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ON GRADED 2-ABSORBING SECOND SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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Abstract. In this paper, we introduce the concepts of graded 2-absorbing second and graded strongly 2-absorbing second submodules. A number of results concerning these classes of graded submodules are given.

1. Introduction and Preliminaries

Throughout this paper all rings are commutative, with identity and all modules are unitary.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a G-*graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g \in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g \in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$ (see [19]).

Let R be a G-graded ring and M an R-module. We say that M is a G-*graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \bigoplus\limits_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup\limits_{g \in G} M_g$ and the elements of

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$h(M)$ are called to be *homogeneous*. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N (see [19]). For more details, one can refer to [16, 20, 21].

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R \mid rM \subseteq N\}$. It is shown in [11, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R \mid rN \subseteq M\}$ is a graded ideal of R. The annihilator of M is defined as $(0 :_R M)$ and is denoted by $Ann_R(M)$.

The notion of graded prime ideals was introduced in [24] and studied in [12, 23, 25]. A proper graded ideal P of R is said to be *a graded prime ideal* if whenever $rs \in P$, we have $r \in P$ or $s \in P$, where $r, s \in h(R)$.

S. E. Atani in [11] extended graded prime ideals to graded prime submodules. A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$. Several authors investigated properties of graded prime submodules, for examples see [3, 6, 7, 15, 22].

The notion of graded 2-absorbing ideals as a generalization of graded prime ideals was introduced and studied in [4, 18]. A proper graded ideal I of R is said to be *a graded 2-absorbing ideal of R* if whenever $r, s, t \in h(R)$ with $rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$.

K. Al-Zoubi and R. Abu-Dawwas in [2] extended graded 2-absorbing ideals to graded 2-absorbing submodules. A proper graded submodule N of M is said to be *a graded 2-absorbing submodule of M* if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$.

The notion of graded second submodules was introduced in [9] and studied in [1, 10, 14]. A non-zero graded submodule N of M is said to be *a graded second* (*gr-second*) if for each homogeneous element r of R, the endomorphism of N given by multiplication by r is either surjective or zero. Recently, H. Ansari-Toroghy and F. Farshadifar, in [8] studied 2-absorbing second and strongly 2-absorbing second submodules.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, we introduced the concepts of graded 2-absorbing second and graded strongly 2-absorbing second submodules, investigate some properties of these graded submodules and give some characterizations of them.

2. Graded 2-Absorbing Second Submodules

Definition 2.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule C of M is said to be *a completely graded irreducible* if $C = \cap_{\alpha\in\Delta} C_\alpha$, where $\{C_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M, implies that $C = C_\beta$ for some $\beta \in \Delta$.

Lemma 2.1. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If $m \in h(M) - N$, then there exists a completely graded irreducible submodule C of M such that $N \subseteq C$ and $m \notin C$.*

Proof. Let $m \in h(M) - N$ and Λ be the set of all graded submodules of M that contains N and not containing m. Then $\Lambda \neq \emptyset$, since $N \in \Lambda$. Order Λ by inclusion, i.e. for K, $L \in \Lambda$ then $K \leq L$ if $K \subseteq L$. Clearly, $(\Lambda, \leq)$ is a partially ordered set. Let $\{C_\alpha\}_{\alpha\in\Omega}$ be any chain in Λ. It is clear that $\cup_{\alpha\in\Omega}C_\alpha$ is an upper bound of $\{C_\alpha\}_{\alpha\in\Omega}$ in Λ. Thus, by Zorn's Lemma, Λ contains a maximal element C. We claim that C is a completely graded irreducible submodule of M. Let $\{L_\beta\}_{\beta\in\Delta}$ be a family of graded submodules of M such that $C = \cap_{\beta\in\Delta}L_\beta$. Suppose to the contrary that $C \neq L_\beta$ for all $\beta \in \Delta$. Then each L_β contain m, it follows that $m \in \cap_{\beta\in\Delta}L_\beta = C$, which is a contradiction. □

Lemma 2.2. *Let R be a G-graded ring, M a graded R-module and K, L be two proper graded submodules of M. Then $K \subseteq L$ if and only if every completely graded irreducible submodule containing L, also contains K.*

Proof. $(\Rightarrow)$ is clear.

$(\Leftarrow)$ Assume that every completely graded irreducible submodule of M containing L, also contains K. Suppose to the contrary that $K \nsubseteq L$. Since K is generated by $K \cap h(M)$, there exists $k \in K \cap h(M) - L$. By Lemma 2.1, there exists a completely graded irreducible submodule C of M such that $L \subseteq C$ and $k \notin C$. This implies that $K \nsubseteq C$, which is a contradiction. □

Theorem 2.1. *Let R be a G-graded ring and M a graded R-module. Then every proper graded submodule of M is the intersection of all completely graded irreducible submodules containing it.*

Proof. Let K be a proper graded submodule of M and $\{C_\beta\}_{\beta\in\Delta}$ be the set of all completely graded irreducible submodules containing K. It is clear that $K \subseteq \bigcap\limits_{\beta\in\Delta} C_\beta$. If $k = \sum_{g\in G} k_g \notin K$, then there exists $h \in G$ such that $k_h \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. By Lemma 2.1, there exists a completely graded irreducible submodule C such that $K \subseteq C$ and $k_h \notin C$. Hence $C = C_\alpha$ for some $\alpha \in \Delta$, it follows that $k_h \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. So, $k \notin \bigcap\limits_{\beta\in\Delta} C_\beta$. Consequently, $\bigcap\limits_{\beta\in\Delta} C_\beta \subseteq K$. □

Definition 2.2. Let R be a G-graded ring and M a graded R-module. A non-zero graded submodule S of M is said to be *a graded* 2*-absorbing second submodule of M* if whenever r, $t \in h(R)$, C is a completely graded irreducible submodule of M and $rtS \subseteq C$, then $rS \subseteq C$ or $tS \subseteq C$ or $rt \in Ann_R(S)$.

Let R be a G-graded ring. The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [23]).

Let M be a non-zero graded R-module. Then M is said to be *a graded secondary* if for each homogeneous element r of R, the endomorphism of M given by multiplication by r is either surjective or nilpotent. This implies that $Gr(Ann_R(M)) = P$ is a graded prime ideal of R. For convenience, a graded submodule of M which is graded secondary, is called a *graded secondary* submodule of M (see [13]).

Lemma 2.3. *Let R be a G-graded ring and M a graded R-module of M.*

(i) *If S is a graded second submodule of M and $rS \subseteq K$, where $r \in h(R)$ and K is a graded submodule of M, then either $rS = 0$ or $S \subseteq K$.*
(ii) *If S is a graded secondary submodule of M and $rS \subseteq K$, where $r \in h(R)$ and K is a graded submodule of M, then either $r^n S = 0$ for some $n \in \mathbb{N}$ or $S \subseteq K$.*

Proof. Straightforward. □

Theorem 2.2. *Let R be a G-graded ring and M a graded R-module. Then the following hold.*

(i) *If either S is a graded second submodule of M or S is a sum of two graded second submodules of M, then S is a graded 2-absorbing second submodule.*
(ii) *If S is a graded secondary submodule of M and $R/Ann_R(S)$ has no non-zero nilpotent homogeneous element, then S is a graded 2-absorbing second submodule.*

Proof. (i) Assume that S is a graded second submodule of M. Let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rtS \subseteq C$. By Lemma 2.3 (i), either $rtS = 0$ or $S \subseteq C$. Thus S is a graded 2-absorbing second submodule. Now assume that $S = S_1 + S_2$, where S_1 and S_2 are graded second submodules of M. Let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rtS \subseteq C$. Since S_1 is a graded second submodule, by Lemma 2.3 (i), we have either $rtS_1 = 0$ or $S_1 \subseteq C$. Similarly, we have $rtS_2 = 0$ or $S_2 \subseteq C$. If $rtS_1 = 0$ and $rtS_2 = 0$, then $rt \in Ann(S_1 + S_2)$, we are done. If $S_1 \subseteq C$ and $S_2 \subseteq C$, then we are done. Assume that $rtS_1 = 0$ and $S_2 \subseteq L$. Then $rt \in Ann_R(S_1)$. By [9, Proposition 3.15], $Ann_R(S_1)$ is a graded prime ideal. This yields that $r \in Ann_R(S_1)$ or $t \in Ann_R(S_1)$. If $r \in Ann_R(S_1)$, then $r(S_1 + S_2) \subseteq rS_1 + S_2 \subseteq S_2 \subseteq C$. Similarly, If $t \in Ann_R(S_1)$, we get $t(S_1 + S_2) \subseteq C$. Also if $rtS_2 = 0$ and $S_1 \subseteq C$, we get either $r(S_1 + S_2) \subseteq C$ or $t(S_1 + S_2) \subseteq C$. Therefore S is a graded 2-absorbing second submodule.

(ii) Since C is a graded secondary submodule of M, $Gr(Ann_R(C))$ is a graded prime ideal. This yields that $Ann_R(C)$ is a graded prime ideal because $R/Ann_R(C)$ has no non-zero nilpotent homogeneous element. By [10, Proposition 2.3 (i)], we have C is a graded second submodule and hence C is a graded 2-absorbing second submodule by part (i). □

Lemma 2.4. *Let R be a G-graded ring, M a graded R-module and S a graded 2-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. If $r \in h(R)$, $g \in G$ and C is a completely graded irreducible submodule of M with $I_g rS \subseteq C$, then either $rS \subseteq C$ or $I_g S \subseteq C$ or $I_g r \subseteq Ann_R(S)$.*

Proof. Assume that $r \in h(R)$, $g \in G$ and C is a completely graded irreducible submodule of M such that $I_g rS \subseteq C$, $rS \nsubseteq C$ and $I_g r \nsubseteq Ann_R(S)$. We have to show that $I_g S \subseteq C$. Assume that $i_g \in I_g$. By assumption there exists $i'_g \in I_g$ such that $ri'_g S \neq 0$. Since S is a graded 2-absorbing second submodule of M, $ri'_g S \subseteq C$, $rS \nsubseteq C$ and $ri'_g \notin Ann_R(S)$, we get $i'_g S \subseteq C$. By $(i_g + i'_g) \in I_g$ it follows that $(i_g + i'_g)rS \subseteq C$. Then either $(i_g + i'_g)S \subseteq C$ or $(i_g + i'_g)r \in Ann_R(S)$ as S is a graded 2-absorbing second submodule of M. If $(i_g + i'_g)S \subseteq C$, then we get $i_g S \subseteq C$. If $(i_g + i'_g)r \in Ann_R(S)$, then $i_g r \notin Ann_R(S)$. Since S is a graded 2-absorbing second, $i_g rS \subseteq C, i_g r \notin Ann_R(S)$ and $rS \nsubseteq C$, we get $i_g S \subseteq C$. Therefore, $I_g S \subseteq C$. □

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statement are equivalent.*

(i) *S is a graded 2-absorbing second submodule of M.*
(ii) *If C is a completely graded irreducible submodule of M and $g, h \in G$ with $I_g J_h S \subseteq C$, then either $I_g S \subseteq C$ or $J_h S \subseteq C$ or $I_g J_h \subseteq Ann_R(S)$.*

Proof. (i)⇒(ii) Assume that S is a graded 2-absorbing second submodule of M. Let C be a completely graded irreducible submodule of M and $g, h \in G$ such that $I_g J_h S \subseteq C, I_g S \nsubseteq C$ and $J_h S \nsubseteq C$. We show that $I_g J_h \subseteq Ann_R(S)$. Assume that $i_g \in I_g$ and $j_h \in J_h$. By assumption there exists $i'_g \in I_g$ such that $i'_g S \nsubseteq L$. Since $i'_g J_h S \subseteq C$, $J_h S \nsubseteq C$ and $i'_g S \nsubseteq C$, by Lemma 2.4 we get $i'_g J_h \subseteq Ann_R(S)$ and hence $(I_g \backslash (C :_R S))J_h \subseteq Ann_R(S)$. Similarly there exists $j'_h \in J_h$ and $j'_h S \nsubseteq C$ such that $j'_h I_g \subseteq Ann_R(S)$ and also $(J_h \backslash (C :_R S))I_g \subseteq Ann_R(S)$. Thus we have $i'_g j'_h \in Ann_R(S)$, $i'_g j_h \in Ann_R(S)$ and $i_g j'_h \in Ann_R(S)$. By $(i_g + i'_g) \in I_g$ and $(j_h + j'_h) \in J_h$ it follows that $(i_g + i'_g)(j_h + j'_h)S \subseteq C$. Since S is a graded 2-absorbing second, we get either $(i_g + i'_g)S \subseteq C$ or $(j_h + j'_h)S \subseteq C$ or $(i_g + i'_g)(j_h + j'_h) \in Ann_R(S)$. If $(i_g + i'_g)S = i_g S + i'_g S \subseteq C$, then $i_g S \nsubseteq C$. So $i_g \in I_g \backslash (C :_R S)$ it follows that $i_g j_h \in Ann_R(S)$. Similarly by $(j_h + j'_h)S \subseteq C$ we get $i_g j_h \in Ann_R(S)$. If $(i_g + i'_g)(j_h + j'_h) \in Ann_R(S)$, then $i_g j_h + i_g j'_h + i'_g j_h + i'_g j'_h \in Ann_R(S)$ and so $i_g j_h \in Ann_R(S)$. Thus $I_g J_h \subseteq Ann_R(S)$.

(ii)⇒ (i) Assume that (ii) holds. Let $r_g, t_h, \in h(R)$ and C be a completely graded irreducible submodule of M with $r_g t_h S \subseteq C$. Let $I = r_g R$ and $J = t_h R$ be a graded ideals of R generated by r_g and t_h, respectively. Then $I_g J_h S \subseteq C$. By our assumption we obtain $I_g S \subseteq C$ or $J_h S \subseteq C$ or $I_g J_h \subseteq Ann_R(S)$. Hence $r_g S \subseteq C$ or $t_h S \subseteq C$ or $r_g t_h \in Ann_R(S)$. Therefore, S is a graded 2-absorbing second submodule of M. □

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and S a graded 2-absorbing second submodule of M. Then we have the following.*

(i) *If* $Ann_R(S)$ *is a graded prime ideal of* R, *then* $(C :_R S)$ *is a graded prime ideal of* R *for all completely graded irreducible submodule* C *of* M *such that* $S \nsubseteq C$.

(ii) *If* $Gr(Ann_R(S)) = P$ *for some graded prime ideal* P *of* R, *then* $Gr((C :_R S))$ *is a graded prime ideal of* R *containing* P *for all completely graded irreducible submodule* C *of* M *such that* $S \nsubseteq C$.

Proof. (i) Let $r, t \in h(R)$, C be a completely graded irreducible submodule of M such that $S \nsubseteq C$ and $rt \in (C :_R S)$. So $rtS \subseteq C$. Since S is a graded 2-absorbing second submodule, we have $rS \subseteq C$ or $tS \subseteq C$ or $rt \in Ann_R(S)$. If $rS \subseteq C$ or $tS \subseteq C$, then we are done. If $rt \in Ann_R(S)$, then $r \in Ann_R(S)$ or $t \in Ann_R(S)$ because $Ann_R(S)$ is a graded prime ideal of R. This yields that $r \in (C :_R S)$ or $t \in (C :_R S)$. Thus $(C :_R S)$ is a graded prime ideal of R.

(ii) Let $r, t \in h(R)$ and $rt \in Gr((C :_R S))$. Then $(rt)^n \in (C :_R S)$ for some $n \in \mathbb{Z}^+$. So $r^n t^n S \subseteq C$. Since S is a graded 2-absorbing second submodule, we have $r^n S \subseteq C$ or $t^n S \subseteq C$ or $r^n t^n \in Ann_R(S)$. If $r^n S \subseteq C$ or $t^n S \subseteq C$, then $r \in Gr((C :_R S))$ or $t \in Gr((C :_R S))$ so we are done. Now assume that $r^n t^n \in Ann_R(S)$ so $rt \in Gr(Ann_R(S))$. Then $r \in Gr(Ann_R(S))$ or $t \in Gr(Ann_R(S))$ as $Gr((Ann_R(S))$ is a graded prime ideal of R. Since $Ann_R(S) \subseteq (C :_R S)$, we have $Gr(Ann_R(S)) \subseteq Gr((C :_R S))$. This yields that $r \in Gr((C :_R S))$ or $t \in Gr((C :_R S))$. Therefore, $Gr((C :_R S))$ is a graded prime ideal of R containing P. □

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$ (see [21].)

Lemma 2.5. *Let* R *be a* G*-graded ring and* M, M' *be two graded* R*-modules and let* $\varphi : M \to M'$ *be a graded monomorphism.*

(i) *If* C *is a completely graded irreducible submodule of* M, *then* $\varphi(C)$ *is a completely graded irreducible submodule of* $\varphi(M)$.

(ii) *If* C' *is a graded completely irreducible submodule of* $\varphi(M)$, *then* $\varphi^{-1}(C')$ *is a completely graded irreducible submodule of* M.

Proof. Straightforward. □

Theorem 2.5. *Let* R *be a* G*-graded ring and* M, M' *be two graded* R*-modules. Let* φ: $M \to M'$ *be a graded monomorphism. Then we have the following.*

(i) *If* S *is a graded* 2*-absorbing second submodule of* M, *then* $\varphi(S)$ *is a graded* 2*-absorbing second submodule of* $\varphi(M)$.

(ii) *If* S' *is a graded* 2*-absorbing second submodule of* $\varphi(M)$, *then* $\varphi^{-1}(S')$ *is a graded* 2*-absorbing second submodule of* M.

Proof. (i) Since $S \neq 0$ and φ is a graded monomorphism, we have $\varphi(S) \neq 0$. Let $r, t \in h(R)$ and C' be a graded completely irreducible submodule of $\varphi(M)$ with $rt\varphi(S) \subseteq C'$. Then $rtS \subseteq \varphi^{-1}(C')$. By Lemma 2.5 (ii), we have $\varphi^{-1}(C')$ is a graded completely irreducible submodule of M. Then either $rS \subseteq \varphi^{-1}(C')$ or $tS \subseteq \varphi^{-1}(C')$

or $rtS = 0$ as S is graded 2-absorbing second submodule of M. If $rtS = 0$, then $rt\varphi(S) = 0$. If $rS \subseteq \varphi^{-1}(C')$, then $r\varphi(S) = \varphi(rS) \subseteq \varphi\varphi^{-1}(C') = C' \cap \varphi(M) = C'$. Similarly, if $tS \subseteq \varphi^{-1}(C')$, we get $t\varphi(S) \subseteq C'$. Therefore, $\varphi(S)$ is a graded 2-absorbing second submodule of $\varphi(M)$.

(ii) If $\varphi^{-1}(S') = 0$, then $\varphi(M) \cap S' = \varphi\varphi^{-1}(S') = \varphi(0) = 0$. Thus $S' = 0$ which is a contradiction. So $\varphi^{-1}(S') \neq 0$. Now let $r, t \in h(R)$ and C be a completely graded irreducible submodule of M with $rt\varphi^{-1}(S') \subseteq C$. Then $rtS' = rt(S' \cap \varphi(M)) = rt\varphi\varphi^{-1}(S') = \varphi(rt\varphi^{-1}(S')) \subseteq \varphi(C)$. By Lemma 2.5(i), we have $\varphi(C)$ is a completely graded irreducible submodule of $\varphi(M)$. Then $rS' \subseteq \varphi(C)$ or $tS' \subseteq \varphi(C)$ or $rtS' = 0$ as S' is a graded 2-absorbing second submodule of $\varphi(M)$. Thus $r\varphi^{-1}(S') \subseteq \varphi^{-1}\varphi(C) = C$ or $t\varphi^{-1}(S') \subseteq \varphi^{-1}\varphi(C) = C$ or $rt\varphi^{-1}(S') = 0$. Therefore, $\varphi^{-1}(S')$ is a graded 2-absorbing second submodule of M. □

3. Graded 2-Absorbing Strongly Second Submodules

Definition 3.1. Let R be a G-graded ring and M a graded R-module. A non-zero graded submodule S of M is said to be *a graded strongly* 2*-absorbing second submodule of* M if whenever r, $t \in h(R)$, C_1, C_2 are completely graded irreducible submodules of M, and $rtS \subseteq C_1 \cap C_2$, then $rS \subseteq C_1 \cap C_2$ or $tS \subseteq C_1 \cap C_2$ or $rt \in Ann_R(S)$.

Clearly every graded strongly 2-absorbing second submodule is a graded 2-absorbing second submodule.

Lemma 3.1. *Let R be a G-graded ring, M a graded R-module and S a graded strongly* 2*-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. If $r \in h(R)$, $g \in G$ and C_1, C_2 are completely graded irreducible submodules of M with $I_g rS \subseteq C_1 \cap C_2$, then either $rS \subseteq C_1 \cap C_2$ or $I_g S \subseteq C_1 \cap C_2$ or $I_g r \subseteq Ann_R(S)$.*

Proof. The proof is similar to the proof of Lemma 2.4, so we omit it. □

Theorem 3.1. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *S is a graded strongly* 2*-absorbing second submodule of M.*
(ii) *If L_1 and L_2 are a completely graded irreducible submodules of M and $g, h \in G$ with $I_g J_h S \subseteq L_1 \cap L_2$, then either $I_g S \subseteq L_1 \cap L_2$ or $J_h S \subseteq L_1 \cap L_2$ or $I_g J_h \subseteq Ann_R(S)$.*

Proof. The proof is similar to the proof of Theorem 2.3, so we omit it. □

Theorem 3.2. *Let R be a G-graded ring, M a graded R-module and S a non-zero graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *S is a graded strongly* 2*-absorbing second submodule of M.*
(ii) *For every graded submodule K of M and $g, h \in G$ such that $I_g J_h S \subseteq K$, either $I_g S \subseteq K$ or $J_h S \subseteq K$ or $I_g J_h \subseteq Ann_R(S)$.*

(iii) *For every graded submodule K of M and every pair of elements r_g, $t_h \in h(R)$ such that $r_g t_h S \subseteq K$, either $r_g S \subseteq K$ or $t_h S \subseteq K$ or $r_g t_h S \subseteq Ann_R(S)$.*
(iv) *For every pair of elements r_g, $t_h \in h(R)$, either $r_g t_h S = r_g S$ or $r_g t_h S = t_h S$ or $r_g t_h S = 0$.*

Proof. (i)⇒(ii) Let $g, h \in G$ and K a graded submodule of M such that $I_g J_h S \subseteq K$ and $I_g J_h \nsubseteq Ann_R(S)$. By Theorem 2.3, for all completely graded irreducible submodule C of M such that $K \subseteq C$, we have either $I_g S \subseteq C$ or $J_h S \subseteq C$ and hence either $I_g S \subseteq K$ or $J_h S \subseteq K$ by Lemma 2.2. If $I_g S \subseteq C$ (resp. $J_h S \subseteq C$) for all completely graded irreducible submodule C of M with $K \subseteq C$, we are done. Now suppose that C_1 and C_2 are two completely graded irreducible submodules of M with $K \subseteq C_1$, $K \subseteq C_2$, $I_g S \nsubseteq C_1$ and $J_h S \nsubseteq C_2$. Since S is a graded 2-absorbing second submodule, $I_g J_h S \subseteq C_1, I_g S \nsubseteq C_1$ and $I_g J_h \nsubseteq Ann_R(S)$, by Theorem 2.3, we have $J_h S \subseteq C_1$. Similarly by $J_h S \nsubseteq C_2$ we get $I_g S \subseteq C_2$. Since S is a graded strongly 2-absorbing second submodule of M, $I_g J_h S \subseteq C_1 \cap C_2$, $I_g J_h \nsubseteq Ann_R(S)$, by Theorem 3.1, we conclude that either $I_g S \subseteq C_1 \cap C_2$ or $J_h S \subseteq C_1 \cap C_2$. Hence, either $I_g S \subseteq C_1$ or $J_h S \subseteq C_2$, which is a contradiction.

(ii)⇒(iii) Assume that $r_g t_h S \subseteq K$ where r_g, $t_h \in h(R)$ and K a graded submodule of M. Let $I = r_g R$, $J = t_h R$ be a graded ideals of R generated by r_g and t_h, respectively. Then $I_g J_j S \subseteq K$. By our assumption we have either $I_g S \subseteq K$ or $J_h S \subseteq K$ or $I_g J_h \subseteq Ann_R(S)$. It follows that either $r_g S \subseteq K$ or $t_h S \subseteq K$ or $r_g t_h S \subseteq Ann_R(S)$.

(iii)⇒(iv) Let r_g, $t_h \in h(R)$. Then $r_g t_h S \subseteq r_g t_h S$ implies that $r_g S \subseteq r_g t_h S$ or $t_h S \subseteq r_g t_h S$ or $r_g t_h \in Ann(S)$. This yields that $r_g S = r_g t_h S$ or $t_h S = r_g t_h S$ or $r_g t_h \in Ann(S)$.

(iv)⇒(i) This is clear. □

Lemma 3.2. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. Then $Ann_R(S)$ is a graded 2-absorbing ideal of R.*

Proof. Let r_g, s_h, $t_\lambda \in h(R)$ such that $r_g s_h t_\lambda \in Ann_R(S)$. Since S a graded strongly 2-absorbing second submodule of M and r_g, $s_h, \in h(R)$, by Theorem 3.2, we get either $r_g S = r_g s_h S$ or $s_h S = r_g s_h S$ or $r_g s_h S = 0$. If $r_g s_h S = 0$, then $r_g s_h \in Ann_R(S)$. If $r_g S = r_g s_h S$, then $t_\lambda r_g S \subseteq t_\lambda r_g s_h S = 0$ and hence $t_\lambda r_g \in Ann_R(S)$. Similarly, by $s_h S = r_g s_h S$ we get $t_\lambda s_h \in Ann_R(S)$. Therefore, $Ann_R(S)$ is a graded 2-absorbing ideal of R. □

Theorem 3.3. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. If K is a graded submodule of M such that $S \nsubseteq K$, then $(K :_R S)$ is a graded 2-absorbing ideal of R.*

Proof. Let r_g, s_h, $t_\lambda \in h(R)$ such that $r_g s_h t_\lambda \in (K :_R S)$. Then $r_g s_h t_\lambda S \subseteq K$. Since S is a graded strongly 2-absorbing second submodule of M and $r_g s_h (t_\lambda S) \subseteq K$, by Theorem 3.2 we conclude that either $r_g t_\lambda S \subseteq K$ or $s_h t_\lambda S \subseteq K$ or $r_g s_h t_\lambda S = 0$, which means $r_g t_\lambda \in (K :_R S)$ or $s_h t_\lambda \in (K :_R S)$ or $r_g s_h t_\lambda \in Ann_R(S)$. If $r_g t_\lambda \in (K :_R S)$

or $s_h t_\lambda \in (K :_R S)$, then we are done. Assume that $r_g s_h t_\lambda \in Ann_R(S)$. Then either $r_g s_h \in Ann_R(S)$ or $r_g t_\lambda \in Ann_R(S)$ or $s_h\ t_\lambda \in Ann_R(S)$ by Lemma 3.2. This yields that either $r_g s_h \in (K :_R S)$ or $r_g t_\lambda \in (K :_R S)$ or $s_h t_\lambda \in (K :_R S)$. Hence, $(K :_R S)$ is a graded 2-absorbing ideal of R. □

Lemma 3.3. *Let R be a G-graded ring, J a graded 2-absorbing ideal of R and $I = \bigoplus_{g\in G} I_g$ a graded ideal of R. If $r,\ s \in h(R)$ and $g \in G$ with $rsI_g \subseteq J$, then either $rI_g \subseteq J$ or $sI_g \subseteq J$ or $rs \in J$.*

Proof. Let $r, s \in h(R)$ and $g \in G$ such that $rsI_g \subseteq J$ and $rs \notin J$. Let $i_g \in I_g$ so $rsi_g \in J$. Then $ri_g \in J$ or $si_g \in J$ as J is a graded 2-absorbing ideal of R. If $ri_g \in J$ for all $i_g \in I_g$, then $rI_g \subseteq J$, we are done. Similarly, if $si_g \in J$ for all $i_g \in I_g$, then $sI_g \subseteq J$, we are done. Suppose that there exist $i_{g1},\ i_{g2} \in I_g$ such that $ri_{g1} \notin J$ and $si_{g2} \notin J$. Since J is a graded 2-absorbing ideal, $rsi_{g1} \in J, ri_{g1} \notin J$ and $rs \notin J$, we conclude that $si_{g1} \in J$. Also $rsi_{g2} \in J$ implies that $ri_{g2} \in J$, since J is a graded 2-absorbing ideal. Since $rs(i_{g1} + i_{g2}) \in J$ and $rs \notin J$, we conclude that either $r(i_{g1} + i_{g2}) \in J$ or $s(i_{g1} + i_{g2}) \in J$ as J is a graded 2-absorbing ideal and hence either $si_{g2} \in J$ or $ri_{g1} \in J$, which is a contradiction. □

Lemma 3.4. *Let R be a G-graded ring and J a graded 2-absorbing ideal of R. Let $I = \bigoplus_{g\in G} I_g$ and $K = \bigoplus_{g\in G} K_g$ be a graded ideals of R. If $r \in h(R)$ and $g, h \in G$ with $rI_gK_h \subseteq J$, then either $rI_g \subseteq J$ or $rK_h \subseteq J$ or $I_gK_h \subseteq J$.*

Proof. Let $r \in h(R)$ and $g,\ h \in G$ such that $rI_gK_h \subseteq J$, $rI_g \nsubseteq J$ and $rK_h \nsubseteq J$. We have to show that $I_gK_h \subseteq J$. Assume that $i_g \in I_g$ and $k_h \in K_h$. By assumption there exist $i'_g \in I_g$ and $k'_h \in K_h$ such that $ri'_g \notin J$ and $rk'_h \notin J$. Since $ri'_gK_h \subseteq J$, $rK_h \nsubseteq J$ and $ri'_g \notin J$, by Lemma 3.3, we get $i'_gK_h \subseteq J$. Also, since $rk'_hI_g \subseteq J$, $rk'_h \notin J$ and $rI_g \nsubseteq J$, by Lemma 3.3, we get $k'_hI_g \subseteq J$. By $(i_g + i'_g) \in I_g$ and $(k'_h + k_h) \in K_h$, we get $r(i_g + i'_g)(k'_h + k_h) \in J$. Then either $r(i_g + i'_g) \in J$ or $r(k'_h + k_h) \in J$ as J is a graded 2-absorbing ideal. If $r(i_g + i'_g) \in J$, then $ri_g \notin J$. Which implies that $i_gk_h \in J$ by Lemma 3.3. Similarly, by $r(k'_h + k_h) \in J$, we conclude that $i_gk_h \in J$. Therefore, $I_gK_h \subseteq J$. □

Theorem 3.4. *Let R be a G-graded ring and J a proper graded ideal of R. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ and $K = \bigoplus_{g\in G} K_g$ be a graded ideals of R. Then the following statements are equivalent.*

(i) *J is a graded 2-absorbing ideal of R.*
(ii) *For every $g, h, \lambda \in G$ with $I_gK_hL_\lambda \subseteq J$, either $I_gL_\lambda \subseteq J$ or $K_hL_\lambda \subseteq J$ or $I_gK_h \subseteq J$.*

Proof. (i)⇒(ii) Assume that J is a graded 2-absorbing ideal of R. Let $g, h, \lambda \in G$ such that $I_gK_hL_\lambda \subseteq J$ and $I_gL_\lambda \nsubseteq J$. Then for all $k_h \in K_h$ either $k_hI_g \subseteq J$ or $k_hL_\lambda \subseteq J$ by Lemma 3.4. If $k_hI_g \subseteq J$ for all $k_h \in K_h$, then $I_gK_h \subseteq J$, we are done. Similarly, if $k_hL_\lambda \subseteq J$ for all $k_h \in K_h$, then $K_hL_\lambda \subseteq J$, we are done. Suppose that $k_{h1},\ k_{h2} \in K_h$ are such that $k_{h1}I_g \nsubseteq J$ and $k_{h2}L_\lambda \nsubseteq J$. It follows that $k_{h1}L_\lambda \subseteq J$ and $k_{h2}I_g \subseteq J$. Since

$(k_{h1}+k_{h2})I_gL_\lambda \subseteq J$, by Lemma 3.4. we have $(k_{h1}+k_{h2})L_\lambda \subseteq J$ or $(k_{h1}+k_{h2})I_g \subseteq J$. By $(k_{h1}+k_{h2})L_\lambda \subseteq J$ it follows that $k_{h2}L_\lambda \subseteq J$, which is a contradiction. Similarly by $(k_{h1}+k_{h2})I_g \subseteq J$ we get a contradiction. Therefore $K_hL_\lambda \subseteq J$ or $I_gK_h \subseteq J$.

(ii)⇒(i) Assume that (ii) holds. Let $r_g, s_h, t_\lambda \in h(R)$ such that $r_gs_ht_\lambda \in J$. Let $I = r_gR$, $K = s_hR$ and $L = t_\lambda R$ be a graded ideals of R generated by r_g, s_h and t_λ, respectively. Then $I_gK_hL_\lambda \subseteq J$. By our assumption we obtain $I_gK_h \subseteq J$ or $I_gL_\lambda \subseteq J$ or $K_hL_\lambda \subseteq J$. Hence, $r_gs_h \in J$ or $r_gt_\lambda \in J$ or $s_ht_\lambda \in J$. Therefore, J is a graded 2-absorbing ideal of R. □

Theorem 3.5. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. Then for each $g \in G$, $I_g^nS = I_g^{n+1}S$ for all $n \geqslant 2$.*

Proof. Let $g \in G$. It is enough to show that $I_g^2S = I_g^3S$. It is clear that $I_g^3S \subseteq I_g^2S$. Since S a graded strongly 2-absorbing second submodule of M, $I_g^3S \subseteq I_g^3S$ implies that $I_g^2S \subseteq I_g^3S$ or $I_gS \subseteq I_g^3S$ or $I_g^3S = 0$ by Theorem 3.2. If $I_gS \subseteq I_g^3S$ or $I_g^2S \subseteq I_g^3S$, then we are done. Assume that $I_g^3S = 0$, hence $I_g^3 \subseteq Ann_R(S)$. By Lemma 3.2 and Theorem 3.4, we get $I_g^2 \subseteq Ann_R(S)$ and hence $I_g^2S \subseteq I_g^3S$. Therefore, $I_g^2S = I_g^3S$. □

Theorem 3.6. *Let R be a G-graded ring, M a graded R-module and S a graded strongly 2-absorbing second submodule of M. If $Gr(Ann_R(S)) = P$ for some graded prime ideal P of R, C_1 and C_2 are completely graded irreducible submodules of M such that $S \nsubseteq C_1$ and $S \nsubseteq C_2$. Then either $Gr((C_1 :_R S)) \subseteq Gr((C_2 :_R S))$ or $Gr((C_2 :_R S)) \subseteq Gr((C_1 :_R S))$.*

Proof. Assume that $Gr((C_1 :_R S))) \nsubseteq Gr((C_2 :_R S))$. Since $Gr((C_1 :_R S))$ is generated by $Gr((C_1 :_R S)) \cap h(R)$, there exists $r \in Gr((C_1 :_R S)) \cap h(R) - Gr((C_2 :_R S))$. Now, let $t \in Gr((C_2 :_R S)) \cap h(R)$. Then there exists a positive integer n such that $t^nS \subseteq C_2$, $r^nS \subseteq C_1$ and $r^nS \nsubseteq C_2$. Hence $t^nr^nS \subseteq C_1 \cap C_2$. So either $t^nS \subseteq C_1 \cap C_2$ or $t^nr^n \subseteq Ann_R(S)$ as S is a graded strongly 2-absorbing second submodule of M. If $t^nS \subseteq C_1 \cap C_2$, then $t^nS \subseteq C_1$, which implies $t \in Gr((C_1 :_R S))$. So, assume that $t^nr^n \subseteq Ann_R(S)$. Then $tr \in Gr(Ann_R(S)) = P$. Since P is a graded prime ideal of R, either $r \in P$ or $t \in P$. If $r \in P$, then $r^mS = 0 \in C_2$ for some $m \in \mathbb{Z}^+$ which is a contradiction. This yields that $t \in P = Gr(Ann_R(S)) \subseteq Gr((C_1 :_R S))$. Thus, $Gr((C_2 :_R S)) \subseteq Gr((C_1 :_R S))$. □

Theorem 3.7. *Let R be a G-graded ring and M, M' be two graded R-modules. Let $\varphi : M \to M'$ be a graded monomorphism. Then the following hold.*

(i) *If S is a graded strongly 2-absorbing second submodule of M, then $\varphi(S)$ is a graded 2-absorbing second submodule of M'.*
(ii) *If S' is a graded strongly 2-absorbing second submodule of M' and $S' \subseteq \varphi(M)$, then $\varphi^{-1}(S')$ is a graded 2-absorbing second submodule of M.*

Proof. By using Theorem 3.2 the proof is similar to that of Theorem 2.5. □

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