Kragujevac Journal of Mathematics
Volume 48(1) (2024), Pages 123–144.

ON THE STRUCTURE OF SOME TYPES OF HIGHER DERIVATIONS

AMIN HOSSEINI[1*] AND NADEEM UR REHMAN[2]

Abstract. In this paper we introduce the concepts of higher $\{L_{g_n}, R_{h_n}\}$-derivation, higher $\{g_n, h_n\}$-derivation and Jordan higher $\{g_n, h_n\}$-derivation. Then we give a characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations in terms of $\{L_g, R_h\}$-derivations and $\{g, h\}$-derivations, respectively. Using this result, we prove that every Jordan higher $\{g_n, h_n\}$-derivation on a semiprime algebra is a higher $\{g_n, h_n\}$-derivation. In addition, we show that every Jordan higher $\{g_n, h_n\}$-derivation of the tensor product of a semiprime algebra and a commutative algebra is a higher $\{g_n, h_n\}$-derivation. Moreover, we show that there is a one to one correspondence between the set of all higher $\{L_{g_n}, R_{h_n}\}$-derivations and the set of all sequences of $\{L_{G_n}, R_{H_n}\}$-derivations. Also, it is presented that if $\mathcal{A}$ is a unital algebra and $\{f_n\}$ is a generalized higher derivation associated with a sequence $\{d_n\}$ of linear mappings, then $\{d_n\}$ is a higher derivation. Some other related results are also discussed.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be an algebra and let $g, h : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is said to be a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) if $f(ab) = g(a)b + ah(b)$ (resp. $f(a) = h(a)b + ag(b)$) for all $a, b \in \mathcal{A}$. By following Brešar [1], a linear mapping f is called a $\{g, h\}$-derivation on $\mathcal{A}$ if it is both a $\{L_g, R_h\}$-derivation and a $\{R_g, L_h\}$-derivation, i.e., $f(ab) = g(a)b + ah(b) = h(a)b + ag(b)$ for all $a, b \in \mathcal{A}$. A linear mapping f is called a Jordan $\{g, h\}$-derivation if $f(a \circ b) = g(a) \circ b + a \circ h(b)$ for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$. We call $a \circ b$ the Jordan product of a and b. It is evident that $a \circ b = b \circ a$ for all $a, b \in \mathcal{A}$. The notion of a Jordan $\{g, h\}$-derivation

Key words and phrases. Higher $\{L_{g_n}, R_{h_n}\}$-derivation, higher $\{g_n, h_n\}$-derivation, Jordan higher $\{g_n, h_n\}$-derivation, generalized higher derivation.
2010 *Mathematics Subject Classification.* Primary: 16W25. Secondary: 47B47.
DOI 10.46793/KgJMat2401.123H
Received: August 20, 2020.
Accepted: March 06, 2021.

is a generalization of what is called a Jordan generalized derivation in [10]. Recall that a linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a *Jordan generalized derivation* if there exists a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $f(a \circ b) = f(a) \circ b + a \circ d(b)$ for all $a, b \in \mathcal{A}$; in this case d is called an associated linear mapping of f. It is clear that $f(a \circ b) = d(a) \circ b + a \circ f(b)$ for all $a, b \in \mathcal{A}$. Obviously, the definition of a generalized Jordan derivation is generally not equivalent to that of Jordan generalized derivation. For more details in this regard, see e.g., [1, 10], and the references therein.

As an important result, Brešar [1, Theorem 4.3] proved that every Jordan $\{g, h\}$-derivation of a semiprime algebra $\mathcal{A}$ is a $\{g, h\}$-derivation. He also showed that every Jordan $\{g, h\}$-derivation of the tensor product of a semiprime algebra and a commutative algebra is a $\{g, h\}$-derivation. It is evident that every $\{g, h\}$-derivation is a Jordan $\{g, h\}$-derivation, but the converse is in general not true, for instance, see [1, Example 2.1].

In this study, we introduce the concepts of higher $\{L_{g_n}, R_{h_n}\}$-derivation, higher $\{R_{g_n}, L_{h_n}\}$-derivation, higher $\{g_n, h_n\}$-derivation, Jordan higher $\{g_n, h_n\}$-derivation and then we present a characterization of these concepts on algebras. Throughout this paper, $\mathcal{A}$ denotes an algebra over a field $\mathbb{F}$ with $char(\mathbb{F}) = 0$ and I denotes the identity mapping on $\mathcal{A}$. Let f be a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) on an algebra $\mathcal{A}$. An easy induction argument implies that $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} g^{n-k}(a)h^k(b)$ (resp. $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}(a)g^k(b)$) (Leibniz rule) for each $a, b \in \mathcal{A}$ and each nonnegative integer n, where $f^0 = g^0 = h^0 = I$. Hence, if f is a $\{g, h\}$-derivation, then $f^n(ab) = \sum_{k=0}^{n} \binom{n}{k} g^{n-k}(a)h^k(b) = \sum_{k=0}^{n} \binom{n}{k} h^{n-k}(a)g^k(b)$ for all $a, b \in \mathcal{A}$. Suppose that f is a $\{L_g, R_h\}$-derivation on $\mathcal{A}$. If we define the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ by $f_n = \frac{f^n}{n!}$, $g_n = \frac{g^n}{n!}$ and $h_n = \frac{h^n}{n!}$, with $f_0 = g_0 = h_0 = I$, then it follows from the Leibniz rule that f_n's, g_n's and h_n's satisfy

$$f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b), \tag{1.1}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Similarly, if f is a $\{R_g, L_h\}$-derivation, then the above f_n, g_n and h_n satisfy

$$f_n(ab) = \sum_{k=0}^{n} h_{n-k}(a)g_k(b), \tag{1.2}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Also, if f is a $\{g, h\}$-derivation, then we have

$$f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b) = \sum_{k=0}^{n} h_{n-k}(a)g_k(b), \tag{1.3}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. This is our motivation to investigate the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ of linear mappings on an algebra $\mathcal{A}$ that satisfy (1.1) or (1.2) or (1.3). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) if there exist two sequences

$\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying (1.1) (resp. (1.2)). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{g_n, h_n\}$-derivation if it is both a higher $\{L_{g_n}, R_{h_n}\}$-derivation and a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. In addition, a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b),$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Notice that if $\{f_n\}$ is a higher $\{f_n, f_n\}$-derivation (resp. Jordan higher $\{f_n, f_n\}$-derivation), then it is an ordinary higher derivation (resp. Jordan higher derivation). We know that if f is a $\{L_g, R_h\}$-derivation, then $\{f_n = \frac{f^n}{n!}\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation, where $g_n = \frac{g^n}{n!}$, $h_n = \frac{h^n}{n!}$ and $f_0 = g_0 = h_0 = I$. We call this kind of higher $\{L_{g_n}, R_{h_n}\}$-derivation an ordinary higher $\{L_{g_n}, R_{h_n}\}$-derivation, but this is not the only example of a higher $\{L_{g_n}, R_{h_n}\}$-derivation. We have the same expression for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations. Using the idea of [10] and to make the article more accurate, we consider generalized derivations as follows: A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called an *l-generalized derivation* (resp. *r-generalized derivation*) associated with a linear mapping $d : \mathcal{A} \to \mathcal{A}$ if f is a $\{L_f, R_d\}$-derivation (resp. $\{R_f, L_d\}$-derivation) on $\mathcal{A}$. Naturally, a linear mapping f is called a *two-sided generalized derivation* if it is both an l-generalized derivation associated with a linear mapping d_1 and a r-generalized derivation associated with a linear mapping d_2 on $\mathcal{A}$. Recently, Hosseini [7] has studied two-sided generalized derivations and in that article he has presented a r-generalized derivation which is not an l-generalized derivation. A sequence $\{f_n\}$ of linear mappings is called a higher l-generalized derivation associated with a sequence $\{d_n\}$ of linear mappings if it is a higher $\{L_{f_n}, R_{d_n}\}$-derivation. Similarly, the concepts of higher r-generalized derivations and two-sided generalized higher derivations are defined. Most authors who have studied generalized higher derivations suppose that these mappings are dependent on higher derivations, see, e.g. [5, 12, 14], and the references therein. In this paper and in the characterization that we offer, we do not use this assumption. In fact, if $\{f_n\}$ is a generalized higher derivation (resp. Jordan generalized higher derivation) associated with a sequence $\{d_n\}$ of linear mappings, we do not assume that the sequence $\{d_n\}$ is necessarily a higher derivation (resp. Jordan higher derivation).

In 2010, Miravaziri [11] characterized all higher derivations on an algebra $\mathcal{A}$ in terms of derivations on $\mathcal{A}$. In this article, by getting idea and using techniques of [11], our aim is to characterize higher $\{L_{g_n}, R_{h_n}\}$-derivations, higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations on an algebra $\mathcal{A}$ in terms of $\{L_g, R_h\}$-derivations, $\{R_g, L_h\}$-derivations and $\{g, h\}$-derivations, respectively. As the main result of this article, we prove that if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$

of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. The same is also true for higher $\{g_n, h_n\}$-derivations. Using this result, if $\{f_n\}$ is a higher l-generalized derivation (resp. higher r-generalized derivation) associated with a sequence $\{d_n\}$, then we characterize $\{f_n\}$ without assuming that $\{d_n\}$ is a higher derivation. Mirzavaziri and Tehrani [12] characterized generalized higher derivations while assuming the associated sequences are higher derivations. So, our results improve their work.

As an application of the main result of this article, we investigate Jordan higher $\{g_n, h_n\}$-derivations on algebras. Let us give a brief background in this regard. It is a classical question in which algebras (or rings) a Jordan derivation is necessarily a derivation. In 1957, Herstein [9] achieved a result which asserts any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [3]. In 1975, Cusack [4] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). Moreover, Vukman [13] investigated generalized Jordan derivations on semiprime rings and he proved that every *generalized Jordan derivation* of a 2-torsion free semiprime ring is a generalized derivation. Recently, the first name author along with Ajda Fošner [6] have studied the same problem for (σ, τ)-derivations from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module $\mathcal{M}$. In this paper, we show that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation of a semiprime algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then it is a higher $\{g_n, h_n\}$-derivation, and further we prove that if $\mathcal{A}$ is a semiprime algebra, $\mathcal{S}$ is a commutative algebra, and $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation of $\mathcal{A} \otimes \mathcal{S}$, with $f_0 = g_0 = h_0 = I$, then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation. Here, $\mathcal{A} \otimes \mathcal{S}$ denotes the tensor product of $\mathcal{A}$ and $\mathcal{S}$. Also, some results related to generalized higher derivations are presented.

2. Main Results

Throughout the article, $\mathcal{A}$ denotes an algebra over a field of characteristic zero, and I is the identity mapping on $\mathcal{A}$. We begin with the following definitions.

Definition 2.1. Let $f, g, h : \mathcal{A} \to \mathcal{A}$ be linear mappings. We say that f is a $\{L_g, R_h\}$-derivation (resp. $\{R_g, L_h\}$-derivation) if $f(ab) = g(a)b + ah(b)$ (resp. $f(ab) = h(a)b + ag(b)$) for all $a, b \in \mathcal{A}$.

Following Brešar [1], a linear mapping f is called a $\{g, h\}$-derivation on $\mathcal{A}$ if it is both a $\{L_g, R_h\}$-derivation and a $\{R_g, L_h\}$-derivation, i.e., $f(ab) = g(a)b + ah(b) =$

$h(a)b + ag(b)$ for all $a, b \in \mathcal{A}$. A linear mapping f is called a Jordan $\{g, h\}$-derivation if $f(a \circ b) = g(a) \circ b + a \circ h(b)$ for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$.

Definition 2.2. A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying (1.1) (resp. (1.2)). A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a higher $\{g_n, h_n\}$-derivation if it is both a higher $\{L_{g_n}, R_{h_n}\}$-derivation and a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. In addition, a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ satisfying

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b), \tag{2.1}$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n.

Before establishing the first result of this paper, we would like to draw your attention to the following discussion that makes clear the process of characterizing of higher $\{L_{g_n}, R_{h_n}\}$-derivations by $\{L_g, R_h\}$-derivations. Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation. So, $f_n(ab) = \sum_{k=0}^{n} g_{n-k}(a)h_k(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. If $f_0 = g_0 = h_0 = I$, then we have $f_1(ab) = g_1(a)b + ah_1(b)$, which means that f_1 is a $\{L_{g_1}, R_{h_1}\}$-derivation. Therefore, we have

$$\begin{aligned} f_1^2(ab) &= f_1\left(g_1(a)b + ah_1(b)\right) \\ &= g_1^2(a)b + g_1(a)h_1(b) + g_1(a)h_1(b) + ah_1^2(b) \\ &= g_1^2(a)b + 2g_1(a)h_1(b) + ah_1^2(b). \end{aligned}$$

Thus,

$$2g_1(a)h_1(b) = f_1^2(ab) - g_1^2(a)b - ah_1^2(b), \quad a, b \in \mathcal{A}. \tag{2.2}$$

Note that $f_2(ab) = g_2(a)b + g_1(a)h_1(b) + ah_2(b)$. So, $2f_2(ab) = 2g_2(a)b + 2g_1(a)h_1(b) + 2ah_2(b)$ holds for all $a, b \in \mathcal{A}$. Putting (2.2) in the previous formula, we deduce that $2f_2(ab) = 2g_2(a)b + f_1^2(ab) - g_1^2(a)b - ah_1^2(b) + 2ah_2(b)$ for all $a, b \in \mathcal{A}$. Hence, we can write

$$2f_2(ab) - f_1^2(ab) = \left(2g_2(a) - g_1^2(a)\right)b + a\left(2h_2(b) - h_1^2(b)\right). \tag{2.3}$$

Letting $F_2 = 2f_2 - f_1^2$, $G_2 = 2g_2 - g_1^2$ and $H_2 = 2h_2 - h_1^2$ in (2.3), we arrive at

$$F_2(ab) = G_2(a)b + aH_2(b), \quad a, b \in \mathcal{A},$$

which means that F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. If we assume that $F_1 = f_1$, $G_1 = g_1$ and $H_1 = h_1$, then we have $f_2 = \frac{1}{2}F_1^2 + \frac{1}{2}F_2$, $g_2 = \frac{1}{2}G_1^2 + \frac{1}{2}G_2$ and $h_2 = \frac{1}{2}H_1^2 + \frac{1}{2}H_2$. Indeed, we characterize f_2 by F_1 and F_2, where F_1 is a $\{L_{G_1}, R_{H_1}\}$-derivation and F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. By a process similar to the one described above, we achieve that $f_3 = \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3$, $g_3 = \frac{1}{6}G_1^3 + \frac{1}{6}G_1G_2 + \frac{1}{3}G_2G_1 + \frac{1}{3}G_3$ and $h_3 = \frac{1}{6}H_1^3 + \frac{1}{6}H_1H_2 + \frac{1}{3}H_2H_1 + \frac{1}{3}H_3$, where F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation on $\mathcal{A}$

for $i \in \{1,2,3\}$. Thus, we can inductively construct a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations characterizing a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$. This inductive method leads us to this idea that every higher $\{L_{g_n}, R_{h_n}\}$-derivation is characterized by a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations. The same is also true for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g,h\}$-derivations. In the following, we show that the characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations is not necessarily unique. In view of the above discussion, if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation with $f_0 = g_0 = h_0 = I$, then we have $f_2 = \frac{1}{2}F_1^2 + \frac{1}{2}F_2$, where $F_1 = f_1$ and F_2 is a $\{L_{G_2}, R_{H_2}\}$-derivation. But we can also characterize the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ in other form. We know that $f_1(ab) = g_1(a)b + ah_1(b)$ for all $a, b \in \mathcal{A}$. Therefore,

$$\begin{aligned} f_1^2(ab) &= f_1\left(g_1(a)b + ah_1(b)\right) \\ &= g_1^2(a)b + g_1(a)h_1(b) + g_1(a)h_1(b) + ah_1^2(b) \\ &= g_1^2(a)b + 2g_1(a)h_1(b) + ah_1^2(b). \end{aligned}$$

Thus,

$$g_1(a)h_1(b) = \frac{1}{2}\left(f_1^2(ab) - g_1^2(a)b - ah_1^2(b)\right), \quad a, b \in \mathcal{A}. \tag{2.4}$$

Also, we know that $f_2(ab) = g_2(a)b + g_1(a)h_1(b) + ah_2(b)$ for all $a, b \in \mathcal{A}$. Putting (2.4) in the previous equation, we deduce that $f_2(ab) = g_2(a)b + \frac{1}{2}f_1^2(ab) - \frac{1}{2}g_1^2(a)b - \frac{1}{2}ah_1^2(b) + ah_2(b)$ for all $a, b \in \mathcal{A}$. Hence, we have

$$f_2(ab) - \frac{1}{2}f_1^2(ab) = \left(g_2(a) - \frac{1}{2}g_1^2(a)\right)b + a\left(h_2(b) - \frac{1}{2}h_1^2(b)\right), \tag{2.5}$$

for all $a, b \in \mathcal{A}$. Letting $\mathfrak{F}_2 = f_2 - \frac{1}{2}f_1^2$, $\mathfrak{G}_2 = g_2 - \frac{1}{2}g_1^2$ and $\mathfrak{H}_2 = h_2 - \frac{1}{2}h_1^2$ in (2.5), we arrive at

$$\mathfrak{F}_2(ab) = \mathfrak{G}_2(a)b + a\mathfrak{H}_2(b), \quad a, b \in \mathcal{A}.$$

Thus, $\mathfrak{F}_2$ is a $\{L_{\mathfrak{G}_2}, R_{\mathfrak{H}_2}\}$-derivation. So, we have $f_2 = \frac{1}{2}f_1^2 + \mathfrak{F}_2$, $g_2 = \frac{1}{2}g_1^2 + \mathfrak{G}_2$ and $h_2 = \frac{1}{2}h_1^2 + \mathfrak{H}_2$. The above expressions show that the term f_2 is characterized by f_1 and $\mathfrak{F}_2$, where f_1 is a $\{L_{g_1}, R_{h_1}\}$-derivation and $\mathfrak{F}_2$ is a $\{L_{\mathfrak{G}_2}, R_{\mathfrak{H}_2}\}$-derivation. Using the above method and doing more calculations, we get

$$\begin{aligned} \left(f_3 - \frac{1}{6}f_1^3 - f_1\mathfrak{F}_2\right)(ab) &= \left(g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2\right)(a)b + a\left(h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2\right)(b) \\ &= \left(h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2\right)(a)b + a\left(g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2\right)(b). \end{aligned}$$

Letting $\mathfrak{F}_3 = f_3 - \frac{1}{6}f_1^3 - f_1\mathfrak{F}_2$, $\mathfrak{G}_3 = g_3 - \frac{1}{6}g_1^3 - g_1\mathfrak{G}_2$ and $\mathfrak{H}_3 = h_3 - \frac{1}{6}h_1^3 - h_1\mathfrak{H}_2$, it is observed that $\mathfrak{F}_3$ is a $\{L_{\mathfrak{G}_3}, R_{\mathfrak{H}_3}\}$-derivation. Thus, we see that the terms f_3, g_3 and h_3 are characterized as follows:

$$\begin{cases} f_3 = \frac{1}{6}f_1^3 + f_1\mathfrak{F}_2 + \mathfrak{F}_3, \\ g_3 = \frac{1}{6}g_1^3 + g_1\mathfrak{G}_2 + \mathfrak{G}_3, \\ h_3 = \frac{1}{6}h_1^3 + h_1\mathfrak{H}_2 + \mathfrak{H}_3. \end{cases}$$

The aforementioned discussion demonstrates that the characterization of higher $\{L_{g_n}, R_{h_n}\}$-derivations is not necessarily unique. Therefore, one can think that if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation with $f_0 = g_0 = h_0 = I$, then there exist two sequences of $\{L_{G_n}, R_{H_n}\}$-derivations and $\{L_{\mathfrak{G}_n}, R_{\mathfrak{H}_n}\}$-derivations characterizing the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. The same is also valid for higher $\{R_{g_n}, L_{h_n}\}$-derivations and higher $\{g_n, h_n\}$-derivations. In particular, if $\{d_n\}_{n=0,1,\dots}$ with $d_0 = I$ is a higher derivation on $\mathcal{A}$, we can obtain two sequences $\{\delta_n\}_{n=0,1,\dots}$ and $\{\Delta_n\}_{n=0,1,\dots}$ of derivations on $\mathcal{A}$ characterizing $\{d_n\}$.

We begin our results with the following lemma which will be used extensively to prove the main theorem of this article. The following lemma has been motivated by [11].

Lemma 2.1. *Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases} (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1} f_{n-k}, \\ (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}, \end{cases}$$

for each nonnegative integer n. The same is also true for higher $\{g_n, h_n\}$-derivations.

Proof. Using induction on n, we prove this lemma. Let $n = 0$. We know that $f_1(ab) = g_1(a)b + ah_1(b)$ for all $a, b \in \mathcal{A}$. Thus, if $F_1 = f_1$, $G_1 = g_1$ and $H_1 = h_1$, then F_1 is a $\{L_{G_1}, R_{H_1}\}$-derivation on $\mathcal{A}$ and further, $(0+1)f_{0+1} = \sum_{k=0}^{0} F_{k+1} f_{0-k}$, $(0+1)g_{0+1} = \sum_{k=0}^{0} G_{k+1} g_{0-k}$ and $(0+1)h_{0+1} = \sum_{k=0}^{0} H_{k+1} h_{0-k}$. As induction assumption, suppose that F_k is a $\{L_{G_k}, R_{H_k}\}$-derivation for any $k \le n$ and further

$$\begin{cases} (r+1)f_{r+1} = \sum_{k=0}^{r} F_{k+1} f_{r-k}, \\ (r+1)g_{r+1} = \sum_{k=0}^{r} G_{k+1} g_{r-k}, \\ (r+1)h_{r+1} = \sum_{k=0}^{r} H_{k+1} h_{r-k}, \end{cases}$$

for $r = 0, 1, \dots, n-1$. Put $F_{n+1} = (n+1)f_{n+1} - \sum_{k=0}^{n-1} F_{k+1} f_{n-k}$, $G_{n+1} = (n+1)g_{n+1} - \sum_{k=0}^{n-1} G_{k+1} g_{n-k}$ and $H_{n+1} = (n+1)h_{n+1} - \sum_{k=0}^{n-1} H_{k+1} h_{n-k}$. Our next task is to show that F_{n+1} is a $\{L_{G_{n+1}}, R_{H_{n+1}}\}$-derivation on $\mathcal{A}$. For $a, b \in \mathcal{A}$, we have

$$\begin{aligned} F_{n+1}(ab) &= (n+1)f_{n+1}(ab) - \sum_{k=0}^{n-1} F_{k+1} f_{n-k}(ab) \\ &= (n+1)\sum_{k=0}^{n+1} g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right). \end{aligned}$$

So, we have

$$\begin{aligned} F_{n+1}(ab) &= \sum_{k=0}^{n+1}(n+1)g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right) \\ &= \sum_{k=0}^{n+1}(k+n+1-k)g_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1} F_{k+1}\left(\sum_{l=0}^{n-k} g_l(a)h_{n-k-l}(b)\right). \end{aligned}$$

Since F_k is a $\{L_{G_k}, R_{H_k}\}$-derivation for each $k = 1, 2, \ldots, n$,

$$\begin{aligned}F_{n+1}(ab) = &\sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) + \sum_{k=0}^{n+1} g_k(a)(n+1-k)h_{n+1-k}(b)\\ &- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} \left[G_{k+1}\left(g_l(a)\right)h_{n-k-l}(b) + g_l(a)H_{k+1}\left(h_{n-k-l}(b)\right)\right].\end{aligned}$$

Letting

$$\begin{aligned}G &= \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} G_{k+1}\left(g_l(a)\right)h_{n-k-l}(b),\\ H &= \sum_{k=0}^{n+1} g_k(a)(n+1-k)h_{n+1-k}(b) - \sum_{k=0}^{n-1}\sum_{l=0}^{n-k} g_l(a)H_{k+l}\left(h_{n-k-l}(b)\right),\end{aligned}$$

we have $F_{n+1}(ab) = G+H$. Here, we compute G and H. In the summation $\sum_{k=0}^{n-1}\sum_{l=0}^{n-k}$, we have $0 \leq k+l \leq n$ and $k \neq n$. Thus if we put $r = k+l$, then we can write it as the form $\sum_{r=0}^{n}\sum_{k+l=r, k\neq n}$. Putting $l = r-k$, we find that

$$\begin{aligned}G &= \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n}\sum_{0\leq k\leq r, k\neq n} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b)\\ &= \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b) - \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b.\end{aligned}$$

It means that

$$G + \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b = \sum_{k=0}^{n+1} kg_k(a)h_{n+1-k}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b).$$

Putting $r+1$ instead of k in the first summation of above, we have

$$\begin{aligned}&G + \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)b\\ &= \sum_{r=0}^{n}(r+1)g_{r+1}(a)h_{n-r}(b) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)h_{n-r}(b)\\ &= \sum_{r=0}^{n-1}\left[(r+1)g_{r+1}(a) - \sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)\right]h_{n-r}(b) + (n+1)g_{n+1}(a)b.\end{aligned}$$

According to the induction hypothesis, $(r+1)g_{r+1}(a) = \sum_{k=0}^{r} G_{k+1}\left(g_{r-k}(a)\right)$ for $r = 0, \ldots, n-1$. So, it is obtained that

$$G = \left[(n+1)g_{n+1}(a) - \sum_{k=0}^{n-1} G_{k+1}\left(g_{n-k}(a)\right)\right]b = G_{n+1}(a)b.$$

Like above, we achieve that

$$H = a\left[(n+1)h_{n+1}(b) - \sum_{k=0}^{n-1} H_{k+1}\left(h_{n-k}(b)\right)\right] = aH_{n+1}(b).$$

Therefore, we have $F_{n+1}(ab) = G + H = G_{n+1}(a)b + aH_{n+1}(b)$. $\square$

Example 2.1. Using Lemma 2.1, the first five terms of a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ are as follows:

$$\begin{aligned}
f_0 =& I,\\
f_1 =& F_1,\\
2f_2 =& F_1f_1 + F_2f_0 = F_1F_1 + F_2,\\
f_2 =& \frac{1}{2}F_1^2 + \frac{1}{2}F_2,\\
3f_3 =& F_1f_2 + F_2f_1 + F_3f_0 = F_1(\frac{1}{2}F_1^2 + \frac{1}{2}F_2) + F_2F_1 + F_3,\\
f_3 =& \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3,\\
4f_4 =& F_1f_3 + F_2f_2 + F_3f_1 + F_4f_0\\
=& F_1\left(\frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3\right) + F_2\left(\frac{1}{2}F_1^2 + \frac{1}{2}F_2\right) + F_3F_1 + F_4,\\
f_4 =& \frac{1}{24}F_1^4 + \frac{1}{24}F_1^2F_2 + \frac{1}{12}F_1F_2F_1 + \frac{1}{12}F_1F_3 + \frac{1}{8}F_2F_1^2 + \frac{1}{8}F_2^2 + \frac{1}{4}F_3F_1 + \frac{1}{4}F_4.
\end{aligned}$$

We are now in a position to present the first main theorem of this article.

Theorem 2.1. *Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases}
f_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) F_{r_1}\cdots F_{r_i}\right),\\
g_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) G_{r_1}\cdots G_{r_i}\right),\\
h_n = \sum_{i=1}^n \left(\sum_{\sum_{j=1}^i r_j = n} \left(\prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}\right) H_{r_1}\cdots H_{r_i}\right),
\end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^i r_j = n$. The same is also valid for higher $\{g_n, h_n\}$-derivations.

Proof. Let $\{f_n\}$ be a higher $\{L_{g_n}, R_{h_n}\}$-derivation. We first show that if f_n, g_n and h_n are of the above forms, then they satisfy the recursive relations of Lemma 2.1. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation, we put $a_{r_1,\ldots,r_i} = \prod_{j=1}^i \frac{1}{r_j+\cdots+r_i}$. Note that if $r_1 + \cdots + r_i = n+1$, then $(n+1)a_{r_1,\ldots,r_i} = a_{r_2,\ldots,r_i}$. Furthermore, $a_{n+1} = \frac{1}{n+1}$. According to the aforementioned assumptions, we have

$$\begin{aligned} f_{n+1} &= \sum_{i=2}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} a_{r_1,\dots,r_i} F_{r_1} \cdots F_{r_i} \right) + a_{n+1} F_{n+1} \\ &= \sum_{i=2}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} a_{r_1,\dots,r_i} F_{r_1} \cdots F_{r_i} \right) + \frac{F_{n+1}}{n+1}. \end{aligned}$$

So,

$$\begin{aligned} (n+1) f_{n+1} &= \sum_{i=2}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1) a_{r_1,\dots,r_i} F_{r_1} \cdots F_{r_i} \right) + F_{n+1} \\ &= \sum_{i=2}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} a_{r_2,\dots,r_i} F_{r_1} \cdots F_{r_i} \right) + F_{n+1} \\ &= \sum_{i=2}^{n+1} \left(\sum_{r_1=1}^{n+2-i} F_{r_1} \sum_{\sum_{j=2}^{i} r_j = n+1-r_1} a_{r_2,\dots,r_i} F_{r_2} \cdots F_{r_i} \right) + F_{n+1} \\ &= \sum_{r_1=1}^{n} F_{r_1} \sum_{i=2}^{n-(r_1-1)} \left(\sum_{\sum_{j=2}^{i} r_j = n-(r_1-1)} a_{r_2,\dots,r_i} F_{r_2} \cdots F_{r_i} \right) + F_{n+1} \\ &= \sum_{r_1=1}^{n} F_{r_1} f_{n-(r_1-1)} + F_{n+1} \\ &= \sum_{k=0}^{n} F_{k+1} f_{n-k}. \end{aligned}$$

Reasoning like above, we get that

$$\begin{cases} (n+1) g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1) h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}, \end{cases}$$

for each nonnegative integer n. Putting $n+1=m$, we find that

$$m f_m = \sum_{k=0}^{m-1} F_{k+1} f_{m-1-k} = \sum_{k=0}^{m-2} F_{k+1} f_{m-1-k} + F_m,$$

and consequently

$$F_m = m f_m - \sum_{k=0}^{m-2} F_{k+1} f_{m-1-k}.$$

Similarly, we have

$$\begin{cases} G_m = mg_m - \sum_{k=0}^{m-2} G_{k+1}g_{m-1-k}, \\ H_m = mh_m - \sum_{k=0}^{m-2} H_{k+1}h_{m-1-k}. \end{cases}$$

Therefore, we can define $F_n, G_n, H_n : \mathcal{A} \to \mathcal{A}$ by $F_0 = G_0 = H_0 = 0$ and

$$\begin{cases} F_n = nf_n - \sum_{k=0}^{n-2} F_{k+1}f_{n-1-k}, \\ G_n = ng_n - \sum_{k=0}^{n-2} G_{k+1}g_{n-1-k}, \\ H_n = nh_n - \sum_{k=0}^{n-2} H_{k+1}h_{n-1-k}, \end{cases}$$

for each positive integer n. It follows from Lemma 2.1 that $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations. In addition, we prove that if f_n, g_n and h_n have the forms

$$\begin{cases} (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1}f_{n-k}, \\ (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1}g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1}h_{n-k}, \end{cases}$$

where $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations, then $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. To see this, we use induction on n. For $n = 0$, we have $f_0(ab) = ab = g_0(a)h_0(b)$. As the inductive hypothesis, assume that

$$f_k(ab) = \sum_{i=0}^{k} g_i(a)h_{k-i}(b), \quad \text{for } k \leq n.$$

Therefore, we have

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{k=0}^{n} F_{k+1}f_{n-k}(ab) \\ &= \sum_{k=0}^{n} F_{k+1} \sum_{i=0}^{n-k} g_i(a)h_{n-k-i}(b) \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} G_{k+1}g_{n-k-i}(a) \right) h_i(b) + \sum_{i=0}^{n} g_i(a) \left(\sum_{k=0}^{n-i} H_{k+1}h_{n-k-i}(b) \right). \end{aligned}$$

According to the above-mentioned recursive relations, we continue the previous expressions as follows:

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{i=0}^{n}(n-i+1)g_{n-i+1}(a)h_i(b) + \sum_{i=0}^{n} g_i(a)(n-i+1)h_{n-i+1}(b) \\ &= \sum_{i=1}^{n+1} ig_i(a)h_{n+1-i}(b) + \sum_{i=0}^{n}(n-i+1)g_i(a)h_{n+1-i}(b) \\ &= (n+1)\sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b), \end{aligned}$$

which means that $f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b)$. Thus, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ which is characterized by the sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations. The same can be proved for higher $\{g_n, h_n\}$-derivations. □

In the next example, using the above theorem, we characterize term f_4 of a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$.

Example 2.2. We compute the coefficients $a_{r_1,\ldots,r_i}$ for the case $n = 4$. First, note that $4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1$. Based on the definition of $a_{r_1,\ldots,r_i}$ we have

$$\begin{aligned}
a_4 &= \frac{1}{4},\\
a_{1,3} &= \frac{1}{1+3}\cdot\frac{1}{3} = \frac{1}{12},\\
a_{3,1} &= \frac{1}{3+1}\cdot\frac{1}{1} = \frac{1}{4},\\
a_{2,2} &= \frac{1}{2+2}\cdot\frac{1}{2} = \frac{1}{8},\\
a_{1,1,2} &= \frac{1}{1+1+2}\cdot\frac{1}{1+2}\cdot\frac{1}{2} = \frac{1}{24},\\
a_{1,2,1} &= \frac{1}{1+2+1}\cdot\frac{1}{2+1}\cdot\frac{1}{1} = \frac{1}{12},\\
a_{2,1,1} &= \frac{1}{2+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{8},\\
a_{1,1,1,1} &= \frac{1}{1+1+1+1}\cdot\frac{1}{1+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{24}.
\end{aligned}$$

Therefore, f_4, g_4 and h_4 are characterized as follows:

$$\begin{aligned}
f_4 =& \frac{1}{4}F_4 + \frac{1}{12}F_1F_3 + \frac{1}{4}F_3F_1 + \frac{1}{8}F_2F_2 + \frac{1}{24}F_1F_1F_2 + \frac{1}{12}F_1F_2F_1\\
&+ \frac{1}{8}F_2F_1F_1 + \frac{1}{24}F_1F_1F_1F_1,\\
g_4 =& \frac{1}{4}G_4 + \frac{1}{12}G_1G_3 + \frac{1}{4}G_3G_1 + \frac{1}{8}G_2G_2 + \frac{1}{24}G_1G_1G_2 + \frac{1}{12}G_1G_2G_1\\
&+ \frac{1}{8}G_2G_1G_1 + \frac{1}{24}G_1G_1G_1G_1,\\
h_4 =& \frac{1}{4}H_4 + \frac{1}{12}H_1H_3 + \frac{1}{4}H_3H_1 + \frac{1}{8}H_2H_2 + \frac{1}{24}H_1H_1H_2 + \frac{1}{12}H_1H_2H_1\\
&+ \frac{1}{8}H_2H_1H_1 + \frac{1}{24}H_1H_1H_1H_1.
\end{aligned}$$

Corollary 2.1. *Let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then there is a sequence $\{F_n\}$ of $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that*

$$\begin{cases}
(i) & (n+1)f_{n+1} = \sum_{k=0}^{n} F_{k+1}f_{n-k},\\
(ii) & (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1}g_{n-k},\\
(iii) & (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1}h_{n-k},
\end{cases}$$

for each nonnegative integer n. Furthermore, we have

$$\begin{cases} (iv) \quad f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ (v) \quad g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ (vi) \quad h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

Proof. According to Lemma 2.1, if $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ satisfying recursive relations (i)-(vi). On the other hand, $\{f_n\}$ is a higher $\{R_{g_n}, L_{h_n}\}$-derivation on $\mathcal{A}$. Hence, there is a sequence $\{\mathfrak{F}_n\}$ of $\{R_{\mathfrak{G}_n}, L_{\mathfrak{H}_n}\}$-derivations on $\mathcal{A}$ satisfying all the equations of (i)-(vi). But, we know that the solution of the recursive relations is unique. Therefore, we infer that $F_n = \mathfrak{F}_n$, $G_n = \mathfrak{G}_n$ and $H_n = \mathfrak{H}_n$ for all positive integers n. □

In [12], Mirzavaziri and Tehrani presented a characterization of generalized higher derivations. They defined a generalized higher derivation as follows. A sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a generalized higher derivation if there exists a higher derivation $\{d_n\}$ on $\mathcal{A}$ such that $f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a) d_k(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. In fact, they assume that each generalized higher derivation is dependent on a higher derivation. In the following corollary, we show that this assumption is unnecessary.

Corollary 2.2. *Let $\{f_n\}$ be a higher $\{L_{f_n}, R_{d_n}\}$-derivation (resp. higher $\{R_{f_n}, L_{d_n}\}$-derivation) on an algebra $\mathcal{A}$ with $f_0 = d_0 = I$. Then there is a sequence $\{F_n\}$ of $\{L_{F_n}, R_{D_n}\}$-derivations (resp. $\{R_{F_n}, L_{D_n}\}$-derivations) on $\mathcal{A}$ such that*

$$\begin{cases} (n+1) f_{n+1} = \sum_{k=0}^{n} F_{k+1} f_{n-k}, \\ (n+1) d_{n+1} = \sum_{k=0}^{n} D_{k+1} d_{n-k}, \end{cases}$$

for each nonnegative integer n. Furthermore, we have

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) D_{r_1} \cdots D_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$.

We are now going to give an example of a generalized higher derivation that does not depend on a higher derivation.

Example 2.3. Let $\mathcal{R}$ be a ring and let

$$\mathfrak{R} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \ : \ a, b, c \in \mathcal{R} \right\}.$$

Clearly, $\mathfrak{R}$ is a ring. Define the additive mappings $f, d : \mathfrak{R} \to \mathfrak{R}$ by

$$f\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}.$$

It is routine to see that

$$f(AB) = f(A)B + Ad(B), \quad A, B \in \mathfrak{R},$$

which means that f is an l-generalized derivation associated with d in which d is not a derivation. Define $f_n = \frac{f^n}{n!}$ and $d_n = \frac{d^n}{n!}$ for each nonnegative integer n with $f^0 = d^0 = I$. A straightforward verification shows that $f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b)$ for each nonnegative integer n, while $\{d_n\}$ is not a higher derivation.

Theorem 2.2. *Let $\{f_n\}$ be a sequence of linear mappings satisfying*

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for each positive integer n with $f_0 = I$, where F_n is a $\{L_{G_n}, R_{H_n}\}$-derivation (resp. $\{R_{G_n}, L_{H_n}\}$-derivation) for each positive integer n. Then there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings such that

$$\begin{cases} g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

for each positive integer n with $g_0 = h_0 = I$, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$ and furthermore, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation (resp. higher $\{R_{g_n}, L_{h_n}\}$-derivation) on $\mathcal{A}$.

Proof. We use induction on n. Suppose that if

$$f_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for $1 \leq k \leq n$, where F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation for each $i \leq k$, then there exist the linear mappings g_k and h_k such that

$$\begin{cases} g_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_k = \sum_{i=1}^{k} \left(\sum_{\sum_{j=1}^{i} r_j = k} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

with $g_0 = h_0 = I$ and further $f_k(ab) = \sum_{i=0}^{k} g_{k-i}(a)h_i(b)$ for all $a, b \in \mathcal{A}$. Based on the assumption, we have the following equation:

$$f_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

in which F_i is a $\{L_{G_i}, R_{H_i}\}$-derivation for each $1 \leq i \leq n+1$. Now, we define

$$\begin{cases} g_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_{n+1} = \sum_{i=1}^{n+1} \left(\sum_{\sum_{j=1}^{i} r_j = n+1} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right). \end{cases}$$

It follows from the proof of Theorem 2.1 that g_{n+1} and h_{n+1} satisfy the following recursive relations:

$$\begin{cases} (n+1)g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1)h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}. \end{cases}$$

Our next task is to show that $f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a)h_{n+1-i}(b)$ for all $a, b \in \mathcal{A}$. Reusing the proof of Theorem 2.1, we have $(n+1)f_{n+1}(ab) = \sum_{k=0}^{n} F_{k+1} f_{n-k}(ab)$ for all $a, b \in \mathcal{A}$. Therefore,

$$\begin{aligned} (n+1)f_{n+1}(ab) &= \sum_{k=0}^{n} F_{k+1} f_{n-k}(ab) \\ &= \sum_{k=0}^{n} F_{k+1} \sum_{i=0}^{n-1} g_i(a) h_{n-k-i}(b) \\ &= \sum_{i=0}^{n} \left(\sum_{k=0}^{n-i} G_{k+1} g_{n-k-i}(a) \right) h_i(b) + \sum_{i=0}^{n} g_i(a) \left(\sum_{k=0}^{n-i} H_{k+1} h_{n-k-i}(b) \right) \\ &= \sum_{i=0}^{n} (n-i+1) g_{n-i+1}(a) h_i(b) + \sum_{i=0}^{n} (n-i+1) g_i(a) h_{n-i+1}(b) \\ &= \sum_{i=1}^{n+1} i g_i(a) h_{n+1-i}(b) + \sum_{i=0}^{n} (n-i+1) g_i(a) h_{n-i+1}(b) \\ &= \sum_{i=0}^{n+1} (n+1) g_i(a) h_{n+1-i}(b), \end{aligned}$$

which means that

$$f_{n+1}(ab) = \sum_{i=0}^{n+1} g_i(a) h_{n+1-i}(b).$$

Thereby, our proof is complete. □

Corollary 2.3. *Let $\{f_n\}$ be a sequence of linear mappings satisfying*

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

for each positive integer n with $f_0 = I$, where F_n is a generalized derivation associated with a linear mapping D_n for each positive integer n. Then there exists a sequence $\{d_n\}$ of linear mappings such that

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) D_{r_1} \cdots D_{r_i} \right), \tag{2.6}$$

for each positive integer n with $d_0 = I$, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$ and furthermore, $\{f_n\}$ is a generalized higher derivation associated with the sequence $\{d_n\}$.

For instance, let F_i be a generalized derivation associated with a linear mapping D_i for $i \in \{1, 2, 3\}$ on $\mathcal{A}$ and let $f_3 = \frac{1}{6}F_1^3 + \frac{1}{6}F_1F_2 + \frac{1}{3}F_2F_1 + \frac{1}{3}F_3$. So we have the following calculations:

$$\begin{aligned} f_3(ab) = &\left(\frac{1}{6}F_1^3(a) + \frac{1}{6}F_1F_2(a) + \frac{1}{3}F_2F_1(a) + \frac{1}{3}F_3(a)\right) b \\ &+ \left(\frac{1}{2}F_1^2(a) + \frac{1}{2}F_2(a)\right) d_1(b) + f_1(a) \left(\frac{1}{2}D_1^2(b) + \frac{1}{2}D_2(b)\right) \\ &+ a \left(\frac{1}{6}D_1^3(a) + \frac{1}{6}D_1D_2(a) + \frac{1}{3}D_2D_1(a) + \frac{1}{3}D_3(a)\right), \end{aligned}$$

for all $a, b \in \mathcal{A}$. Considering $d_2 = \frac{1}{2}D_1^2 + \frac{1}{2}D_2$ and $d_3 = \frac{1}{6}D_1^3 + \frac{1}{6}D_1D_2 + \frac{1}{3}D_2D_1 + \frac{1}{3}D_3$, we see that

$$f_3(ab) = f_3(a)b + f_2(a)d_1(b) + f_1(a)d_2(b) + ad_3(b) = \sum_{k=0}^{3} f_{3-k}(a)d_k(b).$$

This leads us to the sequence $\{d_n\}$ satisfying (2.6) and further

$$f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b).$$

In the following, there are some immediate consequences of the previous results. Before it, recall that a sequence $\{f_n\}$ of linear mappings on $\mathcal{A}$ is called a Jordan higher $\{g_n, h_n\}$-derivation if there exist two sequences $\{g_n\}$ and $\{h_n\}$ of linear mappings on $\mathcal{A}$ such that $f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b)$ holds for each $a, b \in \mathcal{A}$ and each nonnegative integer n. Since the Jordan product is commutative, we have

$$f_n(a \circ b) = f_n(b \circ a) = \sum_{k=0}^{n} g_{n-k}(b) \circ h_k(a) = \sum_{k=0}^{n} g_k(b) \circ h_{n-k}(a) = \sum_{k=0}^{n} h_{n-k}(a) \circ g_k(b).$$

So, it is observed that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation, then

$$f_n(a \circ b) = \sum_{k=0}^{n} g_{n-k}(a) \circ h_k(b) = \sum_{k=0}^{n} h_{n-k}(a) \circ g_k(b),$$

for all $a, b \in \mathcal{A}$.

Corollary 2.4. *Let $\{f_n\}$ be a Jordan higher $\{g_n, h_n\}$-derivation on a semiprime algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation.*

Proof. Using the proof of Theorem 2.1, we can show that if $\{f_n\}$ is a Jordan higher $\{g_n, h_n\}$-derivation on an algebra $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$, then there exists a sequence $\{F_n\}$ of Jordan $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Since $\mathcal{A}$ is a semiprime algebra, [1, Theorem 4.3] proves the corollary. $\square$

In the following, $\mathcal{A} \otimes \mathcal{S}$ denotes the tensor product of two algebras $\mathcal{A}$ and $\mathcal{S}$, where both $\mathcal{A}$ and $\mathcal{S}$ are defined over a field $\mathbb{F}$ of characteristic zero. We know that the tensor product of two vector spaces V and W over a field $\mathbb{F}$ is also a vector space over $\mathbb{F}$.

Corollary 2.5. *Let $\mathcal{A}$ be a semiprime and $\mathcal{S}$ be a commutative algebra, and let $\{f_n\}$ be a Jordan higher $\{g_n, h_n\}$-derivation of $\mathcal{A} \otimes \mathcal{S}$ with $f_0 = g_0 = h_0 = I$. Then $\{f_n\}$ is a higher $\{g_n, h_n\}$-derivation.*

Proof. As stated above, for a Jordan higher $\{g_n, h_n\}$-derivation $\{f_n\}$ of $\mathcal{A} \otimes \mathcal{S}$ with $f_0 = g_0 = h_0 = I$ there exists a sequence $\{F_n\}$ of Jordan $\{G_n, H_n\}$-derivations on the algebra $\mathcal{A} \otimes \mathcal{S}$ such that

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Theorems 3.1 and 4.3 of [1] together show that every Jordan $\{g, h\}$-derivation of the tensor product of a semiprime and a commutative algebra is a $\{g, h\}$-derivation. This fact along with the above-mentioned characterization of $\{f_n\}$ implies that the Jordan higher $\{g_n, h_n\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$ is a higher $\{g_n, h_n\}$-derivation. $\square$

Corollary 2.6. *Let $\mathcal{A}$ be a semiprime and $\mathcal{S}$ be a commutative algebra, and let $\{d_n\}$ be a Jordan higher derivation of $\mathcal{A} \otimes \mathcal{S}$ with $d_0 = I$. Then $\{d_n\}$ is a higher derivation.*

Proof. This is an immediate consequence of [1, Corollary 4.4] and [11, Theorem 2.3]. $\square$

The importance of Corollary 2.5 and 2.6 is that the algebra $\mathcal{A} \otimes \mathcal{S}$ is not semiprime if $\mathcal{S}$ is not semiprime. On the other hand, even the tensor product of semiprime algebras is not always semiprime. So, we are presenting a characterization of higher $\{g_n, h_n\}$-derivations on some algebras which maybe are not semiprime.

Remark 2.1. We know that the notion of a Jordan $\{g, h\}$-derivation is a generalization of Jordan generalized derivations (see Introduction). A sequence $\{f_n\}$ of linear mappings on an algebra $\mathcal{A}$ is called a Jordan generalized higher derivation if there exists a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ such that $f_n(a \circ b) = \sum_{k=0}^{n} f_{n-k}(a) \circ d_k(b)$ for all $a, b \in \mathcal{A}$. So, Corollaries 2.4 and 2.5 are also valid for Jordan generalized higher derivations.

Motivated by [11, Theorem 2.5], we prove the following theorem.

Theorem 2.3. *Let $\mathfrak{f}$ be the set of all higher $\{L_{g_n}, R_{h_n}\}$-derivations $\{f_n\}_{n=0,1,\dots}$ on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$ and $\mathfrak{F}$ be the set of all sequences $\{F_n\}_{n=0,1,\dots}$ of $\{L_{G_n}, R_{H_n}\}$-derivations on $\mathcal{A}$ with $F_0 = G_0 = H_0 = 0$. Then there is a one to one correspondence between $\mathfrak{f}$ and $\mathfrak{F}$. The same is also valid for higher $\{g_n, h_n\}$-derivations.*

Proof. Let $\{f_n\} \in \mathfrak{f}$. We are going to obtain a sequence $\{F_n\}_{n=0,1,\dots}$ of $\{L_{G_n}, R_{H_n}\}$-derivations with $F_0 = G_0 = H_0 = 0$ that characterizes the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. Define $F_n, G_n, H_n : \mathcal{A} \to \mathcal{A}$ by $F_0 = G_0 = H_0 = 0$ and

$$\begin{cases} F_n = nf_n - \sum_{k=0}^{n-2} F_{k+1} f_{n-1-k}, \\ G_n = ng_n - \sum_{k=0}^{n-2} G_{k+1} g_{n-1-k}, \\ H_n = nh_n - \sum_{k=0}^{n-2} H_{k+1} h_{n-1-k}, \end{cases}$$

for each positive integer n. Then it follows from Lemma 2.1 that $\{F_n\}$ is a sequence of $\{L_{G_n}, R_{H_n}\}$-derivations characterizing the higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$. Conversely, suppose that $\{F_n\} \in \mathfrak{F}$ which means that every F_n is a $\{L_{G_n}, R_{H_n}\}$-derivation with $F_0 = G_0 = H_0 = 0$. We will show that there exists a higher $\{L_{g_n}, R_{h_n}\}$-derivation $\{f_n\}$ with $f_0 = g_0 = h_0 = I$ which is characterized by the sequence $\{F_n\}_{n=0,1,\dots}$. We define $f_n, g_n, h_n : \mathcal{A} \to \mathcal{A}$ by $f_0 = g_0 = h_0 = I$ and

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \dots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \dots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \dots + r_i} \right) H_{r_1} \cdots H_{r_i} \right). \end{cases}$$

By Theorem 2.1, f_n, g_n and h_n satisfy the following recursive relations:

$$\begin{cases} (n+1) f_{n+1} = \sum_{k=0}^{n} F_{k+1} f_{n-k}, \\ (n+1) g_{n+1} = \sum_{k=0}^{n} G_{k+1} g_{n-k}, \\ (n+1) h_{n+1} = \sum_{k=0}^{n} H_{k+1} h_{n-k}. \end{cases}$$

Based on the last part of the proof of Theorem 2.1, $\{f_n\}$ is a higher $\{L_{g_n}, R_{h_n}\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. Thus, $\{f_n\} \in \mathfrak{f}$. Now, define $\mathcal{F} : \mathfrak{F} \to \mathfrak{f}$ by

$\mathcal{F}(\{F_n\}) = \{f_n\}$, where

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right). \end{cases}$$

Clearly, $\mathcal{F}$ is a one to one correspondence. This yields the desired result. □

Remark 2.2. Let $\mathcal{A}$ be a unital algebra with the identity element $\mathbf{e}$ and let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on $\mathcal{A}$ with $f_0 = g_0 = h_0 = I$. According to Theorem 2.1, there exists a sequence $\{F_n\}$ of $\{G_n, H_n\}$-derivations on $\mathcal{A}$ such that

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) F_{r_1} \cdots F_{r_i} \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) G_{r_1} \cdots G_{r_i} \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) H_{r_1} \cdots H_{r_i} \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. It follows from [8, Theorem 3.1] that if f is a $\{g, h\}$-derivation on a unital algebra, then f, g and h are generalized derivation associated with the derivation δ. Indeed, we have $f = \delta + L_{f(\mathbf{e})}$, $g = \delta + L_{g(\mathbf{e})}$ and $h = \delta + L_{h(\mathbf{e})}$. Using this fact and that every $\{F_n\}$ is a $\{G_n, H_n\}$-derivation, we deduce that there is a sequence $\{D_n\}$ of derivations such that $F_n = D_n + L_{F_n(\mathbf{e})}$, $G_n = D_n + L_{G_n(\mathbf{e})}$ and $H_n = D_n + L_{H_n(\mathbf{e})}$ for any $n \in \mathbb{N}$. It means that every F_n, G_n and H_n is a generalized derivation associated with the derivation D_n. We can thus infer from [12] that $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are generalized higher derivations. We can see that

$$\begin{cases} f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) (D_{r_1} + L_{F_{r_1}(\mathbf{e})}) \cdots (D_{r_i} + L_{F_{r_i}(\mathbf{e})}) \right), \\ g_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) (D_{r_1} + L_{G_{r_1}(\mathbf{e})}) \cdots (D_{r_i} + L_{G_{r_i}(\mathbf{e})}) \right), \\ h_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) (D_{r_1} + L_{H_{r_1}(\mathbf{e})}) \cdots (D_{r_i} + L_{H_{r_i}(\mathbf{e})}) \right), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$. Easily, we deduce that there is a higher derivation

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \cdots + r_i} \right) D_{r_1} \cdots D_{r_i} \right),$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{i} r_j = n$ on $\mathcal{A}$ such that

$$\begin{cases} f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a) d_k(b), \\ g_n(ab) = \sum_{k=0}^{n} g_{n-k}(a) d_k(b), \\ h_n(ab) = \sum_{k=0}^{n} h_{n-k}(a) d_k(b), \end{cases}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. It follows from [1] that if f is a $\{g, h\}$-derivation on a unital algebra, then $f(\mathbf{e}), g(\mathbf{e}), h(\mathbf{e}) \in Z(\mathcal{A})$. So, we have

$$\begin{aligned} F_n(ab) &= D_n(ab) + L_{F_n(\mathbf{e})}(ab) \\ &= D_n(a)b + aD_n(b) + F_n(\mathbf{e})ab \\ &= D_n(a) + a\left(D_n(b) + F_n(\mathbf{e})b\right) \\ &= D_n(a)b + aF_n(b), \end{aligned}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. Similarly, $G_n(ab) = D_n(a)b + aG_n(b)$ and $H_n(ab) = D_n(a)b + aH_n(b)$ for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. So, one can easily obtain that

$$\begin{cases} f_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)f_k(b), \\ g_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)g_k(b), \\ h_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)h_k(b), \end{cases}$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$.

Proposition 2.1. *Let $\mathfrak{R}$ be a unital ring with the identity element $\boldsymbol{e}$ and let $\{f_n\}$ be a higher $\{g_n, h_n\}$-derivation on $\mathfrak{R}$. Then $f_n(\boldsymbol{e}), g_n(\boldsymbol{e}), h_n(\boldsymbol{e}) \in Z(\mathfrak{R})$ for any nonnegative integer n.*

Proof. Using induction on n, we prove this proposition. According to page 2 of [1], the result is certainly true if $n = 1$. We show that the result is true for $n = 2$. We know that

$$f_2(xy) = g_2(x)y + g_1(x)h_1(y) + xh_2(y) = h_2(x)y + h_1(x)g_1(y) + xg_2(y), \tag{2.7}$$

for all $x, y \in \mathfrak{R}$. Taking $y = \mathbf{e}$ in (2.7), we obtain

$$f_2(x) = g_2(x) + g_1(x)h_1(\mathbf{e}) + xh_2(\mathbf{e}) = h_2(x) + h_1(x)g_1(\mathbf{e}) + xg_2(\mathbf{e}), \tag{2.8}$$

and taking $x = \mathbf{e}$, we get

$$f_2(y) = g_2(\mathbf{e})y + g_1(\mathbf{e})h_1(y) + h_2(y) = h_2(\mathbf{e})y + h_1(\mathbf{e})g_1(y) + g_2(y). \tag{2.9}$$

Comparing (2.8) and (2.9) and using the fact that $h_1(\mathbf{e}), g_1(\mathbf{e}) \in Z(\mathfrak{R})$, we see that $g_2(\mathbf{e}), h_2(\mathbf{e}) \in Z(\mathfrak{R})$ and consequently, $f_2(\mathbf{e}) \in Z(\mathfrak{R})$. As induction hypothesis, assume that the result is true for any $k < n$. We have

$$\begin{aligned} f_n(xy) &= g_n(x)y + g_{n-1}(x)h_1(y) + \cdots + xh_n(y) \\ &= h_n(x)y + h_{n-1}(x)g_1(y) + \cdots + xg_n(y). \end{aligned}$$

Reasoning like above, we have

$$\begin{aligned} f_n(x) &= g_n(x) + g_{n-1}(x)h_1(\mathbf{e}) + \cdots + xh_n(\mathbf{e}) \\ &= h_n(x) + h_{n-1}(x)g_1(\mathbf{e}) + \cdots + xg_n(\mathbf{e}) \end{aligned}$$

and also

$$\begin{aligned} f_n(y) &= g_n(\mathbf{e})y + g_{n-1}(\mathbf{e})h_1(y) + \cdots + h_n(y) \\ &= h_n(\mathbf{e})y + h_{n-1}(\mathbf{e})g_1(y) + \cdots + g_n(y). \end{aligned}$$

Comparing the above equations and using the inductive hypothesis, we get that $g_n(\mathbf{e}), h_n(\mathbf{e}) \in Z(\mathfrak{R})$ and consequently, $f_n(\mathbf{e}) \in Z(\mathfrak{R})$. $\square$

The article ends with the following theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a unital algebra with the identity element $\mathbf{e}$ and let $\{f_n\}$ be a generalized higher derivation associated with a sequence $\{d_n\}$ of linear mappings. Then $\{d_n\}$ is a higher derivation.*

Proof. We use induction to get our goal. The result trivially holds for $n = 1$. Now suppose that $d_k(ab) = \sum_{i=0}^{k} d_{k-i}(a)d_i(b)$ for any $k < n$. We have

$$f_n(ab) = \sum_{k=0}^{n} f_{n-k}(a)d_k(b) = f_n(a)b + ad_n(b) + \sum_{k=1}^{n-1} f_{n-k}(a)d_k(b).$$

Since $\mathcal{A}$ is unital, we get that

$$f_n(b) = f_n(\mathbf{e})b + d_n(b) + \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(b)$$

and consequently, we have

$$d_n(b) = f_n(b) - f_n(\mathbf{e})b - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(b),$$

for all $b \in \mathcal{A}$. Now, we have the following expressions:

$$\begin{aligned}
d_n(ab) =& f_n(ab) - f_n(\mathbf{e})ab - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e})d_k(ab) \\
=& \sum_{k=0}^{n} f_{n-k}(a)d_k(b) - f_n(\mathbf{e})ab - \sum_{k=1}^{n-1} f_{n-k}(\mathbf{e}) \sum_{i=0}^{k} d_{k-i}(a)d_i(b) \\
=& \left[f_n(a) - f_n(\mathbf{e})a - f_{n-1}(\mathbf{e})d_1(a) - \cdots - f_1(\mathbf{e})d_{n-1}(a)\right] b \\
& + \left[f_{n-1}(a) - f_{n-1}(\mathbf{e})a - f_{n-2}(\mathbf{e})d_1(a) - \cdots - f_1(\mathbf{e})d_{n-2}(a)\right] d_1(b) \\
& + \cdots + ad_n(b) \\
=& d_n(a)b + d_{n-1}(a)d_1(b) + \cdots + ad_n(b) \\
=& \sum_{k=0}^{n} d_{n-k}(a)d_k(b).
\end{aligned}$$

It means that $\{d_n\}$ is a higher derivation, as desired. $\square$

Acknowledgements The authors thank the referee for his/her careful reading of the article and suggesting valuable comments that improved the quality of this work. For the second author, this research is supported by the National Board of Higher Mathematics (NBHM), India, Grant No. 02011/16/2020 NBHM (R. P.) R and D II/ 7786.

References

[1] M. Brešar, *Jordan $\{g, h\}$-derivations on tensor products of algebras*, Linear Multilinear Algebra **64**(11) (2016), 2199–2207. https://doi.org/10.1080/03081087.2016.1145184

[2] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **140**(4) (1988), 1003–1006. https://doi.org/10.2307/2047580

[3] M. Brešar and J. Vukman, *Jordan derivations on prime rings*, Bull. Austral. Math. Soc. **37** (1988), 321–322.

[4] J. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), 1104–1110. https://doi.org/10.2307/2040004

[5] A. Fošner, *Generalized higher derivations on algebras*, Ukrainian Math. J. **69** (2018), 1659–1667. https://doi.org/10.1007/s11253-018-1461-8

[6] A. Hosseini and A. Fošner, *Identities related to (σ, τ)-Lie derivations and (σ, τ)-derivations*, Boll. Unione Mat. Ital. **11** (2018), 395–401. https://doi.org/10.1007/s40574-017-0141-1

[7] A. Hosseini, *Characterization of two-sided generalized derivations*, Acta Sci. Math. (Szeged) **86** (2020), 577–600. https://doi.org/10.14232/actasm-020-295-8

[8] A. Hosseini, *A new proof of Singer-Wermer theorem with some results on $\{g, h\}$-derivations*, International Journal of Nonlinear Analysis and Applications **11**(1) (2020), 453–471. https://doi.org/10.22075/IJNAA.2019.17189.1915

[9] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

[10] Y. Li and D. Benkovič. *Jordan generalized derivations on triangular algebras*, Linear Multilinear Algebra **59** (2011), 841–849. https://doi.org/10.1080/03081087.2010.507600

[11] M. Mirzavaziri, *Characterization of higher derivations on algebras*, Comm. Algebra **38**(3) (2010), 75–86. https://doi.org/10.1080/00927870902828751

[12] M. Mirzavaziri and E. O. Tehrani, *Generalized higher derivations are sequences of generalized derivations*, Journal of Advanced Research in Pure Mathematics **3**(1) (2011), 981–987. https://doi.org/10.5373/JARPM.445.052610

[13] J. Vukman, *A note on generalized derivations of semiprime rings*, Taiwanese J. Math. **11** (2007), 367–370. https://doi.org/10.11650/twjm/1500404694

[14] F. Wei and Z. Xiao, *Generalized Jordan derivations on semiprime rings and Its applications in range inclusion problems*, Mediterr. J. Math. **8** (2011), 271–291. https://doi.org/10.1007/s00009-010-0081-9

[1]Department of Mathematics,
Kashmar Higher Education Institute,
Kashmar, Iran
Email address: hosseini.amin82@gmail.com
Email address: a.hosseini@kashmar.ac.ir

[2]Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002 India
Email address: rehman100@gmail.com

[*]Corresponding Author