Erreurv||_{L_1} = \frac{||v_a - v_n||_1}{||v_a||_1}, \tag{5.1}
$$

and

$$
||Erreuru||_{L_\infty} = \frac{||u_a - u_n||_\infty}{||u_a||_\infty}, \quad ||Erreurv||_{L_\infty} = \frac{||v_a - v_n||_\infty}{||v_a||_\infty}, \tag{5.2}
$$

where the pair (u_a,v_a) is the analytical solution (5.3) (see [11, page 581]) for the system (2.3) and the pair (u_n,v_n) is the computed solution (4.7) for system (2.3).

To simulate, we take the following exact solution for system (2.3) in over square domain $\Omega = [0, 1] \times [0, 1]$

$$
\begin{cases}
u_a(x, y, t) = \dfrac{3}{4} - \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]},\\
v_a(x, y, t) = \dfrac{3}{4} + \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]}.
\end{cases} \tag{5.3}
$$

Note that the initial and boundary conditions can be taken from the exact solutions. After computing, we evaluate respectively the relative errors (5.1) and (5.2). We use then the explicit and implicit schemes for the conformable time-derivative 2D heat equation and give the convergence of each scheme in the following Table 1 and Table 2.

Table 1. Relative errors L_1-norm.

Relative error	$\|\|Erreuru\|\|_{L_1}$		$\|\|Erreurv\|\|_{L_1}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.30e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.17e-03$	$2.17e-03$	$1.55e-03$	$1.55e-03$
$\Delta x = \Delta y = 0.05$	$1.46e-03$	$1.53e-03$	$8.19e-04$	$8.02e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.60e-03$	$5.64e-03$	$1.63e-03$	$1.58e-03$
$\Delta x = \Delta y = 0.1$	$4.69e-03$	$4.56e-03$	$1.58e-03$	$1.41e-03$
$\Delta x = \Delta y = 0.05$	$4.48e-03$	$4.52e-03$	$1.46e-03$	$1.37e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.85e-03$	$7.90e-03$	$1.43e-03$	$1.43e-03$
$\Delta x = \Delta y = 0.1$	$7.37e-03$	$7.47e-03$	$1.37e-03$	$1.31e-03$
$\Delta x = \Delta y = 0.05$	$7.26e-03$	$7.35e-03$	$1.06e-03$	$1.29e-03$

We remark that the relative error decreases as time increases in the Table 1.

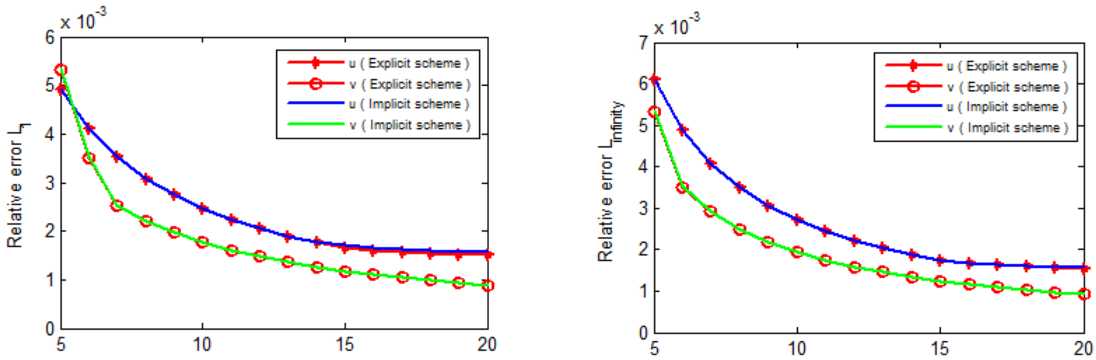

Figure 1. Graphs represent the tendency of the relative error .

We show through the Figure 1 the tendency of the relative errors. Let's give in the Figure 2 the graphs representing the numerical solution for 2D time-fractional heat equation (3.29) by using various values of α as shown in Table 3.

TABLE 2. Relative errors L_∞-norm.

Relative error	$\|\|Erreur u\|\|_{L_\infty}$		$\|\|Erreur v\|\|_{L_\infty}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.35e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.39e-03$	$2.39e-03$	$1.70e-03$	$1.70e-03$
$\Delta x = \Delta y = 0.05$	$1.52e-03$	$1.53e-03$	$8.29e-04$	$8.29e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.62e-03$	$5.64e-03$	$1.69e-03$	$1.69e-03$
$\Delta x = \Delta y = 0.1$	$4.76e-03$	$4.76e-03$	$1.57e-03$	$1.91e-03$
$\Delta x = \Delta y = 0.05$	$4.62e-03$	$4.62e-03$	$1.48e-03$	$1.87e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.88e-03$	$7.90e-03$	$1.68e-03$	$1.68e-03$
$\Delta x = \Delta y = 0.1$	$7.47e-03$	$7.47e-03$	$1.50e-03$	$1.51e-03$
$\Delta x = \Delta y = 0.05$	$7.34e-03$	$7.35e-03$	$1.47e-03$	$1.49e-03$

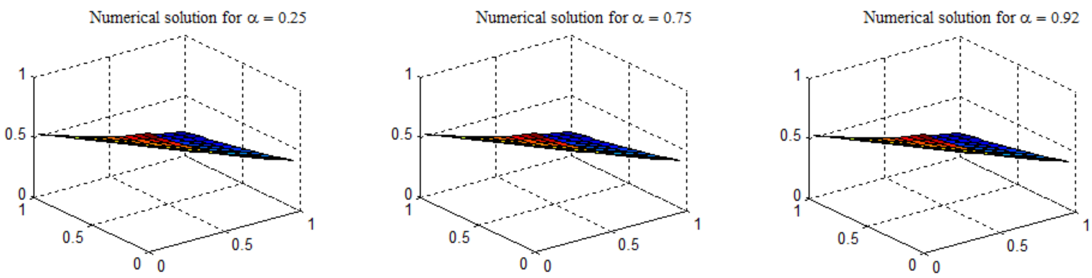

FIGURE 2. Graphs of the numerical solution for 2D time-fractional heat equation, for $r = 0.5, \Delta x = \Delta y = 0.08$ and $\alpha = 0.25, 0.75$ and 0.92.

TABLE 3. The numerical solutions ϕ of heat equation.

Values of α		$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.92$
x	y	Numerical solution ϕ	Numerical solution ϕ	Numerical solution ϕ
0.08	0.72	0.5984	0.5971	0.5969
0.96	0.32	0.3558	0.3550	0.3548
0.48	0.32	0.5339	0.5386	0.5384
0.88	0.64	0.3115	0.3108	0.3107
0.88	0.88	0.268	0.2674	0.2673
0.96	0.96	0.2377	0.2372	0.2371

In same way, we give the graphs of the exact and numerical solutions in Figure 3 for the system (2.3).

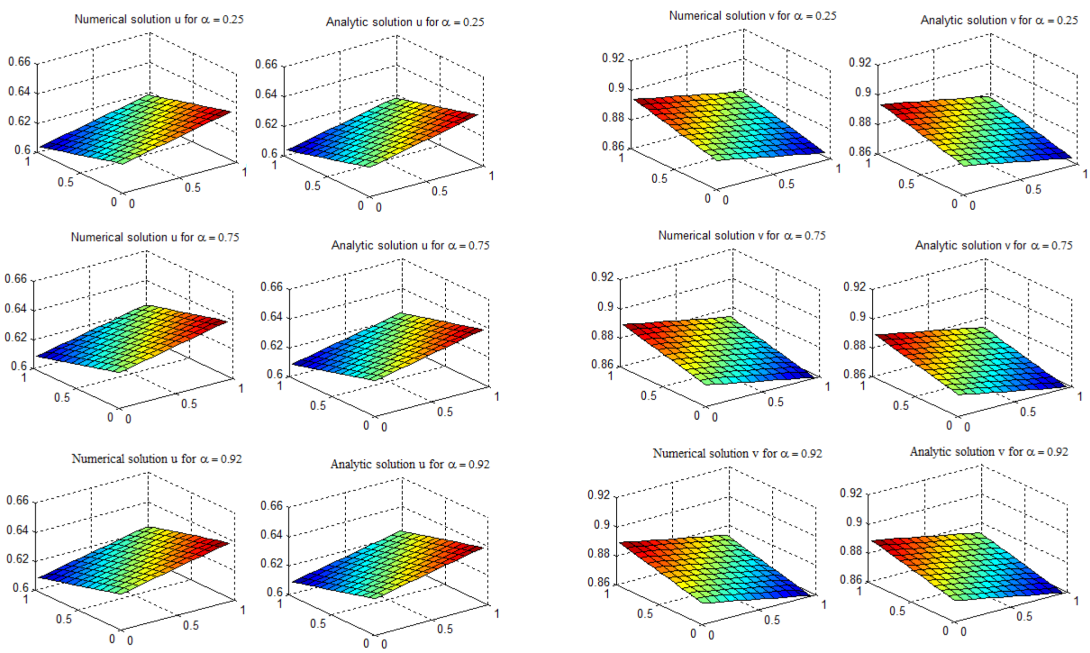

FIGURE 3. Graphs of exact and numerical solution for 2D time-fractional Burgers' equations, for $r = 0.5, \Delta x = \Delta y = 0.08$, and $\alpha = 0.25,\ 0.75$ and 0.92.

It is clear from the graphs that exact and approximate solutions are similar and compatible with each other. Tables 4 and 5 give the comparison of numerical and exact results for varying $\alpha = 0.75$ and 0.92. It is clear that the approximate solutions are accurate.

TABLE 4. Comparison between of the exact and numerical solutions u of the system (2.3).

Values of α		$\alpha = 0.75$		$\alpha = 0.92$	
x	y	Numerical solution	Exact solution	Numerical solution	Exact solution
0.08	0.72	0.6153	0.6149	0.6153	0.615
0.96	0.32	0.6351	0.6348	0.6352	0.6349
0.48	0.32	0.6278	0.6273	0.6278	0.6274
0.88	0.64	0.629	0.6286	0.629	0.6287
0.88	0.88	0.625	0.6248	0.6252	0.6249
0.96	0.96	0.6246	0.6248	0.6247	0.6249

TABLE 5. Comparison between of the exact and numerical solutions v of the system (2.3).

Values of α		$\alpha = 0.75$		$\alpha = 0.92$	
x	y	Numerical solution	Exact solution	Numerical solution	Exact solution
0.08	0.72	0.8857	0.8851	0.8857	0.885
0.96	0.32	0.8649	0.8652	0.8648	0.8651
0.48	0.32	0.8733	0.8727	0.8733	0.8726
0.88	0.64	0.8718	0.8714	0.8719	0.8713
0.88	0.88	0.8755	0.8752	0.8757	0.8751
0.96	0.96	0.8754	0.8752	0.8753	0.8751

6. CONCLUSION

In this study, we considered a coupling system of Burgers' equations with fractional conformable derivative in which involves nonlinearity and dissipation, it is relatively simple in contract with the Navier-Stokes system. It makes suitable model equations to test different numerical algorithms. For this purpose, we have used the Cole-Hopf transformation which shows its efficiency to deal with this class of fractional nonlinear problems. This approach is simple and effective and permits the comparison the obtained results with exact solution of the problem. In the future, we intend in first time to study some concrete examples that illustrate if the conformable derivative is capable or incapable of giving the fractional derivative obtainable from Riemann-Liouville or Caputo derivatives. In a second time, we want to apply such approach to other complex problems such as time-space diffusion equation of the type $\partial^\alpha u/\partial t^\alpha = -k(-\Delta)^\beta u$, where the α, β are changed into $\alpha(x,t), \beta(x,t)$.

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SOME k-FRACTIONAL INTEGRAL INEQUALITIES FOR p-CONVEX FUNCTIONS

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Abstract. In this paper, we use Riemann-Liouville k-fractional and k-fractional confomable integrals to prove Hermite-Hadamard inequality, an identity and Hermite-Hadamard type inequality for p-convex functions. Some special cases are also discussed. Our work is extensions of other related previous results.

1. Introduction

Convex functions have been used to investigate various scientific problems. Many refinements have been built for convex functions in order to study problems of pure and applied sciences (see [3, 4, 8, 14–16].)

The Hermite-Hadamard inequality [6, 7] for a convex function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ on an interval $\mathcal{H}$ is defined by

$$\mathcal{F}\left(\frac{h_1+h_2}{2}\right) \leq \frac{1}{h_2-h_1}\int_{h_1}^{h_2} \mathcal{F}(g)dg \leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}, \tag{1.1}$$

for all $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$. Due to extensive applicability of Hermite-Hadamard type inequalities and fractional integrals, number of researchers expand their research involving generalized fractional integrals for diverse classes of convex functions. For instance see [12, 13, 17–19, 23, 25, 26] etc.

Fractional integral inequalities are helpful in estimating the uniqueness of solutions for specific fractional partial differential equations. These inequalities also ensure upper and lower bounds for solutions of the fractional boundary value problems. Our

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Key words and phrases. Hermite-Hadamard inequality, p-convex function, Riemann-Liouville k-fractional integrals, k-fractional conformable integrals.

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aim is to prove several Hermite-Hadamard type inequalities for p-convex functions via Riemann-Liouville k-fractional and k-fractional confomable integrals.

2. Preliminaries

Here we give some basic definitions from the literature. For $k \in (0, \infty)$ and $h \in \mathbb{C}$, the k-gamma function is given by (see [1, 21])

$$\Gamma_k(h) = \lim_{n\to\infty} \frac{n! k^n n k^{\frac{h}{k}-1}}{h_{n,k}}$$

in terms of

$$\tau_{n,k} = \begin{cases} 1, & n = 0, \\ \tau(\tau + k)\cdots(\tau + (n-1)k), & n \in \mathbb{N}, \end{cases}$$

where the integral representaion of $\Gamma_k(\cdot)$ is given as:

$$\Gamma_k(\beta) = \int_0^\infty t^{\beta-1} e^{-\frac{t^k}{k}} dt.$$

Definition 2.1 ([11])**.** Let $\mathcal{F} \in L_1[h_1, h_2]$. The left and right sided Riemann-Liouville fractional integrals $J^{\alpha}_{h_1+}\mathcal{F}$ and $J^{\alpha}_{h_2-}\mathcal{F}$ of order $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,(\alpha) > 0$ and $h_2 > h_1 \geq 0$ are defined by

$$J^{\alpha}_{h_1+}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)} \int_{h_1}^{g} (g-t)^{\alpha-1}\mathcal{F}(t)dt, \quad g > h_1, \tag{2.1}$$

and

$$J^{\alpha}_{h_2-}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)} \int_{g}^{h_2} (t-g)^{\alpha-1}\mathcal{F}(t)dt, \quad g < h_2, \tag{2.2}$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

Mubeen and Habibullah [20] defined the following generalized fractional integrals.

Definition 2.2 ([20])**.** Let $\mathcal{F} \in L_1[h_1, h_2]$. The left and right sided Riemann-Liouville k-fractional integrals $J^{\alpha}_{k,h_1+}\mathcal{F}$ and $J^{\alpha}_{k,h_2-}\mathcal{F}$ of order $\alpha \in \mathbb{C}$ and $h_2 > h_1 \geq 0$ are defined by

$$J^{\alpha}_{k,h_1+}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{h_1}^{g} (g-t)^{\alpha/k-1}\mathcal{F}(t)dt, \quad g > h_1, \tag{2.3}$$

and

$$J^{\alpha}_{k,h_2-}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{g}^{h_2} (t-g)^{\alpha/k-1}\mathcal{F}(t)dt, \quad g < h_2, \tag{2.4}$$

respectively, with $\mathrm{Re}\,(\alpha)$, $k > 0$.

Definition 2.3 ([10])**.** Let $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) > 0$, then the left and right sided fractional conformable integral operators are characterised as:

$$
\begin{aligned}
&(2.5) \qquad {}^{\beta}_{h_1}\mathcal{J}^{\alpha}\mathcal{F}(g) = \frac{1}{\Gamma(\beta)} \int_{h_1}^{g} \left(\frac{(g-h_1)^{\alpha} - (t-h_1)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(t)}{(t-h_1)^{1-\alpha}} dt,\\
&(2.6) \qquad {}^{\beta}\mathcal{J}^{\alpha}_{h_2}\mathcal{F}(g) = \frac{1}{\Gamma(\beta)} \int_{g}^{h_2} \left(\frac{(h_2-g)^{\alpha} - (h_2-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathcal{F}(t)}{(h_2-t)^{1-\alpha}} dt,
\end{aligned}
$$

respectively, with $\alpha > 0$.

Qi et al. [22] defined k-fractional conformable fractional integrals as follows.

Definition 2.4 ([22])**.** Let $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) > 0$, then the left and right sided k-fractional conformable integrals are characterised as:

$$
\begin{aligned}
&(2.7) \qquad {}^{\beta}_{k,h_1}\mathcal{J}^{\alpha}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\beta)} \int_{h_1}^{g} \left(\frac{(g-h_1)^{\alpha} - (t-h_1)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathcal{F}(t)}{(t-h_1)^{1-\alpha}} dt,\\
&(2.8) \qquad {}^{\beta}\mathcal{J}^{\alpha}_{k,h_2}\mathcal{F}(g) = \frac{1}{k\Gamma_k(\beta)} \int_{g}^{h_2} \left(\frac{(h_2-g)^{\alpha} - (h_2-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathcal{F}(t)}{(h_2-t)^{1-\alpha}} dt,
\end{aligned}
$$

respectively, with $\alpha, k > 0$.

Definition 2.5 ([8])**.** Consider an interval $\mathcal{H} \subset (0,\infty)$ and $p \in \mathbb{R}\backslash\{0\}$. A function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ is called p-convex if

$$
\mathcal{F}\left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \le r\mathcal{F}(h_1) + (1-r)\mathcal{F}(h_2), \tag{2.9}
$$

for all $h_1, h_2 \in \mathcal{H}$ and $r \in [0,1]$. If (2.9) is reversed then $\mathcal{F}$ is called p-concave.

3. Inequalities for k-Fractional Integrals

First we prove the k-fractional Hadamard's inequality for p-convex function.

Theorem 3.1. *Let $\mathcal{F} : [h_1, h_2] \subset (0,\infty) \to \mathbb{R}$ be a p-convex function such that $\mathcal{F} \in L_1[h_1, h_2]$. Then*

(i) for $p > 0$ we have

$$
\begin{aligned}
\mathcal{F}\left(\left[\frac{h_1^p + h_2^p}{2} \right]^{1/p} \right) &\le \frac{\Gamma_k(\alpha+k)}{2(h_2^p - h_1^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p) \right]\\
&\le \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2},
\end{aligned} \tag{3.1}
$$

where $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_1^p, h_2^p]$;

(ii) for $p < 0$ we have

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(h_1^p-h_2^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p-}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p+}(\mathcal{F}\circ\mu)(h_1^p)\right] \leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}, \tag{3.2}$$

where $\mu(g) = g^{\frac{1}{p}}$, $g \in [h_2^p, h_1^p]$.

Proof. Since $\mathcal{F}$ is p-convex on $[h_1, h_2]$, we get

$$\mathcal{F}\left(\left[\frac{u^p+w^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\mathcal{F}(u)+\mathcal{F}(w)}{2}.$$

Taking $u^p = rh_1^p + (1-r)h_2^p$ and $w^p = (1-r)h_1^p + rh_2^p$ with $r \in [0,1]$, we get

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)}{2}. \tag{3.3}$$

Multiplying (3.3) by $r^{\frac{\alpha}{k}-1}$ on both sides with $r \in (0,1)$, $\alpha > 0$, and then integrating along r over $r \in [0,1]$ and using changes of variable, we obtain

$$\begin{aligned}
&\frac{2k}{\alpha}\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\leq \int_0^1 r^{\frac{\alpha}{k}-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr + \int_0^1 r^{\frac{\alpha}{k}-1}\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr\\
&= \int_{h_2^p}^{h_1^p}\left(\frac{h_2^p-w}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_1^p-h_2^p} + \int_{h_1^p}^{h_2^p}\left(\frac{z-h_1^p}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}(\mathcal{F}\circ\mu)(z)\frac{dz}{h_2^p-h_1^p}\\
&= \frac{k\Gamma_k(\alpha)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}}}[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)],
\end{aligned}$$

that is,

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right]. \tag{3.4}$$

This completes the left inequality of (3.1). For the right inequality, we consider

$$\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right) \leq [\mathcal{F}(h_1)+\mathcal{F}(h_2)]. \tag{3.5}$$

Multiplying (3.5) by $r^{\frac{\alpha}{k}-1}$ on both sides with $r \in (0,1)$, $\alpha > 0$, and then integrating along r over $\in [0,1]$ and using changes of variable, we obtain

$$\frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right] \leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}. \tag{3.6}$$

This completes the second inequality of (3.1). Hence, from (3.4) and (3.6), we get (3.1).

(ii) The proof is analogous to (i). □

Remark 3.1. In Theorem 3.1

(i) if $p = 1$, then the inequality (3.1) becomes the inequality (2.1) of Theorem 2.1 in [5];

(ii) if one takes $\alpha = k = 1$, then the inequality (3.1) becomes the inequality (1.11) of Theorem 6 in [8];

(iii) if one takes $k = p = 1$, then the inequality (3.1) becomes the inequality (2.1) of Theorem 2 in [23];

(iv) if one takes $\alpha = k = p = 1$, then the inequality (3.1) becomes the inequality (1.1).

Lemma 3.1. *Consider a differentiable mapping $\mathcal{F} : [h_1, h_2] \to \mathbb{R}$ on (h_1, h_2) with $h_1 < h_2$. If $\mathcal{F}' \in L_1[h_1, h_2]$, then the following equality holds.*

(i) For $p > 0$

$$\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right] \tag{3.7}$$
$$=\frac{h_2^p-h_1^p}{2p}\int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr,$$

where $A_r^{\frac{1}{p}-1} = [rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}$ and $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_1^p, h_2^p]$;

(ii) For $p < 0$

$$\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p-}(\mathcal{F}\circ\mu)(h_2^p) + J^{\alpha}_{k,h_2^p+}(\mathcal{F}\circ\mu)(h_1^p)\right] \tag{3.8}$$
$$=\frac{h_1^p-h_2^p}{2p}\int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)B_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr,$$

where $B_r^{\frac{1}{p}-1} = [rh_2^p+(1-r)k_1^p]^{\frac{1}{p}}$, $\mu(g) = g^{\frac{1}{p}}$ for all $g \in [h_2^p, h_1^p]$.

Proof. First consider

$$\begin{aligned} I =& \int_0^1\left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right)A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \\ =& \left[\int_0^1 (1-r)^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\right] \\ &+ \left[-\int_0^1 r^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\right] \\ =& I_1 + I_2. \end{aligned} \tag{3.9}$$

Integrating by parts, we obtain

$$I_1 = \int_0^1 (1-r)^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{3.10}$$

$$
\begin{aligned}
=&\frac{p(1-r)^{\frac{\alpha}{k}}}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1\\
&+\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\alpha(1-r)^{\frac{\alpha}{k}-1}}{k}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_2)-\frac{\alpha p}{k(h_1^p-h_2^p)}\int_{h_2^p}^{h_1^p}\left(\frac{h_1^p-w}{h_1^p-h_2^p}\right)^{\frac{\alpha}{k}-1}\frac{(\mathcal{F}\circ\mu)(w)}{h_1^p-h_2^p}dw\\
=&\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_2)-\frac{p\Gamma_k(\alpha+k)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}+1}}J^{\alpha}_{h_2^p-}(\mathcal{F}\circ\mu)(h_1^p).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2=&-\int_0^1 r^{\frac{\alpha}{k}}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \qquad (3.11)\\
=&-\frac{pr^{\frac{\alpha}{k}}}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1\\
&+\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\alpha r^{\frac{\alpha}{k}-1}}{k}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
=&\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_1)-\frac{\alpha p}{k(h_2^p-h_1^p)}\int_{h_2^p}^{h_1^p}\left(\frac{h_2^p-w}{h_2^p-h_1^p}\right)^{\frac{\alpha}{k}-1}\frac{(\mathcal{F}\circ\mu)(w)}{h_1^p-h_2^p}dw\\
=&\frac{p}{h_2^p-h_1^p}\mathcal{F}(h_1)-\frac{p\Gamma_k(\alpha+k)}{(h_2^p-h_1^p)^{\frac{\alpha}{k}+1}}J^{\alpha}_{h_1^p+}(\mathcal{F}\circ\mu)(h_2^p).
\end{aligned}
$$

Using (3.10) and (3.11) in (3.9) and then multiplying $\frac{h_2^p-h_1^p}{2p}$ on both sides, we get (3.7).

(ii) Proof is analogous to part (i). □

Remark 3.2. In Lemma 3.1

(i) if $p=1$, then the identity (3.7) becomes the identity (2.6) of Lemma 2.3 in [5];

(ii) if one takes $\alpha=k=1$, then the identity (3.7) becomes the identity (1.12) of Lemma 3 in [8];

(iii) if one takes $k=p=1$, then the identity (3.7) becomes the identity (3.1) of Lemma 2 in [23];

(iv) if one takes $\alpha=k=p=1$, then the identity (3.7) becomes the identity (2.1) of Lemma 2.1 in [2].

Theorem 3.2. *Consider a differentiable mapping $\mathcal{F}:[h_1,h_2]\to\mathbb{R}$ on (h_1,h_2) with $h_1<h_2$ such that $\mathcal{F}'\in L_1[h_1,h_2]$. If $|\mathcal{F}'|^q$ is p-convex on $[h_1,h_2]$ with $q\geq 1$, then the following inequality holds:*

(i) *for $p>1$*

$$
\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(h_2^p-h_1^p)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{k,h_1^p+}(\mathcal{F}\circ\mu)(h_2^p)+J^{\alpha}_{k,h_2^p-}(\mathcal{F}\circ\mu)(h_1^p)\right]\right| \tag{3.12}
$$

$$\leq \frac{k^{\frac{1}{q}}(h_2^p - h_1^p)}{2p} Q_1^{1-\frac{1}{q}} \left(\frac{|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q}{\alpha + k} \right)^{\frac{1}{q}},$$

where $Q_1 = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right)$;

(*ii*) *for* $p < 1$

$$\left| \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha + k)}{2(h_1^p - h_2^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p-}(\mathcal{F} \circ \mu)(h_2^p) + J^{\alpha}_{k,h_2^p+}(\mathcal{F} \circ \mu)(h_1^p) \right] \right| \tag{3.13}$$

$$\leq \frac{k^{\frac{1}{q}}(h_1^p - h_2^p)}{2p} Q_2^{1-\frac{1}{q}} \left(\frac{|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q}{\alpha + k} \right)^{\frac{1}{q}},$$

where $Q_2 = \frac{h_2^{p-1}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_2^p}{h_1^p}\right)$.

Proof. Using Lemma 3.1 and p-convexity of $|\mathcal{F}'|$, we get

(3.14)

$$\left| \frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2} - \frac{\Gamma_k(\alpha + k)}{2(h_2^p - h_1^p)^{\frac{\alpha}{k}}} \left[J^{\alpha}_{k,h_1^p+}(\mathcal{F} \circ \mu)(h_2^p) + J^{\alpha}_{k,h_2^p-}(\mathcal{F} \circ \mu)(h_1^p) \right] \right|$$

$$= \left| \frac{h_2^p - h_1^p}{2} \int_0^1 \left((1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}} \right) A_r^{\frac{1}{p}-1} \mathcal{F}'\left([r h_1^p + (1-r) h_2^p]^{\frac{1}{p}} \right) dr \right|$$

$$\leq \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \left(\int_0^1 \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) \left| \mathcal{F}'\left([r h_1^p + (1-r) h_2^p]^{\frac{1}{p}} \right) \right|^q dr \right)^{\frac{1}{q}}$$

$$\leq \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \left(\int_0^1 \left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) [r|\mathcal{F}'(h_1)|^q + (1-r)|\mathcal{F}'(h_2)|^q] dr \right)^{\frac{1}{q}}$$

$$= \frac{h_2^p - h_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$

$$\times \left(|\mathcal{F}'(h_1)|^q \int_0^1 r\left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) + |\mathcal{F}'(h_2)|^q \int_0^1 (1-r)\left((1-r)^{\frac{\alpha}{k}} + r^{\frac{\alpha}{k}} \right) dr \right)^{\frac{1}{q}}.$$

Since

$$\int_0^1 A_r^{\frac{1}{p}-1} dr = \frac{h_2^{1-p}}{2}\ {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right), \tag{3.15}$$

$$\int_0^1 r(1-r)^{\frac{\alpha}{k}} dr = \int_0^1 (1-r) r^{\frac{\alpha}{k}} dr = \frac{k^2}{(\alpha + k)(\alpha + 2k)} \tag{3.16}$$

and

$$\int_0^1 r^{\frac{\alpha}{k}+1} dr = \int_0^1 (1-r)^{\frac{\alpha}{k}+1} dr = \frac{k}{\alpha + 2k}, \tag{3.17}$$

by using (3.15)−(3.17) in (3.14), we get (3.12). Hence, theorem is proved.

(*ii*) Proof is analogous to part (*i*). □

By taking $p=-1$ in Theroem 3.1, Lemma 3.1 and Theorem 3.2, one can get new results for harminically convex functions via k-fractional integrals.

4. Inequalities for k-Fractional Conformable Integrals

Here our aim is to prove Hadamard's inequalities for p-convex function via k-fractional conformable integrals.

Theorem 4.1. *Let $\mathcal{F}:[h_1,h_2]\subset(0,\infty)\to\mathbb{R}$ be a p-convex function such that $\mathcal{F}\in L_1[h_1,h_2]$.*

(i) Then for $p>0$ we have

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right)\le\frac{\alpha^{\beta/k}\Gamma(\beta+k)}{2(h_2^p-h_1^p)^{\alpha\beta/k}}\left[{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)+{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\right]\le\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2},\tag{4.1}$$

where $\mu(g)=g^{\frac{1}{p}}$ for all $g\in[h_1^p,h_2^p]$.

(ii) Then for $p<0$ we have

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{1/p}\right)\le\frac{\alpha^{\beta/k}\Gamma(\beta+k)}{2(h_1^p-h_2^p)^{\alpha\beta/k}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_1^p}(\mathcal{F}\circ\mu)(h_2^p)+{}^{\beta}_{k,h_2^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_1^p)\right]\le\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2},\tag{4.2}$$

where $\mu(g)=g^{\frac{1}{p}}$, $g\in[h_2^p,h_1^p]$.

Proof. Since $\mathcal{F}$ is p-convex on $[h_1,h_2]$, we can have

$$\mathcal{F}\left(\left[\frac{x^p+u^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{\mathcal{F}(x)+\mathcal{F}(u)}{2}.$$

Taking $x^p=rh_1^p+(1-r)h_2^p$ and $u^p=(1-r)h_1^p+rh_2^p$ with $r\in[0,1]$, we get

$$\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)}{2}.\tag{4.3}$$

Multiplying (4.3) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}$ on both sides with $r\in(0,1)$, $\alpha>0$, and then integrating along r over $r\in[0,1]$, we obtain

$$2\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}dr\le\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\tag{4.4}$$

$$
\begin{aligned}
&+\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)dr\\
=&I_1+I_2.
\end{aligned}
$$

By setting $w=rh_1^p+(1-r)h_2^p$, we have

$$
\begin{aligned}
I_1&=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr\\
&=\int_{h_2^p}^{h_1^p}\left(\frac{1-\left(\frac{w-h_2^p}{h_1^p-h_2^p}\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{w-h_2^p}{h_1^p-h_2^p}\right)^{\alpha-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_1^p-h_2^p}\\
&=\frac{1}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\int_{h_1^p}^{h_2^p}\left(\frac{(h_2^p-h_1^p)^\alpha-(h_2^p-w)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(h_2^p-w)^{\alpha-1}(\mathcal{F}\circ\mu)(w)dw\\
&=\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p).
\end{aligned}\tag{4.5}
$$

Similarly, by setting $w=rh_2^p+(1-r)h_1^p$, we have

$$
\begin{aligned}
I_2&=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}\mathcal{F}\left([(1-r)h_1^p+rh_2^p]^{\frac{1}{p}}\right)dr\\
&=\int_{h_1^p}^{h_2^p}\left(\frac{1-\left(\frac{w-h_1^p}{h_2^p-h_1^p}\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{w-h_1^p}{h_2^p-h_1^p}\right)^{\alpha-1}(\mathcal{F}\circ\mu)(w)\frac{dw}{h_2^p-h_1^p}\\
&=\frac{1}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\int_{h_1^p}^{h_2^p}\left(\frac{(h_2^p-h_1^p)^\alpha-(w-h_1^p)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(w-h_1^p)^{\alpha-1}(\mathcal{F}\circ\mu)(w)dw\\
&=\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p).
\end{aligned}\tag{4.6}
$$

Also, we have

$$\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}r^{\alpha-1}dr=\frac{k}{\beta\alpha^{\beta/k}}.$$

Thus, by putting values of I_1 and I_2 in (4.4), we get

$$\frac{k}{\alpha^{\beta/k}\beta}\mathcal{F}\left(\left[\frac{h_1^p+h_2^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{k\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\alpha\beta/k}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)+{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\right].\tag{4.7}$$

This completes the first inequality of (4.1). For second inequality, we know that

$$\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)+\mathcal{F}\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)\le[\mathcal{F}(h_1)+\mathcal{F}(h_2)].\tag{4.8}$$

Multiplying (4.8) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k-1} r^{\alpha-1}$ on both sides with $r \in (0,1)$, $\alpha > 0$, and then integrating with respect to r on interval $[0,1]$, we obtain the following inequality

$$\frac{k\Gamma_k(\beta)}{(h_2^p - h_1^p)^{\alpha\beta/k}} \left[{}^{\beta}\mathfrak{J}^{\alpha}_{h_2^p}(\mathcal{F}\circ\mu)(h_1^p) + {}^{\beta}_{h_1^p}\mathfrak{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\right] \leq \frac{k}{\alpha^{\beta/k}\beta}(\mathcal{F}(h_1)+\mathcal{F}(h_2)). \tag{4.9}$$

This completes the second inequality of (4.1). Hence, the proof is completed.

(ii) The proof is parallel to (i). □

Remark 4.1. In Theorem 4.1

(i) if we take $k = 1$, then we get Thoerem 2.1 in [18];

(ii) by letting $p = k = 1$, we find Theorem 2.1 in [24];

(iii) by letting $p = k = 1$ and $\alpha = 1$, we obtain Theorem 2 in [23];

(iv) by letting $p = -1$ and $k = \alpha = 1$, we get Theorem 4 in [9].

Corollary 4.1. *With the parallel assumption of Theorem* 4.1*, if we take* $p = -1$*, then we get the following inequality*

(4.10)
$$\begin{aligned}\mathcal{F}\left(\frac{2h_1h_2}{h_1+h_2}\right) &\leq \frac{(h_1h_2)^{\frac{\alpha\beta}{k}}\alpha^{\beta/k}\Gamma_k(\beta+k)}{2(h_2-h_1)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{k,1/h_1}(\mathcal{F}\circ\mu)\left(\frac{1}{h_2}\right) + {}^{\beta}_{k,1/h_2}\mathfrak{J}^{\alpha}(\mathcal{F}\circ\mu)\left(\frac{1}{h_1}\right)\right] \\ &\leq \frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2},\end{aligned}$$

where $\mu(g) = \frac{1}{g}$, $g \in \left[\frac{1}{h_2}, \frac{1}{h_1}\right]$.

Lemma 4.1. *Let* $\mathcal{F} : [h_1, h_2] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* (h_1, h_2) *with* $h_1 < h_2$ *such that* $\mathcal{F}' \in L_1[h_1, h_2]$*, then we have*

(i) *for* $p > 0$

$$\begin{aligned}&\left(\frac{\mathcal{F}(h_1^p)+\mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k,h_1^p}\mathfrak{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p) + {}^{\beta}\mathfrak{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\right] \\ &= \frac{(h_2^p-h_1^p)\alpha^{\beta/k}}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right] \\ &\quad \times A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr,\end{aligned} \tag{4.11}$$

where $A_r = [rh_1^p + (1-r)h_2^p]$;

(ii) *for* $p < 0$

$$\begin{aligned}&\left(\frac{\mathcal{F}(h_1^p)+\mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(h_1^p-h_2^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{k,h_1^p}(\mathcal{F}\circ\mu)(h_2^p) + {}^{\beta}_{k,h_2^p}\mathfrak{J}^{\alpha}(\mathcal{F}\circ\mu)(h_1^p)\right] \\ &= \frac{(h_1^p-h_2^p)\alpha^{\beta/k}}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right]\end{aligned} \tag{4.12}$$

$$\times B_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_2^p+(1-r)h_1^p]^{\frac{1}{p}}\right)dr,$$

where $B_r=[rh_2^p+(1-r)h_1^p]$.

Proof. (i) Consider

$$\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\right]A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{4.13}$$
$$=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$
$$-\int_0^1\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$
$$=I_1-I_2.$$

Then applying by parts integration, we achieve

$$I_1=\int_0^1\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{4.14}$$
$$=\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k}\frac{p}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1$$
$$-\frac{p}{h_2^p-h_1^p}\int_0^1\frac{\beta}{k}\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta/k-1}r^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$
$$=\frac{p}{\alpha^{\beta/k}(h_2^p-h_1^p)}\mathcal{F}(h_2^p)-\frac{p\beta}{(h_2^p-h_1^p)}\frac{\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\alpha\beta}}\,{}^{\beta}\mathcal{J}^{\alpha}_{h_2^p}(\mathcal{F}\circ\mu)(h_1^p)$$
$$=\frac{p}{h_2^p-h_1^p}\left[\frac{\mathcal{F}(h_2^p)}{\alpha^{\beta/k}}-\frac{\Gamma_k(\beta+k)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F}\circ\mu)(h_1^p)\right].$$

Similarly,

$$I_2=\int_0^1\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}A_r^{\frac{1}{p}-1}\mathcal{F}'\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr \tag{4.15}$$
$$=\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\beta/k}\frac{p}{h_1^p-h_2^p}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)\Big|_0^1$$
$$-\frac{p}{h_1^p-h_2^p}\int_0^1\frac{\beta}{k}\left(\frac{1-(1-r)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-r)^{\alpha-1}\mathcal{F}\left([rh_1^p+(1-r)h_2^p]^{\frac{1}{p}}\right)dr$$
$$=-\frac{p}{\alpha^{\beta/k}(h_2^p-h_1^p)}\mathcal{F}(h_1^p)+\frac{p\beta}{h_2^p-h_1^p}\frac{\Gamma_k(\beta)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}_{h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)$$
$$=-\frac{p}{h_2^p-h_1^p}\left[\frac{\mathcal{F}(h_2^p)}{\alpha^{\beta/k}}-\frac{\Gamma_k(\beta+k)}{(h_2^p-h_1^p)^{\frac{\alpha\beta}{k}}}\,{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F}\circ\mu)(h_2^p)\right].$$

Here we apply change of variable by taking $w = 1 - r$. Hence, adding I_1, $-I_2$ and then by multiplying by $\frac{\alpha^{\beta/k}(h_2^p - h_1^p)}{2p}$, on both sides, we get (4.11).

(ii) The proof is similar to (i). □

Remark 4.2. In Lemma 4.1

(i) by letting $k = 1$, then one gets Lemma 2.4 in [18];

(ii) by letting $p = k = 1$, then one gets Lemma 3.1 in [24];

(iii) by letting $p = k = 1$ and $\alpha = 1$, then one gets Lemma 2 in [23].

Theorem 4.2. *Let $\mathcal{F} : [h_1, h_2] \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on (h_1, h_2), $h_1 < h_2$, such that $\mathcal{F}' \in 1[h_1, h_2]$. If $|\mathcal{F}'|^q$, where $q \geq 1$, is p-convex, then*

(i) for $p > 0$

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta + k)\alpha^{\beta/k}}{2(h_2^p - h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{k,h_1^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{k,h_2^p}(\mathcal{F} \circ \mu)(h_1^p)\right]\right| \tag{4.16}$$
$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p}\left(\frac{h_2^{1-p}}{2}\, {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)\left[|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q\right]\right)^{q};$$

(ii) for $p < 0$

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma(\beta + 1)\alpha^{\beta/k}}{2(h_1^p - h_2^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{k,h_1^p}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}_{k,h_2^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_1^p)\right]\right| \tag{4.17}$$
$$\leq \frac{(h_1^p - h_2^p)\alpha^{\beta/k}}{2p}\left(\frac{h_1^{1-p}}{2}\, {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_2^p}{h_1^p}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)\left[|\mathcal{F}'(h_1)|^q + |\mathcal{F}'(h_2)|^q\right]\right)^{q},$$

where B and ${}_2F_1$ are classical Beta and Hypergeometric functions, respectively.

Proof. Applying Lemma 4.1, modulus property, Hölder's inequality and p-convexity of $|\mathcal{F}'|^q$, we achieve

$$\left|\left(\frac{\mathcal{F}(h_1^p) + \mathcal{F}(h_2^p)}{2}\right) - \frac{\Gamma_k(\beta + k)\alpha^{\beta/k}}{2(h_2^p - h_1^p)^{\frac{\alpha\beta}{k}}}\left[{}^{\beta}_{h_1^p}\mathcal{J}^{\alpha}(\mathcal{F} \circ \mu)(h_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{h_2^p}(\mathcal{F} \circ \mu)(h_1^p)\right]\right| \tag{4.18}$$
$$= \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p}\left|\int_0^1 \left[\left(\frac{1 - r^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1 - (1 - r)^\alpha}{\alpha}\right)^{\beta/k}\right]\right.$$
$$\left.\times A_r^{\frac{1}{p}-1}\mathcal{F}'\left([r h_1^p + (1 - r)h_2^p]^{\frac{1}{p}}\right) dr\right|$$

$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left| \int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] \right.$$
$$\left. \times A_r^{\frac{1}{p}-1} \mathcal{F}' \left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \right| dr$$
$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] \left| \mathcal{F}' \left([rh_1^p + (1-r)h_2^p]^{\frac{1}{p}} \right) \right|^q dr \right)^{1/q}$$
$$\leq \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] (r|\mathcal{F}'(h_1)|^q + (1-r)|\mathcal{F}'(h_2)|^q) \, dr \right)^{1/q}$$
$$= \frac{(h_2^p - h_1^p)\alpha^{\beta/k}}{2p} \nu^{1-\frac{1}{q}} \left(|\mathcal{F}'(h_1)|^q \int_0^1 \left[r \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] dr \right.$$
$$\left. + |\mathcal{F}'(h_2)|^q \int_0^1 \left[(1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} + (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} \right] dr \right)^{1/q},$$

where

$$\nu = \int_0^1 A_r^{\frac{1}{p}-1} dr = \frac{h_2^{1-p}}{2} \, {}_2F_1 \left(1 - \frac{1}{p}, 1; 2; 1 - \frac{h_1^p}{h_2^p} \right),$$

and from changes of variables, $x = r^\alpha$ and $y = (1-r)^\alpha$, we find

$$\int_0^1 r \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right),$$
$$\int_0^1 r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} \left[B \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right],$$
$$\int_0^1 (1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} \left[B \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right],$$
$$\int_0^1 (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^{\beta/k} dr = \frac{1}{\alpha^{\frac{\beta}{k}+1}} B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right).$$

Thus, by using above equalities in (4.18), we obtain the inequality (4.16).

(ii) Proof is similar to (i). □

Remark 4.3. In Theorem 4.2, if we take $k = 1$, then we get Thoerem 2.6 in [18].

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INTEGRAL TRANSFORMS AND EXTENDED HERMITE-APOSTOL TYPE FROBENIUS-GENOCCHI POLYNOMIALS

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Abstract. The schemata for applications of the integral transforms of mathematical physics to recurrence relations, differential, integral, integro-differential equations and in the theory of special functions has been developed. The article aims to introduce and present operational representations for a new class of extended Hermite-Apostol type Frobenius-Genocchi polynomials via integral transforms. The recurrence relations and some identities involving these polynomials are established. The article concludes by establishing a determinant form and quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials and for their extended forms.

1. Introduction and preliminaries

The convolution of two or more polynomials in order to introduce the new multi-variable generalized polynomials is a topic of research and is useful from the point of view of applications. These polynomials are important as they possess significant properties including the recurrence and explicit relations, functional and differential equations, summation formulae, symmetric and convolution identities, determinant forms *et cetera.* The usefulness and potential for applications of various properties of multi-variable hybrid special polynomials in certain problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics has given motivation for introducing many new classes of hybrid polynomials.

Key words and phrases. Quasi-monomiality, extended Hermite-Apostol type Frobenius-Genocchi polynomials, fractional operators, operational rules.

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The properties and applications of the hybrid polynomials lie within the parent polynomials. The applications of hybrid Legendre polynomials lie in problems dealing with either gravitational potentials or electrostatic potentials. The hybrid polynomials involving Hermite polynomials occur in quantum mechanical and optical beam transport problems and in probability theory. The hybrid polynomials related with truncated-exponential polynomials appear in the theory of flattened beams, which plays an importance in optics and in super-Gaussian optical resonators and hybrid polynomials associated with Laguerre polynomials occur in physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings.

Certain new classes of hybrid special polynomials associated with the Appell sequences were introduced and studied by Khan et al. [13, 14]. The problems arising in different areas of science and engineering are usually expressed in terms of differential equations which in most of the cases have special functions as their solutions. The differential equations satisfied by the hybrid special polynomials may be used to express the problems arising in new and emerging areas of sciences.

Various forms of the Apostol type polynomials are the generalizations of the Appell family [2]. The Appell polynomial sequences appear in different applications in pure and applied mathematics. These sequences arise in theoretical physics, chemistry [7, 23] and several branches of mathematics [18] such as the study of polynomial expansions of analytic functions, number theory and numerical analysis. The typical examples of Appell polynomial sequences are the Bernoulli, Euler and Genocchi polynomials. These polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis and can be defined by various methods depending on the applications.

Several interesting results related to Frobenius type polynomials and their hybrid forms were obtained by many authors, see [12, 17], which are important from applications point of view. The hybrid class of 3-variable Hermite-Apostol type Frobenius-Genocchi polynomials was introduced in [5] by considering the discrete convolution of the Apostol type Frobenius-Genocchi polynomials $\mathcal{H}_n(x;\lambda;u)$ [5] with the 3-variable Hermite polynomials $H_n(x,y,z)$ [10].

The Apostol type Frobenius-Genocchi polynomials and the 3-variable Hermite polynomials are defined by

$$\left(\frac{(1-u)t}{\lambda e^t - u}\right) e^{xt} = \sum_{n=0}^{\infty} \mathcal{H}_n(x;\lambda;u)\frac{t^n}{n!}, \quad u,\lambda \in \mathbb{C},\, u \neq 1 \tag{1.1}$$

which for $x = 0$ gives the Apostol type Frobenius-Euler numbers $\mathcal{H}_n(u;\lambda)$ and

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!}, \tag{1.2}$$

which for $z=0$ reduce to the 2-variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ [3] and for $z=0,\ x=2x$ and $y=-1$ become the classical Hermite polynomials $H_n(x)$ [1], respectively.

The 3-variable Hermite-Apostol type Frobenius-Genocchi polynomials [5] are defined by means of the following generating function and series expansion:

$$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y,z;\lambda;u)\frac{t^n}{n!},\quad u,\lambda\in\mathbb{C},\ u\neq 1, \tag{1.3}$$

$$_H\mathcal{H}_n(x,y,z;\lambda;u)=n!\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^rH_{k-3r}(x,y)}{(n-k)!r!(k-3r)!}. \tag{1.4}$$

Next, we present certain special cases of $_H\mathcal{H}_n(x,y,z;\lambda;u)$ in Table 1.

TABLE 1. Special cases of $_H\mathcal{H}_n(x,y,z;\lambda;u)$

S.No.	**Cases**	**Name of polynomial**	**Generating function**
I	$\lambda=1$	Hermite Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y,z;u)\frac{t^n}{n!}$
	$u=-1,$ $\lambda=1$	Hermite-Genocchi polynomials [4]	$\left(\frac{2t}{e^t+1}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_HG_n(x,y,z)\frac{t^n}{n!}$
II	$z=0$	2-variable Hermite-Apostol type Frobenius-Genocchi polynomials [5]	$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y;u;\lambda)\frac{t^n}{n!}$
	$z=0,$ $\lambda=1$	2-variable Hermite-Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y;u)\frac{t^n}{n!}$
III	$x=2x,$ $y=-1,\ z=0$	Hermite-Apostol type Frobenius-Genocchi polynomials [5]	$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x;\lambda;u)\frac{t^n}{n!}$
	$x=2x,\ y=-1$ $z=0,\ \lambda=1$	Hermite-Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x;u)\frac{t^n}{n!}$

Fractional calculus is one of the most intensively developing areas of mathematical analysis. Its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. Integration to an arbitrary order named fractional calculus has a long history. The idea of non-integral order of integration is drawn back to the origin of differential calculus. The Newton's rival Leibnitz made some assertions on the meaning and possibility of fractional derivative of order 1/2 in the end of 17th century. However, a precise and rigorous research was first carried out by Liouville. Methods connected with the use of integral transforms have been successfully applied to the solution of differential and integral equations. Fractional operators have been attracting the attention of mathematicians and engineers from long time ago [19,24]. The use of integral transforms to deal with fractional derivatives was originated by Riemann and Liouville [19, 24]. The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives, see for example [6, 11, 15, 16].

The possibility of using integral transforms in a wider context was discussed by Dattoli et al. [11], where by using Euler's integral:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-at} t^{\nu-1} dt, \quad \min\{\mathrm{Re}(\nu), \mathrm{Re}(a)\} > 0, \tag{1.5}$$

it was shown that [11]:

$$\begin{aligned} \left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) =& \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\ e^{t\frac{\partial}{\partial x}} f(x) dt \\ =& \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\ f(x+t) dt, \end{aligned} \tag{1.6}$$

whereas for the cases involving second order derivatives, it was shown that

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1}\ e^{t\frac{\partial^2}{\partial x^2}} f(x) dt. \tag{1.7}$$

The fractional operators can be treated in an efficient way by combining the properties of exponential operators and suitable integral representations.

In this article, the extended Hermite-Apostol type Frobenius Genocchi polynomials are introduced using integral transforms. The recurrence relations and some identities involving these polynomials are also derived. Finally, the quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials and for their extended forms are obtained.

2. Extended Hermite-Apostol Type Frobenius-Genocchi Polynomials

In order to develop extended forms of the Hermite-Apostol type Frobenius-Genocchi polynomials via Euler's integral, we first establish the operational connection for the Hermite-Apostol type Frobenius-Genocchi polynomials.

From generating equation (1.3), we find that the Hermite-Apostol type Frobenius-Genocchi polynomials are the solutions of the following equations:

$$\begin{aligned} \frac{\partial}{\partial y} {}_H\mathcal{H}_n(x,y,z;\lambda;u) =& \frac{\partial^2}{\partial x^2} {}_H\mathcal{H}_n(x,y,z;\lambda;u), \\ \frac{\partial}{\partial z} {}_H\mathcal{H}_n(x,y,z;\lambda;u) =& \frac{\partial^3}{\partial x^3} {}_H\mathcal{H}_n(x,y,z;\lambda;u), \end{aligned}$$

under the following initial condition:

$$ {}_H\mathcal{H}_n(x,0,0;\lambda;u) = \mathcal{H}_n(x;\lambda;u), \tag{2.1} $$

where $\mathcal{H}_n(x;\lambda;u)$ are the Apostol type Frobenius-Genocchi polynomials [5].

Thus, in view of above equation, it follows that, for the Hermite-Apostol type Frobenius-Genocchi polynomials the following operational connection holds true:

$$ {}_H\mathcal{H}_n(x,y,z;\lambda;u) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{\mathcal{H}_n(x;\lambda;u)\}. \tag{2.2} $$

Further by making use of operational rule (2.2) and Euler's integral, we derive the operational relation for the polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$. For this we prove the following result.

Theorem 2.1. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$, the following operational connection holds true:*

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) = {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha). \tag{2.3} $$

Proof. Replacing a by $\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ in integral (1.5) and then operating the resultant equation on $\mathcal{H}_n(x;\lambda;u)$, it follows that

$$ \begin{aligned} \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) =& \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(yt\frac{\partial^2}{\partial x^2} + zt\frac{\partial^3}{\partial x^3}\right) \\ & \times \mathcal{H}_n(x;\lambda;u)dt, \end{aligned} \tag{2.4} $$

which in view of equation (2.2) gives

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathcal{H}_n(x,yt,zt;\lambda;u)dt. \tag{2.5} $$

The transform on the r.h.s of equation (2.5) defines a new family of polynomials as the extended Hermite-Apostol type Frobenius-Genocchi polynomials, i.e.,

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathcal{H}_n(x,yt,zt;\lambda;u)dt. \tag{2.6} $$

Thus, in view of equations (2.5) and (2.6), assertion (2.3) follows. □

Remark 2.1. We know that for $\lambda = 1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ [5] reduce to the Hermite-Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;u)$ [4]. Therefore, taking $\lambda = 1$ in the both sides of equation (2.3), we find the following operational connection between extended Hermite-Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;u;\alpha)$ and the Frobenius-Genocchi polynomials $\mathcal{H}_n(x;u)$ [25]:

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;u) = {}_{\nu H}\mathcal{H}_n(x,y,z;u;\alpha). \tag{2.7} $$

Remark 2.2. For $\lambda = 1$ and $u = -1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Genocchi polynomials ${}_HG_n(x, y, z)$ [4]. Therefore, taking $\lambda = 1$ and $u = -1$ in both sides of equation (2.3), we find the following operational connection between the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$ and the Genocchi polynomials $G_n(x)$ [21]:

$$\left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} G_n(x) = {}_{\nu H}G_n(x, y, z; \alpha). \tag{2.8}$$

Next, we derive the generating function for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$ by proving the following result.

Theorem 2.2. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$, the following generating function holds true:*

$$\frac{(1-u)w \ \exp(xw)}{(\lambda e^w - u) \ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)\frac{w^n}{n!}. \tag{2.9}$$

Proof. Multiplying both sides of equation (2.6) by $\frac{w^n}{n!}$, then summing it over n and making use of equation (1.3) in the r.h.s. of the resultant equation, we find

$$\sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)\frac{w^n}{n!} = \frac{(1-u)w \ \exp(xw)}{(\lambda e^w - u) \ \Gamma(\nu)} \int_0^\infty e^{-\left(\alpha-(yw^2+zw^3)\right)t} \ t^{\nu-1}dt, \tag{2.10}$$

which in view of integral (1.5) yields assertion (2.9). □

Remark 2.3. We know that for $\lambda = 1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; u)$ [4]. Therefore, by taking $\lambda = 1$ in the both sides of equation (2.9), we find the following generating for the extended Hermite-Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; u; \alpha)$:

$$\frac{(1-u)w \ \exp(xw)}{(e^w - u) \ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; u; \alpha)\frac{w^n}{n!}. \tag{2.11}$$

Remark 2.4. We know that for $\lambda = 1$ and $u = -1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Genocchi polynomials ${}_HG_n(x, y, z)$ [4]. Therefore, by taking $\lambda = 1$ and $u = -1$ in both sides of equation (2.9), we find the following generating function for the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$:

$$\frac{2w \exp(xw)}{(e^w + 1) \ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}G_n(x, y, z; \alpha)\frac{w^n}{n!}. \tag{2.12}$$

Now, we derive the recurrence relations for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$ by taking into consideration its generating relation. A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

On differentiating generating function (2.9) with respect to x, y, z and α, we find the following recurrence relations for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$:

$$\frac{\partial}{\partial x}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = n\ {}_{\nu H}\mathcal{H}_{n-1}(x,y,z;\lambda;u;\alpha),$$
$$\frac{\partial}{\partial y}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = \nu\ n(n-1)_{\nu+1 H}\mathcal{H}_{n-2}(x,y,z;\lambda;u;\alpha),$$
$$\frac{\partial}{\partial z}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = \nu\ n(n-1)(n-2)_{\nu+1 H}\mathcal{H}_{n-3}(x,y,z;\lambda;u;\alpha),$$
$$\frac{\partial}{\partial \alpha}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\nu_{\nu+1 H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha). \tag{2.13}$$

In view of the above relations, it follows that

$$\frac{\partial}{\partial y}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\frac{\partial^3}{\partial x^2\partial\alpha}\ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha),$$
$$\frac{\partial}{\partial z}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\frac{\partial^4}{\partial x^3\partial\alpha}\ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha).$$

Theorem 2.3. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials* ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$*, the following explicit series expansion holds true:*

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = n!\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^r H_{k-3r}(x,yt)(\nu)_r}{\alpha^{\nu+r}\ (n-k)!r!(k-3r)!}. \tag{2.14}$$

Proof. Using the series expansion (1.4) in the r.h.s of equation (2.6), we find

$$\begin{aligned} {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) =& \frac{\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\nu+r)}\int_0^\infty e^{-\alpha t}t^{\nu+r-1}n! \\ &\times\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^r H_{k-3r}(x,yt)}{(n-k)!r!(k-3r)!}dt, \end{aligned} \tag{2.15}$$

which in view of integral (1.5) yields assertion (2.14). □

In the next section, we establish the determinant form and quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ and derive corresponding results for their extended forms.

3. Determinant Approach and Quasi-Monomial Properties

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of special functions. The use of operational techniques in the study of hybrid special functions provide explicit solutions for the families of partial differential equations including those of Heat and d'Alembert type and to frame the hybrid special polynomials within the context of linear algebraic approach. We use the operational rules to establish the determinant forms for the special cases of the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$.

We recall the following operational definition and the generating function of the extended 3-variable Hermite polynomials from [11]:

$$ {}_{\nu}H_n(x,y,z;\alpha) = \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} x^n, \tag{3.1} $$

$$ (\alpha - (yt^2 + zt^3))^{-\nu} e^{xt} = \sum_{n=0}^{\infty} {}_{\nu}H_n(x,y,z;\alpha)\frac{t^n}{n!}. \tag{3.2} $$

Theorem 3.1. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$, the following explicit summation formula in terms of the generalized Hermite polynomials ${}_{\nu}H_n(x,y,z;\alpha)$ and Apostol type Frobenius-Genocchi polynomials $\mathcal{H}_n(w;\lambda;u)$ holds true:*

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = \sum_{l=0}^{n}\sum_{p=0}^{n}\binom{n}{l}\binom{n-l}{p}(-w)^l\mathcal{H}_p(w;\lambda;u)\,{}_{\nu}H_{n-l-p}(x,y,z;\alpha). \tag{3.3} $$

Proof. We consider the product of generating equations (3.2) and (1.1) such that

$$ \left(\frac{(1-u)t}{\lambda e^t - u}\right) e^{wt}(\alpha - (yt^2 + zt^3))^{-\nu} e^{xt} = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\mathcal{H}_p(w;\lambda;u)\,{}_{\nu}H_n(x,y,z;\alpha)\frac{t^{n+p}}{n!\,p!}, \tag{3.4} $$

which on rearranging the terms yields

$$ \begin{aligned} \left(\frac{(1-u)t}{\lambda e^t - u}\right)(\alpha - (yt^2 + zt^3))^{-\nu} e^{xt} = & \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{p=0}^{n}\binom{n}{l}\binom{n-l}{p}(-w)^l\mathcal{H}_p(w;\lambda;u) \\ & \times {}_{\nu}H_{n-l-p}(x,y,z;\alpha)\frac{t^n}{n!}. \end{aligned} \tag{3.5} $$

Finally, using generating function (2.9) in the l.h.s. of equation (3.5) and then by equating the coefficients of like powers of t in the resultant equation, assertion (3.3) follows. □

Next, by making use of determinant form of Genocchi polynomials [20], we obtained the determinant form of the extended Hermite-Genocchi polynomials.

Definition 3.1. The Genocchi polynomials $G_n(x)$ of degree n are defined by [20]

$$G_0(x) = 1,$$

$$G_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n} & \frac{1}{2(n+1)} \\ 0 & 1 & \binom{2}{1}\frac{1}{4} & \cdots & \binom{n-1}{1}\frac{1}{2(n-1)} & \binom{n}{1}\frac{1}{2n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2(n-2)} & \binom{n}{2}\frac{1}{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{4} \end{vmatrix}, \tag{3.6}$$

where $n = 1, 2, \ldots$

Now, apply operational rules (3.1) in the r.h.s. and (2.8) in the l.h.s. of determinant form (3.6) of Genocchi polynomials and after simplification, we find the following determinant form for the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$:

$$\begin{aligned} &{}_{\nu H}G_0(x, y, z; \alpha) = 1, \\ &{}_{\nu H}G_n(x, y, z; \alpha) \\ &= (-1)^n \begin{vmatrix} 1 & {}_{\nu}H_1(x,y,z;\alpha) & {}_{\nu}H_2(x,y,z;\alpha) & \cdots & {}_{\nu}H_{n-1}(x,y,z;\alpha) & {}_{\nu}H_n(x,y,z;\alpha) \\ 1 & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n} & \frac{1}{2(n+1)} \\ 0 & 1 & \binom{2}{1}\frac{1}{4} & \cdots & \binom{n-1}{1}\frac{1}{2(n-1)} & \binom{n}{1}\frac{1}{2n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2(n-2)} & \binom{n}{2}\frac{1}{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{4} \end{vmatrix}, \end{aligned} \tag{3.7}$$

where $n = 1, 2, \ldots$

The method proposed in this article can be used in combination with the monomiality principle as a useful tool in analysing the solutions of a wide class of partial differential equations often encountered in physical problems. The combination of monomiality principle along with operational techniques in the case of multi-variable hybrid special polynomials yields new mechanism of analysis for the solutions of a large class of partial differential equations usually experienced in physical problems. The operational methods open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of special polynomials. The concept of monomiality principle arises from the idea of poweroid suggested by Steffensen [22]. This idea was reformulated and systematically used by Dattoli [9]. Ben Cheikh [8] was shown that every polynomial set is quasi-monomial and the properties of a given polynomial set may be deduced from the quasi-monomiality.

In order to frame the polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ within the context of monomiality principle, the following result is proved.

Theorem 3.2. *The Hermite-Apostol type Frobenius-Genocchi polynomials* ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{H\mathcal{H}} = x + 2y\partial_x + 3z\partial_x^2 - \frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u} \tag{3.8}$$

and

$$\hat{P}_{H\mathcal{H}} = \partial_x, \quad \partial_x := \frac{\partial}{\partial x}, \tag{3.9}$$

respectively.

Proof. Differentiating equation (1.3) partially with respect to t, it follows that

$$\left(x + 2yt + 3zt^2 - \frac{\lambda e^t(1-t)-u}{\lambda e^t-u}\right)\left(\frac{(1-u)t}{\lambda e^t-u}\right) = \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n+1}(x,y,z;\lambda;u)\frac{t^n}{n!}. \tag{3.10}$$

Now, using identity

$$\partial_x\{{}_H\mathcal{H}_n(x,y,z;\lambda;u)\} = t\{{}_H\mathcal{H}_n(x,y,z;\lambda;u)\} \tag{3.11}$$

and generating equation (1.3) in the l.h.s of equation (3.10), it follows that

$$\begin{aligned}\left(x + 2y\partial_x + 3z\partial_x^2 - \frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u}\right)&\sum_{n=0}^{\infty} {}_H\mathcal{H}_n(x,y,z;\lambda;u)\\ = \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n+1}(x,y,z;\lambda;u),&\end{aligned} \tag{3.12}$$

which in view of monomiality principle equation $\hat{M}\{p_n(x)\} = p_{n+1}(x)$ and then equating the coefficients of same powers of t in both sides yields assertion (3.8).

Again, in view of generating function (1.3) and identity (3.11), it follows that

$$\partial_x\left\{\sum_{n=0}^{\infty} {}_H\mathcal{H}_n(x,y,z;\lambda;u)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_H\mathcal{H}_{n-1}(x,y,z;\lambda;u)\frac{t^n}{(n-1)!}. \tag{3.13}$$

Rearranging the terms in above equation and using monomiality principle equation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$ and then by equating the coefficients of same powers of t in both sides of the resultant equation, assertion (3.9) follows. □

Remark 3.1. By making use of expressions (3.8) and (3.9) in relation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$, we find that the following differential equation for the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ holds true:

$$\left(x\partial_x + 2y\partial_x^2 + 3z\partial_x^3 - \frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u}\partial_x - n\right) {}_H\mathcal{H}_n(x,y,z;\lambda;u) = 0. \tag{3.14}$$

Next, with the use of integral transforms, we show that the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$ are quasi-monomial.

Consider the operation (Θ): replacement of y by yt and z by zt, multiplication by $\frac{1}{\Gamma(\nu)}e^{-at}t^{\nu-1}$ and then integration with respect to t from $t=0$ to $t=\infty$.

Operating (Θ) on equations (3.8) and (3.9) and then using equation (2.15) and further in view of recurrence relations $\hat{M}\{p_n(x)\} = p_{n+1}(x)$ and $\hat{P}\{p_n(x)\} = np_{n-1}(x)$, we find that the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{\nu H\mathcal{H}} = x + 2y\partial_x\partial_\alpha + 3z\partial_x^2\partial_\alpha - \frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u} \tag{3.15}$$

and

$$\hat{P}_{\nu H\mathcal{H}} = \partial_x, \tag{3.16}$$

respectively.

Further, use of equations (3.15) and (3.16) in relation $\hat{P}\{p_n(x)\} = np_{n-1}(x)$ yields the following differential equation for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$:

$$\left(x\partial_x + 2y\partial_x^2\partial_\alpha + 3z\partial_x^3\partial_\alpha - \frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u}\partial_x - n\right){}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = 0. \tag{3.17}$$

The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional operators. To bolster the contention of using this approach, the extended form of hybrid type polynomials are introduced. The generating function and recurrence relations for the extended hybrid polynomials are derived here. These results may be useful in the investigation of other useful properties of these polynomials and may have applications in physics.

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ON GRADED 2-ABSORBING SECOND SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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Abstract. In this paper, we introduce the concepts of graded 2-absorbing second and graded strongly 2-absorbing second submodules. A number of results concerning these classes of graded submodules are given.

1. Introduction and Preliminaries

Throughout this paper all rings are commutative, with identity and all modules are unitary.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a G-*graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$ (see [19]).

Let R be a G-graded ring and M an R-module. We say that M is a G-*graded* R-*module* (or *graded* R-*module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus\limits_{g\in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup\limits_{g\in G} M_g$ and the elements of

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$h(M)$ are called to be *homogeneous*. Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus_{g \in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N (see [19]). For more details, one can refer to [16, 20, 21].

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R \mid rM \subseteq N\}$. It is shown in [11, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R \mid rN \subseteq M\}$ is a graded ideal of R. The annihilator of M is defined as $(0 :_R M)$ and is denoted by $Ann_R(M)$.

The notion of graded prime ideals was introduced in [24] and studied in [12, 23, 25]. A proper graded ideal P of R is said to be *a graded prime ideal* if whenever $rs \in P$, we have $r \in P$ or $s \in P$, where $r, s \in h(R)$.

S. E. Atani in [11] extended graded prime ideals to graded prime submodules. A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$. Several authors investigated properties of graded prime submodules, for examples see [3, 6, 7, 15, 22].

The notion of graded 2-absorbing ideals as a generalization of graded prime ideals was introduced and studied in [4, 18]. A proper graded ideal I of R is said to be *a graded 2-absorbing ideal of R* if whenever $r, s, t \in h(R)$ with $rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$.

K. Al-Zoubi and R. Abu-Dawwas in [2] extended graded 2-absorbing ideals to graded 2-absorbing submodules. A proper graded submodule N of M is said to be *a graded 2-absorbing submodule of M* if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$.

The notion of graded second submodules was introduced in [9] and studied in [1, 10, 14]. A non-zero graded submodule N of M is said to be *a graded second* (*gr-second*) if for each homogeneous element r of R, the endomorphism of N given by multiplication by r is either surjective or zero. Recently, H. Ansari-Toroghy and F. Farshadifar, in [8] studied 2-absorbing second and strongly 2-absorbing second submodules.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, we introduced the concepts of graded 2-absorbing second and graded strongly 2-absorbing second submodules, investigate some properties of these graded submodules and give some characterizations of them.

2. Graded 2-Absorbing Second Submodules

Definition 2.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule C of M is said to be *a completely graded irreducible* if $C = \cap_{\alpha \in \Delta} C_\alpha$, where $\{C_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M, implies that $C = C_\beta$ for some $\beta \in \Delta$.

Lemma 2.1. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If $m \in h(M) - N$, then there exists a completely graded irreducible submodule C of M such that $N \subseteq C$ and $m \notin C$.*

Proof. Let $m \in h(M) - N$ and Λ be the set of all graded submodules of M that contains N and not containing m. Then $\Lambda \neq \emptyset$, since $N \in \Lambda$. Order Λ by inclusion, i.e. for K, $L \in \Lambda$ then $K \leq L$ if $K \subseteq L$. Clearly, $(\Lambda, \leq)$ is a partially ordered set. Let $\{C_\alpha\}_{\alpha\in\Omega}$ be any chain in Λ. It is clear that $\cup_{\alpha\in\Omega}C_\alpha$ is an upper bound of $\{C_\alpha\}_{\alpha\in\Omega}$ in Λ. Thus, by Zorn's Lemma, Λ contains a maximal element C. We claim that C is a completely graded irreducible submodule of M. Let $\{L_\beta\}_{\beta\in\Delta}$ be a family of graded submodules of M such that $C = \cap_{\beta\in\Delta}L_\beta$. Suppose to the contrary that $C \neq L_\beta$ for all $\beta \in \Delta$. Then each L_β contain m, it follows that $m \in \cap_{\beta\in\Delta}L_\beta = C$, which is a contradiction. □

Lemma 2.2. *Let R be a G-graded ring, M a graded R-module and K, L be two proper graded submodules of M. Then $K \subseteq L$ if and only if every completely graded irreducible submodule containing L, also contains K.*

Proof. $(\Rightarrow)$ is clear.

$(\Leftarrow)$ Assume that every completely graded irreducible submodule of M containing L, also contains K. Suppose to the contrary that $K \nsubseteq L$. Since K is generated by $K \cap h(M)$, there exists $k \in K \cap h(M) - L$. By Lemma 2.1, there exists a completely graded irreducible submodule C of M such that $L \subseteq C$ and $k \notin C$. This implies that $K \nsubseteq C$, which is a contradiction. □

Theorem 2.1. *Let R be a G-graded ring and M a graded R-module. Then every proper graded submodule of M is the intersection of all completely graded irreducible submodules containing it.*

Proof. Let K be a proper graded submodule of M and $\{C_\beta\}_{\beta\in\Delta}$ be the set of all completely graded irreducible submodules containing K. It is clear that $K \subseteq \bigcap\limits_{\beta\in\Delta} C_\beta$. If $k = \sum_{g\in G} k_g \notin K$, then there exists $h \in G$ such that $k_h \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. By Lemma 2.1, there exists a completely graded irreducible submodule C such that $K \subseteq C$ and $k_h \notin C$. Hence $C = C_\alpha$ for some $\alpha \in \Delta$, it follows that $k_h \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. So, $k \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. Consequently, $\bigcap\limits_{\beta\in\Delta} C_\beta \subseteq K$. □

Definition 2.2. Let R be a G-graded ring and M a graded R-module. A non-zero graded submodule S of M is said to be *a graded 2-absorbing second submodule of M* if whenever r, $t \in h(R)$, C is a completely graded irreducible submodule of M and $rtS \subseteq C$, then $rS \subseteq C$ or $tS \subseteq C$ or $rt \in Ann_R(S)$.

Let R be a G-graded ring. The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [23]).

Let M be a non-zero graded R-module. Then M is said to be *a graded secondary* if for each homogeneous element r of R, the endomorphism of M given by multiplication by r is either surjective or nilpotent. This implies that $Gr(Ann_R(M)) = P$ is a graded prime ideal of R. For convenience, a graded submodule of M which is graded secondary, is called a *graded secondary* submodule of M (see [13]).

Lemma 2.3. *Let R be a G-graded ring and M a graded R-module of M.*

(i) *If S is a graded second submodule of M and $rS \subseteq K$, where $r \in h(R)$ and K is a graded submodule of M, then either $rS = 0$ or $S \subseteq K$.*
(ii) *If S is a graded secondary submodule of M and $rS \subseteq K$, where $r \in h(R)$ and K is a graded submodule of M, then either $r^nS = 0$ for some $n \in \mathbb{N}$ or $S \subseteq K$.*

Proof. Straightforward. □

Theorem 2.2. *Let R be a G-graded ring and M a graded R-module. Then the following hold.*

(i) *If either S is a graded second submodule of M or S is a sum of two graded second submodules of M, then S is a graded 2-absorbing second submodule.*
(ii) *If S is a graded secondary submodule of M and $R/Ann_R(S)$ has no non-zero nilpotent homogeneous element, then S is a graded 2-absorbing second submodule.*

Proof. (i) Assume that S is a graded second submodule of M. Let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rtS \subseteq C$. By Lemma 2.3 (i), either $rtS = 0$ or $S \subseteq C$. Thus S is a graded 2-absorbing second submodule. Now assume that $S = S_1 + S_2$, where S_1 and S_2 are graded second submodules of M. Let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rtS \subseteq C$. Since S_1 is a graded second submodule, by Lemma 2.3 (i), we have either $rtS = 0$ or $S_1 \subseteq C$. Similarly, we have $rtS_2 = 0$ or $S_2 \subseteq C$. If $rtS_1 = 0$ and $rtS_2 = 0$, then $rt \in Ann(S_1 + S_2)$, we are done. If $S_1 \subseteq C$ and $S_2 \subseteq C$, then we are done. Assume that $rtS_1 = 0$ and $S_2 \subseteq L$. Then $rt \in Ann_R(S_1)$. By [9, Proposition 3.15], $Ann_R(S_1)$ is a graded prime ideal. This yields that $r \in Ann_R(S_1)$ or $t \in Ann_R(S_1)$. If $r \in Ann_R(S_1)$, then $r(S_1 + S_2) \subseteq rS_1 + S_2 \subseteq S_2 \subseteq C$. Similarly, If $t \in Ann_R(S_1)$, we get $t(S_1 + S_2) \subseteq C$. Also if $rtS_2 = 0$ and $S_1 \subseteq C$, we get either $r(S_1 + S_2) \subseteq C$ or $t(S_1 + S_2) \subseteq C$. Therefore S is a graded 2-absorbing second submodule.

(ii) Since C is a graded secondary submodule of M, $Gr(Ann_R(C))$ is a graded prime ideal. This yields that $Ann_R(C)$ is a graded prime ideal because $R/Ann_R(C)$ has no non-zero nilpotent homogeneous element. By [10, Proposition 2.3 (i)], we have C is a graded second submodule and hence C is a graded 2-absorbing second submodule by part (i). □

Lemma 2.4. *Let R be a G-graded ring, M a graded R-module and S a graded 2-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. If $r \in h(R)$, $g \in G$ and C is a completely graded irreducible submodule of M with $I_g rS \subseteq C$, then either $rS \subseteq C$ or $I_g S \subseteq C$ or $I_g r \subseteq Ann_R(S)$.*

Proof. Assume that $r \in h(R)$, $g \in G$ and C is a completely graded irreducible submodule of M such that $I_g rS \subseteq C$, $rS \nsubseteq C$ and $I_g r \nsubseteq Ann_R(S)$. We have to show that $I_g S \subseteq C$. Assume that $i_g \in I_g$. By assumption there exists $i'_g \in I_g$ such that $ri'_g S \neq 0$. Since S is a graded 2-absorbing second submodule of M, $ri'_g S \subseteq C$, $rS \nsubseteq C$ and $ri'_g \notin Ann_R(S)$, we get $i'_g S \subseteq C$. By $(i_g + i'_g) \in I_g$ it follows that $(i_g + i'_g)rS \subseteq C$. Then either $(i_g+i'_g)S \subseteq C$ or $(i_g+i'_g)r \in Ann_R(S)$ as S is a graded 2-absorbing second submodule of M. If $(i_g+i'_g)S \subseteq C$, then we get $i_g S \subseteq C$. If $(i_g+i'_g)r \in Ann_R(S)$, then $i_g r \notin Ann_R(S)$. Since S is a graded 2-absorbing second, $i_g rS \subseteq C, i_g r \notin Ann_R(S)$ and $rS \nsubseteq C$, we get $i_g S \subseteq C$. Therefore, $I_g S \subseteq C$. $\square$

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statement are equivalent.*

(i) *S is a graded 2-absorbing second submodule of M.*
(ii) *If C is a completely graded irreducible submodule of M and $g, h \in G$ with $I_g J_h S \subseteq C$, then either $I_g S \subseteq C$ or $J_h S \subseteq C$ or $I_g J_h \subseteq Ann_R(S)$.*

Proof. (i)$\Rightarrow$(ii) Assume that S is a graded 2-absorbing second submodule of M. Let C be a completely graded irreducible submodule of M and $g, h \in G$ such that $I_g J_h S \subseteq C, I_g S \nsubseteq C$ and $J_h S \nsubseteq C$. We show that $I_g J_h \subseteq Ann_R(S)$. Assume that $i_g \in I_g$ and $j_h \in J_h$. By assumption there exists $i'_g \in I_g$ such that $i'_g S \nsubseteq L$. Since $i'_g J_h S \subseteq C$, $J_h S \nsubseteq C$ and $i'_g S \nsubseteq C$, by Lemma 2.4 we get $i'_g J_h \subseteq Ann_R(S)$ and hence $(I_g \backslash (C :_R S))J_h \subseteq Ann_R(S)$. Similarly there exists $j'_h \in J_h$ and $j'_h S \nsubseteq C$ such that $j'_h I_g \subseteq Ann_R(S)$ and also $(J_h \backslash (C :_R S))I_g \subseteq Ann_R(S)$. Thus we have $i'_g j'_h \in Ann_R(S)$, $i'_g j_h \in Ann_R(S)$ and $i_g j'_h \in Ann_R(S)$. By $(i_g + i'_g) \in I_g$ and $(j_h + j'_h) \in J_h$ it follows that $(i_g + i'_g)(j_h + j'_h)S \subseteq C$. Since S is a graded 2-absorbing second, we get either $(i_g + i'_g)S \subseteq C$ or $(j_h + j'_h)S \subseteq C$ or $(i_g + i'_g)(j_h + j'_h) \in Ann_R(S)$. If $(i_g + i'_g)S = i_g S + i'_g S \subseteq C$, then $i_g S \nsubseteq C$. So $i_g \in I_g \backslash (C :_R S)$ it follows that $i_g j_h \in Ann_R(S)$. Similarly by $(j_h + j'_h)S \subseteq C$ we get $i_g j_h \in Ann_R(S)$. If $(i_g + i'_g)(j_h + j'_h) \in Ann_R(S)$, then $i_g j_h + i_g j'_h + i'_g j_h + i'_g j'_h \in Ann_R(S)$ and so $i_g j_h \in Ann_R(S)$. Thus $I_g J_h \subseteq Ann_R(S)$.

(ii)$\Rightarrow$ (i) Assume that (ii) holds. Let $r_g, t_h, \in h(R)$ and C be a completely graded irreducible submodule of M with $r_g t_h S \subseteq C$. Let $I = r_g R$ and $J = t_h R$ be a graded ideals of R generated by r_g and t_h, respectively. Then $I_g J_h S \subseteq C$. By our assumption we obtain $I_g S \subseteq C$ or $J_h S \subseteq C$ or $I_g J_h \subseteq Ann_R(S)$. Hence $r_g S \subseteq C$ or $t_h S \subseteq C$ or $r_g t_h \in Ann_R(S)$. Therefore, S is a graded 2-absorbing second submodule of M. $\square$

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and S a graded 2-absorbing second submodule of M. Then we have the following.*

(i) *If $Ann_R(S)$ is a graded prime ideal of R, then $(C :_R S)$ is a graded prime ideal of R for all completely graded irreducible submodule C of M such that $S \nsubseteq C$.*
(ii) *If $Gr(Ann_R(S)) = P$ for some graded prime ideal P of R, then $Gr((C :_R S))$ is a graded prime ideal of R containing P for all completely graded irreducible submodule C of M such that $S \nsubseteq C$.*

Proof. (i) Let $r, t \in h(R)$, C be a completely graded irreducible submodule of M such that $S \nsubseteq C$ and $rt \in (C :_R S)$. So $rtS \subseteq C$. Since S is a graded 2-absorbing second submodule, we have $rS \subseteq C$ or $tS \subseteq C$ or $rt \in Ann_R(S)$. If $rS \subseteq C$ or $tS \subseteq C$, then we are done. If $rt \in Ann_R(S)$, then $r \in Ann_R(S)$ or $t \in Ann_R(S)$ because $Ann_R(S)$ is a graded prime ideal of R. This yields that $r \in (C :_R S)$ or $t \in (C :_R S)$. Thus $(C :_R S)$ is a graded prime ideal of R.

(ii) Let $r, t \in h(R)$ and $rt \in Gr((C :_R S))$. Then $(rt)^n \in (C :_R S)$ for some $n \in \mathbb{Z}^+$. So $r^n t^n S \subseteq C$. Since S is a graded 2-absorbing second submodule, we have $r^n S \subseteq C$ or $t^n S \subseteq C$ or $r^n t^n \in Ann_R(S)$. If $r^n S \subseteq C$ or $t^n S \subseteq C$, then $r \in Gr((C :_R S))$ or $t \in Gr((C :_R S))$ so we are done. Now assume that $r^n t^n \in Ann_R(S)$ so $rt \in Gr(Ann_R(S))$. Then $r \in Gr(Ann_R(S))$ or $t \in Gr(Ann_R(S))$ as $Gr((Ann_R(S))$ is a graded prime ideal of R. Since $Ann_R(S) \subseteq (C :_R S)$, we have $Gr(Ann_R(S)) \subseteq Gr((C :_R S))$. This yields that $r \in Gr((C :_R S))$ or $t \in Gr((C :_R S))$. Therefore, $Gr((C :_R S))$ is a graded prime ideal of R containing P. □

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$ (see [21].)

Lemma 2.5. *Let R be a G-graded ring and M, M' be two graded R-modules and let $\varphi : M \to M'$ be a graded monomorphism.*
(i) *If C is a completely graded irreducible submodule of M, then $\varphi(C)$ is a completely graded irreducible submodule of $\varphi(M)$.*
(ii) *If C' is a graded completely irreducible submodule of $\varphi(M)$, then $\varphi^{-1}(C')$ is a completely graded irreducible submodule of M.*

Proof. Straightforward. □

Theorem 2.5. *Let R be a G-graded ring and M, M' be two graded R-modules. Let φ: $M \to M'$ be a graded monomorphism. Then we have the following.*
(i) *If S is a graded 2-absorbing second submodule of M, then $\varphi(S)$ is a graded 2-absorbing second submodule of $\varphi(M)$.*
(ii) *If S' is a graded 2-absorbing second submodule of $\varphi(M)$, then $\varphi^{-1}(S')$ is a graded 2-absorbing second submodule of M.*

Proof. (i) Since $S \neq 0$ and φ is a graded monomorphism, we have $\varphi(S) \neq 0$. Let $r, t \in h(R)$ and C' be a graded completely irreducible submodule of $\varphi(M)$ with $rt\varphi(S) \subseteq C'$. Then $rtS \subseteq \varphi^{-1}(C')$. By Lemma 2.5 (ii), we have $\varphi^{-1}(C')$ is a graded completely irreducible submodule of M. Then either $rS \subseteq \varphi^{-1}(C')$ or $tS \subseteq \varphi^{-1}(C')$

or $rtS = 0$ as S is graded 2-absorbing second submodule of M. If $rtS = 0$, then $rt\varphi(S) = 0$. If $rS \subseteq \varphi^{-1}(C')$, then $r\varphi(S) = \varphi(rS) \subseteq \varphi\varphi^{-1}(C') = C' \cap \varphi(M) = C'$. Similarly, if $tS \subseteq \varphi^{-1}(C')$, we get $t\varphi(S) \subseteq C'$. Therefore, $\varphi(S)$ is a graded 2-absorbing second submodule of $\varphi(M)$.

(ii) If $\varphi^{-1}(S') = 0$, then $\varphi(M) \cap S' = \varphi\varphi^{-1}(S') = \varphi(0) = 0$. Thus $S' = 0$ which is a contradiction. So $\varphi^{-1}(S') \neq 0$. Now let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rt\varphi^{-1}(S') \subseteq C$. Then $rtS' = rt(S' \cap \varphi(M)) = rt\varphi\varphi^{-1}(S') = \varphi(rt\varphi^{-1}(S')) \subseteq \varphi(C)$. By Lemma 2.5(i), we have $\varphi(C)$ is a completely graded irreducible submodule of $\varphi(M)$. Then $rS' \subseteq \varphi(C)$ or $tS' \subseteq \varphi(C)$ or $rtS' = 0$ as S' is a graded 2-absorbing second submodule of $\varphi(M)$. Thus $r\varphi^{-1}(S') \subseteq \varphi^{-1}\varphi(C) = C$ or $t\varphi^{-1}(S') \subseteq \varphi^{-1}\varphi(C) = C$ or $rt\varphi^{-1}(S') = 0$. Therefore, $\varphi^{-1}(S')$ is a graded 2-absorbing second submodule of M. □

3. Graded 2-Absorbing Strongly Second Submodules

Definition 3.1. Let R be a G-graded ring and M a graded R-module. A non-zero graded submodule S of M is said to be *a graded strongly 2-absorbing second submodule of M* if whenever $r, t \in h(R)$, C_1, C_2 are completely graded irreducible submodules of M, and $rtS \subseteq C_1 \cap C_2$, then $rS \subseteq C_1 \cap C_2$ or $tS \subseteq C_1 \cap C_2$ or $rt \in Ann_R(S)$.

Clearly every graded strongly 2-absorbing second submodule is a graded 2-absorbing second submodule.

Lemma 3.1. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. If $r \in h(R)$, $g \in G$ and C_1, C_2 are completely graded irreducible submodules of M with $I_g rS \subseteq C_1 \cap C_2$, then either $rS \subseteq C_1 \cap C_2$ or $I_g S \subseteq C_1 \cap C_2$ or $I_g r \subseteq Ann_R(S)$.*

Proof. The proof is similar to the proof of Lemma 2.4, so we omit it. □

Theorem 3.1. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *S is a graded strongly 2-absorbing second submodule of M.*
(ii) *If L_1 and L_2 are a completely graded irreducible submodules of M and $g, h \in G$ with $I_g J_h S \subseteq L_1 \cap L_2$, then either $I_g S \subseteq L_1 \cap L_2$ or $J_h S \subseteq L_1 \cap L_2$ or $I_g J_h \subseteq Ann_R(S)$.*

Proof. The proof is similar to the proof of Theorem 2.3, so we omit it. □

Theorem 3.2. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *S is a graded strongly 2-absorbing second submodule of M.*
(ii) *For every graded submodule K of M and $g, h \in G$ such that $I_g J_h S \subseteq K$, either $I_g S \subseteq K$ or $J_h S \subseteq K$ or $I_g J_h \subseteq Ann_R(S)$.*

(iii) *For every graded submodule K of M and every pair of elements r_g, $t_h \in h(R)$ such that $r_g t_h S \subseteq K$, either $r_g S \subseteq K$ or $t_h S \subseteq K$ or $r_g t_h S \subseteq Ann_R(S)$.*

(iv) *For every pair of elements r_g, $t_h \in h(R)$, either $r_g t_h S = r_g S$ or $r_g t_h S = t_h S$ or $r_g t_h S = 0$.*

Proof. (i)⇒(ii) Let $g, h \in G$ and K a graded submodule of M such that $I_g J_h S \subseteq K$ and $I_g J_h \nsubseteq Ann_R(S)$. By Theorem 2.3, for all completely graded irreducible submodule C of M such that $K \subseteq C$, we have either $I_g S \subseteq C$ or $J_h S \subseteq C$ and hence either $I_g S \subseteq K$ or $J_h S \subseteq K$ by Lemma 2.2. If $I_g S \subseteq C$ (resp. $J_h S \subseteq C$) for all completely graded irreducible submodule C of M with $K \subseteq C$, we are done. Now suppose that C_1 and C_2 are two completely graded irreducible submodules of M with $K \subseteq C_1$, $K \subseteq C_2$, $I_g S \nsubseteq C_1$ and $J_h S \nsubseteq C_2$. Since S is a graded 2-absorbing second submodule, $I_g J_h S \subseteq C_1, I_g S \nsubseteq C_1$ and $I_g J_h \nsubseteq Ann_R(S)$, by Theorem 2.3, we have $J_h S \subseteq C_1$. Similarly by $J_h S \nsubseteq C_2$ we get $I_g S \subseteq C_2$. Since S is a graded strongly 2-absorbing second submodule of M, $I_g J_h S \subseteq C_1 \cap C_2$, $I_g J_h \nsubseteq Ann_R(S)$, by Theorem 3.1, we conclude that either $I_g S \subseteq C_1 \cap C_2$ or $J_h S \subseteq C_1 \cap C_2$. Hence, either $I_g S \subseteq C_1$ or $J_h S \subseteq C_2$, which is a contradiction.

(ii)⇒(iii) Assume that $r_g t_h S \subseteq K$ where r_g, $t_h \in h(R)$ and K a graded submodule of M. Let $I = r_g R$, $J = t_h R$ be a graded ideals of R generated by r_g and t_h, respectively. Then $I_g J_j S \subseteq K$. By our assumption we have either $I_g S \subseteq K$ or $J_h S \subseteq K$ or $I_g J_h \subseteq Ann_R(S)$. It follows that either $r_g S \subseteq K$ or $t_h S \subseteq K$ or $r_g t_h S \subseteq Ann_R(S)$.

(iii)⇒(iv) Let r_g, $t_h \in h(R)$. Then $r_g t_h S \subseteq r_g t_h S$ implies that $r_g S \subseteq r_g t_h S$ or $t_h S \subseteq r_g t_h S$ or $r_g t_h \in Ann(S)$. This yields that $r_g S = r_g t_h S$ or $t_h S = r_g t_h S$ or $r_g t_h \in Ann(S)$.

(iv)⇒(i) This is clear. □

Lemma 3.2. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. Then $Ann_R(S)$ is a graded 2-absorbing ideal of R.*

Proof. Let r_g, s_h, $t_\lambda \in h(R)$ such that $r_g s_h t_\lambda \in Ann_R(S)$. Since S a graded strongly 2-absorbing second submodule of M and r_g, $s_h, \in h(R)$, by Theorem 3.2, we get either $r_g S = r_g s_h S$ or $s_h S = r_g s_h S$ or $r_g s_h S = 0$. If $r_g s_h S = 0$, then $r_g s_h \in Ann_R(S)$. If $r_g S = r_g s_h S$, then $t_\lambda r_g S \subseteq t_\lambda r_g s_h S = 0$ and hence $t_\lambda r_g \in Ann_R(S)$. Similarly, by $s_h S = r_g s_h S$ we get $t_\lambda s_h \in Ann_R(S)$. Therefore, $Ann_R(S)$ is a graded 2-absorbing ideal of R. □

Theorem 3.3. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. If K is a graded submodule of M such that $S \nsubseteq K$, then $(K :_R S)$ is a graded 2-absorbing ideal of R.*

Proof. Let r_g, s_h, $t_\lambda \in h(R)$ such that $r_g s_h t_\lambda \in (K :_R S)$. Then $r_g s_h t_\lambda S \subseteq K$. Since S is a graded strongly 2-absorbing second submodule of M and $r_g s_h (t_\lambda S) \subseteq K$, by Theorem 3.2 we conclude that either $r_g t_\lambda S \subseteq K$ or $s_h t_\lambda S \subseteq K$ or $r_g s_h t_\lambda S = 0$, which means $r_g t_\lambda \in (K :_R S)$ or $s_h t_\lambda \in (K :_R S)$ or $r_g s_h t_\lambda \in Ann_R(S)$. If $r_g t_\lambda \in (K :_R S)$

or $s_h t_\lambda \in (K :_R S)$, then we are done. Assume that $r_g s_h t_\lambda \in Ann_R(S)$. Then either $r_g s_h \in Ann_R(S)$ or $r_g t_\lambda \in Ann_R(S)$ or $s_h\, t_\lambda \in Ann_R(S)$ by Lemma 3.2. This yields that either $r_g s_h \in (K :_R S)$ or $r_g t_\lambda \in (K :_R S)$ or $s_h t_\lambda \in (K :_R S)$. Hence, $(K :_R S)$ is a graded 2-absorbing ideal of R. □

Lemma 3.3. *Let R be a G-graded ring, J a graded 2-absorbing ideal of R and $I = \bigoplus_{g\in G} I_g$ a graded ideal of R. If $r,\, s \in h(R)$ and $g \in G$ with $rsI_g \subseteq J$, then either $rI_g \subseteq J$ or $sI_g \subseteq J$ or $rs \in J$.*

Proof. Let $r, s \in h(R)$ and $g \in G$ such that $rsI_g \subseteq J$ and $rs \notin J$. Let $i_g \in I_g$ so $rsi_g \in J$. Then $ri_g \in J$ or $si_g \in J$ as J is a graded 2-absorbing ideal of R. If $ri_g \in J$ for all $i_g \in I_g$, then $rI_g \subseteq J$, we are done. Similarly, if $si_g \in J$ for all $i_g \in I_g$, then $sI_g \subseteq J$, we are done. Suppose that there exist $i_{g1},\, i_{g2} \in I_g$ such that $ri_{g1} \notin J$ and $si_{g2} \notin J$. Since J is a graded 2-absorbing ideal, $rsi_{g1} \in J, ri_{g1} \notin J$ and $rs \notin J$, we conclude that $si_{g1} \in J$. Also $rsi_{g2} \in J$ implies that $ri_{g2} \in J$, since J is a graded 2-absorbing ideal. Since $rs(i_{g1} + i_{g2}) \in J$ and $rs \notin J$, we conclude that either $r(i_{g1} + i_{g2}) \in J$ or $s(i_{g1} + i_{g2}) \in J$ as J is a graded 2-absorbing ideal and hence either $si_{g2} \in J$ or $ri_{g1} \in J$, which is a contradiction. □

Lemma 3.4. *Let R be a G-graded ring and J a graded 2-absorbing ideal of R. Let $I = \bigoplus_{g\in G} I_g$ and $K = \bigoplus_{g\in G} K_g$ be a graded ideals of R. If $r \in h(R)$ and $g, h \in G$ with $rI_gK_h \subseteq J$, then either $rI_g \subseteq J$ or $rK_h \subseteq J$ or $I_gK_h \subseteq J$.*

Proof. Let $r \in h(R)$ and $g,\, h \in G$ such that $rI_gK_h \subseteq J$, $rI_g \nsubseteq J$ and $rK_h \nsubseteq J$. We have to show that $I_gK_h \subseteq J$. Assume that $i_g \in I_g$ and $k_h \in K_h$. By assumption there exist $i'_g \in I_g$ and $k'_h \in K_h$ such that $ri'_g \notin J$ and $rk'_h \notin J$. Since $ri'_gK_h \subseteq J$, $rK_h \nsubseteq J$ and $ri'_g \notin J$, by Lemma 3.3, we get $i'_gK_h \subseteq J$. Also, since $rk'_hI_g \subseteq J$, $rk'_h \notin J$ and $rI_g \nsubseteq J$, by Lemma 3.3, we get $k'_hI_g \subseteq J$. By $(i_g + i'_g) \in I_g$ and $(k'_h + k_h) \in K_h$, we get $r(i_g + i'_g)(k'_h + k_h) \in J$. Then either $r(i_g + i'_g) \in J$ or $r(k'_h + k_h) \in J$ as J is a graded 2-absorbing ideal. If $r(i_g + i'_g) \in J$, then $ri_g \notin J$. Which implies that $i_gk_h \in J$ by Lemma 3.3. Similarly, by $r(k'_h + k_h) \in J$, we conclude that $i_gk_h \in J$. Therefore, $I_gK_h \subseteq J$. □

Theorem 3.4. *Let R be a G-graded ring and J a proper graded ideal of R. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ and $K = \bigoplus_{g\in G} K_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *J is a graded 2-absorbing ideal of R.*
(ii) *For every $g, h, \lambda \in G$ with $I_gK_hL_\lambda \subseteq J$, either $I_gL_\lambda \subseteq J$ or $K_hL_\lambda \subseteq J$ or $I_gK_h \subseteq J$.*

Proof. (i)⇒(ii) Assume that J is a graded 2-absorbing ideal of R. Let $g, h, \lambda \in G$ such that $I_gK_hL_\lambda \subseteq J$ and $I_gL_\lambda \nsubseteq J$. Then for all $k_h \in K_h$ either $k_hI_g \subseteq J$ or $k_hL_\lambda \subseteq J$ by Lemma 3.4. If $k_hI_g \subseteq J$ for all $k_h \in K_h$, then $I_gK_h \subseteq J$, we are done. Similarly, if $k_hL_\lambda \subseteq J$ for all $k_h \in K_h$, then $K_hL_\lambda \subseteq J$, we are done. Suppose that $k_{h1},\, k_{h2} \in K_h$ are such that $k_{h1}I_g \nsubseteq J$ and $k_{h2}L_\lambda \nsubseteq J$. It follows that $k_{h1}L_\lambda \subseteq J$ and $k_{h2}I_g \subseteq J$. Since

$(k_{h1}+k_{h2})I_gL_\lambda \subseteq J$, by Lemma 3.4. we have $(k_{h1}+k_{h2})L_\lambda \subseteq J$ or $(k_{h1}+k_{h2})I_g \subseteq J$. By $(k_{h1}+k_{h2})L_\lambda \subseteq J$ it follows that $k_{h2}L_\lambda \subseteq J$, which is a contradiction. Similarly by $(k_{h1}+k_{h2})I_g \subseteq J$ we get a contradiction. Therefore $K_hL_\lambda \subseteq J$ or $I_gK_h \subseteq J$.

(ii)⇒(i) Assume that (ii) holds. Let $r_g, s_h, t_\lambda \in h(R)$ such that $r_gs_ht_\lambda \in J$. Let $I = r_gR$, $K = s_hR$ and $L = t_\lambda R$ be a graded ideals of R generated by r_g, s_h and t_λ, respectively. Then $I_gK_hL_\lambda \subseteq J$. By our assumption we obtain $I_gK_h \subseteq J$ or $I_gL_\lambda \subseteq J$ or $K_hL_\lambda \subseteq J$. Hence, $r_gs_h \in J$ or $r_gt_\lambda \in J$ or $s_ht_\lambda \in J$. Therefore, J is a graded 2-absorbing ideal of R. □

Theorem 3.5. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. Then for each $g \in G$, $I_g^nS = I_g^{n+1}S$ for all $n \geqslant 2$.*

Proof. Let $g \in G$. It is enough to show that $I_g^2S = I_g^3S$. It is clear that $I_g^3S \subseteq I_g^2S$. Since S a graded strongly 2-absorbing second submodule of M, $I_g^3S \subseteq I_g^3S$ implies that $I_g^2S \subseteq I_g^3S$ or $I_gS \subseteq I_g^3S$ or $I_g^3S = 0$ by Theorem 3.2. If $I_gS \subseteq I_g^3S$ or $I_g^2S \subseteq I_g^3S$, then we are done. Assume that $I_g^3S = 0$, hence $I_g^3 \subseteq Ann_R(S)$. By Lemma 3.2 and Theorem 3.4, we get $I_g^2 \subseteq Ann_R(S)$ and hence $I_g^2S \subseteq I_g^3S$. Therefore, $I_g^2S = I_g^3S$. □

Theorem 3.6. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. If $Gr(Ann_R(S)) = P$ for some graded prime ideal P of R, C_1 and C_2 are completely graded irreducible submodules of M such that $S \nsubseteq C_1$ and $S \nsubseteq C_2$. Then either $Gr((C_1 :_R S)) \subseteq Gr((C_2 :_R S))$ or $Gr((C_2 :_R S)) \subseteq Gr((C_1 :_R S))$.*

Proof. Assume that $Gr((C_1 :_R S))) \nsubseteq Gr((C_2 :_R S))$. Since $Gr((C_1 :_R S))$ is generated by $Gr((C_1 :_R S)) \cap h(R)$, there exists $r \in Gr((C_1 :_R S)) \cap h(R) - Gr((C_2 :_R S))$. Now, let $t \in Gr((C_2 :_R S)) \cap h(R)$. Then there exists a positive integer n such that $t^nS \subseteq C_2$, $r^nS \subseteq C_1$ and $r^nS \nsubseteq C_2$. Hence $t^nr^nS \subseteq C_1 \cap C_2$. So either $t^nS \subseteq C_1 \cap C_2$ or $t^nr^n \subseteq Ann_R(S)$ as S is a graded strongly 2-absorbing second submodule of M. If $t^nS \subseteq C_1 \cap C_2$, then $t^nS \subseteq C_1$, which implies $t \in Gr((C_1 :_R S))$. So, assume that $t^nr^n \subseteq Ann_R(S)$. Then $tr \in Gr(Ann_R(S)) = P$. Since P is a graded prime ideal of R, either $r \in P$ or $t \in P$. If $r \in P$, then $r^mS = 0 \in C_2$ for some $m \in \mathbb{Z}^+$ which is a contradiction. This yields that $t \in P = Gr(Ann_R(S)) \subseteq Gr((C_1 :_R S))$. Thus, $Gr((C_2 :_R S)) \subseteq Gr((C_1 :_R S))$. □

Theorem 3.7. *Let R be a G-graded ring and M, M' be two graded R-modules. Let $\varphi : M \to M'$ be a graded monomorphism. Then the following hold.*

(i) *If S is a graded strongly 2-absorbing second submodule of M, then $\varphi(S)$ is a graded 2-absorbing second submodule of M'.*
(ii) *If S' is a graded strongly 2-absorbing second submodule of M' and $S' \subseteq \varphi(M)$, then $\varphi^{-1}(S')$ is a graded 2-absorbing second submodule of M.*

Proof. By using Theorem 3.2 the proof is similar to that of Theorem 2.5. □

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GEODESIC E-INVEX SETS AND GEODESIC E-PREINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we first introduce two new classes of sets and functions called geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold, respectively. Moreover, we present the definition and properties of geodesic E-quasi-preinvex functions on Riemannian manifolds. Finally, we investigate the properties and characterizations of these two classes of sets and functions.

1. Introduction

Convexity plays an important and significant role in optimization theory. This concept in the linear topological vector spaces relies on the possibility of connecting any two points of the space by the line segment between them. Since convexity is often not enjoyed by the real problems, various approaches have been proposed by several reseachers in order to extend the validity of results to the larger classes of optimization. An important and significant generalization of convexity is invexity, which was introduced by Hanson [8] in 1981. Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the roles and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Ben-Isreal and Mond [5] introduced a new generalization of convex sets and convex functions that called by Craven [6] the invex sets and preinvex functions, respectively, see also [3].

In general, a manifold is not a linear space, but the extension of concepts and techniques from linear spaces to Riemannian manifolds are natural and applicable.

Key words and phrases. Geodesic E-invex set, geodesic E-preinvex function, geodesic E-quasi-preinvex function, Riemannian manifold.

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Rapcsak [18] and Udriste [19] proposed a generalization of convexity, called geodesic convexity, and extended many results of convex analysis and optimization theory to Riemannian manifolds. In this setting, the linear space has been replaced by a Riemannian manifold and the line segment by a geodesic. For more details, we refer the reader to [10–12, 15, 17, 18] and the references therein.

The notion of invex functions on Riemannian manifolds was introduced in [16]. However, its generalization has been investigated by Mititelu [13]. The concept of geodesic invex sets, geodesic invex functions and geodesic preinvex functions on a Riemannian Manifold with respect to the particular mappings have been introduced in [4].

In this paper, we first discuss various concepts, definitions and properties of functions defined on a Riemannian manifold. The notion of invexity and its generalization on Riemannian manifolds are presented in Section 2. In Section 3, we first define the concept of geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold. Next, we investigate their properties and characterizations. The class of geodesic E-quasi-preinvex functions are introduced in Section 4, and we give their characterizations.

2. Preliminaries

We first recall some definitions and known results about η-invex sets and geodesic η-preinvex functions on Riemannian manifolds, which will be used throughout the paper.

Let M be an n-dimensional differentiable manifold, and let T_pM be the tangent space to M at the point $p \in M$. Suppose that at each point $p \in M$, a positive inner product $g_p(x, y)$ on T_pM is given $(x, y \in T_pM)$. Recall that [12], a C^∞ mapping $g : p \to g_p$, which assigns a positive inner product g_p on T_pM to each point $p \in M$, is called a Riemannian metric. A manifold M equipped with the Riemannian metric g is called a Riemannian manifold. We denote by TM the tangent space to M.

Suppose that (M, g) is a complete n-dimensional Riemannian manifold with Riemannian connection ∇ (see [12]). Let x, y be two points in M, and $\gamma_{x,y} : [0, 1] \to M$ be a geodesic joining the points x and y, i.e., $\gamma_{x,y}(0) = y$, $\gamma_{x,y}(1) = x$.

Let us recall that [12] the length of a piecewise C^1 curve $\gamma : [a, b] \to M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\| dt.$$

For any two points $p, q \in M$, we define [12]

$$d(p, q) := \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ and } q\}.$$

Then, d is a distance which induces the original topology on M. We know that on every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection, denoted by ∇_{XY} for any vector fields $X, Y \in M$. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, i.e., γ satisfies the equation $\nabla_{d\gamma(t)/d(t)} d\gamma(t)/d(t)=0$. Any path γ joining p and $q \in M$ such that $L(\gamma) = d(p,q)$ is a geodesic and is called a minimal geodesic.

Definition 2.1 ([9])**.** A subset A of $\mathbb{R}^n$ is called η-invex with respect to the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if $x, y \in A$, $\lambda \in [0,1]$, then $y + \lambda\eta(x,y) \in A$.

It is obvious that Definition 2.1 is a generalization of the notion of a convex set (with $\eta(x,y) := x - y$). Note that any set in $\mathbb{R}^n$ is invex with respect to $\eta(x,y) \equiv 0$, for all $x, y \in \mathbb{R}^n$.

In 1987, Hanson and Mond [9] introduced the notion of preinvex functions. The following definition of a preinvex function has been given by Jeyakumar [19].

Definition 2.2 ([19])**.** Let f be a real valued function defined on an η-invex set $A \subseteq \mathbb{R}^n$. Then, f is said to be preinvex with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$f[y + \lambda\eta(x,y)] \leq \lambda f(x) + (1-\lambda) f(y), \quad \text{for all } x, y \in A,\ \lambda \in [0,1].$$

In the sequel, we consider the function $E : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.3. ([7, Definition 2.2]). A subset A of $\mathbb{R}^n$ is said to be E-invex with respect to a given mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$E(y) + \lambda\eta(E(x), E(y)) \in A, \quad \text{for all } x, y \in A,\ \lambda \in [0,1].$$

Definition 2.4. ([7, Definition 2.3]). Let $A \subseteq \mathbb{R}^n$ be an E-invex set with respect to a given mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be E-preinvex on A with respect to η if

$$f(E(y) + \lambda\eta(E(x), E(y)) \leq \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \text{for all } x, y \in A,\ \lambda \in [0,1].$$

The concept of geodesic invex sets and the invexity of a function f defined on an open geodesic invex subset of a Riemannian manifold were given in [4].

Definition 2.5. ([4, Definition 3.1]). Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x,y) \in T_y M$ for each $x, y \in M$. A nonempty subset S of M is said to be geodesic invex with respect to η if for each $x, y \in S$ there exists exactly one geodesic $\alpha_{x,y} : [0,1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x,y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0,1].$$

Recall that a subset S of a Riemannian manifold is called geodesic convex if any two points $x, y \in S$ can be joined by exactly one geodesic of length $d(x,y)$, which belongs entirely to S.

Definition 2.6. ([4, Definition 3.3]). Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x,y) \in T_y M$ for each $x, y \in M$. Let $S \subseteq M$ be a geodesic invex set with respect to η. We say that a function $f : S \to \mathbb{R}$ is geodesic η-preinvex if

$$f(\alpha_{x,y}(t)) \leq t f(x) + (1-t) f(y), \quad \text{for all } t \in [0,1],\ x, y \in S,$$

where $\alpha_{x,y}$ is the unique geodesic which defined by Definition 2.5. If the inequality is strict, then we say that f is a strictly geodesic η-preinvex function.

3. Geodesic E-Invex Sets and Geodesic E-Preinvex Functions

The definition of a preinvex function on $\mathbb{R}^n$ was given in [20], see also [3, 14, 21] for the properties of preinvex functions. Fulga and Preda [7] introduced the class of E-preinvex and E-quasi-preinvex functions defined on $\mathbb{R}^n$. In [4, 10, 11], this notion has been extended for Reimannian manifolds.

Throughout the paper, let $E : M \to M$ and $\eta : M \times M \to TM$ be fixed mappings. We now introduce the concept of geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold as follows.

Definition 3.1. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x, y) \in T_yM$ for each $x, y \in M$. A nonempty subset S of M is said to be geodesic E-invex with respect to η if for each $x, y \in S$ there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0, 1] \to M$ such that

$$\begin{aligned} &\alpha_{E(x),E(y)}(0) = E(y), \quad \alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y)),\\ &\alpha_{E(x),E(y)}(t) \in S, \quad \text{for all } t \in [0, 1]. \end{aligned}$$

Note that, in the special case, let $M := \mathbb{R}^n$, $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. Consider $\alpha_{x,y} : [0, 1] \to \mathbb{R}^n$ is defined by $\alpha_{x,y}(t) := y + t\eta(x, y)$ for all $t \in [0, 1]$. Then

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \lim_{t\to 0} \frac{\alpha_{x,y}(t) - \alpha_{x,y}(0)}{t} = \lim_{t\to 0} \frac{y + t\eta(x, y) - y}{t} = \eta(x, y),$$

and $\alpha_{x,y}(t) \in S$ for all $t \in [0, 1]$ because S is invex with respect to η. Therefore, the definition of geodesic invexity and geodesic E-invexity coincide in $\mathbb{R}^n$.

Definition 3.2. Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic E-invex set with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be geodesic E-preinvex with respect to η if

$$f(\alpha_{E(x),E(y)}(t)) \leq tf(E(x)) + (1 - t)f(E(y)), \quad \text{for all } t \in [0, 1],\ x, y \in S,$$

where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 3.1. If the inequality is strict, then we say that f is strictly geodesic E-preinvex with respect to η.

Let $M := \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ be a geodesic invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Consider $\alpha_{x,y}(t) = y + t\eta(x, y)$ for all $t \in [0, 1]$. Then $f(\alpha_{x,y}(t)) = f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y)$, i.e., the definition of geodesic preinvex and geodesic E-preinvex coincide for a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ whenever $M = \mathbb{R}^n$.

From now on, for simplicity, we will call geodesic E-invex set with respect to η, geodesic E-quasi-preinvex set with respect to η, geodesic E-preinvex function with respect to η and geodesic E-quasi-preinvex function with respect to η by geodesic E-invex set, geodesic E-quasi-preinvex set, geodesic E-preinvex function and gedesic E-quasi-preinvex function, respectively.

We now give some results related to geodesic E-convex sets on Riemannian manifolds (see also [1]).

Proposition 3.1. *Every geodesic invex set $A \subseteq M$ is geodesic E-invex.*

Proof. The proof is obvious by taking the mapping $E : M \to M$ as the identity map. $\square$

Proposition 3.2. *Let A be a subset of M. If A is a geodesic E-invex set, then $E(A) \subseteq A$.*

Proof. Since A is geodesic E-invex set, then for each $x, y \in A$ there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A$ for all $t \in [0,1]$. Put $t := 0$, then $E(y) = \alpha_{E(x),E(y)}(0) \in A$, so, $E(A) \subseteq A$. $\square$

Proposition 3.3. *Let $E(A)$ be an invex set. If $E(A) \subseteq A$, then A is a geodesic E-invex set.*

Proof. Let $x, y \in A$ be arbitrary. Then $E(x), E(y) \in E(A)$. Since $E(A)$ is invex with respect to η, thus there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in E(A) \subseteq A$ for all $t \in [0,1]$, hence, A is a geodesic E-invex set. $\square$

Proposition 3.4. *If $\{A_i\}_{i\in I}$ is an arbitrary collection of geodesic E-invex subsets of M with respect to the mapping $E : M \to M$, then $\cap_{i\in I} A_i$ is a geodesic E-invex subset of M.*

Proof. Let $\{A_i\}_{i\in I}$ be a collection of geodesic E-invex subsets of M with respect to the mapping $E : M \to M$. If $\cap_{i\in I} A_i = \emptyset$, we are done. Let $x, y \in \cap_{i\in I} A_i$ be arbitrary. Then $x, y \in A_i$ for all $i \in I$. By the geodesic E-invexity of A_i, there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A_i$ for all $t \in [0,1]$ and all $i \in I$, which implies that $\alpha_{E(x),E(y)}(t) \in \cap_{i\in I} A_i$ for all $t \in [0,1]$, and hence, $\cap_{i\in I} A_i$ is a geodesic E-invex set. $\square$

Lemma 3.1. *Let $A \subseteq M$ be a geodesic E_1-invex and E_2-invex set. Then A is a geodesic $E_1 \circ E_2$-invex and $E_2 \circ E_1$-invex set, where $E_1, E_2 : M \to M$ are arbitrary mappings.*

Proof. By the hypothesis, since $A \subseteq M$ is a geodesic E_1-invex and E_2-invex set, then for each $x, y \in A$ there exist exactly one geodesic $\alpha_{E_1(x),E_1(y)} : [0,1] \to M$ such that $\alpha_{E_1(x),E_1(y)}(0) = E_1(y)$, $\alpha'_{E_1(x),E_1(y)}(0) = \eta(E_1(x), E_1(y))$, $\alpha_{E_1(x),E_1(y)}(t) \in A$, and exactly one geodesic $\alpha_{E_2(x),E_2(y)} : [0,1] \to M$ such that $\alpha_{E_2(x),E_2(y)}(0) = E_2(y)$, $\alpha'_{E_2(x),E_2(y)}(0) = \eta(E_2(x), E_2(y))$ and $\alpha_{E_2(x),E_2(y)}(t) \in A$ for all $t \in [0,1]$. Now, let $x, y \in A$ be arbitrary. Put $x_0 := E_2(x)$ and $y_0 := E_2(y)$. Thus, in view of Proposition 3.2, we conclude that $x_0, y_0 \in A$. Therefore,

$$\alpha_{E_1\circ E_2(x),E_1\circ E_2(y)}(0) = \alpha_{E_1(x_0),E_1(y_0)}(0) = E_1(y_0) = E_1 \circ E_2(y)$$

and

$$\alpha'_{E_1\circ E_2(x),E_1\circ E_2(y)}(0) = \alpha'_{E_1(x_0),E_1(y_0)} = \eta(E_1(x_0), E_1(y_0)) = \eta(E_1 \circ E_2(x), E_1 \circ E_2(y))$$

and

$$\alpha_{E_1\circ E_2(x),E_1\circ E_2(y)}(t) = \alpha_{E_1(x_0),E_1(y_0)}(t) \in A, \quad \text{for all } t \in [0,1],$$

so, $A \subseteq M$ is a geodesic $E_1 \circ E_2$-invex set. Similarly, $A \subseteq M$ is a geodesic $E_2 \circ E_1$-invex set. □

Theorem 3.1. *Let $A \subseteq M$ be a geodesic invex set with respect to the function $\eta : M \times M \to TM$ and $f : A \to \mathbb{R}$ be a geodesic η-preinvex function. If $g : I \subseteq \mathbb{R} \to M$ is an increasing* (*strictly increasing*) *convex function such that $ran(f) \subseteq I$, then $g \circ f$ is geodesic* (*strictly geodesic*) *η-preinvex function on A.*

Proof. Since f is a geodesic η-preinvex functin, we have $f(\alpha_{x,y}(t)) \leq tf(x)+(1-t)f(y)$ for all $x, y \in A$ and all $t \in [0,1]$, where $\alpha_{x,y}$ is the unique geodesic which defined by Definition 2.5. Since g is an increasing convex function, we get

$$\begin{aligned} g[f(\alpha_{x,y}(t))] &\leq g[(1-t)f(y) + tf(x)] \\ &\leq (1-t)g(f(y)) + tg(f(x)) \\ &= (1-t)(g \circ f)(y) + t(g \circ f)(x), \end{aligned}$$

which shows that $g \circ f$ is a geodesic η-preinvex function on A. Similarly, we can show that $g \circ f$ is a strictly geodesic η-preinvex function if g is a strictly increasing convex function. □

Theorem 3.2. *Let $A \subseteq M$ be a geodesic E-invex set, and let $f_i : A \to \mathbb{R}$, $i = 1, \dots, p$ be a geodesic E-preinvex function. Then, $f := \sum_{i=1}^{p} \lambda_i f_i$ is a geodesic E-preinvex function on A with respect to the function η, where $\lambda_i \in \mathbb{R}$ with $\lambda_i \geq 0$, $i = 1, \dots, p$.*

Proof. By the hypothesis, for each $i = 1, \dots, p$, one has

$$f_i(\alpha_{E(x),E(y)}(t)) \leq (1-t)f_i(E(y)) + tf_i(E(x)),$$

where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 2.5. It follows that

$$\lambda_i f_i(\alpha_{E(x),E(y)}(t)) \leq (1-t)\lambda_i f_i(E(y)) + t\lambda_i f_i(E(x)),$$

and hence

$$\sum_{i=1}^{p} \lambda_i f_i(\alpha_{E(x),E(y)}(t)) \leq (1-t)\sum_{i=1}^{p} \lambda_i f_i(E(y)) + t\sum_{i=1}^{p} \lambda_i f_i(E(x)),$$

which completes the proof. □

Proposition 3.5. *Let M be a Riemannian manifold and $A \subseteq M$ be a geodesic E-invex set. Assume that $E : M \to M$ is an idempotent mapping* (*i.e., $E^2 = E$*)*. Suppose that $f \circ E : A \to \mathbb{R}$ is a geodesic E-preinvex function. Then the following holds.*

(*i*) *Every lower level set of $f \circ E$ which defined by $S(f \circ E, \lambda) := \{x \in A : (f \circ E)(x) \leq \lambda\}$, $\lambda \in \mathbb{R}$, is a geodesic E-invex set with respect to the function $\eta : M \times M \to TM$.*

(ii) The solution set K of the following optimization problem:

$$(P) \qquad \min(f \circ E)(x) \text{ subject to } x \in A,$$

is a geodesic E-invex set.

Moreover, if f is a strictly geodesic E-preinvex function, then K contains at most one point.

Proof. (i) Let $x, y \in S(f \circ E, \lambda) \subseteq A$ be arbitrary. Since A is a geodesic E-invex set with respect to the function η, then there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A$ for all $t \in [0,1]$. By the geodesic E-preinvexity of $f \circ E$, we have

$$\begin{aligned}
(f \circ E)(\alpha_{E(x),E(y)}(t)) &\leq tf(E(E(x))) + (1-t)f(E(E(y))) \\
&= t(f \circ E^2)(x) + (1-t)(f \circ E^2)(y) \\
&= tf(E(x)) + (1-t)f(E(y)) \\
&\leq t\lambda + (1-t)\lambda \\
&= \lambda, \quad \text{for all } t \in [0,1].
\end{aligned}$$

Therefore, $\alpha_{E(x),E(y)}(t) \in S(f \circ E, \lambda)$ for all $t \in [0,1]$, and so, $S(f \circ E, \lambda)$ is a geodesic E-invex set with respect to the function η.

(ii) Put $\alpha := \inf_{x \in A}(f \circ E)(x)$. Then, clearly $K = \cap_{\lambda > \alpha} S(f \circ E, \lambda)$, i.e., K is an intersection of geodesic E-invex sets, and so in view of Proposition 3.4, it is a geodesic E-invex set.

Now, suppose that f is a strictly geodesic E-preinvex function. If $K = \emptyset$, we are done. Assume that $K \neq \emptyset$. We claim that K has only one point. Assume if possible that there exist $x, y \in K$ such that $x \neq y$. Then, by the geodesic E-invexity of K with respect to the function η, there exists exactly one geodesic $\beta_{E(x),E(y)} : [0,1] \to M$ such that

$$\beta_{E(x),E(y)}(0) = E(y), \quad \beta'_{E(x),E(y)}(0) = \eta(E(x), E(y)),$$

and $\beta_{E(x),E(y)}(t) \in K$ for all $t \in [0,1]$. Since f is a strictly E-preinvex function, thus

$$\begin{aligned}
\alpha &= f(\beta_{E(x),E(y)}(t)) \\
&< tf(E(x)) + (1-t)f(E(y)) \\
&\leq t\alpha + (1-t)\alpha \\
&= \alpha, \quad \text{for all } t \in [0,1],
\end{aligned}$$

which is a contradiction. □

4. Generalized Geodesic E-preinvex Functions

In [16], it has been introduced the notion of η-quasi-preinvex functions on an invex set. In [2], this notion extended to geodesic η-quasi-preinvexity on a geodesic invex set by replacing the line segments with geodesics. In this section, we extend this concept and define geodesic E-quasi-preinvex functions. Moreover, some properties and characterizations of this class of functions are presented.

Definition 4.1. Let $A \subseteq M$ be a nonempty geodesic E-invex set with respect to $\eta : M \times M \to TM$. A function $f : A \to \mathbb{R}$ is said to be

(i) geodesic E-quasi-preinvex if

$$f(\alpha_{E(x),E(y)}(t)) \leq \max\{f(E(x)), f(E(y))\},$$

for all $x, y \in A$ and all $t \in [0, 1]$;

(ii) strictly geodesic E-quasi-preinvex if for all $x, y \in A$ with $E(x) \neq E(y)$ and all $t \in (0, 1)$, $f(\alpha_{E(x),E(y)}(t)) < \max\{f(E(x)), f(E(y))\}$.

Theorem 4.1. *Let $A \subseteq M$ be a geodesic E-invex set and let $\{f_i\}_{i \in I}$ be a collection of real valued functions defined on A such that $\sup_{i \in I} f_i(x)$ is finite for each $x \in A$. Let $f : A \to \mathbb{R}$ be defined by $f(x) := \sup_{i \in I} f_i(x)$ for each $x \in A$.*

(i) If $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-preinvex function on A with respect to the function $\eta : M \times M \to TM$, then the function f is geodesic E-preinvex on A.

(ii) If $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-quasi-preinvex function on A, then the function f is geodesic E-quasi-preinvex on A.

Proof. (i) Let $f_i : A \to \mathbb{R}$, $i \in I$, be a geodesic E-preinvex function on A. Then, for each $x, y \in A$ and each $t \in [0, 1]$, we have

$$f_i(\alpha_{E(x),E(y)}(t)) \leq (1-t)f_i(E(x)) + tf_i(E(y)), \quad \text{for all } i \in I,$$

and so

$$\begin{aligned} f(\alpha_{E(x),E(y)}(t)) &= \sup_{i \in I} f_i(\alpha_{E(x),E(y)}(t)) \\ &\leq \sup_{i \in I}[(1-t)f_i(E(x)) + tf_i(E(y))] \\ &\leq (1-t)\sup_{i \in I} f_i(E(x)) + t\sup_{i \in I} f_i(E(y)) \\ &= (1-t)f(E(x)) + tf(E(y)). \end{aligned}$$

So, f is a geodesic E-preinvex function on A.

(ii) Suppose that $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-quasi-preinvex function on A. Therefore, by Definition 4.1, for each $x, y \in A$ and each $t \in [0, 1]$, one has

$$\begin{aligned} f(\alpha_{E(x),E(y)}(t)) &= \sup_{i \in I} f_i(\alpha_{E(x),E(y)}(t)) \\ &\leq \sup_{i \in I} \max\{f_i(E(x)), f_i(E(y))\} \\ &\leq \max\{\sup_{i \in I} f_i(E(x)), \sup_{i \in I} f_i(E(y))\} \\ &= \max\{f(E(x)), f(E(y))\}, \end{aligned}$$

and hence, f is a geodesic E-quasi-preinvex function on A. □

Let $A \subseteq M$ be a nonempty geodesic E-invex set. It follows from Proposition 3.2 that $E(A) \subseteq A$. Hence, for any function $f : A \to \mathbb{R}$, define the restriction $\tilde{f}$ of f to $E(A)$ by $\tilde{f}(\tilde{x}) := f(\tilde{x})$ for all $\tilde{x} \in E(A)$.

Theorem 4.2. *Let $A \subseteq M$ be a geodesic E-invex set and let $f : A \to \mathbb{R}$ be a geodesic E-quasi-preinvex function on A. Then the restriction $\tilde{f} : C \to \mathbb{R}$ of f to any nonempty invex subset C of $E(A)$ is a geodesic η-quasi-preinvex function on C.*

Proof. Let $x, y \in C \subseteq E(A)$ be arbitrary. Then there exist $x_1, y_1 \in A$ such that $x = E(x_1)$ and $y = E(y_1)$. Since C is an invex set, there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0, 1] \to M$ such that $\alpha_{x,y}(0) = y$, $\alpha'_{x,y}(0) = \eta(x, y)$ and $\alpha_{x,y}(t) \in C$ for all $t \in [0, 1]$. But, $\alpha_{E(x_1),E(y_1)}(t) = \alpha_{x,y}(t) \in C$ for all $t \in [0, 1]$. Therefore, since f is a geodesic E-quasi-preinvex function on A, we conclude that

$$\begin{aligned}
\tilde{f}(\alpha_{x,y}(t)) &= f(\alpha_{E(x_1),E(y_1)}(t)) \\
&\leq \max\{f(E(x_1)), f(E(y_1))\} \\
&= \max\{f(x), f(y)\} \\
&= \max\{\tilde{f}(x), \tilde{f}(y)\},
\end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $A \subseteq M$ be a geodesic E-invex set, $f : A \to \mathbb{R}$ be a real valued function and $E(A)$ be an invex set. Then, f is geodesic E-quasi-preinvex on A if and only if its restriction $\tilde{f}$ to $E(A)$ is geodesic E-quasi-preinvex on $E(A)$.*

Proof. Let $x, y \in A$ be arbitrary. So, $E(x), E(y) \in E(A)$. By the hypothesis, $E(A)$ is an invex set. Therefore, by the definition, we have $\alpha_{E(x),E(y)}(t) \in E(A)$ for all $t \in [0, 1]$, where $\alpha_{E(x),E(y)}$ is the unique geodesic function corresponding to $E(A)$. Since $E(A) \subseteq A$ (because A is a geodesic E-invex set and using Proposition 3.4), it follows that

$$\alpha_{E(x),E(y)}(t) \in A, \quad \text{for all } t \in [0, 1], x, y \in A. \tag{4.1}$$

Now, suppose that f is a geodesic E-quasi-preinvex function on A. Then

$$\begin{aligned}
\tilde{f}(\alpha_{E(x),E(y)}(t)) &= f(\alpha_{E(x),E(y)}(t)) \\
&\leq \max\{f(E(x)), f(E(y))\} \\
&= \max\{\tilde{f}(E(x)), \tilde{f}(E(y))\},
\end{aligned}$$

i.e., $\tilde{f}$ is geodesic E-quasi-preinvex on $E(A)$.

Conversely, assume that $\tilde{f}$ is a geodesic E-quasi-preinvex function on $E(A)$. Then, by (4.1), for each $x, y \in A$ and each $t \in [0, 1]$, one has

$$\begin{aligned}
f(\alpha_{E(x),E(y)}(t)) &= \tilde{f}(\alpha_{E(x),E(y)}(t)) \\
&\leqslant \max\{\tilde{f}(E(x)), \tilde{f}(E(y))\} \\
&= \max\{f(E(x)), f(E(y))\},
\end{aligned}$$

and the proof is complete. ☐

An analogous result to Theorem 4.2 for the geodesic E-preinvex functions is presented as follows. The proof is similar to the one of Theorem 4.2.

Theorem 4.4. *Let $A \subseteq M$ be a geodesic E-invex set and $f : A \to \mathbb{R}$ be a geodesic E-preinvex function on A. Then, the restriction $\tilde{f} : C \to \mathbb{R}$ of f to any nonempty invex subset C of $E(A)$ is a geodesic invex function.*

An analogous result to Theorem 4.3 for the geodesic E-preinvex functions is presented as follows. The proof is similar to the one of Theorem 4.3.

Theorem 4.5. *Let $A \subseteq M$ be a geodesic E-invex set, $f : A \to \mathbb{R}$ be a real valued function and $E(A)$ be an invex set. Then, f is a geodesic E-preinvex function on A if and only if its restriction $\tilde{f}$ to $E(A)$ is a geodesic E-preinvex function on $E(A)$.*

We now characterize geodesic E-quasi-preinvex functions in terms of their lower level sets. For any real number $r \in \mathbb{R}$, the lower level set of the function $f \circ E : A \to \mathbb{R}$ is defined by $L_r(f \circ E) := \{x \in A : (f \circ E)(x) = f(E(x)) \leqslant r\}$. Moreover, the lower level set of the function $\tilde{f} : E(A) \to \mathbb{R}$ is defined by $L_r(\tilde{f}) := \{\tilde{x} \in E(A) : \tilde{f}(\tilde{x}) = f(\tilde{x}) \leqslant r\}$.

Theorem 4.6. *Let $E(A)$ be an invex set and $f : A \to \mathbb{R}$ be a real valued function. A function f is geodesic E-quasi-preinvex if and only if the lower level set $L_r(\tilde{f})$ is an invex set for each $r \in \mathbb{R}$.*

Proof. Suppose that f is a geodesic E-quasi-preinvex function. Since $E(A)$ is an invex set, for each $x, y \in A$, we have $E(x), E(y) \in E(A)$ and $\alpha_{E(x),E(y)}(t) \in E(A) \subseteq A$, where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 2.5. Let $r \in \mathbb{R}$ and $E(x), E(y) \in L_r(\tilde{f})$ be arbitrary. Put $\tilde{x} := E(x)$ and $\tilde{y} := E(y)$. Then, $\tilde{x}, \tilde{y} \in L_r(\tilde{f})$, and so, $f(\tilde{x}) \leqslant r$ and $f(\tilde{y}) \leqslant r$. Thus,

$$\tilde{f}(\alpha_{\tilde{x},\tilde{y}}(t)) = f(\alpha_{E(x),E(y)}(t)) \leqslant \max\{f(E(x)), f(E(y))\} = \max\{f(\tilde{x}), f(\tilde{y})\} \leqslant r,$$

which shows that $\alpha_{\tilde{x},\tilde{y}}(t) \in L_r(\tilde{f})$ for all $t \in [0,1]$. Moreover, one has $\alpha_{\tilde{x},\tilde{y}}(0) = \alpha_{E(x),E(y)}(0) = E(y) = \tilde{y}$ and $\alpha'_{\tilde{x},\tilde{y}}(0) = \alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y)) = \eta(\tilde{x}, \tilde{y})$ because $E(A)$ is an invex set. Hence, $L_r(\tilde{f})$ is an invex set.

Conversely, assume that $L_r(\tilde{f})$ is an invex set for each $r \in \mathbb{R}$. Let $x, y \in A$ and $t \in [0,1]$ be arbitrary. Take $r := \max\{f(E(x)), f(E(y))\}$ and $\tilde{x} := E(x)$, $\tilde{y} := E(y)$. Therefore, $\tilde{f}(\tilde{x}) = f(E(x)) \leq r$ and $\tilde{f}(\tilde{y}) = f(E(y)) \leq r$ because $E(x), E(y) \in E(A)$. This implies that $\tilde{x}, \tilde{y} \in L_r(\tilde{f})$. Since, by the hypothesis, $L_r(\tilde{f})$ is an invex set, so there exists exactly one geodesic $\alpha_{\tilde{x},\tilde{y}} : [0,1] \to M$ such that $\alpha_{\tilde{x},\tilde{y}}(0) = \tilde{y}$, $\alpha'_{\tilde{x},\tilde{y}}(0) = \eta(\tilde{x}, \tilde{y})$ and $\alpha_{\tilde{x},\tilde{y}}(t) \in L_r(\tilde{f})$ for all $t \in [0,1]$. Then, since $L_r(\tilde{f}) \subseteq E(A)$, it follows that

$$f(\alpha_{E(x),E(y)}(t)) = f(\alpha_{\tilde{x},\tilde{y}}(t)) = \tilde{f}(\alpha_{\tilde{x},\tilde{y}}(t)) \leqslant r = \max\{f(E(x)), f(E(y))\},$$

and so, f is a geodesic E-quasi-preinvex function. ☐

The geodesic E-quasi-preinvexity preserves under nondecreasing functions.

Theorem 4.7. *Let $A \subseteq M$ be a nonempty geodesic E-invex set and let $f : A \to \mathbb{R}$ be a geodesic E-quasi-preinvex function. Suppose that $\Phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Then $\Phi \circ f$ is a geodesic E-quasi-preinvex function on A.*

Proof. Since the function $f : A \to \mathbb{R}$ is geodesic E-quasi-preinvex and $\Phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, then, for all $x, y \in A$ and all $t \in [0, 1]$, it follows that

$$\begin{aligned}(\Phi \circ f)(\alpha_{E(x),E(y)}(t)) &= \Phi(f(\alpha_{E(x),E(y)}(t)))\\ &\leq \Phi\{\max\{f(E(x)), f(E(y))\}\\ &\leq \max\{\Phi(f(E(x)), \Phi(f(E(y)))\}\\ &= \max\{(\Phi \circ f)(E(x)), (\Phi \circ f)(E(y))\},\end{aligned}$$

which shows that $\Phi \circ f$ is a geodesic E-quasi-preinvex function on A. □

Theorem 4.8. *If the function $f : A \to \mathbb{R}$ is geodesic E-preinvex on A, then f is a geodesic E-quasi-preinvex function on A.*

Proof. Let f be geodesic E-preinvex on A. Then, for all $x, y \in A$ and all $t \in [0, 1]$, we have

$$\begin{aligned}f(\alpha_{E(x),E(y)}(t)) \leq& (1-t)f(E(y)) + tf(E(x))\\ \leq& (1-t)\max\{f(E(x)), f(E(y))\},\\ &+ t\max\{f(E(x)), f(E(y))\}\\ =& \max\{f(E(x)), f(E(y))\},\end{aligned}$$

and hence, f is a geodesic E-quasi-preinvex function on A. □

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ON UNIFORMLY STRONGLY PRIME Γ-SEMIHYPERRING

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Abstract. The Γ-semihyperring is a generalization of the concepts of a semiring, a semihyperring and a Γ-semiring. The concepts of uniformly strongly (weak) prime Γ-semihyperring and essential extension for the Γ-semihyperring are introduced and studied some important properties in this respect. It is proved that any essential extension of a uniformly strongly prime Γ-semihyperring is a uniformly weak prime Γ-semihyperring. Also strongly prime radical of a Γ-semihyperring is introduced and its characterization is made with the help of a super sp-system. A necessary and sufficient condition for a ideal of Γ-semihyperring to be a right strongly prime ideal is provided with the help of sp-system and super sp-system.

1. Introduction and Preliminaries

In 1975, Hadelman and Lawrence [4] introduced the notion of strongly prime ring motivated by the notion of primitive group ring and proved some properties of strongly prime rings. In 2006, Dutta and Das [2] introduced the notion of strongly prime ideal in a semiring and strongly prime semiring. Again in 2006, Dutta and Dhara [3] introduced the concept of uniformly strongly prime Γ-semirings and studied uniformly strongly prime k-radical of a Γ-semiring as special class via its operator semiring. The notion of essential ideal and essential extension for semirings was introduced and studied some important properties in this respect by Pawar and Deore [7].

The notion of hypergroup was introduced by Marty [5] in 1934. After that, many authors studied algebraic hyperstructure which are generalization of classical algebraic structure. In classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure composition of two elements is a

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set. Let H be a non-empty set. Then, the map $\circ : H \times H \to \wp^*(H)$ is called a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H and the couple $(H, \circ)$ is called a hypergroupoid. Moreover, the couple $(H, \circ)$ is called a semihypergroup if for every $a, b, c \in H$ we have $(a \circ b) \circ c = a \circ (b \circ c)$. The notion of Γ-semihyperrings as a generalization of semiring, semihyperring and Γ-semiring was introduced by Dehkordi and Davvaz [8]. Also, Pawar et al. [6] introduced regular (strongly regular) Γ-semihyperrings and made it's characterization with the help of ideals in Γ-semihyperrings.

In now days hyperstructure theory was studied widely as it has vast applications in various streams of sciences. In this paper, we extended various concepts of classical algebraic structure to a Γ-semihyperring. In Section 2, we introduced the notion of uniformly strongly prime Γ-semihyperring, essential ideal and essential extension for Γ-semihyperring and proved some important properties. In Section 3, we introduced the notion of right strongly prime ideal and super sp-system. These concepts are studied analogously with the concepts of classical algebraic structures which are studied in [2,3].

Here are some useful definitions and the readers are requested to refer [8].

Definition 1.1. Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$ for all $a, b \in R$ and $\alpha \in \Gamma$) and $\wp^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$;
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$;
(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$;
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Definition 1.2. A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 1.3. A Γ-semihyperring R is said to be with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$. Then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\},$$
$$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 1.4. A non-empty subset R_1 of Γ-semihyperring R is said to be a Γ-subsemihyperring if it is closed with respect to the addition and multiplication, that is, $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 1.5. A right (left) ideal I of a Γ-semihyperring R is an additive sub semihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I(R\Gamma I \subseteq I)$. If I is both right and left ideal of R, then we say that I is a two sided ideal or simply an ideal of R.

2. Uniformly Strongly (Weak) Prime Γ-Semihyperrings

Definition 2.1. A Γ-semihyperring R is said to be a finitely multiplicative if F and G are finite subsets of R and Δ is finite subset of Γ, then $F\Delta G$ is finite subset of R.

Definition 2.2. A Γ-semihyperring R is said to be a finitely additive if F and G are finite subsets of R, then $F+G$ is finite subset of R.

Example 2.1 ([6]). Consider the following:

$$\begin{aligned} R &= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{R} \right\}, \\ \Gamma &= \{z \mid z \in \mathbb{Z}\}, \\ A_\alpha &= \left\{ \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \mid a, b \in \mathbb{R}, \alpha \in \Gamma \right\}. \end{aligned}$$

Then, R is a Γ-semihyperring under the matrix addition with hyperoperation $M\alpha N \mapsto MA_\alpha N$ for all $M, N \in R$ and $\alpha \in \Gamma$. Here R is a finitely additive but not finitely multiplicative.

Example 2.2. Let X be a non-empty set and τ is a topology on X. We define the hyperoperation of the addition and the multiplication on τ as $A, B \in \tau, A + B = A \cup B, A \cdot B = A \cap B$. Then τ is a Γ-semihyperring, where Γ is a commutative group, if we define $x\alpha y \mapsto x \cdot y$ for every $x, y \in \tau, \alpha \in \Gamma$. Here τ is a finitely additive as well as finitely multiplicative.

Throughout this paper we consider that a Γ-semihyperring R is always finitely multiplicative, finitely additive and contains a zero element.

Definition 2.3. A Γ-semihyperring R is called uniformly right strongly prime if there exist a finite subset F of R and a finite subset Δ of Γ if $0 \notin A \subseteq R$ and $0 \in A\delta_1 f\delta_2 B$ for all $\delta_1, \delta_2 \in \Delta$ and $f \in F$ implies that $0 \in B$. The pair (F, Δ) is called a uniform right insulator for R.

Definition 2.4. A Γ-semihyperring R is called uniformly right weak prime if there exist a finite subset F of R and a finite subset Δ of Γ if $x(\neq 0) \in R$ and $0 \in x\delta_1 f\delta_2 y$ for all $\delta_1, \delta_2 \in \Delta$ and $f \in F$ implies that $y = 0$. The pair (F, Δ) is called a uniform right insulator for R.

Analogously we can define uniformly left strongly (weak) prime Γ-semihyperring. It is obvious that a uniformly right (left) strongly prime Γ-semihyperring R is uniformly right (left) weak prime.

Definition 2.5 ([1])**.** A Γ-semihyperring R with zero is called prime if $0 \in x\alpha r\beta y$ for all $r \in R$ and $\alpha, \beta \in \Gamma$ implies that either $x = 0$ or $y = 0$.

Theorem 2.1. *A Γ-semihyperring R is uniformly right weak prime if and only if there exist finite subsets F of R and Δ of Γ such that for any two nonzero elements x and y of R, there exists $f \in F$ and $\delta_1, \delta_2 \in \Gamma$ such that $0 \notin x\delta_1 f\delta_2 y$.*

Proof. Let R be a uniformly right weak prime Γ-semihyperring and (F, Δ) be a uniform right insulator for R. Suppose x and y be a two nonzero elements of R and $0 \in x\delta_1 f\delta_2 y$, for all $\delta_1, \delta_2 \in \Gamma$ and $f \in F$. Then we get $y = 0$, a contradiction. So there exist $f \in F$ and $\delta_1, \delta_2 \in \Gamma$ such that $0 \notin x\delta_1 f\delta_2 y$.

Conversely, let for any two nonzero elements x and y of R there exist $f \in F$ and $\delta_1, \delta_2 \in \Gamma$ such that $0 \notin x\delta_1 f\delta_2 y$. Consider $a(\neq 0) \in R$ and $0 \in a\delta_1 f\delta_2 b$, for all $\delta_1, \delta_2 \in \Gamma$ and $f \in F$ so by our hypothesis b must be 0. Therefore, by definition Γ-semihyperring R is uniformly right weak prime. $\square$

Theorem 2.2. *A Γ-semihyperring R is uniformly right strongly prime if and only if there exist finite subsets F of R and Δ of Γ such that for any two non-empty subsets A and B of R and $0 \notin A, 0 \notin B$, there exist $f \in F$ and $\delta_1, \delta_2 \in \Delta$ such that $0 \notin A\delta_1 f\delta_2 B$.*

Corollary 2.1. *A Γ-semihyperring R is uniformly right weak (strongly) prime if and only if R is uniformly left weak (strongly) prime.*

So, we can use uniformly strongly (weak) prime instead of uniformly right (left) strongly (weak) prime and uniform insulator instead of uniform right (left) insulator.

Proposition 2.1. *A uniformly weak prime Γ-semihyperring is prime.*

Proof. Let R be a uniformly weak prime Γ-semihyperring and (F, Δ) is a uniform insulator for R. Let $x(\neq 0) \in R$ and $0 \in x\alpha r\beta y$ for all $\alpha, \beta \in \Gamma$ and $r \in R$. Now, $F \subseteq R$ and $\Delta \subseteq \Gamma$, so $0 \in x\delta_1 f\delta_2 y$, for all $\delta_1, \delta_2 \in \Delta$ and $f \in F$. Since R is a uniformly weak prime Γ-semihyperring and (F, Δ) is a uniform insulator for R, then $y = 0$. Therefore, by definition, R is a prime Γ-semihyperring. $\square$

Proposition 2.2. *If R is uniformly weak prime Γ-semihyperring, then for nonzero ideal I of R, there exist finite subsets F of I and Δ of Γ such that $0 \in f\delta y$ for all $f \in F$ and $\delta \in \Delta$, then $y = 0$.*

Proof. Let I be a nonzero ideal of a uniformly weak prime Γ-semihyperring R and (F, Δ) is a uniform insulator for R. Let $x(\neq 0) \in I$. Then $F' = x\Delta F$ is finite subset of I. Also if $0 \in x\delta_1 f\delta_2 y$ for all $\delta_1, \delta_2 \in \Delta$ and $f \in F$, then $y = 0$. Then $0 \in f^{'}\delta y$, for all $f^{'} \in F^{'}, \delta \in \Delta$ implies that $0 \in x\delta_1 f\delta_2 y$, for all $\delta_1, \delta_2 \in \Delta$ and $f \in F$ gives that $y = 0$. This complete the proof. $\square$

Definition 2.6. A nonzero ideal I of a Γ-semihyperring R is called an essential ideal of R if for any nonzero ideal J of R, $x(\neq 0) \in I \cap J$.

Definition 2.7. A Γ-semihyperring T is said to be an essential extension of a Γ-semihyperring R if R is an essential ideal of T.

Definition 2.8. Let A be a non-empty subset of a Γ-semihyperring R. Right annihilator of A in R, denoted by $ann_r(A)$, is defined as $ann_r(A) = \{x \in R \mid 0 \in a\alpha x$ for all $a \in A, \alpha \in \Gamma\}$.

Similarly, we can define left annihilator of A in R, i.e., $ann_l(A)$.

Lemma 2.1. *Let R be a Γ-semihyperring and T be its essential extension. If R is a uniformly strongly prime Γ-semihyperring, then for each nonzero x of T, $0 \in x\alpha f$ for all $\alpha \in \Gamma, f \in F$ implies that $x \in ann_r(R)$ and $0 \in f\alpha x$ for all $\alpha \in \Gamma, f \in F$ implies that $x \in ann_l(R)$, where (F, Δ) is a uniform insulator for R.*

Proof. Let T be an essential extension of a uniformly strongly prime Γ-semihyperring R and (F, Δ) is uniform insulator for R. Let $x(\neq 0) \in T$ and $0 \in x\alpha f$ for all $\alpha \in \Gamma, f \in F$. Then $0 \in (k\gamma x)\delta_1 f \delta_2 (k\gamma x)$ for all $\delta_1, \delta_2 \in \Delta, \gamma \in \Gamma, f \in F$ and $k \in R$. Since R is a uniformly strongly prime Γ-semihyperring and (F, Δ) is a uniform insulator for $R, 0 \in k\gamma x$ for all $k \in R, \gamma \in \Gamma$, i.e., $x \in ann_r(R)$.

On similar lines, we can prove $0 \in f\alpha x$ for all $\alpha \in \Gamma$, $f \in F$ implies that $x \in ann_l(R)$. □

Lemma 2.2. *If R is a uniformly strongly prime Γ-semihyperring and I is an ideal of R, then I is a uniformly weak prime Γ-subsemihyperring.*

Proof. Let R be a uniformly strongly prime Γ-semihyperring and (F, Δ) be a uniform insulator for R. If I is zero ideal, then obviously I is a uniformly weak prime Γ-subsemihyperring. Suppose $I \neq 0$ and r be a fixed nonzero element of I. Let $F^{'} = \{x \in f_1\alpha r\beta f_2 \mid f_1, f_2 \in F, \alpha, \beta \in \Delta\}$. Since I is an ideal of R and F, Δ are finite subsets, $F^{'}$ is finite subset of I. Let $x(\neq 0) \in I$ and $y \in I$. If $0 \in x\delta_1 f^{'} \delta_2 y$ for all $\delta_1, \delta_2 \in \Delta$ and $f^{'} \in F^{'}$, then $0 \in x\delta_1 f_1 \alpha r \beta f_2 \delta_2 y$ for all $f_1, f_2 \in F$ and for all $\delta_1, \delta_2, \alpha, \beta \in \Delta$, i.e., $0 \in x\delta_1 f_1 \alpha(r\beta f_2 \delta_2 y)$ for all $f_1, f_2 \in F$ and for all $\delta_1, \delta_2, \alpha, \beta \in \Delta$. Since $r\beta f_2 \delta_2 y \subseteq R$, for all $f_2 \in F$ and for all $\beta, \delta_2 \in \Delta$ and R is a uniformly strongly prime Γ-semihyperring with $x \neq 0$, then $0 \in r\beta f_2 \delta_2 y$ for all $f_2 \in F$ and for all $\beta, \delta_2 \in \Delta$. But as $r \neq 0$ it gives $y = 0$. Hence, I is a uniformly weak prime Γ-semihyperring and $(F^{'}, \Delta)$ is uniform insulator for I. □

Definition 2.9. An element k of a Γ-semihyperring R is additively aggressive with respect to subset A of R if k belongs to $a\alpha b(b\alpha a)$ and $a\alpha c(c\alpha a)$ for all $a \in A$ and $\alpha \in \Gamma$, then for any $p \in b + c$, $k \in a\alpha p$ $(k \in p\alpha a)$ for all $a \in A$ and $\alpha \in \Gamma$.

Definition 2.10. An element k of a Γ-semihyperring R is multiplicatively aggressive with respect to subset A of R if k belongs to $a\alpha b(b\alpha a)$ for all $a \in A$ and $\alpha \in \Gamma$, then for any $p \in b\alpha t(p \in t\alpha b)$, where $\alpha \in \Gamma$, $t \in R$, we have $k \in a\alpha p(k \in p\alpha a)$ for all $a \in A$ and $\alpha \in \Gamma$.

Example 2.1 zero element (zero matrix) is a multiplicatively aggressive.

Definition 2.11. An element k of a Γ-semihyperring R is additively and multiplicatively aggressive with respect to all subset A of R, then k is aggressive element of a Γ-semihyperring R.

Example 2.3 ([6]). Let $R = \{a, b, c, d\}$. Then R is commutative semihypergroup with following hyperoperations

$+$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$	$\{b, d\}$
c	$\{a, c\}$	$\{b, c\}$	$\{c\}$	$\{c, d\}$
d	$\{a, d\}$	$\{b, d\}$	$\{c, d\}$	$\{d\}$

,

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c, d\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$	$\{b, c, d\}$
c	$\{a, b, c\}$	$\{b, c\}$	$\{c\}$	$\{c, d\}$
d	$\{a, b, c, d\}$	$\{b, c, d\}$	$\{c, d\}$	$\{d\}$

.

Then R be a Γ-semihyperring, where Γ-is any commutative group with operation $x\alpha y \mapsto x \cdot y$ for $x, y \in R$ and $\alpha \in \Gamma$. Here a is a aggressive element of R.

Theorem 2.3. *If zero is an aggressive element of a Γ-semihyperring R, then $ann_r(A)$ is a right ideal of R and $ann_l(A)$ is a left ideal of R. If A is an ideal of Γ-semihyperring R, then both annihilators are ideals of R.*

Now in the rest part of the given section we consider zero as aggressive element of Γ-semihyperring R.

Lemma 2.3. *Let R be a uniformly weak prime Γ-semihyperring and T be its essential extension. Then both annihilators of R in T are zero.*

Proof. Let (F, Δ) is a uniform insulator for R. If possible let $ann_r(R) \neq 0$. Then $ann_r(R)$ is nonzero ideal of T. Since R is an essential ideal of T, $ann_r(R) \cap R \neq 0$. Let $x(\neq 0) \in ann_r(R) \cap R$. Then $0 \in k\alpha x$, for all $k \in R, \alpha \in \Gamma$. As $\Delta \subseteq \Gamma$ and $F \subseteq R$, it gives $0 \in x\delta_1 r\delta_2 x$ for all $\delta_1, \delta_2 \in \Delta$ and $r \in F$. Since R is a uniformly weak prime Γ-semihyperring, $x = 0$, a contradiction. Therefore $ann_r(R) = 0$.

Similarly, we can prove that $ann_l(R) = 0$. □

Lemma 2.4. *Let R be a uniformly strongly prime Γ-semihyperring with pair (F, Δ) be a uniform insulator for R and T be its essential extension. Then for any nonzero element x of T there exist some $f \in F$, $\delta \in \Delta$ such that $0 \notin x\delta f$.*

Proof. Let (F, Δ) be a uniform insulator for R and T be an essential extension of R. Let x be a nonzero element of T. Suppose that $0 \in x\delta f$ for all $\delta \in \Delta$, $f \in F$, then by Lemma 2.1, $x \in ann_r(R)$. Also, by Lemma 2.3, $ann_r(R) = 0$, which implies that $x = 0$, a contradiction. Therefore, $0 \notin x\delta f$ for some $f \in F, \delta \in \Delta$. □

Lemma 2.5. *Let R be a uniformly strongly prime Γ-semihyperring with pair (F, Δ) be a uniform insulator for R and T be its essential extension. Then for any nonzero element x of T there exist some $f \in F$, $\delta \in \Delta$ such that $0 \notin f\delta x$.*

Theorem 2.4. *Any essential extension of a uniformly strongly prime* Γ*-semihyperring* R *is a uniformly weak prime* Γ*-semihyperring.*

Proof. Let (F, Δ) be a uniform insulator for R and T be an essential extension of R. Let y, z be two nonzero elements of T. Then by Lemmas 2.4 and 2.5, there exist $f_1, f_2 \in F$ and $\delta_1, \delta_2 \in \Delta$ such that $0 \notin y\delta_1 f_1$ and $0 \notin f_2\delta_2 z$. Since R is an ideal of T, so $y\delta_1 f_1$ and $f_2\delta_2 z$ are subsets of R. Again since R is uniformly strongly prime and (F, Δ) be a uniform insulator for R, then by Theorem 2.2, there exist $\alpha, \beta \in \Delta$ and $f \in F$ such that $0 \notin y\delta_1 f_1 \alpha f \beta f_2 \delta_2 z$. Let $F' = \{k \in f_1 \alpha f \beta f_2 \mid 0 \notin y\delta_1 f_1 \alpha f \beta f_2 \delta_2 z; f_1, f, f_2 \in F, \alpha, \beta, \delta_1, \delta_2 \in \Delta, y, z \in T\}$. Then $F' \subseteq T$ is finite subset, since F and Δ are finite subset. Hence, by Theorem 2.1, T is uniformly weak prime Γ-semihyperring with insulator (F', Δ). □

3. Right Uniformly Strongly Prime Radical

Definition 3.1. An ideal I of a Γ-semihyperring R is said to be right strongly prime if $a \notin I$, then there are two finite sets $F \subseteq < a >$ and $\Delta \subseteq \Gamma$ such that $F\Delta b \subseteq I$ implies that $b \in I$.

Definition 3.2. A subset G of a Γ-semihyperring R is called an sp-system if for any $g \in G$ there are two finite sets $F \subseteq < g >$ and $\Delta \subseteq \Gamma$ such that $(f\delta z) \cap G \neq \emptyset$ for all $f \in F$, $\delta \in \Delta$ and $z \in G$.

Proposition 3.1. *An ideal* I *of a* Γ*-semihyperring* R *is a right strongly prime if and only if* $R \setminus I$ *is an sp-system.*

Proof. Let I be a right strongly prime ideal of R and let $g \in R \setminus I$. Then $g \notin I$. So there exists a finite subsets F of $< g >$ and Δ of Γ such that $F\Delta b \subseteq I$ implies that $b \in I$, i.e., $(f\delta z) \cap (R \setminus I) \neq \emptyset$ for all $f \in F$, $\delta \in \Delta$ and $z \in R \setminus I$. Therefore, $R \setminus I$ is an sp-system.

Conversely, suppose $R \setminus I$ is an sp-system. Let $a \notin I$. Then $a \in R \setminus I$. So there exists a finite subsets F of $< a >$ and Δ of Γ such that $(f\delta z) \cap (R \setminus I) \neq \emptyset$ for all $f \in F, \delta \in \Delta$ and $z \in R \setminus I$. Let $F\Delta b \subseteq I$. Then $F\Delta b \cap (R \setminus I) = \emptyset$. If possible let $b \notin I$. Then $b \in R \setminus I$ which implies that $(f\delta b) \cap (R \setminus I) \neq \emptyset$ for all $f \in F$, $\delta \in \Delta$, a contradiction. Hence, $b \in I$. Therefore, I is a right strongly prime ideal of R. □

Definition 3.3. Right strongly prime radical of a Γ-semihyperring R is a defined by $SP(R) = \cap\{I \mid I$ is a right strongly prime ideal of $R\}$.

Definition 3.4. A pair of subsets (G, P) where P is an ideal of a Γ-semihyperring R and G is a non-empty subset of R is called a super sp-system of R if $G \cap P$ contains no nonzero element of R and for any $g \in G$ there are finite subsets F of $< g >$ and Δ of Γ such that $(f\delta z) \cap G \neq \emptyset$ for all $f \in F$, $\delta \in \Delta$ and $z \notin P$.

Remark 3.1. An ideal I of a Γ-semihyperring R is a right strongly prime ideal if and only if $(R \setminus I, I)$ is super sp-system.

Theorem 3.1. *Let any* Γ*-semihyperring* R*. Then* $x \in SP(R)$ *if and only if whenever* $x \in G$ *and* (G, P) *is super sp-system for some ideal* P *of* R*, then* $0 \in G$*.*

Proof. Let $x \in SP(R)$. If possible let $x \in G$ where (G, P) is a super sp-system and $0 \notin G$. Then $G \cap P = \emptyset$. By Zorn's Lemma choose an ideal Q with $P \subseteq Q$ and Q is maximal with respect to $G \cap Q = \emptyset$. We now prove that Q is a right strongly prime ideal of R. Let $a \notin Q$. Then there is a $g \in G$ such that $< g > \subseteq Q + < a >$. Since (G, P) is a super sp-system there exists a finite subsets $F = \{f_1, f_2, \ldots, f_m\} \subseteq < g >$ and $\Delta \subseteq \Gamma$ such that $f_i \delta z \cap G \neq \emptyset$ for all $f_i \in F$, $\delta \in \Delta$ and $z \notin P$. Since $F \subseteq Q + < a >$ each $f_i \in q_i + a_i$ for some $q_i \in Q$ and $a_i \in < a >$. Let $F^* = \{a_1, a_2, \ldots, a_m\}$. Then $F^* \subseteq < a >$. Let $z \in R$ such that $f_i^* \delta z \subseteq Q$ for all $f_i^* \in F^*$, $\delta \in \Delta$. Then $f_i \delta z \subseteq (q_i + ai)\delta z \subseteq Q$ for all $f_i \in F$, $\delta \in \Delta$, i.e., $F \Delta z \subseteq Q$. If $z \notin Q$, then $f_i \delta z \cap G \neq \emptyset$, because $P \subseteq Q$. But this contradict $G \cap Q = \emptyset$. Hence, $z \in Q$ must hold. So, Q is a right strongly prime ideal. But $x \notin Q$, since $x \in G$, which is a contradiction. Hence, $0 \in G$.

Conversely, let whenever $x \in G$ and (G, P) is super sp-system for some ideal P of R, then $0 \in G$. Then there exists a right strongly prime ideal I of R such that $x \notin I$. Then $(R \setminus I, I)$ is a super sp-system where $x \in R \setminus I$ but $0 \notin R \setminus I$, a contradiction. Hence, converse follows. ☐

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APPLICATIONS POISSON DISTRIBUTION AND RUSCHEWEYH DERIVATIVE OPERATOR FOR BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we establish upper bounds for the second and third coefficients of holomorphic and bi-univalent functions in a new family which involve the Bazilevič functions and β-pseudo-starlike functions under a new operator joining Poisson distribution with Ruscheweyh derivative operator. Also, we discuss Fekete-Szegö problem of functions in this family.

1. INTRODUCTION

Let $\mathcal{A}$ be the collection of functions f that are holomorphic in the unit disk $\mathbb{D} = \{|z| < 1\}$ in the complex plane $\mathbb{C}$ and that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Further, let $\mathcal{S}$ be the sub-collection of $\mathcal{A}$ containing of functions which are univalent in $\mathbb{D}$. According to the Koebe one-quarter theorem (see [3]), every function $f \in \mathcal{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and $f(f^{-1}(w)) = w$, $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. If f is of the form (1.1), then

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots, \quad |w| < r_0(f). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. We denote by Σ the set of bi-univalent functions in $\mathbb{D}$. Srivastava et al. [19] have apparently resuscitated the study of holomorphic and bi-univalent functions in

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recent years. It was followed by such works as those by Frasin and Aouf [5], Goyal and Goswami [6], Srivastava and Bansal [15] and others (see, for example [2, 16–18, 20]).

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the (M,N)-Lucas Polynomials $L_{M,N,k}(x)$ are defined by the following recurrence relation (see [8]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x), \quad k \geq 2,$$

with

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \text{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x). \tag{1.3}$$

The Lucas Polynomials play an important role in a diversity of disciplines in the mathematical, statistical, physical and engineering sciences (see, for example [4, 9, 21]). The generating function of the (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ (see [9]) is given by

$$T_{M(x),N(x)}(z) = \sum_{k=2}^{\infty} L_{M,N,k}(x)z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}. \tag{1.4}$$

Let the functions f and g be holomorphic in $\mathbb{D}$, we say that the function f is subordinate to g, if there exists a function w, holomorphic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathbb{D}$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence (see [10])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

A function $f \in \mathcal{A}$ is called Bazilevič function of order α, $\alpha \geq 0$, if (see [14])

$$\operatorname{Re}\left\{\frac{z^{1-\alpha}f'(z)}{(f(z))^{1-\alpha}}\right\} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called β-pseudo-starlike function of order β, $\beta \geq 1$, if (see [1])

$$\operatorname{Re}\left\{\frac{z\,(f'(z))^{\beta}}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

Recall that a random variable X has the Poisson distribution with parameter θ, if

$$P(X = r) = \frac{\theta^r e^{-\theta}}{r!}, \quad r = 0, 1, 2, 3, \ldots$$

Recently, Porwal [11] introduced a power series whose coefficients are probabilities of "Poisson distribution"

$$K(\theta, z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!} e^{-\theta} z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$. By ratio test the radius of convergence of the above series is infinity.

In 2016, Porwal and Kumar [12] introduced and investigated a linear operator $I(\theta, z) : \mathcal{A} \to \mathcal{A}$, $\theta > 0$, by using the Hadamard product (or convolution) and defined as follows

$$I(\theta, z)f(z) = K(\theta, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}}{(n-1)!} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where "$*$" indicate the Hadamard product (or convolution) of two power series.

In this paper, for $f \in \mathcal{A}$ we introduce a new linear operator $\mathfrak{J}_\theta^\delta : \mathcal{A} \to \mathcal{A}$ by

$$\mathfrak{J}_\theta^\delta f(z) = I(\theta, z) * \mathcal{R}^\delta, \tag{1.5}$$

where $\mathcal{R}^\delta$, $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, denote the Ruscheweyh derivative operator [13] given by

$$\mathcal{R}^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta + n)}{\Gamma(\delta + 1)\Gamma(n)} a_n z^n, \quad z \in \mathbb{D}.$$

It is easy to obtain from (1.5) that

$$\mathfrak{J}_\theta^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\theta^{n-1}\Gamma(\delta + n)}{\Gamma(\delta + 1)\left(\Gamma(n)\right)^2} e^{-\theta} a_n z^n, \quad z \in \mathbb{D},$$

where $\theta > 0$, $\delta \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

2. Main Results

We begin this section by defining the family $\Upsilon_\Sigma(\lambda, \alpha, \beta, \delta, \theta; h)$ as follows.

Definition 2.1. Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$ and h is analytic in $\mathbb{D}$, $h(0) = 1$. The function $f \in \Sigma$ is in the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ if it fulfills the subordinations:

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\left(\mathfrak{J}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda \frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda \frac{w\left(\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f^{-1}(w)} \prec 1 + e_1 z + e_2 z^2 + \cdots,$$

where f^{-1} is given by (1.2).

In particular, if we choose $\lambda = 1$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$ reduces to the family $\mathcal{L}_\Sigma(\beta, \delta, \theta; h)$ of β-pseudo bi-starlike functions which satisfying the following subordinations:

$$\frac{z\left(\left(\mathfrak{J}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$\frac{w\left(\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{J}_\theta^\delta f^{-1}(w)} \prec h(w).$$

If we choose $\lambda = 0$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;h)$ reduces to the family $\mathcal{B}_\Sigma(\alpha,\delta,\theta;h)$ of Bazilevič bi-univalent functions which satisfying the following subordinations:

$$\frac{z^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\left(\mathfrak{J}_\theta^\delta f(z)\right)^{1-\alpha}} \prec h(z)$$

and

$$\frac{w^{1-\alpha}\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} \prec h(w).$$

If we choose $\lambda = \beta = 1$ in Definition 2.1, the family $\Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;h)$ reduces to the family $\mathcal{S}_\Sigma(\delta,\theta;h)$ of bi-starlike functions which satisfying the following subordinations:

$$\frac{z\left(\mathfrak{J}_\theta^\delta f(z)\right)'}{\mathfrak{J}_\theta^\delta f(z)} \prec h(z)$$

and

$$\frac{w\left(\mathfrak{J}_\theta^\delta f^{-1}(w)\right)'}{\mathfrak{J}_\theta^\delta f^{-1}(w)} \prec h(w).$$

Theorem 2.1. *Assume that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$ and $\theta > 0$. If $f \in \Sigma$ of the form* (1.1) *is in the class $\Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;h)$, with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then*

$$|a_2| \leq \frac{|e_1|}{\left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta}} = \frac{|e_1|}{A} \tag{2.1}$$

and

$$|a_3| \leq \min\left\{\max\left\{\left|\frac{e_1}{B}\right|, \left|\frac{e_2}{B} - \frac{Ce_1^2}{A^2B}\right|\right\}, \max\left\{\left|\frac{e_1}{B}\right|, \left|\frac{e_2}{B} - \frac{(2B+C)e_1^2}{A^2B}\right|\right\}\right\}, \tag{2.2}$$

where

$$\begin{aligned} A &= \left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta},\\ B &= \frac{1}{4}\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right](\delta^2+3\delta+2)\theta^2 e^{-\theta},\\ C &= \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1)+\lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2 e^{-2\theta}. \end{aligned} \tag{2.3}$$

Proof. Suppose that $f \in \Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;;e_1;e_2)$. Then there exist two holomorphic functions $\phi, \psi : \mathbb{D} \to \mathbb{D}$ given by

$$\phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{2.4}$$

and

$$\psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots, \quad w \in \mathbb{D}, \tag{2.5}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{I}_\theta^\delta f(z)\right)'}{\left(\mathfrak{I}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda\frac{z\left(\left(\mathfrak{I}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{I}_\theta^\delta f(z)} = 1 + e_1\phi(z) + e_2\phi^2(z) + \cdots \tag{2.6}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda\frac{w\left(\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{I}_\theta^\delta f^{-1}(w)} = 1 + e_1\psi(w) + e_2\psi^2(w) + \cdots. \tag{2.7}$$

Combining (2.4), (2.5), (2.6) and (2.7), yield

$$(1-\lambda)\frac{z^{1-\alpha}\left(\mathfrak{I}_\theta^\delta f(z)\right)'}{\left(\mathfrak{I}_\theta^\delta f(z)\right)^{1-\alpha}} + \lambda\frac{z\left(\left(\mathfrak{I}_\theta^\delta f(z)\right)'\right)^\beta}{\mathfrak{I}_\theta^\delta f(z)} = 1 + e_1 r_1 z + \left[e_1 r_2 + e_2(x) r_1^2\right] z^2 + \cdots \tag{2.8}$$

and

$$(1-\lambda)\frac{w^{1-\alpha}\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)'}{\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)^{1-\alpha}} + \lambda\frac{w\left(\left(\mathfrak{I}_\theta^\delta f^{-1}(w)\right)'\right)^\beta}{\mathfrak{I}_\theta^\delta f^{-1}(w)} = 1 + e_1 s_1 w + \left[e_1 s_2 + e_2 s_1^2\right] w^2 + \cdots. \tag{2.9}$$

It is quite well-known that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we get

$$|r_j| \le 1 \quad \text{and} \quad |s_j| \le 1, \quad j \in \mathbb{N}. \tag{2.10}$$

In the light of (2.8) and (2.9), after simplifying, we find that

$$[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)](\delta+1)\theta e^{-\theta} a_2 = e_1 r_1, \tag{2.11}$$

$$\begin{aligned} &\frac{1}{4}[(1-\lambda)(\alpha+2) + \lambda(3\beta-1)](\delta^2+3\delta+2)\theta^2 e^{-\theta} a_3 \\ &+ \left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1) + \lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2 e^{-2\theta} a_2^2 \\ =&\, e_1 r_2 + e_2 r_1^2, \end{aligned} \tag{2.12}$$

$$-[(1-\lambda)(\alpha+1) + \lambda(2\beta-1)](\delta+1)\theta e^{-\theta} a_2 = e_1 s_1 \tag{2.13}$$

and

$$\frac{1}{4}\left[(1-\lambda)(\alpha+2)+\lambda(3\beta-1)\right](\delta^2+3\delta+2)\theta^2e^{-\theta}\left(2a_2^2-a_3\right)$$
$$+\left[\frac{1}{2}(1-\lambda)(\alpha+2)(\alpha-1)+\lambda\left(2\beta(\beta-2)+1\right)\right](\delta+1)^2\theta^2e^{-2\theta}a_2^2$$
$$=e_1s_2+e_2s_1^2. \tag{2.14}$$

Inequality (2.1) follows from (2.11) and (2.13). If we apply notation (2.3), then (2.11) and (2.12) become

$$Aa_2=e_1r_1, \quad Ba_3+Ca_2^2=e_1r_2+e_2r_1^2. \tag{2.15}$$

This gives

$$\frac{B}{e_1}a_3=r_2+\left(\frac{e_2}{e_1}-\frac{Ce_1}{A^2}\right)r_2^2, \tag{2.16}$$

and on using the known sharp result [7, page 10]:

$$|r_2-\mu r_1^2|\le\max\{1,|\mu|\}, \tag{2.17}$$

for all $\mu\in\mathbb{C}$, we obtain

$$\left|\frac{B}{e_1}\right||a_3|\le\max\left\{1,\left|\frac{e_2}{e_1}-\frac{Ce_1}{A^2}\right|\right\}. \tag{2.18}$$

In the same way, (2.13) and (2.14) become

$$-Aa_2=e_1s_1, \quad B(2a_2^2-a_3)+Ca_2^2=e_1s_2+e_2s_1^2. \tag{2.19}$$

This gives

$$-\frac{B}{e_1}a_3=s_2+\left(\frac{e_2}{e_1}-\frac{(2B+C)e_1}{A^2}\right)s_2^2. \tag{2.20}$$

Applying (2.17), we obtain

$$\left|\frac{B}{e_1}\right||a_3|\le\max\left\{1,\left|\frac{e_2}{e_1}-\frac{(2B+C)e_1}{A^2}\right|\right\}. \tag{2.21}$$

Inequality (2.2) follows from (2.18) and (2.21). □

If we take the generating function (1.4) of the (M,N)-Lucas polynomials $L_{M,N,k}(x)$ as $h(z)+1$, then from (1.3), we have $e_1=M(x)$ and $e_2=M^2(x)+2N(x)$ and Theorem 2.1 becomes the following corollary.

Corollary 2.1. *If $f\in\Sigma$ of the form* (1.1) *is in the class $\Upsilon_\Sigma(\alpha,\beta,\delta,\lambda,\theta;T_{M(x),N(x)}-1)$, then*

$$|a_2|\le\frac{|M(x)|}{\left[(1-\lambda)(\alpha+1)+\lambda(2\beta-1)\right](\delta+1)\theta e^{-\theta}}$$

and

$$|a_3| \le \min\left\{\max\left\{\left|\frac{M(x)}{B}\right|, \left|\frac{M^2(x)+2N(x)}{B} - \frac{CM^2(x)}{A^2B}\right|\right\},\right.$$
$$\left.\max\left\{\left|\frac{M(x)}{B}\right|, \left|\frac{M^2(x)+2N(x)}{B} - \frac{(2B+C)M^2(x)}{A^2B}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, x$ *such that* $\alpha \ge 0$, $\beta \ge 1$, $\delta \in \mathbb{N}_0$, $0 \le \lambda \le 1$, $\theta > 0$ *and* $x \in \mathbb{R}$, *where* A, B, C *are given by* (2.3) *and* $T_{M(x),N(x)}$ *is given by* (1.4).

In the next theorem, we discuss a bound for $|a_3 - \eta a_2^2|$ called "the Fekete-Szegö problem".

Theorem 2.2. *If* $f \in \Sigma$ *of the form* (1.1) *is in the class* $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; h)$, *then*

(2.22)
$$\left|a_3 - \eta a_2^2\right|$$
$$\le \frac{|e_1|}{B}\min\left\{\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right|\right\}, \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta$ *such that* $\alpha \ge 0$, $\beta \ge 1$, $\delta \in \mathbb{N}_0$, $0 \le \lambda \le 1$, $\theta > 0$ *and* $\eta \in \mathbb{C}$, *where* A, B, C *are given by* (2.3).

Proof. We apply the notations from the proof of Theorem 2.1. From (2.15) and from (2.16), we have

$$a_3 - \eta a_2^2 = \frac{e_1}{B}\left(r_2 + \left(\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right) r_1^2\right) \tag{2.23}$$

and on using the known sharp result $|r_2 - \mu r_1^2| \le \max\{1, |\mu|\}$, we get

$$|a_3 - \eta a_2^2| \le \frac{|e_1|}{B}\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(C-\eta B)e_1}{A^2}\right|\right\}. \tag{2.24}$$

In the same way, from (2.19) and from (2.20), we have

$$a_3 - \eta a_2^2 = -\frac{e_1}{B}\left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right) s_1^2\right) \tag{2.25}$$

and on using $|s_2 - \mu s_1^2| \le \max\{1, |\mu|\}$, we get

$$|a_3 - \eta a_2^2| \le \frac{|e_1|}{B}\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2B+C-\eta B)e_1}{A^2}\right|\right\}. \tag{2.26}$$

Inequality (2.22) follows from (2.24) and (2.26). □

Corollary 2.2. *If $f \in \Sigma$ of the form* (1.1) *is in the class $\Upsilon_\Sigma(\alpha, \beta, \delta, \lambda, \theta; T_{M(x),N(x)} - 1)$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|M(x)|}{B} \min\left\{\max\left\{1, \left|\frac{M^2(x) + 2N(x)}{M(x)} - \frac{(C - \eta B)M(x)}{A^2}\right|\right\}, \max\left\{1, \left|\frac{M^2(x) + 2N(x)}{M(x)} - \frac{(2B + C - \eta B)M(x)}{A^2}\right|\right\}\right\},$$

for all $\alpha, \beta, \delta, \lambda, \theta, \eta, x$ such that $\alpha \geq 0$, $\beta \geq 1$, $\delta \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\theta > 0$, $\eta \in \mathbb{C}$ and $x \in \mathbb{R}$, where A, B, C are given by (2.3) *and $T_{M(x),N(x)}$ is given by* (1.4).

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HALF LIGHTLIKE SUBMANIFOLDS OF A GOLDEN SEMI-RIEMANNIAN MANIFOLD

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Abstract. We present half lightlike submanifolds of a golden semi-Riemannian manifold. We prove that there is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold. We get results for screen semi-invariant half lightlike submanifolds of a golden semi-Riemannian manifold. We prove the conditions for integrability of distributions on screen semi-invariant half lightlike submanifolds and investigate the geometry of leaves of distributions. Moreover, we study screen conformal half lightlike submanifolds of a golden semi-Riemannian manifold.

1. Introduction

The theory of lightlike submanifolds is a significant topic of research in modern differential geometry. Lightlike submanifolds were developed by Duggal and Bejancu [5] and Duggal and Şahin [9]. Class of lightlike submanifolds of codimension 2 is called half lightlike or coisotropic submanifolds the according to the rank of its radical distribution. This class is composed of two subclasses [6]. Half lightlike submanifold is a special case of the general r-lightlike submanifold such that r = 1 and its geometry is more general form than that of coisotropic submanifold or lightlike hypersurface [5]. Screen semi-invariant half lightlike submanifolds of a semi-Riemannian product manifold were studied in [3]. Real half lightlike submanifolds of an indefinite Kaehler manifold were studied in [17]. Semi-invariant lightlike submanifolds of a semi-Riemannian product manifold were presented in [2].

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Notion of a f-structure which is a (1,1)-tensor field of constant rank on $\tilde{N}$ and satisfies the equality $f^3 + f = 0$. It is a generalization of almost complex and almost contact structures. This notion is presented in [22]. It has been generalized by Goldberg and Yano. They defined a polynomial structure of degree d which is a (1,1)-tensor field f of constant rank on $\tilde{N}$ and satisfies the equation $Q(f) = f_d + a_d f_{d-1} + \cdots + a_2 f + a_1 I = 0$, where $a_1,\ a_2, \ldots, a_d$ are real numbers and I is the identity tensor of type $(1,1)$ [12]. The number $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ which is a solution of the equation $x^2 - x - 1 = 0$ represents the golden proportion. The golden proportion has been used in many different areas such as in architecture, music, arts and philosophy. Using the golden proportion, Crasmareanu and Hretcanu defined a golden manifold $\tilde{N}$ by a tensor field $\tilde{P}$ on $\tilde{N}$ satisfies $\tilde{P}^2 = \tilde{P} + I$ in [4]. They also defined golden Riemannian manifolds and studied their submanifolds in [15]. Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic maps [21]. Gök, Keleş and Kılıç studied some characterizations for any submanifold of a golden Riemannian manifold to be semi-invariant in terms of canonical structures on the submanifold, induced by the golden structure of the ambient manifold [13]. Poyraz and Yaşar introduced lightlike hypersurfaces of a golden semi-Riemannian manifold [20]. Moreover several works in this direction are studied [1, 10, 11].

In this paper, we introduce half lightlike submanifolds of a golden semi-Riemannian manifold. In Section 2, we give basic concepts. In Section 3, we introduce half lightlike submanifolds of a golden semi-Riemannian manifold. We define invariant, screen semi-invariant and radical anti-invariant half lightlike submanifolds. Moreover, we prove that there is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold. In Section 4, we obtain results for screen semi-invariant half lightlike submanifolds. We prove the conditions for integrability of distributions on screen semi-invariant half lightlike submanifolds and investigate the geometry of leaves of distributions. We also give two examples. We find condition for its Ricci tensor to be symmetric. In Section 5, we investigate screen conformal half lightlike submanifolds of a golden semi-Riemannian manifold.

2. Preliminaries

Let $\tilde{N}$ be an n-dimensional differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is called a golden structure on $\tilde{N}$, where I is the identity transformation [14].

Let $(\tilde{N}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{N}$. If $\tilde{P}$ satisfies the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y), \tag{2.2}$$

then $(\tilde{N}, \tilde{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [19].

Let $(\tilde{N}, \tilde{g}, \tilde{P})$ be a golden semi-Riemannian manifold. Then the equation (2.2) is equivalent to

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y), \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{N})$.

If F is an almost product structure on $\tilde{N}$, then

$$\tilde{P} = \frac{1}{2}(I + \sqrt{5}F)$$

is a golden structure on $\tilde{N}$. Conversely, if $\tilde{P}$ is a golden structure on $\tilde{N}$, then

$$F = \frac{1}{\sqrt{5}}(2\tilde{P} - I)$$

is an almost product structure on $\tilde{N}$ [4].

Let $\acute{N}_p$ and $\acute{N}_q$ be real space forms with constant sectional curvatures c_p and c_q, respectively. Then similar calculations of semi-Riemannian product real space form (see [23]), one obtains the Riemannian curvature tensor $\tilde{R}$ of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$ as the following

$$\begin{aligned} \tilde{R}(X,Y)Z = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(\tilde{P}Y,Z)\tilde{P}X \\ &- \tilde{g}(\tilde{P}X,Z)\tilde{P}Y\} + \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y,Z)X \\ &- \tilde{g}(\tilde{P}X,Z)Y + \tilde{g}(Y,Z)\tilde{P}X - \tilde{g}(X,Z)\tilde{P}Y\}. \end{aligned} \tag{2.4}$$

Let $(\tilde{N}, \tilde{g})$ be an $(n+3)$-dimensional semi-Riemannian manifold of index $q \geq 1$ and $\acute{N}$ be a lightlike submanifold of codimension 2 of $\tilde{N}$. Then the radical distribution $Rad(T\acute{N}) = T\acute{N} \cap T\acute{N}^{\perp}$ of $\acute{N}$ is a vector subbundle of the tangent bundle $T\acute{N}$ and the normal bundle $T\acute{N}^{\perp}$ of rank 1 or 2. If $rank(Rad(T\acute{N})) = 1$, then $\acute{N}$ is called half lightlike submanifold of $\tilde{N}$. Then there exist complementary non-degenerate distributions $S(T\acute{N})$ and $S(T\acute{N}^{\perp})$ of $Rad(T\acute{N})$ in $T\acute{N}$ and $T\acute{N}^{\perp}$, which are called the screen and the screen transversal distribution on $\acute{N}$ respectively. Thus, we have

$$T\acute{N} = Rad(T\acute{N}) \perp S(T\acute{N}), \; T\acute{N}^{\perp} = Rad(T\acute{N}) \perp S(T\acute{N}^{\perp}). \tag{2.5}$$

Choose $L \in \Gamma(S(T\acute{N}^{\perp}))$ as a unit vector field with $\tilde{g}\,(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(T\acute{N})^{\perp}$ to $S(T\acute{N})$ in $T\tilde{N}$. Then ξ and L belong to $\Gamma(S(T\acute{N})^{\perp})$. Thus, we obtain

$$S(T\acute{N})^{\perp} = S(T\acute{N}^{\perp}) \perp S(T\acute{N}^{\perp})^{\perp},$$

where $S(T\acute{N}^{\perp})^{\perp}$ is the orthogonal complementary to $S(T\acute{N}^{\perp})$ in $S(T\acute{N})^{\perp}$. For any null section $\xi \in Rad(T\acute{N})$ on a coordinate neighborhood $U \subset \acute{N}$, there exists a uniquely determined null vector field $N \in \Gamma(ltr(T\acute{N}))$ satisfying

$$\tilde{g}\,(N, \xi) = 1, \quad \tilde{g}\,(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \quad \text{for all } X \in \Gamma(T\acute{N}).$$

We call N, $ltr(T\acute{N})$ and $tr(T\acute{N}) = S(T\acute{N}^{\perp}) \perp ltr(T\acute{N})$ the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of $\acute{N}$ with respect to $S(T\acute{N})$, respectively. Hence we have

$$\begin{aligned} T\tilde{N} &= T\acute{N} \oplus tr(T\acute{N}) \\ &= \{Rad(T\acute{N}) \oplus ltr(T\acute{N})\} \perp S(T\acute{N}) \perp S(T\acute{N}^{\perp}). \end{aligned} \tag{2.6}$$

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{N}$. Using (2.6) we define the projection morphism $Q : \Gamma(T\acute{N}) \to \Gamma(S(T\acute{N}))$. Hence we derive

$$\tilde{\nabla}_X Y = \nabla_X Y + D_1(X,Y)N + D_2(X,Y)L, \tag{2.7}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.9}$$

$$\tilde{\nabla}_X L = -A_L X + \psi(X)N, \tag{2.10}$$

$$\nabla_X QY = \nabla^*_X QY + E(X,QY)\xi, \tag{2.11}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.12}$$

for any $X, Y \in \Gamma(T\acute{N})$, $\xi \in \Gamma(Rad(T\acute{N}))$, $U \in \Gamma(tr(T\acute{N}))$, $N \in \Gamma(ltr(T\acute{N}))$ and $L \in \Gamma(S(T\acute{N}^{\perp}))$. Then ∇ and ∇^* are called induced linear connections on $T\acute{N}$ and $S(T\acute{N})$ respectively, D_1 and D_2 are called the local second fundamental forms of $\acute{N}$, C is called the local second fundamental form on $S(T\acute{N})$. A_N, A^*_ξ and A_L are called linear operators on $T\acute{N}$. Also τ, ρ and ψ are called 1-forms on $T\acute{N}$. Since the connection $\tilde{\nabla}$ of $\tilde{N}$ is torsion-free, ∇ of $\acute{N}$ is also torsion-free and D_1 and D_2 are symmetric on $T\acute{N}$. D_1 and D_2 satisfy

$$D_1(X,\xi) = 0, \quad D_2(X,\xi) = -\epsilon\psi(X), \tag{2.13}$$

for all $X \in \Gamma(T\acute{N})$.

The induced connection ∇ of $\acute{N}$ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = D_1(X,Y)\eta(Z) + D_1(X,Z)\eta(Y),$$

for any $X, Y, Z \in \Gamma(T\acute{N})$, where η is a $1-$form defined by

$$\eta(X) = \tilde{g}(X,N),$$

for all $X \in \Gamma(T\acute{N})$. Therefore, one obtains

$$D_1(X,Y) = g(A^*_\xi X, Y), \quad g(A^*_\xi X, N) = 0, \tag{2.14}$$

$$E(X,QY) = g(A_N X, QY), \quad g(A_N X, N) = 0, \tag{2.15}$$

$$\epsilon D_2(X,QY) = g(A_L X, QY), \quad g(A_L X, N) = \epsilon\rho(X), \tag{2.16}$$

$$\epsilon D_2(X,Y) = g(A_L X, Y) - \psi(X)\eta(Y), \tag{2.17}$$

for all $X, Y \in \Gamma(T\acute{N})$. By (2.14) and (2.15), A^*_ξ and A_N are $\Gamma(S(T\acute{N}))$-valued shape operators related to D_1 and E, respectively and $A^*_\xi \xi = 0$.

Using (2.7), (2.12) and (2.13), one derives

$$\tilde{\nabla}_X\xi = -A^*_\xi X - \tau(X)\xi - \epsilon\psi(X)L, \tag{2.18}$$

for any $X \in \Gamma(T\acute{N})$.

Definition 2.1. A half lightlike submanifold $(\acute{N}, g)$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is said to be irrotational [18] if $\tilde{\nabla}_X\xi \in \Gamma(T\acute{N})$ for any $X \in \Gamma(T\acute{N})$. From (2.13) and (2.18), definition of irrotational is equivalent to the condition $\psi(X) = 0$, that is, $D_2(X, \xi) = 0$ for any $X \in \Gamma(T\acute{N})$.

Definition 2.2. A half lightlike submanifold $(\acute{N}, g)$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is called totally umbilical in $\tilde{N}$, if there is a smooth vector field $H \in \Gamma(tr(T\acute{N}))$ on any coordinate neighborhood U such that

$$h(X, Y) = Hg(X, Y),$$

for any $X, Y \in \Gamma(T\acute{N})$, where

$$h(X, Y) = D_1(X, Y)N + D_2(X, Y)L \tag{2.19}$$

is the global second fundamental form tensor of $\acute{N}$. In case $H = 0$ on U, we say that $\acute{N}$ is totally geodesic [6].

It is easy to see that $\acute{N}$ is totally umbilical iff, on each coordinate neighborhood U, there exist smooth vector functions λ and δ such that

$$D_1(X, Y) = \lambda g(X, Y), D_2(X, Y) = \delta g(X, Y), \tag{2.20}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Definition 2.3. We say that the screen distribution $S(T\acute{N})$ of $\acute{N}$ is totally umbilical [6] in $\acute{N}$ if there is a smooth function γ on any coordinate neighborhood $U \subset \acute{N}$ such that

$$E(X, QY) = \gamma g(X, Y), \tag{2.21}$$

for any $X, Y \in \Gamma(T\acute{N})$. If $\gamma = 0$ on U, then we say that $S(T\acute{N})$ is totally geodesic in $\acute{N}$.

We indicate by $\tilde{R}$, R and R^* the curvature tensors of $\tilde{\nabla}$, ∇ and ∇^*, respectively. From (2.7)–(2.12), we derive the Gauss-Codazzi equations for $\acute{N}$ and $S(T\acute{N})$:

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, QW) =& g(R(X, Y)Z, QW) \\ &+ D_1(X, Z)E(Y, QW) - D_1(Y, Z)E(X, QW) \\ &+ \epsilon \left\{D_2(X, Z)D_2(Y, QW) - D_2(Y, Z)D_2(X, QW)\right\}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, \xi) =& (\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z) \\ &+ \tau(X)D_1(Y, Z) - \tau(Y)D_1(X, Z) \\ &+ \psi(X)D_2(Y, Z) - \psi(Y)D_2(X, Z), \end{aligned} \tag{2.23}$$

$$\tilde{g}(\tilde{R}(X,Y)Z,N) = g(R(X,Y)Z,N) + \epsilon\{\rho(Y)D_2(X,Z) - \rho(X)D_2(Y,Z)\}, \tag{2.24}$$

$$\tilde{g}(\tilde{R}(X,Y)\xi,N) = g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X,Y) + \rho(X)\psi(Y) - \rho(Y)\psi(X), \tag{2.25}$$

$$g(R(X,Y)QZ,QW) = g(R^*(X,Y)Z,QW) + D_1(Y,QW)E(X,QZ) - D_1(X,QW)E(Y,QZ), \tag{2.26}$$

$$\tilde{g}(R(X,Y)QZ,N) = (\nabla_X E)(Y,QZ) - (\nabla_Y E)(X,QZ) + \tau(Y)E(X,QZ) - \tau(X)E(Y,QZ), \tag{2.27}$$

for any $X, Y, Z \in \Gamma(T\acute{N})$.

3. Half Lightlike Submanifolds of a Golden Semi-Riemannian Manifold

Let $(\tilde{N}, \tilde{g}, \tilde{P})$ be a golden semi-Riemannian manifold and $\acute{N}$ be a half lightlike submanifold of $\tilde{N}$. For any $X \in \Gamma(T\acute{N})$, $N \in \Gamma(ltr(T\acute{N}))$ and $L \in \Gamma(S(T\acute{N}^\perp))$, we can write

$$\tilde{P}X = PX + \theta_1(X)N + \theta_2(X)L, \tag{3.1}$$

$$\tilde{P}N = U + \theta_1(N)N + \theta_2(N)L, \tag{3.2}$$

$$\tilde{P}L = W + \theta_1(L)N + \theta_2(L)L, \tag{3.3}$$

where $PX, U, W \in \Gamma(T\acute{N})$ and θ_1 and θ_2 are 1-forms defined by

$$\theta_1(\cdot) = g(\cdot, \tilde{P}\xi), \quad \theta_2(\cdot) = \epsilon g(\cdot, \tilde{P}L).$$

Lemma 3.1. *Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, we have*

$$P^2X = PX + X - \theta_1(X)U - \theta_2(X)W, \tag{3.4}$$

$$\theta_1(PX) = \theta_1(X)(1 - \theta_1(N)) - \theta_2(X)\theta_1(L), \tag{3.5}$$

$$\theta_2(PX) = \theta_2(X)(1 - \theta_2(L)) - \theta_1(X)\theta_2(N), \tag{3.6}$$

$$PU = U(1 - \theta_1(N)) - \theta_2(N)W, \tag{3.7}$$

$$\theta_1(U) = 1 + \theta_1(N) - (\theta_1(N))^2 - \theta_2(N)\theta_1(L), \tag{3.8}$$

$$\theta_2(U) = \theta_2(N)(1 - \theta_1(N)) - \theta_2(N)\theta_2(L), \tag{3.9}$$

$$PW = (1 - \theta_2(L))W - \theta_1(L)U, \tag{3.10}$$

$$\theta_1(W) = \theta_1(L)(1 - \theta_1(N) - \theta_2(L)), \tag{3.11}$$

$$\theta_2(W) = 1 + \theta_2(L) - (\theta_2(L))^2 - \theta_1(L)\theta_2(N), \tag{3.12}$$

$$g(PX,Y) - g(X,PY) = (-\theta_1 \otimes \eta + \eta \otimes \theta_1)(X,Y), \tag{3.13}$$

$$g(PX,PY) = g(PX,Y) + g(X,Y) + \theta_1(X)\eta(Y) - \eta(PX)\theta_1(Y) \tag{3.14}$$

$$- \theta_1(X)\eta(PY) - \epsilon\theta_2(X)\theta_2(Y),$$

for any $X, Y \in \Gamma(T\acute{N})$.

Proof. Applying $\tilde{P}$ to (3.1), using (2.1) and taking tangential, lightlike transversal and screen transversal parts of the resulting equation, we derive (3.4), (3.5) and (3.6). Similarly, applying $\tilde{P}$ to (3.2) and (3.3), using (2.1), we get (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12). Using (2.2), (2.3) and (3.1), we obtain (3.13) and (3.14). □

Lemma 3.2. *Let* $\acute{N}$ *be a half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$ *with* $\tilde{\nabla}\tilde{P} = 0$. *Then, we have*

$$(\nabla_X P)Y = \theta_1(Y)A_N X + \theta_2(Y)A_L X + D_1(X,Y)U + D_2(X,Y)W, \tag{3.15}$$

$$(\nabla_X \theta_1)Y = - D_1(X, PY) - \tau(X)\theta_1(Y) - \phi(X)\theta_2(Y) + D_1(X,Y)\theta_1(N) + D_2(X,Y)\theta_1(L), \tag{3.16}$$

$$(\nabla_X \theta_2)Y = - D_2(X, PY) - \rho(X)\theta_1(Y) + D_1(X,Y)\theta_2(N) + D_2(X,Y)\theta_2(L), \tag{3.17}$$

$$\nabla_X U = - PA_N X + \tau(X)U + \rho(X)W + \theta_1(N)A_N X + \theta_2(N)A_L X, \tag{3.18}$$

$$D_1(X,U) = - X(\theta_1(N)) - \phi(X)\theta_2(N) - \theta_1(A_N X) + \rho(X)\theta_1(L), \tag{3.19}$$

$$D_2(X,U) = - X(\theta_2(N)) - \rho(X)\theta_1(N) - \theta_2(A_N X) + \tau(X)\theta_2(N) + \rho(X)\theta_2(L), \tag{3.20}$$

$$\nabla_X W = - PA_L X + \theta_1(L)A_N X + \theta_2(L)A_L X + \phi(X)U, \tag{3.21}$$

$$D_1(X,W) = - \tau(X)\theta_1(L) - \phi(X)\theta_2(L) - \theta_1(A_L X) + \phi(X)\theta_1(N) - X(\theta_1(L)), \tag{3.22}$$

$$D_2(X,W) = - \rho(X)\theta_1(L) - X(\theta_2(L)) - \theta_2(A_L X) + \phi(X)\theta_2(N), \tag{3.23}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Proof. Since $\tilde{\nabla}\tilde{P} = 0$, we obtain $\tilde{\nabla}_X \tilde{P}Y = \tilde{P}\tilde{\nabla}_X Y$ for any $X, Y \in \Gamma(T\acute{N})$. Taking tangential, lightlike transversal and screen transversal parts of the resulting equation, we get (3.15), (3.16) and (3.17). Similarly, replacing Y with N and L respectively we obtain (3.18), (3.19), (3.20), (3.21), (3.22) and (3.23). □

Throughout this paper, we assume that $\tilde{\nabla}\tilde{P} = 0$.

Definition 3.1. Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$.

i) We say that $\acute{N}$ is an invariant half lightlike submanifold if $\tilde{P}(T\acute{N}) = T\acute{N}$.
ii) We say that $\acute{N}$ is a screen semi-invariant half lightlike submanifold if $\tilde{P}(Rad(T\acute{N})) \subset S(T\acute{N})$ and $\tilde{P}(ltr(T\acute{N})) \subset S(T\acute{N})$.
iii) We say that $\acute{N}$ is a radical anti-invariant half lightlike submanifold if $\tilde{P}(Rad(T\acute{N})) = ltr(T\acute{N})$.

Theorem 3.1. *Let $\acute{N}$ be a half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *$\acute{N}$ is invariant.*
ii) *θ_1 and θ_2 vanish on $\acute{N}$.*
iii) *P is a golden structure on $\acute{N}$.*

Proof. $\acute{N}$ is invariant if and only if $\tilde{P}X = PX$ for any $X \in \Gamma(T\acute{N})$. Then $\theta_1(X) = \theta_2(X) = 0$ and we obtain i)⇔ii).

θ_1 and θ_2 vanish on $\acute{N}$ if and only if $\tilde{P}X = PX$ for any $X \in \Gamma(T\acute{N})$. Then $P^2X = PX + X$ and $g(PX, Y) = g(X, PY)$ for any $X, Y \in \Gamma(T\acute{N})$. Thus, P is a golden structure on $\acute{N}$ and we get ii)⇔iii). □

Theorem 3.2. *There is no radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold.*

Proof. Suppose on the contrary that $\acute{N}$ is a radical anti-invariant half lightlike submanifold of a golden semi-Riemannian manifold $\tilde{N}$. By the definition of radical anti-invariant for $\xi \in \Gamma(Rad(T\acute{N}))$, $\tilde{P}\xi \in \Gamma(ltr(T\acute{N}))$. Using (2.3), we obtain

$$\begin{aligned} \tilde{g}(\tilde{P}\xi, \tilde{P}\xi) &= \tilde{g}(\tilde{P}\xi, \xi) + \tilde{g}(\xi, \xi), \\ 0 &= \tilde{g}(\tilde{P}\xi, \xi) + 0. \end{aligned}$$

Thus, $\tilde{g}(\tilde{P}\xi, \xi) = 0$ and $\tilde{P}\xi \notin \Gamma(ltr(T\acute{N}))$ which is a contradiction. □

4. Screen Semi-invariant Half Lightlike Submanifolds of a Golden Semi-Riemannian Manifold

Let $(\acute{N}, g, S(T\acute{N}))$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If we take $L_1 = \tilde{P}(Rad(T\acute{N}))$, $L_2 = \tilde{P}(ltr(T\acute{N}))$ and $L_3 = \tilde{P}(S(T\acute{N}^{\perp}))$, then we can write

$$S(T\acute{N}) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3,$$

where L_0 is a $(n-3)$-dimensional distribution. Therefore, we have

$$T\tilde{N} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp \{Rad(T\acute{N}) \oplus ltr(T\acute{N})\} \perp S(T\acute{N}^{\perp}).$$

If we set

$$L = L_0 \perp Rad(T\acute{N}) \perp \tilde{P}(Rad(T\acute{N})) \quad \text{and} \quad L^{\perp} = L_2 \perp L_3,$$

we can write

$$T\acute{N} = L \oplus L^{\perp}, \quad T\tilde{N} = \{L \oplus L^{\perp}\} \oplus ltr(T\acute{N}) \perp S(T\acute{N}^{\perp}). \tag{4.1}$$

Let U, V and W be vector fields defined by

$$U = \tilde{P}N, \quad V = \tilde{P}\xi, \quad W = \tilde{P}L. \tag{4.2}$$

From Lemma 3.1, Lemma 3.2, differentiating (4.2) with X and using Gauss-Weingarten formulas we obtain

$$P^2X = PX + X - \theta_1(X)U - \theta_2(X)W, \tag{4.3}$$

$$\theta_1(PX) = \theta_1(X), \quad \theta_2(PX) = \theta_2(X), \quad PU = U, \quad PW = W, \tag{4.4}$$

$$\theta_1(U) = 1, \quad \theta_2(U) = 0, \quad \theta_1(W) = 0, \quad \theta_2(W) = 1, \tag{4.5}$$

$$g(PX,Y) - g(X,PY) = (-\theta_1 \otimes \eta + \eta \otimes \theta_1)(X,Y), \tag{4.6}$$

$$\begin{aligned} g(PX,PY) =& g(PX,Y) + g(X,Y) + \theta_1(X)\eta(Y) - \eta(PX)\theta_1(Y) \\ & - \theta_1(X)\eta(PY) - \epsilon\theta_2(X)\theta_2(Y), \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\nabla_X P)Y =& \theta_1(Y)A_N X + \theta_2(Y)A_L X + D_1(X,Y)U \\ & + D_2(X,Y)W, \end{aligned} \tag{4.8}$$

$$(\nabla_X \theta_1)Y = - D_1(X,PY) - \tau(X)\theta_1(Y) - \psi(X)\theta_2(Y), \tag{4.9}$$

$$(\nabla_X \theta_2)Y = - D_2(X,PY) - \rho(X)\theta_1(Y), \tag{4.10}$$

$$\nabla_X U = - PA_N X + \tau(X)U + \rho(X)W, \tag{4.11}$$

$$\nabla_X V = - PA^*_\xi X - \tau(X)V - \epsilon\psi(X)W, \tag{4.12}$$

$$\nabla_X W = - PA_L X + \psi(X)U, \tag{4.13}$$

$$D_1(X,U) = - E(X,V), \quad D_1(X,W) = -\epsilon D_2(X,V), \tag{4.14}$$

$$\epsilon D_2(X,U) = - E(X,W),$$

$$D_1(X,V) = E(X,U) = D_2(X,W) = 0, \tag{4.15}$$

for any $X, Y \in \Gamma(T\acute{N})$.

Corollary 4.1. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, we have*

$$D_1(X,V) = 0,$$

that is, vector field V degenerates local second fundamental form of $\acute{N}$.

Corollary 4.2. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_2-component of A^*_ξ.*

Proof. From (2.14)-1 and (4.15), we get $D_1(X,V) = g(A^*_\xi X, V) = 0$. Thus, the proof is completed. □

Corollary 4.3. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_1-component of A_N.*

Proof. From (2.15)-1 and (4.15), we obtain $E(X,U) = g(A_N X, U) = 0$, which proves the assertion. □

Corollary 4.4. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, there is no L_3-component of A_L.*

Proof. From (2.16)-1 and (4.15), we have $D_2(X,W) = g(A_L X, W) = 0$. Thus, the proof is completed. $\square$

Proposition 4.1. *The distribution L_0 and L are invariant distributions with respect to $\tilde{P}$.*

Example 4.1. Let $(\tilde{N} = \mathbb{R}^7_3, \tilde{g})$ be a 7-dimensional semi-Euclidean space with signature $(-,+,-,+,+,+,-)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ be the standard coordinate system of $\mathbb{R}^7_3$. If we define a mapping $\tilde{P}$ by $\tilde{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = (\phi x_1, \phi x_2, (1-\phi)x_3, (1-\phi)x_4, \phi x_5, (1-\phi)x_6, \phi x_7)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{N}$. Let $\acute{N}$ be a half lightlike submanifold in $\tilde{N}$ given by the equations

$$x_1 = t_1 + \phi t_2 - \frac{\phi}{2(2+\phi)} t_3, \quad x_2 = t_1 + \phi t_2 + \frac{\phi}{2(2+\phi)} t_3,$$
$$x_3 = \phi t_1 - t_2 + \frac{1}{2(2+\phi)} t_3, \quad x_4 = \phi t_1 - t_2 - \frac{1}{2(2+\phi)} t_3,$$
$$x_5 = \sqrt{2}\phi t_4 + t_5, \quad x_6 = -t_4, \quad x_7 = \phi t_4 + \sqrt{2} t_5,$$

where t_i, $1 \le i \le 5$, are real parameters. Thus, $T\acute{N} = \mathrm{Span}\{U_1,U_2,U_3,U_4,U_5\}$, where

$$U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \phi\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_4}, \quad U_2 = \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4},$$
$$U_3 = -\frac{1}{2(2+\phi)}\left(\phi\frac{\partial}{\partial x_1} - \phi\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right), \quad U_4 = \sqrt{2}\phi\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \phi\frac{\partial}{\partial x_7},$$
$$U_5 = \frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_7}.$$

We easily check that the vector U_1 is a degenerate vector, $\acute{N}$ is a 1-lightlike submanifold of $\tilde{N}$. We set $\xi = U_1$, then we have $Rad(T\acute{N}) = \mathrm{Span}\{\xi\}$ and $S(T\acute{N}) = \mathrm{Span}\{U_2,U_3,U_4,U_5\}$. We can easily obtain

$$ltr(T\acute{N}) = \mathrm{Span}\left\{N = -\frac{1}{2(2+\phi)}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \phi\frac{\partial}{\partial x_3} - \phi\frac{\partial}{\partial x_4}\right)\right\}$$

and

$$S(T\acute{N}^\perp) = \mathrm{Span}\left\{L = \sqrt{2}\frac{\partial}{\partial x_5} + \phi\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}\right\}.$$

Thus, $\acute{N}$ is a half lightlike submanifold of $\tilde{N}$. We also get

$$\tilde{P}\xi = U_2, \quad \tilde{P}N = U_3, \quad \tilde{P}L = U_4.$$

If we set $L_0 = \mathrm{Span}\{U_5\}$, $L_1 = \mathrm{Span}\{U_2\}$, $L_2 = \mathrm{Span}\{U_3\}$, $L_3 = \mathrm{Span}\{U_4\}$, then $\acute{N}$ is a screen semi-invariant half lightlike submanifold of $\tilde{N}$.

Example 4.2. Let $(\tilde{N} = \mathbb{R}^8_2, \tilde{g})$ be a 8-dimensional semi-Euclidean space with signature $(+,+,-,+,-,+,+,+)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ be the standard coordinate system of $\mathbb{R}^8_2$. If we define a mapping $\tilde{P}$ by $\tilde{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) =$

$(\phi x_1, \phi x_2, \phi x_3, \phi x_4, (1-\phi)x_5, \phi x_6, (1-\phi)x_7, (1-\phi)x_8)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{N}$. Let $\acute{N}$ be a half lightlike submanifold in $\tilde{N}$ given by the equations

$$\begin{aligned}
&x_1 = t_1 + t_4 + \phi t_5 + t_6, \quad x_2 = -t_2 + t_4 + \phi t_5,\\
&x_3 = \frac{1}{\sqrt{2}}t_1 + \frac{1}{\sqrt{2}}t_2 + \sqrt{2}t_4 + \sqrt{2}\phi t_5 + \frac{1}{\sqrt{2}}t_6, \quad x_4 = \frac{1}{2}\log(1 + (t_1 - t_2)^2),\\
&x_5 = (1-\phi)t_2 + \phi t_4 - t_5, \quad x_6 = \phi t_3, \quad x_7 = -(1-\phi)t_2 + \phi t_4 - t_5, \quad x_8 = t_3,
\end{aligned}$$

where t_i, $1 \leq i \leq 6$, are real parameters. Thus, $T\acute{N} = \mathrm{Span}\{U_1, U_2, U_3, U_4, U_5, U_6\}$, where

$$\begin{aligned}
&U_1 = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{(t_1 - t_2)}{(1 + (t_1 - t_2)^2)}\frac{\partial}{\partial x_4},\\
&U_2 = -\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} - \frac{(t_1 - t_2)}{(1 + (t_1 - t_2)^2)}\frac{\partial}{\partial x_4} + (1-\phi)\frac{\partial}{\partial x_5} - (1-\phi)\frac{\partial}{\partial x_7},\\
&U_3 = \phi\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, \quad U_4 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_5} + \phi\frac{\partial}{\partial x_7},\\
&U_5 = \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \quad U_6 = \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3}.
\end{aligned}$$

It follows that $Rad(T\acute{N}) = \mathrm{Span}\{U_4\}$ and $S(T\acute{N}) = \mathrm{Span}\{W_1 = U_1, W_2 = U_5, W_3 = -\frac{\phi}{2(2+\phi)}(U_1 + U_2), W_4 = U_3, W_5 = U_6\}$. By direct calculations we obtain

$$ltr(T\acute{N}) = \mathrm{Span}\left\{N = -\frac{1}{2(2+\phi)}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_5} - \phi\frac{\partial}{\partial x_7}\right)\right\}$$

and

$$S(T\acute{N}^{\perp}) = \mathrm{Span}\left\{L = \frac{\partial}{\partial x_6} - \phi\frac{\partial}{\partial x_8}\right\}.$$

Thus, $\acute{N}$ is a half lightlike submanifold of $\tilde{N}$. We also get

$$\tilde{P}\xi = W_2, \quad \tilde{P}N = W_3, \quad \tilde{P}L = W_4.$$

If we set $L_0 = \mathrm{Span}\{W_1, W_5\}$, $L_1 = \mathrm{Span}\{W_2\}$, $L_2 = \mathrm{Span}\{W_3\}$, $L_3 = \mathrm{Span}\{W_4\}$, then $\acute{N}$ is a screen semi-invariant half lightlike submanifold of $\tilde{N}$.

Theorem 4.1. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then L_0 is integrable if and only if*

$$\begin{aligned}
D_1(\tilde{P}X, \tilde{P}Y) &= D_1(\tilde{P}X, Y) + D_1(X, Y),\\
E(\tilde{P}X, \tilde{P}Y) &= E(Y, \tilde{P}X) + E(Y, X),\\
D_2(\tilde{P}X, \tilde{P}Y) &= D_2(\tilde{P}X, Y) + D_2(X, Y),\\
E(\tilde{P}X, Y) &= E(Y, \tilde{P}X),
\end{aligned}$$

for any $X, Y \in \Gamma(L_0)$.

Proof. Since L_0 is invariant, if $X \in \Gamma(L_0)$, then $\tilde{P}X \in \Gamma(L_0)$. The distribution L_0 is integrable if and only if

$$\theta_1([\tilde{P}X, Y]) = \theta_3([\tilde{P}X, Y]) = \theta_2([\tilde{P}X, Y]) = \eta([\tilde{P}X, Y]) = 0,$$

for any $X, Y \in \Gamma(L_0)$, where θ_3 is 1-form defined by

$$\theta_3(X) = g(X, \tilde{P}N).$$

Then from (2.2), (2.3), (2.7) and (2.11) we derive

$$\theta_1([\tilde{P}X, Y]) = D_1(\tilde{P}X, \tilde{P}Y) - D_1(Y, \tilde{P}X) - D_1(Y, X), \tag{4.16}$$

$$\theta_3([\tilde{P}X, Y]) = E(\tilde{P}X, \tilde{P}Y) - E(Y, \tilde{P}X) - E(Y, X), \tag{4.17}$$

$$\theta_2([\tilde{P}X, Y]) = \epsilon D_2(\tilde{P}X, \tilde{P}Y) - \epsilon D_2(Y, \tilde{P}X) - \epsilon D_2(Y, X), \tag{4.18}$$

$$\eta([\tilde{P}X, Y]) = E(\tilde{P}X, Y) - E(Y, \tilde{P}X). \tag{4.19}$$

From (4.16), (4.17), (4.18) and (4.19) we derive our theorem. □

Theorem 4.2. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the distribution L is integrable if and only if*

$$D_1(\tilde{P}X, \tilde{P}Y) = D_1(\tilde{P}X, Y) + D_1(X, Y),$$
$$D_2(\tilde{P}X, \tilde{P}Y) = D_2(\tilde{P}X, Y) + D_2(X, Y),$$

for any $X, Y \in \Gamma(L)$.

Proof. L is integrable if and only if

$$\theta_1([\tilde{P}X, Y]) = \theta_2([\tilde{P}X, Y]) = 0,$$

for any $X, Y \in \Gamma(L)$. Then using (4.16) and (4.18) we obtain our assertion. □

Theorem 4.3. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If L_0 is integrable, then leaves of L_0 have a golden structure.*

Proof. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold and $\acute{N}'$ be a leaf of L_0. Then for any $p \in \acute{N}'$ we obtain $T_p\acute{N}' = (L_0)_p$. Since $X \in \Gamma(L_0)$, then $\theta_1(X) = \theta_2(X) = 0$. Therefore, from (3.1) we get $\tilde{P}X = PX$.

Letting $P' = P_{|_{L_0}}$, we say that P' defines an $(1,1)$-tensor field on $\acute{N}'$ because L_0 is $\tilde{P}$-invariant. For any $X \in \Gamma(L_0)$ we derive$P'^2X = P'X + X$, which proves the assertion. □

From Theorem 4.3 we derive Corollary 4.5.

Corollary 4.5. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If L is integrable, then the leaves of L have a golden structure.*

Theorem 4.4. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *The distribution L is parallel.*
ii) $D_1(X, \tilde{P}Y) = D_2(X, \tilde{P}Y) = 0$ *for any* $X, Y \in \Gamma(L)$.
iii) $(\nabla_X P)Y = 0$ *for any* $X, Y \in \Gamma(L)$.

Proof. Using (4.1)-1, L is parallel if and only if $\theta_1(\nabla_X Y) = \theta_2(\nabla_X Y) = 0$, for any $X, Y \in \Gamma(L)$. Then from (2.7), we derive

$$\theta_1(\nabla_X Y) = D_1(X, \tilde{P}Y),$$
$$\theta_2(\nabla_X Y) = D_2(X, \tilde{P}Y).$$

Thus, we derive i)⇔ii). For any $Y \in \Gamma(L)$, then $\theta_1(Y) = \theta_2(Y) = 0$. From (4.8), we derive

$$(\nabla_X P)Y = D_1(X, Y)U + D_2(X, Y)W.$$

Hence, we have ii)⇔iii). □

Theorem 4.5. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is totally geodesic if and only if*

$$(\nabla_X P)Y = 0, \tag{4.20}$$
$$(\nabla_X P)U = A_N X, \tag{4.21}$$
$$(\nabla_X P)W = A_L X, \tag{4.22}$$

for any $X \in \Gamma(T\acute{N})$ and $Y \in \Gamma(L)$.

Proof. Let $\acute{N}$ be totally geodesic. For any $Y \in \Gamma(L)$, we have $\theta_1(Y) = \theta_2(Y) = 0$ and thus, from (4.8), we get $(\nabla_X P)Y = 0$. Similarly, letting $Y = U$ in (4.8), we get $(\nabla_X P)U = A_N X$. Similarly, letting $Y = W$ in (4.8), we get $(\nabla_X P)W = A_L X$.

Conversely, we suppose that the conditions (4.20), (4.21) and (4.22) hold. If $Y \in \Gamma(T\acute{N})$, using (4.1)-1, we can write $Y = Y_l + fU + hW$ for any $Y \in \Gamma(T\acute{N})$. Thus, we obtain

$$D_1(X, Y) = D_1(X, Y_l) + fD_1(X, U) + hD_1(X, W), \tag{4.23}$$
$$D_2(X, Y) = D_2(X, Y_l) + fD_2(X, U) + hD_2(X, W). \tag{4.24}$$

Using (4.20) and replacing Y by Y_l in (4.8), we find $D_1(X, Y_l)U + D_2(X, Y_l)W = -\theta_1(Y_l)A_N X - \theta_2(Y_l)A_L X = 0$. From this fact we get $D_1(X, Y_l) = D_2(X, Y_l) = 0$. Using (4.21) and replacing Y by U in (4.8), we derive $D_1(X, U)U + D_2(X, U)W = 0$. From this we obtain $D_1(X, U) = D_2(X, U) = 0$. From (4.15) we have $D_2(X, W) = 0$. Moreover, replacing Y by W in (4.8), using (4.15) and (4.22) we derive $D_1(X, W)U = 0$, which implies $D_1(X, W) = 0$. Considering (4.23) and (4.24) we obtain $D_1 = D_2 = 0$. Hence, the claim holds. □

Definition 4.1. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is mixed totally geodesic if and only if

$$D_1(X, Y) = D_2(X, Y) = 0,$$

for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.

Theorem 4.6. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then the following assertions are equivalent.*

i) *$\acute{N}$ is mixed totally geodesic.*
ii) *$\nabla_Y \tilde{P}X$ has no component in $\Gamma(L^{\perp})$ for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.*
iii) *$(\nabla_Y P)X = 0$ for any $X \in \Gamma(L)$ and $Y \in \Gamma(L^{\perp})$.*

Proof. $\acute{N}$ is mixed totally geodesic if and only if for any $X \in \Gamma(L)$, $Y \in \Gamma(L^{\perp})$, $D_1(X, Y) = D_2(X, Y) = 0$. Since D_1 and D_2 are symmetric and using (2.3) and (2.7) we get

$$D_1(X, Y) = D_1(Y, X) = g(\nabla_Y \tilde{P}X, \tilde{P}\xi) - D_1(Y, \tilde{P}X) \tag{4.25}$$

and

$$\epsilon D_2(X, Y) = \epsilon D_2(Y, X) = g(\nabla_Y \tilde{P}X, \tilde{P}L) - \epsilon D_2(Y, \tilde{P}X). \tag{4.26}$$

From (4.25) and (4.26) we derive $D_1(Y, X) + D_1(Y, \tilde{P}X) = D_1(Y, \tilde{P}^2 X) = g(\nabla_Y \tilde{P}X, \tilde{P}\xi)$ and $\epsilon D_2(Y, X) + \epsilon D_2(Y, \tilde{P}X) = \epsilon D_2(Y, \tilde{P}^2 X) = g(\nabla_Y \tilde{P}X, \tilde{P}L)$, respectively. Thus, we have i)⇔ii). For any $X \in \Gamma(L)$, $Y \in \Gamma(L^{\perp})$, $\theta_1(X) = \theta_2(X) = 0$ and from (4.8) we derive

$$(\nabla_Y P)X = D_1(Y, X)U + D_2(Y, X)W.$$

Thus, we have i)⇔iii). □

Theorem 4.7. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then $\acute{N}$ is totally geodesic.*

Proof. Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $\tilde{N}$. From (4.14)-2, we have $D_1(X, W) = -\epsilon D_2(X, V)$. Since $\acute{N}$ is totally umbilical, using (2.20) we derive $\lambda g(X, W) = -\epsilon\delta g(X, V)$. Replacing X by U and W in this equation, respectively, we obtain $\lambda = \delta = 0$. Thus, the proof is completed. □

Theorem 4.8. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then we have $c_p = c_q = 0$.*

Proof. From (2.4) we get

$$\tilde{g}(\tilde{R}(X, Y)Z, \xi) = \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\left\{g(\tilde{P}Y, Z)\theta_1(X) - g(\tilde{P}X, Z)\theta_1(Y)\right\}$$

$$+\left(-\frac{(1-\phi)c+\phi c_q}{4}\right)\{g(Y,Z)\theta_1(X)-g(X,Z)\theta_1(Y)\}, \tag{4.27}$$

for any $X, Y, Z \in \Gamma(T\acute{N})$. Using Theorem 4.7 in (2.23) we derive $\tilde{g}(\tilde{R}(X,Y)Z,\xi)=0$. Moreover, letting $X=U$, $Y=\xi$, $Z=U$ in (4.27), we obtain

$$\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}=0. \tag{4.28}$$

Similarly, if we let $X=U$, $Y=V$, $Z=U$ in (4.27), we get

$$\left(-\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}\right)+\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)=0. \tag{4.29}$$

From (4.28) and (4.29), we obtain $c_p=c_q=0$. Thus, the proof is completed. □

Theorem 4.9. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{P})$. If $S(T\acute{N})$ is totally umbilical, then $S(T\acute{N})$ is totally geodesic.*

Proof. Let $S(T\acute{N})$ be a totally umbilical. From (2.21) and (4.15) we get

$$E(X,U)=\gamma g(X,U)=0,$$

for any $X\in\Gamma(T\acute{N})$. Letting $X=V$ in last equation, we obtain $\gamma=0$, i.e., $E=0$. Thus, the proof is completed. □

Theorem 4.10. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{P})$. Then we have the following assertions.*

i) *If P is parallel with respect to ∇ on $\acute{N}$, then $\rho(X)=\psi(X)=D_2(X,U)=0$,*

$$E(X,Z)\theta_1(Y)+\epsilon D_2(X,Z)\theta_2(Y)+D_1(X,Y)\theta_3(Z)+D_2(X,Y)\theta_2(Z)=0 \tag{4.30}$$

and

$$D_1(X,Y)=-E(X,V)\theta_1(Y)-\epsilon D_2(X,V)\theta_2(Y), \tag{4.31}$$

$$D_2(X,Y)=-E(X,W)\theta_1(Y), \tag{4.32}$$

for any $X,Y\in\Gamma(T\acute{N})$ and $Z\in\Gamma(S(T\acute{N}))$.

ii) *If V is parallel with respect to ∇ on $\acute{N}$, then $\tau(X)=0$,*

$$A^*_\xi X=\theta_2(A^*_\xi X)W \quad \text{and} \quad \theta_2(A^*_\xi X)=-\epsilon\psi(X),$$

for any $X\in\Gamma(T\acute{N})$.

iii) *If U is parallel with respect to ∇ on $\acute{N}$, then*

$$A_N X=\theta_1(A_N X)U+\theta_2(A_N X)W, \quad \theta_1(A_N X)=\tau(X) \quad \text{and} \quad \theta_2(A_N X)=\rho(X),$$

for any $X\in\Gamma(T\acute{N})$.

iv) *If W is parallel with respect to ∇ on $\acute{N}$, then $\rho(X)=0$,*

$$A_L X=\psi(X)U \quad \text{and} \quad \theta_1(A_L X)=\psi(X),$$

for any $X \in \Gamma(T\acute{N})$.

Moreover, if all of V, U *and* W *are parallel with respect to* ∇ *on* $\acute{N}$, *then* $S(T\acute{N})$ *is totally geodesic in* $\acute{N}$ *and* $\tau = \rho = 0$.

Proof. Let P be parallel with respect to ∇. Then taking the scalar product with for any $Z \in \Gamma(T\acute{N})$ and V in (4.8) we obtain (4.30) and (4.31), respectively. Taking the scalar product with W in (4.8) and using (4.15) we derive (4.32). Moreover, taking the scalar product with for any N in (4.8) we get

$$\epsilon\rho(X)\theta_2(Y) = 0. \tag{4.33}$$

Taking $Y = W$ in (4.33) we get $\rho(X) = 0$. Similarly, taking the scalar product with for any U in (4.8) and using (4.15) we get $D_2(X, U) = 0$. Moreover letting $Y = \xi$ in (4.32) we get $\psi(X) = 0$.

If V is parallel with respect to ∇ on $\acute{N}$, then from (4.12) we obtain

$$-\tilde{P}A^*_\xi X + \theta_1(A^*_\xi X)N + \theta_2(A^*_\xi X)L - \tau(X)V - \epsilon\psi(X)W = 0.$$

Using (2.14)-1 and (4.15), we derive $D_1(X, V) = \theta_1(A^*_\xi X) = 0$. Thus, we get

$$-\tilde{P}A^*_\xi X + \theta_2(A^*_\xi X)L - \tau(X)V - \epsilon\psi(X)W = 0, \tag{4.34}$$

for any $X \in \Gamma(T\acute{N})$. Applying $\tilde{P}$ to (4.34) and from (2.1), (3.1) and (4.2), we derive

$$\begin{aligned} &- PA^*_\xi X - A^*_\xi X - \tau(X)V + (\theta_2(A^*_\xi X) - \epsilon\psi(X))W - \tau(X)\xi - \theta_2(A^*_\xi X)L \\ &- \epsilon\psi(X)L = 0, \end{aligned} \tag{4.35}$$

for any $X \in \Gamma(T\acute{N})$. Then subtracting (4.12) from (4.35) and taking tangential and normal part of the resulting equation, we get ii). Similarly, by using (2.1), (3.1), (4.2), (4.11) and (4.13), we have iii) and iv).

Suppose that all of V, U and W are parallel with respect to ∇ on $\acute{N}$. Then from iii) we have $A_N X = \theta_1(A_N X)U + \theta_2(A_N X)W$. From ii) and iii) we get $\theta_1(A_N X) = \tau(X) = 0$ and from iii) and iv) we obtain $\rho(X) = \theta_2(A_N X) = 0$. Thus, $A_N = 0$, that is, $S(T\acute{N})$ is totally geodesic in $\acute{N}$. □

From Theorem 4.10 i) we have Corollary 4.6.

Corollary 4.6. *Let* $\acute{N}$ *be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$. *If* P *is parallel with respect to* ∇ *on* $\acute{N}$, *then* $\acute{N}$ *is irrotational.*

Theorem 4.11. *Let* $\acute{N}$ *be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold* $(\tilde{N}, \tilde{g}, \tilde{P})$. *If* P *and* V *are parallel with respect to* ∇ *on* $\acute{N}$, *then* $\acute{N}$ *is totally geodesic in* $\tilde{N}$ *and the* 1*-forms* ρ, ψ *and* τ *vanish.*

Proof. Suppose that P and V are parallel with respect to ∇ on $\acute{N}$. Then from Theorem 4.10 i) and ii) we have $\rho(X) = \psi(X) = \tau(X) = 0$ and $A^*_\xi X = -\epsilon\psi(X)W$. From this fact, we get $A^*_\xi = 0$, i.e., $D_1 = 0$.

For any $Y \in \Gamma(T\acute{N})$, we have (4.24). Using (4.32) with $Y = Y_l \in \Gamma(L)$, we find $D_2(X, Y_l) = -E(X, W)\theta_1(Y_l) = 0$. From this fact we get $D_2(X, Y_l) = 0$. From (4.15) and Theorem 4.10 i) we have $D_2(X, U) = D_2(X, W) = 0$. Using (4.24) we obtain $D_2 = 0$. Thus, we get $D = 0$, which completes the proof. □

Theorem 4.12. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If P is parallel with respect to ∇ on $\acute{N}$, then L and $L^{\perp}$ are parallel and integrable distributions with respect to ∇ and $\acute{N}$ is locally a product manifold $\acute{N}_1 \times \acute{N}_2$, where $\acute{N}_1$ and $\acute{N}_2$ are leaves of L and $L^{\perp}$, respectively.*

Proof. Since $\tilde{\nabla}$ is a metric connection, from (2.2), (2.3), (2.7) and (2.18), we derive

$$\begin{aligned} &g(\nabla_X \xi, V) = D_1(X, V), \quad g(\nabla_X V, V) = 0, \quad g(\nabla_X Y, V) = D_1(X, \tilde{P}Y),\\ &g(\nabla_X \xi, W) = \epsilon D_2(X, V), \quad g(\nabla_X V, W) = \epsilon D_2(X, V) - \psi(X),\\ &g(\nabla_X Y, W) = \epsilon D_2(X, \tilde{P}Y), \end{aligned} \tag{4.36}$$

for any $X \in \Gamma(L)$ and $Y \in \Gamma(L_0)$.

Since $\tilde{\nabla}$ is a metric connection, using (2.2), (2.3), (2.7), (2.9) and (2.11), we derive

$$\begin{aligned} &g(\nabla_Z W, N) = -\epsilon D_2(Z, U), \quad g(\nabla_Z W, U) = -\epsilon D_2(Z, U) - \epsilon\rho(Z),\\ &g(\nabla_Z W, Y) = -\epsilon D_2(Z, \tilde{P}Y), \quad g(\nabla_Z U, N) = E(Z, U),\\ &g(\nabla_Z U, U) = 0, \quad g(\nabla_Z U, Y) = -E(Z, \tilde{P}Y), \end{aligned} \tag{4.37}$$

for any $Z \in \Gamma(L^{\perp})$ and $Y \in \Gamma(L_0)$.

From (4.15) we have $D_1(X, V) = 0$. Letting $Y = V$ in equation (4.32) we obtain $D_2(X, V) = 0$ for any $X \in \Gamma(T\acute{N})$. If we replace Y by $\tilde{P}Y \in \Gamma(L_0)$ in equation (4.31) and (4.32) then we derive $D_1(X, \tilde{P}Y) = D_2(X, \tilde{P}Y) = 0$. Also, from (4.15) and Theorem 4.10 i) we have $E(X, U) = \rho(X) = \psi(X) = D_2(X, U) = 0$ for any $X \in \Gamma(T\acute{N})$. Replacing X, Y, Z by $Z \in \Gamma(L^{\perp}), U, \tilde{P}Y \in \Gamma(L_0)$, respectively, in equation (4.30) and if $D_2(X, U) = 0$ is used in this equation, we get $E(Z, \tilde{P}Y) = 0$. Thus, we prove our theorem. □

Theorem 4.13. *Let $\acute{N}$ be a totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then L is a parallel and integrable distribution with respect to ∇ and $\acute{N}$ is locally a product manifold $R_u \times R_w \times \acute{N}_1$, where R_u and R_w are null and non-null curves tangent to $\tilde{P}(ltr(T\acute{N}))$ and $\tilde{P}(S(T\acute{N}^{\perp}))$, respectively, and $\acute{N}_1$ is a leaf of L.*

Proof. Suppose that $\acute{N}$ is totally umbilical, then $\acute{N}$ is totally geodesic and $D_1 = D_2 = \psi = 0$. All terms of (4.36) are zero. Hence, L is a parallel and integrable distribution with respect to ∇. Moreover, $\tilde{P}(ltr(T\acute{N}))$ and $\tilde{P}(S(T\acute{N}^{\perp}))$ are integrable distributions. Hence, the proof is completed. □

Theorem 4.14. *Let $\acute{N}$ be a half lightlike submanifold of a semi-Remannian manifold $(\tilde{N}, \tilde{g})$. Then the screen transversal distribution $S(T\acute{N}^{\perp})$ is parallel with respect to $\tilde{\nabla}$ if and only if $A_L = 0$ on $\Gamma(T\acute{N})$* [16].

Theorem 4.15. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$ with a parallel screen transversal distribution. If $S(T\acute{N})$ is totally umbilicial, then $c_p = c_q = 0$.*

Proof. Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$, $c_p, c_q \neq 0$, with a parallel screen transversal distribution. From (2.4) we derive

$$\tilde{g}(\tilde{R}(\xi, Y)QZ, N) = \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y, QZ) - \theta_1(QZ)\theta_3(Y)\}$$
$$+\left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y, QZ) - \theta_1(QZ)\eta(Y)\}. \tag{4.38}$$

Since $S(T\acute{N})$ is totally umbilicial and screen transversal distribution is parallel, using Theorem 4.9 and Theorem 4.14 in (2.24) and (2.27) we obtain

$$\tilde{g}(\tilde{R}(X, Y)QZ, N) = 0. \tag{4.39}$$

If we put $Y = V$, $Z = U$ in (4.38), we obtain

$$\frac{(1-\phi)c_p + \phi c_q}{2\sqrt{5}} = 0. \tag{4.40}$$

Similarly, if we put $Y = U$, $Z = V$ in (4.38), we get

$$\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right) + \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right) = 0. \tag{4.41}$$

From (4.40) and (4.41), we obtain $c_p = c_q = 0$, which proves the assertion. □

The induced Ricci type tensor $R^{(0,2)}$ of $\acute{N}$ is defined by

$$R^{(0,2)} = trace\{Z \to R(Z, X)Y\},$$

for any $X, Y, Z \in \Gamma(T\acute{N})$, where

$$R^{(0,2)}(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(E_i, X)Y, E_i) + \bar{g}(R(\xi, X)Y, N),$$

for the quasi-orthonormal frame $\{E_1, \ldots, E_n, \xi\}$ of $T_p\acute{N}$ and where $\epsilon_i = g(E_i, E_i)$ is the sign of E_i. Generally, the induced Ricci type tensor R is not symmetric [5–7]. A tensor field $R^{(0,2)}$ of lightlike submanifold M is called its induced Ricci tensor if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be indicated by Ric.

If $\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q)$ is a locally golden product space form, then we have

$$R^{(0,2)}(X, Y) = \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{(n-1)\tilde{g}(X, Y) - \theta_1(Y)\theta_3(X)$$

$$
\begin{aligned}
&+ ((tr\tilde{P}) - 1 + \theta_3(\xi))\tilde{g}(\tilde{P}X, Y)\} \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{(n-1)\tilde{g}(\tilde{P}X, Y) \\
&+ ((tr\tilde{P}) + \theta_3(\xi))\tilde{g}(X, Y) - \theta_1(Y)\eta(X)\} \\
&+ D_1(X, Y)trA_N + D_2(X, Y)trA_L - g(A_N X, A^*_\xi Y) \\
&- \epsilon g(A_L X, A_L Y) + \rho(X)\psi(Y).
\end{aligned} \tag{4.42}
$$

From (4.42), we have

$$
\begin{aligned}
R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = &\left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)(\theta_1(X)\theta_3(Y) - \theta_1(Y)\theta_3(X)) \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)(\theta_1(X)\eta(Y) - \theta_1(Y)\eta(X)) \\
&+ g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) \\
&+ \rho(X)\psi(Y) - \rho(Y)\psi(X).
\end{aligned} \tag{4.43}
$$

From (2.4) and (2.25) we get

$$
\begin{aligned}
2d\tau(X, Y) = &- \frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}(\theta_1(X)\theta_3(Y) - \theta_1(Y)\theta_3(X)) \\
&+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)(\theta_1(X)\eta(Y) - \theta_1(Y)\eta(X)) \\
&+ g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) + \rho(X)\psi(Y) - \rho(Y)\psi(X).
\end{aligned} \tag{4.44}
$$

Thus, from (4.43) and (4.44) we obtain

$$
R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y). \tag{4.45}
$$

From (4.45), we obtain Theorem 4.16.

Theorem 4.16. *Let $\acute{N}$ be a screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then, $R^{(0,2)}$ is a symmetric if and only if τ is closed.*

5. Screen Conformal Screen Semi-invariant Half Lightlike Submanifolds Of A Golden Semi-Riemannian Manifold

A half lightlike submanifold $(\acute{N}, g, S(T\acute{N}))$ of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is screen conformal if the shape operators A_N and A^*_ξ of $\acute{N}$ and $S(T\acute{N})$, respectively are related by $A_N = \varphi A^*_\xi$, or equivalently

$$
E(X, QY) = \varphi D_1(X, Y), \tag{5.1}
$$

for all $X, Y \in \Gamma(T\acute{N})$, where φ is a non-vanishing smooth function on a neighborhood U in $\acute{N}$. In particular, if φ is a non-zero constant, then $\acute{N}$ is called screen homothetic [8].

Remark 5.1. If $\acute{N}$ is a screen conformal half lightlike submanifold, then E is symmetric on $\Gamma(S(T\acute{N}))$. Thus, $S(T\acute{N})$ is integrable distribution and $\acute{N}$ is locally a product manifold $R_\xi \times \acute{N}^*$ where R_ξ is a null curve tangent to $Rad(T\acute{N})$ and $\acute{N}^*$ is a leaf of $S(T\acute{N})$ [5].

Theorem 5.1. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. Then, $\acute{N}$ and $S(T\acute{N})$ are totally geodesic.*

Proof. Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of $\tilde{N}$. Then from Theorem 4.7 we have $D_1 = D_2 = 0$. Since $\acute{N}$ is screen conformal, $E(X, QY) = \varphi D_1(X, Y) = 0$, which proves the assertion. □

Theorem 5.2. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If P is parallel with respect to ∇ on $\acute{N}$, then $\acute{N}$ and $S(T\acute{N})$ are totally geodesic in $\tilde{N}$ and $\rho = \psi = 0$.*

Proof. Suppose that P is parallel with respect to ∇ on $\acute{N}$. For any $Y \in \Gamma(T\acute{N})$, we have (4.23) and (4.24). Replacing Y by Y_l in (4.31) and (4.32), we find $D_1(X, Y_l) = D_2(X, Y_l) = 0$. From (4.15) and (5.1) we have $E(X, U) = \varphi D_1(X, U) = 0$. Taking $Y = V$ in (4.32) we get $D_2(X, V) = 0$ and from (4.14)-2 we obtain $D_1(X, W) = -\epsilon D_2(X, V) = 0$. From (4.15) and Theorem 4.10 i) we have $D_2(X, U) = D_2(X, W) = 0$. Considering (4.23) and (4.24) we obtain $D_1 = D_2 = 0$. Since $\acute{N}$ is conformal, $E = 0$. Also, from Theorem 4.10 i) $\rho(X) = \psi(X) = 0$, which proves the assertion. □

Theorem 5.3. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$. If U or W is parallel with respect to ∇ on $\acute{N}$, then L and $L^\perp$ are parallel and integrable distribution with respect to ∇ and $\acute{N}$ is locally a product manifold $\acute{N}_1 \times \acute{N}_2$, where $\acute{N}_1$ is a leaf of L and $\acute{N}_2$ is a leaf of $L^\perp$.*

Proof. Let $\acute{N}$ be totally umbilical. From Theorem 5.2 $\acute{N}$ and $S(T\acute{N})$ are totally geodesic and all terms of (4.36) and (4.37) are zero except $\rho(Z)$. Since $S(T\acute{N})$ is totally geodesic and U is parallel, using Theorem 4.10 iii) we obtain $\theta_2(A_N X) = \epsilon E(X, W) = \rho(X) = 0$.

If W is parallel, from Theorem 4.10 (iv), we have $\rho(X) = 0$. Hence L and $L^\perp$ are parallel and integrable distributions on $\acute{N}$. This completes the proof. □

From (2.19), (4.14), (4.15) and (5.1), we have

$$\begin{aligned} h(X, U) =& D_1(X, U)N + D_2(X, U)L = -E(X, V)N - \epsilon E(X, W)L \\ =& -\varphi D_1(X, V)N - \epsilon\varphi D_1(X, W)L \\ =& \varphi D_1(X, V)N + \varphi D_2(X, V)L = \varphi h(X, V), \end{aligned}$$

for any $X \in \Gamma(T\acute{N})$. Thus, we have

$$h(X, U - \varphi V) = 0. \tag{5.2}$$

Since $\{U, V\}$ is a basis for $\Gamma(\tilde{P}(Rad(T\acute{N})) \oplus \tilde{P}(ltr(T\acute{N})))$, $\{\omega_1, \omega_2\}$ is an orthogonal basis of $\Gamma(\tilde{P}(Rad(T\acute{N})) \oplus \tilde{P}(ltr(T\acute{N})))$, where

$$\omega_1 = U - \varphi V, \quad \omega_2 = U + \varphi V.$$

Let $R(\omega_1) = \text{Span}\{\omega_1\}$. Then $S(\omega_1) = L_0 \perp \text{Span}\{\omega_2, W\}$ is a complementary vector subbundle to $R(\omega_1)$ in $S(T\acute{N})$. Thus, we have

$$S(T\acute{N}) = R(\omega_1) \perp S(\omega_1). \tag{5.3}$$

Theorem 5.4. *Let $\acute{N}$ be a screen conformal totally umbilical screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. Then the non-null vector field ω_1 is parallel with respect to ∇ if and only if the 1-forms ρ, τ and ψ vanish and $\acute{N}$ is screen homothetic.*

Proof. Since P is linear and using (4.11), (4.12), $A_N = \varphi A^*_\xi$, we obtain

$$\nabla_X \omega_1 = \tau(X)U + (\varphi\tau(X) - X[\varphi])V + (\rho(X) + \epsilon\varphi\psi(X))W,$$

for any $X \in \Gamma(T\acute{N})$. Thus, we say that ω_1 is parallel if and only if

$$\tau(X)U + (\varphi\tau(X) - X[\varphi])V + (\rho(X) + \epsilon\varphi\psi(X))W = 0.$$

If we take the scalar product with U, V and W, respectively, we obtain $\tau(X) = \varphi\tau(X) - X[\varphi] = \rho(X) + \epsilon\varphi\psi(X) = 0$. Since $\tau(X) = 0$, then $X[\varphi] = 0$, i.e., $\acute{N}$ is screen homothetic. If $S(T\acute{N}^{\perp})$ is parallel, then we obtain $\rho(X) = 0$. Thus, $\psi(X) = 0$. ☐

Theorem 5.5. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. If ω_1 is parallel with respect ∇, then $\acute{N}$ is locally a product manifold $R_\xi \times R_{\omega_1} \times \acute{N}_1$, where R_ξ is null curve tangent to $T\acute{N}^{\perp}$, R_{ω_1} is non-null geodesic tangent to $R(\omega_1)$ and $\acute{N}_1$ is a leaf of $S(\omega_1)$. Also, $\acute{N}$ is screen homothetic.*

Proof. For any $X \in \Gamma(S(\omega_1))$ and $Y \in \Gamma(L_0)$, we have

$$\begin{aligned}
g(\nabla_X Y, \omega_1) &= g(\tilde{\nabla}_X Y, \omega_1) = -g(Y, \tilde{\nabla}_X \omega_1) = -g(Y, \nabla_X \omega_1) = 0,\\
g(\nabla_X \omega_2, \omega_1) &= g(\tilde{\nabla}_X \omega_2, \omega_1) = -g(\omega_2, \nabla_X \omega_1) = X[\varphi] - 2\varphi\tau(X),\\
g(\nabla_X W, \omega_1) &= g(\tilde{\nabla}_X W, \omega_1) = -g(W, \nabla_X \omega_1) = -\rho(X) - \epsilon\varphi\psi(X).
\end{aligned} \tag{5.4}$$

From Theorem 5.4, the 1-forms ρ, τ and ψ vanish and $\acute{N}$ is screen homothetic. Then all equations in (5.4) are zero. Thus, the distribution $S(\omega_1)$ is a parallel and integrable distribution. Using this fact and Remark 5.1, we derive our assertion. ☐

Theorem 5.6. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{P})$ such that $S(T\acute{N}^{\perp})$ is parallel distribution with respect to $\tilde{\nabla}$. If ω_1 is parallel with respect to ∇ on $\acute{N}$, then $\acute{N}$ is locally a product manifold $R_{\omega_1} \times \acute{N}_1$, where R_{ω_1} is non-null geodesic tangent to $R(\omega_1)$ and $\acute{N}_1$ is a leaf of $G(\omega_1) = L_0 \perp \mathrm{Span}\{\xi, \omega_2, W\}$ respectively. Furthermore, $\acute{N}$ is screen homothetic.*

Proof. From (2.5) and (5.3), we derive $T\acute{N} = R(\omega_1) \oplus_{orth} G(\omega_1)$. For any $X \in \Gamma(G(\omega_1))$ and $Y \in \Gamma(L_0)$, we derive

$$\begin{aligned} g(\nabla_X Y, \omega_1) &= g(\tilde{\nabla}_X Y, \omega_1) = -g(Y, \tilde{\nabla}_X \omega_1) = -g(Y, \nabla_X \omega_1) = 0, \\ g(\nabla_X \xi, \omega_1) &= g(\tilde{\nabla}_X \xi, \omega_1) = -g(\xi, \tilde{\nabla}_X \omega_1) = -D_1(X, \omega_1) = 0, \\ g(\nabla_X \omega_2, \omega_1) &= g(\tilde{\nabla}_X \omega_2, \omega_1) = -g(\omega_2, \nabla_X \omega_1) = X[\varphi] - 2\varphi\tau(X), \\ g(\nabla_X W, \omega_1) &= g(\tilde{\nabla}_X W, \omega_1) = -g(W, \nabla_X \omega_1) = -\rho(X) - \epsilon\varphi\psi(X). \end{aligned} \tag{5.5}$$

From Theorem 5.4, all equations in (5.5) is zero. Thus, distribution $G(\omega_1)$ is a parallel and integrable. Thus, we derive our assertion. ☐

Theorem 5.7. *Let $\acute{N}$ be a screen conformal screen semi-invariant half lightlike submanifold of a locally golden product space form $(\tilde{N} = \acute{N}_p(c_p) \times \acute{N}_q(c_q), \tilde{g}, \tilde{P})$. Then, we have $c_p = (\phi + 1)c_q$.*

Proof. From (2.4) and (2.23), we derive

$$\begin{aligned} &(\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z) + \tau(X)D_1(Y, Z) - \tau(Y)D_1(X, Z) \\ &+ \psi(X)D_2(Y, Z) - \psi(Y)D_2(X, Z) \\ =& \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(\tilde{P}Y, Z)\theta_1(X) - \tilde{g}(\tilde{P}X, Z)\theta_1(Y)\} \\ &+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(Y, Z)\theta_1(X) - \tilde{g}(X, Z)\theta_1(Y)\}, \end{aligned} \tag{5.6}$$

for any $X, Y, Z \in \Gamma(T\acute{N})$. Using (2.4), (2.24), (2.27) and (5.1) we get

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)QZ, N) =& \left(-\frac{(1-\phi)c_p - \phi c_q}{2\sqrt{5}}\right)\{\tilde{g}(Y, QZ)\eta(X) - \tilde{g}(X, QZ)\eta(Y) \\ &+ \tilde{g}(\tilde{P}Y, QZ)\theta_3(X) - \tilde{g}(\tilde{P}X, QZ)\theta_3(Y)\} \\ &+ \left(-\frac{(1-\phi)c_p + \phi c_q}{4}\right)\{\tilde{g}(\tilde{P}Y, QZ)\eta(X) \\ &- \tilde{g}(\tilde{P}X, QZ)\eta(Y) + \tilde{g}(Y, QZ)\theta_3(X) - \tilde{g}(X, QZ)\theta_3(Y)\} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)QZ, N) =& \varphi((\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z)) + \varphi\tau(Y)D_1(X, QZ) \\ &- \varphi\tau(X)D_1(Y, QZ) + X[\varphi]\, D_1(Y, QZ) - Y[\varphi]\, D_1(X, QZ) \end{aligned}$$

$$+\epsilon\left\{\rho(Y)D_2(X,QZ)-\rho(X)D_2(Y,QZ)\right\}. \tag{5.8}$$

Thus, from (5.6), (5.7) and (5.8), we derive

$$\begin{aligned}
&\left(-\frac{(1-\phi)c_p-\phi c_q}{2\sqrt{5}}\right)\{\varphi\tilde{g}(\tilde{P}Y,QZ)\theta_1(X)-\varphi\tilde{g}(\tilde{P}X,QZ)\theta_1(Y)\\
&\quad-\tilde{g}(Y,QZ)\eta(X)+\tilde{g}(X,QZ)\eta(Y)-\tilde{g}(\tilde{P}Y,QZ)\theta_3(X)+\tilde{g}(\tilde{P}X,QZ)\theta_3(Y)\}\\
&\quad+\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)\{\varphi\tilde{g}(Y,QZ)\theta_1(X)-\varphi\tilde{g}(X,QZ)\theta_1(Y)\\
&\quad-\tilde{g}(\tilde{P}Y,QZ)\eta(X)+\tilde{g}(\tilde{P}X,QZ)\eta(Y)-\tilde{g}(Y,QZ)\theta_3(X)+\tilde{g}(X,QZ)\theta_3(Y)\}\\
&=\left[-X\left[\varphi\right]+2\varphi\tau(X)\right]D_1(Y,QZ)+\left[Y\left[\varphi\right]-2\varphi\tau(Y)\right]D_1(X,QZ)\\
&\quad-\left[\varphi\psi(Y)+\epsilon\rho(Y)\right]D_2(X,QZ)+\left[\varphi\psi(X)+\epsilon\rho(X)\right]D_2(Y,QZ).
\end{aligned}\tag{5.9}$$

Replacing QZ by ω_1 in (5.9) and using (5.2), we obtain

$$\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)\{-\varphi\theta_1(X)\eta(Y)+\varphi\theta_1(Y)\eta(X)+\theta_3(X)\eta(Y)-\theta_3(Y)\eta(X)\}=0.$$

Letting $X=V$, $Y=\xi$ in last equation we get

$$\left(-\frac{(1-\phi)c_p+\phi c_q}{4}\right)=0.$$

From this, we see that $c_p=(\phi+1)c_q$, which completes the proof. $\square$

Corollary 5.1. *There is no screen conformal screen semi-invariant half lightlike submanifold of a locally golden product space form* $(\tilde{N}=\acute{N}_p(c_p)\times\acute{N}_q(c_q),\tilde{g},\tilde{P})$ *with* $c_p\neq(\phi+1)c_q$.

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ON THE STRUCTURE OF SOME TYPES OF HIGHER DERIVATIONS

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Abstract. In this paper we introduce the concepts of higher $\{L_{g_n}, R_{h_n}\}$-derivation, higher $\{g_n, h_n\}$-derivation and Jordan higher $\{g_n, h_n\}$-derivation. Then we give a characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations in terms of $\{L_g, R_h\}$-derivations and $\{g, h\}$-derivations, respectively. Using this result, we prove that every Jordan higher $\{g_n, h_n\}$-derivation on a semiprime algebra is a higher $\{g_n, h_n\}$-derivation. In addition, we show that every Jordan higher $\{g_n, h_n\}$-derivation of the tensor product of a semiprime algebra and a commutative algebra is a higher $\{g_n, h_n\}$-derivation. Moreover, we show that there is a one to one correspondence between the set of all higher $\{L_{g_n}, R_{h_n}\}$-derivations and the set of all sequences of $\{L_{G_n}, R_{H_n}\}$-derivations. Also, it is presented that if $\mathcal{A}$ is a unital algebra and $\{f_n\}$ is a generalized higher derivation associated with a sequence $\{d_n\}$ of linear mappings, then $\{d_n\}$ is a higher derivation. Some other related results are also discussed.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be an algebra and let $g, h : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is said to be a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) if $f(ab) = g(a)b + ah(b)$ (resp. $f(a) = h(a)b + ag(b)$) for all $a, b \in \mathcal{A}$. By following Brešar [1], a linear mapping f is called a $\{g, h\}$-derivation on $\mathcal{A}$ if it is both a $\{L_g, R_h\}$-derivation and a $\{R_g, L_h\}$-derivation, i.e., $f(ab) = g(a)b + ah(b) = h(a)b + ag(b)$ for all $a, b \in \mathcal{A}$. A linear mapping f is called a Jordan $\{g, h\}$-derivation if $f(a \circ b) = g(a) \circ b + a \circ h(b)$ for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$. We call $a \circ b$ the Jordan product of a and b. It is evident that $a \circ b = b \circ a$ for all $a, b \in \mathcal{A}$. The notion of a Jordan $\{g, h\}$-derivation

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is a generalization of what is called a Jordan generalized derivation in [10]. Recall that a linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *Jordan generalized derivation* if there exists a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $f(a \circ b) = f(a) \circ b + a \circ d(b)$ for all $a, b \in \mathcal{A}$; in this case d is called an associated linear mapping of f. It is clear that $f(a \circ b) = d(a) \circ b + a \circ f(b)$ for all $a, b \in \mathcal{A}$. Obviously, the definition of a generalized Jordan derivation is generally not equivalent to that of Jordan generalized derivation. For more details in this regard, see e.g., [1, 10], and the references therein.

As an important result, Brešar [1, Theorem 4.3] proved that every Jordan $\{g, h\}$-derivation of a semiprime algebra $\mathcal{A}$ is a $\{g, h\}$-derivation. He also showed that every Jordan $\{g, h\}$-derivation of the tensor product of a semiprime algebra and a commutative algebra is a $\{g, h\}$-derivation. It is evident that every $\{g, h\}$-derivation is a Jordan $\{g, h\}$-derivation, but the converse is in general not true, for instance, see [1, Example 2.1].

In this study, we introduce the concepts of higher $\{L_{g_n}, R_{h_n}\}$-derivation, higher $\{R_{g_n}, L_{h_n}\}$-derivation, higher $\{g_n, h_n\}$-derivation, Jordan higher $\{g_n, h_n\}$-derivation and then we present a characterization of these concepts on algebras. Throughout this paper, $\mathcal{A}$ denotes an algebra over a field $\mathbb{F}$ with $char(\mathbb{F}) = 0$ and I denotes the identity mapping on $\mathcal{A}$. Let f be a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) on an algebra $\mathcal{A}$. An easy induction argument implies that $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} g^{n-k}(a)h^k(b)$ (resp. $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}(a)g^k(b)$) (Leibniz rule) for each $a, b \in \mathcal{A}$ and each nonnegative integer n, where $f^0 = g^0 = h^0 = I$. Hence, if f is a $\{g, h\}$-derivation, then $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} g^{n-k}(a)h^k(b) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}(a)g^k(b)$ for all $a, b \in \mathcal{A}$. Suppose that f is a $\{L_g, R_h\}$-derivation on $\mathcal{A}$. If we define the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ by $f_n = \frac{f^n}{n!}$, $g_n = \frac{g^n}{n!}$ and $h_n = \frac{h^n}{n!}$, with $f_0 = g_0 = h_0 = I$, then it follows from the Leibniz rule that f_n's, g_n's and h_n's satisfy

$$f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b), \tag{1.1}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Similarly, if f is a $\{R_g, L_h\}$-derivation, then the above f_n, g_n and h_n satisfy

$$f_n(ab) = \sum_{k=0}^{n} h_{n-k}(a)g_k(b), \tag{1.2}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Also, if f is a $\{g, h\}$-derivation, then we have

$$f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b) = \sum_{k=0}^{n} h_{n-k}(a)g_k(b), \tag{1.3}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. This is our motivation to investigate the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ of linear mappings on an algebra $\mathcal{A}$ that satisfy (1.1) or (1.2) or (1.3). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) if there exist two sequences

$\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying (1.1) (resp. (1.2)). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{g_n, h_n\}$-derivation if it is both a higher $\{L_{g_n}, R_{h_n}\}$-derivation and a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. In addition, a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b),$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Notice that if $\{f_n\}$ is a higher $\{f_n, f_n\}$-derivation (resp. Jordan higher $\{f_n, f_n\}$-derivation), then it is an ordinary higher derivation (resp. Jordan higher derivation). We know that if f is a $\{L_g, R_h\}$-derivation, then $\{f_n = \frac{f^n}{n!}\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation, where $g_n = \frac{g^n}{n!}$, $h_n = \frac{h^n}{n!}$ and $f_0 = g_0 = h_0 = I$. We call this kind of higher $\{L_{g_n}, R_{h_n}\}$-derivation an ordinary higher $\{L_{g_n}, R_{h_n}\}$-derivation, but this is not the only example of a higher $\{L_{g_n}, R_{h_n}\}$-derivation. We have the same expression for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations. Using the idea of [10] and to make the article more accurate, we consider generalized derivations as follows: A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called an *l-generalized derivation* (resp. *r-generalized derivation*) associated with a linear mapping $d : \mathcal{A} \to \mathcal{A}$ if f is a $\{L_f, R_d\}$-derivation (resp. $\{R_f, L_d\}$-derivation) on $\mathcal{A}$. Naturally, a linear mapping f is called a *two-sided generalized derivation* if it is both an l-generalized derivation associated with a linear mapping d_1 and a r-generalized derivation associated with a linear mapping d_2 on $\mathcal{A}$. Recently, Hosseini [7] has studied two-sided generalized derivations and in that article he has presented a r-generalized derivation which is not an l-generalized derivation. A sequence $\{f_n\}$ of linear mappings is called a higher l-generalized derivation associated with a sequence $\{d_n\}$ of linear mappings if it is a higher $\{L_{f_n}, R_{d_n}\}$-derivation. Similarly, the concepts of higher r-generalized derivations and two-sided generalized higher derivations are defined. Most authors who have studied generalized higher derivations suppose that these mappings are dependent on higher derivations, see, e.g. [5, 12, 14], and the references therein. In this paper and in the characterization that we offer, we do not use this assumption. In fact, if $\{f_n\}$ is a generalized higher derivation (resp. Jordan generalized higher derivation) associated with a sequence $\{d_n\}$ of linear mappings, we do not assume that the sequence $\{d_n\}$ is necessarily a higher derivation (resp. Jordan higher derivation).

In 2010, Miravaziri [11] characterized all higher derivations on an algebra $\mathcal{A}$ in terms of derivations on $\mathcal{A}$. In this article, by getting idea and using techniques of [11], our aim is to characterize higher $\{L_{g_n}, R_{h_n}\}$-derivations, higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations on an algebra $\mathcal{A}$ in terms of $\{L_g, R_h\}$-derivations, $\{R_g, L_h\}$-derivations and $\{g, h\}$-derivations, respectively. As the main result of this article, we prove that if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$

of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. The same is also true for higher $\{g_n, h_n\}$-derivations. Using this result, if $\{f_n\}$ is a higher l-generalized derivation (resp. higher r-generalized derivation) associated with a sequence $\{d_n\}$, then we characterize $\{f_n\}$ without assuming that $\{d_n\}$ is a higher derivation. Mirzavaziri and Tehrani [12] characterized generalized higher derivations while assuming the associated sequences are higher derivations. So, our results improve their work.

As an application of the main result of this article, we investigate Jordan higher $\{g_n, h_n\}$-derivations on algebras. Let us give a brief background in this regard. It is a classical question in which algebras (or rings) a Jordan derivation is necessarily a derivation. In 1957, Herstein [9] achieved a result which asserts any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [3]. In 1975, Cusack [4] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). Moreover, Vukman [13] investigated generalized Jordan derivations on semiprime rings and he proved that every *generalized Jordan derivation* of a 2-torsion free semiprime ring is a generalized derivation. Recently, the first name author along with Ajda Fošner [6] have studied the same problem for (σ, τ)-derivations from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module $\mathcal{M}$. In this paper, we show that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation of a semiprime algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then it is a higher $\{g_n, h_n\}$-derivation, and further we prove that if $\mathcal{A}$ is a semiprime algebra, $\mathcal{S}$ is a commutative algebra, and $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation of $\mathcal{A} \otimes \mathcal{S}$, with $f_0 = g_0 = h_0 = I$, then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation. Here, $\mathcal{A} \otimes \mathcal{S}$ denotes the tensor product of $\mathcal{A}$ and $\mathcal{S}$. Also, some results related to generalized higher derivations are presented.

2. Main Results

Throughout the article, $\mathcal{A}$ denotes an algebra over a field of characteristic zero, and I is the identity mapping on $\mathcal{A}$. We begin with the following definitions.

Definition 2.1. Let $f, g, h : \mathcal{A} \to \mathcal{A}$ be linear mappings. We say that f is a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) if $f(ab) = g(a)b + ah(b)$ (resp. $f(ab) = h(a)b + ag(b)$) for all $a, b \in \mathcal{A}$.

Following Brešar [1], a linear mapping f is called a $\{g, h\}$-derivation on $\mathcal{A}$ if it is both a $\{L_g, R_h\}$-derivation and a $\{R_g, L_h\}$-derivation, i.e., $f(ab) = g(a)b + ah(b) =$

$h(a)b + ag(b)$ for all $a, b \in \mathcal{A}$. A linear mapping f is called a Jordan $\{g, h\}$-derivation if $f(a \circ b) = g(a) \circ b + a \circ h(b)$ for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$.

Definition 2.2. A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying (1.1) (resp. (1.2)). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{g_n, h_n\}$-derivation if it is both a higher $\{L_{g_n}, R_{h_n}\}$-derivation and a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. In addition, a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b), \tag{2.1}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n.

Before establishing the first result of this paper, we would like to draw your attention to the following discussion that makes clear the process of characterizing of higher $\{L_{g_n}, R_{h_n}\}$-derivations by $\{L_g, R_h\}$-derivations. Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation. So, $f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. If $f_0 = g_0 = h_0 = I$, then we have $f_1(ab) = g_1(a)b + ah_1(b)$, which means that f_1 is a $\{L_{g_1}, R_{h_1}\}$-derivation. Therefore, we have

$$\begin{aligned} f_1^2(ab) &= f_1\left(g_1(a)b + ah_1(b)\right) \\ &= g_1^2(a)b + g_1(a)h_1(b) + g_1(a)h_1(b) + ah_1^2(b) \\ &= g_1^2(a)b + 2g_1(a)h_1(b) + ah_1^2(b). \end{aligned}$$

Thus,

$$2g_1(a)h_1(b) = f_1^2(ab) - g_1^2(a)b - ah_1^2(b), \quad a, b \in \mathcal{A}. \tag{2.2}$$

Note that $f_2(ab) = g_2(a)b + g_1(a)h_1(b) + ah_2(b)$. So, $2f_2(ab) = 2g_2(a)b + 2g_1(a)h_1(b) + 2ah_2(b)$ holds for all $a, b \in \mathcal{A}$. Putting (2.2) in the previous formula, we deduce that $2f_2(ab) = 2g_2(a)b + f_1^2(ab) - g_1^2(a)b - ah_1^2(b) + 2ah_2(b)$ for all $a, b \in \mathcal{A}$. Hence, we can write

$$2f_2(ab) - f_1^2(ab) = \left(2g_2(a) - g_1^2(a)\right) b + a \left(2h_2(b) - h_1^2(b)\right). \tag{2.3}$$

Letting $F_2 = 2f_2 - f_1^2$, $G_2 = 2g_2 - g_1^2$ and $H_2 = 2h_2 - h_1^2$ in (2.3), we arrive at

$$F_2(ab) = G_2(a)b + aH_2(b), \quad a, b \in \mathcal{A},$$

which means that F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. If we assume that $F_1 = f_1$, $G_1 = g_1$ and $H_1 = h_1$, then we have $f_2 = \frac{1}{2}F_1^2 + \frac{1}{2}F_2$, $g_2 = \frac{1}{2}G_1^2 + \frac{1}{2}G_2$ and $h_2 = \frac{1}{2}H_1^2 + \frac{1}{2}H_2$. Indeed, we characterize f_2 by F_1 and F_2, where F_1 is a $\{L_{G_1}, R_{H_1}\}$-derivation and F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. By a process similar to the one described above, we achieve that $f_3 = \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3$, $g_3 = \frac{1}{6}G_1^3 + \frac{1}{6}G_1G_2 + \frac{1}{3}G_2G_1 + \frac{1}{3}G_3$ and $h_3 = \frac{1}{6}H_1^3 + \frac{1}{6}H_1H_2 + \frac{1}{3}H_2H_1 + \frac{1}{3}H_3$, where F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation on $\mathcal{A}$

for $i \in \{1,2,3\}$. Thus, we can inductively construct a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations characterizing a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$. This inductive method leads us to this idea that every higher $\{L_{g_n}, R_{h_n}\}$-derivation is characterized by a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations. The same is also true for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g, h\}$-derivations. In the following, we show that the characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations is not necessarily unique. In view of the above discussion, if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation with $f_0 = g_0 = h_0 = I$, then we have $f_2 = \frac{1}{2}F_1^2 + \frac{1}{2}F_2$, where $F_1 = f_1$ and F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. But we can also characterize the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ in other form. We know that $f_1(ab) = g_1(a)b + ah_1(b)$ for all $a, b \in \mathcal{A}$. Therefore,

$$\begin{aligned} f_1^2(ab) &= f_1\left(g_1(a)b + ah_1(b)\right) \\ &= g_1^2(a)b + g_1(a)h_1(b) + g_1(a)h_1(b) + ah_1^2(b) \\ &= g_1^2(a)b + 2g_1(a)h_1(b) + ah_1^2(b). \end{aligned}$$

Thus,

$$g_1(a)h_1(b) = \frac{1}{2}\left(f_1^2(ab) - g_1^2(a)b - ah_1^2(b)\right), \quad a, b \in \mathcal{A}. \tag{2.4}$$

Also, we know that $f_2(ab) = g_2(a)b + g_1(a)h_1(b) + ah_2(b)$ for all $a, b \in \mathcal{A}$. Putting (2.4) in the previous equation, we deduce that $f_2(ab) = g_2(a)b + \frac{1}{2}f_1^2(ab) - \frac{1}{2}g_1^2(a)b - \frac{1}{2}ah_1^2(b) + ah_2(b)$ for all $a, b \in \mathcal{A}$. Hence, we have

$$f_2(ab) - \frac{1}{2}f_1^2(ab) = \left(g_2(a) - \frac{1}{2}g_1^2(a)\right)b + a\left(h_2(b) - \frac{1}{2}h_1^2(b)\right), \tag{2.5}$$

for all $a, b \in \mathcal{A}$. Letting $\mathfrak{F}_2 = f_2 - \frac{1}{2}f_1^2$, $\mathfrak{G}_2 = g_2 - \frac{1}{2}g_1^2$ and $\mathfrak{H}_2 = h_2 - \frac{1}{2}h_1^2$ in (2.5), we arrive at

$$\mathfrak{F}_2(ab) = \mathfrak{G}_2(a)b + a\mathfrak{H}_2(b), \quad a, b \in \mathcal{A}.$$

Thus, $\mathfrak{F}_2$ is a $\{L_{\mathfrak{G}_2}, R_{\mathfrak{H}_2}\}$-derivation. So, we have $f_2 = \frac{1}{2}f_1^2 + \mathfrak{F}_2$, $g_2 = \frac{1}{2}g_1^2 + \mathfrak{G}_2$ and $h_2 = \frac{1}{2}h_1^2 + \mathfrak{H}_2$. The above expressions show that the term f_2 is characterized by f_1 and $\mathfrak{F}_2$, where f_1 is a $\{L_{g_1}, R_{h_1}\}$-derivation and $\mathfrak{F}_2$ is a $\{L_{\mathfrak{G}_2}, R_{\mathfrak{H}_2}\}$-derivation. Using the above method and doing more calculations, we get

$$\begin{aligned} \left(f_3 - \frac{1}{6}f_1^3 - f_1\mathfrak{F}_2\right)(ab) &= \left(g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2\right)(a)b + a\left(h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2\right)(b) \\ &= \left(h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2\right)(a)b + a\left(g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2\right)(b). \end{aligned}$$

Letting $\mathfrak{F}_3 = f_3 - \frac{1}{6}f_1^3 - f_1\mathfrak{F}_2$, $\mathfrak{G}_3 = g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2$ and $\mathfrak{H}_3 = h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2$, it is observed that $\mathfrak{F}_3$ is a $\{L_{\mathfrak{G}_3}, R_{\mathfrak{H}_3}\}$-derivation. Thus, we see that the terms f_3, g_3 and h_3 are characterized as follows:

$$\begin{cases} f_3 = \frac{1}{6}f_1^3 + f_1\mathfrak{F}_2 + \mathfrak{F}_3, \\ g_3 = \frac{1}{6}g_1^3 + g_1\mathfrak{G}_2 + \mathfrak{G}_3, \\ h_3 = \frac{1}{6}h_1^3 + h_1\mathfrak{H}_2 + \mathfrak{H}_3. \end{cases}$$

The aforementioned discussion demonstrates that the characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations is not necessarily unique. Therefore, one can think that if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation with $f_0 = g_0 = h_0 = I$, then there exist two sequences of $\{L_{G_n}, R_{H_n}\}$-derivations and $\{L_{\mathfrak{G}_n}, R_{\mathfrak{H}_n}\}$-derivations characterizing the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. The same is also valid for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations. In particular, if $\{d_n\}_{n=0,1,\ldots}$ with $d_0 = I$ is a higher derivation on $\mathcal{A}$, we can obtain two sequences $\{\delta_n\}_{n=0,1,\ldots}$ and $\{\Delta_n\}_{n=0,1,\ldots}$ of derivations on $\mathcal{A}$ characterizing $\{d_n\}$.

We begin our results with the following lemma which will be used extensively to prove the main theorem of this article. The following lemma has been motivated by [11].

Lemma 2.1. *Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases} (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1}f_{n-k}, \\ (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1}g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1}h_{n-k}, \end{cases}$$

for each nonnegative integer n. The same is also true for higher $\{g_n, h_n\}$-derivations.

Proof. Using induction on n, we prove this lemma. Let $n = 0$. We know that $f_1(ab) = g_1(a)b + ah_1(b)$ for all $a, b \in \mathcal{A}$. Thus, if $F_1 = f_1$, $G_1 = g_1$ and $H_1 = h_1$, then F_1 is a $\{L_{G_1}, R_{H_1}\}$-derivation on $\mathcal{A}$ and further, $(0+1)f_{0+1} = \sum_{k=0}^{0} F_{k+1}f_{0-k}$, $(0+1)g_{0+1} = \sum_{k=0}^{0} G_{k+1}g_{0-k}$ and $(0+1)h_{0+1} = \sum_{k=0}^{0} H_{k+1}h_{0-k}$. As induction assumption, suppose that F_k is a $\{L_{G_k}, R_{H_k}\}$-derivation for any $k \leq n$ and further

$$\begin{cases} (r+1)f_{r+1} = \sum_{k=0}^{r} F_{k+1}f_{r-k}, \\ (r+1)g_{r+1} = \sum_{k=0}^{r} G_{k+1}g_{r-k}, \\ (r+1)h_{r+1} = \sum_{k=0}^{r} H_{k+1}h_{r-k}, \end{cases}$$

for $r = 0, 1, \ldots, n-1$. Put $F_{n+1} = (n+1)f_{n+1} - \sum_{k=0}^{n-1} F_{k+1}f_{n-k}$, $G_{n+1} = (n+1)g_{n+1} - \sum_{k=0}^{n-1} G_{k+1}g_{n-k}$ and $H_{n+1} = (n+1)h_{n+1} - \sum_{k=0}^{n-1} H_{k+1}h_{n-k}$. Our next task is to show that F_{n+1} is a $\{L_{G_{n+1}}, R_{H_{n+1}}\}$-derivation on $\mathcal{A}$. For $a, b \in \mathcal{A}$, we have

$$\begin{aligned} F_{n+1}(ab) &= (n+1)f_{n+1}(ab) - \sum_{k=0}^{n-1} F_{k+1}f_{n-k}(ab) \\ &= (n+1)\sum_{k=0}^{n+1} g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right). \end{aligned}$$

So, we have

$$\begin{aligned} F_{n+1}(ab) &= \sum_{k=0}^{n+1}(n+1)g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right) \\ &= \sum_{k=0}^{n+1}(k+n+1-k)g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right). \end{aligned}$$

Since F_k is a $\{L_{G_k}, R_{H_k}\}$-derivation for each $k = 1, 2, \ldots, n$,

$$F_{n+1}(ab) = \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) + \sum_{k=0}^{n+1} g_k(a)(n+1-k)h_{n+1-k}(b)$$
$$- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} \left[G_{k+1}\left(g_l(a)\right)h_{n-k-l}(b) + g_l(a)H_{k+1}\left(h_{n-k-l}(b)\right)\right].$$

Letting

$$G = \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} G_{k+1}\left(g_l(a)\right)h_{n-k-l}(b),$$
$$H = \sum_{k=0}^{n+1} g_k(a)(n+1-k)h_{n+1-k}(b) - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} g_l(a)H_{k+l}\left(h_{n-k-l}(b)\right),$$

we have $F_{n+1}(ab) = G+H$. Here, we compute G and H. In the summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, we have $0 \leq k+l \leq n$ and $k \neq n$. Thus if we put $r = k+l$, then we can write it as the form $\sum_{r=0}^{n}\sum_{k+l=r,k\neq n}$. Putting $l = r-k$, we find that

$$\begin{aligned} G &= \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n}\sum_{0\leq k\leq r,k\neq n} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b) \\ &= \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b) - \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b. \end{aligned}$$

It means that

$$G + \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b = \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b).$$

Putting $r+1$ instead of k in the first summation of above, we have

$$\begin{aligned} &G + \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b \\ &= \sum_{r=0}^{n}(r+1)g_{r+1}(a)h_{n-r}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b) \\ &= \sum_{r=0}^{n-1}\left[(r+1)g_{r+1}(a) - \sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)\right]h_{n-r}(b) + (n+1)g_{n+1}(a)b. \end{aligned}$$

According to the induction hypothesis, $(r+1)g_{r+1}(a) = \sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)$ for $r = 0, \ldots, n-1$. So, it is obtained that

$$G = \left[(n+1)g_{n+1}(a) - \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)\right]b = G_{n+1}(a)b.$$

Like above, we achieve that

$$H = a\left[(n+1)h_{n+1}(b) - \sum_{k=0}^{n-1} H_{k+1}\left(h_{n-k}(b)\right)\right] = aH_{n+1}(b).$$

Therefore, we have $F_{n+1}(ab) = G + H = G_{n+1}(a)b + aH_{n+1}(b)$. □

Example 2.1. Using Lemma 2.1, the first five terms of a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ are as follows:

$$\begin{aligned}
f_0 =& I,\\
f_1 =& F_1,\\
2f_2 =& F_1f_1 + F_2f_0 = F_1F_1 + F_2,\\
f_2 =& \frac{1}{2}F_1^2 + \frac{1}{2}F_2,\\
3f_3 =& F_1f_2 + F_2f_1 + F_3f_0 = F_1(\frac{1}{2}F_1^2 + \frac{1}{2}F_2) + F_2F_1 + F_3,\\
f_3 =& \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3,\\
4f_4 =& F_1f_3 + F_2f_2 + F_3f_1 + F_4f_0\\
=& F_1\left(\frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3\right) + F_2\left(\frac{1}{2}F_1^2 + \frac{1}{2}F_2\right) + F_3F_1 + F_4,\\
f_4 =& \frac{1}{24}F_1^4 + \frac{1}{24}F_1^2F_2 + \frac{1}{12}F_1F_2F_1 + \frac{1}{12}F_1F_3 + \frac{1}{8}F_2F_1^2 + \frac{1}{8}F_2^2 + \frac{1}{4}F_3F_1 + \frac{1}{4}F_4.
\end{aligned}$$

We are now in a position to present the first main theorem of this article.

Theorem 2.1. *Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases}
f_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n}\left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) F_{r_1}\cdots F_{r_i}\right),\\
g_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n}\left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) G_{r_1}\cdots G_{r_i}\right),\\
h_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n}\left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) H_{r_1}\cdots H_{r_i}\right),
\end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^i r_j = n$. The same is also valid for higher $\{g_n, h_n\}$-derivations.

Proof. Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation. We first show that if f_n, g_n and h_n are of the above forms, then they satisfy the recursive relations of Lemma 2.1. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation, we put $a_{r_1,\ldots,r_i} = \prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}$. Note that if $r_1 + \cdots + r_i = n+1$, then $(n+1)a_{r_1,\ldots,r_i} = a_{r_2,\ldots,r_i}$. Furthermore, $a_{n+1} = \frac{1}{n+1}$. According to the aforementioned assumptions, we have

$$
\begin{aligned}
f_{n+1} &= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1} a_{r_1,\ldots,r_i}F_{r_1}\cdots F_{r_i}\right)+a_{n+1}F_{n+1}\\
&= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1} a_{r_1,\ldots,r_i}F_{r_1}\cdots F_{r_i}\right)+\frac{F_{n+1}}{n+1}.
\end{aligned}
$$

So,

$$
\begin{aligned}
(n+1)f_{n+1} &= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1} (n+1)a_{r_1,\ldots,r_i}F_{r_1}\cdots F_{r_i}\right)+F_{n+1}\\
&= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j=n+1} a_{r_2,\ldots,r_i}F_{r_1}\cdots F_{r_i}\right)+F_{n+1}\\
&= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}F_{r_1}\sum_{\sum_{j=2}^{i} r_j=n+1-r_1} a_{r_2,\ldots,r_i}F_{r_2}\cdots F_{r_i}\right)+F_{n+1}\\
&= \sum_{r_1=1}^{n}F_{r_1}\sum_{i=2}^{n-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j=n-(r_1-1)} a_{r_2,\ldots,r_i}F_{r_2}\cdots F_{r_i}\right)+F_{n+1}\\
&= \sum_{r_1=1}^{n}F_{r_1}f_{n-(r_1-1)}+F_{n+1}\\
&= \sum_{k=0}^{n}F_{k+1}f_{n-k}.
\end{aligned}
$$

Reasoning like above, we get that

$$
\begin{cases}
(n+1)g_{n+1}=\sum_{k=0}^{n}G_{k+1}g_{n-k},\\
(n+1)h_{n+1}=\sum_{k=0}^{n}H_{k+1}h_{n-k},
\end{cases}
$$

for each nonnegative integer n. Putting $n+1=m$, we find that

$$
mf_m=\sum_{k=0}^{m-1}F_{k+1}f_{m-1-k}=\sum_{k=0}^{m-2}F_{k+1}f_{m-1-k}+F_m,
$$

and consequently

$$
F_m=mf_m-\sum_{k=0}^{m-2}F_{k+1}f_{m-1-k}.
$$

Similarly, we have

$$\begin{cases} G_m = mg_m - \sum_{k=0}^{m-2} G_{k+1}g_{m-1-k}, \\ H_m = mh_m - \sum_{k=0}^{m-2} H_{k+1}h_{m-1-k}. \end{cases}$$

Therefore, we can define $F_n, G_n, H_n : \mathcal{A} \to \mathcal{A}$ by $F_0 = G_0 = H_0 = 0$ and

$$\begin{cases} F_n = nf_n - \sum_{k=0}^{n-2} F_{k+1}f_{n-1-k}, \\ G_n = ng_n - \sum_{k=0}^{n-2} G_{k+1}g_{n-1-k}, \\ H_n = nh_n - \sum_{k=0}^{n-2} H_{k+1}h_{n-1-k}, \end{cases}$$

for each positive integer n. It follows from Lemma 2.1 that $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations. In addition, we prove that if f_n, g_n and h_n have the forms

$$\begin{cases} (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1}f_{n-k}, \\ (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1}g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1}h_{n-k}, \end{cases}$$

where $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations, then $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. To see this, we use induction on n. For $n = 0$, we have $f_0(ab) = ab = g_0(a)h_0(b)$. As the inductive hypothesis, assume that

$$f_k(ab) = \sum_{i=0}^{k} g_i(a)h_{k-i}(b), \quad \text{for } k \le n.$$

Therefore, we have

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{k=0}^{n} F_{k+1}f_{n-k}(ab) \\ &= \sum_{k=0}^{n} F_{k+1} \sum_{i=0}^{n-k} g_i(a)h_{n-k-i}(b) \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} G_{k+1}g_{n-k-i}(a) \right) h_i(b) + \sum_{i=0}^{n} g_i(a) \left(\sum_{k=0}^{n-i} H_{k+1}h_{n-k-i}(b) \right). \end{aligned}$$

According to the above-mentioned recursive relations, we continue the previous expressions as follows:

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{i=0}^{n}(n-i+1)g_{n-i+1}(a)h_i(b) + \sum_{i=0}^{n} g_i(a)(n-i+1)h_{n-i+1}(b) \\ &= \sum_{i=1}^{n+1} ig_i(a)h_{n+1-i}(b) + \sum_{i=0}^{n}(n-i+1)g_i(a)h_{n+1-i}(b) \\ &= (n+1)\sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b), \end{aligned}$$

which means that $f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b)$. Thus, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ which is characterized by the sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations. The same can be proved for higher $\{g_n, h_n\}$-derivations. □

In the next example, using the above theorem, we characterize term f_4 of a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$.

Example 2.2. We compute the coefficients $a_{r_1,\dots,r_i}$ for the case $n = 4$. First, note that $4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1$. Based on the definition of $a_{r_1,\dots,r_i}$ we have

$$\begin{aligned}
a_4 &= \frac{1}{4},\\
a_{1,3} &= \frac{1}{1+3}\cdot\frac{1}{3} = \frac{1}{12},\\
a_{3,1} &= \frac{1}{3+1}\cdot\frac{1}{1} = \frac{1}{4},\\
a_{2,2} &= \frac{1}{2+2}\cdot\frac{1}{2} = \frac{1}{8},\\
a_{1,1,2} &= \frac{1}{1+1+2}\cdot\frac{1}{1+2}\cdot\frac{1}{2} = \frac{1}{24},\\
a_{1,2,1} &= \frac{1}{1+2+1}\cdot\frac{1}{2+1}\cdot\frac{1}{1} = \frac{1}{12},\\
a_{2,1,1} &= \frac{1}{2+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{8},\\
a_{1,1,1,1} &= \frac{1}{1+1+1+1}\cdot\frac{1}{1+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{24}.
\end{aligned}$$

Therefore, f_4, g_4 and h_4 are characterized as follows:

$$\begin{aligned}
f_4 =& \frac{1}{4}F_4 + \frac{1}{12}F_1F_3 + \frac{1}{4}F_3F_1 + \frac{1}{8}F_2F_2 + \frac{1}{24}F_1F_1F_2 + \frac{1}{12}F_1F_2F_1\\
&+ \frac{1}{8}F_2F_1F_1 + \frac{1}{24}F_1F_1F_1F_1,\\
g_4 =& \frac{1}{4}G_4 + \frac{1}{12}G_1G_3 + \frac{1}{4}G_3G_1 + \frac{1}{8}G_2G_2 + \frac{1}{24}G_1G_1G_2 + \frac{1}{12}G_1G_2G_1\\
&+ \frac{1}{8}G_2G_1G_1 + \frac{1}{24}G_1G_1G_1G_1,\\
h_4 =& \frac{1}{4}H_4 + \frac{1}{12}H_1H_3 + \frac{1}{4}H_3H_1 + \frac{1}{8}H_2H_2 + \frac{1}{24}H_1H_1H_2 + \frac{1}{12}H_1H_2H_1\\
&+ \frac{1}{8}H_2H_1H_1 + \frac{1}{24}H_1H_1H_1H_1.
\end{aligned}$$

Corollary 2.1. *Let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases}
(i) & (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1}f_{n-k},\\
(ii) & (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1}g_{n-k},\\
(iii) & (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1}h_{n-k},
\end{cases}$$

for each nonnegative integer n. Furthermore, we have

$$\begin{cases} (iv) \quad f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ (v) \quad g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ (vi) \quad h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Proof. According to Lemma 2.1, if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ satisfying recursive relations (i)-(vi). On the other hand, $\{f_n\}$ is a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. Hence, there is a sequence $\{\mathfrak{F}_n\}$ of $\{R_{\mathfrak{G}_n}, L_{\mathfrak{H}_n}\}$-derivations on $\mathcal{A}$ satisfying all the equations of (i)-(vi). But, we know that the solution of the recursive relations is unique. Therefore, we infer that $F_n = \mathfrak{F}_n$, $G_n = \mathfrak{G}_n$ and $H_n = \mathfrak{H}_n$ for all positive integers n. □

In [12], Mirzavaziri and Tehrani presented a characterization of generalized higher derivations. They defined a generalized higher derivation as follows. A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a generalized higher derivation if there exists a higher derivation $\{d_n\}$ on $\mathcal{A}$ such that $f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a) d_k(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. In fact, they assume that each generalized higher derivation is dependent on a higher derivation. In the following corollary, we show that this assumption is unnecessary.

Corollary 2.2. *Let $\{f_n\}$ be a higher $\{L_{f_n}, R_{d_n}\}$-derivation (resp. higher $\{R_{f_n}, L_{d_n}\}$-derivation) on an algebra $\mathcal{A}$ with $f_0 = d_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{F_n}, R_{D_n}\}$-derivations (resp. $\{R_{F_n}, L_{D_n}\}$-derivations) on $\mathcal{A}$ such that*

$$\begin{cases} (n+1) f_{n+1} = \sum_{k=0}^{n} F_{k+1} f_{n-k}, \\ (n+1) d_{n+1} = \sum_{k=0}^{n} D_{k+1} d_{n-k}, \end{cases}$$

for each nonnegative integer n. Furthermore, we have

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) D_{r_1} \cdots D_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

We are now going to give an example of a generalized higher derivation that does not depend on a higher derivation.

Example 2.3. Let $\mathcal{R}$ be a ring and let

$$\mathfrak{R} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \; : \; a, b, c \in \mathcal{R} \right\}.$$

Clearly, $\mathfrak{R}$ is a ring. Define the additive mappings $f, d : \mathfrak{R} \to \mathfrak{R}$ by

$$f\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}.$$

It is routine to see that

$$f(AB) = f(A)B + Ad(B), \quad A, B \in \mathfrak{R},$$

which means that f is an l-generalized derivation associated with d in which d is not a derivation. Define $f_n = \frac{f^n}{n!}$ and $d_n = \frac{d^n}{n!}$ for each nonnegative integer n with $f^0 = d^0 = I$. A straightforward verification shows that $f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a) d_k(b)$ for each nonnegative integer n, while $\{d_n\}$ is not a higher derivation.

Theorem 2.2. *Let $\{f_n\}$ be a sequence of linear mappings satisfying*

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for each positive integer n with $f_0 = I$, where F_n is a $\{L_{G_n}, R_{H_n}\}$-derivation (resp. $\{R_{G_n}, L_{H_n}\}$-derivation) for each positive integer n. Then there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings such that

$$\begin{cases} g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

for each positive integer n with $g_0 = h_0 = I$, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$ and furthermore, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) on $\mathcal{A}$.

Proof. We use induction on n. Suppose that if

$$f_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for $1 \le k \le n$, where F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation for each $i \le k$, then there exist the linear mappings g_k and h_k such that

$$\begin{cases} g_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

with $g_0 = h_0 = I$ and further $f_k(ab) = \sum_{i=0}^{k} g_{k-i}(a)h_i(b)$ for all $a, b \in \mathcal{A}$. Based on the assumption, we have the following equation:

$$f_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

in which F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation for each $1 \le i \le n+1$. Now, we define

$$\begin{cases} g_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right). \end{cases}$$

It follows from the proof of Theorem 2.1 that g_{n+1} and h_{n+1} satisfy the following recursive relations:

$$\begin{cases} (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}. \end{cases}$$

Our next task is to show that $f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b)$ for all $a, b \in \mathcal{A}$. Reusing the proof of Theorem 2.1, we have $(n+1)f_{n+1}(ab) = \sum_{k=0}^{n} F_{k+1} f_{n-k}(ab)$ for all $a, b \in \mathcal{A}$. Therefore,

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{k=0}^{n} F_{k+1} f_{n-k}(ab) \\ &= \sum_{k=0}^{n} F_{k+1} \sum_{i=0}^{n-1} g_i(a)h_{n-k-i}(b) \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} G_{k+1} g_{n-k-i}(a) \right) h_i(b) + \sum_{i=0}^{n} g_i(a) \left(\sum_{k=0}^{n-i} H_{k+1} h_{n-k-i}(b) \right) \\ &= \sum_{i=0}^{n} (n-i+1) g_{n-i+1}(a)h_i(b) + \sum_{i=0}^{n} (n-i+1) g_i(a)h_{n-i+1}(b) \\ &= \sum_{i=1}^{n+1} i g_i(a)h_{n+1-i}(b) + \sum_{i=0}^{n} (n-i+1) g_i(a)h_{n-i+1}(b) \\ &= \sum_{i=0}^{n+1} (n+1) g_i(a)h_{n+1-i}(b), \end{aligned}$$

which means that

$$f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b).$$

Thereby, our proof is complete. □

Corollary 2.3. *Let $\{f_n\}$ be a sequence of linear mappings satisfying*

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for each positive integer n with $f_0 = I$, where F_n is a generalized derivation associated with a linear mapping D_n for each positive integer n. Then there exists a sequence $\{d_n\}$ of linear mappings such that

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) D_{r_1} \cdots D_{r_i} \right), \tag{2.6}$$

for each positive integer n with $d_0 = I$, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$ and furthermore, $\{f_n\}$ is a generalized higher derivation associated with the sequence $\{d_n\}$.

For instance, let F_i be a generalized derivation associated with a linear mapping D_i for $i \in \{1, 2, 3\}$ on $\mathcal{A}$ and let $f_3 = \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3$. So we have the following calculations:

$$\begin{aligned} f_3(ab) =& \left(\frac{1}{6}F_1^3(a) + \frac{1}{6}F_1F_2(a) + \frac{1}{3}F_2F_1(a) + \frac{1}{3}F_3(a)\right) b \\ &+ \left(\frac{1}{2}F_1^2(a) + \frac{1}{2}F_2(a)\right) d_1(b) + f_1(a) \left(\frac{1}{2}D_1^2(b) + \frac{1}{2}D_2(b)\right) \\ &+ a \left(\frac{1}{6}D_1^3(a) + \frac{1}{6}D_1D_2(a) + \frac{1}{3}D_2D_1(a) + \frac{1}{3}D_3(a)\right), \end{aligned}$$

for all $a, b \in \mathcal{A}$. Considering $d_2 = \frac{1}{2}D_1^2 + \frac{1}{2}D_2$ and $d_3 = \frac{1}{6}D_1^3 + \frac{1}{6}D_1D_2 + \frac{1}{3}D_2D_1 + \frac{1}{3}D_3$, we see that

$$f_3(ab) = f_3(a)b + f_2(a)d_1(b) + f_1(a)d_2(b) + ad_3(b) = \sum_{k=0}^{3} f_{3-k}(a)d_k(b).$$

This leads us to the sequence $\{d_n\}$ satisfying (2.6) and further

$$f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b).$$

In the following, there are some immediate consequences of the previous results. Before it, recall that a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ such that $f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b)$ holds for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Since the Jordan product is commutative, we have

$$f_n(a \circ b) = f_n(b \circ a) = \sum_{k=0}^{n} g_{n-k}(b) \circ h_k(a) = \sum_{k=0}^{n} g_k(b) \circ h_{n-k}(a) = \sum_{k=0}^{n} h_{n-k}(a) \circ g_k(b).$$

So, it is observed that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation, then

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b) = \sum_{k=0}^{n} h_{n-k}(a) \circ g_k(b),$$

for all $a, b \in \mathcal{A}$.

Corollary 2.4. *Let $\{f_n\}$ be a Jordan higher $\{g_n, h_n\}$-derivation on a semiprime algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation.*

Proof. Using the proof of Theorem 2.1, we can show that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$ of Jordan $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Since $\mathcal{A}$ is a semiprime algebra, [1, Theorem 4.3] proves the corollary. □

In the following, $\mathcal{A} \otimes \mathcal{S}$ denotes the tensor product of two algebras $\mathcal{A}$ and $\mathcal{S}$, where both $\mathcal{A}$ and $\mathcal{S}$ are defined over a field $\mathbb{F}$ of characteristic zero. We know that the tensor product of two vector spaces V and W over a field $\mathbb{F}$ is also a vector space over $\mathbb{F}$.

Corollary 2.5. *Let $\mathcal{A}$ be a semiprime and $\mathcal{S}$ be a commutative algebra, and let $\{f_n\}$ be a Jordan higher $\{g_n, h_n\}$-derivation of $\mathcal{A} \otimes \mathcal{S}$ with $f_0 = g_0 = h_0 = I$. Then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation.*

Proof. As stated above, for a Jordan higher $\{g_n, h_n\}$-derivation $\{f_n\}$ of $\mathcal{A} \otimes \mathcal{S}$ with $f_0 = g_0 = h_0 = I$ there exists a sequence $\{F_n\}$ of Jordan $\{G_n, H_n\}$-derivations on the algebra $\mathcal{A} \otimes \mathcal{S}$ such that

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Theorems 3.1 and 4.3 of [1] together show that every Jordan $\{g, h\}$-derivation of the tensor product of a semiprime and a commutative algebra is a $\{g, h\}$-derivation. This fact along with the above-mentioned characterization of $\{f_n\}$ implies that the Jordan higher $\{g_n, h_n\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$ is a higher $\{g_n, h_n\}$-derivation. □

Corollary 2.6. *Let $\mathcal{A}$ be a semiprime and $\mathcal{S}$ be a commutative algebra, and let $\{d_n\}$ be a Jordan higher derivation of $\mathcal{A} \otimes \mathcal{S}$ with $d_0 = I$. Then $\{d_n\}$ is a higher derivation.*

Proof. This is an immediate consequence of [1, Corollary 4.4] and [11, Theorem 2.3]. □

The importance of Corollary 2.5 and 2.6 is that the algebra $\mathcal{A} \otimes \mathcal{S}$ is not semiprime if $\mathcal{S}$ is not semiprime. On the other hand, even the tensor product of semiprime algebras is not always semiprime. So, we are presenting a characterization of higher $\{g_n, h_n\}$-derivations on some algebras which maybe are not semiprime.

Remark 2.1. We know that the notion of a Jordan $\{g, h\}$-derivation is a generalization of Jordan generalized derivations (see Introduction). A sequence $\{f_n\}$ of linear mappings on an algebra $\mathcal{A}$ is called a Jordan generalized higher derivation if there exists a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ such that $f_n(a \circ b) = \sum_{k=0}^{n} f_{n-k}(a) \circ d_k(b)$ for all $a, b \in \mathcal{A}$. So, Corollaries 2.4 and 2.5 are also valid for Jordan generalized higher derivations.

Motivated by [11, Theorem 2.5], we prove the following theorem.

Theorem 2.3. *Let $\mathfrak{f}$ be the set of all higher $\{L_{g_n}, R_{h_n}\}$-derivations $\{f_n\}_{n=0,1,\ldots}$ on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$ and $\mathfrak{F}$ be the set of all sequences $\{F_n\}_{n=0,1,\ldots}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ with $F_0 = G_0 = H_0 = 0$. Then there is a one to one correspondence between $\mathfrak{f}$ and $\mathfrak{F}$. The same is also valid for higher $\{g_n, h_n\}$-derivations.*

Proof. Let $\{f_n\} \in \mathfrak{f}$. We are going to obtain a sequence $\{F_n\}_{n=0,1,\ldots}$ of $\{L_{G_n}, R_{H_n}\}$-derivations with $F_0 = G_0 = H_0 = 0$ that characterizes the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. Define $F_n, G_n, H_n : \mathcal{A} \to \mathcal{A}$ by $F_0 = G_0 = H_0 = 0$ and

$$\begin{cases} F_n = nf_n - \sum_{k=0}^{n-2} F_{k+1} f_{n-1-k}, \\ G_n = ng_n - \sum_{k=0}^{n-2} G_{k+1} g_{n-1-k}, \\ H_n = nh_n - \sum_{k=0}^{n-2} H_{k+1} h_{n-1-k}, \end{cases}$$

for each positive integer n. Then it follows from Lemma 2.1 that $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations characterizing the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. Conversely, suppose that $\{F_n\} \in \mathfrak{F}$ which means that every F_n is a $\{L_{G_n}, R_{H_n}\}$-derivation with $F_0 = G_0 = H_0 = 0$. We will show that there exists a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$ which is characterized by the sequence $\{F_n\}_{n=0,1,\ldots}$. We define $f_n, g_n, h_n : \mathcal{A} \to \mathcal{A}$ by $f_0 = g_0 = h_0 = I$ and

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right). \end{cases}$$

By Theorem 2.1, f_n, g_n and h_n satisfy the following recursive relations:

$$\begin{cases} (n+1) f_{n+1} = \sum_{k=0}^{n} F_{k+1} f_{n-k}, \\ (n+1) g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1) h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}. \end{cases}$$

Based on the last part of the proof of Theorem 2.1, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Thus, $\{f_n\} \in \mathfrak{f}$. Now, define $\mathcal{F} : \mathfrak{F} \to \mathfrak{f}$ by

$\mathcal{F}(\{F_n\}) = \{f_n\}$, where

$$\begin{cases} f_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) F_{r_1}\cdots F_{r_i}\right), \\ g_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) G_{r_1}\cdots G_{r_i}\right), \\ h_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) H_{r_1}\cdots H_{r_i}\right). \end{cases}$$

Clearly, $\mathcal{F}$ is a one to one correspondence. This yields the desired result. $\square$

Remark 2.2. Let $\mathcal{A}$ be a unital algebra with the identity element $\mathbf{e}$ and let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. According to Theorem 2.1, there exists a sequence $\{F_n\}$ of $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that

$$\begin{cases} f_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) F_{r_1}\cdots F_{r_i}\right), \\ g_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) G_{r_1}\cdots G_{r_i}\right), \\ h_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) H_{r_1}\cdots H_{r_i}\right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^i r_j = n$. It follows from [8, Theorem 3.1] that if f is a $\{g, h\}$-derivation on a unital algebra, then f, g and h are generalized derivation associated with the derivation δ. Indeed, we have $f = \delta + L_{f(\mathbf{e})}$, $g = \delta + L_{g(\mathbf{e})}$ and $h = \delta + L_{h(\mathbf{e})}$. Using this fact and that every $\{F_n\}$ is a $\{G_n, H_n\}$-derivation, we deduce that there is a sequence $\{D_n\}$ of derivations such that $F_n = D_n + L_{F_n(\mathbf{e})}$, $G_n = D_n + L_{G_n(\mathbf{e})}$ and $H_n = D_n + L_{H_n(\mathbf{e})}$ for any $n \in \mathbb{N}$. It means that every F_n, G_n and H_n is a generalized derivation associated with the derivation D_n. We can thus infer from [12] that $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are generalized higher derivations. We can see that

$$\begin{cases} f_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) (D_{r_1} + L_{F_{r_1}(\mathbf{e})})\cdots(D_{r_i} + L_{F_{r_i}(\mathbf{e})})\right), \\ g_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) (D_{r_1} + L_{G_{r_1}(\mathbf{e})})\cdots(D_{r_i} + L_{G_{r_i}(\mathbf{e})})\right), \\ h_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) (D_{r_1} + L_{H_{r_1}(\mathbf{e})})\cdots(D_{r_i} + L_{H_{r_i}(\mathbf{e})})\right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^i r_j = n$. Easily, we deduce that there is a higher derivation

$$d_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) D_{r_1}\cdots D_{r_i}\right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^i r_j = n$ on $\mathcal{A}$ such that

$$\begin{cases} f_n(ab) = \sum_{k=0}^n f_{n-k}(a)d_k(b), \\ g_n(ab) = \sum_{k=0}^n g_{n-k}(a)d_k(b), \\ h_n(ab) = \sum_{k=0}^n h_{n-k}(a)d_k(b), \end{cases}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. It follows from [1] that if f is a $\{g, h\}$-derivation on a unital algebra, then $f(\mathbf{e}), g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$. So, we have

$$\begin{aligned}F_n(ab) &= D_n(ab) + L_{F_n(\mathbf{e})}(ab)\\ &= D_n(a)b + aD_n(b) + F_n(\mathbf{e})ab\\ &= D_n(a) + a\left(D_n(b) + F_n(\mathbf{e})b\right)\\ &= D_n(a)b + aF_n(b),\end{aligned}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. Similarly, $G_n(ab) = D_n(a)b + aG_n(b)$ and $H_n(ab) = D_n(a)b + aH_n(b)$ for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. So, one can easily obtain that

$$\begin{cases} f_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)f_k(b),\\ g_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)g_k(b),\\ h_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)h_k(b),\end{cases}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$.

Proposition 2.1. *Let $\mathfrak{R}$ be a unital ring with the identity element $\boldsymbol{e}$ and let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on $\mathfrak{R}$. Then $f_n(\boldsymbol{e}), g_n(\boldsymbol{e}), h_n(\boldsymbol{e}) \in Z(\mathfrak{R})$ for any nonnegative integer n.*

Proof. Using induction on n, we prove this proposition. According to page 2 of [1], the result is certainly true if $n = 1$. We show that the result is true for $n = 2$. We know that

$$f_2(xy) = g_2(x)y + g_1(x)h_1(y) + xh_2(y) = h_2(x)y + h_1(x)g_1(y) + xg_2(y), \tag{2.7}$$

for all $x, y \in \mathfrak{R}$. Taking $y = \mathbf{e}$ in (2.7), we obtain

$$f_2(x) = g_2(x) + g_1(x)h_1(\mathbf{e}) + xh_2(\mathbf{e}) = h_2(x) + h_1(x)g_1(\mathbf{e}) + xg_2(\mathbf{e}), \tag{2.8}$$

and taking $x = \mathbf{e}$, we get

$$f_2(y) = g_2(\mathbf{e})y + g_1(\mathbf{e})h_1(y) + h_2(y) = h_2(\mathbf{e})y + h_1(\mathbf{e})g_1(y) + g_2(y). \tag{2.9}$$

Comparing (2.8) and (2.9) and using the fact that $h_1(\mathbf{e}), g_1(\mathbf{e}) \in Z(\mathfrak{R})$, we see that $g_2(\mathbf{e}), h_2(\mathbf{e}) \in Z(\mathfrak{R})$ and consequently, $f_2(\mathbf{e}) \in Z(\mathfrak{R})$. As induction hypothesis, assume that the result is true for any $k < n$. We have

$$\begin{aligned}f_n(xy) &= g_n(x)y + g_{n-1}(x)h_1(y) + \cdots + xh_n(y)\\ &= h_n(x)y + h_{n-1}(x)g_1(y) + \cdots + xg_n(y).\end{aligned}$$

Reasoning like above, we have

$$\begin{aligned}f_n(x) &= g_n(x) + g_{n-1}(x)h_1(\mathbf{e}) + \cdots + xh_n(\mathbf{e})\\ &= h_n(x) + h_{n-1}(x)g_1(\mathbf{e}) + \cdots + xg_n(\mathbf{e})\end{aligned}$$

and also

$$\begin{aligned}f_n(y) &= g_n(\mathbf{e})y + g_{n-1}(\mathbf{e})h_1(y) + \cdots + h_n(y)\\ &= h_n(\mathbf{e})y + h_{n-1}(\mathbf{e})g_1(y) + \cdots + g_n(y).\end{aligned}$$

Comparing the above equations and using the inductive hypothesis, we get that $g_n(\mathbf{e}), h_n(\mathbf{e}) \in Z(\mathfrak{R})$ and consequently, $f_n(\mathbf{e}) \in Z(\mathfrak{R})$. $\square$

The article ends with the following theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a unital algebra with the identity element $\mathbf{e}$ and let $\{f_n\}$ be a generalized higher derivation associated with a sequence $\{d_n\}$ of linear mappings. Then $\{d_n\}$ is a higher derivation.*

Proof. We use induction to get our goal. The result trivially holds for $n = 1$. Now suppose that $d_k(ab) = \sum_{i=0}^{k} d_{k-i}(a)d_i(b)$ for any $k < n$. We have

$$f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b) = f_n(a)b + ad_n(b) + \sum_{k=1}^{n-1} f_{n-k}(a)d_k(b).$$

Since $\mathcal{A}$ is unital, we get that

$$f_n(b) = f_n(\mathbf{e})b + d_n(b) + \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(b)$$

and consequently, we have

$$d_n(b) = f_n(b) - f_n(\mathbf{e})b - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(b),$$

for all $b \in \mathcal{A}$. Now, we have the following expressions:

$$\begin{aligned}
d_n(ab) =& f_n(ab) - f_n(\mathbf{e})ab - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(ab) \\
=& \sum_{k=0}^{n} f_{n-k}(a)d_k(b) - f_n(\mathbf{e})ab - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e}) \sum_{i=0}^{k} d_{k-i}(a)d_i(b) \\
=& \left[f_n(a) - f_n(\mathbf{e})a - f_{n-1}(\mathbf{e})d_1(a) - \cdots - f_1(\mathbf{e})d_{n-1}(a)\right] b \\
& + \left[f_{n-1}(a) - f_{n-1}(\mathbf{e})a - f_{n-2}(\mathbf{e})d_1(a) - \cdots - f_1(\mathbf{e})d_{n-2}(a)\right] d_1(b) \\
& + \cdots + ad_n(b) \\
=& d_n(a)b + d_{n-1}(a)d_1(b) + \cdots + ad_n(b) \\
=& \sum_{k=0}^{n} d_{n-k}(a)d_k(b).
\end{aligned}$$

It means that $\{d_n\}$ is a higher derivation, as desired. $\square$

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SOME PROPERTIES OF NEW HYPERGEOMETRIC FUNCTIONS IN FOUR VARIABLES

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Abstract. In this paper, we introduce ten new quadruple hypergeometric series. We also obtain their various properties such that integral representations, fractional derivatives, N-fractional connections, operational relations and generating functions.

1. Introduction

In recent years, several interesting and useful properties of certain multiple hypergeometric functions have been investigated by many authors (see, e.g., [1, 3–9, 11, 12, 14, 15, 17, 21, 22, 25, 26]). In a sequel of such type of works mentioned above in this paper, we introduce ten new hypergeometric series of four variables as below

$$\begin{aligned} & X_{70}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u\right) \\ & = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} & X_{71}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, y, z, u\right) \\ & = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \end{aligned} \tag{1.2}$$

$$X_{72}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u\right)$$

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$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.3}$$

$$X_{73}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.4}$$

$$X_{74}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.5}$$

$$X_{75}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.6}$$

$$X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.7}$$

$$X_{77}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.8}$$

$$X_{78}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.9}$$

$$X_{79}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.10}$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1),$$

for $m \geq 1$, $(a)_0 = 1$, Γ being the well-known Gamma function.

The present paper aims at introducing and investigating certain properties of hypergeometric series $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$. The structure of this paper is as follows. In Section 2, integral representations of Laplace-type are obtained. In Section 3, we establish some fractional derivatives for our series. Section 4 presents certain connections by means of N-fractional operator. Section 5 deals with the derivation of operational relations between the quadruple functions $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$ and triple hypergeometric functions. The generating functions are given in the last section of this paper.

2. Integral Representations of Laplace-Type

In this section, we present certain integrals of Laplace-type involving the quadruple series $X_i^{(4)}$, $i = 70, 71, \ldots, 79$. For our purpose, we begin by recalling the following confluent hypergeometric functions [23]:

$$ {}_0F_1\left(-;c;x\right) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \cdot \frac{x^m}{m!}, \tag{2.1} $$

$$ {}_1F_1\left(a;c;x\right) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \cdot \frac{x^m}{m!}, \tag{2.2} $$

$$ \Phi_3\left(a;c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}, \tag{2.3} $$

$$ \mathsf{H}_6\left(a;c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}, \tag{2.4} $$

$$ \mathsf{H}_7\left(a;b,c;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_m (c)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}. \tag{2.5} $$

Now, if we consider the definitions of the confluent hypergeometric functions ${}_0F_1, {}_1F_1, \Phi_3, \mathsf{H}_6$ and H_7, we can derive the following integral representations:

$$ \begin{aligned} & X_{70}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right) \\ =& \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathsf{H}_7\left(a_1;c_1,c_2;x,sy\right) {}_0F_1\left(-;c_3;s^2 z\right) {}_1F_1\left(a_3;c_4;su\right) ds, \\ & \mathrm{Re}\left(a_2\right) > 0, \end{aligned} \tag{2.6} $$

$$ \begin{aligned} & X_{71}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_1,c_2,c_3;x,y,z,u\right) \\ =& \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathsf{H}_6\left(a_1;c_1;x,sy\right) {}_0F_1\left(-;c_2;s^2 z\right) {}_1F_1\left(a_3;c_3;su\right) ds, \\ & \mathrm{Re}\left(a_2\right) > 0, \end{aligned} \tag{2.7} $$

$$ \begin{aligned} & X_{72}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right) \\ =& \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1}\ t^{a_2-1} {}_0F_1\left(-;c_1;s^2 x + t^2 z\right) \\ & \times {}_0F_1\left(-;c_2;sty\right) {}_1F_1\left(a_3;c_3;tu\right) dsdt, \quad \mathrm{Re}\left(a_1\right) > 0, \mathrm{Re}\left(a_2\right) > 0, \end{aligned} \tag{2.8} $$

$$ \begin{aligned} & X_{73}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_2,c_1,c_1,c_3;x,y,z,u\right) \\ =& \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1}\ t^{a_2-1} {}_0F_1\left(-;c_1;sty + t^2 z\right) \\ & \times {}_0F_1\left(-;c_2;s^2 x\right) {}_1F_1\left(a_3;c_3;tu\right) dsdt, \quad \mathrm{Re}\left(a_1\right) > 0, \mathrm{Re}\left(a_2\right) > 0, \end{aligned} \tag{2.9} $$

$$\begin{aligned}
&X_{74}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty+t^2z\right)\\
&\times{}_1F_1\left(a_3;c_2;tu\right)dsdt,\quad \mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0, \qquad (2.10)\\
&X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_2)}\int_0^\infty e^{-s}s^{a_2-1}\mathsf{H}_7\left(a_1;c_1,c_2;x,sy\right){}_1F_1\left(a_3;c_3;sz\right){}_1F_1\left(a_4;c_4;su\right)ds,\\
&\mathrm{Re}\left(a_2\right)>0, \qquad (2.11)\\
&X_{76}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_2)}\int_0^\infty e^{-s}s^{a_2-1}\mathsf{H}_6\left(a_1;c_1;x,sy\right){}_1F_1\left(a_3;c_2;sz\right){}_1F_1\left(a_4;c_3;su\right)ds,\\
&\mathrm{Re}\left(a_2\right)>0, \qquad (2.12)\\
&X_{77}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}\Phi_3\left(a_3;c_1;tz,s^2x\right){}_0F_1\left(-;c_2;sty\right)\\
&\times{}_1F_1\left(a_4;c_3;tu\right)dsdt,\quad \mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0, \qquad (2.13)\\
&X_{78}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_2,c_1,c_1,c_3;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}\Phi_3\left(a_3;c_1;tz,sty\right){}_0F_1\left(-;c_2;s^2x\right)\\
&\times{}_1F_1\left(a_4;c_3;tu\right)dsdt,\quad \mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0, \qquad (2.14)\\
&X_{79}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_1,c_1,c_2;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}\ t^{a_2-1}\ v^{a_3-1}\\
&\times{}_0F_1\left(-;c_1;s^2x+sty+tvz\right){}_1F_1\left(a_4;c_2;tu\right)dsdtdv,\\
&\mathrm{Re}\left(a_1\right)>0,\mathrm{Re}\left(a_2\right)>0,\mathrm{Re}\left(a_3\right)>0. \qquad (2.15)
\end{aligned}$$

Proof. To establish (2.6), denote by $\mathcal{I}$ the right side of the relation (2.6). Then, by substituting the expression of the confluent hypergeometric functions (2.1), (2.2) and (2.5) into the right hand side of (2.6), we have

$$\mathcal{I}=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_3)_q}{(c_1)_m(c_1)_n(c_1)_p(c_1)_q\Gamma(a_2)}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\cdot\frac{u^q}{q!}\int_0^\infty e^{-s}s^{a_2+2p+n+q-1}ds,$$

by using the known equality (see [23])

$$\Gamma(a)=\int_0^\infty e^{-s}\ s^{a-1}ds,\quad \mathrm{Re}\left(a\right)>0,$$

we get the result after some simplifications. Similarly, one can proof the relations (2.6) to (2.15). $\square$

3. Fractional Derivatives

The fractional derivative operator D_w^k that was introduced by Miller and Ross [16] is given as

$$D_w^k w^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} w^{a-k}, \quad \mathrm{Re}\,(a) > -1. \tag{3.1}$$

Now, by using the above operator, we aim in this section at establishing the following fractional derivative formulae:

$$\begin{aligned} &D_w^{a_1-c}\left[w^{a_1-1} X_{70}^{(4)}\left(c,c,a_2,a_2,c,a_2,a_2,a_3;c_1,c_2,c_3,c_4;w^2x,wy,z,u\right)\right] \\ &=\frac{\Gamma(a_1)}{\Gamma(c)} w^{c-1} X_{70}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_2,c_3,c_4;w^2x,wy,z,u\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &D_w^{a_2-c}\left[w^{a_2-1} X_{71}^{(4)}\left(a_1,a_1,c,c,a_1,c,c,a_3;c_1,c_1,c_2,c_3;x,wy,w^2z,wu\right)\right] \\ &=\frac{\Gamma(a_2)}{\Gamma(c)} w^{c-1} X_{71}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_1,c_2,c_3;x,wy,w^2z,wu\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'}\left[w_1^{a_1-1} w_2^{a_2-1} X_{72}^{(4)}\left(c,c,c',c',c,c',c',a_3;c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,\right.\right. \\ &\left.\left.w_2^2z,w_2u\right)\right] \\ &=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{72}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,\ a_2,a_3;c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,\right. \\ &\left.w_2^2z,w_2u\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_3-c'}\left[w_1^{a_1-1} w_2^{a_3-1} X_{73}^{(4)}\left(c,c,a_2,a_2,c,a_2,a_2,c';c_2,c_1,c_1,c_3;w_1^2x,w_1y,z,w_2u\right)\right] \\ &=\frac{\Gamma(a_1)\Gamma(a_3)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{73}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,\ a_2,a_2,a_3;c_2,c_1,c_1,c_3;w_1^2x,w_1y,z,w_2u\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} D_{w_3}^{a_3-c''}\left[w_1^{a_1-1} w_2^{a_2-1} w_3^{a_3-1} X_{74}^{(4)}\left(c,c,c',c',c,c',c',c'';c_1,c_1,c_1,c_2;w_1^2x,\right.\right. \\ &\left.\left.w_1w_2y,w_2^2z,w_2w_3u\right)\right] \\ &=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')} w_1^{c-1} w_2^{c'-1} w_3^{c''-1} \\ &\quad\times X_{74}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_1,c_1,c_1,c_2;w_1^2x,w_1w_2y,w_2^2z,w_2w_3u\right), \end{aligned} \tag{3.6}$$

$$D_{w_1}^{a_3-c} D_{w_2}^{a_4-c'}\left[w_1^{a_3-1} w_2^{a_4-1} X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,c,c';c_1,c_2,c_3,c_4;x,y,w_1z,w_2u\right)\right]$$

(3.7)
$$=\frac{\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;\ c_2,c_1,c_1,c_3;x,y,w_1z,w_2u\right),$$

$$D_w^{a_2-c}\left[w^{a_2-1}X_{76}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,c,c';c_1,c_1,c_2,c_3;x,wy,wz,wu\right)\right]$$

(3.8)
$$=\frac{\Gamma(a_2)}{\Gamma(c)}w^{c-1}X_{76}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_2,c_1,c_1,c_3;x,wy,wz,wu\right),$$

(3.9)
$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}D_{w_3}^{a_3-c''}D_{w_4}^{a_4-c'''}\left[w_1^{a_1-1}w_2^{a_2-1}w_3^{a_3-1}w_4^{a_4-1}X_{77}^{(4)}\left(c,c,c',c',c,c',c'',c''';\right.\right.$$
$$\left.\left.c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,w_2w_3z,w_2w_4u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma(c''')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}w_4^{c'''-1}$$

(3.10)
$$\times X_{77}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,w_2w_3z,w_2w_4u\right),$$

$$D_{w_1}^{a_2-c}D_{w_2}^{a_3-c'}D_{w_3}^{a_4-c''}\left[w_1^{a_2-1}w_2^{a_3-1}w_3^{a_4-1}X_{78}^{(4)}\left(a_1,a_1,c,c,a_1,c,c',c'';c_2,c_1,c_1,c_3;\right.\right.$$
$$\left.x,w_1y,w_1w_2z,w_1w_3u)\right]$$
$$=\frac{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}$$

(3.11)
$$\times X_{78}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_2,c_1,c_1,c_3;x,w_1y,w_1w_2z,w_1w_3u\right),$$

$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}\left[w_1^{a_1-1}w_2^{a_2-1}X_{79}^{(4)}\left(c,c,c',c',c,c',a_3,a_4;c_1,c_1,c_1,c_2;w_1^2x,w_1w_2y,\right.\right.$$
$$\left.w_2z,w_2u)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}$$

(3.12)
$$\times X_{79}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_1,c_1,c_2;w_1^2x,w_1w_2y,w_2z,w_2u\right).$$

Proof. We have

$$D_w^{a_1-c}\left[w^{a_1-1}X_{70}^{(4)}\left(c,c,a_2,a_2,c,a_2,a_2,a_3;c_1,c_2,c_3,c_4;w^2x,wy,z,u\right)\right]$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(c)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\cdot\frac{u^q}{q!}D_w^{a_1-c}w^{a_1+2m+n-1}.$$

Now, with the help of (3.1) and Definition 1.1, the proof of the first fractional derivative formula is completed. The proofs of the assertions (3.3) to (3.12) run parallel to that of the assertion (3.2) then we skip the details. $\square$

4. N-Fractional Connections

First, by recalling the N-fractional operator due to Bin-Saad [4]:

$$\mathcal{M}_w^{a,c,b} = \left[w^{a-1}(1-w)^{-b}\right]_{a-c} = \frac{\Gamma(a-c)}{2\pi i}\int_C \frac{\eta^{a-1}(1-\eta)^{-b}}{(\eta-z)^{a-c}}d\eta, \tag{4.1}$$

where $a, b, c \in \mathbb{C}$ and $(a-c) \notin \mathbb{Z}$, we aim in this section to investigate the following relationships:

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{12}\left(a_1, b; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&A X_{70}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_3, c; x, y, z, u\right),\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{10}\left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&A X_{71}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_2, c; x, y, z, u\right),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{11}\left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&A X_{72}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{10}\left(b, a_1; c_1, c_2; \frac{x}{(1-u)^2}, \frac{y}{(1-u)}, z\right)\\ =&A X_{73}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_2, c_1, c_1, c; z, y, x, u\right),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{9}\left(a_1, b; c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&A X_{74}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{17}\left(a_1, b, a_2; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&A X_{75}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_3, c; x, y, z, u\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{14}\left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&A X_{76}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_1, c_2, c; x, y, z, u\right),\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{16}\left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&A X_{77}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.9}$$

$$\mathcal{M}_u^{a,c,b} X_{15}\left(a_1, b, a_2; c_2, c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)$$

$$=AX_{78}^{(4)}\left(a_1,a_1,b,b,a_1,b,a_2,a;c_2,c_1,c_1,c;x,y,z,u\right), \tag{4.10}$$

$$\mathcal{M}_u^{a,c,b}X_{13}\left(a_1,b,a_2;c_1;x,\frac{y}{(1-u)},\frac{z}{(1-u)}\right)$$
$$=AX_{79}^{(4)}\left(a_1,a_1,b,b,a_1,b,a_2,a;c_1,c_1,c_1,c;x,y,z,u\right), \tag{4.11}$$

where $A=e^{-\pi i(a-c)}\frac{\Gamma(1-c)}{\Gamma(1-a)}u^{c-1}$ and $X_9,X_{10},\ldots,X_{17}$ are Exton's hypergeometric functions of three variables [10] defined by

$$X_9\left(a_1,a_2;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c)_{m+n+p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.12}$$

$$X_{10}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_{m+n}(c_2)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.13}$$

$$X_{11}\left(a_1,a_2;c_1,c_2;x,y,\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_{m+p}(c_2)_n}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.14}$$

$$X_{12}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.15}$$

$$X_{13}\left(a_1,a_2,a_3;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c)_{m+n+p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.16}$$

$$X_{14}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{m+n}(c_2)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.17}$$

$$X_{15}\left(a_1,a_2,a_3;c_2,c_1;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{n+p}(c_2)_m}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.18}$$

$$X_{16}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_{m+p}(c_2)_n}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.19}$$

$$X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}. \tag{4.20}$$

Proof. To prove (4.2), from the equality (4.15), we can write

$$\mathcal{M}_u^{a,c,b}X_{12}\left(a_1,b;c_1,c_2,c_3;x,\frac{y}{(1-u)},\frac{z}{(1-u)^2}\right)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(b)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\mathcal{M}_u^{a,c,b}(1-u)^{-(n+2p)}.$$

By applying the formula (4.1) and in view of the relation (1.1) one can get the result with direct calculations. The proofs of the remaining relations run in the same way. □

5. Operational Relations

Here, in this section, we shall discuss some operational relations by means of the following operational formulas (see [3, 20]):

$$D_{\alpha}^{k}\alpha^{a} = \frac{\Gamma(a+1)}{\Gamma(a-k+1)}\alpha^{a-k}, \tag{5.1}$$

$$D_{\alpha}^{-k}\alpha^{a} = \frac{\Gamma(a+1)}{\Gamma(a+k+1)}\alpha^{a+k}, \tag{5.2}$$

$k \in \mathbb{N} \cup \{0\}, a \in \mathbb{C} - \{-1,-2,\dots\}$, where D_{α} denotes the derivative operator and D_{α}^{-1} denotes the inverse of the derivative.

In the following, certain operational connections among the hypergeometric series of three and four variables as:

(5.3)
$$\begin{aligned}&\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a} X_{8}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,\alpha y,z\right)\left(\alpha^{a_2-1}\beta^{c_4-1}\gamma^{a-1}\right)\\ =&\alpha^{a_2-1}\beta^{c_4-1}\gamma^{a-1}X_{70}^{(4)}\left(a_2,a_2,a_1,a_1,a_2,a_1,a_1,a_3;c_4,c_2,c_1,c_3;u,\alpha y,x,z\right),\end{aligned}$$

(5.4)
$$\begin{aligned}&\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a} X_{14}\left(a_1,a_2,a_3;c_1,c_2;x,\alpha y,\alpha z\right)\left(\alpha^{a_2-1}\beta^{c_3-1}\gamma^{a-1}\right)\\ =&\alpha^{a_2-1}\beta^{c_3-1}\gamma^{a-1}X_{71}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_3,c_2;x,\alpha y,u,\alpha z\right),\end{aligned}$$

(5.5)
$$\begin{aligned}&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a} X_{20}\left(a_1,a_2,\frac{a_3}{2},\frac{a_3+1}{2};c_1,c_2;\alpha_1^2x,\alpha_1 y,\right.\\ &\left.4\alpha_2^2 z\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1}\right)\\ =&\alpha_1^{a_1-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1}X_{72}^{(4)}\left(a_3,a_3,a_1,a_1,a_3,a_1,a_1,a_2;c_1,c_3,c_1,c_2;\alpha_2^2z,u,\alpha_1^2x,\alpha_1y\right),\end{aligned}$$

(5.6)
$$\begin{aligned}&\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a} X_{6}\left(a_1,a_2,a_3;c_1,c_2;x,\alpha y,z\right)\left(\alpha^{a_2-1}\beta^{c_3-1}\gamma^{a-1}\right)\\ =&\alpha^{a_2-1}\beta^{c_3-1}\gamma^{a-1}X_{73}^{(4)}\left(a_2,a_2,a_1,a_1,a_2,a_1,a_1,a_3;\ c_3,c_1,c_1,c_2;u,\alpha y,x,z\right),\end{aligned}$$

(5.7)
$$\begin{aligned}&\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a} X_{20}\left(a_1,a_2,\frac{a_3}{2},\frac{a_3+1}{2};c_1,c_2;\alpha_1^2\beta x,\right.\\ &\left.\alpha_1 y,4\alpha_2^2\beta z\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_3-1}\beta^{c_1-1}\gamma^{a-1}\right)\\ =&\alpha_1^{a_1-1}\alpha_2^{a_3-1}\beta^{c_1-1}\gamma^{a-1}\\ &\times X_{74}^{(4)}\left(a_3,a_3,a_1,a_1,a_3,a_1,a_1,a_2;c_1,c_1,c_1,c_2;\alpha_2^2\beta z,u,\alpha_1^2\beta x,\alpha_1 y\right),\end{aligned}$$

(5.8)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,\alpha_1 y,\alpha_1 z\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_4-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_4-1}\gamma^{a-1}X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,\alpha_1 y,\alpha_1 z,u\right),$$
$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^2\right)u\right]^{-a}F(3)_A\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;\alpha\beta x,y,z\right)$$
$$\times\left(\alpha^{a_2-1}\beta^{c_1-1}\gamma^{a-1}\right)$$
$$=\alpha^{a_2-1}\beta^{c_1-1}\gamma^{a-1}X_{76}^{(4)}\left(a_2,a_2,a_1,a_1,a_2,a_1,a_3,a_4;\ c_1,c_1,c_2,c_3;u,\alpha\beta x,y,z\right),$$

(5.9)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{16}\left(a_1,a_2,a_3;c_1,c_2;x,\alpha_1 y,\alpha_1 z\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}X_{77}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,\alpha_1 y,\alpha_1 z,u\right),$$

(5.10)

$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^2\right)u\right]^{-a}F_G\left(a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_2,c_2;x,\alpha y,z\right)$$
$$\times\left(\alpha^{a_3-1}\beta^{c_3-1}\gamma^{a-1}\right)$$
$$=\alpha^{a_3-1}\beta^{c_3-1}\gamma^{a-1}X_{78}^{(4)}\left(a_3,a_3,a_1,a_1,a_3,a_1,a_4,a_2;c_3,c_2,\ c_2,c_1;u,\alpha y,z,x\right),$$

(5.11)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}$$
$$\times F_N\left(a_1,\frac{a_2}{2},a_3,a_4,\frac{a_2+1}{2},a_4;c_1,c_2,c_2;\alpha_2 x,4\alpha_1^2\beta y,\alpha_2\beta z\right)\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}$$
$$\times X_{79}^{(4)}\left(a_2,a_2,a_4,a_4,a_2,a_4,a_3,a_1;c_2,c_2,c_2,c_1;\alpha_1^2\beta y,u,\alpha_2\beta z,\alpha_2 x\right),$$

where X_6, X_8 and X_{20} are the Exton's triple hypergeometric series defined by [10]

$$X_6\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},\tag{5.12}$$

(5.13)

$$X_8\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},$$

(5.14)

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},$$

and $F_A^{(3)}$, F_G, F_N denote the Lauricella's series of three variables (see [13]).

Proof. To solve equation (5.3), first we assume the left hand side of (5.3) by the notation $\mathcal{I}$, then expressing the Exton's function X_8 as a series in the left hand side of (5.3) and using the binomial theorem, it follows that:

$$\begin{aligned}\mathcal{I} = &\sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p(a)_q}{(c_1)_m(c_2)_n(c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!} \\ &\times \beta^{-q}\gamma^{-q} D_{\alpha}^{2q} D_{\beta}^{-q} D_{\gamma}^{-q} \left(\alpha^{a_2+n+2q-1}\beta^{c_4-1}\gamma^{a-1}\right).\end{aligned}$$

Now, we use the above formulas in (5.1) and (5.2), then in view of Definition 1.1, we arrive at the desired result (5.3). In a similar manner, one can prove the relations (5.4) to (5.11). □

6. Generating Functions

In this section, we will consider some generating functions for our quadruple series. Because the proofs of the following relations are similar to the proofs of results in [2, 18, 19, 23, 24], we omit these proofs.

The generating relations of series $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$ given as below

(6.1)
$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} X_{70}^{(4)} \left(a_1+k, a_1+k, a_2, a_2, a_1+k, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u\right) t^k \\ =&(1-t)^{-a_1} X_{70}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; \frac{x}{(1-t)^2}, \frac{y}{(1-t)}, z, u\right),\end{aligned}$$

(6.2)
$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(a_2)_k}{k!} X_{71}^{(4)} \left(a_1, a_1, a_2+k, a_2+k, a_1, a_2+k, a_2+k, a_3; c_1, c_1, c_2, c_3; x, y, z, u\right) t^k \\ =&(1-t)^{-a_2} X_{71}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, \frac{y}{(1-t)}, \frac{z}{(1-t)^2}, \frac{u}{(1-t)}\right),\end{aligned}$$

(6.3)
$$\begin{aligned}&\sum_{k_1,k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_2)_{k_2}}{k_1!k_2!} X_{72}^{(4)} \left(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2, a_2+k_2,\right. \\ &\left. a_3; c_1, c_2, c_1, c_3; x, y, z, u\right) t_1^{k_1} t_2^{k_2} \\ =&(1-t_1)^{-a_1}(1-t_2)^{-a_2} X_{72}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; \frac{x}{(1-t_1)^2},\right. \\ &\left. \frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)^2}, \frac{u}{(1-t_2)}\right),\end{aligned}$$

(6.4)
$$\sum_{k_1,k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_3)_{k_2}}{k_1!k_2!} X_{73}^{(4)}(a_1+k_1, a_1+k_1, a_2, a_2, a_1+k_1, a_2, a_2, a_3+k_2;$$

$$c_2, c_1, c_1, c_3; x, y, z, u)t_1^{k_1}t_2^{k_2}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_3}X_{73}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; \frac{x}{(1-t_1)^2},$$
$$\frac{y}{(1-t_1)}, z, \frac{u}{(1-t_2)}\Bigg),$$

(6.5)
$$\sum_{k=0}^{\infty}\frac{(a_3)_k}{k!}X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3+k; c_1, c_1, c_1, c_2; x, y, z, u\right)t^k$$
$$=(1-t)^{-a_3}X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, \frac{u}{(1-t)}\right),$$

(6.6)
$$\sum_{k=0}^{\infty}\frac{(a_2)_k}{k!}X_{75}^{(4)}\left(a_1, a_1, a_2+k, a_2+k, a_1, a_2+k, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u\right)t^k$$
$$=(1-t)^{-a_2}X_{75}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, \frac{y}{(1-t)}, \frac{z}{(1-t)}, \frac{u}{(1-t)}\right),$$

(6.7)
$$\sum_{k_1,k_2=0}^{\infty}\frac{(a_1)_{k_1}(a_2)_{k_2}}{k_1!k_2!}X_{76}^{(4)}(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2, a_3, a_4;$$
$$c_1, c_1, c_2, c_3; x, y, z, u)t_1^{k_1}t_2^{k_2}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2}X_{76}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; \frac{x}{(1-t_1)^2},$$
$$\frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)}, \frac{u}{(1-t_2)}\Bigg),$$

(6.8)
$$\sum_{k_1,k_2=0}^{\infty}\frac{(a_3)_{k_1}(a_4)_{k_2}}{k_1!k_2!}X_{77}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3+k_1, a_4+k_2; c_1, c_2, c_1, c_3;$$
$$x, y, z, u\Bigg)t_1^{k_1}t_2^{k_2}$$
$$=(1-t_1)^{-a_3}(1-t_2)^{-a_4}X_{77}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y,$$
$$\frac{z}{(1-t_1)}, \frac{u}{(1-t_2)}\Bigg),$$

(6.9)
$$\sum_{k_1,k_2,k_3=0}^{\infty}\frac{(a_1)_{k_1}(a_2)_{k_2}(a_3)_{k_3}}{k_1!k_2!k_3!}X_{78}^{(4)}(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2,$$

$$a_3+k_3,a_4;c_2,c_1,c_1,c_3;x,y,z,u)t_1^{k_1}t_2^{k_2}t_3^{k_3}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2}(1-t_3)^{-a_3}X_{78}^{(4)}\Bigg(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_2,c_1,c_1,c_3;$$
$$\frac{x}{(1-t_1)^2},\frac{y}{(1-t_1)(1-t_2)},\frac{z}{(1-t_2)(1-t_3)},\frac{u}{(1-t_2)}\Bigg),$$

(6.10)
$$\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\frac{(a_1)_{k_1}(a_2)_{k_2}(a_3)_{k_3}(a_4)_{k_4}}{k_1!k_2!k_3!k_4!}X_{79}^{(4)}(a_1+k_1,a_1+k_1,a_2+k_2,a_2+k_2,a_1+k_1,$$
$$a_2+k_2,a_3+k_3,a_4+k_4;c_1,c_1,c_1,c_2;x,y,z,u)t_1^{k_1}t_2^{k_2}t_3^{k_3}t_4^{k_4}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2}(1-t_3)^{-a_3}(1-t_4)^{-a_4}X_{79}^{(4)}\,(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;$$
$$c_1,c_1,c_1,c_2;\frac{x}{(1-t_1)^2},\frac{y}{(1-t_1)(1-t_2)},\frac{z}{(1-t_2)(1-t_3)},\frac{u}{(1-t_2)(1-t_4)}\Bigg).$$

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