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INTEGRAL TRANSFORMS AND EXTENDED HERMITE-APOSTOL TYPE FROBENIUS-GENOCCHI POLYNOMIALS

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Abstract. The schemata for applications of the integral transforms of mathematical physics to recurrence relations, differential, integral, integro-differential equations and in the theory of special functions has been developed. The article aims to introduce and present operational representations for a new class of extended Hermite-Apostol type Frobenius-Genocchi polynomials via integral transforms. The recurrence relations and some identities involving these polynomials are established. The article concludes by establishing a determinant form and quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials and for their extended forms.

1. Introduction and preliminaries

The convolution of two or more polynomials in order to introduce the new multi-variable generalized polynomials is a topic of research and is useful from the point of view of applications. These polynomials are important as they possess significant properties including the recurrence and explicit relations, functional and differential equations, summation formulae, symmetric and convolution identities, determinant forms *et cetera.* The usefulness and potential for applications of various properties of multi-variable hybrid special polynomials in certain problems of number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory and other fields of pure and applied mathematics has given motivation for introducing many new classes of hybrid polynomials.

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The properties and applications of the hybrid polynomials lie within the parent polynomials. The applications of hybrid Legendre polynomials lie in problems dealing with either gravitational potentials or electrostatic potentials. The hybrid polynomials involving Hermite polynomials occur in quantum mechanical and optical beam transport problems and in probability theory. The hybrid polynomials related with truncated-exponential polynomials appear in the theory of flattened beams, which plays an importance in optics and in super-Gaussian optical resonators and hybrid polynomials associated with Laguerre polynomials occur in physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings.

Certain new classes of hybrid special polynomials associated with the Appell sequences were introduced and studied by Khan et al. [13, 14]. The problems arising in different areas of science and engineering are usually expressed in terms of differential equations which in most of the cases have special functions as their solutions. The differential equations satisfied by the hybrid special polynomials may be used to express the problems arising in new and emerging areas of sciences.

Various forms of the Apostol type polynomials are the generalizations of the Appell family [2]. The Appell polynomial sequences appear in different applications in pure and applied mathematics. These sequences arise in theoretical physics, chemistry [7, 23] and several branches of mathematics [18] such as the study of polynomial expansions of analytic functions, number theory and numerical analysis. The typical examples of Appell polynomial sequences are the Bernoulli, Euler and Genocchi polynomials. These polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis and can be defined by various methods depending on the applications.

Several interesting results related to Frobenius type polynomials and their hybrid forms were obtained by many authors, see [12, 17], which are important from applications point of view. The hybrid class of 3-variable Hermite-Apostol type Frobenius-Genocchi polynomials was introduced in [5] by considering the discrete convolution of the Apostol type Frobenius-Genocchi polynomials $\mathcal{H}_n(x;\lambda;u)$ [5] with the 3-variable Hermite polynomials $H_n(x,y,z)$ [10].

The Apostol type Frobenius-Genocchi polynomials and the 3-variable Hermite polynomials are defined by

$$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{xt}=\sum_{n=0}^{\infty}\mathcal{H}_n(x;\lambda;u)\frac{t^n}{n!},\quad u,\lambda\in\mathbb{C},\ u\neq 1 \tag{1.1}$$

which for $x=0$ gives the Apostol type Frobenius-Euler numbers $\mathcal{H}_n(u;\lambda)$ and

$$e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}, \tag{1.2}$$

which for $z=0$ reduce to the 2-variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ [3] and for $z=0,\ x=2x$ and $y=-1$ become the classical Hermite polynomials $H_n(x)$ [1], respectively.

The 3-variable Hermite-Apostol type Frobenius-Genocchi polynomials [5] are defined by means of the following generating function and series expansion:

$$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y,z;\lambda;u)\frac{t^n}{n!},\quad u,\lambda\in\mathbb{C},\ u\neq 1,\tag{1.3}$$

$$ {}_H\mathcal{H}_n(x,y,z;\lambda;u)=n!\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^r H_{k-3r}(x,y)}{(n-k)!r!(k-3r)!}.\tag{1.4}$$

Next, we present certain special cases of ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ in Table 1.

TABLE 1. Special cases of ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$

S.No.	**Cases**	**Name of polynomial**	**Generating function**
I	$\lambda=1$	Hermite Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y,z;u)\frac{t^n}{n!}$
	$u=-1$, $\lambda=1$	Hermite-Genocchi polynomials [4]	$\left(\frac{2t}{e^t+1}\right)e^{xt+yt^2+zt^3}=\sum_{n=0}^{\infty}{}_HG_n(x,y,z)\frac{t^n}{n!}$
II	$z=0$	2-variable Hermite-Apostol type Frobenius-Genocchi polynomials [5]	$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y;u;\lambda)\frac{t^n}{n!}$
	$z=0$, $\lambda=1$	2-variable Hermite-Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x,y;u)\frac{t^n}{n!}$
III	$x=2x$, $y=-1,\ z=0$	Hermite-Apostol type Frobenius-Genocchi polynomials [5]	$\left(\frac{(1-u)t}{\lambda e^t-u}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x;\lambda;u)\frac{t^n}{n!}$
	$x=2x,\ y=-1$ $z=0,\ \lambda=1$	Hermite-Frobenius-Genocchi polynomials [4]	$\left(\frac{(1-u)t}{e^t-u}\right)e^{2xt-t^2}=\sum_{n=0}^{\infty}{}_H\mathcal{H}_n(x;u)\frac{t^n}{n!}$

Fractional calculus is one of the most intensively developing areas of mathematical analysis. Its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. Integration to an arbitrary order named fractional calculus has a long history. The idea of non-integral order of integration is drawn back to the origin of differential calculus. The Newton's rival Leibnitz made some assertions on the meaning and possibility of fractional derivative of order 1/2 in the end of 17th century. However, a precise and rigorous research was first carried out by Liouville. Methods connected with the use of integral transforms have been successfully applied to the solution of differential and integral equations. Fractional operators have been attracting the attention of mathematicians and engineers from long time ago [19,24]. The use of integral transforms to deal with fractional derivatives was originated by Riemann and Liouville [19, 24]. The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives, see for example [6, 11, 15, 16].

The possibility of using integral transforms in a wider context was discussed by Dattoli et al. [11], where by using Euler's integral:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-at} t^{\nu-1} dt, \quad \min\{\mathrm{Re}(\nu), \mathrm{Re}(a)\} > 0, \tag{1.5}$$

it was shown that [11]:

$$\begin{aligned} \left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) =& \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \ e^{t\frac{\partial}{\partial x}} f(x) dt \\ =& \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \ f(x+t) dt, \end{aligned} \tag{1.6}$$

whereas for the cases involving second order derivatives, it was shown that

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \ e^{t\frac{\partial^2}{\partial x^2}} f(x) dt. \tag{1.7}$$

The fractional operators can be treated in an efficient way by combining the properties of exponential operators and suitable integral representations.

In this article, the extended Hermite-Apostol type Frobenius Genocchi polynomials are introduced using integral transforms. The recurrence relations and some identities involving these polynomials are also derived. Finally, the quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials and for their extended forms are obtained.

2. Extended Hermite-Apostol Type Frobenius-Genocchi Polynomials

In order to develop extended forms of the Hermite-Apostol type Frobenius-Genocchi polynomials via Euler's integral, we first establish the operational connection for the Hermite-Apostol type Frobenius-Genocchi polynomials.

From generating equation (1.3), we find that the Hermite-Apostol type Frobenius-Genocchi polynomials are the solutions of the following equations:

$$\begin{aligned} \frac{\partial}{\partial y} {}_H\mathcal{H}_n(x,y,z;\lambda;u) =& \frac{\partial^2}{\partial x^2} {}_H\mathcal{H}_n(x,y,z;\lambda;u), \\ \frac{\partial}{\partial z} {}_H\mathcal{H}_n(x,y,z;\lambda;u) =& \frac{\partial^3}{\partial x^3} {}_H\mathcal{H}_n(x,y,z;\lambda;u), \end{aligned}$$

under the following initial condition:

$$ {}_H\mathcal{H}_n(x,0,0;\lambda;u) = \mathcal{H}_n(x;\lambda;u), \tag{2.1} $$

where $\mathcal{H}_n(x;\lambda;u)$ are the Apostol type Frobenius-Genocchi polynomials [5].

Thus, in view of above equation, it follows that, for the Hermite-Apostol type Frobenius-Genocchi polynomials the following operational connection holds true:

$$ {}_H\mathcal{H}_n(x,y,z;\lambda;u) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{\mathcal{H}_n(x;\lambda;u)\}. \tag{2.2} $$

Further by making use of operational rule (2.2) and Euler's integral, we derive the operational relation for the polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$. For this we prove the following result.

Theorem 2.1. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$, the following operational connection holds true:*

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) = {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha). \tag{2.3} $$

Proof. Replacing a by $\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ in integral (1.5) and then operating the resultant equation on $\mathcal{H}_n(x;\lambda;u)$, it follows that

$$ \begin{aligned} \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) =& \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(yt\frac{\partial^2}{\partial x^2} + zt\frac{\partial^3}{\partial x^3}\right) \\ & \times \mathcal{H}_n(x;\lambda;u)dt, \end{aligned} \tag{2.4} $$

which in view of equation (2.2) gives

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;\lambda;u) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathcal{H}_n(x,yt,zt;\lambda;u)dt. \tag{2.5} $$

The transform on the r.h.s of equation (2.5) defines a new family of polynomials as the extended Hermite-Apostol type Frobenius-Genocchi polynomials, i.e.,

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_H\mathcal{H}_n(x,yt,zt;\lambda;u)dt. \tag{2.6} $$

Thus, in view of equations (2.5) and (2.6), assertion (2.3) follows. □

Remark 2.1. We know that for $\lambda = 1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ [5] reduce to the Hermite-Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;u)$ [4]. Therefore, taking $\lambda = 1$ in the both sides of equation (2.3), we find the following operational connection between extended Hermite-Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;u;\alpha)$ and the Frobenius-Genocchi polynomials $\mathcal{H}_n(x;u)$ [25]:

$$ \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu}\mathcal{H}_n(x;u) = {}_{\nu H}\mathcal{H}_n(x,y,z;u;\alpha). \tag{2.7} $$

Remark 2.2. For $\lambda = 1$ and $u = -1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Genocchi polynomials ${}_HG_n(x, y, z)$ [4]. Therefore, taking $\lambda = 1$ and $u = -1$ in both sides of equation (2.3), we find the following operational connection between the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$ and the Genocchi polynomials $G_n(x)$ [21]:

$$\left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} G_n(x) = {}_{\nu H}G_n(x, y, z; \alpha). \tag{2.8}$$

Next, we derive the generating function for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$ by proving the following result.

Theorem 2.2. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$, the following generating function holds true:*

$$\frac{(1-u)w\ \exp(xw)}{(\lambda e^w - u)\ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)\frac{w^n}{n!}. \tag{2.9}$$

Proof. Multiplying both sides of equation (2.6) by $\frac{w^n}{n!}$, then summing it over n and making use of equation (1.3) in the r.h.s. of the resultant equation, we find

$$\sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)\frac{w^n}{n!} = \frac{(1-u)w\ \exp(xw)}{(\lambda e^w - u)\ \Gamma(\nu)} \int_0^\infty e^{-\left(\alpha-(yw^2+zw^3)\right)t}\ t^{\nu-1}dt, \tag{2.10}$$

which in view of integral (1.5) yields assertion (2.9). □

Remark 2.3. We know that for $\lambda = 1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; u)$ [4]. Therefore, by taking $\lambda = 1$ in the both sides of equation (2.9), we find the following generating for the extended Hermite-Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; u; \alpha)$:

$$\frac{(1-u)w\ \exp(xw)}{(e^w - u)\ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}\mathcal{H}_n(x, y, z; u; \alpha)\frac{w^n}{n!}. \tag{2.11}$$

Remark 2.4. We know that for $\lambda = 1$ and $u = -1$, the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x, y, z; \lambda; u)$ [5] reduce to the Hermite-Genocchi polynomials ${}_HG_n(x, y, z)$ [4]. Therefore, by taking $\lambda = 1$ and $u = -1$ in both sides of equation (2.9), we find the following generating function for the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$:

$$\frac{2w \exp(xw)}{(e^w + 1)\ (\alpha - (yw^2 + zw^3))^\nu} = \sum_{n=0}^{\infty} {}_{\nu H}G_n(x, y, z; \alpha)\frac{w^n}{n!}. \tag{2.12}$$

Now, we derive the recurrence relations for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x, y, z; \lambda; u; \alpha)$ by taking into consideration its generating relation. A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

On differentiating generating function (2.9) with respect to x, y, z and α, we find the following recurrence relations for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$:

$$\begin{aligned}
&\frac{\partial}{\partial x}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = n\ {}_{\nu H}\mathcal{H}_{n-1}(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial y}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = \nu\ n(n-1)_{\nu+1 H}\mathcal{H}_{n-2}(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial z}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = \nu\ n(n-1)(n-2)_{\nu+1 H}\mathcal{H}_{n-3}(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial \alpha}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\nu_{\nu+1 H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha).
\end{aligned} \tag{2.13}$$

In view of the above relations, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial y}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\frac{\partial^3}{\partial x^2\partial\alpha}\ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha),\\
&\frac{\partial}{\partial z}\Big({}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)\Big) = -\frac{\partial^4}{\partial x^3\partial\alpha}\ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha).
\end{aligned}$$

Theorem 2.3. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials* ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$*, the following explicit series expansion holds true:*

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = n!\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^r H_{k-3r}(x,yt)(\nu)_r}{\alpha^{\nu+r}\ (n-k)!r!(k-3r)!}. \tag{2.14}$$

Proof. Using the series expansion (1.4) in the r.h.s of equation (2.6), we find

$$\begin{aligned}
{}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) =& \frac{\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\nu+r)}\int_0^\infty e^{-\alpha t}t^{\nu+r-1}n!\\
&\times\sum_{k=0}^{n}\sum_{r=0}^{[k/3]}\frac{\mathcal{H}_{n-k}(\lambda;u)z^r H_{k-3r}(x,yt)}{(n-k)!r!(k-3r)!}dt,
\end{aligned} \tag{2.15}$$

which in view of integral (1.5) yields assertion (2.14). □

In the next section, we establish the determinant form and quasi-monomial properties for the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ and derive corresponding results for their extended forms.

3. Determinant Approach and Quasi-Monomial Properties

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of special functions. The use of operational techniques in the study of hybrid special functions provide explicit solutions for the families of partial differential equations including those of Heat and d'Alembert type and to frame the hybrid special polynomials within the context of linear algebraic approach. We use the operational rules to establish the determinant forms for the special cases of the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$.

We recall the following operational definition and the generating function of the extended 3-variable Hermite polynomials from [11]:

$$ {}_{\nu}H_n(x,y,z;\alpha) = \left(\alpha - \left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\right)^{-\nu} x^n, \tag{3.1}$$

$$ (\alpha - (yt^2+zt^3))^{-\nu} e^{xt} = \sum_{n=0}^{\infty} {}_{\nu}H_n(x,y,z;\alpha)\frac{t^n}{n!}. \tag{3.2}$$

Theorem 3.1. *For the extended Hermite-Apostol type Frobenius-Genocchi polynomials* ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$*, the following explicit summation formula in terms of the generalized Hermite polynomials* ${}_{\nu}H_n(x,y,z;\alpha)$ *and Apostol type Frobenius-Genocchi polynomials* $\mathcal{H}_n(w;\lambda;u)$ *holds true:*

$$ {}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha) = \sum_{l=0}^{n}\sum_{p=0}^{n}\binom{n}{l}\binom{n-l}{p}(-w)^l\mathcal{H}_p(w;\lambda;u)\,{}_{\nu}H_{n-l-p}(x,y,z;\alpha). \tag{3.3}$$

Proof. We consider the product of generating equations (3.2) and (1.1) such that

$$ \left(\frac{(1-u)t}{\lambda e^t - u}\right) e^{wt}(\alpha - (yt^2+zt^3))^{-\nu}e^{xt} = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\mathcal{H}_p(w;\lambda;u)\,{}_{\nu}H_n(x,y,z;\alpha)\frac{t^{n+p}}{n!\,p!}, \tag{3.4}$$

which on rearranging the terms yields

$$ \begin{aligned} \left(\frac{(1-u)t}{\lambda e^t - u}\right)(\alpha - (yt^2+zt^3))^{-\nu}e^{xt} &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{p=0}^{n}\binom{n}{l}\binom{n-l}{p}(-w)^l\mathcal{H}_p(w;\lambda;u) \\ &\quad \times {}_{\nu}H_{n-l-p}(x,y,z;\alpha)\frac{t^n}{n!}. \end{aligned} \tag{3.5}$$

Finally, using generating function (2.9) in the l.h.s. of equation (3.5) and then by equating the coefficients of like powers of t in the resultant equation, assertion (3.3) follows. □

Next, by making use of determinant form of Genocchi polynomials [20], we obtained the determinant form of the extended Hermite-Genocchi polynomials.

Definition 3.1. The Genocchi polynomials $G_n(x)$ of degree n are defined by [20]

$$G_0(x) = 1,$$

$$G_n(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n} & \frac{1}{2(n+1)} \\ 0 & 1 & \binom{2}{1}\frac{1}{4} & \cdots & \binom{n-1}{1}\frac{1}{2(n-1)} & \binom{n}{1}\frac{1}{2n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2(n-2)} & \binom{n}{2}\frac{1}{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{4} \end{vmatrix}, \tag{3.6}$$

where $n = 1, 2, \ldots$

Now, apply operational rules (3.1) in the r.h.s. and (2.8) in the l.h.s. of determinant form (3.6) of Genocchi polynomials and after simplification, we find the following determinant form for the extended Hermite-Genocchi polynomials ${}_{\nu H}G_n(x, y, z; \alpha)$:

$$\begin{aligned} &{}_{\nu H}G_0(x, y, z; \alpha) = 1, \\ &{}_{\nu H}G_n(x, y, z; \alpha) \\ &= (-1)^n \begin{vmatrix} 1 & {}_{\nu}H_1(x,y,z;\alpha) & {}_{\nu}H_2(x,y,z;\alpha) & \cdots & {}_{\nu}H_{n-1}(x,y,z;\alpha) & {}_{\nu}H_n(x,y,z;\alpha) \\ 1 & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n} & \frac{1}{2(n+1)} \\ 0 & 1 & \binom{2}{1}\frac{1}{4} & \cdots & \binom{n-1}{1}\frac{1}{2(n-1)} & \binom{n}{1}\frac{1}{2n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2(n-2)} & \binom{n}{2}\frac{1}{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{4} \end{vmatrix}, \end{aligned} \tag{3.7}$$

where $n = 1, 2, \ldots$

The method proposed in this article can be used in combination with the monomiality principle as a useful tool in analysing the solutions of a wide class of partial differential equations often encountered in physical problems. The combination of monomiality principle along with operational techniques in the case of multi-variable hybrid special polynomials yields new mechanism of analysis for the solutions of a large class of partial differential equations usually experienced in physical problems. The operational methods open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of special polynomials. The concept of monomiality principle arises from the idea of poweroid suggested by Steffensen [22]. This idea was reformulated and systematically used by Dattoli [9]. Ben Cheikh [8] was shown that every polynomial set is quasi-monomial and the properties of a given polynomial set may be deduced from the quasi-monomiality.

In order to frame the polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ within the context of monomiality principle, the following result is proved.

Theorem 3.2. *The Hermite-Apostol type Frobenius-Genocchi polynomials* ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{H\mathcal{H}} = x + 2y\partial_x + 3z\partial_x^2 - \frac{\lambda e^{\partial_x}(1-\partial_x) - u}{\lambda e^{\partial_x} - u} \tag{3.8}$$

and

$$\hat{P}_{H\mathcal{H}} = \partial_x, \quad \partial_x := \frac{\partial}{\partial x}, \tag{3.9}$$

respectively.

Proof. Differentiating equation (1.3) partially with respect to t, it follows that

$$\left(x + 2yt + 3zt^2 - \frac{\lambda e^t(1-t) - u}{\lambda e^t - u}\right)\left(\frac{(1-u)t}{\lambda e^t - u}\right) = \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n+1}(x,y,z;\lambda;u)\frac{t^n}{n!}. \tag{3.10}$$

Now, using identity

$$\partial_x\{{}_H\mathcal{H}_n(x,y,z;\lambda;u)\} = t\{{}_H\mathcal{H}_n(x,y,z;\lambda;u)\} \tag{3.11}$$

and generating equation (1.3) in the l.h.s of equation (3.10), it follows that

$$\begin{aligned}&\left(x + 2y\partial_x + 3z\partial_x^2 - \frac{\lambda e^{\partial_x}(1-\partial_x) - u}{\lambda e^{\partial_x} - u}\right)\sum_{n=0}^{\infty} {}_H\mathcal{H}_n(x,y,z;\lambda;u)\\ &\quad = \sum_{n=0}^{\infty} {}_H\mathcal{H}_{n+1}(x,y,z;\lambda;u),\end{aligned} \tag{3.12}$$

which in view of monomiality principle equation $\hat{M}\{p_n(x)\} = p_{n+1}(x)$ and then equating the coefficients of same powers of t in both sides yields assertion (3.8).

Again, in view of generating function (1.3) and identity (3.11), it follows that

$$\partial_x\left\{\sum_{n=0}^{\infty} {}_H\mathcal{H}_n(x,y,z;\lambda;u)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_H\mathcal{H}_{n-1}(x,y,z;\lambda;u)\frac{t^n}{(n-1)!}. \tag{3.13}$$

Rearranging the terms in above equation and using monomiality principle equation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$ and then by equating the coefficients of same powers of t in both sides of the resultant equation, assertion (3.9) follows. □

Remark 3.1. By making use of expressions (3.8) and (3.9) in relation $\hat{P}\{p_n(x)\} = n\ p_{n-1}(x)$, we find that the following differential equation for the Hermite-Apostol type Frobenius-Genocchi polynomials ${}_H\mathcal{H}_n(x,y,z;\lambda;u)$ holds true:

$$\left(x\partial_x + 2y\partial_x^2 + 3z\partial_x^3 - \frac{\lambda e^{\partial_x}(1-\partial_x) - u}{\lambda e^{\partial_x} - u}\partial_x - n\right){}_H\mathcal{H}_n(x,y,z;\lambda;u) = 0. \tag{3.14}$$

Next, with the use of integral transforms, we show that the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$ are quasi-monomial.

Consider the operation (Θ): replacement of y by yt and z by zt, multiplication by $\frac{1}{\Gamma(\nu)}e^{-at}t^{\nu-1}$ and then integration with respect to t from $t=0$ to $t=\infty$.

Operating (Θ) on equations (3.8) and (3.9) and then using equation (2.15) and further in view of recurrence relations $\hat{M}\{p_n(x)\}=p_{n+1}(x)$ and $\hat{P}\{p_n(x)\}=np_{n-1}(x)$, we find that the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\hat{M}_{\nu H\mathcal{H}}=x+2y\partial_x\partial_\alpha+3z\partial_x^2\partial_\alpha-\frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u} \tag{3.15}$$

and

$$\hat{P}_{\nu H\mathcal{H}}=\partial_x, \tag{3.16}$$

respectively.

Further, use of equations (3.15) and (3.16) in relation $\hat{P}\{p_n(x)\}=np_{n-1}(x)$ yields the following differential equation for the extended Hermite-Apostol type Frobenius-Genocchi polynomials ${}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)$:

$$\left(x\partial_x+2y\partial_x^2\partial_\alpha+3z\partial_x^3\partial_\alpha-\frac{\lambda e^{\partial_x}(1-\partial_x)-u}{\lambda e^{\partial_x}-u}\partial_x-n\right){}_{\nu H}\mathcal{H}_n(x,y,z;\lambda;u;\alpha)=0. \tag{3.17}$$

The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional operators. To bolster the contention of using this approach, the extended form of hybrid type polynomials are introduced. The generating function and recurrence relations for the extended hybrid polynomials are derived here. These results may be useful in the investigation of other useful properties of these polynomials and may have applications in physics.

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