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SOME PROPERTIES OF NEW HYPERGEOMETRIC FUNCTIONS IN FOUR VARIABLES

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Abstract. In this paper, we introduce ten new quadruple hypergeometric series. We also obtain their various properties such that integral representations, fractional derivatives, N-fractional connections, operational relations and generating functions.

1. Introduction

In recent years, several interesting and useful properties of certain multiple hypergeometric functions have been investigated by many authors (see, e.g., [1, 3–9, 11, 12, 14, 15, 17, 21, 22, 25, 26]). In a sequel of such type of works mentioned above in this paper, we introduce ten new hypergeometric series of four variables as below

$$\begin{aligned} &X_{70}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u\right) \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &X_{71}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, y, z, u\right) \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \end{aligned} \tag{1.2}$$

$$X_{72}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u\right)$$

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$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.3}$$

$$X_{73}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.4}$$

$$X_{74}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.5}$$

$$X_{75}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.6}$$

$$X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.7}$$

$$X_{77}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.8}$$

$$X_{78}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.9}$$

$$X_{79}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \tag{1.10}$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1),$$

for $m \geq 1$, $(a)_0 = 1$, Γ being the well-known Gamma function.

The present paper aims at introducing and investigating certain properties of hypergeometric series $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$. The structure of this paper is as follows. In Section 2, integral representations of Laplace-type are obtained. In Section 3, we establish some fractional derivatives for our series. Section 4 presents certain connections by means of N-fractional operator. Section 5 deals with the derivation of operational relations between the quadruple functions $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$ and triple hypergeometric functions. The generating functions are given in the last section of this paper.

2. Integral Representations of Laplace-Type

In this section, we present certain integrals of Laplace-type involving the quadruple series $X_i^{(4)}$, $i = 70, 71, \ldots, 79$. For our purpose, we begin by recalling the following confluent hypergeometric functions [23]:

$$ {}_0F_1(-; c; x) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \cdot \frac{x^m}{m!}, \tag{2.1} $$

$$ {}_1F_1(a; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \cdot \frac{x^m}{m!}, \tag{2.2} $$

$$ \Phi_3(a; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}, \tag{2.3} $$

$$ \mathsf{H}_6(a; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}, \tag{2.4} $$

$$ \mathsf{H}_7(a; b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_m (c)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}. \tag{2.5} $$

Now, if we consider the definitions of the confluent hypergeometric functions ${}_0F_1, {}_1F_1, \Phi_3, \mathsf{H}_6$ and H_7, we can derive the following integral representations:

$$ \begin{aligned} & X_{70}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u) \\ = & \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathsf{H}_7(a_1; c_1, c_2; x, sy)\, {}_0F_1\left(-; c_3; s^2 z\right) {}_1F_1(a_3; c_4; su)\, ds, \\ & \operatorname{Re}(a_2) > 0, \end{aligned} \tag{2.6} $$

$$ \begin{aligned} & X_{71}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, y, z, u) \\ = & \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathsf{H}_6(a_1; c_1; x, sy)\, {}_0F_1\left(-; c_2; s^2 z\right) {}_1F_1(a_3; c_3; su)\, ds, \\ & \operatorname{Re}(a_2) > 0, \end{aligned} \tag{2.7} $$

$$ \begin{aligned} & X_{72}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u) \\ = & \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1}\, t^{a_2-1} {}_0F_1\left(-; c_1; s^2 x + t^2 z\right) \\ & \times {}_0F_1(-; c_2; sty)\, {}_1F_1(a_3; c_3; tu)\, ds dt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \end{aligned} \tag{2.8} $$

$$ \begin{aligned} & X_{73}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, z, u) \\ = & \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1}\, t^{a_2-1} {}_0F_1\left(-; c_1; sty + t^2 z\right) \\ & \times {}_0F_1\left(-; c_2; s^2 x\right) {}_1F_1(a_3; c_3; tu)\, ds dt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \end{aligned} \tag{2.9} $$

$$\begin{aligned}
&X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1}\ t^{a_2-1}{}_0F_1\left(-; c_1; s^2x+sty+t^2z\right)\\
&\times {}_1F_1\left(a_3; c_2; tu\right) dsdt, \quad \operatorname{Re}(a_1)>0, \operatorname{Re}(a_2)>0, \qquad (2.10)
\end{aligned}$$

$$\begin{aligned}
&X_{75}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_2)}\int_0^\infty e^{-s} s^{a_2-1}\mathsf{H}_7\left(a_1; c_1, c_2; x, sy\right){}_1F_1\left(a_3; c_3; sz\right){}_1F_1\left(a_4; c_4; su\right) ds,\\
&\operatorname{Re}(a_2)>0, \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
&X_{76}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_2)}\int_0^\infty e^{-s} s^{a_2-1}\mathsf{H}_6\left(a_1; c_1; x, sy\right){}_1F_1\left(a_3; c_2; sz\right){}_1F_1\left(a_4; c_3; su\right) ds,\\
&\operatorname{Re}(a_2)>0, \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
&X_{77}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1}\ t^{a_2-1}\Phi_3\left(a_3; c_1; tz, s^2x\right){}_0F_1\left(-; c_2; sty\right)\\
&\times {}_1F_1\left(a_4; c_3; tu\right) dsdt, \quad \operatorname{Re}(a_1)>0, \operatorname{Re}(a_2)>0, \qquad (2.13)
\end{aligned}$$

$$\begin{aligned}
&X_{78}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)} s^{a_1-1}\ t^{a_2-1}\Phi_3\left(a_3; c_1; tz, sty\right){}_0F_1\left(-; c_2; s^2x\right)\\
&\times {}_1F_1\left(a_4; c_3; tu\right) dsdt, \quad \operatorname{Re}(a_1)>0, \operatorname{Re}(a_2)>0, \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
&X_{79}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; x, y, z, u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)} s^{a_1-1}\ t^{a_2-1}\ v^{a_3-1}\\
&\times {}_0F_1\left(-; c_1; s^2x+sty+tvz\right){}_1F_1\left(a_4; c_2; tu\right) dsdtdv,\\
&\operatorname{Re}(a_1)>0, \operatorname{Re}(a_2)>0, \operatorname{Re}(a_3)>0. \qquad (2.15)
\end{aligned}$$

Proof. To establish (2.6), denote by $\mathcal{I}$ the right side of the relation (2.6). Then, by substituting the expression of the confluent hypergeometric functions (2.1), (2.2) and (2.5) into the right hand side of (2.6), we have

$$\mathcal{I}=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_3)_q}{(c_1)_m(c_1)_n(c_1)_p(c_1)_q\Gamma(a_2)}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\cdot\frac{u^q}{q!}\int_0^\infty e^{-s}s^{a_2+2p+n+q-1}ds,$$

by using the known equality (see [23])

$$\Gamma(a)=\int_0^\infty e^{-s}\ s^{a-1}ds, \quad \operatorname{Re}(a)>0,$$

we get the result after some simplifications. Similarly, one can proof the relations (2.6) to (2.15). □

3. Fractional Derivatives

The fractional derivative operator D_w^k that was introduced by Miller and Ross [16] is given as

$$D_w^k w^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} w^{a-k}, \quad \mathrm{Re}\,(a) > -1. \tag{3.1}$$

Now, by using the above operator, we aim in this section at establishing the following fractional derivative formulae:

$$\begin{aligned} &D_w^{a_1-c}\left[w^{a_1-1} X_{70}^{(4)}\left(c, c, a_2, a_2, c, a_2, a_2, a_3; c_1, c_2, c_3, c_4; w^2x, wy, z, u\right)\right] \\ &= \frac{\Gamma(a_1)}{\Gamma(c)} w^{c-1} X_{70}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; w^2x, wy, z, u\right), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &D_w^{a_2-c}\left[w^{a_2-1} X_{71}^{(4)}\left(a_1, a_1, c, c, a_1, c, c, a_3; c_1, c_1, c_2, c_3; x, wy, w^2z, wu\right)\right] \\ &= \frac{\Gamma(a_2)}{\Gamma(c)} w^{c-1} X_{71}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, wy, w^2z, wu\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'}\left[w_1^{a_1-1} w_2^{a_2-1} X_{72}^{(4)}\left(c, c, c', c', c, c', c', a_3; c_1, c_2, c_1, c_3; w_1^2x, w_1w_2y,\right.\right. \\ &\left.\left. w_2^2z, w_2u\right)\right] \\ &= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{72}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; w_1^2x, w_1w_2y,\right. \\ &\left. w_2^2z, w_2u\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_3-c'}\left[w_1^{a_1-1} w_2^{a_3-1} X_{73}^{(4)}\left(c, c, a_2, a_2, c, a_2, a_2, c'; c_2, c_1, c_1, c_3; w_1^2x, w_1y, z, w_2u\right)\right] \\ &= \frac{\Gamma(a_1)\Gamma(a_3)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{73}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; w_1^2x, w_1y, z, w_2u\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} D_{w_3}^{a_3-c''}\left[w_1^{a_1-1} w_2^{a_2-1} w_3^{a_3-1} X_{74}^{(4)}\left(c, c, c', c', c, c', c', c''; c_1, c_1, c_1, c_2; w_1^2x,\right.\right. \\ &\left.\left. w_1w_2y, w_2^2z, w_2w_3u\right)\right] \\ &= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')} w_1^{c-1} w_2^{c'-1} w_3^{c''-1} \\ &\times X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; w_1^2x, w_1w_2y, w_2^2z, w_2w_3u\right), \end{aligned} \tag{3.6}$$

$$D_{w_1}^{a_3-c} D_{w_2}^{a_4-c'}\left[w_1^{a_3-1} w_2^{a_4-1} X_{75}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, c, c'; c_1, c_2, c_3, c_4; x, y, w_1z, w_2u\right)\right]$$

(3.7)
$$=\frac{\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;\; c_2,c_1,c_1,c_3;x,y,w_1z,w_2u\right),$$

$$D_w^{a_2-c}\left[w^{a_2-1}X_{76}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,c,c';c_1,c_1,c_2,c_3;x,wy,wz,wu\right)\right]$$

(3.8)
$$=\frac{\Gamma(a_2)}{\Gamma(c)}w^{c-1}X_{76}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_2,a_3;c_2,c_1,c_1,c_3;x,wy,wz,wu\right),$$

(3.9)
$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}D_{w_3}^{a_3-c''}D_{w_4}^{a_4-c'''}\left[w_1^{a_1-1}w_2^{a_2-1}w_3^{a_3-1}w_4^{a_4-1}X_{77}^{(4)}\left(c,c,c',c',c,c',c'',c''';\right.\right.$$
$$\left.\left.c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,w_2w_3z,w_2w_4u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma(c''')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}w_4^{c'''-1}$$

(3.10)
$$\times X_{77}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_3;w_1^2x,w_1w_2y,w_2w_3z,w_2w_4u\right),$$

$$D_{w_1}^{a_2-c}D_{w_2}^{a_3-c'}D_{w_3}^{a_4-c''}\left[w_1^{a_2-1}w_2^{a_3-1}w_3^{a_4-1}X_{78}^{(4)}\left(a_1,a_1,c,c,a_1,c,c',c'';c_2,c_1,c_1,c_3;\right.\right.$$
$$\left.\left.x,w_1y,w_1w_2z,w_1w_3u\right)\right]$$
$$=\frac{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')}w_1^{c-1}w_2^{c'-1}w_3^{c''-1}$$

(3.11)
$$\times X_{78}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_2,c_1,c_1,c_3;x,w_1y,w_1w_2z,w_1w_3u\right),$$

$$D_{w_1}^{a_1-c}D_{w_2}^{a_2-c'}\left[w_1^{a_1-1}w_2^{a_2-1}X_{79}^{(4)}\left(c,c,c',c',c,c',a_3,a_4;c_1,c_1,c_1,c_2;w_1^2x,w_1w_2y,\right.\right.$$
$$\left.\left.w_2z,w_2u\right)\right]$$
$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')}w_1^{c-1}w_2^{c'-1}$$

(3.12)
$$\times X_{79}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_1,c_1,c_2;w_1^2x,w_1w_2y,w_2z,w_2u\right).$$

Proof. We have

$$D_w^{a_1-c}\left[w^{a_1-1}X_{70}^{(4)}\left(c,c,a_2,a_2,c,a_2,a_2,a_3;c_1,c_2,c_3,c_4;w^2x,wy,z,u\right)\right]$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(c)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\cdot\frac{u^q}{q!}D_w^{a_1-c}w^{a_1+2m+n-1}.$$

Now, with the help of (3.1) and Definition 1.1, the proof of the first fractional derivative formula is completed. The proofs of the assertions (3.3) to (3.12) run parallel to that of the assertion (3.2) then we skip the details. □

4. N-Fractional Connections

First, by recalling the N-fractional operator due to Bin-Saad [4]:

$$\mathcal{M}_w^{a,c,b} = \left[w^{a-1}(1-w)^{-b}\right]_{a-c} = \frac{\Gamma(a-c)}{2\pi i}\int_C \frac{\eta^{a-1}(1-\eta)^{-b}}{(\eta-z)^{a-c}}d\eta, \tag{4.1}$$

where $a, b, c \in \mathbb{C}$ and $(a-c) \notin \mathbb{Z}$, we aim in this section to investigate the following relationships:

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{12}\left(a_1, b; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&AX_{70}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_3, c; x, y, z, u\right),\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{10}\left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&AX_{71}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_2, c; x, y, z, u\right),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{11}\left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&AX_{72}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{10}\left(b, a_1; c_1, c_2; \frac{x}{(1-u)^2}, \frac{y}{(1-u)}, z\right)\\ =&AX_{73}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_2, c_1, c_1, c; z, y, x, u\right),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{9}\left(a_1, b; c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2}\right)\\ =&AX_{74}^{(4)}\left(a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{17}\left(a_1, b, a_2; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&AX_{75}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_3, c; x, y, z, u\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{14}\left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&AX_{76}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_1, c_2, c; x, y, z, u\right),\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\mathcal{M}_u^{a,c,b} X_{16}\left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)\\ =&AX_{77}^{(4)}\left(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_1, c; x, y, z, u\right),\end{aligned} \tag{4.9}$$

$$\mathcal{M}_u^{a,c,b} X_{15}\left(a_1, b, a_2; c_2, c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)}\right)$$

$$=AX^{(4)}_{78}\left(a_1,a_1,b,b,a_1,b,a_2,a;c_2,c_1,c_1,c;x,y,z,u\right), \tag{4.10}$$

$$\mathcal{M}^{a,c,b}_{u}X_{13}\left(a_1,b,a_2;c_1;x,\frac{y}{(1-u)},\frac{z}{(1-u)}\right)$$
$$=AX^{(4)}_{79}\left(a_1,a_1,b,b,a_1,b,a_2,a;c_1,c_1,c_1,c;x,y,z,u\right), \tag{4.11}$$

where $A=e^{-\pi i(a-c)}\frac{\Gamma(1-c)}{\Gamma(1-a)}u^{c-1}$ and $X_9,X_{10},\ldots,X_{17}$ are Exton's hypergeometric functions of three variables [10] defined by

$$X_9\left(a_1,a_2;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c)_{m+n+p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.12}$$

$$X_{10}\left(a_1,a_2;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_{m+n}(c_2)_{p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.13}$$

$$X_{11}\left(a_1,a_2;c_1,c_2;x,y,\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_{m+p}(c_2)_{n}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.14}$$

$$X_{12}\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+2p}}{(c_1)_{m}(c_2)_{n}(c_3)_{p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.15}$$

$$X_{13}\left(a_1,a_2,a_3;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p}}{(c)_{m+n+p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.16}$$

$$X_{14}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p}}{(c_1)_{m+n}(c_2)_{p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.17}$$

$$X_{15}\left(a_1,a_2,a_3;c_2,c_1;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p}}{(c_1)_{n+p}(c_2)_{m}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.18}$$

$$X_{16}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p}}{(c_1)_{m+p}(c_2)_{n}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}, \tag{4.19}$$

$$X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_{p}}{(c_1)_{m}(c_2)_{n}(c_3)_{p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}. \tag{4.20}$$

Proof. To prove (4.2), from the equality (4.15), we can write

$$\mathcal{M}^{a,c,b}_{u}X_{12}\left(a_1,b;c_1,c_2,c_3;x,\frac{y}{(1-u)},\frac{z}{(1-u)^2}\right)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(b)_{n+2p}}{(c_1)_{m}(c_2)_{n}(c_3)_{p}}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!}\mathcal{M}^{a,c,b}_{u}(1-u)^{-(n+2p)}.$$

By applying the formula (4.1) and in view of the relation (1.1) one can get the result with direct calculations. The proofs of the remaining relations run in the same way. □

5. Operational Relations

Here, in this section, we shall discuss some operational relations by means of the following operational formulas (see [3, 20]):

$$D_\alpha^k \alpha^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} \alpha^{a-k}, \tag{5.1}$$

$$D_\alpha^{-k} \alpha^a = \frac{\Gamma(a+1)}{\Gamma(a+k+1)} \alpha^{a+k}, \tag{5.2}$$

$k \in \mathbb{N} \cup \{0\}, a \in \mathbb{C} - \{-1, -2, \ldots\}$, where D_α denotes the derivative operator and D_α^{-1} denotes the inverse of the derivative.

In the following, certain operational connections among the hypergeometric series of three and four variables as:

(5.3)
$$\left[1 - \left(D_\alpha^2 \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha^2\right) u\right]^{-a} X_8 \left(a_1, a_2, a_3; c_1, c_2, c_3; x, \alpha y, z\right) \left(\alpha^{a_2-1} \beta^{c_4-1} \gamma^{a-1}\right)$$
$$= \alpha^{a_2-1} \beta^{c_4-1} \gamma^{a-1} X_{70}^{(4)} \left(a_2, a_2, a_1, a_1, a_2, a_1, a_1, a_3; c_4, c_2, c_1, c_3; u, \alpha y, x, z\right),$$
(5.4)
$$\left[1 - \left(D_\alpha^2 \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha^2\right) u\right]^{-a} X_{14} \left(a_1, a_2, a_3; c_1, c_2; x, \alpha y, \alpha z\right) \left(\alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1}\right)$$
$$= \alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1} X_{71}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_3; c_1, c_1, c_3, c_2; x, \alpha y, u, \alpha z\right),$$
(5.5)
$$\left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha_1 \alpha_2\right) u\right]^{-a} X_{20} \left(a_1, a_2, \frac{a_3}{2}, \frac{a_3+1}{2}; c_1, c_2; \alpha_1^2 x, \alpha_1 y, 4\alpha_2^2 z\right) \left(\alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_3-1} \gamma^{a-1}\right)$$
$$= \alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_3-1} \gamma^{a-1} X_{72}^{(4)} \left(a_3, a_3, a_1, a_1, a_3, a_1, a_1, a_2; c_1, c_3, c_1, c_2; \alpha_2^2 z, u, \alpha_1^2 x, \alpha_1 y\right),$$
(5.6)
$$\left[1 - \left(D_\alpha^2 \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha^2\right) u\right]^{-a} X_6 \left(a_1, a_2, a_3; c_1, c_2; x, \alpha y, z\right) \left(\alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1}\right)$$
$$= \alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1} X_{73}^{(4)} \left(a_2, a_2, a_1, a_1, a_2, a_1, a_1, a_3;\ c_3, c_1, c_1, c_2; u, \alpha y, x, z\right),$$
(5.7)
$$\left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha_1 \alpha_2\right) u\right]^{-a} X_{20} \left(a_1, a_2, \frac{a_3}{2}, \frac{a_3+1}{2}; c_1, c_2; \alpha_1^2 \beta x, \alpha_1 y, 4\alpha_2^2 \beta z\right) \left(\alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_1-1} \gamma^{a-1}\right)$$
$$= \alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_1-1} \gamma^{a-1}$$
$$\times X_{74}^{(4)} \left(a_3, a_3, a_1, a_1, a_3, a_1, a_1, a_2; c_1, c_1, c_1, c_2; \alpha_2^2 \beta z, u, \alpha_1^2 \beta x, \alpha_1 y\right),$$

(5.8)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,\alpha_1 y,\alpha_1 z\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_4-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_4-1}\gamma^{a-1}X_{75}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,\alpha_1 y,\alpha_1 z,u\right),$$
$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^2\right)u\right]^{-a}F(3)_A\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;\alpha\beta x,y,z\right)$$
$$\times\left(\alpha^{a_2-1}\beta^{c_1-1}\gamma^{a-1}\right)$$
$$=\alpha^{a_2-1}\beta^{c_1-1}\gamma^{a-1}X_{76}^{(4)}\left(a_2,a_2,a_1,a_1,a_2,a_1,a_3,a_4;\ c_1,c_1,c_2,c_3;u,\alpha\beta x,y,z\right),$$

(5.9)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{16}\left(a_1,a_2,a_3;c_1,c_2;x,\alpha_1 y,\alpha_1 z\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}X_{77}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,\alpha_1 y,\alpha_1 z,u\right),$$

(5.10)

$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^2\right)u\right]^{-a}F_G\left(a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_2,c_2;x,\alpha y,z\right)$$
$$\times\left(\alpha^{a_3-1}\beta^{c_3-1}\gamma^{a-1}\right)$$
$$=\alpha^{a_3-1}\beta^{c_3-1}\gamma^{a-1}X_{78}^{(4)}\left(a_3,a_3,a_1,a_1,a_3,a_1,a_4,a_2;c_3,c_2,\ c_2,c_1;u,\alpha y,z,x\right),$$

(5.11)

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}$$
$$\times F_N\left(a_1,\frac{a_2}{2},a_3,a_4,\frac{a_2+1}{2},a_4;c_1,c_2,c_2;\alpha_2 x,4\alpha_1^2\beta y,\alpha_2\beta z\right)\left(\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_2-1}\alpha_2^{a_4-1}\beta^{c_2-1}\gamma^{a-1}$$
$$\times X_{79}^{(4)}\left(a_2,a_2,a_4,a_4,a_2,a_4,a_3,a_1;c_2,c_2,c_2,c_1;\alpha_1^2\beta y,u,\alpha_2\beta z,\alpha_2 x\right),$$

where X_6, X_8 and X_{20} are the Exton's triple hypergeometric series defined by [10]

$$X_6\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},\tag{5.12}$$

(5.13)

$$X_8\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},$$

(5.14)

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n}\cdot\frac{x^m}{m!}\cdot\frac{y^n}{n!}\cdot\frac{z^p}{p!},$$

and $F_A^{(3)}$, F_G, F_N denote the Lauricella's series of three variables (see [13]).

Proof. To solve equation (5.3), first we assume the left hand side of (5.3) by the notation $\mathfrak{I}$, then expressing the Exton's function X_8 as a series in the left hand side of (5.3) and using the binomial theorem, it follows that:

$$\mathfrak{I} = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p(a)_q}{(c_1)_m(c_2)_n(c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}$$
$$\times \beta^{-q}\gamma^{-q} D_\alpha^{2q} D_\beta^{-q} D_\gamma^{-q} \left(\alpha^{a_2+n+2q-1}\beta^{c_4-1}\gamma^{a-1}\right).$$

Now, we use the above formulas in (5.1) and (5.2), then in view of Definition 1.1, we arrive at the desired result (5.3). In a similar manner, one can prove the relations (5.4) to (5.11). □

6. Generating Functions

In this section, we will consider some generating functions for our quadruple series. Because the proofs of the following relations are similar to the proofs of results in [2, 18, 19, 23, 24], we omit these proofs.

The generating relations of series $X_{70}^{(4)}, X_{72}^{(4)}, \ldots, X_{79}^{(4)}$ given as below

(6.1)
$$\sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} X_{70}^{(4)}\left(a_1+k, a_1+k, a_2, a_2, a_1+k, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u\right) t^k$$
$$= (1-t)^{-a_1} X_{70}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; \frac{x}{(1-t)^2}, \frac{y}{(1-t)}, z, u\right),$$

(6.2)
$$\sum_{k=0}^{\infty} \frac{(a_2)_k}{k!} X_{71}^{(4)}\left(a_1, a_1, a_2+k, a_2+k, a_1, a_2+k, a_2+k, a_3; c_1, c_1, c_2, c_3; x, y, z, u\right) t^k$$
$$= (1-t)^{-a_2} X_{71}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, \frac{y}{(1-t)}, \frac{z}{(1-t)^2}, \frac{u}{(1-t)}\right),$$

(6.3)
$$\sum_{k_1,k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_2)_{k_2}}{k_1!k_2!} X_{72}^{(4)}\left(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2, a_2+k_2,\right.$$
$$\left. a_3; c_1, c_2, c_1, c_3; x, y, z, u\right) t_1^{k_1} t_2^{k_2}$$
$$= (1-t_1)^{-a_1}(1-t_2)^{-a_2} X_{72}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; \frac{x}{(1-t_1)^2},\right.$$
$$\left.\frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)^2}, \frac{u}{(1-t_2)}\right),$$

(6.4)
$$\sum_{k_1,k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_3)_{k_2}}{k_1!k_2!} X_{73}^{(4)}(a_1+k_1, a_1+k_1, a_2, a_2, a_1+k_1, a_2, a_2, a_3+k_2;$$

$$c_2, c_1, c_1, c_3; x, y, z, u)t_1^{k_1} t_2^{k_2}$$

$$=(1-t_1)^{-a_1}(1-t_2)^{-a_3} X_{73}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; \frac{x}{(1-t_1)^2},$$

$$\frac{y}{(1-t_1)}, z, \frac{u}{(1-t_2)}\Bigg),$$

(6.5)

$$\sum_{k=0}^{\infty} \frac{(a_3)_k}{k!} X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3+k; c_1, c_1, c_1, c_2; x, y, z, u\right) t^k$$

$$=(1-t)^{-a_3} X_{74}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, \frac{u}{(1-t)}\right),$$

(6.6)

$$\sum_{k=0}^{\infty} \frac{(a_2)_k}{k!} X_{75}^{(4)}\left(a_1, a_1, a_2+k, a_2+k, a_1, a_2+k, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u\right) t^k$$

$$=(1-t)^{-a_2} X_{75}^{(4)}\left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, \frac{y}{(1-t)}, \frac{z}{(1-t)}, \frac{u}{(1-t)}\right),$$

(6.7)

$$\sum_{k_1,k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_2)_{k_2}}{k_1! k_2!} X_{76}^{(4)}(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2, a_3, a_4;$$

$$c_1, c_1, c_2, c_3; x, y, z, u)t_1^{k_1} t_2^{k_2}$$

$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2} X_{76}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; \frac{x}{(1-t_1)^2},$$

$$\frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)}, \frac{u}{(1-t_2)}\Bigg),$$

(6.8)

$$\sum_{k_1,k_2=0}^{\infty} \frac{(a_3)_{k_1}(a_4)_{k_2}}{k_1! k_2!} X_{77}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3+k_1, a_4+k_2; c_1, c_2, c_1, c_3;$$

$$x, y, z, u\Bigg) t_1^{k_1} t_2^{k_2}$$

$$=(1-t_1)^{-a_3}(1-t_2)^{-a_4} X_{77}^{(4)}\Bigg(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y,$$

$$\frac{z}{(1-t_1)}, \frac{u}{(1-t_2)}\Bigg),$$

(6.9)

$$\sum_{k_1,k_2,k_3=0}^{\infty} \frac{(a_1)_{k_1}(a_2)_{k_2}(a_3)_{k_3}}{k_1! k_2! k_3!} X_{78}^{(4)}(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2,$$

$$a_3+k_3,a_4;c_2,c_1,c_1,c_3;x,y,z,u)t_1^{k_1}t_2^{k_2}t_3^{k_3}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2}(1-t_3)^{-a_3}X_{78}^{(4)}\Big(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;c_2,c_1,c_1,c_3;$$
$$\frac{x}{(1-t_1)^2},\frac{y}{(1-t_1)(1-t_2)},\frac{z}{(1-t_2)(1-t_3)},\frac{u}{(1-t_2)}\Big),$$

(6.10)
$$\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\frac{(a_1)_{k_1}(a_2)_{k_2}(a_3)_{k_3}(a_4)_{k_4}}{k_1!k_2!k_3!k_4!}X_{79}^{(4)}(a_1+k_1,a_1+k_1,a_2+k_2,a_2+k_2,a_1+k_1,$$
$$a_2+k_2,a_3+k_3,a_4+k_4;c_1,c_1,c_1,c_2;x,y,z,u)t_1^{k_1}t_2^{k_2}t_3^{k_3}t_4^{k_4}$$
$$=(1-t_1)^{-a_1}(1-t_2)^{-a_2}(1-t_3)^{-a_3}(1-t_4)^{-a_4}X_{79}^{(4)}\,(a_1,a_1,a_2,a_2,a_1,a_2,a_3,a_4;$$
$$c_1,c_1,c_1,c_2;\frac{x}{(1-t_1)^2},\frac{y}{(1-t_1)(1-t_2)},\frac{z}{(1-t_2)(1-t_3)},\frac{u}{(1-t_2)(1-t_4)}\Big).$$

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