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A NUMERICAL SOLUTION OF A COUPLING SYSTEM OF CONFORMABLE TIME-DERIVATIVE TWO-DIMENSIONAL BURGERS' EQUATIONS

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Abstract. In this paper, we deal with a numerical solution of a coupling system of fractional conformable time-derivative two-dimensional (2D) Burgers' equations. The presence of both the fractional time derivative and the nonlinear terms in this system of equations makes solving it more difficult. Firstly, we use the Cole-Hopf transformation in order to reduce the coupling system of equations to a conformable time-derivative 2D heat equation for which the numerical solution is calculated by the explicit and implicit schemes. Secondly, we calculate the numerical solution of the proposed system by using both the obtained solution of the conformable time-derivative heat equation and the inverse Cole-Hopf transformation. This approach shows its efficiency to deal with this class of fractional nonlinear problems. Some numerical experiments are displayed to consolidate our approach.

1. Introduction

In the last two decades, the fractional derivatives regained an important interest, and have been widely used in various fields, such as modelling viscoelastic problems, signal processing, control theory, finance, etc. Thus, many classical mathematical models have been reformulated into new models with fractional-order derivatives for their important numerous applications (see [7, 8, 10, 13, 16]). As a result, the scientists introduced different fractional derivative definitions (see [4, 5, 10]): Caputo

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fractional derivative, Riemann-Liouville fractional derivative, Grünwald–Letnikov fractional derivative and others. We give for example two popular definitions below. For $\alpha \in [n-1, n)$, the α-derivatives of the function f are given as

(i) Riemann-Liouville definition

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx;$$

(ii) Caputo definition

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

All definitions including (i) and (ii) satisfy the property that the fractional derivative is linear. We cite for example some research works linked with the subject. Tarasov [16] investigated some properties of the chain rule and Leibniz rule for fractional derivatives. Khalil et al. [9] introduced a new definition of a fractional derivative and fractional integral (called also fractional conformable derivative and fractional integral) for which there are large number of numerous works done (see [1, 4, 5, 9, 17, 18]). Anderson et al. [4] introduced more precise definition of the conformable derivative motiving by a proportional-derivative controller. Ortigueira et al. [14] analyzed the definitions of the Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives. For instance, Abdeljawad [1] gave conformable versions of the chain rule, integration by parts, Taylor power series expansions and Laplace transform. In [2], the authors introduced the fractional conformable semi-group of operators whose generator will be the fractional derivative of the semigroup at $t = 0$. In [3] the authors studied the fractional logistic models in the frame of fractional operator generated by conformable derivatives. Yavuz et al. [19] introduced the conformable derivative operator in modelling neuronal dynamics.

In this work, we are interested in studying a coupling system of the fractional conformable derivative 2D Burgers' equations which incorporate the interaction between the nonlinear convection processes and the diffusive viscous processes. Many works studied the one/two viscous Burgers' equation (with integer-order derivative) using the Cole-Hopf transformation [11, 15]. It is known that the Burgers' equation has been used as a mathematical model in various areas such as number theory, gas dynamics, heat conduction, elasticity theory, etc. It has a lot of similarity to the famous Navier-Stokes equations [6, 12] and has often been used as a simple model equation for comparing the accuracy of different computational algorithms. However the inviscid Burgers' equation lacks one most important property attributed to turbulence since the solutions do no exhibit chaotic features like sensitivity with respect to initial conditions. The purpose of the current study focuses in the use of the Cole-Hopf transformation for this class of the fractional nonlinear problems. So, we transform with the help of Cole-Hopf transformation the coupling system of the conformable time-derivative 2D Burgers' equations into conformable time-derivative heat equation. The numerical solution of the latter is obtained by the explicit and implicit schemes. Therefore, we can easily calculate the solution of the system of the conformable time-derivative 2D Burgers' equations via the inverse Cole-Hopf transformation. For illustration, some numerical experiments are displayed to show the efficiency of this approach.

The paper is organized as follows. Section 2 gives some useful materiel and position of the problem. Section 3 uses the 2D Cole-Hopf transformation. Section 4 proposes the calculation of numerical solutions to heat equation by the explicit and implicit schemes and gives the required solutions for the coupling of 2D Burgers' equations. The last section displays the numerical results.

2. Preliminaries and Position of the Problem

Let us recall below a definition and a theorem which summarizes some important properties.

Definition 2.1 ([5, 9])**.** Given a function $f : [0, \infty) \to \mathbb{R}$, then the conformable fractional derivative of f with order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in (0.1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

Theorem 2.1 ([5, 9])**.** *Let $0 < \alpha \le 1$ and f, g be α-differentiable at a point $t > 0$. Then*

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$;
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$;
3. $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$;
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$;
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$;
6. *in addition, if f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

2.1. **Coupling system of the conformable derivatives 2D Burgers' equations.** Let us consider the following coupling system of 2D Burgers' equations

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right), \end{cases} \tag{2.3}$$

where $\alpha \in (0, 1)$, $r > 0$ the diffusion coefficient, $(x, y) \in \Omega$ (a rectangular domain), $t > 0$ and $\partial^\alpha u/\partial t^\alpha$, $\partial^\alpha v/\partial t^\alpha$ mean conformable derivatives respectively of the functions $u(x, y, t)$ and $v(x, y, t)$.

Subject to the initial conditions

$$\begin{cases} u(x,y,0) = u_0(x,y), & \text{for any } (x,y) \in \Omega, \\ v(x,y,0) = v_0(x,y), & \text{for any } (x,y) \in \Omega, \end{cases} \tag{2.4}$$

and the boundary conditions

$$\begin{cases} u(x,y,t) = f(x,y,t), & \text{for any } (x,y) \in \partial\Omega,\ t > 0, \\ v(x,y,t) = g(x,y,t), & \text{for any } (x,y,t) \in \partial\Omega,\ t > 0, \end{cases} \tag{2.5}$$

where f, g are two given functions.

We need later to use the following potential symmetry condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2.6}$$

3. Linearizing System (2.3) by the Cole-Hopf Transformation

Using the property 6 of Theorem 2.1, we can rewrite system (2.3) as follows

$$\begin{cases} t^{(1-\alpha)}\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), \\ t^{(1-\alpha)}\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right). \end{cases} \tag{3.1}$$

The Cole-Hopf transformation is performed in two steps.

First step. Suppose that $u = \psi_x$ and $v = \psi_y$, thus the system (3.1) becomes

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \psi_x\psi_{xx} + \psi_y\psi_{xy} = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ t^{(1-\alpha)}\psi_{yt} + \psi_x\psi_{yx} + \psi_y\psi_{yy} = r\left(\psi_{yxx} + \psi_{yyy}\right), \end{cases} \tag{3.2}$$

which can be rewritten as

$$\begin{cases} t^{(1-\alpha)}\psi_{xt} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xxx} + \psi_{xyy}\right), \\ t^{(1-\alpha)}\psi_{yt} + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_x^2\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{yxx} + \psi_{yyy}\right). \end{cases} \tag{3.3}$$

Integrating respectively the first equation of system (3.3) with respect to x and the second with respect to y, we obtain

$$\begin{cases} t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_1(y,t), \\ t^{(1-\alpha)}\psi_t + \left(\dfrac{1}{2}\psi_x^2\right) + \left(\dfrac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta_2(x,t), \end{cases} \tag{3.4}$$

where $\eta_1(y,t)$ and $\eta_2(x,t)$ are arbitrary functions depending respectively of y and x. Using the condition (2.6), we can combine two equations of system (3.4) and conclude that ψ satisfies the following equation (see [11])

$$t^{(1-\alpha)}\psi_t + \left(\frac{1}{2}\psi_x^2\right) + \left(\frac{1}{2}\psi_y^2\right) = r\left(\psi_{xx} + \psi_{yy}\right) + \eta(t). \tag{3.5}$$

Second step. Introducing the transformation as $\psi = -2r \ln \phi$, we have

$$u = -2r\frac{\phi_x}{\phi} \quad \text{and} \quad v = -2r\frac{\phi_y}{\phi}. \tag{3.6}$$

Both the derivatives of function ψ are

$$\psi_t = -2r\frac{\phi_t}{\phi}, \quad \psi_x = -2r\frac{\phi_x}{\phi}, \quad \psi_y = -2r\frac{\phi_y}{\phi}, \tag{3.7}$$

$$\psi_{xx} = -2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}, \quad \psi_{yy} = -2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}. \tag{3.8}$$

Inserting the derivatives ψ_t, ψ_x and ψ_y in the left side of (3.5) and the derivatives ψ_{xx} and ψ_{yy} in the right side, we obtain

$$\begin{aligned} &-2rt^{(1-\alpha)}\frac{\phi_t}{\phi} + \frac{1}{2}\left(-2r\frac{\phi_x}{\phi}\right)^2 + \frac{1}{2}\left(-2r\frac{\phi_y}{\phi}\right)^2 \\ =& r\left(-2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2} - 2r\frac{\phi_{yy}}{\phi} + 2r\frac{\phi_y^2}{\phi^2}\right) + \eta(t). \end{aligned} \tag{3.9}$$

Equation (3.9) can be reduced to

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}) + \zeta(t)\phi, \quad \text{where } \zeta(t) = \frac{-\eta(t)}{2r}. \tag{3.10}$$

We now state the following theorem in order to show that the calculus of the functions $u(x, y, t)$ and $v(x, y, t)$ is independent of the function $\zeta(t)$.

Theorem 3.1. *Let $\phi(x, y, t)$ be the solution of (3.10), $u(x, y, t)$ and $v(x, y, t)$ are defined in (3.6), then the solution u and v are independent of $\zeta(t)$.*

Proof. Let

$$\beta(t) = \int \frac{1}{t^{1-\alpha}}\zeta(t)dt,$$

then

$$\beta'(t) = \frac{1}{t^{1-\alpha}}\zeta(t).$$

Multiply by $e^{-\beta(t)}$ the two sides of (3.10), yields

$$\frac{\partial^\alpha \phi}{\partial t^\alpha}e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)} + \zeta(t)\phi e^{-\beta(t)}. \tag{3.11}$$

By using the property 6 of Theorem 2.1, (3.11) becomes

$$t^{1-\alpha}\frac{\partial \phi}{\partial t}e^{-\beta(t)} - \zeta(t)\phi e^{-\beta(t)} = r(\phi_{xx} + \phi_{yy})e^{-\beta(t)}. \tag{3.12}$$

Then

$$t^{1-\alpha}\frac{\partial}{\partial t}\left(e^{-\beta(t)}\phi\right) = r((e^{-\beta(t)}\phi)_{xx} + (e^{-\beta(t)}\phi)_{yy}). \tag{3.13}$$

Now, let $\psi(x,y,t) = e^{-\beta(t)}\phi(x,y,t)$. Then $\psi(x,y,t)$ satisfies the following 2D heat equation

$$t^{1-\alpha}\frac{\partial \psi}{\partial t} = r(\psi_{xx} + \psi_{yy}), \tag{3.14}$$

which rewrites in other form

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = r(\psi_{xx} + \psi_{yy}). \tag{3.15}$$

Note that the difference between the solution of (3.10) and (3.15) is the factor $e^{-\beta(t)}$. Therefore,

$$u(x,y,t) = \frac{\phi_x}{\phi} = \frac{e^{-\beta(t)}\phi_x}{e^{-\beta(t)}\phi} = \frac{\psi_x}{\psi}, \tag{3.16}$$

$$v(x,y,t) = \frac{\phi_y}{\phi} = \frac{e^{-\beta(t)}\phi_y}{e^{-\beta(t)}\phi} = \frac{\psi_y}{\psi}. \tag{3.17}$$

It is clear that the solutions u and v are independent of the function $\zeta(t)$. □

For simplicity of the present study, we can take for example $\zeta(t) \equiv 0$. Then we get the diffusion equation

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}). \tag{3.18}$$

3.1. **Initial and boundary conditions.** We now try to determine a new derivation of the initial and boundary conditions which correspond to (3.18). For the sake of simplicity, let us take

$$\Omega = [a,b] \times [a,b], \quad \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

with

$$\Gamma_1 = \{a \le x \le b,\ y = a\}, \quad \Gamma_2 = \{a \le x \le b,\ y = b\},$$
$$\Gamma_3 = \{x = a,\ a \le y \le b\} \quad \text{and} \quad \Gamma_4 = \{x = b,\ a \le y \le b\}.$$

Initial condition (IC). From (3.6), we can rewrite

$$\frac{\phi_x}{\phi} = \frac{u(x,y,t)}{-2r}. \tag{3.19}$$

Integrating the left and right sides of (3.19) with respect to x, we obtain

$$\ln(\phi) = \frac{-1}{2r}\int_a^x u(s,y,t)ds + \ln(\phi(a,y,t)).$$

Then we get

$$\phi(x,y,t) = \phi(a,y,t)\exp\left(\frac{-1}{2r}\int_a^x u(s,y,t)ds\right). \tag{3.20}$$

On the other hand, we rearrange the second term of (3.6) as follows

$$\frac{\phi_y}{\phi} = \frac{v(x,y,t)}{-2r}. \tag{3.21}$$

Integration of the above equation with respect to y, then we obtain

$$\ln(\phi) = \frac{-1}{2r}\int_a^y u(x,s,t)ds + \ln(\phi(x,a,t)),$$

which yields

$$\phi(x,y,t) = \phi(x,a,t)\exp\left(\frac{-1}{2r}\int_a^y v(x,s,t)ds\right). \tag{3.22}$$

At $x = a$, (3.22) gives

$$\phi(a,y,t) = \phi(a,a,t)\exp\left(\frac{-1}{2r}\int_a^y v(a,s,t)ds\right). \tag{3.23}$$

Inserting (3.23) into (3.20), yields

$$\phi(x,y,t) = \phi(a,a,t)\exp\left(-\frac{1}{2r}\int_a^y v(a,s,t)ds - \frac{1}{2r}\int_a^x u(s,y,t)ds\right), \tag{3.24}$$

and at $t = 0$ in (3.24), then the initial condition is written as

$$\phi(x,y,0) = \phi(a,a,0)\exp\left(-\frac{1}{2r}\int_a^y v(a,s,0)ds - \frac{1}{2r}\int_a^x u(s,y,0)ds\right). \tag{3.25}$$

From (3.6), it is clear that $\phi(a,a,0)$ has no effect on the final solution of system (2.3). In our case, we can consider for example $\phi(a,a,0) = 1$. It yields

$$\phi_0(x,y) = \exp\left(-\frac{1}{2r}\int_a^y v(a,s,0)ds - \frac{1}{2r}\int_a^x u(s,y,0)ds\right). \tag{3.26}$$

Boundary conditions (BC). Using (3.6), the boundary conditions are reduced to

$$\begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), & (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), & (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{3.27}$$

Therefore, the time-conformable diffusion equation with the initial and Neumann boundary conditions is given by

$$\begin{cases} \text{Eq. :} & \dfrac{\partial^\alpha \phi}{\partial t^\alpha} = r(\phi_{xx} + \phi_{yy}), \\ \text{IC :} & \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\int_a^y v(a,s,0)ds - \dfrac{1}{2r}\int_a^x u(s,y,0)ds\right), \\ \text{BC :} & \begin{cases} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)), \\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t), \ (x,y,t) \in (\partial\Omega \times (0,T)). \end{cases} \end{cases} \tag{3.28}$$

Reformulating problem (3.28) by using the property 6 of Theorem 2.1, it yields

$$
\left\{
\begin{array}{ll}
\text{Eq. :} & t^{(1-\alpha)}\dfrac{\partial \phi}{\partial t} = r(\phi_{xx}+\phi_{yy}),\\
\text{IC :} & \phi_0(x,y) = \exp\left(-\dfrac{1}{2r}\int_a^y v(a,s,0)ds - \dfrac{1}{2r}\int_a^x u(s,y,0)ds\right),\\
\text{BC:} & \left\{\begin{array}{l} \phi_x = -\dfrac{1}{2r}u(x,y,t)\phi(x,y,t),\ (x,y,t)\in(\partial\Omega\times(0,T)),\\ \phi_y = -\dfrac{1}{2r}v(x,y,t)\phi(x,y,t),\ (x,y,t)\in(\partial\Omega\times(0,T)).\end{array}\right.
\end{array}
\right. \tag{3.29}
$$

The solution of the problem (3.29) can be found in [13]. Finally, once the solution of the problem (3.29) is known, we can easily obtain the solution of the coupled problem (2.3)-(2.5) via the formula (3.6).

4. Numerical Study of the Problem (3.29)

We discretize the domain Ω by the finite difference method (FDM) into nx each of length $\Delta x = (b-a)/nx$ and into ny each of length $\Delta y = (b-a)/ny$ along, respectively the x-axis and y-axis. We define then the discrete mesh points (x_i, y_j, t_n) by $(a + i\Delta x, a + j\Delta y, n\Delta t)$, where $i = 0,\dots,nx$, $j = 0,\dots,ny$, $n = 0,\dots,T$.

4.1. **An explicit scheme.** By using a simple forward in time and centered in space discretization at point (x_i, y_j, t_n), the explicit scheme of (3.29) is given by

$$
t_n^{(1-\alpha)}\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t} = r\left(\frac{\phi_{i+1,j}^{n}-2\phi_{i,j}^{n}+\phi_{i-1,j}^{n}}{\Delta x^2}+\frac{\phi_{i,j+1}^{n}-2\phi_{i,j}^{n}+\phi_{i,j-1}^{n}}{\Delta y^2}\right).
$$

For every interior point (x_i, y_j, t_n), with $i = 1,\dots,nx-1$, $j = 1,\dots,ny-1$, we have

$$
\phi_{i,j}^{n+1} = A\phi_{i,j}^{n} + B(\phi_{i+1,j}^{n}+\phi_{i-1,j}^{n}) + C(\phi_{i,j+1}^{n}-\phi_{i,j-1}^{n}), \tag{4.1}
$$

where

$$
\begin{aligned}
A =& 1 - \frac{2r\Delta t}{\Delta x^2\, t_n^{(1-\alpha)}} - \frac{2r\Delta t}{\Delta y^2 t_n^{(1-\alpha)}},\\
B =& \frac{r\Delta t}{\Delta x^2\, t_n^{(1-\alpha)}}, \quad C = \frac{r\Delta t}{\Delta y^2\, t_n^{(1-\alpha)}}.
\end{aligned}
$$

Now, let us consider the so-called BC described as

$$
\left\{
\begin{array}{l}
\phi_x(x_i,y_j,t_n) \simeq \dfrac{\phi_{i+1,j}^{n}-\phi_{i-1,j}^{n}}{2\Delta x} = -\dfrac{1}{2r}u_{i,j}^{n}\phi_{i,j}^{n},\\
\phi_y(x_i,y_j,t_n) \simeq \dfrac{\phi_{i,j+1}^{n}-\phi_{i,j-1}^{n}}{2\Delta y} = -\dfrac{1}{2r}v_{i,j}^{n}\phi_{i,j}^{n},
\end{array}
\right. \tag{4.2}
$$

which can be rewritten as

$$
\left\{
\begin{array}{l}
\phi_{i+1,j}^{n} = \phi_{i-1,j}^{n} - \dfrac{\Delta x}{r}u_{i,j}^{n}\phi_{i,j}^{n},\\
\phi_{i,j+1}^{n} = \phi_{i,j-1}^{n} - \dfrac{\Delta y}{r}v_{i,j}^{n}\phi_{i,j}^{n}.
\end{array}
\right. \tag{4.3}
$$

Thus, we give the details for each discrete side as follows.

On the side Γ_1, let $j=0$ in (4.3), then we have

$$\begin{cases} \phi^n_{i+1,0} = \phi^n_{i-1,0} - \dfrac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}, \\ \phi^n_{i,1} = \phi^n_{i,-1} - \dfrac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}. \end{cases} \tag{4.4}$$

Substituting this constraint into (4.1) at the boundary points, for $i = 1, \dots, nx$, we obtain

$$\begin{aligned} \phi^{n+1}_{i,0} =& A\phi^n_{i,0} + B(\phi^n_{i+1,0} + \phi^n_{i-1,0}) + C(\phi^n_{i,1} + \phi^n_{i,-1}) \\ =& A\phi^n_{i,0} + B\left(2\phi^n_{i-1,0} - \frac{\Delta x}{r} u^n_{i,0}\phi^n_{i,0}\right) + C\left(2\phi^n_{i,1} - \frac{\Delta y}{r} v^n_{i,0}\phi^n_{i,0}\right). \end{aligned}$$

In same way as previously, we can calculate respectively the expressions both of the side Γ_2 for $j = ny, \Gamma_3$ for $i = 0$ and Γ_4 for $i = nx$, for all $i = 1, \dots, nx$,

$$\phi^{n+1}_{i,ny} = A\phi^n_{i,ny} + B\left(2\phi^n_{i-1,ny} - \frac{\Delta x}{r} u^n_{i,ny}\phi^n_{i,ny}\right) + C\left(2\phi^n_{i,ny-1} - \frac{\Delta y}{r} v^n_{i,ny}\phi^n_{i,ny}\right). \tag{4.5}$$

And for $j = 1, \dots, ny$,

$$\begin{aligned} \phi^{n+1}_{0,j} =& A\phi^n_{0,j} + B\left(2\phi^n_{1,j} + \frac{\Delta x}{r} u^n_{0,j}\phi^n_{0,j}\right) + C\left(2\phi^n_{0,j-1} - \frac{\Delta y}{r} v^n_{0,j}\phi^n_{0,j}\right), \\ \phi^{n+1}_{nx,j} =& A\phi^n_{nx,j} + B\left(2\phi^n_{nx-1,j} - \frac{\Delta x}{r} u^n_{nx,j}\phi^n_{nx,j}\right) + C\left(2\phi^n_{nx,j-1} - \frac{\Delta y}{r} v^n_{nx,j}\phi^n_{nx,j}\right). \end{aligned}$$

Adding the left-lower corner point (x_0, y_0), we obtain

$$\phi^{n+1}_{0,0} = A\phi^n_{0,0} + B\left(2\phi^n_{1,0} + \frac{\Delta x}{r} u^n_{0,0}\phi^n_{0,0}\right) + C\left(2\phi^n_{0,1} + \frac{\Delta y}{r} v^n_{0,0}\phi^n_{0,0}\right).$$

4.2. **An implicit scheme.** By using a simple forward in time and centered in space (FTCS) discretization at point (x_i, y_j, t_n), the implicit scheme for (3.29) is given by

$$t_n^{1-\alpha}\frac{\phi^{n+1}_{i,j} - \phi^n_{i,j}}{\Delta t} = r\left(\frac{\phi^{n+1}_{i+1,j} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i-1,j}}{\Delta x^2} + \frac{\phi^{n+1}_{i,j+1} - 2\phi^{n+1}_{i,j} + \phi^{n+1}_{i,j-1}}{\Delta y^2}\right),$$

which can rewrite as

$$-\alpha(\phi^{n+1}_{i+1,j} + \phi^{n+1}_{i-1,j}) + \gamma\phi^{n+1}_{i,j} - \beta(\phi^{n+1}_{i,j+1} + \phi^{n+1}_{i,j-1}) = \phi^n_{i,j}, \tag{4.6}$$

where

$$\alpha = \frac{r\Delta t}{\Delta x^2 t_n^{1-\alpha}}, \quad \beta = \frac{r\Delta t}{\Delta y^2 t_n^{1-\alpha}}, \quad \gamma = 1 + 2\alpha + 2\beta,$$

or in matrix form

$$\mathcal{A}.\mathcal{X} = \mathcal{B},$$

where

$$\mathcal{A} = \begin{pmatrix} A & B & 0 & \cdots & 0 \\ C & D & K & 0 \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & C & D & K \\ 0 & \cdots & 0 & L & M \end{pmatrix}_{(nx\times ny, nx\times ny)} ,$$

$\mathcal{X}^t = (\phi_{0,0}^{n+1}, \phi_{0,1}^{n+1}, ..., \phi_{nx,ny}^{n+1})$ and $\mathcal{B} = (\phi_{0,0}^{n}, \phi_{0,1}^{n}, ..., \phi_{nx,ny}^{n})$, A, B, C, D, K, L and M are the submatrices with dimension (nx, ny) and are defined respectively by

$$A = \begin{pmatrix} a & -2\beta & 0 & \cdots & 0 \\ -2\beta & \gamma & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & \gamma \end{pmatrix}, \quad B = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & b_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & b_n \end{pmatrix},$$

$$C = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & -2\beta & 0 & \cdots & 0 \\ -\beta & \gamma & -\beta & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma & -\beta \\ 0 & \cdots & 0 & -2\beta & d_{ny} \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad L = \begin{pmatrix} -2\alpha & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -2\alpha \end{pmatrix},$$

$$M = \begin{pmatrix} m' & -2\beta & 0 & \cdots & 0 \\ -2\beta & m_1 & 0 & \cdots & \vdots \\ 0 & \ddots & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -2\beta & m_{ny} \end{pmatrix},$$

with

$$\begin{aligned} a =& -\frac{\alpha\Delta x}{r} u_{0,0}^n + \gamma - \frac{\beta\Delta y}{r} v_{0,0}^n, \\ b_j =& -2\alpha - \frac{\alpha\Delta x}{r} u_{0,j}^n + \frac{\beta\Delta y}{r} v_{0,j}^n, \quad j = 1, \dots, ny, \end{aligned}$$

$$
\begin{aligned}
d_1 &= -\frac{\alpha\Delta x}{r}u_{1,0}^n + \gamma - \frac{\beta\Delta y}{r}v_{1,0}^n,\\
d_{ny} &= -\frac{\alpha\Delta x}{r}u_{nx,ny}^n + \gamma + \frac{\beta\Delta y}{r}v_{nx,ny}^n,\\
m' &= d_1, \quad m_i = -\frac{\alpha\Delta x}{r}u_{i,ny}^n + \gamma + \frac{\beta\Delta y}{r}v_{i,ny}^n, \quad \text{for } i = 1,\dots,nx.
\end{aligned}
$$

4.3. **Calculating the required solution.** The calculation of solution to system (2.3) can be obtained by the inverse Cole-Hopf transformation.

Let $D_x\phi_{i,j}^n$ and $D_y\phi_{i,j}^n$ denote the derivative of ϕ, respectively at point (x_i, y_j, t_n) with respect to x and y. The $D_x\phi_{i,j}^n$ and $D_y\phi_{i,j}^n$ can be calculated from the first order centered difference formula, for $i = 1,\dots,nx-1$, $j = 1,\dots,ny-1$,

$$
\begin{aligned}
D_x\phi_{i,j}^n &= \frac{\partial\phi}{\partial x} \simeq \frac{\phi_{i+1,j}^n - \phi_{i-1,j}^n}{2\Delta x},\\
D_y\phi_{i,j}^n &= \frac{\partial\phi}{\partial y} \simeq \frac{\phi_{i,j+1}^n - \phi_{i,j-1}^n}{2\Delta y}.
\end{aligned}
$$

Note that the derivatives $D_y\phi_{0,j}^n, D_y\phi_{nx,j}^n, D_x\phi_{i,0}^n$ and $D_x\phi_{i,ny}^n$ at the end points are known. Once the approximated values of ϕ, ϕ_x and ϕ_y are known at any discrete point (x_i, y_j, t_n), then the approximated values of u and v at discrete points can be calculated from the following discrete version, for $i = 1,\dots,nx$, $j = 1,\dots,ny$,

$$
\begin{cases}
u_{i,j}^n = -2r\dfrac{D_x\phi_{i,j}^n}{\phi_{i,j}^n},\\[2mm]
v_{i,j}^n = -2r\dfrac{D_y\phi_{i,j}^n}{\phi_{i,j}^n}.
\end{cases} \tag{4.7}
$$

5. Numerical Experiments

For illustration of the proposed method, we will report the accuracy of the method based on relative error L_1-norm and L_∞-norm which are defined by:

$$
||Erreuru||_{L_1} = \frac{||u_a - u_n||_1}{||u_a||_1}, \quad ||Erreurv||_{L_1} = \frac{||v_a - v_n||_1}{||v_a||_1}, \tag{5.1}
$$

and

$$
||Erreuru||_{L_\infty} = \frac{||u_a - u_n||_\infty}{||u_a||_\infty}, \quad ||Erreurv||_{L_\infty} = \frac{||v_a - v_n||_\infty}{||v_a||_\infty}, \tag{5.2}
$$

where the pair (u_a,v_a) is the analytical solution (5.3) (see [11, page 581]) for the system (2.3) and the pair (u_n,v_n) is the computed solution (4.7) for system (2.3).

To simulate, we take the following exact solution for system (2.3) in over square domain $\Omega = [0,1]\times[0,1]$

$$
\begin{cases}
u_a(x,y,t) = \dfrac{3}{4} - \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]},\\[2mm]
v_a(x,y,t) = \dfrac{3}{4} + \dfrac{1}{4[1 + exp((-4x\alpha + 4y\alpha - t^\alpha)/32r\alpha)]}.
\end{cases} \tag{5.3}
$$

Note that the initial and boundary conditions can be taken from the exact solutions. After computing, we evaluate respectively the relative errors (5.1) and (5.2). We use then the explicit and implicit schemes for the conformable time-derivative 2D heat equation and give the convergence of each scheme in the following Table 1 and Table 2.

TABLE 1. Relative errors L_1-norm.

Relative error	$\|\|Erreuru\|\|_{L_1}$		$\|\|Erreurv\|\|_{L_1}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.30e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.17e-03$	$2.17e-03$	$1.55e-03$	$1.55e-03$
$\Delta x = \Delta y = 0.05$	$1.46e-03$	$1.53e-03$	$8.19e-04$	$8.02e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.60e-03$	$5.64e-03$	$1.63e-03$	$1.58e-03$
$\Delta x = \Delta y = 0.1$	$4.69e-03$	$4.56e-03$	$1.58e-03$	$1.41e-03$
$\Delta x = \Delta y = 0.05$	$4.48e-03$	$4.52e-03$	$1.46e-03$	$1.37e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.85e-03$	$7.90e-03$	$1.43e-03$	$1.43e-03$
$\Delta x = \Delta y = 0.1$	$7.37e-03$	$7.47e-03$	$1.37e-03$	$1.31e-03$
$\Delta x = \Delta y = 0.05$	$7.26e-03$	$7.35e-03$	$1.06e-03$	$1.29e-03$

We remark that the relative error decreases as time increases in the Table 1.

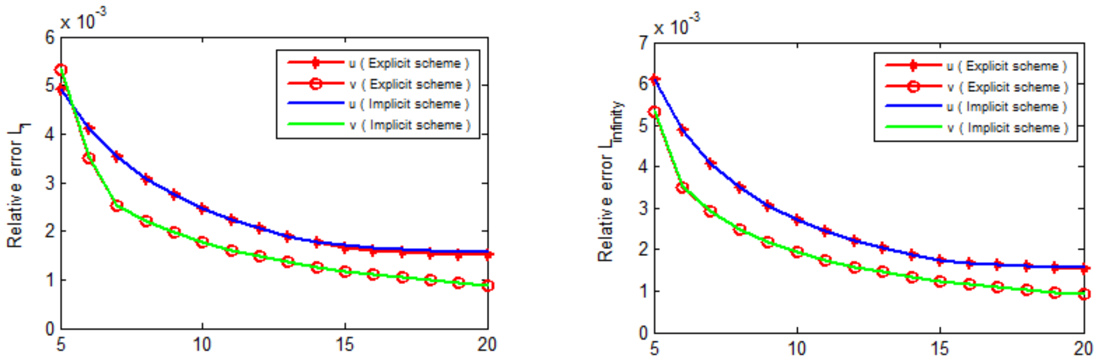

FIGURE 1. Graphs represent the tendency of the relative error .

We show through the Figure 1 the tendency of the relative errors. Let's give in the Figure 2 the graphs representing the numerical solution for 2D time-fractional heat equation (3.29) by using various values of α as shown in Table 3.

TABLE 2. Relative errors L_∞-norm.

Relative error	$\|\|Erreur u\|\|_{L_\infty}$		$\|\|Erreur v\|\|_{L_\infty}$	
Scheme	Explicit	Implicit	Explicit	Implicit
	T=0.1			
$\Delta x = \Delta y = 0.2$	$3.35e-03$	$3.34e-03$	$3.20e-03$	$3.22e-03$
$\Delta x = \Delta y = 0.1$	$2.39e-03$	$2.39e-03$	$1.70e-03$	$1.70e-03$
$\Delta x = \Delta y = 0.05$	$1.52e-03$	$1.53e-03$	$8.29e-04$	$8.29e-04$
	T=0.5			
$\Delta x = \Delta y = 0.2$	$5.62e-03$	$5.64e-03$	$1.69e-03$	$1.69e-03$
$\Delta x = \Delta y = 0.1$	$4.76e-03$	$4.76e-03$	$1.57e-03$	$1.91e-03$
$\Delta x = \Delta y = 0.05$	$4.62e-03$	$4.62e-03$	$1.48e-03$	$1.87e-03$
	T=1			
$\Delta x = \Delta y = 0.2$	$7.88e-03$	$7.90e-03$	$1.68e-03$	$1.68e-03$
$\Delta x = \Delta y = 0.1$	$7.47e-03$	$7.47e-03$	$1.50e-03$	$1.51e-03$
$\Delta x = \Delta y = 0.05$	$7.34e-03$	$7.35e-03$	$1.47e-03$	$1.49e-03$

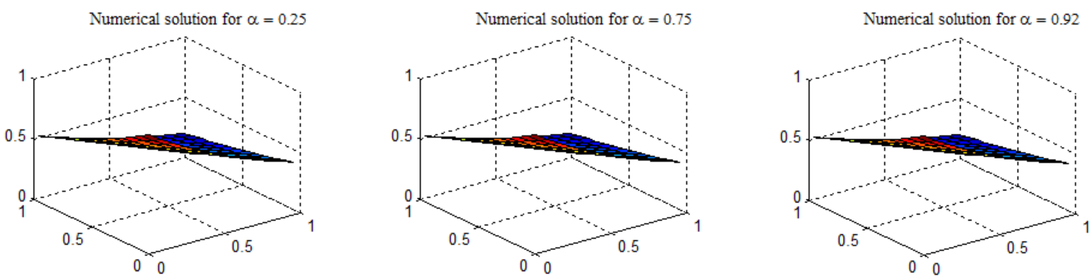

FIGURE 2. Graphs of the numerical solution for 2D time-fractional heat equation, for $r = 0.5, \Delta x = \Delta y = 0.08$ and $\alpha = 0.25, 0.75$ and 0.92.

TABLE 3. The numerical solutions ϕ of heat equation.

Values of α		$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.92$
x	y	Numerical solution ϕ	Numerical solution ϕ	Numerical solution ϕ
0.08	0.72	0.5984	0.5971	0.5969
0.96	0.32	0.3558	0.3550	0.3548
0.48	0.32	0.5339	0.5386	0.5384
0.88	0.64	0.3115	0.3108	0.3107
0.88	0.88	0.268	0.2674	0.2673
0.96	0.96	0.2377	0.2372	0.2371

In same way, we give the graphs of the exact and numerical solutions in Figure 3 for the system (2.3).

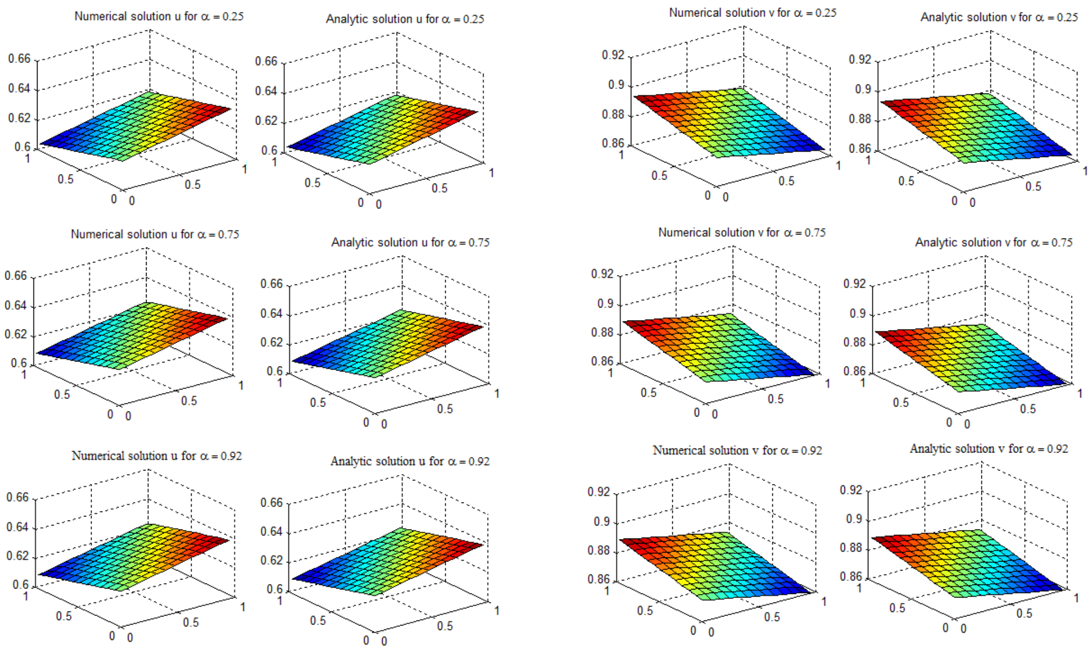

FIGURE 3. Graphs of exact and numerical solution for 2D time-fractional Burgers' equations, for $r = 0.5, \Delta x = \Delta y = 0.08$, and $\alpha = 0.25,\ 0.75$ and 0.92.

It is clear from the graphs that exact and approximate solutions are similar and compatible with each other. Tables 4 and 5 give the comparison of numerical and exact results for varying $\alpha = 0.75$ and 0.92. It is clear that the approximate solutions are accurate.

Table 4. Comparison between of the exact and numerical solutions u of the system (2.3).

Values of α		$\alpha = 0.75$		$\alpha = 0.92$	
x	y	Numerical solution	Exact solution	Numerical solution	Exact solution
0.08	0.72	0.6153	0.6149	0.6153	0.615
0.96	0.32	0.6351	0.6348	0.6352	0.6349
0.48	0.32	0.6278	0.6273	0.6278	0.6274
0.88	0.64	0.629	0.6286	0.629	0.6287
0.88	0.88	0.625	0.6248	0.6252	0.6249
0.96	0.96	0.6246	0.6248	0.6247	0.6249

Table 5. Comparison between of the exact and numerical solutions v of the system (2.3).

Values of α		$\alpha = 0.75$		$\alpha = 0.92$	
x	y	Numerical solution	Exact solution	Numerical solution	Exact solution
0.08	0.72	0.8857	0.8851	0.8857	0.885
0.96	0.32	0.8649	0.8652	0.8648	0.8651
0.48	0.32	0.8733	0.8727	0.8733	0.8726
0.88	0.64	0.8718	0.8714	0.8719	0.8713
0.88	0.88	0.8755	0.8752	0.8757	0.8751
0.96	0.96	0.8754	0.8752	0.8753	0.8751

6. Conclusion

In this study, we considered a coupling system of Burgers' equations with fractional conformable derivative in which involves nonlinearity and dissipation, it is relatively simple in contract with the Navier-Stokes system. It makes suitable model equations to test different numerical algorithms. For this purpose, we have used the Cole-Hopf transformation which shows its efficiency to deal with this class of fractional nonlinear problems. This approach is simple and effective and permits the comparison the obtained results with exact solution of the problem. In the future, we intend in first time to study some concrete examples that illustrate if the conformable derivative is capable or incapable of giving the fractional derivative obtainable from Riemann-Liouville or Caputo derivatives. In a second time, we want to apply such approach to other complex problems such as time-space diffusion equation of the type $\partial^\alpha u/\partial t^\alpha = -k(-\Delta)^\beta u$, where the α, β are changed into $\alpha(x,t), \beta(x,t)$.

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