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LORENTZIAN PARA-SASAKIAN MANIFOLDS AND $*$-RICCI SOLITONS

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Abstract. We study the properties of Lorentzian para-Sasakian manifolds endowed with $*$-Ricci solitons and gradient $*$-Ricci solitons. Finally, the existence of $*$-Ricci soliton on a 4-dimensional Lorentzian para-Sasakian manifold is proved by constructing a non-trivial example.

1. Introduction

A Ricci soliton (g, F, λ) [12] on a semi-Riemannian manifold (M, g) is a generalization of Einstein metric such that

$$\frac{1}{2}\pounds_F g + S + \lambda g = 0,$$

where S is the Ricci tensor, $\pounds_F$ is the Lie derivative operator along the vector field F on M, g represents the semi-Riemannian metric of M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according to λ being less than 0, 0 and greater than 0, respectively.

In 1959, the notion of $*$-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [23] and further studied by Hamada [11] on real hypersurfaces of non-flat complex space forms. A semi-Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton [16] if there exists a smooth vector field F (called soliton vector field) and a real number λ, such that

$$\pounds_F g + 2S^* = -2\lambda g, \tag{1.1}$$

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where

$$S^*(U,V) = g(Q^*U,V) = \text{Trace}\{\phi \circ R(U,\phi V)\},$$

for all vector fields U,V on M [6]. Here, ϕ is the $(1,1)$ tensor field and Q^* is the $(1,1)$ $*$-Ricci operator. If we choose λ as a smooth function in (1.1), then the soliton (g,F,λ) satisfying equation (1.1) is known as an almost $*$-Ricci soliton on M. In this connection, we recommend the papers [4, 10, 13, 15, 17, 21, 22, 24, 25] for more details about the study of Ricci solitons, η-Ricci solitons and $*$-Ricci solitons in the context of contact Riemannian geometry. As far as our knowledge goes, the study of $*$-Ricci solitons in the context of Lorentzian para-Sasakian manifolds is left. The main motive of this article is to fill this gap.

In 1989, K. Matsumoto [18] introduced the notion of $LP-$Sasakian manifolds, while in 1992, the same notion was independently studied by I. Mihai and R. Rosca [19] and they obtained several results on this manifold. The Lorentzian para-Sasakian manifolds have also been studied by various authors such as [1, 2, 7–9, 14, 26] and many others.

We present our work as follows. In Section 2, we collect the basic results and some basic definitions of Lorentzian para-Sasakian manifolds. The $*$-Ricci solitons and gradient $*$-Ricci solitons on Lorentzian para-Sasakian manifolds are discussed in Section 3 and Section 4, respectively. We present a 4-dimensional non-trivial example of Lorentzian para-Sasakian manifold admitting a $*$-Ricci soliton in Section 5.

2. Preliminaries

Let M be an n-dimensional smooth manifold equipped with a quartet (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is the unit timelike vector field, η is a 1-form and a Lorentzian metric g on M such that [5, 20]

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi U) = 0, \quad \text{rank}(\phi) = n-1, \tag{2.2}$$

for all $U \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields of M. The manifold M is said to have an almost para-contact metric structure (ϕ,ξ,η,g) when it admits a Lorentzian metric g, such that

$$g(\phi U,\phi V) = g(U,V) + \eta(U)\eta(V), \quad g(U,\xi) = \eta(U), \tag{2.3}$$

for all $U,V \in \mathfrak{X}(M)$.

If moreover,

$$(\nabla_U\phi)V = \eta(V)\phi^2U + g(\phi U,\phi V)\xi, \tag{2.4}$$

$$\nabla_U\xi = \phi X \Leftrightarrow (\nabla_U\eta)V = g(\phi U,V) = g(U,\phi V), \tag{2.5}$$

where ∇ denotes the Levi-Civita connection of the manifold.

An n-dimensional Lorentzian para-Sasakian manifold satisfies the following relations (see [9]):

$$g(R(U,V)W,\xi) = g(V,W)\eta(U) - g(U,W)\eta(V), \tag{2.6}$$

$$R(U,V)\xi = \eta(V)U - \eta(U)V, \tag{2.7}$$

$$S(U,\xi) = (n-1)\eta(U) \Leftrightarrow Q\xi = (n-1)\xi, \tag{2.8}$$

for all $U,V,W \in \mathfrak{X}(M)$, where R denotes the curvature tensor and S denotes the Ricci tensor of M such that $S(U,V) = g(QU,V)$ for all $U,V \in \mathfrak{X}(M)$.

A Lorentzian para-Sasakian manifold M is said to be a generalized η-Einstein [3] if its non-vanishing Ricci tensor S is of the form

$$S(U,V) = \rho_1 g(U,V) + \rho_2\eta(U)\eta(V) + \rho_3 g(\phi U,V), \tag{2.9}$$

where ρ_1, ρ_2 and ρ_3 are smooth functions on M. If $\rho_3 = 0$ (resp. $\rho_2 = \rho_3 = 0$), then M is called an η-Einstein (resp. Einstein) manifold.

Lemma 2.1. *An n-dimensional Lorentzian para-Sasakian manifold satisfies the following relations*

$$(\nabla_U Q)\xi = (n-1)\phi U - Q\phi U, \tag{2.10}$$

$$(\nabla_\xi Q)U = -2Q\phi U + 2aU + 2a\eta(U)\xi, \tag{2.11}$$

where Q is the Ricci operator.

Proof. Differentiating $Q\xi = (n-1)\xi$ along U and using (2.5), we get (2.10). Next differentiating (2.7) then using (2.5), we find

$$(\nabla_E R)(V,W)\xi = -R(V,W)\phi E + g(\phi E,W)V - g(\phi E,V)W. \tag{2.12}$$

Let $\{e_i\}_{i=1}^n$ be a local orthonormal basis on M. Putting $V = E = e_i$ in (2.12) and summing over i leads to

$$\begin{aligned}\sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} R)(e_i,W)\xi,U) =& S(W,\phi U) + (n-1)g(\phi W,U) \\ &- 2ag(W,U) - 2a\eta(V)\eta(W),\end{aligned} \tag{2.13}$$

where $\epsilon_i = g(e_i,e_i)$ and $a = \operatorname{tr}\phi$. Here tr stands for trace. From Bianchi's second identity, we can easily obtain that

$$\sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} R)(U,\xi)W), e_i) = (\nabla_U S)(\xi,W) - (\nabla_\xi S)(U,W). \tag{2.14}$$

By considering (2.13) in (2.14), equation (2.11) follows. $\square$

On a Lorentzian para-Sasakian manifold (M,ϕ,ξ,η,g), we have the following lemmas.

Lemma 2.2. *On a Lorentzian para-Sasakian manifold (M,ϕ,ξ,η,g), we have*

$$\begin{aligned}\bar{R}(U,V,\phi W,\phi E) =&\bar{R}(U,V,W,E) - g(U,W)g(V,E) + g(V,W)g(U,E)\\ &+ 2[g(V,W)\eta(U)\eta(E) - g(U,W)\eta(V)\eta(E)\\ &+ g(U,E)\eta(V)\eta(W) - g(V,E)\eta(U)\eta(W)]\\ &+ g(U,\phi W)g(V,\phi E) - g(V,\phi W)g(U,\phi E),\end{aligned} \tag{2.15}$$

for any U,V,W,E on M, where $\bar{R}(U,V,W,E) = g(R(U,V)W,E)$.

Proof. By virtue of the well-known definition of curvature tensor, we can write

$$\bar{R}(U,V,\phi W,\phi E) =g(\nabla_U\nabla_V\phi W,\phi E) - g(\nabla_V\nabla_U\phi W,\phi E) - g(\nabla_{[U,V]}\phi W,\phi E). \tag{2.16}$$

By making use of (2.2), (2.4) and (2.5), (2.16) takes the form

$$\begin{aligned}\bar{R}(U,V,\phi W,\phi W) =&g(R(U,V)W,E) + \eta(R(U,V)W)\eta(E)\\ &+ g(V,W)g(\phi U,\phi E) - g(U,W)g(\phi V,\phi E)\\ &+ 2g(U,E)\eta(V)\eta(W) - 2g(V,E)\eta(U)\eta(W)\\ &+ g(U,\phi W)g(V,\phi E) - g(V,\phi W)g(U,\phi E),\end{aligned}$$

which in view of (2.3) and (2.6) leads to (2.15). This completes the proof. □

Lemma 2.3. *The $*$-Ricci tensor of an n-dimensional Lorentzian para-Sasakian manifold (M,ϕ,ξ,η,g) is given by*

$$S^*(V,W) = S(V,W) + (n-2)g(V,W) - g(V,\phi W)a + (2n-3)\eta(V)\eta(W), \tag{2.17}$$

for any $V,W \in \mathfrak{X}(M)$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of the tangent space at each point of the manifold. By the definition of $*$-Ricci tensor, from (2.15), we have

$$\begin{aligned}S^*(V,W) =&\sum_{i=1}^n \epsilon_i\bar{R}(e_i,V,\phi W,\phi e_i)\\ =&\sum_{i=1}^n \epsilon_i\bar{R}(e_i,V,W,e_i) + \sum_{i=1}^n \epsilon_i[g(V,W)g(e_i,e_i) - g(e_i,W)g(V,e_i)]\\ &+ 2\sum_{i=1}^n \epsilon_i[g(V,W)\eta(e_i)\eta(e_i) - g(e_i,W)\eta(V)\eta(e_i)\\ &+ g(e_i,e_i)\eta(V)\eta(W) - g(V,e_i)\eta(e_i)\eta(W)]\\ &+ \sum_{i=1}^n \epsilon_i[g(e_i,\phi W)g(V,\phi e_i) - g(V,\phi W)g(e_i,\phi e_i)],\end{aligned}$$

which leads to (2.17), where $\epsilon_i = g(e_i,e_i)$, i.e., $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{n-1} = 1$, $\epsilon_n = -1$. □

3. Lorentzian Para-Sasakian Manifolds Admitting $*$-Ricci Solitons

In this section, we characterize the properties of Lorentzian para-Sasakian manifold endowed with $*$-Ricci solitons. Now, we prove the following.

Theorem 3.1. *If an n-dimensional Lorentzian para-Sasakian manifold admits a $*$-Ricci soliton (g, F, λ), then the $*$-Ricci soliton is steady.*

Proof. By using (2.17) in (1.1), we have

$$(\pounds_F g)(U,V) = -2S(U,V) - 2[\lambda + (n-2)]g(U,V) - 2(2n-3)\eta(U)\eta(V) + 2g(U,\phi V)a. \tag{3.1}$$

Taking covariant differentiation of (3.1) with respect to W, we get

$$\begin{aligned}(\nabla_W \pounds_F g)(U,V) =& -2(\nabla_W S)(U,V) - 2(2n-3)[g(\phi W,U)\eta(V) \\ &+ g(\phi W,V)\eta(U)] + 2[g(V,W)\eta(U) \\ &+ g(U,W)\eta(V) + 2\eta(U)\eta(V)\eta(W)]a.\end{aligned} \tag{3.2}$$

Following Yano [27], the following formula

$$(\pounds_F \nabla_U g - \nabla_U \pounds_F g - \nabla_{[F,U]} g)(V,W) = -g((\pounds_F \nabla)(U,V),W) - g((\pounds_F \nabla)(U,W),V)$$

is well-known for any U, V, W on M. As g is parallel with respect to ∇, the above relation becomes

$$(\nabla_U \pounds_F g)(V,W) - g((\pounds_F \nabla)(U,V),W) - g((\pounds_F \nabla)(U,W),V) = 0, \tag{3.3}$$

for any U, V, W. Since $\pounds_F \nabla$ is a symmetric tensor of type $(1,2)$, then from (3.3) it follows that

$$g((\pounds_F \nabla)(U,V),W) = \frac{1}{2}(\nabla_V \pounds_F g)(U,W) + \frac{1}{2}(\nabla_U \pounds_F g)(V,W) - \frac{1}{2}(\nabla_W \pounds_F g)(U,V). \tag{3.4}$$

Using (3.2) in (3.4), we have

$$\begin{aligned}g((\pounds_F \nabla)(U,V),W) =&(\nabla_W S)(U,V) - (\nabla_V S)(W,U) - (\nabla_U S)(V,W) \\ &- 2(2n-3)g(\phi U,V)\eta(W) + 2g(\phi U,\phi V)\eta(W)a,\end{aligned}$$

which by putting $V = \xi$ reduces to

$$g((\pounds_F \nabla)(U,\xi),W) = (\nabla_W S)(U,\xi) - (\nabla_U S)(\xi,W) - (\nabla_\xi S)(W,U). \tag{3.5}$$

By considering (2.10) and (2.11) in (3.5), we obtain

$$(\pounds_F \nabla)(U,\xi) = 2Q\phi U - 2aU - 2a\eta(U)\xi. \tag{3.6}$$

Taking the covariant derivative of (3.6) with respect to V, we have

$$\begin{aligned}(\nabla_V \pounds_F \nabla)(U,\xi) =&2(\nabla_V Q)\phi U - (\pounds_F \nabla)(U,\phi V) + 2Q(\nabla_V \phi)U \\ &- 2ag(U,\phi V)\xi - 2a\eta(U)\phi V.\end{aligned}$$

Again from [27], we have

$$(\pounds_F R)(U,V)W + (\nabla_V \pounds_F \nabla)(U,W) - (\nabla_U \pounds_F \nabla)(V,W) = 0.$$

Thus the last two equations give

$$\begin{aligned}(\pounds_F R)(U,V)\xi =& 2(\nabla_U Q)\phi V - 2(\nabla_V Q)\phi U \\ &+ 2Q(\eta(V)U - \eta(U)V) + 2a(\eta(U)\phi V - \eta(V)\phi U) \\ &+ (\pounds_F \nabla)(U,\phi V) - (\pounds_F \nabla)(V,\phi U).\end{aligned} \tag{3.7}$$

Setting $V = \xi$ in (3.7) and making use of (2.11), it follows that

$$(\pounds_F R)(U,\xi)\xi = 2QU + 2Q\eta(U)\xi - 2a\phi U - (\pounds_F \nabla)(\xi,\phi U). \tag{3.8}$$

Taking the Lie derivative of $R(U,\xi)\xi = -U - \eta(U)\xi$ along F, we have

$$(\pounds_F R)(U,\xi)\xi - g(U,\pounds_F \xi)\xi + 2\eta(\pounds_F \xi)U = -(\pounds_F \eta)(U)\xi. \tag{3.9}$$

By using (3.9), (3.8) takes the form

$$\begin{aligned}(\pounds_F \eta)(U)\xi =& - 2QU - 2Q\eta(U)\xi + 2a\phi U + (\pounds_F \nabla)(\xi,\phi U) + g(U,\pounds_F \xi)\xi \\ &- 2\eta(\pounds_F \xi)U.\end{aligned} \tag{3.10}$$

Now taking the Lie derivative of $g(U,\xi) = \eta(U)$, we find

$$(\pounds_F \eta)U = g(U,\pounds_F \xi) + (\pounds_F g)(U,\xi). \tag{3.11}$$

By putting $V = \xi$ in (3.1) and using (2.1)–(2.3), we find

$$(\pounds_F g)(U,\xi) = -2\lambda\eta(U). \tag{3.12}$$

Again putting $U = \xi$ in (3.12), we arrive

$$\eta(\pounds_F \xi) = -\lambda. \tag{3.13}$$

By making use of (3.11)-(3.13), we get from (3.10) that

$$(\lambda I - Q)\phi^2 U = -a\phi U - \frac{1}{2}(\pounds_F \nabla)(\xi,\phi U),$$

which by virtue of (3.6) leads to $\lambda = 0$, where $\phi^2 U \neq 0$. This shows that $*$-Ricci soliton on M is steady. This completes the proof. □

Theorem 3.2. *An n-dimensional Lorentzian para-Sasakian manifold endowed with an almost $*$-Ricci soliton (g,ξ,λ) is a generalized η-Einstein. Also, the soliton is steady.*

Proof. Let the Lorentzian metric of an n-dimensional Lorentzian para-Sasakian manifold be an almost $*$-Ricci soliton (g,ξ,λ), then (1.1)) turns into

$$g(\nabla_U \xi, V) + g(U,\nabla_V \xi) + 2S^*(U,V) + 2\lambda g(U,V) = 0, \tag{3.14}$$

for all vector fields U and V on M. By making use of equations (2.5) and (2.17), equation (3.14) transforms to

$$S = \rho_1 g + \rho_2 \eta \otimes \eta + \rho_3 g(\cdot,\phi\,\cdot),$$

where $\rho_1 = -(\lambda + n - 2)$, $\rho_2 = -(2n-3)$ and $\rho_3 = a - 1$. Also, in view of (2.1)–(2.3), (2.8) and the above equation, we can easily find that $\lambda = 0$. This gives the statement of Theorem 3.2. $\square$

Particularly, if we suppose that $a = \operatorname{tr}\phi = 1$, then from Theorem 3.2, we infer that

$$S = \rho_1 g + \rho_2 \eta \otimes \eta. \tag{3.15}$$

Let us consider an orthonormal frame field on a Lorentzian para-Sasakian manifold and contracting (3.15), we lead

$$r = n\rho_1 - \rho_2 = -n^2 + 4n - 3.$$

Now, we state the following.

Corollary 3.1. *If an n-dimensional Lorentzian para-Sasakian manifold admits an almost $*$-Ricci soliton (g, ξ, λ), with $\operatorname{tr}\phi = 1$, then it has constant scalar curvature.*

A non-flat semi-Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the non-zero Ricci tensor S of type $(0,2)$ of the manifold satisfies the condition [28]

$$(\nabla_U S)(V,W) = 2A(U)S(V,W) + A(V)S(U,W) + A(W)S(U,V), \tag{3.16}$$

where A is a non-zero 1-form such that $g(U,\sigma) = A(U)$, for all vector fields U; σ being the vector field corresponding to the associated 1-form A. In partcular, if $A = 0$, then the manifold is called Ricci symmetric.

Taking the covariant derivative of (3.15) leads to

$$(\nabla_U S)(V,W) = \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)]. \tag{3.17}$$

Now using (3.15) and (3.17), (3.16) becomes

$$\begin{aligned} \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)] =& 2A(U)[\rho_1 g(V,W) + \rho_2 \eta(V)\eta(W)] \\ &+ A(V)[\rho_1 g(U,W) + \rho_2 \eta(U)\eta(W)] \\ &+ A(W)[\rho_1 g(U,V) + \rho_2 \eta(U)\eta(V)]. \end{aligned} \tag{3.18}$$

Taking $U = W = \xi$ in (3.18), we get $A(V) = 3A(\xi)\eta(V)$, which by putting $V = \xi$ gives $A(\xi) = 0$. This implies that $A(V) = 0$. Thus we have the following.

Theorem 3.3. *A pseudo Ricci symmetric Lorentzian para-Sasakian manifold admitting an almost $*$-Ricci soliton (g, ξ, λ), with $\operatorname{tr}\phi = 1$ is Ricci symmetric.*

4. Gradient $*$-Ricci Solitons on η-Einstein Lorentzian Para-Sasakian Manifolds

This section is concerned with the study of gradient $*$-Ricci solitons within the context of η-Einstein Lorentzian para-Sasakian manifolds.

Let an n-dimensional Lorentzian para-Sasakian manifold be η-Einstein, then it is noticed that the equation (2.9) takes the form

$$S = \rho_1 g(U,V) + \rho_2 \eta(U) \otimes \eta(V). \tag{4.1}$$

Setting $V = U = e_i$ in (4.1), where $\{e_i\}_{i=1}^{n}$ represents a set of orthonormal frame field of M, and taking the summation over i, $1 \leq i \leq n$, we have

$$r = \rho_1 n - \rho_2. \tag{4.2}$$

On the other hand, putting $U = V = \xi$ in (4.1) and making use of (2.1) and (2.3), we also have

$$-(n-1) = -\rho_1 + \rho_2. \tag{4.3}$$

Hence, it follows from (4.2) and (4.3) that

$$\rho_1 = \frac{r}{n-1} - 1, \quad \rho_2 = \frac{r}{n-1} - n.$$

Thus, the Ricci tensor S of an η-Einstein Lorentzian para-Sasakian manifold is given by

$$S(U,V) = \left(\frac{r}{n-1} - 1\right) g(U,V) + \left(\frac{r}{n-1} - n\right) \eta(U)\eta(V). \tag{4.4}$$

Definition 4.1. A semi-Riemannian metric g of a semi-Riemannian manifold M is called a gradient $*$-Ricci soliton if it satisfies

$$\mathrm{Hess} f + S^* + \lambda g = 0, \tag{4.5}$$

for some smooth function f, where Hess f (Hessian f) is defined by Hess $f = \nabla\nabla f$. It is noticed that if we choose $F = Df$ in equation (1.1), where D denotes the gradient operator of g, then we get (4.5).

Let the η-Einstein Lorentzian para-Sasakian manifold M admit a gradient $*$-Ricci soliton. Then from (4.5) it follows that

$$\nabla_U Df + Q^* U + \lambda U = 0, \tag{4.6}$$

for all U on M. First we prove the following lemmas for later use.

Lemma 4.1. *An n-dimensional η-Einstein Lorentzian para-Sasakian manifold satisfies*

$$(\nabla_U Q^*)\xi - (\nabla_\xi Q^*)U = -\left(\frac{r}{n-1} + n - 3\right)\phi U + \left(a - \frac{\xi(r)}{n-1}\right)(U + \eta(U)\xi), \tag{4.7}$$

for all X on M.

Proof. By using (4.4) in (2.17), we find

$$S^*(V,W) = \left(\frac{r}{n-1} + n - 3\right)(g(V,W) + \eta(V)\eta(W)) - g(V,\phi W)a.$$

It yields

$$Q^* V = \left(\frac{r}{n-1} + n - 3\right)(V + \eta(V)\xi) - \phi V a. \tag{4.8}$$

Differentiating (4.8) along U, we get

$$(\nabla_U Q^*)V = \left(\frac{r}{n-1} + n - 3\right) [(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi] \\ - (g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi)a + \frac{U(r)}{n-1}(V + \eta(V)\xi), \tag{4.9}$$

which by replacing V by ξ and using (2.1), (2.3) and (2.5) reduces to

$$(\nabla_U Q^*)\xi = -\left(\frac{r}{n-1} + n - 3\right)\phi U + (U + \eta(U)\xi)a. \tag{4.10}$$

Again replacing U by ξ in (4.9) and using same equations, we find

$$(\nabla_\xi Q^*)U = \frac{\xi r}{n-1}(U - \eta(U)\xi). \tag{4.11}$$

By subtracting (4.11) from (4.10), (4.7) follows. □

Lemma 4.2. *If an η-Einstein Lorentzian para-Sasakian manifold admits a gradient $*$-Ricci soliton, then we have*

$$R(U,V)Df = (\nabla_V Q^*)U - (\nabla_U Q^*)V. \tag{4.12}$$

Proof. Differentiating (4.6) covariantly along Y, we have

$$\nabla_V \nabla_U Df + \nabla_V Q^* U + \lambda \nabla_V U = 0, \tag{4.13}$$

which by interchanging U and V becomes

$$\nabla_U \nabla_V Df + \nabla_U Q^* V + \lambda \nabla_U V = 0. \tag{4.14}$$

Also from (4.6), we find

$$\nabla_{[U,V]} Df = -Q^*[U,V] - \lambda[U,V]. \tag{4.15}$$

By making use of (4.13)–(4.15), Lemma 4.2 follows. □

Theorem 4.1. *Let the metric of an η-Einstein Lorentzian para-Sasakian manifold M admit a gradient $*$-Ricci soliton. Then the gradient of the potential function is pointwise collinear with the potential vector field of M.*

Proof. Putting $U = \xi$ in (4.12), we have

$$R(\xi,V)Df = (\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V,$$

which by virtue of the Lemma 4.1 leads to

$$g(R(\xi,V)Df, \xi) = 0. \tag{4.16}$$

By using (2.8), we have

$$g(R(\xi,V)Df, \xi) = -(Vf) - \eta(V)(\xi f). \tag{4.17}$$

From (4.16) and (4.17), we find $(Vf) = -\eta(V)(\xi f)$. This implies that

$$Df = -(\xi f)\xi.$$

This completes the proof. □

Taking the covariant derivative of $Df = -(\xi f)\xi$ along U, we have

$$\nabla_U Df = -(U(\xi f))\xi - (\xi f)\phi U, \tag{4.18}$$

which gives

$$g(\nabla_U Df, \xi) = U(\xi f),$$

where (2.1) and (2.2) are used. Using the last equation in (4.18), we obtain

$$\nabla_U Df = -g(\nabla_U Df, \xi)\xi - (\xi f)\phi U. \tag{4.19}$$

From equations (2.17) and (4.6), we conclude that

$$\nabla_U Df = -QU - (\lambda + n - 2)U - (2n-3)\eta(U)\xi + \phi U a, \tag{4.20}$$

which implies that

$$g(\nabla_U Df, \xi) = -\lambda\eta(U). \tag{4.21}$$

Thus from the equations (2.1), (2.2), (2.8), and (4.19)–(4.21), we obtain

$$QU = -(\lambda + n - 2)U - (\lambda + 2n - 3)\eta(U)\xi + (a + (\xi f))\phi U,$$

which informs that the manifold M under the consideration is generalized η-Einstein. Hence, we can state the following.

Corollary 4.1. *Every η-Einstein Lorentzian para-Sasakian manifold of dimension n endowed with a gradient $*$-Ricci metric is generalized η-Einstein.*

5. Example

In this section, we construct a non-trivial example of a Lorentzian para-Sasakian manifold.

We consider the 4-dimensional manifold $M = \{(u, v, w, t) \in \mathbb{R}^4\}$, where (u, v, w, t) are the standard coordinates in $\mathbb{R}^4$. Let ζ_1, ζ_2, ζ_3 and ζ_4 be the vector fields on M given by

$$\zeta_1 = e^t \frac{\partial}{\partial u}, \quad \zeta_2 = e^t \frac{\partial}{\partial v}, \quad \zeta_3 = e^t \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right), \quad \zeta_4 = -\frac{\partial}{\partial t}.$$

Let g be the semi-Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & 1 \leq i = j \leq 3, \\ -1, & i = j = 4, \\ 0, & 1 \leq i \neq j \leq 4. \end{cases}$$

Let η be the 1-form on M defined by $\eta(U) = g(U, \zeta_4) = g(U, \xi)$ for all $U \in \mathfrak{X}(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi\zeta_1 = \zeta_1, \quad \phi\zeta_2 = \zeta_2, \quad \phi\zeta_3 = \zeta_3, \quad \phi\zeta_4 = 0.$$

By applying the linearity of ϕ and g, we have

$$\begin{aligned} \eta(\xi) =& -1, \quad \phi^2 U = U + \eta(U)\xi, \quad \eta(\phi U) = 0, \\ g(U,\xi) =& \eta(U), \quad g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \end{aligned}$$

for all $U, V \in \mathfrak{X}(M)$. Then we have

$$\begin{aligned} [\zeta_1,\zeta_2] =& [\zeta_1,\zeta_3] = [\zeta_2,\zeta_3] = 0, \\ [\zeta_1,\zeta_4] =& \zeta_1, \quad [\zeta_2,\zeta_4] = \zeta_2, \quad [\zeta_3,\zeta_4] = \zeta_3. \end{aligned}$$

Using Koszul's formula, we can easily calculate

$$\begin{aligned} \nabla_{\zeta_1}\zeta_1 =& \zeta_4, \quad \nabla_{\zeta_1}\zeta_2 = 0, \quad \nabla_{\zeta_1}\zeta_3 = 0, \quad \nabla_{\zeta_1}\zeta_4 = \zeta_1, \\ \nabla_{\zeta_2}\zeta_1 =& 0, \quad \nabla_{\zeta_2}\zeta_2 = \zeta_4, \quad \nabla_{\zeta_2}\zeta_3 = 0, \quad \nabla_{\zeta_2}\zeta_4 = \zeta_2, \\ \nabla_{\zeta_3}\zeta_1 =& 0, \quad \nabla_{\zeta_3}\zeta_2 = 0, \quad \nabla_{\zeta_3}\zeta_3 = \zeta_4, \quad \nabla_{\zeta_3}\zeta_4 = \zeta_3, \\ \nabla_{\zeta_4}\zeta_1 =& 0, \quad \nabla_{\zeta_4}\zeta_2 = 0, \quad \nabla_{\zeta_4}\zeta_3 = 0, \quad \nabla_{\zeta_4}\zeta_4 = 0. \end{aligned}$$

From the above values it can be easily verified that for $\zeta_4 = \xi$, M is a Lorentzian para-Sasakian manifold. We found that the non-vanishing components of curvature tensor are given by

$$\begin{aligned} R(\zeta_1,\zeta_2)\zeta_1 =& -\zeta_2, \quad R(\zeta_1,\zeta_3)\zeta_1 = -\zeta_3, \quad R(\zeta_1,\zeta_4)\zeta_1 = -\zeta_4, \\ R(\zeta_1,\zeta_2)\zeta_2 =& \zeta_1, \quad R(\zeta_2,\zeta_3)\zeta_2 = -\zeta_3, \quad R(\zeta_2,\zeta_4)\zeta_2 = -\zeta_4, \\ R(\zeta_1,\zeta_3)\zeta_1 =& \zeta_1, \quad R(\zeta_2,\zeta_3)\zeta_3 = \zeta_2, \quad R(\zeta_3,\zeta_4)\zeta_3 = -\zeta_4, \\ R(\zeta_1,\zeta_4)\zeta_4 =& -\zeta_1, \quad R(\zeta_2,\zeta_4)\zeta_4 = -\zeta_2, \quad R(\zeta_3,\zeta_4)\zeta_3 = -\zeta_3. \end{aligned}$$

From the above expressions of curvature tensors, we obtain

$$S(\zeta_{1,}\zeta_1) = S(\zeta_2,\zeta_2) = S(\zeta_3,\zeta_3) = 3, \quad S(\zeta_4,\zeta_4) = -3.$$

In view of 2.17, L.H.S. of (1.1) can be expressed as

$$\begin{aligned} (\pounds_F g)(V,W) + 2S^*(V,W) + 2\lambda g(V,W) =& g(\nabla_V F, W) + g(V, \nabla_W F) \\ &+ 2S(V,W) + 4g(V,W) \\ &- 6g(V,\phi W)a + 10\eta(V)\eta(W). \end{aligned}$$

Let $V = \sum_{i=1}^4 V^i e_i$, $W = \sum_{i=1}^4 W^i e_i$ and $F = \sum_{i=1}^4 F^i e_i$, where V^i, W^i and F^i are scalars for $i = 1,2,3,4$ such that

$$F^4 = \frac{F^1(V^1W^4 + W^1V^4) + F^2(V^2W^4 + W^2V^4) + F^3(V^3W^4 + W^3V^4)}{2(V^1W^1 + V^2W^2 + V^3W^3)} - 2,$$

provided $V^1W^1 + V^2W^2 + V^3W^3 \neq 0$. Then by the straight forward calculations, we can notice that

$$\begin{aligned} &2(V^1W^1F^4 + V^2W^2F^4 + V^3W^3F^4) - (V^1F^1W^4 + V^2F^2W^4 + V^3F^3W^4 \\ &+ W^1F^1V^4 + W^2F^2V^4 + W^3F^3V^4) + 4(V^1W^1 + V^2W^2 + V^3W^3) = 0, \end{aligned}$$

for $a = 3$ and hence we have $\pounds_F g + 2S^* + 2\lambda g = 0$, provided $\lambda = 0$. Thus, we can say that the Lorentzian para-Sasakian manifold of dimension 4 admits a steady type $*$-Ricci soliton, which proves Theorem 3.1.

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WOVEN (WEAVING) FRAMES IN BANACH SPACES

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Abstract. Banach frames are defined by the straightforward generalization of Hilbert space frames. Woven (weaving) frames are the recent generalization of standard frames which appeared in the applications of distributed signal processing. In this paper, we introduce the concepts of woven (weaving) Bessel and frame sequences in Banach spaces and characterize the woven frames in terms of bounded operators. We also give some equivalent conditions for woven X_d-frame in Banach spaces.

1. Introduction

The origin of frame theory can be traced back to the early 1950s with the seminal work of Duffin and Schaeffer [13] in nonharmonic Fourier series. Today, the theory of frames has expanded into an independent and broad field of research with widespread applications to signal processing, image processing, data compression, pattern matching, sampling theory, spherical codes, wavelet analysis, communication and data transmission [4,8,11,18,19]. Inspired by a problem raised in distributed signal processing, Bemrose et al. [1] introduced the concept of weaving frames in separable Hilbert spaces and observed that the weaving frames may be applied in sensor networks which requires distributed processing under different frames. In recent years, a considerable amount of research has been conducted to extend the notion of weaving frames to different settings which include weaving frames in Banach spaces, continuous weaving frames, generalized weaving frames, weaving Riesz bases, weaving fusion frames, weaving controlled frames and weaving vector-valued frames [5,6,20,22,24–26,31–34].

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Frames in Hilbert spaces were extended to Banach spaces by Feichtinger and Gröchenig [15] who introduced the concept of atomic decompositions in Banach spaces. Later on, Gröchenig [17] laid down the foundations for the theory of coherent Banach frames and constructed Banach frames for a wide class of Banach spaces, the so-called coorbit spaces. Keeping in view the fact that the weaving frames have potential applications in wireless sensor networks and other allied areas, we are deeply motivated to extend the concept of woven (weaving) frames to Banach spaces by invoking certain fundamental concepts of operator theory.

This article is organized as follows. Section 2 contains basic definitions and results regarding frames and weaving frames in Hilbert spaces. In Section 3, we introduce the notion of weaving frames in Banach spaces and then generalize the definitions of X_d-frame and p-frame for the woven.

2. Frames and Woven Frames in Hilbert Spaces

In this section, we give a short review of the concept of frames and woven frames in Hilbert spaces and make some preparatory observations. For a complete treatment of frame theory, we recommend the excellent book of Christensen [8], the tutorials of Casazza [2,3] and the memoir of Han and Larson [21]. Throughout this paper, H denotes a separable infinite-dimensional Hilbert space, X, Y, Z the separable Banach spaces with dual X^*, Y^*, Z^*, X_d a Banach sequence space and $\mathbb{I}$ an index set which is finite or countable. Let $\mathbb{N}$ be the set of all positive integers and let $m \in \mathbb{N}$ be fixed. Then for this choice of m, we set $[m] = \{1, 2, \dots, m\}$ and $[m]^c = \mathbb{I} \setminus [m] = \{m+1, m+2, \dots\}$. Let us start with the well-known notion of Hilbert space frames.

2.1. Discrete frame in Hilbert spaces.

In this section, we give a short review of the concept of frames in Hilbert spaces, and make some preparatory observations. Let us start with the well known notion of Hilbert space frames.

Definition 2.1. A family of vectors $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ in a Hilbert space H is said to be a frame if there exist constants $0 < A \leq B < \infty$ so that for all $x \in H$

$$A\|x\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2,$$

where A and B are lower and upper frame bounds, respectively. If only B is assumed, then it is called B-Bessel sequence. If $A = B$, it is said to be a tight frame and if $A = B = 1$, it is called a Parseval frame.

If $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ is a Bessel sequence for H, then the synthesis operator of Φ defined as

$$T : l^2(\mathbb{I}) \to H, \quad T\{c_i\} := \sum_{i\in\mathbb{I}} c_i\varphi_i,$$

and the adjoint of T is the analysis operator

$$T^* : H \to l^2(\mathbb{I}), \quad T^*x := \{\langle x, \varphi_i\rangle\}_{i\in\mathbb{I}}.$$

The frame operator $S : H \to H$ is defined by $S := TT^*$

$$Sx = TT^*x = \sum_{i\in\mathbb{I}} \langle x, \varphi_i\rangle\varphi_i, \quad \text{for all } x \in H.$$

The operator S is positive, self-adjoint, invertible and $AI \leq S \leq BI$. Any $x \in H$ has an expansion

$$x = \sum_{i\in\mathbb{I}} \langle S^{-1}\varphi_i, x\rangle\varphi_i = \sum_{i\in\mathbb{I}} \langle\varphi_i, x\rangle S^{-1}\varphi_i. \tag{2.1}$$

The family $\{S^{-1}\varphi_i\}_{i\in\mathbb{I}}$ is also a frame with bounds B^{-1}, A^{-1} and this frame is called the canonical dual or reciprocal frame of $\{\varphi_i\}_{i\in\mathbb{I}}$.

Definition 2.2. A family of vectors $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ in a Hilbert space H is said to be a Riesz sequence if there exist constants $0 \leq A \leq B < \infty$ so that for all $\{c_i\}_{i\in\mathbb{I}} \in l^2(\mathbb{I})$

$$A\sum_{i\in\mathbb{I}} |c_i|^2 \leq \left\|\sum_{i\in\mathbb{I}} c_i\varphi_i\right\|^2 \leq B\sum_{i\in\mathbb{I}} |c_i|^2,$$

where A and B are the lower Riesz bound and upper Riesz bound, respectively. If in addition, Φ is complete in H, then it is called as the Riesz basis for H.

2.2. **Woven Frame in Hilbert spaces.** Woven frames in Hilbert spaces were introduced by Bemros et al. [1,6] in 2015. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. In this subsection, we review the notions of woven and weaving frames in Hilbert spaces and present certain new examples.

Definition 2.3. A family of frames $\{f_{ij}\}_{i\in\mathbb{I}}$ with $j \in [m]$ for a Hilbert space H is said to be woven if there exist universal constants A and B so that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{f_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively. For every $j \in [m]$, the frames $\{f_{ij}\}_{i\in\sigma_j}$ are called weaving frames.

The following proposition shows that every weaving frame has always a universal upper frame bound.

Proposition 2.1. *If each $\phi = \{\varphi_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ is a Bessel sequence for H with bounds B_j for all $j \in [m]$, then every weaving frame is a Bessel sequence with $\sum_{j=1}^m B_j$ as a Bessel bound.*

Proof. For every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$ and every $x \in H$, the inequality

$$\sum_{j=1}^m \sum_{i\in\sigma_j} |\langle x, \varphi_{ij}\rangle|^2 \leq \sum_{j=1}^m \sum_{i\in\mathbb{I}} |\langle x, \varphi_{ij}\rangle|^2 \leq \|x\|^2 \sum_{j=1}^m B_j,$$

yields the desired bound. □

Example 2.1. There exist two Parseval frames that yield weaving frames with arbitrary weaving bounds. For showing this, assume $\varepsilon > 0$, set $\delta = (1+\varepsilon^2)^{-\frac{1}{2}}$, and let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of $\mathbb{R}^3$. Then the sets $\phi = \{\varphi_i\}_{i=1}^n = \{\delta e_1+\delta\varepsilon e_1, \delta e_2+\delta\varepsilon e_2, \delta e_3+\delta\varepsilon e_3\}$ and $\psi = \{\psi_i\}_{i=1}^n = \{\delta\varepsilon e_2+\delta e_2, \delta\varepsilon e_1+\delta e_1, \delta\varepsilon e_3+\delta e_3\}$, are Parseval frames, which are woven since any choice of σ gives the spaning set. Since they are Parseval, as a consequence of Proposition 2.1, the universal frame bound for every weaving frame can be chosen to be n. For $\sigma = \{2,4,6\}$, we have

$$
\begin{aligned}
&\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle x, \psi_i\rangle|^2 \\
=&|\langle x, \delta e_1 + \delta\varepsilon e_1\rangle|^2 + |\langle x, \delta e_2 + \delta\varepsilon e_2\rangle|^2 \\
&+ |\langle x, \delta e_3 + \delta\varepsilon e_3\rangle|^2 + |\langle x, \delta\varepsilon e_2 + \delta e_2\rangle|^2 + |\langle x, \delta\varepsilon e_1 + \delta e_1\rangle|^2 + |\langle x, \delta\varepsilon e_3 + \delta e_3\rangle|^2 \\
=&2\left(\delta^2 + \delta^2\varepsilon^2\right)|\langle x, e_1\rangle|^2 + 2\left(\delta^2\varepsilon^2 + \delta^2\right)|\langle x, e_2\rangle|^2 + 2\left(\delta^2\varepsilon^2 + \delta^2\right)|\langle x, e_3\rangle|^2 \\
=&2\delta^2\left(1+\varepsilon^2\right)\|x\|^2 = \frac{2\varepsilon^2}{1+\varepsilon^2}\|x\|^2,
\end{aligned}
$$

which lies between 0, 3 for arbitrary choice of $\varepsilon \in (0,\infty)$.

The following proposition demonstrates that the perturbed frames are obtained as the image of a bounded and invertible operator of a given frame.

Proposition 2.2. *Let $\{\varphi_i\}_{i\in\mathbb{I}}$ be a frame with bounds A, B and V be a bounded operator. If $\|Id - V\|^2 \le \frac{A}{B}$ and $\|V - V^2\|^2 \le \frac{A}{B}$, then the frames $\{\varphi_i\}_{i\in\mathbb{I}}$, $\{V\varphi_i\}_{i\in\mathbb{I}}$ and $\{V^2\varphi_i\}_{i\in\mathbb{I}}$ are woven.*

Proof. Note that by Neumann's Theorem V is invertible and thus $\{V\varphi_i\}_{i\in\mathbb{I}}$ and $\{V^2\varphi_i\}_{i\in\mathbb{I}}$ automatically constitute frames. For every partitions $\sigma, \Delta \subset \mathbb{I}$ and every $x \in H$ by using *Minkowski's* inequality:

$$
\begin{aligned}
&\left(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, V\varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle x, V^2\varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}} \\
=&\left(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 - \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle V^*x, \varphi_i\rangle|^2\right. \\
&\left.+ \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^*x, \varphi_i\rangle|^2 - \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^*x, \varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle (V^2)^*x, \varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}} \\
=&\left(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 - \sum_{i\in\Delta} |\langle (I - V^*)x, \varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^*x, \varphi_i\rangle|^2\right. \\
&\left.- \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle (V^* - (V^2)^*)x, \varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \left(\sum_{i\in\mathbb{I}} |\langle x,\varphi_i\rangle|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\Delta} |\langle (I-V^*)x,\varphi_i\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in\Delta\cup(\mathbb{I}\setminus(\sigma\cup\Delta))} |\langle V^*x,\varphi_i\rangle|^2\right)^{\frac{1}{2}} \\
&\quad - \left(\left|\left\langle \left(V^* - (V^2)^*\right)x,\varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}} \\
&\geq \sqrt{A}\,\|x\| - \sqrt{B}\|(I-V^*)x\| + \sqrt{B}\|V^*x\| - \sqrt{B}\|(V^*-(V^2)^*)x\| \\
&\geq \left(\sqrt{A} - \sqrt{B}\|I-V^*\| + \sqrt{B}\|V^*\| - \sqrt{B}\|V^*\|\|I-V^*\|\right)\|x\|.
\end{aligned}
$$

Thus, $\{\varphi\}_{i\in\sigma}\cup\{V\varphi_i\}_{i\in\Delta}\cup\{V^2\varphi_i\}_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)}$ forms a woven frames having

$$\left(\sqrt{A} - \sqrt{B}\|I-V^*\| + \sqrt{B}\|V^*\| - \sqrt{B}\|V^*\|\|I-V^*\|\right)^2 > 0.$$

□

3. Woven Frames in Banach Space

3.1. **Frames in Banach Space.** Frames were extended to Banach spaces by Feichtinger and Gröchenig [15] who introduced the notion of atomic decompositions for Banach spaces. Later, Gröchenig [17] introduced a more general concept called Banach frame. Banach frames were further studied in [4]. An analysis of p-frames in general Banach spaces first appeared in [9]. The aim of an atomic decomposition for a space of functions or distributions is to represent every element as a sum of simple functions usually called atoms. If this is possible, some properties of these function spaces, such as duality, interpolation, or operator theory for them, can be understood better by means of the atomic decomposition. Decomposition methods have been used for many important theoretical contributions. A Banach space of scalar valued sequences (often called BK-space) is a linear space of sequences equipped with a norm under which it constitutes a Banach Space (i.e., it is complete in the norm) and for which the coordinate functionals are continuous. In a Banach space of scalar valued sequences, the unit vectors are the elements e_i's defined by $e_i(j) = \delta_{ij}$ (δ_{ij} the Kronechker delta).

Definition 3.1. A sequence space X_d is called BK-space, if it is a Banach space and the coordinate functionals $\{a_k\}\to a_k$ are continuous on X_d, that is, the relations $x_n = \{\alpha_j^{(n)}\}$, $x=\{\alpha_j\}\in X_d$, $\lim_{n\to\infty} x_n = x$ imply

$$\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j, \quad j=1,2,\dots$$

A BK-space is called solid if whenever $\{a_k\}$ and $\{b_k\}$ are sequences with $\{b_k\}\in X_d$ and $|a_k|\leq|b_k|$ for each $k\in\mathbb{I}$, then it follows that $\{a_k\}\in X_d$ and

$$\|\{a_k\}\|_{X_d} \leq \|\{b_k\}\|_{X_d}.$$

A sequence space X_d is called an AK-space if it is a topological vector space and

$$\{a_k\} = \lim_n \rho_n(\{a_k\}), \quad \text{for all } \{a_k\}\in X_d,$$

where $\rho_n(\{a_k\}) = (a_1, a_2, \dots, a_n, 0, \dots)$.

If the canonical vectors form a Schauder basis for X_d, then X_d is called a CB-space and its canonical basis is denoted by $\{e_j\}_{i=1}^{\infty}$. If X_d is reflexive and a CB-space, then X_d is called an RCB-space. Also, the dual of X_d is denoted by X_d^*.

Definition 3.2. Let X be a Banach space and X_d be a BK-space. A countable family $\{g_i\}_{i\in\mathbb{I}}$ in the dual X^* is called an X_d-frame for X if

(a) $\{g_i(f)\}_{i\in\mathbb{I}} \in X_d$ for all $f \in X$;

(b) the norms $\|f\|_X$ and $\|\{g_i(f)\}_{i\in\mathbb{I}}\|_{X_d}$ are equivalent, that is, there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{g_i(f)\}_{i\in\mathbb{I}}\|_{X_d} \leq B\|f\|_X, \quad \text{for all } f \in X.$$

A and B are called X_d-frame bounds.

If at least (a) and the upper condition in (b) are satisfied, $\{g_i\}_{i\in\mathbb{I}}$ is called an X_d-Bessel sequence for X. In case $X_d = \ell^p$, the X_d-frame is called p-frame which is introduced by Christensen and Stoeva [9, 30].

Definition 3.3. A countable family $\{g_i\}_{i\in\mathbb{I}} \subset X^*$ is a p-frame for X, $1 < p < \infty$, if there exist $A, B > 0$ such that

$$A\|f\|_X \leq \left(\sum_{i\in\mathbb{I}} |g_i(f)|^p\right)^{\frac{1}{p}} \leq B\|f\|_X, \quad \text{for all } f \in X.$$

The family $\{g_i\}_{i\in\mathbb{I}}$ is a p-Bessel sequence if at least the upper p-frame condition is satisfied.

Lemma 3.1 ([28])**.** *If X is a Banach space and $\{f_n\} \subset X^*$ is total over X, then X is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in X\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_X$ for $x \in X$.*

Definition 3.4. Let X be a Banach space and let X_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S : X_d \to X$ be given. The pair $(\{f_n\}, S)$ is called a Banach frame for X with respect to X_d if

(a) $\{f_n(x)\} \in X_d$ for each $x \in X$;

(b) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \quad \text{for all } x \in X; \tag{3.1}$$

(c) S is a bounded linear operator such that $S(\{f_n(x)\}) = x$ for all $x \in X$.

The positive constants A and B are called the lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : X_d \to X$ is called the reconstruction operator (or the pre-frame operator). The inequality (b) is called the frame inequality. The Banach frame $(\{f_n\}, S)$ is called tight if $A = B$ and normalized tight if $A = B = 1$.

Example 3.1. Let $X = l^p$ and $\{e_n\}$ be the sequence of unit vectors in X. Define $\{f_n\} \subset X^*$ by

$$f_n = f_{n+2} = e_n, \quad n \in \mathbb{I}.$$

Then by Lemma 3.1, there exists an associated Banach space $X_d = \{\{f_n(x)\} : x \in X\}$ and a reconstruction operator $S : X_d \to X$ such that $(\{f_n\}, S)$ is a Banach frame for X.

3.2. **Woven in Banach spaces.** As we mentioned earlier, Bemrose, Casazza et al. in [1, 6] proposed weaving frames in a separable Hilbert space. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, frames in Hilbert spaces. Improving and extending this notion on Hilbert spaces, we generalize the concept of woven (weaving) on Banach spaces.

Definition 3.5. Let X be a Banach space and X_d be a BK-space. The family of Banach frames $\{g_{ij}\}_{i\in\mathbb{I}}$ for $j \in [m]$ is woven X_d-frame for dual X^* with universal bounds A, B if

(a) $\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]} \in X_d$, $f \in X$;
(b) the norms $\|f\|_X$ and $\|\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]}\|_{X_d}$ are equivalent, that is, there exist constants A, $B > 0$ such that

$$A\|f\|_X \leq \|\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]}\|_{X_d} \leq \ B\|f\|_X, \quad f \in X.$$

The constants A and B are called woven X_d-frame bounds. If at least (a) and the upper condition in (b) are satisfied, $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is called a woven X_d-Bessel for X.

Definition 3.6. Let X be a Banach space and let X_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{I}$. Let $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ and $S : X_d \to X$ be given. The pair $\Big(\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}, S\Big)$ is called a woven Banach frame for X with respect to X_d if the pair $\Big(\{f_{ij}\}_{i\in\sigma_j,j\in[m]}, S\Big)$ is a Banach frame for each partitions $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$.

The lack of an inner-product in Banach spaces led G. Lumer [23] in 1961 to introducing the theory of semi-inner product spaces. His procedure suggested the existence of a general theory which it seemed should be useful in the study of operator (normed) algebras by providing better insight on known facts, a more adequate language to "classify" special types of operators, as well as new techniques. This notion was further modified by J. R. Giles [16] and other researchers thereon, and the same is presented below.

Definition 3.7 ([16])**.** Let X be a complex (real) vector space. A complex (real) semi-inner product defined on X is a function from $[\cdot, \cdot] : X \times X \to \mathbb{C}$ such that for all $f, g, h \in X$, $\lambda \in \mathbb{C}$ complex (real)

(a) $[\lambda f + g, h] = \lambda\, [f, h] + [g, h]$, $[f, \lambda g] = \overline{\lambda}\, [f, g]$;
(b) $[f, f] \geq 0$ for $f \in X$ and $[f, f] = 0$ implies $f = 0$;
(c) $|\,[f, g]\,|^2 \leq [f, f]\,[g, g]$.

We call X a complex (real) semi-inner product space, abbreviated with S.I.P.S. An S.I.P.S need not satisfy the following properties

(a) $[f, g] = \overline{[g, f]}$;
(b) $[f, g+h] = [f, g] + [f, h]$.

If $[\cdot,\cdot]$ is a S.I.P.S. on X then $\|f\| := [f, f]^{\frac{1}{2}}$ is a norm on X. Conversely, if X is a normed vector space then it has a S.I.P.S. that induces its norm in this manner which is called the compatible semi-inner product [23]. Let X be a Banach space. We define a duality map $\Phi_X : X \to X^*$ as follows. Given $f \in X$, by the Hahn-Banach theorem, there exists an $f^* \in X^*$ such that $\|f\| = \|f^*\|$ and $f^*(f) = \|f\|^2$. Set $\Phi_X(f) = f^*$, and $\Phi_X(\lambda f) = \overline{\lambda} f^*$, and define Φ_X on the rest of X in the same manner. In general, Φ_X is not unique, linear or continuous. The duality map Φ_X induces a semi-inner product $[\cdot,\cdot]$ if we set $[f, g] = g^*(f)$ [29]. We shall use this definition for $g^*, g \in X$. Note that if X is a Banach space, then the duality map is unique [29]. Recall that a Banach space X is called strictly convex, if for any pair of vectors $f, g \neq 0$ in X, the equation $\|f + g\| = \|f\|_X + \|g\|_X$, implies that there exists a $\lambda > 0$ such that $f = \lambda g$ [12]. In these spaces, the duality mapping from X to X^* is unique and bijective when X is reflexive [12, 14].

In 2011, H. Zhang and J. Zhang [35] introduced frames in Banach space X via S.I.P.S. that is presented in the following definition. The extra condition in Definition 3.5 means that S is a left-inverse of U and thus US is a bounded linear projection of X_d onto the range $R(U)$ of the operator U.

Lemma 3.2 ([10]). *If X_d is a CB-space with the canonical unit vectors e_i, $i \in J$, then the space $X_d^{\circledast} := \{\{G(e_i)\}_{i=1}^{\infty} : G \in X_d^*\}$ with the norm $\|\{G(e_i)\}_{i=1}^{\infty}\|_{X_d^{\circledast}} := \|G\|_{X_d^*}$ is a BK-space isometrically isomorphic to X_d^*. Also, every continuous linear functional Ψ on X_d has the form*

$$\Psi(\{c_j\}) = \sum_j c_j d_j,$$

where $\{d_j\} \in X_d^{\circledast}$ is uniquely determined by $d_j = \Psi(e_j)$, $\|\Psi\| = \|\{\Psi(e_i)\}_{i=1}^{\infty}\|_{X_d^{\circledast}}$. When X_d^* is a CB-space then its canonical basis is denoted by $\{e_j^*\}$.

Remark 3.1. It is easy to see that Lemma 3.2 holds in the following more general case: If Y is a Banach space and $\{y_i\}_{i=1}^{\infty}$ is a complete system in Y, then $Y^{\circledast} := \{\{Gy_i\}_{i=1}^{\infty} : G \in Y^*\}$ normed by $\|\{Gy_i\}_{i=1}^{\infty}\|_{Y^{\circledast}} := \|G\|_{Y^*}$ is a BK-space, isometrically isomorphic to Y^*. Thus, the dual of every separable Banach space can be considered as a BK-space, because every separable Banach space has a complete system [28].

In the following theorem, we will see that the Bessel woven condition can be expressed in terms of the synthesis operator T on X_d. As a prerequisite for analysis, synthesis and frame operators of weaving frames, we define the following space.

For $j \in [m]$, let $(X_d)_j := \left\{\{c_{ij}\}_{i\in\sigma_j} : \sigma_j \subset \mathbb{I}, \|\{c_{ij}\}_{i\in\sigma_j}\|_{X_d} < \infty\right\}$. Define the space

$$\left(\sum_{j\in[m]} \oplus (X_d)_j\right) = \left\{\{c_{ij}\}_{i\in\mathbb{I}, j\in[m]} : \{c_{ij}\}_{i\in\mathbb{I}} \in (X_d)_j \text{ for all } j \in [m]\right\},$$

with the semi-inner product

$$\left[\{c_{ij}\}_{i\in\mathbb{I}, j\in[m]}, \{c'_{ij}\}_{i\in\mathbb{I}, j\in[m]}\right] = \sum_{i\in\mathbb{I}, j\in[m]} \left|c_{ij}\overline{c'_{ij}}\right|.$$

The following proposition characterizes a woven Bessel in term of a bounded operator.

Theorem 3.1. *Let $\{(X_d)_1, (X_d)_2, \dots\}$ be a sequence of Banach spaces. $(X_d)_i$ and $(X_d^*)_i$'s are BK-spaces. Then,*

$$((X_d)_1 \oplus (X_d)_2 \oplus \cdots)^*_{X_d} = ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}.$$

Proof. We shall establish the result when X_d, X_d^* are BK-space. Assume that

$$C = (\{c_{i1}\}, \{c_{i2}\}, \dots) \in ((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$$

and

$$C^* = (\{c_{i1}^*\}, \{c_{i2}^*\}, \dots) \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}.$$

Then the mapping $C^* \mapsto \varphi_{C^*}$, where

$$\varphi_{c^*}(\{c_{i1}\}, \{c_{i2}\}, \dots) = \sum_{i=1}^{\infty} c_{in}^*(c_{in}),$$

is an isometry from $((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$ onto $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$. Fix $C^* \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$. For each $C = (\{c_{i1}\}, \{c_{i2}\}, \dots)$ in $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$, the mapping $\varphi_{C^*}(\{c_{i1}\}, \{c_{i2}\}, \dots) = \sum_{i=1}^{\infty} c_{in}^*(c_{in})$ defines a continuous linear functional on $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$ satisfying $\|\varphi_{C^*}\| \le \|C^*\|_{X_d^*}$, since using Lemma 3.2 we have

$$\begin{aligned}
\|\varphi_{C^*}(\{c_{i1}\}, \{c_{i2}\}, \dots)\| &= \left\|\sum c_{in}^*(c_{in})\right\| \\
&= \sup_{g\in X^*, \|g\|\le 1} \left|g\left(\sum c_{in}^*(c_{in})\right)\right| \\
&= \sup_{g\in X^*, \|g\|\le 1} \left|G_g\left(\sum c_{in}^*(c_{in})\right)\right| \\
&\le \sup_{g\in X^*, \|g\|\le 1} \|\{g(c_{in})\}\|_{X_d} \|\{c_{in}^*(c_{in})\}\|_{X_d^*} \\
&\le \|g\| \|\{c_{in}^*(c_{in})\}\|_{X_d^*}.
\end{aligned}$$

Thus,

$$\|\varphi_{C^*}\| \le \|C^*\|_{X_d^*}, \tag{3.2}$$

for all $C^* \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$.

Fix $0 < \varepsilon < 1$. For each n pick some $\{c_{in}\} \in (X_d)$ with $\|c_{in}\| = 1$ and $c_{in}^*(c_{in}) \ge \varepsilon \|c_{in}^*\|$.

Using Lemma 3.2, we have

$$\begin{aligned}\varepsilon\|c_{in}^*\|_{X_d^*} &= \varepsilon \sup_{g\in X^*,\|g\|\leq 1} |g(c_{in}^*)| = \varepsilon \sup_{g\in X^*,\|g\|\leq 1} |G_g(c_{in}^*)| \\ &\leq \varepsilon\|g\|\|\{c_{in}^*(c_{in})\}\| \leq \varepsilon\|g\|\|\sum_{i=1}^{\infty} c_{in}^*(c_{in})\|.\end{aligned}$$

This implies that $C^* = (\{c_{i1}^*\}, \{c_{i2}^*\}, \dots) \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$, $\|C^*\|_{X_d^*} \leq \|\varphi_{C^*}\|$. Finally, as a consequence of (3.2), we conclude that $C^* \mapsto \varphi_{C^*}$ is an onto linear isometry. □

Proposition 3.1. *Suppose that X_d is a BK-space, for which the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ forms a Schauder basis. Then $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subseteq X^*$ is an X_d^*-Bessel woven for X with universal bound B if and only if the operator*

$$T : \{\ c_{ij}\} \to \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$$

is well defined (hence bounded) from $\sum \oplus X_d$ into X^* and $\|T\| \leq B$.

Proof. Let $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ be an X_d^*-Bessel woven for X with universal bound B and let $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ be the canonical unit vector basis of X_d. Define

$$R : X \to \sum \oplus (X_d)^*$$

by

$$R(g) = \{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]}.$$

We have

$$\begin{aligned}\|R(g)\| = \|\{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]}\| &= \sup|f_{ij}(g(f))| = \sup_{g\in X^*,\|g\|=1} |G_g(f_{ij}(g(f)))| \\ &\leq \sup\|G_g\|\|f_{ij}(g(f))\|.\end{aligned}$$

Then $\|R\| \leq B$, the linear bounded operator $R^* : \sum \oplus (X_d)^{**} \to X^*$ satisfies:

$$R^*(e_{ij})(g) = e_{ij}(R(g)) = f_{ij}(g), \quad \text{for all } g \in X,$$

and thus $R^* e_{ij} = f_{ij}$. Letting $T = R^*|_{\sum \oplus X_d}$, we have

$$\|T\| \leq \|R^*\| = \|R\| \leq B.$$

Finally, $T\left(\{c_{ij}\}_{i\in\mathbb{I},j\in[m]}\right) = T\left(\sum_{i\in\mathbb{I},j\in[m]} c_{ij} e_{ij}\right) = \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$.

Now suppose that $T : \sum \oplus X_d \longrightarrow X^*$ given by $T(\{c_{ij}\}) = \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$ is well defined and thus bounded by the Banach-Steinhaus theorem. Then $T(e_{ij}) = f_{ij}$ and for every $g \in X$ the operator

$$T^* : X^{**} \to \sum \oplus (X_d)^*, \quad T^*(g)(e_{ij}) = g(T(e_{ij})) = g(f_{ij}),$$

is bounded. That is, $\{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]} = \{T^*(g)(e_{ij})\}_{i\in\mathbb{I},j\in[m]}$ which is identified with $T^*(g)$ (by Lemma 3.2). So, $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is a X_d^*- Bessel sequence for X with a bound $\|T^*\| = \|T\| \leq B$. □

Theorem 3.2. *The family* $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ *is a Bessel woven with Bessel bound* B *if and only if the operator*

$$T : \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \to \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}, \quad \textit{for all } \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right),$$

is a well-defined bounded operator from $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ *into* X *and* $\|T\| \le \sqrt{B}$.

Proof. First assume that $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is a Bessel woven with bound B.

Let $\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ be in $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$. We show that $T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ is well-defined, that is, $\sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}$ is convergent. Consider $n, m \in \mathbb{I}$, $n > m$. Then

$$\begin{aligned}\left\|\sum_{i=1,j\in[m]}^{n} l_{i,j}\varphi_{ij} - \sum_{i=1,j\in[m]}^{m} l_{ij}\varphi_{ij}\right\| &= \left\|\sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij}\right\| \\ &= \sup_{\|g^*\|=1, g\in X} g^*\left(\sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij}\right) = *.\end{aligned}$$

Using the duality mappings Φ_X and its induced semi-inner product $[f,g] = g^*(f)$ we have

$$\begin{aligned}* = \sup_{\|g\|=1}\left|\left[\sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij}, g\right]\right| &\le \sup_{\|g\|=1}\sum_{i=m+1,j\in[m]}^{n} |l_{ij}\,[\varphi_{ij}, g]\,| \\ \le \sup \|\{l_{ij}\}\|_{X_d}\|\,[\varphi_{ij}, g]\,\|_{X_d^*} &\le \sup \|\{l_{ij}\}\|_{X_d} B\|g\|_X.\end{aligned}$$

Since $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$, we know that $\|\{l_{ij}\}_{i=1,j\in[m]}^{n}\|_{X_d}$ is a Cauchy sequence in $\mathbb{C}$, The above calculation shows that $\{\sum_{i=1,j\in[m]}^{n} l_{ij}\varphi_{ij}\}_{n=1}^{\infty}$ is a Cauchy sequence in X, and therefore convergent. Thus, $T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ is well-defined. Clearly T is linear, and

$$\left\|T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}\right\| = \sup_{\|f\|=1}\left|\left[T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}, f\right]\right|,$$

that is, $\|T\| \le \sqrt{B}$.

Conversely, suppose T well-defined and that $\|T\| \le \sqrt{B}$, for every $f \in X$, we have

$$[T\{l_{ij}\}, f] = \left[\sum l_{ij}f_{ij}, f\right] = [\{l_{ij}\}, \{[f, f_{ij}]\}],$$

therefore

$$T^* f = \{[f, f_{ij}]\}_{i\in\mathbb{I},j\in[m]}$$

and

$$\sum_{i\in\mathbb{I},j\in[m]} |\,[f, f_{ij}]\,|^2 = \|T^* f\|^2 \le \|T^*\|^2\|f\|^2 \le \sqrt{B}\|f\|^2.$$

Hence, we conclude that the family $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is Bessel woven. □

Theorem 3.3. *Let the sequence* $\{\varphi_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ *in* X *be woven for* X*, and the series* $\sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}$ *converges for all* $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$*. Then the operator*

$$T : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X, \quad T\{l_{ij}\}_{i=1,j\in[m]}^{\infty} := \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij},$$

defines a bounded linear operator. The adjoint operator is given by

$$T^* : X^* \to \left(\sum_{j\in[m]} \oplus (X_d)_j\right)^*, \quad T^*\varphi = \{[\varphi, \varphi_{ij}]\}_{i=1,j\in[m]}^{\infty}.$$

Furthermore,

$$\sum_{i=1,j\in[m]}^{\infty} |\,[\varphi, \varphi_{ij}]\,|^2 \le \|T\|^2\|\varphi\|^2.$$

Proof. Consider the sequence of bounded linear operators

$$T_n : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X, \quad T_n\{l_{ij}\}_{i=1,j\in[m]}^{\infty} := \sum_{i=1,j\in[m]}^{n} l_{ij}\varphi_{ij}.$$

Clearly $T_n \to T$ pointwise as $n \to \infty$, so T is bounded. In order to find the expression for T^*, let $f, \varphi \in X$, $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$. Then

$$\left[\varphi, T\,\{l_{ij}\}_{i=1,j\in[m]}^{\infty}\right]_X = \left[\varphi, \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}\right] = \sum_{i=1,j\in[m]}^{\infty} [\varphi, \varphi_{ij}]\,\overline{l_{ij}}.$$

Alternatively, when $T : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X$ is bounded, then clearly T^* is a bounded operator from X^* to $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)^*$. Therefore, the i-th coordinate function is bounded from X to $\mathbb{C}$; by Riesz representation theorem, T^* has the form

$$T^*\varphi = \{[\varphi, \varphi_{ij}]\}_{i=1,j\in[m]}^{\infty},$$

for some $\{\varphi_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ in X. By definition of T^*, we conclude

$$\sum_{i=1,j\in[m]}^{\infty} [\varphi, f_{ij}]\,\overline{l_{ij}} = \sum_{i=1,j\in[m]}^{\infty} [\varphi, \varphi_{ij}]\,\overline{l_{ij}}, \quad \text{for all } \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right), f \in X.$$

It follows from here that $f_{ij} = \varphi_{ij}$. The adjoint of a bounded operator T is itself bounded, and $\|T\| = \|T^*\|$. Under the assumption in Theorem 3.2, we have

$$\|T^*\varphi\|^2 \le \|T\|^2\|\varphi\|^2, \quad \text{for all } \varphi \in X,$$

which leads to

$$\sum_{i=1,j\in[m]}^{\infty} |\,[\varphi, \varphi_{ij}]\,|^2 \le \|T\|^2\|\varphi\|^2, \quad \text{for all } \varphi \in X. \qquad \square$$

Definition 3.8. Let X be a Banach space and X_d a sequence space. Given a bounded linear operator $S : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X$ and a $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$-woven $\{g_{ij}\} \subset X^*$, we say that $(\{g_{ij}\}, S)$ is a Banach frame for X with respect to $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ if

$$S(\{g_{ij}(f)\}) = f, \quad \text{for all } f \in X. \tag{3.3}$$

Note that (3.3) can be considered as some kind of generalized reconstruction formula, in the sense that it tells how to come back to $f \in X$ via the coefficients $\{g_{ij}(f)\}$.

The condition, however, does not imply reconstruction via an infinite series, as we will see later. For more information on Banach frames we refer to [7, 17].

The woven X_d-frame condition implies that one can define the following isomorphism

$$U : X \to \left(\sum_{j\in[m]} \oplus (X_d)_j\right), \quad Uf := \{g_{ij}(f)\}, \quad f \in X.$$

The extra condition in Definition 3.8 means that S is a left-inverse of U, and thus US is a bounded linear projection of $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ onto the range $R(U)$ of the operator U.

Proposition 3.2. *Suppose that X_d is a BK-space and $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ is a woven X_d-frame for X. Then, the following conditions are equivalent.*

(a) *$R(U)$ is complemented in X_d.*

(b) *The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : X_d \to X$.*

(c) *There exists a linear bounded operator S, such that $\left(\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}, S\right)$ is a Banach woven for X with respect to X_d.*

Also, the condition

(d) *there exists a family $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X$ such that $\{\sum c_{ij} f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is convergent for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$ and*

$$f = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}, \quad \text{for all } f \in X;$$

implies each of (a)-(c).

If we also assume that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d, (d) *is equivalent to* (a)-(c).

(e) *There exists an X_d^*-Bessel woven $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X \subseteq X^{**}$ for X^* such that*

$$f = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}, \quad \text{for all } f \in X.$$

If the canonical unit vectors form a basis for both X_d and X_d^*, (a)-(e) *is equivalent to*

(f) *there exists an X_d^*-Bessel woven $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X \subset X^{**}$ for X^* such that*

$$g = \sum_{i\in\mathbb{I},j\in[m]} g(f_{ij}) g_{ij}, \quad \text{for all } g \in X^*.$$

In each of the cases (e) *and* (f), $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *is actually an* X_d^**-woven for* X^*.

Proof. For convenience, we index $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ and $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$ by the natural numbers. Suppose that X_d is a BK-space. $(a)\to(b)$ is trivial. For the converse, assume (b) and let $V: X_d \to X$ be a linear bounded extension of U^{-1}. Now consider the bounded operator $P: X_d \to R(U)$ defined by $P = UV$. Using the fact that $VU = I$ (on X), we get $P^2 = P$. For every $f \in X$, we have

$$Uf = UVUf = P(Uf) \in R(P).$$

Hence $R(P) = R(U)$, i.e., the range of U equals the range of a bounded projection. Thus, $R(U)$ is complemented (see [27, page 127]). The equivalence $(b)\leftrightarrow(c)$ is clear. We now relate the condition (d) to (a)-(c). First, assume that (d) is satisfied. By assumption, we can define an operator

$$V: X_d \to X, \quad V: \{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \to \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}.$$

By the Banach-Steinhaus theorem, V is bounded. Let $\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]} \in R(U)$. Furthermore,

$$V(g_{ij}(f)) = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij} = f = U^{-1}Uf = U^{-1}\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]},$$

that is, V is an extension of U^{-1}. That is, (b) holds, according to the equivalences proved so far, this means that (a)-(c) holds.

Assume now that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d. Assuming that (b) is satisfied, we show that (d) holds. Let $f_{ij} := Ve_{ij}$. Since V is linear and bounded, for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$, we have

$$\sum_{i=1,j\in[m]}^{n} c_{ij} f_{ij} = V\left(\sum_{i=1,j\in[m]}^{n} c_{ij} e_{ij}\right) \to V(c_{ij}).$$

That is, $\sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$ is convergent. Also, by construction, for all $f \in X$ we have

$$f = VUf = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}. \tag{3.4}$$

Thus, (d) holds as claimed.

Under the assumption that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d, we now prove the equivalence of (d) and (e). First, assume that (d) holds. Due to the equivalence of (b) and (d), we can define $f_{ij} := Le_{ij}$, and (3.4) is available. By Lemma 3.2, for every $g \in X^*$ we have

$$\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]} = \{gV(e_{ij})\}_{i\in\mathbb{I},j\in[m]} \in X_d^*$$

and

$$\|\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]}\|_{X_d^*} = \|gV\| \le \|V\| \, \|g\|_{X^*},$$

hence $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$, considered as a sequence in X^{**}, is an X_d^*-Bessel sequence for X^*. Thus, we have proved the claims in (e). On the other hand, if (e) is valid, then Proposition 3.1 shows that $\sum_{i\in\mathbb{I},j\in[m]} c_{ij}f_{ij}$ is convergent for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$ and hence (d) holds.

Now, assume that the canonical unit vectors form a basis for both X_d and X_d^*; in this case, we want to prove the equivalence of (e) and (f). Let B denote a Bessel bound for the X_d-Bessel sequence $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$. Denote the canonical basis for X_d by $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ and the canonical basis for X_d^* by $\{z_{ij}\}_{i\in\mathbb{I},j\in[m]}$. Assume that (e) is valid. Let $g \in X^*$. For given $n \in \mathbb{N}$

$$\begin{aligned}
\left\| g - \sum_{i=1,j\in[m]}^{n} g\left(f_{ij}\right) g_{ij} \right\|_{X^*} &= \sup_{f\in X,\|f\|=1} \left| g(f) - \sum_{i=1,j\in[m]}^{n} g\left(f_{ij}\right) g_{ij}\left(f\right) \right| \\
&= \sup_{f\in X,\|f\|=1} \left| \sum_{i=1,j\in[m]}^{\infty} g\left(f_{ij}\right) g_{ij}\left(f\right) - \sum_{i=1,j\in[m]}^{n} g\left(f_{ij}\right) g_{ij}\left(f\right) \right| \\
&= \sup_{f\in X,\|f\|=1} \left| \sum_{i=n+1,j\in[m]}^{\infty} g(f_{ij}) g_{ij}\left(f\right) \right| \\
&\le B \left\| \sum_{i=n+1,j\in[m]}^{\infty} g\left(f_{ij}\right) z_{ij} \right\| \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

and hence (f) holds. Assume (f) and let K be an X_d^*-Bessel bound for $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$. For every $g \in X^*$, $\{g\left(f_{ij}\right)\}_{i\in\mathbb{I},j\in[m]}$ belongs to X_d^*, which by Lemma 3.2 is isometrically isomorphic to the space $\{\{G\left(e_{ij}\right)\}_{i\in\mathbb{I},j\in[m]} | G \in X_d^*\}$, and hence $\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]}$ can be identified with $\{G_g(e_{ij})\}_{i\in\mathbb{I},j\in[m]}$ for a unique $G_g \in X_d^*$. Then for every $f \in X$

$$\begin{aligned}
\left\| f - \sum_{i=1,j\in[m]}^{n} g_{ij}\left(f\right) f_{ij} \right\|_{X} &= \sup_{g\in X^*,\|g\|=1} \left| g\left(f\right) - \sum_{i=1,j\in[m]}^{n} g\left(f_{ij}\right) g_{ij}\left(f\right) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left| \sum_{i=1,j\in[m]}^{\infty} g\left(f_{ij}\right) g_{ij}\left(f\right) - \sum_{i=1}^{n} g\left(f_{ij}\right) g_{ij}\left(f\right) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left| \sum_{i=n+1,j\in[m]}^{\infty} g\left(f_{ij}\right) g_{ij}\left(f\right) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left\| G_g \left(\sum_{i=n+1,j\in[m]}^{\infty} g_{ij}\left(f\right) e_{ij} \right) \right\| \\
&\le \sup_{g\in X^*,\|g\|=1} \|G_g\| \left\| \sum_{i=n+1,j\in[m]}^{\infty} g_{ij}(f) e_{ij} \right\| \\
&= \sup_{g\in X^*,\|g\|=1} \left\| \{g\left(f_{ij}\right)\} \sum_{i=n+1,j\in[m]}^{\infty} g_{ij}\left(f\right) e_{ij} \right\|
\end{aligned}$$

$$\leq K \left\| \sum_{i=n+1, j\in[m]}^{\infty} g_{ij}(f)\, e_{ij} T \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, (f) is valid. Moreover, by a similar calculations as above, for every $g \in X^*$ we have

$$\|g\| = \sup_{f\in X^*, \|f\|=1} |g(f)| = \sup_{f\in X^*, \|f\|=1} \left| \sum_{i\in\mathbb{I}, j\in[m]} g(f_{ij})\, g_{ij}(f) \right| \leq B \left\| \{g(f_{ij})\}_{i\in\mathbb{I}, j\in[m]} \right\|_{X_d^*},$$

and hence $\{f_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ is a woven X_d^*-frame for X^*. □

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A NEW INERTIAL-PROJECTION METHOD FOR SOLVING SPLIT GENERALIZED MIXED EQUILIBRIUM AND HIERARCHICAL FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new iterative algorithm of inertial form for approximating the common solution of Split Generalized Mixed Equilibrium Problem (SGMEP) and Hierarchical Fixed Point Problem (HFPP) in real Hilbert spaces. Motivated by the subgradient extragradient method, we incorporate the inertial technique to accelerate the convergence of the proposed method. Under standard and mild assumption of monotonicity and lower semicontinuity of the SGMEP and HFPP associated mappings, we establish the strong convergence of the iterative algorithm. Some numerical experiments are presented to illustrate the performance and behaviour of our method as well as comparing it with some related methods in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a nonlinear mapping. T is said to be:

(i) firmly nonexpansive, if for each $x, y \in C$

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle;$$

(ii) a contraction, if for every $x, y \in C$ and $c \in (0, 1)$

$$\|Tx - Ty\| \leq c\|x - y\|.$$

If $c = 1$, then T is called nonexpansive.

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We denote by $Fix(T)$, the set of fixed points of the mapping T, that is $Fix(T) = \{x \in C : x = Tx\}$. The mapping T is called quasi nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \text{for all } p \in Fix(T),\ x \in C.$$

It is known that if T is quasinonexpansive, then $Fix(T)$ is closed and convex (see [45]).

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The Equilibrium Problem (EP) in the sense of Blum and Oetlli [9], is to find a point $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

We denote by $EP(F, C)$, the set of solutions of EP (1.1). The EP unifies many important mathematical problems, such as optimization problems, complementary problems, fixed point problems, variational inequality problems, see [4, 6, 9, 25, 36, 37]. Let $B : C \to H$ be a nonlinear mapping. The Variational Inequality Problem (VIP) is to obtain a point $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

The set of solutions of the VIP is denoted $VIP(B, C)$. Solution to these class of problems, fixed point problems and related optimization problems have been investigated and iterative algorithm for approximating them have been proposed and studied by several authors, see [2, 5, 10, 14, 15, 17, 19, 20, 27, 28, 32, 35]. Let $\phi : C \to \mathbb{R}$ be a real valued function, then the Minimization Problem (MP), consists of finding a point $x^* \in C$ such that

$$\phi(x^*) \leq \phi(y), \quad \text{for all } y \in C. \tag{1.3}$$

The set of solutions of MP (1.3) will be denoted by $MP(\phi, C)$. For more on MP (see [1, 8, 23, 42]) and the references therein.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $B : C \to H$ a nonlinear mapping and $\phi : C \to \mathbb{R}$ a proper, convex and lower semicontinuous function. The Generalized Mixed Equilibrium Problem (GMEP) [10, 24, 26, 33, 38, 48] is the problem of finding a point $x^* \in C$ such that

$$F(x^*, y) + \langle Bx^*, y - x^* \rangle + \phi(y) - \phi(x^*) \geq 0, \quad \text{for all } y \in C. \tag{1.4}$$

We use $GMEP(F, B, \phi)$ to denote the set of solutions of GMEP (1.4). The GMEP includes several optimization problems as special cases. The relationship with the VIP and MP are easily observed by setting some maps to the zero map in inequality (1.4). Numerous problems in economics, science and engineering can be reduced to the problem of finding a solution to the GMEP (see [26, 34, 37]).

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $L : H_1 \to H_2$ a bounded linear operator. In 1994, Censor and Elfving [12] introduced the notion of Split Feasibility Problem (SFP), which is defined as follows: find a point

$$x^* \in C \text{ such that } Lx^* \in Q. \tag{1.5}$$

The SFP is a special case of the Split Inverse Problem (SIP) first studied by Censor et al. [13]. In SIP, there are two given vector spaces X and Y and a linear operator $L : X \to Y$. The first Inverse Problem, IP_X say, is formulated in space X and the second one IP_Y formulated in space Y. Given this information, the SIP is formulated as follows: find $x^* \in X$ that solves IP_X, such that $y^* = Lx^* \in Y$ solves IP_Y. The SIP is used as a model for sensor networks, radiation therapy treatment planning, color imaging and other image restoration problems, see [11].

Furthermore, SFP over EP have been studied by some authors in the literature. For example, Moudafi [30] considered a SFP over EP and called this the Split Equilibrium Problem (SEP), see [22]. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions and $L : H_1 \to H_2$ be a bounded linear operator. The SEP is given as follows: find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \text{for all } x \in C,$$

and such that

$$y^* = Lx^* \in Q \text{ solves } G(y^*, y) \geq 0, \quad \text{for all } y \in Q.$$

For more, see [37, 46] and the references therein.

Since then, there have been several research in this direction where both bifunctions have same mononotonicity property and others with different monotonicity properties. Dinh et al. [16], studied the SEP involving pseudomonotone and monotone bifunctions. Also, in 2017 Rattanaseeha et al. [40], studied a split generalized equilibrium problem which involves both pseudomonotone bifunction and a monotone bifunction. For more literature on this class of problems (see [16, 40, 43]) and the references therein.

Moudafi and Mainge [31] introduced and studied the following Hierarchical Fixed Point Problem (HFPP) for a nonexpansive mapping S with respect to another nonexpansive mapping T on C. The HFPP consists of finding a point $x^* \in Fix(S)$ such that

$$\langle (I - T)x^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in Fix(S). \tag{1.6}$$

It is easy to see that the HFPP is equivalent to the problem of finding the fixed point of a map $A = P_{Fix(S)} \circ T$. Let Ω denote the solution set of the HFPP (1.6). Note that if $\Omega \neq \emptyset$, then Ω is closed and convex. The HFPP is general in the sense that it includes as special case the monotone VIP on fixed point sets, MP over equilibrium constraints, hierarchical MP... Very recently, Alansari et al. [7], studied an hybrid iterative scheme for approximating a common solution of a split EP involving both monotone and pseudomonotone bifiunction and a HFPP for a nonexpansive and quasi nonexpansive mappings. They proved a weak convergence theorem for their proposed algorithm.

Inspired by the works above and current research interest in this direction, in particular, in order to provide a partial answer to the future research posed by Alansari et al. [7] in conclusion of their work. We propose an iterative algorithm which combines the inertial technique, projection method, diagonal subgradient method and viscosity approach [8, 24], see Section 3. We prove a strong convergence theorem using the proposed algorithm to a solution of a SGMEP involving a pseudomonotone bifunction and a monotone bifunction which is also a solution of a HFPP. In our proposed method, the inertial extrapolation step was included to accelerate the rate of convergence of the algorithm, (see [1, 3, 25, 39]) for more literature on inertial algorithms. We present some numerical examples to illustrate the behaviour and performance of our method as well as comparing it with some related methods in the literature.

2. PRELIMINARIES

We denote by $x_n \rightharpoonup v$ and $x_n \to v$ the weak and strong convergence respectively of a sequence $\{x_n\}$ in H to a point $v \in H$.

For each $x \in H$, there exists a unique nearest point $y = P_C x \in C$ such that

$$\|x - y\| \leq \|x - z\|, \quad \text{for all } z \in C.$$

The mapping $P_C : H \to C$ is called the metric projection from H onto C. It is well known that P_C satisfies the following conditions.

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$.

(ii) For $x \in H$ and $y \in C$, $y = P_C x$ if and only if

$$\langle x - y, y - z\rangle \geq 0, \quad \text{for all } z \in C. \tag{2.1}$$

Definition 2.1. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to $x \in C$ with $\|x_n - Tx_n\| = 0$, then $Tx = x$.

It is well known (see [21]) that the nonexpansive mapping is demiclosed.

Definition 2.2. A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(a) strongly monotone on C, if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \quad \text{for all } x, y \in C;$$

(b) monotone on C, if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(c) pseudomonotone on C, if $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in C$.

It is obvious from above that a strongly pseudomonotone bifunction is contained in the class of monotone bifunctions and a monotone bifunction is pseudomonotone.

Definition 2.3 ([18])**.** Let $f : C \times C \to \mathbb{R}$ be a bifunction, where $f(x, \cdot)$ is convex for each $x \in C$. Then for $\epsilon \geq 0$ the ϵ-subdifferential (ϵ-diagonal subdifferential) of f at x, denoted by $\partial_\epsilon f(x, \cdot)(x)$ is given by

$$\partial_\epsilon f(x, \cdot)(x) = \{z \in H_1 : f(x, y) + \epsilon \geq f(x, x) + \langle z, y - x\rangle \text{ for all } y \in C\}.$$

For solving the GMEP, we assume $\phi : Q \to \mathbb{R}$ is proper, convex and lower semicontinuous, the nonlinear mapping, $B : Q \to H_2$ is continuous and monotone and the bifunction $F : Q \times Q \to \mathbb{R}$ satisfies the following restrictions:

(R1) $F(x,x) = 0$ for all $x \in Q$;
(R2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in Q$;
(R3) $\lim_{t\downarrow 0} F(x + t(z-x), y) \leq F(x,y)$ for all $x, y, z \in Q$;
(R4) for each $x \in Q$, the function $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemmas are used in the sequel.

Lemma 2.1 ([44])**.** *In a real Hilbert space H, the following hold:*

(i) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$;
(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *for all* $x, y \in H$;
(iii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $x, y \in H$ *and* $t \in (0,1)$.

Lemma 2.2 ([48])**.** *Let $B : Q \to H_2$ be a continuous and monotone mapping, $\phi : Q \to \mathbb{R}$ be a proper, lower semicontinuous and convex function, and $F : Q \times Q \to \mathbb{R}$ be a bifunction satisfying the conditions* (R1)-(R4)*. Let $r > 0$ be any given number and $x \in H_2$ be any given point. Then, the following hold.*

(i) *There exists $w \in Q$ such that*

$$F(w,y) + \langle B(w), y-w\rangle + \phi(y) - \phi(w) + \frac{1}{r}\langle y-w, w-x\rangle \geq 0, \quad \text{for all } y \in Q.$$

(ii) *Define a mapping $K_r^{F,B,\phi} : Q \to Q$ by $K_r^{F,B,\phi}(x) = \Big\{w \in Q : F(w,y) + \langle B(w), y-w\rangle + \phi(y) - \phi(w) + \frac{1}{r}\langle y-w, w-x\rangle \geq 0, y \in Q\Big\}$, $x \in Q$.*
The mapping $K_r^{F,B,\phi}$ satisfies the following characteristics:
(a) *$K_r^{F,B,\phi}$ is single valued;*
(b) *$K_r^{F,B,\phi}$ is fimrly nonexpansive, i.e., for all $z, y \in H$*

$$\|K_r^{F,B,\phi}z - K_r^{F,B,\phi}y\|^2 \leq \langle K_r^{F,B,\phi}z - K_r^{F,B,\phi}y, z-y\rangle;$$

(c) *$Fix(K_r^{F,B,\phi}) = GMEP(F,B,\phi)$;*
(d) *$GMEP(F,B,\phi)$ is a closed and convex subset of Q.*

The following restrictions are assumed to be satisfied by the pseudomonotone bifunction $f : C \times C \to \mathbb{R}$:

(F1) $f(x,x) = 0$ for all $x \in C$;
(F2) f is pseudomonotone on C with respect $x \in EP(f,C)$, that is, for $x \in EP(f,C)$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$ for all $y \in C$;
(F3) f is strict paramonotone, that is the following holds

$$x \in EP(f,C), \quad y \in C, \quad f(y,x) \leq 0 \text{ implies } y \in EP(f,C);$$

(F4) f is jointly weakly upper semicontinuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subseteq C$ converges weakly to x and y, respectively, then $f(x_n, y_n) \to f(x,y)$ as $n \to +\infty$.

The following lemmas are very useful in obtaining the strong convergence of the sequence considered in this work.

Lemma 2.3 ([47])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the restrictions:

(i) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$, $\lim_{n\to+\infty} \alpha_n = 0$;
(ii) $\limsup_{n\to+\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\sum_{n=1}^{+\infty} \gamma_n < +\infty$.

Then $\lim_{n\to+\infty} a_n = 0$.

Lemma 2.4 ([29, 41])**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} \leq a_{n_j+1}$ for all $j \in \mathbb{N}$. Consider the integer $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{j \leq n : a_j \leq a_{j+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence satisfying $\lim_{n\to+\infty} \tau(n) = +\infty$ *and for all $n \geq n_0$, the following estimates hold:*

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \text{and} \quad a_n \leq a_{\tau(n)+1}.$$

3. Main Result

In this section, we state and prove our main result. First, we give an explicit statement of the proposed problem in this study. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $L : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be pseudomonotone and monotone bifunctions respectively satisfying restrictions (F1)-(F4) and (R1)-(R4). Let $B : C \to H_2$ be a nonlinear mapping and $\phi : Q \to \mathbb{R}$ a proper, convex and lower semicontinuous function. Let S be a nonexpansive mapping and T a quasinonexpansive mapping such that $I - T$ is monotone. We consider the problem of finding a point $x^* \in C$ such that

$$x^* \in EP(f, C) \cap Fix(P_{Fix(S)} \circ T) \tag{3.1}$$

and such that

$$y^* = Lx^* \in Q \text{ solves } GMEP(F, B, \phi). \tag{3.2}$$

We assume that the solution set of Problem (3.1)–(3.2) denoted by Γ is nonempty.

Remark 3.1 ([18]). If f is pseudomonotone on C with respect to $EP(f, C)$, then by restrictions (F1) and (F4), $EP(f, C)$ is closed and convex. From Lemma 2.2 (d), we have that $GMEP(F, B, \phi)$ is closed and convex. Also, if $Fix(P_{Fix(S)} \circ T) \neq \emptyset$, then the solution set of the HFPP is closed and convex see [31]. We assume $\Gamma \neq \emptyset$, hence Γ is well defined.

Algorithm 3.1. **Initialization.** Choose $x_0, x_1 \in C$. Take the sequence of real numbers $\{\mu_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfying

(i) $0 < r < r_n$, $0 < a < \alpha_n < b < 1$, $0 < \acute{a} < \lambda_n < \acute{b} < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, $\beta_n \geq 0$, $\gamma_n \in (0, 2/\|L\|^2)$ and $\epsilon_n \to 0$ as $n \to +\infty$;
(ii) $\sum_{n=1}^{+\infty} \mu_n^2 < +\infty$;
(iii) $\sum_{n=1}^{+\infty} \beta_n = +\infty$, $\lim_{n\to+\infty} \beta_n = 0$;
(iv) $\{\theta_n\} \subset [0, \theta]$, where $\theta \in [0, 1)$ and $\sum_{n=1}^{+\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$;
(v) $\lim_{n\to+\infty} \frac{\theta_n}{\beta_n} = 0$.

[**Step 1.** Given x_{n-1} and x_n, $n \geq 1$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.3}$$

Step 2. Take $g(w_n) \in \partial_{\epsilon_n}(f(w_n, \cdot))(w_n)$, $n \geq 1$. Calculate $\eta_n = \max\{1, \|g(w_n)\|\}$, $\lambda_n = \frac{\mu_n}{\eta_n}$ and

$$z_n = P_C(w_n - \lambda_n g(w_n)). \tag{3.4}$$

Step 3. If $w_n = z_n$ ($w_n \in EP(f, C)$), then stop. Otherwise, evaluate

$$\begin{cases} t_n = (1 - \sigma_n)Tz_n + \sigma_n z_n \;\; y_n = (1 - \alpha_n)w_n + \alpha_n St_n, \\ u_n = K_{r_n}^{F,B,\phi} Ly_n, \\ v_n = y_n + \gamma_n L^*(u_n - Ly_n). \end{cases} \tag{3.5}$$

Step 4. Compute

$$x_{n+1} = \beta_n h(x_n) + (1 - \beta_n)v_n, \tag{3.6}$$

where h is a contraction.

Step 5. Set $n := n + 1$ and go to step 1.

Lemma 3.1. *Let $\{x_n\}$ be the sequence given by Algorithm* 3.1*, then $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded.*

Proof. Let $u \in \Gamma$, then from Lemma 2.1 (i) and (3.3), we have

$$\begin{aligned} \|w_n - u\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - u\|^2 \\ &= \|x_n - u\|^2 + 2\theta_n\langle x_n - u, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - u\|^2 + 2\theta_n\|x_n - u\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &= \|x_n - u\|^2 + \theta_n\|x_n - x_{n-1}\|(2\|x_n - u\| + \theta_n\|x_n - x_{n-1}\|). \end{aligned} \tag{3.7}$$

It also follows from Lemma 2.1 (iii), that

$$\begin{aligned} \|t_n - u\|^2 &= \|(1 - \sigma_n)(Tz_n - u) + \sigma_n(z_n - u)\|^2 \\ &= (1 - \sigma_n)\|Tz_n - u\|^2 + \sigma_n\|z_n - u\|^2 - \sigma_n(1 - \sigma_n)\|Tz_n - z_n\|^2 \\ &\leq \|z_n - u\|^2 - \sigma_n(1 - \sigma_n)\|Tz_n - z_n\|^2 \\ &\leq \|z_n - u\|^2. \end{aligned} \tag{3.8}$$

Next,

$$\|y_n - u\|^2 = \|(1 - \alpha_n)(w_n - u) + \alpha_n(St_n - u)\|^2$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|St_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\le (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|t_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\le (1-\alpha_n)\|w_n-u\|^2+\alpha_n\|z_n-u\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\le \|w_n-u\|^2+2\alpha_n\langle w_n-z_n,u-z_n\rangle-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2, \quad (3.9)
\end{aligned}
$$

but from the definition of z_n, we get

$$\langle w_n-z_n,u-z_n\rangle \le \lambda_n\langle g(w_n),u-z_n\rangle.$$

Using this in (3.9), we obtain

$$
\begin{aligned}
\|y_n-u\|^2 &\le \|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-z_n\rangle-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&= \|w_n-u\|^2+2\lambda_n\alpha_n[\langle g(w_n),u-w_n\rangle+\langle g(w_n),w_n-z_n\rangle]\\
&\quad -\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&\le \|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-w_n\rangle+2\lambda_n\alpha_n\|g(w_n)\|\|w_n-z_n\|\\
&\quad -\alpha_n(1-\alpha_n)\|St_n-w_n\|^2. \quad (3.10)
\end{aligned}
$$

Note that by the definition of z_n and $w_n\in C$, we have

$$\|w_n-z_n\|^2 \le \lambda_n\langle g(w_n),w_n-z_n\rangle \le \lambda_n\|g(w_n)\|\|w_n-z_n\|,$$

thus $\|w_n-z_n\|\le\lambda_n\|g(w_n)\|$ and

$$
\begin{aligned}
\lambda_n\|g(w_n)\|\|w_n-z_n\| &\le \lambda_n^2\|g(w_n)\|^2\\
&= \left(\frac{\mu_n}{\eta_n}\right)^2\|g(w_n)\|^2=\mu_n^2\left(\frac{\|g(w_n)\|}{\max(1,\|g(w_n)\|)}\right)^2\\
&\le \mu_n^2, \quad (3.11)
\end{aligned}
$$

which implies

$$\|w_n-z_n\|^2\le\mu_n^2. \quad (3.12)$$

Since $\sum_{n=1}^{+\infty}\mu_n^2<+\infty$, we obtain from above inequality, that

$$\|w_n-z_n\|\to 0 \quad \text{as} \quad n\to+\infty. \quad (3.13)$$

Using (3.11) in (3.10), we have

(3.14)

$$\|y_n-u\|^2 \le \|w_n-u\|^2+2\lambda_n\alpha_n\langle g(w_n),u-w_n\rangle+2\alpha_n\mu_n^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2.$$

By using Lemma 2.2, we have

$$
\begin{aligned}
\|u_n-Lu\|^2 &= \|K_{r_n}^{F,B,\phi}Ly_n-Lu\|^2=\|K_{r_n}^{F,B,\phi}Ly_n-K_{r_n}^{F,B,\phi}Lu\|^2\\
&\le \langle K_{r_n}^{F,B,\phi}Ly_n-K_{r_n}^{F,B,\phi}Lu,Ly_n-Lu\rangle\\
&= \langle K_{r_n}^{F,B,\phi}Ly_n-Lu,Ly_n-Lu\rangle\\
&= \frac{1}{2}\left(\|K_{r_n}^{F,B,\phi}Ly_n-Lu\|^2+\|Ly_n-Lu\|^2-\|K_{r_n}^{F,B,\phi}Ly_n-Ly_n\|^2\right).
\end{aligned}
$$

Hence, $\|u_n - Lu\|^2 \leq \|Ly_n - Lu\|^2 - \|u_n - Ly_n\|^2$, which implies

$$2\langle Ly_n - Lu, u_n - Ly_n\rangle \leq -2\|u_n - Ly_n\|^2. \tag{3.15}$$

Now, from (3.5) and (3.15), we have

$$\begin{aligned}
\|v_n - u\|^2 =&\|y_n + \gamma_n L^*(u_n - Ly_n) - u\|^2\\
=&\|y_n - u\|^2 + 2\gamma_n\langle y_n - u, L^*(u_n - Ly_n)\rangle + \gamma_n^2\|L^*(u_n - Ly_n)\|^2\\
=&\|y_n - u\|^2 + 2\gamma_n\langle Ly_n - Lu, u_n - Ly_n\rangle + \gamma_n^2\|L^*(u_n - Ly_n)\|^2\\
\leq&\|y_n - u\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2, \qquad (3.16)
\end{aligned}$$

which implies

$$\begin{aligned}
\|v_n - u\|^2 \leq&\|w_n - u\|^2 + 2\lambda_n\alpha_n\langle g(w_n), u - w_n\rangle + 2\alpha_n\mu_n^2\\
&- \alpha_n(1 - \alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2. \qquad (3.17)
\end{aligned}$$

Since $w_n \in C$ and $g(w_n) \in \partial_{\epsilon_n} f(w_n, \cdot)(w_n)$, we obtain

$$f(w_n, u) + \epsilon_n = f(w_n, u) - f(w_n, w_n) + \epsilon_n \geq \langle g(w_n), u - w_n\rangle.$$

Using this in (3.17), we get

$$\begin{aligned}
\|v_n - u\|^2 \leq&\|w_n - u\|^2 + 2\lambda_n\alpha_n(f(w_n, u) + \epsilon_n) + 2\alpha_n\mu_n^2\\
&- \alpha_n(1 - \alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2.
\end{aligned}$$

From the definition of λ_n and η_n, we obtain

$$\lambda_n = \frac{\mu_n}{\eta_n} \leq \mu_n.$$

Therefore, we get from above, that

$$\begin{aligned}
\|v_n - u\|^2 \leq&\|w_n - u\|^2 + 2\lambda_n\alpha_n f(w_n, u) + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2)\\
&- \alpha_n(1 - \alpha_n)\|St_n - w_n\|^2 - \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2. \qquad (3.18)
\end{aligned}$$

Since $u \in \Gamma$ and $w_n \in C$, we have $f(u, w_n) \geq 0$, then it follows from the monotonicity of f that $f(w_n, u) \leq 0$ and

$$\begin{aligned}
\|v_n - u\|^2 \leq&\|w_n - u\|^2 + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2) - \alpha_n(1 - \alpha_n)\|St_n - w_n\|^2\\
&- \gamma_n(2 - \gamma_n\|L\|^2)\|u_n - Ly_n\|^2\\
\leq&\|w_n - u\|^2 + 2\alpha_n(\mu_n\epsilon_n + \mu_n^2), \qquad (3.19)
\end{aligned}$$

which implies that

$$\|v_n - u\| \leq \|w_n - u\| + \sqrt{(2\alpha_n(\mu_n\epsilon_n + \mu_n^2)}. \tag{3.20}$$

Furthermore, we have from (3.6) and some $M_1, M_2 > 0$, that

$$\begin{aligned}
\|x_{n+1} - u\| =&\|\beta_n h(x_n) + (1 - \beta_n)v_n - u\|\\
\leq&\beta_n\|h(x_n) - u\| + (1 - \beta_n)\|v_n - u\|\\
\leq&\beta_n\|h(x_n) - h(u)\| + \beta_n\|h(u) - u\| + (1 - \beta_n)\|v_n - u\|
\end{aligned}$$

$$
\begin{aligned}
&\leq c\beta_n\|x_n - u\| + \beta_n\|h(u) - u\| + (1-\beta_n)\Big(\|w_n - u\| \\
&\quad + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)}\Big) \\
&\leq c\beta_n\|x_n - u\| + (1-\beta_n)\Big(\|x_n - u\| + \theta_n\|x_n - x_{n-1}\| \\
&\quad + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)}\Big) + \beta_n\|h(u) - u\| \\
&\leq [1 - \beta_n(1-c)]\|x_n - u\| + \theta_n(1-\beta_n)\|x_n - x_{n-1}\| + \beta_n\|h(u) - u\| \\
&\quad + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)} \\
&= [1 - \beta_n(1-c)]\|x_n - u\| + \theta_n(1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + \frac{\beta_n(1-c)}{1-c}\|h(u) - u\| + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)} \\
&\leq \max\left\{\|x_n - u\|, \frac{\|h(u) - u\|}{(1-c)}\right\} + \theta_n(1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)} \\
&\leq \max\left\{\max\left\{\|x_{n-1} - u\|, \frac{\|h(u) - u\|}{(1-c)}\right\}\right\} \\
&\quad + \theta_{n-1}(1-\beta_{n-1})\|x_{n-1} - x_{n-2}\| \\
&\quad + \sqrt{2\alpha_{n-1}(\mu_{n-1}\epsilon_{n-1} + \mu_{n-1}^2)} + \theta_n(1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + \sqrt{2\alpha_n(\mu_n\epsilon_n + \mu_n^2)} \\
&\vdots \\
&\leq \max\left\{\|x_1 - u\|, \frac{\|h(u) - u\|}{1-c}\right\} + M_1 + M_2 \\
&< +\infty,
\end{aligned}
$$

where

$$M_1 = \sum_{i=1}^{n} \theta_i(1-\beta_i)\|x_i - x_{i-1}\| < +\infty,$$

by condition (iv) and

$$M_2 = \sum_{i=1}^{n} \sqrt{2\alpha_i(\mu_i\epsilon_i + \mu_i^2)}.$$

Hence, $\{x_n\}$ is bounded. Consequently, all other sequences in Algorithm 3.1 are bounded. □

Lemma 3.2. *The following inequality is satisfied from* (3.6) *and all* $u \in \Gamma$

$$\|x_{n+1} - u\|^2 \leq \left(1 - \frac{2\beta_n(1-c)}{1 - c\beta_n}\right)\|x_n - u\|^2$$

$$+\frac{2\beta_n(1-c)}{1-c\beta_n}\left(\frac{\langle h(u)-u, x_{n+1}-u\rangle}{1-c}+\frac{\beta_n M_3}{1-c}\right)$$
$$+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}(\|x_n-x_{n-1}\|)(M_4+\|x_n-x_{n-1}\|)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2),$$

for some $M_3, M_4 > 0$.

Proof. Let $u \in \Gamma$, then from Lemma 2.1 (ii), (3.4) and some $M_3, M_4 > 0$, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 =& \|\beta_n(h(x_n)-u)+(1-\beta_n)(v_n-u)\|^2\\
\leq& (1-\beta_n)^2\|v_n-u\|^2+2\beta_n\langle h(x_n)-u, x_{n+1}-u\rangle\\
\leq& (1-\beta_n)^2\|y_n-u\|^2+2\beta_n\langle h(x_n)-h(u), x_{n+1}-u\rangle\\
&+2\beta_n\langle h(u)-u, x_{n+1}-u\rangle\\
\leq& (1-\beta_n)^2\|w_n-u\|^2+2\beta_n\|h(x_n)-h(u)\|\|x_{n+1}-u\|\\
&+2\beta_n\langle h(u)-u, x_{n+1}-u\rangle+(1-\beta_n)^2(2\alpha_n(\mu_n\epsilon_n+\mu_n^2))\\
\leq& (1-\beta_n)^2\left(\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|)\right)\\
&+2\beta_n\langle h(u)-u, x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\
&+c\beta_n(\|x_n-u\|^2+\|x_{n+1}-u\|^2)\\
=& [1-2\beta_n+c\beta_n]\|x_n-u\|^2+c\beta_n\|x_{n+1}-u\|^2+\beta_n^2\|x_n-u\|^2\\
&+2\beta_n\langle h(u)-u, x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\
&+\theta_n(1-\beta_n)^2\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
(1-c\beta_n)\|x_{n+1}-u\|^2 \leq& (1-2\beta_n+c\beta_n)\|x_n-u\|^2+\beta_n^2\|x_n-u\|^2\\
&+2\beta_n\langle h(u)-u, x_{n+1}-u\rangle+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\
&+\theta_n(1-\beta_n)\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-u\|^2 \leq& \left(\frac{1-2\beta_n+c\beta_n}{1-c\beta_n}\right)\|x_n-u\|^2\\
&+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\\
&+\frac{\beta_n^2}{1-c\beta_n}\|x_n-u\|^2+\frac{2\beta_n}{1-c\beta_n}\langle h(u)-u, x_{n+1}-u\rangle\\
&+\frac{2\alpha_n}{1-c\beta_n}(\mu_n\epsilon_n+\mu_n^2)\\
\leq& \left(1-\frac{2\beta_n(1-c)}{1-c\beta_n}\right)\|x_n-u\|^2
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&+\frac{2\beta_n(1-c)}{1-c\beta_n}\left(\frac{\langle h(u)-u, x_{n+1}-u\rangle}{1-c}+\frac{\beta_n M_3}{1-c}\right)\\
&+\frac{\theta_n(1-\beta_n)}{1-c\beta_n}(\|x_n-x_{n-1}\|)\left(M_4+\theta_n\|x_n-x_{n-1}\|\right)\\
&+2\alpha_n\left(\mu_n\epsilon_n+\mu_n^2\right). \qquad \square
\end{aligned}
$$

Theorem 3.2. *Let $\{x_n\}$ be given by Algortihm 3.1, then $\{x_n\}$ converges strongly to $u = P_\Gamma h(u)$, where P_Γ is the metric projection of H_1 onto Γ.*

Proof. We consider the following two possible cases for the sequence $\{\|x_n - u\|\}$.

Case 1. Suppose there exists $n \in \mathbb{N}$ such that $\{\|x_n - u\|^2\}$ is nonincreasing. Then $\{\|x_n - u\|^2\}$ converges and

$$\|x_n-u\|^2-\|x_{n+1}-u\|^2 \to 0 \quad \text{as} \quad n \to +\infty.$$

From (3.3) and condition (iv), we get

$$\|w_n-x_n\| =\|x_n+\theta_n(x_n-x_{n-1})-x_n\| \le \theta_n\|x_n-x_{n-1}\| \to 0 \quad \text{as} \quad n\to+\infty. \tag{3.22}$$

Observe from (3.7) and (3.18), that

$$
\begin{aligned}
\|v_n-u\|^2 \le&\|w_n-u\|^2+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\
&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2\\
\le&\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\\
&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\
&-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2+2\alpha_n(\mu_n\epsilon_n+\mu_n^2),
\end{aligned}
$$

using this in (3.21), we get

$$
\begin{aligned}
\|x_{n+1}-u\|^2 \le&(1-\beta_n)\Big(\|x_n-u\|^2+\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|)\\
&+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big)+2\beta_n\langle h(x_n)-h(u), x_{n+1}-u\rangle\\
&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\alpha_n(1-\alpha_n)(1-\beta_n)\|St_n-w_n\|^2 \le&(1-\beta_n)\Big(\theta_n\|x_n-x_{n-1}\|(2\|x_n-u\|\\
&+\theta_n\|x_n-x_{n-1}\|)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big)\\
&+2\beta_n\langle h(x_n)-h(u), x_{n+1}-u\rangle\\
&+\|x_n-u\|^2-\|x_{n+1}-u\|^2-\beta_n\|x_n-u\|^2.
\end{aligned}
$$

Using conditions (i)-(iv), we get

$$\lim_{n\to+\infty}\|St_n-w_n\|=0. \tag{3.23}$$

Similarly, one gets,

$$\begin{aligned}\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2 \leq &(1-\beta_n)\Big[\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|\\&+\theta_n\|x_n-x_{n-1}\|\Big)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\Big]\\&+2\beta_n\langle h(x_n)-h(u),x_{n+1}-u\rangle+\|x_n-u\|^2\\&-\|x_{n+1}-u\|^2-\beta_n\|x_n-u\|^2.\end{aligned}$$

Since $\gamma_n \in \Big(0, \frac{2}{\|L\|^2}\Big)$, we have

$$\|u_n-Ly_n\| \to 0 \quad \text{as} \quad n\to+\infty. \tag{3.24}$$

Recall from (3.18) that

$$\begin{aligned}\|v_n-u\|^2 \leq &\|w_n-u\|^2+2\lambda_n\alpha_n f(w_n,u)+2\alpha_n(\mu_n\epsilon_n+\mu_n^2)\\&-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2,\end{aligned}$$

using this in (3.21), we obtain

$$\begin{aligned}2(1-\beta_n)\lambda_n\alpha_n(f(-w_n,u)) \leq &\|x_n-u\|^2-\|x_{n+1}-u\|^2\\&+(1-\beta_n)\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\Big)\\&-\beta_n\|x_n-u\|^2+2\beta_n\langle h(x_n)-u,x_{n+1}-u\rangle\\&+2\alpha_n(\mu_n\epsilon_n+\mu_n^2).\end{aligned}$$

Taking limit as $n\to+\infty$ and using (iv), we get

$$2\lim_{n\to+\infty}(1-\beta_n)\lambda_n\alpha_n(-f(w_n,u))=0.$$

Since $0<\acute{a}<\lambda_n<\acute{b}<1$, $0<a<\alpha_n<b<1$ and $-f(w_n,u)\geq 0$, we have that

$$\limsup_{n\to+\infty} f(w_n,u)=0. \tag{3.25}$$

Next we show $\|Tz_n-z_n\|\to 0$ as $n\to+\infty$. Observe that

$$\|z_n-u\|^2=\|z_n-w_n+w_n-u\|^2\leq\|w_n-u\|^2+2\langle z_n-u,z_n-w_n\rangle.$$

It follows from this, (3.8), (3.9) and (3.16), that

$$\begin{aligned}\|v_n-u\|^2 \leq &\|w_n-u\|^2+2\alpha_n\langle z_n-u,z_n-w_n\rangle\\&-\alpha_n\sigma_n(1-\sigma_n)\|Tz_n-z_n\|^2-\alpha_n(1-\alpha_n)\|St_n-w_n\|^2\\&-\gamma_n(2-\gamma_n\|L\|^2)\|u_n-Ly_n\|^2.\end{aligned}\tag{3.26}$$

Substituting (3.26) into (3.21), we get

$$\begin{aligned}\sigma_n\alpha_n(1-\beta_n)(1-\alpha_n)\|Tz_n-z_n\|^2 \leq &\|x_n-u\|^2-\|x_{n+1}-u\|^2\\&+(1-\beta_n)\theta_n\|x_n-x_{n-1}\|\Big(2\|x_n-u\|\\&+\theta_n\|x_n-x_{n-1}\|\Big)-\beta_n\|x_n-u\|^2\\&+2\beta_n\langle h(x_n)-u,x_{n+1}-u\rangle\end{aligned}$$

$$+ 2\alpha_n\|z_n - u\|\|z_n - w_n\|.$$

Again, since $0 < a < \alpha_n < b < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, it follows that

$$\lim_{n\to+\infty} \|Tz_n - z_n\| = 0. \tag{3.27}$$

Observe that

$$\begin{aligned}\|St_n - z_n\|^2 \leq& \|St_n - w_n\|^2 + 2\langle w_n - z_n, St_n - z_n\rangle \\ \leq& \|St_n - w_n\|^2 + 2\|w_n - z_n\|\|St_n - z_n\|,\end{aligned}$$

which implies by condition, (3.13) and (3.23), that

$$\lim_{n\to+\infty} \|St_n - z_n\|^2 = 0. \tag{3.28}$$

The following holds by triangular inequality, (3.27) and (3.28)

$$\|Tz_n - St_n\| \leq \|Tz_n - z_n\| + \|z_n - St_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.29}$$

Also,

$$\lim_{n\to+\infty} \|t_n - z_n\| = \lim_{n\to+\infty} (1 - \sigma_n)\|z_n - Tz_n\| = 0. \tag{3.30}$$

It follows again by triangular inequality, that

(3.31)

$$\begin{cases}\lim_{n\to+\infty} \|t_n - w_n\| \leq \lim_{n\to+\infty}(\|t_n - z_n\| + \|z_n - w_n\|) = 0, \\ \lim_{n\to+\infty} \|St_n - t_n\| \leq \lim_{n\to+\infty}(\|St_n - z_n\| + \|z_n - t_n\|) = 0, \\ \lim_{n\to+\infty} \|y_n - t_n\| \leq \lim_{n\to+\infty}(1 - \alpha_n)\|w_n - t_n\| + \lim_{n\to+\infty} \alpha_n\|St_n - t_n\| = 0, \\ \lim_{n\to+\infty} \|y_n - w_n\| \leq \lim_{n\to+\infty}(\|y_n - t_n\| + \|t_n - w_n\|) = 0, \\ \lim_{n\to+\infty} \|y_n - x_n\| \leq \lim_{n\to+\infty}(\|y_n - w_n\| + \|x_n - w_n\|) = 0.\end{cases}$$

Again,

$$\begin{aligned}\|Sz_n - z_n\|^2 =& \|Sz_n - St_n + St_n - z_n\|^2 \\ \leq& \|Sz_n - St_n\|^2 + 2\langle St_n - z_n, Sz_n - z_n\rangle \\ \leq& \|z_n - t_n\|^2 + 2(\|St_n - z_n\| \times \|Sz_n - z_n\|),\end{aligned}$$

we obtain by (3.28) and (3.30), that

$$\|Sz_n - z_n\|^2 \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.32}$$

Finally, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. Indeed, we have from (3.5) and (3.24), that

$$\begin{aligned}\lim_{n\to+\infty} \|v_n - y_n\| =& \lim_{n\to+\infty} \|y_n + \gamma_n L^*(u_n - Ly_n) - y_n\| \\ \leq& \lim_{n\to+\infty} \gamma_n\|L\|\|u_n - Ly_n\| = 0 \qquad (3.33)\end{aligned}$$

and

$$\|x_{n+1} - v_n\| = \|\beta_n h(x_n) + (1 - \beta_n)v_n - v_n\| \leq \beta_n\|h(x_n) - v_n\|,$$

which by condition (iii), implies that

$$\|x_{n+1} - v_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.34}$$

Hence, by (3.22), (3.31), (3.33) and (3.34), we obtain

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| \le \lim_{n\to+\infty} \Big(\|x_{n+1} - v_n\| + |v_n - y_n\| + \|y_n - w_n\| + \|w_n - x_n\|\Big) = 0.$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup v$ and $\limsup_{n\to+\infty} f(x_n, u) = \lim_{j\to+\infty} f(x_{n_j}, u)$. It follows from (3.13), (3.23), (3.30) and (3.31), that the sequences $\{w_n\}$, $\{t_n\}$, $\{z_n\}$ and $\{y_n\}$ all converge weakly to v. Consequently, $Lz_{n_j} \rightharpoonup Lv$ and $Ly_{n_j} \rightharpoonup Lv$. It follows from the demiclosedness of $I - S$ and (3.32), that $v \in Fix(S)$. Next we show that $v = (P_{Fix(S)} \circ T)v$. It follows from (3.5), that

$$t_n - St_n = \sigma_n(I - T)z_n + (Tz_n - St_n),$$

which implies

$$\frac{1}{\sigma_n}(t_n - St_n) = (I - T)z_n + \frac{1}{\sigma_n}(Tz_n - St_n), \tag{3.35}$$

thus for all $w \in Fix(S)$, the monotonicity of $(I - T)$ and (3.35), we have

$$\begin{aligned}\left\langle \frac{t_n - St_n}{\sigma_n}, z_n - w \right\rangle =& \langle (I - T)z_n - (I - T)w, z_n - w\rangle \\ &+ \langle (I - T)w, z_n - w\rangle + \frac{1}{\sigma_n}\langle Tz_n - St_n, z_n - w\rangle \\ \ge& \langle (I - T)w, z_n - w\rangle + \frac{1}{\sigma_n}\langle Tz_n - St_n, z_n - w\rangle. \end{aligned} \tag{3.36}$$

Since $\{z_n\}$ and $\{z_n - w\}$ are bounded, it follows from (3.29), (3.31) and (3.36), that

$$\limsup_{n\to+\infty} \langle (I - T)w, z_n - w\rangle \le 0, \quad \text{for all } w \in Fix(S). \tag{3.37}$$

Replacing n with n_j and letting $j \to +\infty$ in (3.37), we obtain

$$\langle (I - T)w, v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Note that $tw + (1 - t)v \in$ F(S) for $t \in (0, 1)$, since $Fix(S)$ is convex. Hence,

$$\langle (I - T)(tw + (1 - t)v), v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Setting $t \to 0_+$ and using the continuity of $(I - T)$, we obtain

$$\langle (I - T)v, v - w\rangle \le 0, \quad \text{for all } w \in Fix(S).$$

Thus $v \in F(P_{Fix(S)} \circ T)$. Next, we show that $v \in EP(f, C)$. Since $x_{n_j} \rightharpoonup v$, $\|w_{n_j} - x_{n_j}\| \to 0$ and $\limsup_{n\to+\infty} f(w_n, u) = \lim_{j\to+\infty} f(w_{n_j}, u)$, by the upper weakly continuity of $f(\cdot, u)$ and (3.25), we have

$$f(v, u) \ge \limsup_{j\to+\infty} f(w_{n_j}, u) = \lim_{j\to+\infty} f(w_{n_j}, u) = \limsup_{n\to+\infty} f(w_n, u) = 0.$$

Since $u \in \Gamma$ and $v \in C$, we have $f(u,v) \geq 0$. By the pseudomonotone property of f, we have $f(v,u) \leq 0$. Consequently, we obtain $f(v,u) = 0$, and by restriction $F3$, we get $v \in EP(f,C)$. Furthermore, we show that $Lv \in Fix(K_{r_n}^{F,B,\phi}) = GMEP(F,B,\phi)$. Since $\lim_{n\to+\infty} \|y_n - x_n\| = 0$ and $x_{n_j} \rightharpoonup v$, it is easy to see that $y_{n_j} \rightharpoonup v$. It therefore follows from the continuity of L, that $Ly_{n_j} \rightharpoonup Lv$ and by (3.24), we get $u_{n_j} \rightharpoonup Lv$.

Now since $u_n = K_{r_n}^{F,B,\phi} Ly_n$, we have

$$F(u_n,w)+\langle B(u_n),w-u_n\rangle+\phi(w)-\phi(u_n)+\frac{1}{r_n}\langle w-u_n,u_n-Ly_n\rangle \geq 0, \quad \text{for all } w \in Q.$$

It follows from the monotonicity of F, that

$$\phi(w) - \phi(u_n) + \langle B(u_n), w - u_n\rangle + \frac{1}{r_n}\langle w - u_n, u_n - Ly_n\rangle \geq F(w,u_n), \quad \text{for all } w \in Q,$$

and

$$\phi(w) - \phi(u_{n_j}) + \langle B(u_{n_j}), w - u_{n_j}\rangle + \left\langle w - u_{n_j}, \frac{u_{n_j} - Ly_{n_j}}{r_{n_j}}\right\rangle \geq F(w,u_{n_j}), \tag{3.38}$$

for all $w \in Q$. This implies

$$\begin{aligned}\langle B(Ly_{n_j}), w - u_{n_j}\rangle \geq &\phi(u_{n_j}) - \phi(w) + \langle B(Ly_{n_j}), w - u_{n_j}\rangle - \langle B(u_{n_j}), w - u_{n_j}\rangle \\ &- \left\langle w - u_{n_j}, \frac{u_{n_j} - L_{n_j}}{r_{n_j}}\right\rangle + F(w,u_{n_j}) \\ = &\phi(u_{n_j}) - \phi(w) + \langle B(Ly_{n_j}) - B(u_{n_j}), w - u_{n_j}\rangle \\ &- \left\langle w - u_{n_j}, \frac{u_{n_j} - Ly_{n_j}}{r_{n_j}}\right\rangle + F(w,u_{n_j}).\end{aligned} \tag{3.39}$$

Since B is continuous and $\lim_{n\to+\infty} \|Ly_n - u_n\| = 0$, it follows that $\lim_{n\to+\infty} \|B(Ly_n) - B(u_n)\| = 0$. From the monotonicity of B, the weakly lower semicontinuity of ϕ and $u_{n_j} \rightharpoonup Lv$, it follows from (3.39), that

$$\langle B(Lv), w - Lv\rangle \geq \phi(Lv) - \phi(w) + F(w,Lv), \quad \text{for all } w \in Q. \tag{3.40}$$

For any $t \in (0,1]$ and $w \in Q$, set $z_t = tw + (1-t)Lv$ we have $z_t \in Q$ and thus satisfies (3.40). Using assumptions (R1) and (R4), we get

$$\begin{aligned}0 = &F(z_t,z_t) + \phi(z_t) - \phi(z_t) \\ \leq &tF(z_t,w) + (1-t)F(z_t,Lv) + t\phi(w) + (1-t)\phi(Lv) - \phi(z_t) \\ = &t[F(z_t,w) + \phi(w) - \phi(z_t)] + (1-t)[F(z_t,Lv) + \phi(Lv) - \phi(z_t)] \\ \leq &t[F(z_t,w) + \phi(w) - \phi(z_t)] + (1-t)t\langle B(Lv), w - Lv\rangle.\end{aligned}$$

This implies

$$F(z_t,w) + \phi(w) - \phi(z_t) + (1-t)\langle B(Lv), w - Lv\rangle \geq 0.$$

Letting $t \to 0_+$, we get

$$F(Lv,w) + \phi(w) - \phi(Lv) + \langle B(Lv), w - Lv\rangle \geq 0, \quad \text{for all } w \in C,$$

which implies $Lv \in GMEP(F, B, \phi)$.

To end Case 1, we show that $\{x_n\}$ converges strongly to $u = P_\Gamma h(u)$. To do this, it suffices to show that $\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle \le 0$ and apply Lemma 2.3. Indeed, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup v \in H_1$ and

$$\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle = \lim_{j\to+\infty}\langle h(u) - u, x_{n_j+1} - u\rangle.$$

We have that $x_{n+1} \rightharpoonup v$ since $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. By applying (2.1), we have

$$\limsup_{n\to+\infty}\langle h(u) - u, x_{n+1} - u\rangle = \lim_{j\to+\infty}\langle h(u) - u, x_{n_j+1} - u\rangle = \langle h(u) - u, v - u\rangle \le 0.$$

Using (2.1), (3.21) and Lemma 2.3, we conclude that $\|x_n - u\| \to 0$ as $n \to +\infty$.

Case 2. Assume that $\{\|x_n - u\|\}$ is non monotone. For some n_0 large enough, let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ by

$$\tau(n) := \max\{j \in \mathbb{N} : j \le n, \|x_j - u\| \le \|x_{j+1} - u\|\}.$$

By Lemma 2.4, $\tau(n)$ is nondecreasing sequence such that $\tau(n) \to +\infty$ as $n \to +\infty$ and $0 \le \|x_{\tau(n)} - u\| \le \|x_{\tau(n)+1} - u\|$ for all $n \ge n_0$. Just by using similar argument as in Case 1, we have

$$\begin{aligned}\lim_{n\to+\infty}\|w_{\tau(n)} - x_{\tau(n)}\| &= \lim_{n\to+\infty}\|z_{\tau(n)} - w_{\tau(n)}\| = \lim_{n\to+\infty}\|y_{\tau(n)} - w_{\tau(n)}\| \\ &= \lim_{n\to+\infty}\|u_{\tau(n)} - Ly_{\tau(n)}\| = \lim_{n\to+\infty}\|Tz_{\tau(n)} - z_{\tau(n)}\| \\ &= \lim_{n\to+\infty}\|Sz_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to+\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0\end{aligned}$$

and

$$\lim_{n\to+\infty}\langle h(u) - u, x_{\tau(n)+1} - u\rangle \le 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$ still denoted by $\{x_{\tau(n)}\}$ such that $x_{\tau(n)} \rightharpoonup v \in C$. Following similar argument as in Case 1, we obtain $v \in \Gamma$.

From (3.21), we get

$$\begin{aligned}\|x_{\tau(n)+1} - u\|^2 \le& \left(1 - \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\right)\|x_{\tau(n)} - u\|^2 \\ &+ \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\left(\frac{\langle h(u) - u, x_{\tau(n)+1} - u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right) \\ &+ \frac{\theta_{\tau(n)}(1 - \beta_{\tau(n)})}{1 - c\beta_{\tau(n)}}\left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right)\left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right) \\ &+ 2\alpha_{\tau(n)}\left(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu^2_{\tau(n)}\right).\end{aligned}$$

Since $\|x_{\tau(n)} - u\| \le \|x_{\tau(n)+1} - u\|$ and $\beta_{\tau(n)} > 0$, we have

$$\frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\|x_{\tau(n)} - u\|^2 \le \frac{2\beta_{\tau(n)}(1-c)}{1 - c\beta_{\tau(n)}}\left(\frac{\langle h(u) - u, x_{\tau(n)+1} - u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right)$$

$$+ 2\alpha_{\tau(n)}\left(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu^2_{\tau(n)}\right) + \frac{\theta_{\tau(n)}(1-\beta_{\tau(n)})}{1-c\beta_{\tau(n)}}$$
$$\times \left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right)\left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right).$$

Hence,

$$\begin{aligned}\frac{2(1-c)}{1-c\beta_{\tau(n)}}\|x_{\tau(n)} - u\|^2 \leq & \frac{2(1-c)}{1-c\beta_{\tau(n)}}\left(\frac{\langle h(u)-u, x_{\tau(n)+1}-u\rangle}{1-c} + \frac{\beta_{\tau(n)}M_3}{1-c}\right) \\ & + \frac{2\alpha_{\tau(n)}}{\beta_{\tau(n)}}(\mu_{\tau(n)}\epsilon_{\tau(n)} + \mu^2_{\tau(n)}) \\ & + \frac{\theta_{\tau(n)}(1-\beta_{\tau(n)})}{\beta_{\tau(n)}(1-c\beta_{\tau(n)})}\left(\|x_{\tau(n)} - x_{\tau(n)-1}\|\right) \\ & \times \left(M_4 + \theta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|\right).\end{aligned}$$

This implies that $\limsup_{n\to+\infty}\|x_{\tau(n)} - u\|^2 \leq 0$ and

$$\lim_{n\to+\infty}\|x_{\tau(n)} - u\| = 0. \tag{3.41}$$

From (3.34) and (3.41), we obtain

$$\|x_{\tau(n)+1} - u\| \leq \|x_{\tau(n)} - u\| + \|x_{\tau(n)} - x_{\tau(n)+1}\| \to 0 \quad \text{as} \quad n \to +\infty.$$

Furthermore, for $n \geq n_0$, it is obvious that $\|x_n - u\| \leq \|x_{\tau(n)} - u\|$. Consequently, we get for all $n \geq n_0$, that

$$0 \leq \|x_n - u\| \leq \max\{\|x_{\tau(n)} - u\|, \|x_{\tau(n)+1} - u\|\} = \|x_{\tau(n)+1} - u\|.$$

Therefore, $\|x_n - u\| \to 0$ as $n \to +\infty$, that is $x_n \to u$. Thus completing the proof. □

If we set $B = \phi = 0$ in (3.1)–(3.2), we obtain the following method for obtaining a common solution of split EP and HFPP considered in [7].

Algorithm 3.3. **Initialization.** Choose $x_0, x_1 \in C$. Take the sequence of real numbers $\{\mu_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$, $\{\epsilon_n\}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfying

(i) $0 < r < r_n$, $0 < a < \alpha_n < b < 1$, $0 < \acute{a} < \lambda_n < \acute{b} < 1$, $0 < \bar{a} < \sigma_n < \bar{b} < 1$, $\beta_n \geq 0$, $\gamma_n \in (0, 2/\|L\|^2)$ and $\epsilon_n \to 0$ as $n \to +\infty$;
(ii) $\sum_{n=1}^{+\infty}\mu_n^2 < +\infty$;
(iii) $\sum_{n=1}^{+\infty}\beta_n = +\infty$, $\lim_{n\to+\infty}\beta_n = 0$;
(iv) $\{\theta_n\} \subset [0,\theta]$, where $\theta \in [0,1)$ and $\sum_{n=1}^{+\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$;
(v) $\lim_{n\to+\infty}\frac{\theta_n}{\beta_n} = 0$.

Step 1. Given x_{n-1} and x_n, $n \geq 1$, compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.42}$$

Step 2. Take $g(w_n) \in \partial_{\epsilon_n}(f(w_n,\cdot))(w_n)$, $n \geq 1$. Calculate $\eta_n = \max\{1, \|g(w_n)\|\}$, $\lambda_n = \frac{\mu_n}{\eta_n}$ and $z_n = P_C(w_n - \lambda_n g(w_n))$.

Step 3. If $w_n = z_n$ ($w_n \in EP(f, C)$), then go to step 3. Otherwise, evaluate

$$\begin{cases} t_n = (1-\sigma_n)Tz_n + \sigma_n z_n, \\ y_n = (1-\alpha_n)w_n + \alpha_n Stn, \\ u_n = K^F_{r_n} Ly_n, \\ v_n = y_n + \gamma_n L^*(u_n - Ly_n). \end{cases}$$

Step 4. Compute $x_{n+1} = \beta_n h(x_n) + (1-\beta_n)v_n$, where h is a contraction.
Step 5. Set $n := n+1$ and go to step 1.

We therefore give the following result as a consequence of our main theorem.

Corollary 3.4. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $L : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be bifunctions satisfying restrictions $F1$-$F4$ and $R1$-$R4$ respectively. Let $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a quasinonexpansive mapping such that $I - T$ is monotone. Assume that $\Gamma = EP(f, C) \cap EP(F, Q) \cap \Omega \neq \emptyset$. Then the sequence $\{x_n\}$ given by Algorithm 3.3 converges strongly to $u = P_\Gamma h(u)$, where P_Γ is the metric projection of H_1 onto Γ.

4. Numerical Examples

We give some numerical examples to illustrate the behaviour and performance of our method as well as comparing it with some related methods in the literature.

Example 4.1. Let $H_1 = H_2 = C = Q = \mathbb{R}$ with inner product $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the induced usual norm $|\cdot|$. Let $f : C \times C \to \mathbb{R}$ be defined by $f(x, y) = 2xy(y-x) + xy|y-x|$, for all $x, y \in H_1$. Define the bifunction $F : Q \times Q \to \mathbb{R}$ by $F(u, v) = -u^2 + v^2$, for all $u, v \in Q$, $B : Q \to H_2$ by $B(u) = \frac{u}{5}$ for all $u \in Q$ and $\phi : Q \to \mathbb{R}$ by $\phi(u) = 0$ for all $u \in Q$. For each $x \in H_1$, define the mapping $L : H_1 \to H_2$ by $Lx = x$ for all $x \in H_1$. Also define the mappings S and T respectively by $Sx = \frac{x}{2}$ and $Tx = x$. It is easy to see that f, F, S and T satisfy the conditions of Theorem 3.2 and that $\Gamma = \{0\}$. From Theorem 3.2, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to 0. Let $r_n = 1$, for all $n \geq 1$, it is easy to find that the resolvent $K^{F,B,\phi}_{r_n} Ly_n = \frac{5y_n}{16}$.

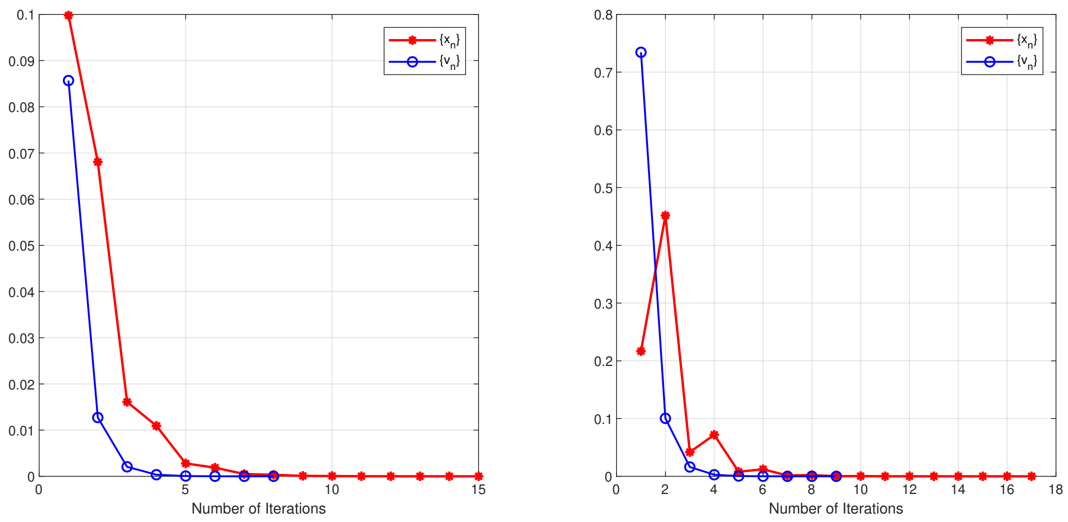

FIGURE 1. Numerical results for Example 4.1. Left: $x_0 = 0.8$, $x_1 = 0.5$; right: $x_0 = -1$, $x_1 = -3$.

Set $\sigma_n = \frac{1}{2n^2+3}$, $\alpha_n = \frac{1}{2n^2+5}$, $\epsilon_n = 0$, $\gamma_n = \frac{1}{5}$, $\lambda_n = \frac{1}{2}$. Also, let $\mu_n = \frac{1}{n}$, $\beta_n = \frac{1}{n+1}$ and $\theta_n = \frac{1}{4n^2+1}$. Then, after simplification Algorithm 3.1, becomes

$$\begin{cases} \text{Given } x_0 \text{ and } x_1 \in H_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n \in H_1 \text{ such that } g(w_n) \in \partial_{\epsilon_n} f(w_n, \cdot)(w_n) = [w_n^2, 3w_n^2], \\ z_n = P_C(w_n - \lambda_n g(w_n)), \\ t_n = (1-\sigma_n)Tz_n + \sigma_n z_n, \\ y_n = (1-\alpha_n)w_n + \alpha_n S t_n, \\ u_n = \frac{5y_n}{16}, \\ v_n = \frac{1}{15}(3u_n + 2y_n), \\ x_{n+1} = \beta_n h(x_n) + (1-\beta_n)v_n. \end{cases}$$

We test our algorithm with varying values of initial terms x_0 and x_1, see Figure 1.

In this example, we set $B = \phi = 0$.

Example 4.2. Let $H_1 = H_2 = Q = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^{+\infty}$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \dots, x_i \dots),\ x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{+\infty} |x_i|^2 < +\infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{+\infty} x_i y_i$, where $x = \{x_i\}_{i=1}^{+\infty}$, $y = \{y_i\}_{i=1}^{+\infty}$ and the norm $\| \cdot \| : \ell_2 \to \mathbb{R}$ by $\|x\|_2 := (\sum_{i=1}^{+\infty} |x_i|^2)^{\frac{1}{2}}$, where $x = \{x_i\}_{i=1}^{+\infty}$. Let $C = \{z \in \ell_2(\mathbb{R}) : \langle a, z\rangle \leq b\}$, where $0 \neq a \in \ell_2$ and $b \in \mathbb{R}$. Let $f : C \times C \to \mathbb{R}$ be defined by $f(x, y) = 2xy(y - x) + xy\|y - x\|$ for all $x = \{x_i\}_{i=1}^{+\infty}, y = \{y_i\}_{i=1}^{+\infty} \in \ell_2$. Define the bifunction $F : Q \times Q \to \mathbb{R}$ by $F(u, v) = u(v - u)$ for all $u = \{u_i\}_{i=1}^{+\infty}, v = \{v_i\}_{i=1}^{+\infty} \in \ell_2$. For each $x \in \ell_2$, define the mapping $L : \ell_2 \to \ell_2$ by $Lx = (x_1, x_2, \dots, x_i, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$. Let $r_n = 0.5$ for all $n \geq 1$, then it is easy to see that $K_{r_n}^F Ly_n = \frac{2Ly_n}{3}$. Also, define the mappings S and T ,respectively by $Sx = (\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_i}{2}, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$ and $Tx = (x_1 \cos x_1, x_2 \cos x_2, \dots, x_i \cos x_i, \dots)$ for all $x = \{x_i\}_{i=1}^{+\infty} \in \ell_2$. It is easy to see that f, F, S and T satisfy the conditions of Corollary 3.4 and that $\Gamma = \{0\}$. We define the control parameters as in Example 4.1 above and obtain the figures for varying initial values. Using $\|x_{n+1} - x_n\|_{\ell_2} < 10^{-3}$ as the stopping criterion, we compare our Algorithm 3.3 with Algorithm Theorem 3.1 in [7], see Figure 2.

Case (i) $x_1 = (3.568, -5.8091, 0, \dots, 0, \dots)^T$, $x_0 = (1.521, -7.5647, 0, \dots, 0, \dots)^T$.

Case (ii) $x_1 = (1.7601, -2.1594, 0, \dots, 0, \dots)^T$, $x_0 = (0.3456, -4.1031, 0, \dots, 0, \dots)^T$.

Case (iii) $x_1 = (10.5613, 7.2610, 0, \dots, 0, \dots)^T$, $x_0 = (5.1063, 2.1687, 0, \dots, 0, \dots)^T$.
We then plot the graphs of error $\|x_{n+1} - x_n\|_{\ell_2}$ against the number of iteration in each case.

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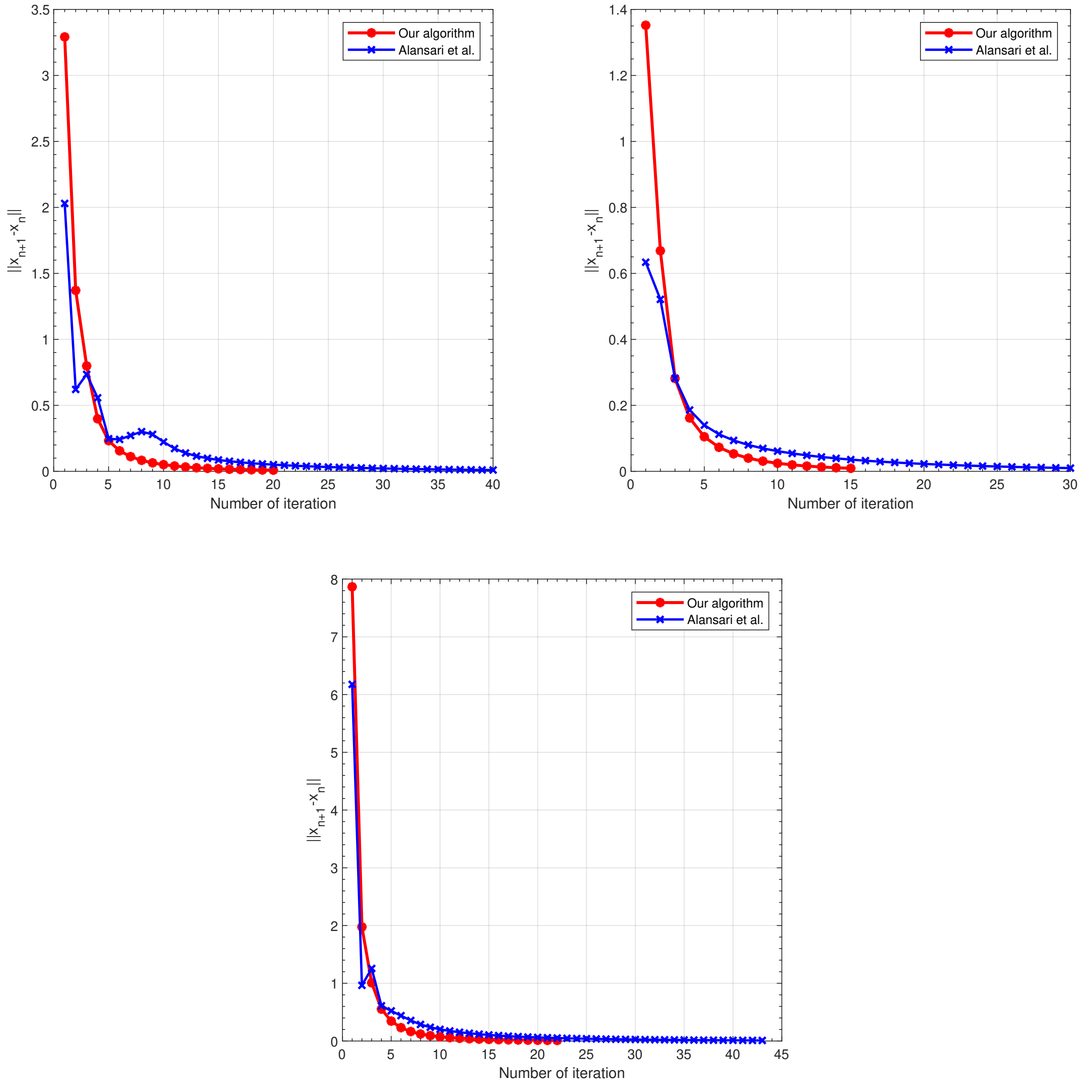

FIGURE 2. Numerical results for Example 4.2. Top left: Case (i); top right: Case (ii); bottom: Case (iii).

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ON VERTEX-EDGE AND EDGE-VERTEX CONNECTIVITY INDICES OF GRAPHS

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The vertex-edge degree of the vertex v, $d_G^e(v)$, equals to the number of different edges that are incident to any vertex from the open neighborhood of v. Also, the edge-vertex degree of the edge $e = uv$, $d_G^v(e)$, equals to the number of vertices of the union of the open neighborhood of u and v. In this paper, the vertex-edge connectivity index, ϕ_v, and the edge-vertex connectivity index, ϕ_e, of a graph G were introduced. These are defined as $\phi_v(G) = \sum_{v\in V(G)} d_G^e(v)d_G(v)$ and $\phi_e(G) = \sum_{e=uv\in E(G)} d_G(e)d_G^v(e)$, where $d_G(v)$ is the degree of a vertex $v \in V(G)$ and $d_G(e)$ is the number of edges in $E(G)$ that are adjacent to e. In this paper, we will study the main properties of $\phi_v(G)$, $\phi_e(G)$ and establish some upper and lower bounds for them. The numbers ϕ_v and ϕ_e for titania nanotubes are also computed.

1. Basic Definitions and Notations

In this paper we study some aspects of the vertex-edge degree of a vertex and we are concerned only with simple graphs, i.e., finite graphs having no loops, multiple and directed edges. Let $G = (V(G), E(G))$ be such a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, the number of vertices and edges in G are denoted by $n = |V|$ and $m = |E|$, respectively. The distance $d_G(u, v)$ between two vertices u and v of a graph G is equal to the length of (number of edges in) a shortest path connecting them. For a vertex $v \in V(G)$, the open neighborhood of v is denoted by $N(v, G)$ and is defined as $N(v, G) = \{u \in V(G) \mid uv \in E(G)\}$. The degree of a vertex

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v in G is denoted by $d_G(v)$ and is defined as the number of neighbours of the vertex v in G, i.e., $\deg_G(v) = |N(v, G)|$. The minimum and maximum degree of vertices in the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any terminology or notation not mention here, we refer to [17].

A topological index of a graph is a graph invariant calculated from a graph representing a molecule and applicable in chemistry. The Zagreb indices have been introduced, more than fifty years ago, by Gutman and Trinajestić [15], in 1972, and elaborated in [16]. They are defined as $M_1(G) = \sum_{uv\in E(G)}[d_G(u) + d_G(v)] = \sum_{v\in V(G)} d_G(v)^2$ and $M_2(G) = \sum_{uv\in E} d_G(u)d_G(v)$. Furtula and Gutman [12] introduced the forgotten index of G, $F(G)$, as $F(G) = \sum_{uv\in E(G)}[d_G(u)^2 + d_G(v)^2] = \sum_{v\in V(G)} d_G(v)^3$. For properties of the two Zagreb indices see [3, 7, 14, 15, 24, 25, 30] and the references therein.

In recent years, some novel variants of ordinary Zagreb indices introduced and studied, such as Zagreb coincides [1, 16], multiplicative Zagreb indices [13, 29, 30], multiplicative sum Zagreb index [10] and multiplicative Zagreb coincides [31].

In 2017, Naji et al. [22], have introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices (number of their second neighbours), and are so-called leap Zagreb indices of a graph G. For properties and more detail on leap Zagreb indices, we refer to [2, 22, 23] and [26].

For a vertex v in $V(G)$ the ve-dominates are every edge incident to v as well as every edge adjacent to these incident edges. Also, for an edge $e = uv$ in $E(G)$, the ev-dominates are the vertices of the set $N(v, G) \cup N(u, G)$. There is a natural duality between ve-dominates and ev-dominates for any graph G: a vertex $v \in V$ is an ev-dominates for edge $e \in E$ if and only if the edge e is an ve-dominates for vertex v [6].

Definition 1.1 ([4])**.** Let G be a connected graph and $v \in V(G)$. The vertex-edge degree of the vertex v, $d^e_G(v)$, equals the number of different edges that incident to any vertex from the open neighborhood of v. Also, the edge-vertex degree of the edge $e = uv$, $d^v_G(e)$, equals the number of vertices of the union of the open neighborhoods of u and v.

The concepts of vertex-edge domination and edge-vertex domination were introduced by Peters [21] in his Ph.D. thesis and studied further in [4, 9, 18, 19, 27]. The following fundamental results which will be used in many of our subsequent considerations are found in the earlier papers [28] and [32].

Let G be a graph. The total ev-degree, T_e, total ve-degree, T_v, ev-degree Zagreb index, S, first ve-degree Zagreb alpha index, S^α, first ve-degree Zagreb beta index, S^β, second ve-degree Zagreb index, S^μ, of graph G are defined by Chellali et al. [6] as:

$$T_e(G) = \sum_{e\in E(G)} d^v_G(e), \quad T_v(G) = \sum_{v\in V(G)} d^e_G(v),$$

$$S(G) = \sum_{e\in E(G)} d_G^v(e)^2, \quad S^\alpha(G) = \sum_{v\in V(G)} d_G^e(v)^2,$$
$$S^\beta(G) = \sum_{e=uv\in E(G)} [d_G^e(v) + d_G^e(u)], \quad S^\mu(G) = \sum_{e=uv\in E(G)} d_G^e(v) d_G^e(u).$$

Let $\eta(G)$ be the number of triangles in graph G. Authors in [6] have proved that:

$$\begin{aligned} &T_e(G) = T_v(G) = M_1(G) - 3\eta(G), \quad \text{where } G \text{ is an arbitrary graph},\\ &S(G) = F(G) + 2M_2(G), \quad \text{where } G \text{ is a triangle free connected graph},\\ &S^\beta(T) = 2M_2(T), \quad \text{where } T \text{ is an arbitrary tree.}\end{aligned} \tag{1.1}$$

In [8], Ediz defined ve-degree atom-bond connectivity, ve-degree geometric - arithmetic, ve-degree harmonic and ve-degree sum-connectivity indices as parallel to their corresponding classical degree versions. Moreover, the mathematical properties were studied in it.

Titania nanotubes which have been produced fifteen years ago have many applications on the very broad of science from medicine to electronics [20]. Computing certain topological indices of titania nanotubes have been started recently. Since 2015, there are many studies to compute the exact value of some topological indices of titania nanotubes [5, 11].

2. Main Results

Define the ev-degree connectivity index, ϕ_e, and ve-degree connectivity index, ϕ_v, of a graph G as:

$$\phi_e(G) = \sum_{e=uv\in E(G)} d_G(e) d_G^v(e),$$
$$\phi_v(G) = \sum_{v\in V(G)} d_G(v) d_G^e(v),$$

where for $e = uv \in E(G)$, $d_G(e) = d_G(u) + d_G(v) - 2$.

Proposition 2.1. *Let P_n, C_n, S_n, K_n and $K_{a,b}$ be path, cycle, star, complete and bipartite graphs on $n \geq 4$ vertices, respectively. Then $(a + b = n)$*

$$\phi_e(P_n) = 8n - 18, \quad \phi_v(P_n) = 8(n-2), \quad \phi_e(C_n) = \phi_v(C_n) = 8n,$$
$$\phi_e(S_n) = n(n-1)(n-2), \quad \phi_v(S_n) = 2(n-1)^2,$$
$$\phi_e(K_n) = n^2(n-1)(n-2), \quad \phi_v(K_n) = \frac{n^2(n-1)^2}{2},$$
$$\phi_e(K_{a,b}) = ab(n^2 - 2n), \quad \phi_v(K_{a,b}) = 2a^2b^2.$$

Proof. By definitions,

$$\phi_e(P_n) = \sum_{e=uv\in E(P_n)} d_{P_n}(e) d_{P_n}^v(e) = 2(1\times 3) + (n-3)(2\times 4) = 8n - 18,$$

$$\phi_v(P_n) = \sum_{v\in V(P_n)} d_{P_n}(v)d^e_{P_n}(v) = 2(1\times 2) + 2(2\times 3) + (n-4)(2\times 4) = 8(n-2).$$

The proof of other cases are similar and we omit them. □

Proposition 2.2. *Let G be a triangle free graph. Then*

$$\phi_e(G) = F(G) + 2M_2(G) - 2M_1(G) \quad \text{and} \quad \phi_v(G) = 2M_2(G).$$

Proof. By definitions,

$$\begin{aligned}
\phi_e(G) &= \sum_{e=uv\in E(G)} d_G(e)d^v_G(e) = \sum_{e=uv\in E(G)} d_G(e)[d_G(u)+d_G(v)]\\
&= \sum_{e=uv\in E(G)} [d_G(u)+d_G(v)-2][d_G(u)+d_G(v)]\\
&= F(G) + 2M_2(G) - 2M_1(G),\\
\phi_v(G) &= \sum_{v\in V(G)} d_G(v)d^e_G(v) = \sum_{v\in V(G)} d_G(v) \sum_{uv\in E(G)} d_G(u)\\
&= \sum_{v\in V(G)} d_G(v) \sum_{uv\in E(G)} d_G(u) = 2\sum_{uv\in E(G)} d_G(u)d_G(v)\\
&= 2M_2(G).
\end{aligned}$$

Hence, the result is obtained. □

Let G be a graph with n vertices and m edges and let $n_i = |\{v \in V(G) \mid d_G(v) = i\}|$, for all integers i, $1 \le i \le n-1$. By definition,

$$n = n_1 + n_2 + \cdots + n_{n-1}. \tag{2.1}$$

Also, it is well-known,

$$2m = n_1 + 2n_2 + \cdots + (n-1)n_{n-1}. \tag{2.2}$$

Therefore, by (2.1), (2.2) and some simple calculations,

$$n_1 = 2n - 2m + \sum_{i=3}^{n-1}(i-2)n_i. \tag{2.3}$$

Theorem 2.1. *Let G be a triangle free graph. Then $\phi_e(G) - \phi_v(G) \ge 2(m-n)$ and equality holds if and only if $\{d_G(v) \mid v \in V(G)\} \subseteq \{1,2\}$.*

Proof. By Proposition 2.2,

$$\begin{aligned}
\phi_e(G) - \phi_v(G) = F(G) - 2M_1(G) &= \sum_{v\in V(G)} d_G(v)^2[d_G(v)-2]\\
&= \sum_{i=1}^{n-1} i^2(i-2)n_i = -n_1 + \sum_{i=3}^{n-1} i^2(i-2)n_i,
\end{aligned}$$

and by (2.3),

$$\phi_e(G) - \phi_v(G) = 2m - 2n - \sum_{i=3}^{n-1}(i-2)n_i + \sum_{i=3}^{n-1} i^2(i-2)n_i$$

$$= 2m - 2n + \sum_{i=3}^{n-1} (i-1)(i-2)(i+1)n_i.$$

Therefore, $\phi_e(G) - \phi_v(G) \geq 2(m-n)$ and equality holds if and only if $\{d_G(v) \mid v \in V(G)\} \subseteq \{1, 2\}$. ☐

Proposition 2.3. *Let G be a triangle free connected graph with n vertices and m edges. Then $\phi_e(G) \leq mn(n-2)$ and equality holds if and only if $G \cong K_{k,n-k}$.*

Proof. By definition of triangle free graph G, $d_G(u)+d_G(v) \leq n$ for all $e = uv \in E(G)$. Thus,

$$\begin{aligned}\phi_e(G) &= \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big)\Big(d_G(u) + d_G(v) - 2\Big)\\ &\leq \sum_{uv \in E(G)} n(n-2) = mn(n-2).\end{aligned}$$

Equality holds if and only if $G \cong K_{k,n-k}$. ☐

A graph G is said to be ve-regular graph if and only if $|\{d_G^e(v) \mid v \in V(G)\}| = 1$ and is said to be ev-regular graph if and only if $|\{d_G^v(e) \mid e \in E(G)\}| = 1$.

Theorem 2.2. *For any graph G with n vertices and m edges*

$$S^\alpha(G) \geq \frac{(M_1(G) - 3\eta(G))^2}{n}. \tag{2.4}$$

Equality holds if and only if G is a ve-regular graph. Moreover,

$$\phi_v(G) \leq \sqrt{S^\alpha(G) M_1(G)}. \tag{2.5}$$

Equality holds if and only if there exists a real number c such that $d_G(v) = c d_G^e(v)$ for all $v \in V(G)$ and

$$\phi_e(G) \leq \sqrt{S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)}. \tag{2.6}$$

Equality holds if and only if there exists a real number l such that $d_G(e) = l d_G^v(e)$ for all $e \in E(G)$.

Proof. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Nest we will use Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right). \tag{2.7}$$

To prove (2.4), we put in (2.7), $a_i = d_G^e(v_i)$ and $b_i = 1$. Then by (1.1)

$$(M_1(G) - 3\eta(G))^2 = T_v(G)^2 = \left(\sum_{i=1}^n d_G^e(v_i)\right)^2 \leq \left(\sum_{i=1}^n d_G^e(v_i)^2\right)\left(\sum_{i=1}^n 1\right) = S^\alpha(G) n.$$

Therefore, $S^\alpha(G) \geq \frac{(M_1(G)-3\eta(G))^2}{n}$ and equality holds in Cauchy-Schwartz inequality if and only if $(a_1, a_2, \ldots, a_n) = c(b_1, b_2, \ldots, b_n)$, where c is a real number. Hence equality holds in (2.4) if and only if G is a ve-regular graph.

To prove (2.5), we put in (2.7), $a_i = d_G^e(v_i)$ and $b_i = d_G(v_i)$. Then we obtain

$$\phi_v(G)^2 = \left(\sum_{i=1}^n d_G^e(v_i) d_G(v_i)\right)^2 \leq \left(\sum_{i=1}^n d_G^e(v_i)^2\right)\left(\sum_{i=1}^n d_G(v_i)^2\right) = S^\alpha(G) M_1(G).$$

Therefore, $\phi_v(G) \leq \sqrt{S^\alpha(G) M_1(G)}$ and equality holds in Cauchy-Schwartz inequality if and only if $(a_1, a_2, \ldots, a_n) = c(b_1, b_2, \ldots, b_n)$, where c is a real number. Hence equality holds in Equation (2.5) if and only if there exists real number c such that $d_G(v) = c d_G^e(v)$ for all $v \in V(G)$.

To prove (2.6), again by Cauchy-Schwartz inequality,

$$\begin{aligned}
\phi_e^2(G) &= \left(\sum_{e=uv\in E(G)} d_G^v(e) d_G(e)\right)^2 \leq \left(\sum_{e=uv\in E(G)} d_G^v(e)^2\right)\left(\sum_{e=uv\in E(G)} d_G(e)^2\right) \\
&= S(G) \sum_{e=uv\in E(G)} \Big(d_G(u) + d_G(v) - 2\Big)^2 \\
&= S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big).
\end{aligned}$$

Thus $\phi_e(G) \leq \sqrt{S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)}$ and equality holds in (2.6) if and only if there exists real number l such that $d_G(e) = l d_G^v(e)$ for all $e \in E(G)$. $\square$

If G is a triangle free r-regular graph, then for all $v \in V(G)$, $d_G^e(v) = \sum_{uv\in E(G)} r = r^2$ and for all $e \in E(G)$, $d_G^v(e = uv) = d_G(u) + d_G(v) = 2d_G(v)$. If G is a complete graph then $d_G^e(v) = n(n-1)/2$, $v \in V(G)$ and $d_G^v(e = uv) = n$ for all $e \in E(G)$. Therefore, the Equalities (2.4), (2.5) and (2.6) hold for triangle free regular graphs and also complete graphs.

Theorem 2.3. *Let G be an r-regular graph. Then*

$$\phi_v(G) = r\Big[M_1(G) - 3\eta(G)\Big] \quad \text{and} \quad \phi_e(G) = 2(r-1)\Big[M_1(G) - 3\eta(G)\Big].$$

Proof. Let G be an r-regular graph. Then (1.1) gives

$$\begin{aligned}
\phi_v(G) &= \sum_{v\in V(G)} d_G^e(v) d_G(v) = \sum_{v\in V(G)} d_G^e(v) r \\
&= r \sum_{v\in V(G)} d_G^e(v) = r\Big[M_1(G) - 3\eta(G)\Big], \\
\phi_e(G) &= \sum_{e\in E(G)} d_G^v(e) d_G(e) = \sum_{e\in E(G)} d_G^v(e) 2(r-1) \\
&= 2(r-1) \sum_{e\in E(G)} d_G^v(e) = 2(r-1)\Big[M_1(G) - 3\eta(G)\Big],
\end{aligned}$$

as desired. ☐

Theorem 2.4. *Let G be graph.*

(a) *If G is a ve-regular graph with $d^e_G(v) = c$ for all $v \in V(G)$, then $\phi_v(G) = 2cm$.*
(b) *If G is an ev-regular graph with $d^v_G(e) = k$ for all $e \in E(G)$, then $\phi_e(G) = k[M_1(G) - 2m]$.*

Proof. Let G be ve-regular graph with $d^e_G(v) = c$ for all $v \in V(G)$. Then

$$\phi_v(G) = \sum_{v\in V(G)} d^e_G(v) d_G(v) = c \sum_{v\in V(G)} d_G(v) = 2cm.$$

Now, let G be ev-regular graph, with $d^v_G(e) = k$, for all $e \in E(G)$. Then

$$\phi_e(G) = \sum_{e\in E(G)} d^v_G(e) d_G(e) = k \sum_{e=uv\in E(G)} [d_G(u) + d_G(v) - 2] = k[M_1(G) - 2m].$$

This completes our argument. ☐

Lemma 2.1. *Let G be a connected graph with given vertices u and v such that $uv \notin E(G)$. If $G' = G + uv$, then $T_v(G) = T_e(G) \le T_v(G') = T_e(G') - 2$.*

Proof. Let $x = M_1(G') - 3\eta(G')$ and $y = M_1(G) - 3\eta(G)$. By definition,

$$\begin{aligned} x - y =&(d_G(u) + 1)^2 + (d_G(v) + 1)^2 - 3\Big(\eta(G) + |N(u, G) \cap N(v, G)|\Big) \\ &- \Big[d_G(u)^2 + d_G(v)^2 - 3\eta(G)\Big] \\ =&2d_G(u) + 2d_G(v) + 2 - 3|N(u, G) \cap N(v, G)| \\ \ge&4|N(u, G) \cap N(v, G)| + 2 - 3|N(u, G) \cap N(v, G)| \ge 2. \end{aligned}$$

The proof follows from (1.1). ☐

Let G be a graph. The path $P_k := v_0v_2 \dots v_k$ is called a pendant path in G if $\{v_0, v_1, \dots, v_k\} \subseteq V(G)$, $d_G(v_0) \ge 3$, $d_G(v_k) = 1$, $\{v_iv_{i+1} \mid 0 \le i \le k-1\} \subseteq E(G)$, and $d_G(v_1) = \dots = d_G(v_{k-1}) = 2$, when $k \ge 2$.

Lemma 2.2. *Let G be a graph with two pendant paths $P_k := v_0v_2 \dots v_k$ and $Q_l := u_0u_2 \dots u_l$. If $G' = G - v_0v_1 + u_lv_1$, then $T_v(G') = T_e(G') < T_v(G) = T_e(G) - 2$.*

Proof. Let $x = M_1(G') - 3\eta(G')$ and $y = M_1(G) - 3\eta(G)$. By definition,

$$y - x = d_G(v_0)^2 + 1 - \Big[(d_G(v_0) - 1)^2 + 4\Big] = 2d_G(v_0) - 4 \ge 2,$$

and (1.1) gives the result. ☐

Lemmas 2.1 and 2.2 give the following result.

Corollary 2.1. *Let G be a connected graph with n vertices. Then*

$$4n - 6 \le T_v(G) = T_e(G) \le \frac{1}{2}n^2(n-1).$$

Equality in left holds if and only if $G \cong P_n$ and equality in right holds if and only if $G \cong K_n$.

Corollary 2.2. *Let G be a connected graph with n vertices. Then*
$$\phi_v(G) \leq \frac{n^2(n-1)^2}{2} \quad \text{and} \quad \phi_e(G) \leq n^2(n-1)(n-2).$$
Equalities hold if and only if $G \cong K_n$.

Proof. By definitions,
$$\phi_v(G) = \sum_{v \in V(G)} d_G^e(v) d_G(v) \leq (n-1) \sum_{v \in V(G)} d_G^e(v) = (n-1)T_v(G),$$
$$\phi_e(G) = \sum_{e \in E(G)} d_G^v(e) d_G(e) \leq (2n-4) \sum_{e \in E(G)} d_G^v(e) = (2n-4)T_e(G).$$
Now, Corollary 2.1 gives the results. □

For positive integer $n \geq 4$, let $C_3 := v_1v_2v_3v_1$ and $P_{n-3} := u_1u_2 \dots u_{n-3}$ be cycle and path graph on 3 and $n-3$ vertices, respectively. Then the graph C_3^{n-3} is obtained from C_3 and P_{n-3} by attaching vertices v_1 and u_1. By (1.1),
$$T_v(C_3^{n-3}) = T_e(C_3^{n-3}) = 4n - 1. \tag{2.8}$$

Lemma 2.3. *Let G be a graph with $n \geq 4$ vertices and minimum degree at least 2. Then $T_v(G) = T_e(G) \geq 4n$, with equality if and only if $G \cong C_n$.*

Proof. If $G \cong C_n$, then $T_v(G) = T_e(G) = 4n$ and lemma holds. Otherwise, by using Lemmas 2.1, 2.2 and (2.8), $T_v(G) = T_e(G) \geq 4n + 1$ which gives the lemma. □

Corollary 2.3. *Let G be a graph with $n \geq 4$ vertices and minimum degree at least 2. Then*
$$\phi_v(G) \geq 8n \quad \text{and} \quad \phi_e(G) \geq 8n.$$
Equalities hold if and only if $G \cong C_n$.

Proof. By definitions,
$$\phi_v(G) = \sum_{v \in V(G)} d_G^e(v) d_G(v) \geq 2 \sum_{v \in V(G)} d_G^e(v) = 2T_v(G),$$
$$\phi_e(G) = \sum_{e \in E(G)} d_G^v(e) d_G(e) \geq 2 \sum_{e \in E(G)} d_G^v(e) = 2T_e(G).$$
Now, Lemma 2.3 gives the results. □

Lemma 2.4 (Diaz-Metcalf inequality)**.** *Let the real numbers $a_i \neq 0$, b_i, $1 \leq i \leq n$, satisfy*
$$l \leq \frac{b_i}{a_i} \leq L.$$
Then
$$\sum_{i=1}^n b_i^2 + lL \sum_{i=1}^n a_i^2 \leq (L + l) \sum_{i=1}^n a_i b_i.$$
Equality holds if and only if $b_i = la_i$ or $b_i = La_i$.

Theorem 2.5. *Let G be a graph with n vertices, m edges, minimum degree $\delta \geq 1$ and maximum degree Δ. Then*

(i) $\phi_v(G) \geq \frac{1}{2\Delta+\delta+1}\left[2S^{\alpha}(G)+(\delta+1)\Delta M_1(G)\right]$ *and equality holds if and only if* $d_G^e(v) = \frac{1}{2}(\delta+1)d_G(v)$ *or* $d_G^e(v) = \Delta\, d_G(v)$ *for all* $v \in V(G)$;

(ii) $\phi_e(G) \geq \frac{1}{3}\left[S(G)+2F(G)+4M_2(G)-6M_1(G)+18\eta(G)\right]$ *and equality holds if and only if* $d_G^v(e) = d_G(e)+2$ *or* $2d_G^v(e) = d_G(e)+2$ *for all* $e \in E(G)$.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To prove (i), by setting $a_i = d_G(v_i)$ and $b_i = d_G^e(v_i)$ for all $i = 1, 2, \ldots, n$, $L = \Delta$ and $l = \frac{1}{2}(\delta+1)$ in Diaz-Metcalf inequality we get

$$\sum_{i=1}^{n} d_G^e(v_i)^2 + \frac{1}{2}(\delta+1)\Delta \sum_{i=1}^{n} d_G(v_i)^2 \leq \Big(\frac{1}{2}(\delta+1)+\Delta\Big)\sum_{i=1}^{n} d_G(v_i)d_G^e(v_i),$$

which implies that

$$S^{\alpha}(G) + \frac{1}{2}(\delta+1)\Delta M_1(G) \leq \Big(\frac{1}{2}(\delta+1)+\Delta\Big)\phi_v(G).$$

Therefore,

$$\phi_v(G) \geq \frac{1}{2\Delta+\delta+1}\Big[2S^{\alpha}(G)+(\delta+1)\Delta M_1(G)\Big],$$

and equality holds if and only if $d_G^e(v) = \frac{1}{2}(\delta+1)d_G(v)$ or $d_G^e(v) = \Delta\, d_G(v)$ for all $v \in V(G)$.

To prove (ii), setting $a_i = d_G^v(e_i)$ and $b_i = d_G(e_i)+2$ for all $i = 1, 2, \ldots, m$, $L = 2$ and $l = 1$ in Diaz-Metcalf inequality we get

$$\sum_{i=1}^{m} d_G^v(e_i)^2 + 2\sum_{i=1}^{m}(d_G(e_i)+2)^2 \leq 3\sum_{i=1}^{m}(d_G(e_i)+2)d_G^v(e_i),$$

which implies that

$$S(G) + 2\Big(F(G)+2M_2(G)\Big) \leq 3\phi_e(G) + 6T_e(G).$$

Therefore, by (1.1),

$$\phi_e(G) \geq \frac{1}{3}\Big[S(G)+2F(G)+4M_2(G)-6M_1(G)+18\eta(G)\Big],$$

and equality holds if and only if $d_G^v(e) = d_G(e)+2$ or $2d_G^v(e) = d_G(e)+2$ for all $e \in E(G)$. This completes the proof. □

If G is a triangle free r-regular graph, then for all $v \in V(G)$, $d_G^e(v) = r^2$ and for all $e = uv \in E(G)$, $d_G^v(e) = d_G(e)+2$. Therefore, by Theorem 2.5,

$$\phi_v(G) = \frac{1}{3r+1}\Big[2S^{\alpha}(G)+(r+1)rM_1(G)\Big],$$
$$\phi_e(G) = \frac{1}{3}\Big[S(G)+2F(G)+4M_2(G)-6M_1(G)\Big].$$

Theorem 2.6. *Let G be a graph with n vertices and m edges. Then*
(i) $\phi_v(G) \geq \frac{2m}{n}\left[M_1(G) - 3\eta(G)\right]$*;*
(ii) $\phi_e(G) \geq \frac{1}{m}[M_1(G) - 2m][M_1(G) - 3\eta(G)]$.
The bounds attain on the cycle C_n, $n \geq 3$, and the star $K_{1,n-1}$, $n \geq 2$.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Chebyshev's inequality states that, for any non-increasing sequences $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, we have

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Suppose $a_i = d_G(v_i)$ and $b_i = d_G^e(v_i)$, for $i = 1, 2, \ldots, n$. By (1.1), we obtain

$$n\sum_{i=1}^{n} d_G(v_i) d_G^e(v_i) \geq \sum_{i=1}^{n} d_G(v_i) \sum_{i=1}^{n} d_G^e(v_i),$$

and hence, $\phi_v(G) \geq \frac{2m}{n}[M_1(G) - 3\eta(G)]$. This proves (i).

To prove (ii), we define $a_i = d_G(e_i)$ and $b_i = d_G^v(e_i)$, for $i = 1, 2, \ldots, m$. By (1.1), we obtain

$$m\sum_{i=1}^{m} d_G(e_i) d_G^v(e_i) \geq \sum_{i=1}^{m} d_G(e_i) \sum_{i=1}^{m} d_G^v(e_i),$$

and hence, $\phi_e(G) \geq \frac{1}{m}[M_1(G) - 2m][M_1(G) - 3\eta(G)]$. ☐

It is well-known that $M_1(G) \geq 4n - 6$, with equality if and only if $G \cong P_n$. Therefore, Theorem 2.6, Corollary 2.1 and $M_1(G) \geq 4n - 6$ give the following results.

Corollary 2.4. *Let G be a graph with n vertices and m edges. Then*

$$\phi_v(G) \geq \frac{2m}{n}(4n - 6) \ \textit{and} \ \phi_e(G) \geq \frac{1}{m}[4n - 2m - 6][4n - 6].$$

Lemma 2.5 (Ozeki-Izumino-Mori-Seo type inequality). *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers satisfying $0 \leq r_1 \leq a_i \leq R_1$ and $0 \leq r_2 \leq b_i \leq R_2$, $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{3}\Big(R_1R_2 - r_1r_2\Big)^2.$$

Theorem 2.7. *Let G be a connected graph with n vertices and m edges. Then*
(i) $\phi_v(G) \geq \sqrt{M_1(G)S^{\alpha}(G) - \frac{n^2}{3}\Big(\Delta^3 - \delta(\delta+1)\Big)^2}$*;*
(ii) $\phi_e(G) \geq \sqrt{\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)S(G) - \frac{16}{3}m^2\Big(\Delta(\Delta-1) - \delta(\delta-1)\Big)^2}$.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To prove (i), we put $a = \Big(d_G(v_1), d_G(v_2), \ldots, d_G(v_n)\Big)$, $b = \Big(d_G^e(v_1), d_G^e(v_2), \ldots, d_G^e(v_n)\Big)$, $r_1 = \delta$, $R_1 = \Delta$, $r_2 = \delta + 1$ and $R_2 = \Delta^2$. By Ozeki-Izumino-Mori-Seo type inequality we get

$$M_1(G)S^{\alpha}(G) - \phi_v(G)^2 \leq \frac{n^2}{3}\Big(\Delta^3 - \delta(\delta+1)\Big)^2,$$

which implies that

$$\phi_v(G) \geq \sqrt{M_1(G)S^\alpha(G) - \frac{n^2}{3}\Big(\Delta^3 - \delta(\delta+1)\Big)^2}.$$

To prove (ii), we set $a = \Big(d_G(e_1), d_G(e_2), \ldots, d_G(e_m)\Big)$, $b = \Big(d_G^v(e_1), d_G^v(e_2), \ldots, d_G^v(e_m)\Big)$, $r_1 = 2(\delta - 1)$, $R_1 = 2(\Delta - 1)$, $r_2 = 2\delta$ and $R_2 = 2\Delta$. Again by Ozeki-Izumino-Mori-Seo type inequality we get

$$\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)S(G) - \phi_e(G)^2 \leq \frac{m^2}{3}\Big(4\Delta(\Delta-1) - 4\delta(\delta-1)\Big)^2,$$

which implies that

$$\phi_e(G) \geq \sqrt{\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)S(G) - \frac{16}{3}m^2\Big(\Delta(\Delta-1) - \delta(\delta-1)\Big)^2}.$$

This completes our argument. □

Corollary 2.5. *Let G be a connected graph with n vertices and m edges. Then*

$$\phi_v(G) \geq \frac{1}{3}\sqrt{\frac{9(4n-6)^3}{n} - 3n^2\Big(n^3 - 3n^2 + 3n - 3\Big)^2}.$$

Proof. By (1.1), (2.4) and Corollary 2.1, $S^\alpha \geq \frac{(4n-6)^2}{n}$. Therefore, by $M_1(G) \geq 4n - 6$ and Theorem 2.7,

$$\phi_v(G) \geq \frac{1}{3}\sqrt{\frac{9(4n-6)^3}{n} - 3n^2\Big(n^3 - 3n^2 + 3n - 3\Big)^2},$$

as desired. □

3. Examples

Let G be a simple graph. The notation $m_{i,j}$, $1 \leq i \leq j \leq n-1$, denote the number of edges of G connecting a vertex of degree i with a vertex of degree j.

It is preferred to show titania nanotubes as $TiO_2[m,n]$, where m and n denote the number of octagons in a row and in a column, respectively. See Figure 1 for details. The $TNT_3[m,n]$ is the two-parametric chemical graph of three-layered titania nanotubes, where m and n represent the number of titanium atoms in each row and column, respectively, Figure 2. Finally, $TNT_6[m,n]$ is the two-parametric chemical graph of a six-layered single-walled titania nanotube, where m and n represent the number of titanium atoms in each column and row, respectively, Figure 3.

The following proposition is a result of Table 1 and Proposition 2.2 in which the *ve*-degree and *ev*-degree connectivity indices of $TiO_2[m,n]$, $TNT_3[m,n]$ and $TNT_6[m,n]$ are given.

Proposition 3.1. *The following hold:*

$$\phi_v\Big(TiO_2[m,n]\Big) = 4m(65n+31), \quad \phi_e\Big(TiO_2[m,n]\Big) = 4m(107n+47),$$

TABLE 1. End point degree edges distributions of $TiO_2[m,n]$, $TNT_3[m,n]$ and $TNT_6[m,n]$

symbol	$m_{2,2}$	$m_{2,3}$	$m_{2,4}$	$m_{2,5}$	$m_{2,6}$	$m_{3,4}$	$m_{3,5}$	$m_{3,6}$
$TiO_2[m,n]$	0	0	$6m$	$4mn+2m$	0	$2m$	$6mn-2m$	0
$TNT_3[m,n]$	0	0	$4m$	0	$4m$	$4m$	0	$2m(6n-5)$
$TNT_6[m,n]$	$2m$	$2m$	$6m$	$8mn$	0	$2m$	$2m(6n-5)$	0

$$\phi_v\Big(TNT_3[m,n]\Big) = 8m(54n-13), \quad \phi_e\Big(TNT_3[m,n]\Big) = 2m(378n-101),$$
$$\phi_v\Big(TNT_6[m,n]\Big) = 4m(130n-29), \quad \phi_e\Big(TNT_6[m,n]\Big) = 4m(214n-55).$$

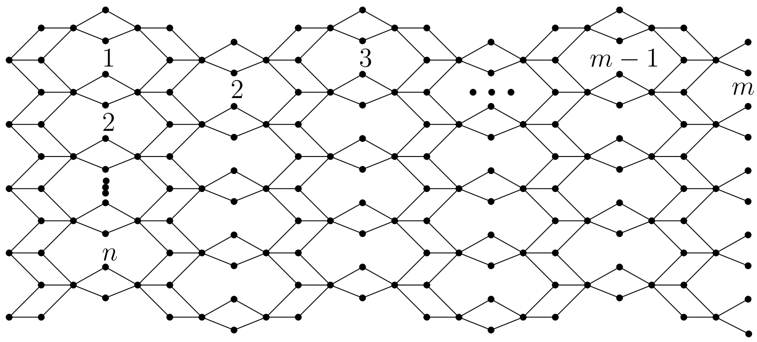

FIGURE 1. The molecular graph of titania nanotubes.

4. Concluding Remarks

In this paper, two graph invariants of the vertex-edge connectivity index and the edge-vertex connectivity index of a graph G were introduced. The main properties

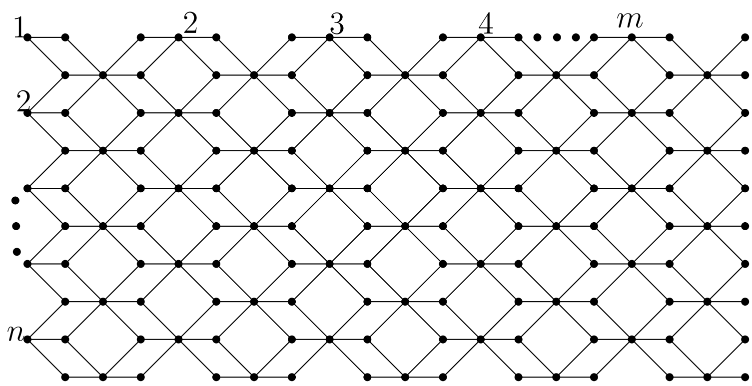

FIGURE 2. The graph of 3-layered titania nanotube.

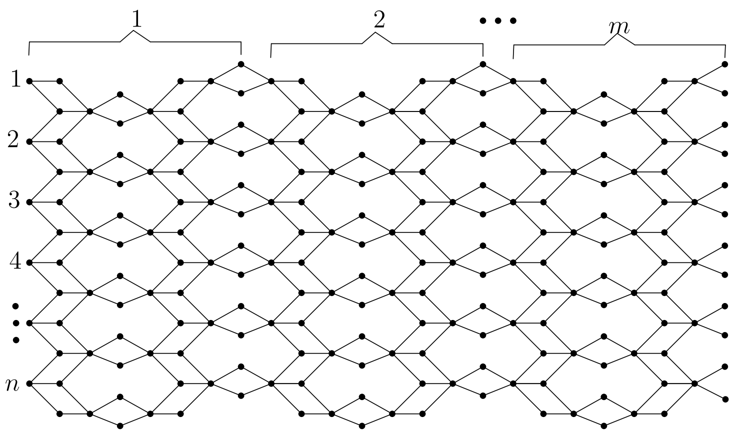

FIGURE 3. The graph of six-layered single walled titania nanotubes.

of these invariants were studied and we established some upper and lower bounds for them. These numbers for titania nanotubes are also computed.

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ON THE ZAGREB INDEX OF TOURNAMENTS

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Abstract. A tournament is an orientation of a complete simple graph. The score of a vertex in a tournament is the out degree of the vertex. The Zagreb index of a tournament is defined as the sum of the squares of the scores of its vertices. In this paper, we obtain various lower and upper bounds for the Zagreb index of a tournament.

1. Introduction

A tournament is an orientation of a complete simple graph. Let T be a tournament with order n and having vertex set $\{v_1, v_2, \ldots, v_n\}$. The score of a vertex v_i, $1 \leq i \leq n$, denoted by s_{v_i} (or simply by s_i), is defined as the out degree of v_i. Clearly, $0 \leq s_i \leq n-1$ for all i, $1 \leq i \leq n$. The sequence $[s_1, s_2, \ldots, s_n]$ in non-decreasing order is called the score sequence of the tournament T. A regular tournament on n (odd) vertices is a tournament in which score of every vertex is $\frac{n-1}{2}$. Many of the important properties of tournaments were first investigated by Landau [5] (1953) in order to model dominance relations in flocks of chickens. Current applications of tournaments include the study of voting theory and social choice theory among other things. Other undefined notations and terminology can be seen in [8].

The following result [5], also called Landau's theorem, gives a necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

Theorem 1.1 (Landau [5])**.** *A sequence $[s_1, s_2, \ldots, s_n]$ of non-negative integers in non-decreasing order is a score sequence of some tournament if and only if*

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$$(1.1) \qquad \sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}, \quad \textit{for } 1 \leq k \leq n,$$

with equality when $k = n$.

Several results for the scores in a tournament can be seen in [3, 6, 7, 9, 13]. Also, stronger inequalities for scores in tournaments can be found in [2]. Further the extension of scores to oriented graphs and digraphs can be seen in [10–12].

For any two distinct vertices u and v of a tournament T, we have one of the following possibilities:

(i) there is an arc directed from u to v which is denoted by $u(1-0)v$;
(ii) there is an arc directed from v to u which is denoted by $u(0-1)v$.

One of the oldest graph invariants is the well-known Zagreb index first introduced by Gutman and Trinajstić [4], where they examined the dependence of total π-electron energy on molecular structure. Some recent work can be seen in [1]. The (first) Zagreb index $M_1(G)$ of a graph G is defined as the sum of the squares of the degrees of the vertices of G and the second Zagreb index $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices. These two topological indices (M_1 and M_2) reflect the extent of branching of the molecular carbon-atom skeleton. Determining the extremal values or bounds of these two topological indices of graphs, as well as characterizing the corresponding extremal graphs, has attracted the attention of many researchers. Analogous to this, we define the Zagreb index $M(T)$ of a tournament T as the sum of the scores of the vertices of T. That is, $M(T) = \sum_{i=1}^{n} s_i^2$.

The rest of the paper is organized as follows. In Section 2, we obtain the lower bounds for the Zagreb index $M(T)$ of a tournament T. In Section 3, we compute the upper bounds for $M(T)$.

2. Lower Bounds for the Zagreb Index $M(T)$

The following result gives the best general lower bound for $M(T)$.

Theorem 2.1. *If* $[s_1, s_2, \ldots, s_n]$ *is the score sequence of a tournament* T*, then*

$$(2.1) \qquad M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n}{2}\Big\{2m(n-m-2) + n - 1\Big\}, \quad \textit{where } m = \left\lfloor \frac{n-1}{2} \right\rfloor,$$

with equality if and only if $s_i - s_j \leq 1$ *for all* i, j, $1 \leq i, j \leq n$, *where* $\lfloor \cdot \rfloor$ *denotes the floor function.*

Proof. Let v_i and v_j be two vertices of the tournament T with their respective scores as s_i and s_j such that $s_i \geq s_j$. Also, assume that $M(T) = \sum_{r=1}^{n} {s_r}^2$ is minimum.

We claim that $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. To prove the claim, we assume to the contrary that $s_i - s_j > 1$ for some i, j, $1 \leq i, j \leq n$. Then there exists a vertex v_k with score s_k such that $v_i(1-0)v_k$ and $v_k(1-0)v_j$. Now, reversing the orientation of these arcs to $v_i(0-1)v_k$ and $v_k(0-1)v_j$ respectively, we get a new tournament T_1

with the score sequence $[t_1, t_2, \dots, t_n]$, where $t_i = s_i - 1$, $t_j = s_j + 1$, $t_r = s_r$ for all r, $1 \le r \le n$ with $r \ne i, j$.

Thus,

$$\begin{aligned}\sum_{r=1}^{n} {t_r}^2 &= \sum_{r=1}^{j-1} {t_r}^2 + t_j^2 + \sum_{r=j+1}^{i-1} {t_r}^2 + t_i^2 + \sum_{r=i+1}^{n} {t_r}^2\\ &= \sum_{r=1}^{j-1} {s_r}^2 + (s_j+1)^2 + \sum_{r=j+1}^{i-1} {s_r}^2 + (s_i-1)^2 + \sum_{r=i+1}^{n} {s_r}^2\\ &= \sum_{r=1}^{n} {s_r}^2 - 2(s_i - s_j - 1).\end{aligned}$$

As $s_i - s_j > 1$, so we obtain

$$\sum_{r=1}^{n} {t_r}^2 < \sum_{r=1}^{n} {s_r}^2,$$

which is a contradiction, since $M(T) = \sum_{r=1}^{n} {s_r}^2$ is minimum. Hence, $s_i - s_j \le 1$ for all i, j, $1 \le i, j \le n$. This means that some of the vertices of T have score m and the remaining vertices (if any) have score $m+1$. If x vertices of T have score m and y vertices have score $m+1$, then

$$x + y = n \tag{2.2}$$

and by (1.1), we have

$$mx + (m+1)y = \frac{n(n-1)}{2}. \tag{2.3}$$

Solving (2.2) and (2.3), we get $x = \frac{n}{2}(2m - n + 3)$ and $y = \frac{n}{2}(n - 2m - 1)$. Therefore,

$$\begin{aligned}\min M(T) &= \min \sum_{i=1}^{n} {s_i}^2 = \min\{s_1^2 + s_2^2 + \cdots + s_n^2\}\\ &= \underbrace{m^2 + m^2 + \cdots + m^2}_{\frac{n}{2}(2m-n+3)-\text{times}} + \underbrace{(m+1)^2 + (m+1)^2 + \cdots + (m+1)^2}_{\frac{n}{2}(n-2m-1)-\text{times}}\\ &= \frac{n}{2}(2m - n + 3)m^2 + \frac{n}{2}(n - 2m - 1)(m+1)^2\\ &= \frac{n}{2}\{2m(n - m - 2) + n - 1\}.\end{aligned}$$

That is,

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \ge \frac{n}{2}\{2m(n - m - 2) + n - 1\}.$$

Now, assume that equality holds in (2.1). Since $M(T)$ is minimal, so some of the vertices of T have score m and the remaining vertices (if any) have score $m+1$, where $m = \lfloor \frac{n-1}{2} \rfloor$. Therefore, $s_i - s_j \le 1$ for all i, j, $1 \le i, j \le n$.

Conversely, assume that $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. Then as above, we have

$$\begin{aligned}
M(T) &= \sum_{i=1}^{n} {s_i}^2 = s_1^2 + s_2^2 + \cdots + s_n^2 \\
&= \underbrace{m^2 + m^2 + \cdots + m^2}_{\frac{n}{2}(2m-n+3)-\text{times}} + \underbrace{(m+1)^2 + (m+1)^2 + \cdots + (m+1)^2}_{\frac{n}{2}(n-2m-1)-\text{times}} \\
&= \frac{n}{2}(2m-n+3)m^2 + \frac{n}{2}(n-2m-1)(m+1)^2 \\
&= \frac{n}{2}\{2m(n-m-2) + n - 1\}.
\end{aligned}$$

Therefore equality holds in (2.1). ☐

Theorem 2.2. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T and $m = \lfloor \frac{n-2}{2} \rfloor$ and $x = \frac{n-1}{2}(n-2m-2)$. Then the following hold.*

(i) *For $s_n > x$, we have*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n-1}{2}\{(2m+1)(n-m) - m\} + s_n^2 - x(2m+1).$$

(ii) *For $s_n \leq x$, we have*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n-1}{2}\{(2m+1)(n-m) - m\} + s_n^2 + 2x - s_n(2m+3).$$

Proof. Let v_n be the vertex of the tournament T with score s_n. Deleting the vertex v_n, we obtain a new tournament $T_1 = T - \{v_n\}$ with score sequence $[t_1, t_2, \ldots, t_{n-1}]$. By Theorem 2.1, the minimum value of $M(T)$ is attained in terms of n if and only if $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. Using this result, we conclude that the value of $\sum_{i=1}^{n-1} {t_i}^2$ (in terms of the number of vertices) will be minimum if the value of $M(T)$ (in terms of n and s_n) is minimum. So, we have to find the minimum value of $M(T)$ in terms of n and s_n. For this, first we find the minimum value of $\sum_{i=1}^{n-1} {t_i}^2$ in terms of the number of vertices.

As the tournament T_1 has $n-1$ vertices, therefore, by using Theorem 2.1, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} {t_i}^2 &\geq \frac{n-1}{2}\{2m(n-1-m-2) + (n-1) - 1\} \\
&= \frac{n-1}{2}\{2m(n-m-3) + (n-2)\},
\end{aligned}$$

where $m = \lfloor \frac{(n-1)-1}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor$ and $t_i - t_j \leq 1$ for all i, j, $1 \leq i, j \leq n-1$.

If x vertices of T_1 have score $m+1$ and y vertices have score m, then we have

$$x + y = n - 1. \tag{2.4}$$

Also, by (1.1), we have

$$(m+1)x+my=\frac{(n-1)(n-2)}{2}. \tag{2.5}$$

Solving (2.4) and (2.5) for x, we have $x=\frac{n-1}{2}(n-2m-2)$. So, T_1 has $x=\frac{n-1}{2}(n-2m-2)$ vertices of score $m+1$ and $n-1-x$ vertices of score m.

Now, we add the vertex v_n of score s_n and join it to the other vertices of the tournament T_1 by arcs, such that $M(T)=\sum_{i=1}^{n} {s_i}^2$ is minimum. This can be done as follows. Let $v_n(1-0)u$ to as many vertices u of score $m+1$ as possible and then $v_n(1-0)v$ to the remaining vertices v of score m till the score s_n is exhausted. Note that other arcs are directed towards v_n in order to complete the tournament. Now, we consider the following two cases.

Case (i). When $s_n>x$, then

$$\min M(T)=\min\sum_{i=1}^{n} {s_i}^2=\min\sum_{i=1}^{n-1} {t_i}^2+s_n^2+(n-1-x)(2m+1),$$

that is,

$$\begin{aligned} M(T)&\geq\frac{n-1}{2}\{2m(n-m-3)+n-2\}+s_n^2+(n-1)(2m+1)-x(2m+1)\\ &=\frac{n-1}{2}\{(2m+1)(n-m)-m\}+s_n^2-x(2m+1), \end{aligned}$$

where $m=\lfloor\frac{n-2}{2}\rfloor$ and $x=\frac{n-1}{2}(n-2m-2)$.

Case (ii). When $s_n\leq x$, then

$$\begin{aligned} \min M(T)&=\min\sum_{i=1}^{n} {s_i}^2\\ &=\min\sum_{i=1}^{n-1} {t_i}^2+s_n^2+(n-1-x)(2m+1)+(x-s_n)\{2(m+1)+1\}, \end{aligned}$$

that is,

$$\begin{aligned} M(T)\geq&\frac{n-1}{2}\{2m(n-m-3)+n-2\}+s_n^2+(n-1)(2m+1)-x(2m+1)\\ &+(x-s_n)(2m+3)\\ =&\frac{n-1}{2}\{(2m+1)(n-m)-m\}+s_n^2+2x-s_n(2m+3), \end{aligned}$$

where $m=\lfloor\frac{n-2}{2}\rfloor$ and $x=\frac{n-1}{2}(n-2m-2)$. $\square$

Remark 2.1. The lower bounds given by Theorems 2.1 and 2.2 are best possible, since these bounds hold for every score sequence $[s_1,s_2,\ldots,s_n]$ of a tournament. In particular, these hold for a regular tournament on n (odd) vertices having score sequence $\left[\frac{n-1}{2},\frac{n-1}{2},\ldots,\frac{n-1}{2}\right]$. Clearly $\sum_{i=1}^{n} s_i^2$ is minimum and so the equality in Theorems 2.1 and 2.2 hold for regular tournaments.

Theorem 2.3. *If* $[s_1, s_2, \dots, s_n]$ *is the score sequence of a tournament* T*, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq s_1^2 + s_n^2 + \frac{1}{n-2}\left\{\frac{n(n-1)}{2} - s_1 - s_2\right\}^2,$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$.

Proof. Consider $s_2, s_3, \dots, s_{n-1}$ as the weights assigned to the scores $s_2, s_3, \dots, s_{n-1}$, respectively. Since the arithmetic mean is greater than or equal to the harmonic mean, therefore

$$\frac{\sum\limits_{i=2}^{n-1} s_i s_i}{\sum\limits_{i=2}^{n-1} s_i} \geq \frac{\sum\limits_{i=2}^{n-1} s_i}{\sum\limits_{i=2}^{n-1} \frac{s_i}{s_i}},$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$. That is,

$$\sum_{i=2}^{n-1} {s_i}^2 \geq \frac{1}{n-2}\left(\sum_{i=2}^{n-1} s_i\right)^2,$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$. After simplification, it is easy to see that

$$\sum_{i=1}^{n} {s_i}^2 - s_1^2 - s_n^2 \geq \frac{1}{n-2}\left(\sum_{i=1}^{n} s_i - s_1 - s_n\right)^2.$$

By using (1.1), we have

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq s_1^2 + s_n^2 + \frac{1}{n-2}\left\{\frac{n(n-1)}{2} - s_1 - s_n\right\}^2,$$

equality holds if and only if $s_2 = s_3 = \cdots = s_{n-1}$. □

Theorem 2.4. *If* $[s_1, s_2, \dots, s_n]$ *is the score sequence of a tournament* T*, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n}{4}(n-1)^2,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$.

Proof. Applying the Cauchy-Schwartz inequality, we have

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i \cdot 1 \leq \left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}},$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. This is equivalent to

$$\sum_{i=1}^{n} s_i \leq \left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} n^{\frac{1}{2}},$$

which after simplification gives

$$\left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} \geq \frac{1}{n^{\frac{1}{2}}}\sum_{i=1}^{n} s_i,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. Now, by using (1.1), we have

$$\sum_{i=1}^{n} {s_i}^2 \geq \frac{1}{n}\left\{\frac{n(n-1)}{2}\right\}^2 = \frac{n}{4}(n-1)^2,$$

where equality occurs if and only if $s_1 = s_2 = \cdots = s_n$. Thus,

$$M(T) \geq \frac{n}{4}(n-1)^2,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. □

3. Upper Bounds for the Zagreb Index $M(T)$

In this section, we obtain the upper bounds for the Zagreb index $M(T)$. In a tournament, we denote with N_i^+ the out-neighbor set of the vertex v_i.

Theorem 3.1. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament and $M(T) = \sum\limits_{i=1}^{n} {s_i}^2$ be maximum. Then*

- **(a)** *$N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$ if and only if $s_i = s_j$;*
- **(b)** *$N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$ if and only if $s_i > s_j$, and*
- **(c)** *$s_i < s_j$ if $v_i \in N_k^+$ and $v_j \in (N_k^+)^c - \{v_k\}$, where s_i and s_j are the scores of the two vertices v_i and v_j respectively.*

Proof. **(a)** Let $s_i = s_j$. Assume to the contrary that $N_i^+ - \{v_j\} \neq N_j^+ - \{v_i\}$. Since $s_i = s_j$, therefore there exist at least two vertex v_p and v_q with their respective scores s_p and s_q such that $v_i(1-0)v_p$, $v_p(1-0)v_j$, $v_j(1-0)v_q$ and $v_q(1-0)v_i$. Now, we consider two cases.

Case (i). When $s_p \geq s_q$. By changing the arcs $v_i(1-0)v_p$ and $v_q(1-0)v_i$ to $v_i(0-1)v_p$ and $v_q(0-1)v_i$ respectively, we get a new score sequence $[t_1, t_2, \ldots, t_n]$, where $t_p = s_p + 1$, $t_q = s_q - 1$ and $t_r = s_r$ for all r, $1 \leq r \leq n$ with $r \neq p, q$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} {t_i}^2 &= \sum_{\substack{i=1\\ i\neq p,q}}^{n} {t_i}^2 + t_p^2 + t_q^2 = \sum_{\substack{i=1\\ i\neq p,q}}^{n} {s_i}^2 + (s_p+1)^2 + (s_q-1)^2\\ &= \sum_{i=1}^{n} {s_i}^2 + 2(s_p - s_q + 1) > \sum_{i=1}^{n} {s_i}^2,\end{aligned}$$

since $s_p \geq s_q$, which is a contradiction, since $M(T) = \sum_{i=1}^{n} {s_i}^2$ was assumed to be maximum.

Case (ii). When $s_p < s_q$. By changing the arcs $v_p(1-0)v_j$ and $v_j(1-0)v_q$ to $v_p(0-1)v_j$ and $v_j(0-1)v_q$, respectively and proceeding as in case (i), we arrive at a contradiction. Hence, $N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$.

Conversely, if $N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$, then $s_i = s_j$.

(b) Let $s_i > s_j$. Assume to the contrary that $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$ is not true. Then there exists a vertex $v_p \in N_j^+ - \{v_i\}$, but $v_p \notin N_i^+ - \{v_j\}$. This means that $v_j(1-0)v_p$ and $v_p(1-0)v_i$, and by changing these arcs to $v_j(0-1)v_p$ and $v_p(0-1)v_i$ respectively, we get a new score sequence $[t_1, t_2, \ldots, t_n]$, where $t_i = s_i + 1$, $t_j = s_j - 1$ and $t_r = s_r$ for all r, $1 \le r \le n$ with $r \ne i, j$. Then

$$\begin{aligned}\sum_{r=1}^{n} {t_r}^2 &= \sum_{\substack{r=1\\ r\neq i,j}}^{n} {t_r}^2 + t_i^2 + t_j^2 = \sum_{\substack{r=1\\ r\neq i,j}}^{n} {s_r}^2 + (s_i+1)^2 + (s_j-1)^2\\ &= \sum_{r=1}^{n} {s_r}^2 + 2(s_i - s_j + 1) > \sum_{r=1}^{n} {s_r}^2,\end{aligned}$$

since $s_i > s_j$, which is a contradiction, since $M(T)$ was assumed to be maximum. Hence, $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$.

Conversely, if $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$, then $s_i > s_j$.

(c) Assume to the contrary that $s_i \ge s_j$. Then, by using parts **(a)** and **(b)**, we have $N_i^+ - \{v_j\} \supseteq N_j^+ - \{v_i\}$. Since $v_i \in N_k^+$ and $v_j \in (N_k^+)^c - \{v_k\}$, so $v_k(1-0)v_i$ and $v_j(1-0)v_k$. Therefore,

$$\{v_k\} \subseteq N_j^+ - \{v_i\} \subseteq N_i^+ - \{v_j\},$$

that is, $v_k \in N_i^+ - \{v_j\}$. Thus, we obtain $v_i(1-0)v_k$, which is a contradiction. Hence, the result follows. □

Lemma 3.1. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament and let m_i be the average of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 = \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i.$$

Proof. Since

$$s_i m_i = s_i \frac{1}{s_i} \sum_{j=1}^{n} \{s_j : \ v_i(1-0)v_j\} = \sum_{j=1}^{n} \{s_j : v_i(1-0)v_j\},$$

therefore, by using (1.1), we have

$$\begin{aligned}\sum_{i=1}^{n} s_i m_i &= \sum_{i=1}^{n}\sum_{j=1}^{n} \{s_j : \ v_i(1-0)v_j\} = \sum_{j=1}^{n}\sum_{i=1}^{n} \{s_j : \ v_i(1-0)v_j\}\\ &= \sum_{j=1}^{n} s_j(n-1-s_j) = (n-1)\sum_{j=1}^{n} s_j - \sum_{j=1}^{n} s_j^2\\ &= (n-1)\frac{n(n-1)}{2} - \sum_{j=1}^{n} s_j^2.\end{aligned}$$

Hence,

$$M(T) = \sum_{i=1}^{n} s_i^2 = \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i.$$

□

Theorem 3.2. *If $[s_1, s_2, \ldots, s_n]$ is the score sequence of a tournament T, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \leq \frac{n(n-1)}{2} s_n, \tag{3.1}$$

with equality if and only if the tournament is regular.

Proof. Let m_i be the average of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then, by using (1.1), we have

$$\begin{aligned} s_i m_i =& s_i \frac{1}{s_i} \sum_{j=1}^{n} \{s_j : v_i(1-0)v_j\} \geq \sum_{j=1}^{n} s_j - s_i - (n-1-s_i)s_n \\ =& \frac{n(n-1)}{2} - s_i - (n-1-s_i)s_n, \end{aligned} \tag{3.2}$$

with equality if and only if $s_i = \frac{n-1}{2}$ for all i, $1 \leq i \leq n$.

Now, by Lemma 3.1, (3.2) and (1.1), we have

$$\begin{aligned} \sum_{i=1}^{n} s_i^2 &= \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i \leq \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} \left(\frac{n(n-1)}{2} - s_i - (n-1-s_i)s_n \right) \\ &= \frac{n(n-1)^2}{2} - \frac{n^2(n-1)}{2} + \sum_{i=1}^{n} s_i + n(n-1)s_n - s_n \sum_{i=1}^{n} s_i \\ &= \frac{n(n-1)^2}{2} - \frac{n^2(n-1)}{2} + \frac{n(n-1)}{2} + n(n-1)s_n - s_n \frac{n(n-1)}{2} \\ &= \frac{n(n-1)}{2} s_n. \end{aligned}$$

Therefore,

$$M(T) \leq \frac{n(n-1)}{2} s_n.$$

Now suppose that equality holds in (3.1). Then, $s_i = \frac{n-1}{2}$ for all i, $1 \leq i \leq n$, that is, the tournament is regular.

Conversely, suppose that the tournament is regular. Then, it can be easily checked that equality holds in (3.1). □

Theorem 3.3. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament with vertex set V and let m_i be the average of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then*

$$M(T) = \sum_{j=1}^{n} s_j^2 \leq \frac{n(n-1)}{4} \Big(n - 1 + \max\{s_j - m_j : v_j \in V\} \Big), \tag{3.3}$$

with equality if and only if $\max\{s_j - m_j : v_j \in V\} = n - 1 - 2m_i$*, where* $1 \leq i, j \leq n$*, with* $i \neq j$*.*

Proof. Applying Lemma 3.1, we have

$$\begin{aligned}
2\sum_{j=1}^{n} s_j^2 &= \sum_{j=1}^{n} s_j^2 + \sum_{j=1}^{n} s_j^2 = \sum_{j=1}^{n} s_j^2 + \frac{n(n-1)^2}{2} - \sum_{j=1}^{n} s_j m_j \\
&= \frac{n(n-1)^2}{2} + \sum_{j=1}^{n} s_j(s_j - m_j) \leq \frac{n(n-1)^2}{2} + \max\{s_j - m_j : v_j \in V\} \sum_{j=1}^{n} s_j \\
&= \frac{n(n-1)}{2}\Big(n - 1 + \max\{s_j - m_j : v_j \in V\}\Big).
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} s_j^2 \leq \frac{n(n-1)}{4}\Big(n - 1 + \max\{s_j - m_j : v_j \in V\}\Big)$$

Equality holds in (3.3) if and only if

$$\sum_{j=1}^{n} s_j^2 = \frac{n(n-1)^2}{4} + \frac{n(n-1)}{4}p, \tag{3.4}$$

where $p = \max\{s_j - m_j : v_j \in V\}$. By Lemma 3.1, (3.4) is equivalent to

$$\frac{n(n-1)^2}{2} - \sum_{j=1}^{n} s_j m_j = \frac{n(n-1)^2}{4} + \frac{n(n-1)}{4}p,$$

which after simplification gives

$$\frac{n(n-1)}{2}\left(\frac{p-n+1}{2}\right) + \sum_{j=1}^{n} s_j m_j = 0.$$

By (1.1), this implies that

$$\sum_{j=1}^{n} s_j\left(\frac{p-n+1}{2}\right) + \sum_{j=1}^{n} s_j m_j = 0,$$

that is,

$$\sum_{j=1}^{n} s_j\left(\frac{p-n+1}{2} + m_j\right) = 0.$$

Finally, after simplification, we have

$$\sum_{j=1}^{n} s_j(p - n + 1 + 2m_j) = 0. \tag{3.5}$$

Now, assume that equality holds in (3.3). Then (3.5) holds. Since each term in this summation is non-negative and sum is equal to zero, therefore for each v_i either

$s_i = 0$ or $\max\{s_j - m_j : v_j \in V\} = p = n-1-2m_i$. But $s_i = 0$ is not possible for each v_i in any tournament (except the tournament with only one vertex), therefore

$$\max\{s_j - m_j : v_j \in V\} = n-1-2m_i.$$

Conversely, assume that $\max\{s_j - m_j : v_j \in V\} = n-1-2m_i$. Then, by Lemma 3.1, we have

$$\begin{aligned}\frac{n(n-1)}{4}\Big(n-1+\max\{s_j-m_j:\ v_j\in V\}\Big) &= \frac{n(n-1)^2}{4}+\frac{n(n-1)}{4}(n-1-2m_i)\\ &= \frac{n(n-1)^2}{4}+\frac{n(n-1)^2}{4}-\frac{n(n-1)}{2}m_i\\ &= \frac{n(n-1)^2}{2}-\sum_{i=1}^{n}s_im_i=\sum_{i=1}^{n}s_i^2.\end{aligned}$$

Therefore, equality holds in (3.3). □

Theorem 3.4. *If $[s_1, s_2, \ldots, s_n]$ is the score sequence of a tournament T, then*

$$M(T) = \sum_{i=1}^{n} s_i^2 \le \frac{n(n-1)}{2}(s_1+s_n) - ns_1s_n, \tag{3.6}$$

with equality if and only if the tournament has only two types of scores s_1 and s_n.

Proof. By using (1.1), we have

$$\begin{aligned}M(T) &= \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n}(s_i^2 - s_is_1 + s_is_1) = \sum_{i=1}^{n}\{s_i(s_i-s_1)+s_is_1\}\\ &\le \sum_{i=1}^{n}\{s_n(s_i-s_1)+s_is_1\} = \sum_{i=1}^{n}(s_ns_i - s_ns_1 + s_is_1)\\ &= \sum_{i=1}^{n}(s_n+s_1)s_i - \sum_{i=1}^{n}s_ns_1 = \frac{n(n-1)}{2}(s_1+s_n) - ns_1s_n.\end{aligned}$$

Equality holds if and only if

$$\sum_{i=1}^{n}\{s_i(s_i-s_1)\} = \sum_{i=1}^{n}\{s_n(s_i-s_1)\}$$

or

$$\sum_{i=1}^{n}\{s_n(s_i-1)-s_i(s_i-1)\} = 0$$

or

$$\sum_{i=1}^{n}\{(s_n-s_i)(s_i-s_1)\} = 0. \tag{3.7}$$

Now, assume that equality holds in (3.6). Then equality holds in (3.7). Since each term in this summation is non-negative and sum is equal to zero, therefore either $s_i = s_1$ or $s_i = s_n$ for $i = 1, 2, \ldots, n$. So the tournament has only two types of scores s_1 and s_n.

Conversely, suppose that the tournament has only two types of scores s_1 and s_n. Then $\sum_{i=1}^{n}\{(s_n - s_i)(s_i - s_1)\} = 0$. Hence, the equality holds. $\square$

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ON BOUNDS FOR NORMS OF SINE AND COSINE ALONG A CIRCLE ON THE COMPLEX PLANE

FENG QI[1,2,3]

Dedicated to Dr. Prof. Aliakbar Montazer Haghighi at Prairie View A&M University in USA

Abstract. In the paper, the author presents lower and upper bounds for norms of the sine and cosine functions along a circle on the complex plane.

1. Motivations

This paper is a companion of the formally published article [6].

In the theory of complex functions, the sine and cosine functions $\sin z$ and $\cos z$ on the complex plane $\mathbb{C}$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

respectively, where $z = x + iy$, $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ is the imaginary unit. They have the least positive periodicity 2π, that is,

$$\sin(z + 2k\pi) = \sin z \quad \text{and} \quad \cos(z + 2k\pi) = \cos z,$$

for $k \in \mathbb{Z}$.

When restricting $z = x \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ become $\sin x$ and $\cos x$ and satisfy

$$0 \le |\sin x| \le 1 \quad \text{and} \quad 0 \le |\cos x| \le 1.$$

Key words and phrases. Bound, norm, sine, cosine, double inequality, circle, complex plane, difference, open problem.

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When restricting $z = iy$ for $y \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ reduce to

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y \to \pm i\infty$$

and

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y \to +\infty,$$

as $y \to \pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

In the textbook [9, page 93], Exercise 6 states that, if $z \in \mathbb{C}$ and $|z| \le R$, then

$$|\sin z| \le \cosh R \quad \text{and} \quad |\cos z| \le \cosh R.$$

In [7], a criterion to justify a holomorphic function was discussed.

In [6], the author discussed and computed bounds of the sine and cosine functions $\sin z$ and $\cos z$ along straight lines on the complex plane $\mathbb{C}$. The main results in the paper [6] can be recited as follows.

(a) The complex functions $\sin z$ and $\cos z$ are bounded along straight lines parallel to the real x-axis on the complex plane $\mathbb{C}$:
  (i) along the horizontal straight line $y = \alpha$ on the complex plane $\mathbb{C}$

$$|\sinh \alpha| \le |\sin(x + i\alpha)| \le \cosh \alpha \tag{1.1}$$

  and

$$|\sinh \alpha| \le |\cos(x + i\alpha)| \le \cosh \alpha, \tag{1.2}$$

  where $\alpha \in \mathbb{R}$ is a constant and $x \in \mathbb{R}$;
  (ii) the equalities in the left hand side of (1.1) and in the right hand side of (1.2) hold if and only if $x = k\pi$ for $k \in \mathbb{Z}$;
  (iii) the equalities in the right hand side of (1.1) and in the left hand side of (1.2) hold if and only if $x = k\pi + \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

(b) The complex functions $\sin z$ and $\cos z$ are unbounded along straight lines whose slopes are not horizontal:
  (i) along the sloped straight line $y = \alpha + \beta x$ on the complex plane $\mathbb{C}$

$$|\sin z| \ge |\sinh(\alpha + \beta x)| \quad \text{and} \quad |\cos z| \ge |\sinh(\alpha + \beta x)|,$$

  where $\alpha \in \mathbb{R}$ and $\beta \ne 0$ are constants;
  (ii) along the vertical straight line $x = \gamma$ on the complex plane $\mathbb{C}$

$$|\sin z| \ge |\sinh y| \quad \text{and} \quad |\cos z| \ge |\sinh y|,$$

  where $\gamma \in \mathbb{R}$ is a constant.

In this paper, we present bounds for norms $|\sin(re^{i\theta})|$ and $|\cos(re^{i\theta})|$ of the sine and cosine functions $\sin z$ and $\cos z$ along a circle $C(0, r)$ centered at the origin $z = 0$ of radius $r > 0$ on the complex plane $\mathbb{C}$ in terms of two double inequalities.

2. A Double Inequality for the Norm of Sine Along a Circle

In this section, we present a double inequality for the norm $|\sin(re^{i\theta})|$ of the sine function $\sin z$ along a circle $C(0,r)$ centered at the origin $z=0$ of radius $r>0$ on the complex plane $\mathbb{C}$.

Theorem 2.1. *Let $r>0$ be a constant and let $C(0,r): z=re^{i\theta}$ for $\theta\in[0,2\pi)$ denote a circle centered at the origin $z=0$ of radius r. Then*

$$|\sin r| \le |\sin(re^{i\theta})| \le \sinh r, \quad \theta\in[0,2\pi). \tag{2.1}$$

The left equality is valid if and only if $\theta=0,\pi$, while the right equality is valid if and only if $\theta=\frac{\pi}{2},\frac{3\pi}{2}$.

Proof. The circle $C(0,r)$ can be represented by

$$z=re^{i\theta}, \quad \theta\in[0,2\pi).$$

It is not difficult to see that, for fixed $r>0$, $|\sin(re^{i\theta})|=|\sin r|$ for $\theta=0,\pi$, $|\sin(re^{i\theta})|=\sinh r$ for $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, and $|\sin(re^{i\theta})|$ has a least positive periodicity π with respect to the argument θ.

Straightforward computation yields

(2.2)

$$\begin{aligned}
\sin(re^{i\theta}) &= \sin(r\cos\theta+ir\sin\theta)\\
&= \frac{e^{i(r\cos\theta+ir\sin\theta)}-e^{-i(r\cos\theta+ir\sin\theta)}}{2i}\\
&= \frac{e^{-(r\sin\theta-ir\cos\theta)}-e^{r\sin\theta-ir\cos\theta}}{2i}\\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta)+i\sin(r\cos\theta)]-e^{r\sin\theta}[\cos(r\cos\theta)-i\sin(r\cos\theta)]}{2i}\\
&= \frac{(e^{-r\sin\theta}-e^{r\sin\theta})\cos(r\cos\theta)+i(e^{-r\sin\theta}+e^{r\sin\theta})\sin(r\cos\theta)]}{2i}\\
&= \cosh(r\sin\theta)\sin(r\cos\theta)+i\sinh(r\sin\theta)\cos(r\cos\theta)
\end{aligned}$$

and

$$|\sin(re^{i\theta})|=\sqrt{[\cosh(r\sin\theta)\sin(r\cos\theta)]^2+[\sinh(r\sin\theta)\cos(r\cos\theta)]^2}\,.$$

In Figure 1, we plot the 3D graph of $|\sin(re^{i\theta})|$ for $r\in[0,5]$ and $\theta\in[0,2\pi)$. In Figure 2, we plot the polarized 3D graph of the norm $|\sin(re^{i\theta})|$ for $r\in[0,4]$ and $\theta\in[0,2\pi)$. In Figure 3, we plot the graph of $|\sin(\pi e^{i\theta})|$ for $\theta\in[0,2\pi)$. These three figures are helpful for analyzing and understanding the behaviour of the sine function $\sin z$ along the circle $C(0,r)$ centered at the origin $z=0$ of radius r.

From Figure 3, we can see that the norm $|\sin(\pi e^{i\theta})|$ has only two maximums at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, while it has only two minimums at $\theta=0,\pi$ on the interval $[0,2\pi)$.

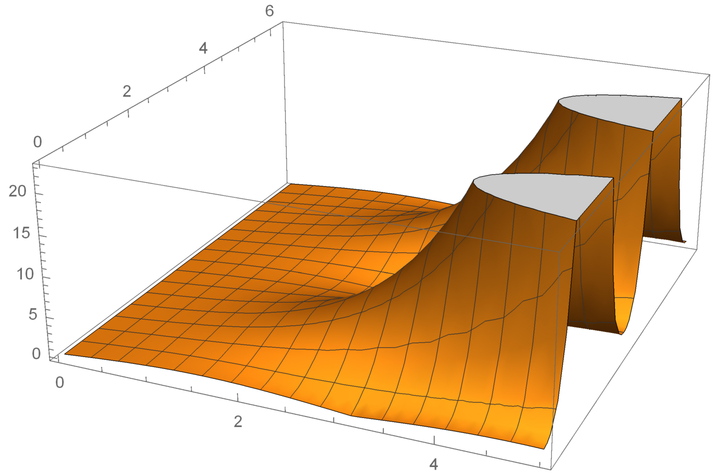

FIGURE 1. The 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$

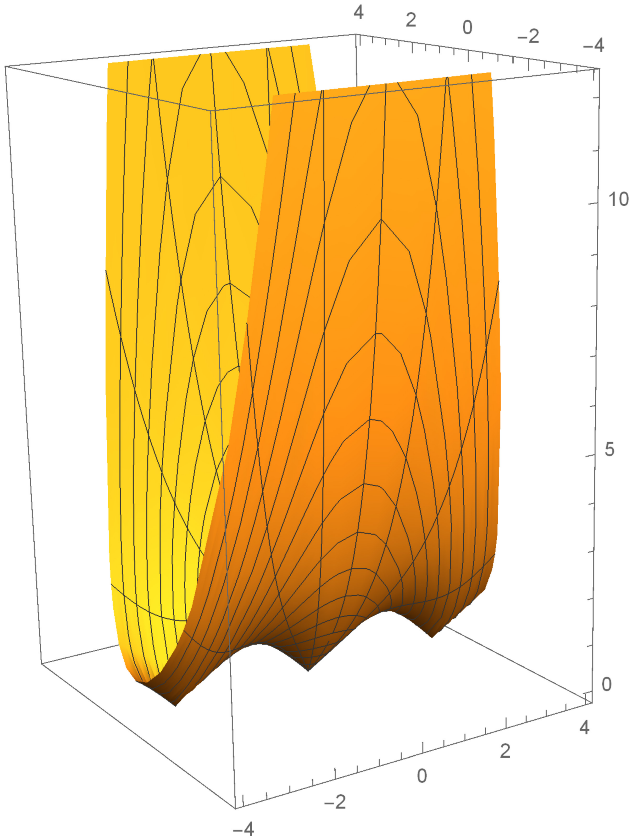

FIGURE 2. The polarized 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$

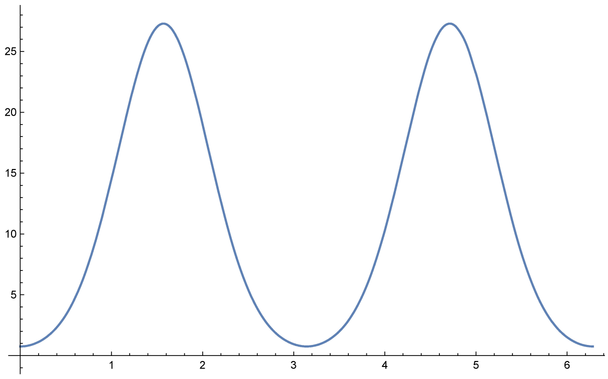

FIGURE 3. The graph of $|\sin(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$

Differentiating the square of $|\sin(re^{i\theta})|$ yields

$$\begin{aligned}\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta} &= r[\cos\theta\sinh(2r\sin\theta) - \sin\theta\sin(2r\cos\theta)]\\ &= r[\sinh(2r\sin\theta) - \tan\theta\sin(2r\cos\theta)]\cos\theta\\ &= r[\cot\theta\sinh(2r\sin\theta) - \sin(2r\cos\theta)]\sin\theta\\ &= r^2\left[\frac{\sinh(2r\sin\theta)}{2r\sin\theta} - \frac{\sin(2r\cos\theta)}{2r\cos\theta}\right]\sin(2\theta).\end{aligned}$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $(k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ for $k = 0, 1, 2, 3$, in order that $\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta} \ne 0$ for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$, it is sufficient to find

$$\frac{\sinh(2r\sin\theta)}{2r\sin\theta} > 1 \tag{2.3}$$

and

$$\frac{\sin(2r\cos\theta)}{2r\cos\theta} < 1, \tag{2.4}$$

for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$. Then, for fixed $r > 0$, the square $|\sin(re^{i\theta})|^2$ and the norm $|\sin(re^{i\theta})|$ have only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while they have only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. At $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, the values of $|\sin(re^{i\theta})|$ are both $\sinh r$; at $\theta = 0, \pi$, the values of $|\sin(re^{i\theta})|$ are both $|\sin r|$.

Considering the odevity of $\sinh t$ and $\sin t$, we see that two inequalities in (2.3) and (2.4) are equivalent to

$$\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} < 1, \tag{2.5}$$

for $t \in (0, \infty)$. The first inequality in (2.5) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 \tag{2.6}$$

in [2, page 270, 3.6.9]. When $t \in (0, \frac{\pi}{2})$, the second inequality in (2.5) follows from the right hand side of the Jordan inequality

$$\frac{\pi}{2} \le \frac{\sin t}{t} < 1, \quad 0 < |t| \le \frac{\pi}{2}, \tag{2.7}$$

in [2, Section 2.3] and the papers [1,3,4,8]. When $t > \frac{\pi}{2}$, the second inequality in (2.5) follows from $\sin t \le 1$ on $(0, \infty)$ and standard argument. The double inequality (2.1) is thus proved. The proof of Theorem 2.1 is complete. □

3. A Double Inequality for the Norm of Cosine Along a Circle

In this section, we present a double inequality for the norm $|\cos(re^{i\theta})|$ of the cosine function $\cos z$ along a circle $C(0, r)$ centered at the origin $z = 0$ of radius $r > 0$ on the complex plane $\mathbb{C}$.

Theorem 3.1. *Let $r > 0$ be a constant and let $C(0, r) : z = re^{i\theta}$ for $\theta \in [0, 2\pi)$ denote a circle centered at the origin $z = 0$ of radius r. Then*

$$|\cos r| \le |\cos(re^{i\theta})| \le \cosh r, \quad \theta \in [0, 2\pi). \tag{3.1}$$

The left equality is valid if and only if $\theta = 0, \pi$, while the right equality is valid if and only if $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Proof. It is easy to see that, for fixed $r > 0$, $|\cos(re^{i\theta})| = |\cos r|$ for $\theta = 0, \pi$, $|\cos(re^{i\theta})| = \cosh r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $|\cos(re^{i\theta})|$ has a least positive periodicity π with respect to the argument θ.

Direct calculation yields

(3.2)

$$\begin{aligned}
\cos(re^{i\theta}) &= \cos(r\cos\theta + ir\sin\theta) \\
&= \frac{e^{i(r\cos\theta + ir\sin\theta)} + e^{-i(r\cos\theta + ir\sin\theta)}}{2} \\
&= \frac{e^{-(r\sin\theta - ir\cos\theta)} + e^{r\sin\theta - ir\cos\theta}}{2} \\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta) + i\sin(r\cos\theta)] + e^{r\sin\theta}[\cos(r\cos\theta) - i\sin(r\cos\theta)]}{2} \\
&= \frac{(e^{-r\sin\theta} + e^{r\sin\theta})\cos(r\cos\theta) + i(e^{-r\sin\theta} - e^{r\sin\theta})\sin(r\cos\theta)]}{2} \\
&= \cosh(r\sin\theta)\cos(r\cos\theta) - i\sinh(r\sin\theta)\sin(r\cos\theta)
\end{aligned}$$

and

$$|\cos(re^{i\theta})| = \sqrt{[\cosh(r\sin\theta)\cos(r\cos\theta)]^2 + [\sinh(r\sin\theta)\sin(r\cos\theta)]^2}\,.$$

In Figure 4, we plot the 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$. In

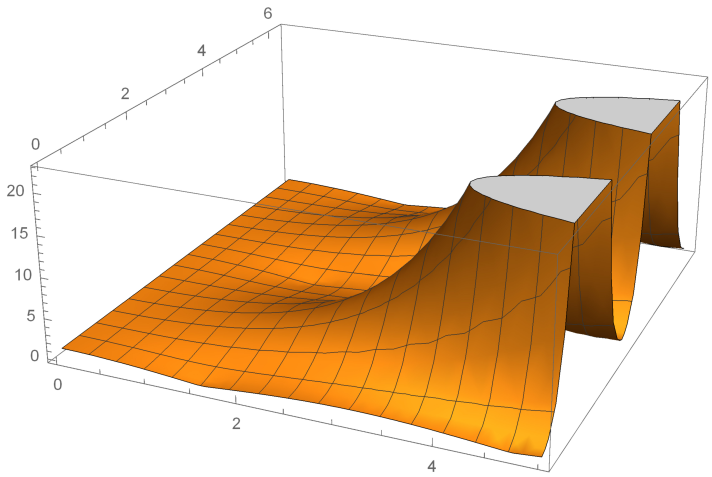

FIGURE 4. The 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$

Figure 5, we plot the polarized 3D graph of the norm $|\cos(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$. In Figure 6, we plot the graph of $|\cos(re^{i\theta})|$ for $r=\pi$ and $\theta \in [0,2\pi)$. These three figures are helpful for analyzing and understanding the behaviour of the cosine function $\cos z$ along the circle $C(0,r)$ centered at the origin $z=0$ of radius r.

From Figure 6, we can see that the norm $|\cos(\pi e^{i\theta})|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0,2\pi)$.

Differentiating the square of $|\cos(re^{i\theta})|$ with respect to θ gives

$$\begin{aligned}
\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta} &= r[\sin\theta\sin(2r\cos\theta) + \cos\theta\sinh(2r\sin\theta)]\\
&= r[\tan\theta\sin(2r\cos\theta) + \sinh(2r\sin\theta)]\cos\theta\\
&= r[\sin(2r\cos\theta) + \cot\theta\sinh(2r\sin\theta)]\sin\theta\\
&= r^2\left[\frac{\sin(2r\cos\theta)}{2r\cos\theta} + \frac{\sinh(2r\sin\theta)}{2r\sin\theta}\right]\sin(2\theta).
\end{aligned}$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $(k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ for $k = 0,1,2,3$, in order that $\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta} \ne 0$, it is sufficient to show

$$\frac{\sinh(2r\sin\theta)}{2r\sin\theta} > 1 \tag{3.3}$$

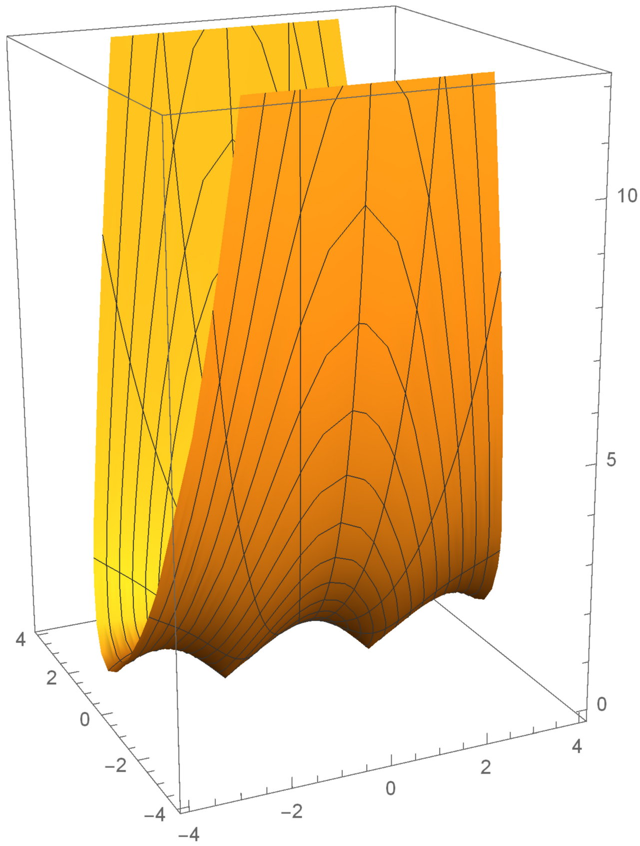

FIGURE 5. The polarized 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$

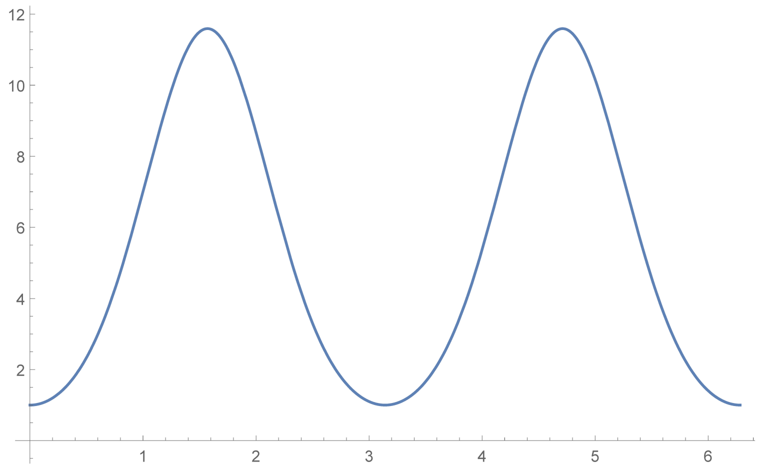

FIGURE 6. The graph of $|\cos(\pi e^{i\theta})|$ for $\theta \in [0,2\pi)$

and

$$\frac{\sin(2r\cos\theta)}{2r\cos\theta} > -1, \tag{3.4}$$

for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$. Then, for fixed $r > 0$, the square $|\cos(re^{i\theta})|^2$ and the norm $|\cos(re^{i\theta})|$ have only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while they have only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. At $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, the values of $|\cos(re^{i\theta})|$ are both $\cosh r$, at $\theta = 0, \pi$ the values of $|\cos(re^{i\theta})|$ are both $|\cos r|$.

Considering odevity of $\sinh t$ and $\sin t$, two inequalities in (3.3) and (3.4) are equivalent to

$$\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} > -1, \tag{3.5}$$

for $t \in (0, \infty)$. The first inequality in (3.5) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality (2.6). When $t \in (0, \frac{\pi}{2})$, the second inequality in (3.5) follows from the left hand side of the Jordan inequality (2.7). When $t > \frac{\pi}{2}$, the second inequality in (3.5) follows from $\sin t \geq -1$ on $(0, \infty)$ and simple argument. The double inequality (3.1) is thus proved. The proof of Theorem 3.1 is complete. □

4. Remarks

In this final section, we list several remarks on our main results in this paper.

Remark 4.1. Comparing Figure 1 and 4, it is not easy to see the difference between $|\sin(re^{i\theta})|$ and $|\cos(re^{i\theta})|$. However, the difference $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r \in [0, 2\pi]$ and $\theta \in [0, 2\pi)$ can be showed by Figure 7.

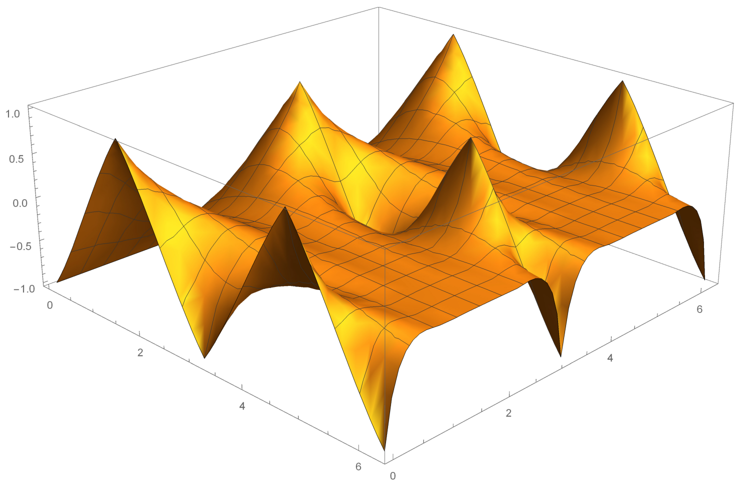

FIGURE 7. The 3D graph of $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r, \theta \in [0, 2\pi)$

Comparing Figure 2 and 5, it is not easy to find the difference between $|\sin(\pi e^{i\theta})|$ and $|\cos(\pi e^{i\theta})|$ yet. However, the difference $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$ can be presented by Figure 8.

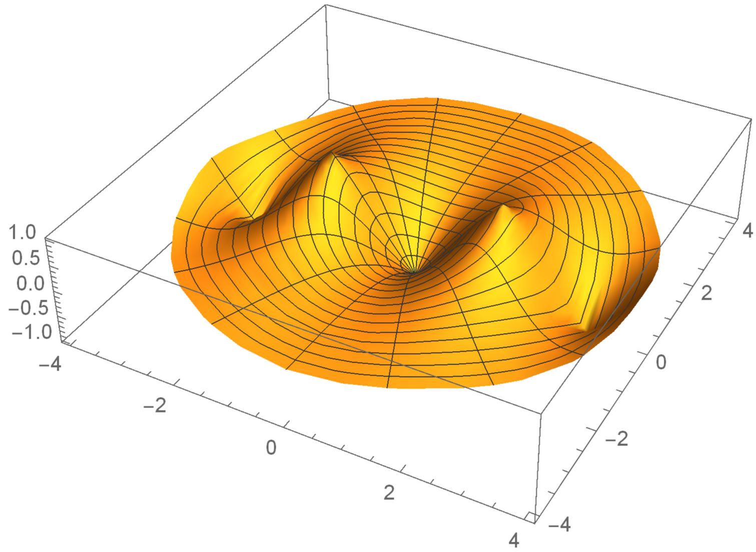

FIGURE 8. The polarized 3D graph of $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$

Comparing Figure 3 and 6, it is also not easy to see the difference between $|\sin(\pi e^{i\theta})|$ and $|\cos(\pi e^{i\theta})|$. However, the difference $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0,2\pi)$ can be demonstrated by Figure 9.

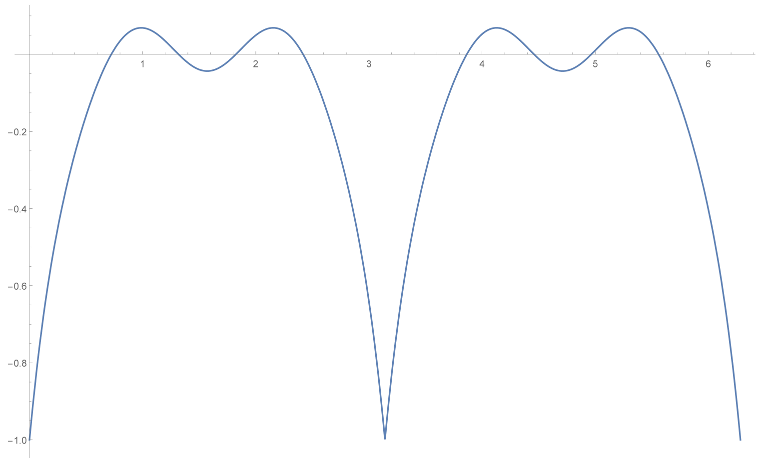

FIGURE 9. The graph of $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0,2\pi)$

Remark 4.2. From Figure 7, 8, and 9, we can guess that the double inequality

$$-1 \le |\sin(re^{i\theta})| - |\cos(re^{i\theta})| \le 1 \tag{4.1}$$

is seemingly valid for all $r > 0$ and $\theta \in [0,2\pi)$. Can one verify, deny, or strengthen this guess?

Remark 4.3. It is standard that

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2=|[\sin(re^{i\theta})-\cos(re^{i\theta})]^2|=|1-\sin(2re^{i\theta})|. \tag{4.2}$$

From (4.2), it follows that

$$|1-|\sin(2re^{i\theta})||\le|\sin(re^{i\theta})-\cos(re^{i\theta})|^2\le 1+|\sin(2re^{i\theta})|.$$

Further by virtue of the double inequality (2.1) in Theorem 2.1, we obtain

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2\le 1+|\sin(2re^{i\theta})|\le 1+\sinh(2r).$$

This means that

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|\le\sqrt{1+\sinh(2r)}, \tag{4.3}$$

for $r>0$ and $\theta\in[0,2\pi)$.

Motivated by the guess expressed in terms of the double inequality (4.1) and by the inequality (4.3), we pose an open problem: what are the nontrivial lower and upper bounds of the norm $|\sin(re^{i\theta})-\cos(re^{i\theta})|$ for $r>0$ and $\theta\in[0,2\pi)$?

Remark 4.4. From (2.2) and (3.2), it follows that

$$\begin{aligned}\sin(re^{i\theta})-\cos(re^{i\theta})&=\cosh(r\sin\theta)[\sin(r\cos\theta)-\cos(r\cos\theta)]\\&\quad+i[\cos(r\cos\theta)+\sin(r\cos\theta)]\sinh(r\sin\theta).\end{aligned}$$

Hence, we have

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|=\sqrt{\sinh^2(r\sin\theta)-\sin(2r\cos\theta)+\cosh^2(r\sin\theta)},$$

which is equivalent to

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2=\cosh(2r\sin\theta)-\sin(2r\cos\theta). \tag{4.4}$$

From (4.4), it follows that

$$\begin{aligned}\frac{\mathrm{d}\,|\sin(re^{i\theta})-\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta}&=2r[\sin\theta\cos(2r\cos\theta)+\cos\theta\sinh(2r\sin\theta)]\\&=2r[\cos(2r\cos\theta)+\cot\theta\sinh(2r\sin\theta)]\sin\theta\\&=2r[\tan\theta\cos(2r\cos\theta)+\sinh(2r\sin\theta)]\cos\theta\\&=2r^2\left[\frac{\cos(2r\cos\theta)}{2r\cos\theta}+\frac{\sinh(2r\sin\theta)}{2r\sin\theta}\right]\sin(2\theta),\end{aligned}$$

which is clearly equal to 0 at $\theta=0,\pi$ for all $r>0$. The function $\frac{\sinh t}{t}$ is even and not less than 1 on $(-\infty,\infty)$. The function $\frac{\cos t}{t}$ is odd on $(-\infty,\infty)$. By finding the set of all zeros of the function

$$\frac{\cos t}{t}+\frac{\sinh\sqrt{4r^2-t^2}}{\sqrt{4r^2-t^2}},\quad t\ne 0,\ r>0,$$

we can obtain sharp bounds of $|\sin(re^{i\theta})-\cos(re^{i\theta})|$ for $r>0$ and $\theta\in[0,2\pi)$. This is a hint, clue, sketch, or approach to solve the above open problem.

Remark 4.5. To the best of my knowledge, the double inequalities (2.1) and (3.1) in Theorems 2.1 and 3.1 are fundamental and new in the literature.

Remark 4.6. This paper is a revised version of the preprint [5].

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A STUDY OF MULTI-TERM TIME-FRACTIONAL DELAY DIFFERENTIAL SYSTEM WITH MONOTONIC CONDITIONS

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ABSTRACT. In this paper, the existence and uniqueness of mild solution for a class of multi-term time-fractional delay differential system have been discussed in ordered Banach space by enforcing monotone iterative technique. The generalized semigroup theory, fractional calculus and measure of noncompactness have been implemented to obtain the required results. A new set of sufficient conditions with the coefficients in the equations satisfying some monotonic properties has been obtained. Finally, an application is given to illustrate the obtained results.

1. INTRODUCTION

The fractional differential equations (in brief, FDEs) including Riemann-Liouville and Caputo fractional derivatives have been magnetizing the interest of many researchers, due to demonstrating applications in widespread areas of sciences and engineering such as mathematical modeling, thermal systems, acoustics, modeling of materials or rheology and mechanical systems. The FDEs have been viewed as a beneficial tool, which may describe dynamical behavior of real life phenomena more precisely. In addition, due to the memory and hereditary properties of various materials and processes, in many areas of science like identification systems, signal processing, robotics or control theory, fractional differential operators seem more appropriate in modeling than the classical integer operators. One can also find the various applications of FDEs in models of medicine (modeling of human tissue under

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mechanical loads), electrical engineering (transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) etc. For more knowledge regarding to fractional systems see the papers [2, 8, 9, 11, 12, 28, 32], the monographs [24, 31, 33] and references therein. In addition, fractional delay differential equations have been used frequently in various fields of science and engineering such as panorama of natural phenomena, modeling of equations and porous media etc. For more detail, see the cited papers [2, 3, 19].

It is very difficult to obtain the exact solutions for the nonlinear fractional differential systems in closed forms. To overcome this difficulty, many analytical and numerical techniques have been developed for instance, the Adomian decomposition method [21] and the homotopy analysis method [36], have been applied to investigate various systems of fractional or non-fractional ordered. However, in recent years, considerable work has been reported in the literature by applying monotone iterative technique, which is a flexible and very effective mechanism to study the existence results in a closed set governed by the lower and upper solutions, to investigate the existence of solutions for a class of fractional differential systems. In monotone iterative technique, we construct two monotone sequences by choosing upper and lower solutions as two initial iterations, which converge uniformly to a extremal mild solution of the system between the lower and upper solutions. Due to monotone behavior, the constructed sequences of iterations play an important role in the study of numerical solutions of various initial value and boundary value problems.

From the last few years, multi-term time-fractional differential equations have been generating great interest among the mathematicians and engineers. In [23, 28, 34], a two-term time-fractional differential equation has been studied in the abstract context, which include a concrete example of fractional diffusion-wave problem. In [13] and [29], the multi-term time-fractional diffusion wave equation have been considered with constant and variable coefficients, respectively. Moreover, in [22, 27], the analytical and numerical solutions of multi-term time-fractional diffusion equation have been discussed. In [32], Pardo and Lizama studied the existence of mild solutions of multi-term time-fractional differential equations with nonlocal initial conditions by using Caratheodory type conditions and measure of noncompactness technique. In last few years, many authors repeatedly apply the monotone iterative technique coupled with lower and upper solutions to various functional differential equations of integer order as well as fractional order, see [4–7, 25, 26, 35] and the references therein. However, in the best of authors' knowledge, no work is reported to the multi-term time-fractional differential system in the literature, by enforcing monotone iterative technique.

In this paper, monotone iterative technique coupled with method of lower and upper solutions has been applied to analyze the existence of mild solution for the following multi-term time-fractional delay differential system

$$(1.1)\quad \begin{cases} {}^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}y(t) = Ay(t) + F\left(t, y_t, \int_0^t h(t,s,y_s)ds\right), & t \in \mathfrak{I}, \\ y(t) = \phi(t) \in \mathfrak{B}, \quad t \in (-\infty, 0], \quad y'(0) = \chi, \end{cases}$$

where ${}^cD^\eta$ stands for the Caputo fractional derivative of order $\eta > 0$ and operational interval $\mathcal{J} = [0, T]$, $T < \infty$. $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator on a Banach space $(\mathbb{X}, \|\cdot\|)$. All γ_j, $j = 1, 2, \ldots, n$, $n \in \mathbb{N}$, are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. The nonlinear functions $F : \mathcal{J} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$ and $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ satisfies some suitable conditions, which will be mentioned later. $\Delta := \{(t, s) : 0 \leq s \leq t \leq T\}$. The delay function $y_t : (-\infty, 0] \to \mathbb{X}$ is characterized by $y_t(s) = y(t + s)$ for $s \in (-\infty, 0]$.

The system (1.1) is a general system, which includes recent investigations in this subject [13, 23, 28, 29, 32, 34]. Anticipating a great interest in the problems modeled as the system (1.1), this paper contributes in study of the existence results for mild solutions by applying monotone iterative technique coupled with the method of lower and upper solutions. It should be noticed that, the semigroup theory may not be directly used to solve problem (1.1). However, we construct a mild solution, which is based on the theory of resolvent families [32], which will provide an effective way to deal such problems.

This paper is organized as follows: In Section 2, some basics of fractional calculus and measure of noncompactness have been discussed which will be employed to obtain mains outcomes. In Section 3, the existence and uniqueness results are obtained for the mild solutions of the system (1.1). In Section 4, an example is provided to show the feasibility of the theory discussed in this paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers, respectively. Let us denote $\mathcal{D}(A)$, $\mathcal{R}(A)$ and $\rho(A)$ by the domain, range and resolvent of a linear operator A on $\mathbb{X}$, respectively. Define a partial ordering in $\mathbb{X}$ introduced by a positive cone $\mathbb{P} = \{y \in \mathbb{X} : y \geq \theta\}$ (where θ symbolizes the zero element of $\mathbb{X}$) such that $x \leq y$ if and only if $y - x \in \mathbb{P}$. If $x \leq y$ and $x \neq y$, then $x < y$. A cone $\mathbb{P}$ is called a normal cone if there exists a constant $N > 0$ (called normal constant) such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. A cone $\mathbb{P} \subset \mathbb{X}$ is said to be regular cone if every increasing, bounded above sequence is convergent, i.e., if $\{w_n\}$ be a sequence such that

$$w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq z,$$

for some $z \in \mathbb{X}$, then there is a $w \in \mathbb{X}$ such that $\|w_n - w\| \to 0$ as $n \to \infty$. Equivalently, a cone $\mathbb{P} \subset \mathbb{X}$ is said to be regular if every bounded below and decreasing sequence is convergent. It should be notice that a regular cone is a normal cone. For more details regarding to the cone $\mathbb{P}$, see [14]. The Banach space of all continuous $\mathbb{X}$-valued functions is represented by $\mathcal{C}(\mathcal{J}, \mathbb{X})$, on the interval $\mathcal{J}$ equipped with norm $\|u\|_{\mathcal{C}} = \sup_{t \in \mathcal{J}} \|u(t)\|$.

To facilitate the discussion, due to infinite delay an axiomatic definition of the phase space $\mathfrak{B}$ has been introduced by Hale and Kato [16]. Recall, the axioms of the phase space $\mathfrak{B}$, by following the terminology used by Hino et al. in [19] so, we omit the details here.

A linear space $\mathfrak{B}$ consists of all functions defined from $(-\infty, 0]$ into $\mathbb{X}$ equipped with the seminorm $\|\cdot\|_{\mathfrak{B}}$ satisfying the following axioms.

(a) If $y : (-\infty, T] \to \mathbb{X}$, $T > 0$ is continuous on $\mathfrak{I}$ and $y_0 \in \mathfrak{B}$, then for every $t \in \mathfrak{I}$ the accompanying conditions hold:
   (i) y_t is a $\mathfrak{B}$-valued continuous function;
   (ii) $\|y(t)\| \leq K\|y_t\|_{\mathfrak{B}}$;
   (iii) $\|y_t\|_{\mathfrak{B}} \leq K_1(t)\sup_{s\in[0,t]}\|y(s)\| + K_2(t)\|y_0\|_{\mathfrak{B}}$, where $K \geq 0$ is a constant and $K_1(\cdot) : [0,\infty) \to [0,\infty)$ is continuous, $K_2(\cdot) : [0,\infty) \to [0,\infty)$ is locally bounded and K_1, K_2 are independent of $y(\cdot)$.

(b) The space $\mathfrak{B}$ is complete.

Now, recall some definitions and basic results on fractional calculus. Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where Γ denotes gamma function. The function g_η has the properties $(g_a * g_b)(t) = g_{a+b}(t)$ for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and $\operatorname{Re}\lambda > 0$, where $\widehat{(\cdot)}$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively.

Definition 2.1. The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}$ $([0,\infty), \mathbb{X})$ of order $\eta > 0$ with lower limit zero is defined as follows

$$I^\eta f(t) = \int_0^t g_\eta(t-s)f(s)ds, \quad t > 0,$$

and $I^0 f(t) = f(t)$.

This fractional integral satisfies the properties $I^\eta \circ I^b = I^{\eta+b}$ for $b > 0$, $I^\eta f(t) = (g_\eta * f)(t)$ and $\widehat{I^\eta f}(\lambda) = \frac{1}{\lambda^\eta}\widehat{f}(\lambda)$ for $\operatorname{Re}\lambda > 0$.

Definition 2.2. Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f : [0,\infty) \to \mathbb{X}$ with lower limit zero is given by

$${}^cD^\eta f(t) = I^{m-\eta}D^m f(t) = \int_0^t g_{m-\eta}(t-s)D^m f(s)ds,$$

and ${}^cD^0 f(t) = f(t)$, where $D^m = \frac{d^m}{dt^m}$. In addition, we have ${}^cD^\eta f(t) = (g_{m-\eta} * D^m f)(t)$ and the Laplace transformation of Caputo fractional derivative is given by

$$\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda) - \sum_{d=0}^{m-1} f^{(d)}(0)\lambda^{\eta-1-d}, \quad \lambda > 0. \tag{2.1}$$

Remark 2.1. Let $m - 1 < \eta \leq m,\ m \in \mathbb{N}$, then

$$(I^\eta \circ {}^cD^\eta)f(t) = f(t) - \sum_{d=0}^{m-1} f^{(d)}(0)g_{d+1}(t), \quad t > 0. \tag{2.2}$$

If $f^{(d)}(0) = 0$, for $d = 1, 2, 3, \ldots, m-1$, then $(I^\eta \circ {}^cD^\eta)f(t) = f(t)$ and $\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda)$.

To give a appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define following family of operators.

Definition 2.3 ([32]). Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and let $\beta > 0$, γ_j, α_j be the real positive numbers. Then A is called the generator of a $(\beta, \gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : [0, \infty) \to \mathcal{L}(\mathbb{X})$ (the space of bounded linear operators on $\mathbb{X}$) such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^\beta \left(\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t} \mathcal{S}_{\beta,\gamma_j}(t) y dt, \quad \operatorname{Re}\lambda > \omega,\ y \in \mathbb{X}. \tag{2.3}$$

The following result guarantees the existence of (β, γ_j)-resolvent family under some suitable conditions.

Theorem 2.1 ([32]). *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a strongly continuous and bounded cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates a bounded (β, γ_j)-resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.*

Let Ω be the set defined by

$$\Omega = \{y \in \mathcal{C}((-\infty, T], \mathbb{X}) : \text{ such that } y_{|_{(-\infty,0]}} \in \mathfrak{B} \text{ and } y_{|_{[0,T]}} \in \mathbb{X}\}.$$

In order to define the mild solution for the system (1.1), we associate system (1.1) with an integral equation, by comparison with the fractional differential system given in [32]. Consider the following definition of mild solution for the system (1.1).

Definition 2.4. Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a bounded (β, γ_j)-resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$. Then a function $y \in \Omega$ is called the mild solution of the system (1.1) if $y'(0) = \chi$ and satisfies the equation

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0) ds & \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, y_s, \int_0^s h(s,\tau,y_\tau)d\tau\right) ds, & t \in \mathcal{J}, \end{cases} \tag{2.4}$$

where $\mathcal{T}_{\beta,\gamma_j}(t) = (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t)$.

Definition 2.5. The resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is said to be positive on $\mathbb{X}$, if the order inequality $\mathcal{S}_{\beta,\gamma_j}(t)y \geq \theta$ holds for all $y \geq \theta$, $y \in \mathbb{X}$ and $t \geq 0$.

Lemma 2.1 ([17]). (Generalized Gronwall inequality). *Assume $\gamma \geq 0$, $\delta > 0$ and $c(t)$ is a nonnegative and locally integrable function on $0 \leq t < T < +\infty$ and let $z(t)$ be nonnegative and locally integrable on $0 \leq t < T + \infty$ such that*

$$z(t) \leq c(t) + \gamma \int_0^t (t-s)^{\delta-1} z(s) ds,$$

then

$$z(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\gamma\Gamma(\delta))^n}{\Gamma(n\delta)} (t-s)^{n\delta-1} c(s) \right] ds, \quad 0 \leq t < T.$$

Let $\mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X}) = \{y \in \mathcal{C}((-\infty,T],\mathbb{X}) : {}^cD^{1+\beta}y(t)$ exists and continuous on $\mathcal{J}$ and $y(t) \in \mathcal{D}(A)$ for all $t \geq 0\}$. An abstract function $y(t) \in \mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ is said to be a solution of (1.1) of if $y(t)$ satisfies the system (1.1).

Definition 2.6. The function $y^{(0)} \in \mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ is said to be a lower solution of the system (1.1), if it satisfies the following inequalities

$$\begin{cases} {}^cD^{1+\beta}y^{(0)}(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y^{(0)}(t) \leq Ay^{(0)}(t) & \\ +F\left(t, y_t^{(0)}, \int_0^t h(t,s,y_s^{(0)})ds\right), & t \in \mathcal{J}, \\ y^{(0)}(t) \leq \phi(t) \in \mathfrak{B}, & t \in (-\infty,0],\ y'^{(0)}(0) \leq \chi. \end{cases} \tag{2.5}$$

If all the inequalities of (2.5) are reversed, then solution is called upper solution denoted by $z^{(0)}$.

Now, we recall some basic definitions and properties of Kuratowski measure of noncompactness. For more details, we refer to the monograph [14] and paper [10, 18].

Definition 2.7. Let $\mathbb{F}$ be a bounded subset of a Banach space $\mathbb{X}$. The Kuratowski measure of noncompactness denoted by $\mu(\cdot)$ of $\mathbb{F}$ is defined by

$$\mu(\mathbb{F}) := \inf\{\delta > 0 : \mathbb{F} = \cup_{i=1}^{n} \mathbb{F}_i \text{ with } \operatorname{diam}(\mathbb{F}_i) \leq \delta \text{ for } i = 1,2,3,\ldots,n\}.$$

Lemma 2.2. *Let $\mathbb{X}$ be a Banach space, and let $\mathbb{F} \subset \mathcal{C}([a_1,a_2],\mathbb{X})$ be bounded and equicontinuous. Then $\mu(\mathbb{F}(t))$ is continuous on $[a_1,a_2]$ and*

$$\mu_{\mathcal{C}}(\mathbb{F}) = \sup_{t \in [a_1,a_2]} \mu(\mathbb{F}(t)).$$

Lemma 2.3. *Let $\{y_n\}_{n=1}^{\infty} \subset L^1(\mathcal{J},\mathbb{X})$ be a sequence and there exists $g \in L^1(\mathcal{J},\mathbb{X})$ such that $\|y_n(t)\| \leq g(t)$, a.e. $t \in \mathcal{J}$, then $\mu(\{y_n(t)\}_{n=1}^{\infty})$ is integrable and*

$$\mu\left(\left\{\int_0^t y_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \mu(\{y_n(s)\}_{n=1}^{\infty}ds.$$

Lemma 2.4. *If $\mathbb{F}$ is bounded subset of $\mathbb{X}$, then there exists $\{y_n\}_{n=1}^{\infty} \subset \mathbb{F}$, such that $\mu(\mathbb{F}) \leq 2\mu(\{y_n\}_{n=1}^{\infty})$.*

3. Main Results

Throughout in this section, we denote $S_0 = \sup_{t\in[0,T]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|$. We consider the following assumptions.

(A1) The functions $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ and $F : \mathcal{J} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$, satisfy Carathéodory type conditions, i.e.,

(i) $h(t,s,\cdot) : \mathfrak{B} \to \mathbb{X}$ is continuous for $(t,s) \in \Delta$ and $h(\cdot,\cdot,v) : \Delta \to \mathbb{X}$ is strongly measurable for all $v \in \mathfrak{B}$;

(ii) $F(t,\cdot,\cdot) : \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$ is continuous for each $t \in \mathcal{J}$ and $F(\cdot,u,v) : \mathcal{J} \to \mathbb{X}$ is strongly measurable for all $(u,v) \in \mathfrak{B} \times \mathbb{X}$.

(A2) For lower and upper solutions $y^{(0)}, z^{(0)} \in \mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ of the system (1.1) such that $y^{(0)} \le z^{(0)}$ the following conditions hold:

(i) $F(t,v_1,w_1) \le F(t,v_2,w_2)$ for all $t \in \mathcal{J}$, and $v_1,v_2 \in \mathfrak{B}$ satisfying $y_t^{(0)} \le v_1 \le v_2 \le z_t^{(0)}$ and $w_1,w_2 \in \mathbb{X}$ such that $\int_0^t h(t,s,y_s^{(0)})ds \le w_1 \le w_2 \le \int_0^t h(t,s,z_s^{(0)})ds$;

(ii) $h(t,s,v_1) \le h(t,s,v_2)$ for all $(t,s) \in \Delta$ and $v_1,v_2 \in \mathfrak{B}$ such that $y_t^{(0)} \le v_1 \le v_2 \le z_t^{(0)}$.

(A3) The functions F,h satisfy the followings conditions.

(i) For $G \subset \mathfrak{B}$ and $H \subset \mathbb{X}$, where $G(r) = \{\varphi(r) : r \in (-\infty,0], \varphi \in G\}$ there exists a constant $L > 0$ such that

$$\mu(F(t,G,H)) \le L\left[\sup_{-\infty<r\le 0}\mu(G(r)) + \mu(H)\right], \quad \text{a.e. } t \in \mathcal{J}.$$

(ii) For each bounded set $G \subset \mathfrak{B}$, there exists an integrable function $\xi : \Delta \to [0,\infty)$ such that

$$\mu(h(t,s,G)) \le \xi(t,s) \sup_{-\infty<r\le 0}\mu(G(r)),$$

for a.e. $(t,s) \in \Delta$. For convenience, we denote $\xi^* = \max \int_0^t \xi(t,s)ds$.

In order to give operator theoretical approach, we define a operator $Q : \Omega \to \Omega$ by

$$(3.1)\quad (Qy)(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds & \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s,y_s,\int_0^s h(s,\tau,y_\tau)d\tau\right)ds, & t \in \mathcal{J}. \end{cases}$$

It is clear to see that Q is well defined.

Let us define a function $u(\cdot) : (-\infty,T] \to \mathbb{X}$ by

$$u(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ 0, & t \in \mathcal{J}. \end{cases}$$

For a function $v : (-\infty,T] \to \mathbb{X}$ such that $v(0) = 0$, we define the function $\overline{v}$ by

$$\overline{v}(t) = \begin{cases} 0, & t \in (-\infty,0], \\ v(t), & t \in \mathcal{J}. \end{cases}$$

If $y(\cdot)$ is a solution of (2.4), then it can be decompose $y(\cdot)$ as $y(t) = u(t) + \overline{v}(t)$, $t \in (-\infty,T]$ and $v(\cdot)$ satisfies

$$\begin{aligned} v(t) =& \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau)d\tau\right)ds. \end{aligned}$$

Define $\mathbb{X}_0 = \{v \in \Omega : v_0 = 0\}$. For any $v \in \mathbb{X}_0$,

$$\|v\|_{\mathbb{X}_0} = \sup_{t\in\mathcal{J}} \|v(t)\| + \|v_0\|_{\mathfrak{B}} = \sup_{t\in\mathcal{J}} \|v(t)\|.$$

Clearly, $\mathbb{X}_0$ is a Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}_0}$. We assume that $(\mathbb{X}_0, \|\cdot\|_{\mathbb{X}_0})$ stands for a ordered Banach space with partial order $\leq$ induced by a positive normal cone $\mathbb{P}_0 = \{v \in \mathbb{X}_0 : v(t) \geq \theta\}$ with the normal constant N_0. Evidently $\mathcal{C}((-\infty, T], \mathbb{X}_0)$ is also an ordered Banach space with the partial order $\leq$ reduced by a positive normal cone $\mathbb{P}_0 = \{v \in \mathbb{X}_0 : v(t) \geq \theta, t \in (-\infty, T]\}$ with normal constant N_0. For $v, w \in \mathcal{C}((-\infty, T], \mathbb{X}_0)$ such that $v \leq w$, $[v, w]$ denotes a ordered interval $\{x \in \mathcal{C}((-\infty, T], \mathbb{X}_0) : v \leq x \leq w\}$ in $\mathcal{C}((-\infty, T], \mathbb{X}_0)$ and $[v(t), w(t)]$ denotes the ordered interval $\{x \in \mathcal{C}((-\infty, T], \mathbb{X}_0) : v(t) \leq x(t) \leq w(t)\}$ in $\mathbb{X}_0$.

Theorem 3.1. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive normal cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is a positive operator and the assumptions* (A1)-(A3) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Let $D = [v^{(0)}, w^{(0)}] = \{u \in \mathcal{C}(\mathcal{J}, \mathbb{X}_0) : v^{(0)} \leq u \leq w^{(0)}\}$. Define a map $\tilde{Q} : D \to \mathbb{X}_0$ by

(3.2)
$$(\tilde{Q}v)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds & \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, u_s + \overline{v}_s, \int_0^s h(s, \tau, u_\tau + \overline{v}_\tau)d\tau\right) ds, & t \in \mathcal{J}. \end{cases}$$

From (A1)-(A2) for any $v \in D$ we have

$$\begin{aligned} F\left(t, u_t + v_t^{(0)}, \int_0^t h(t, \tau, u_\tau + v_\tau^{(0)})d\tau\right) \leq & F\left(t, u_t + \overline{v}_t, \int_0^t h(t, \tau, u_\tau + \overline{v}_\tau)d\tau\right) \\ \leq & F\left(t, u_t + w_t^{(0)}, \int_0^t h(t, \tau, u_\tau + w_\tau^{(0)})d\tau\right). \end{aligned}$$

Now, using normality of the cone $\mathbb{P}_0$, there exists a constant $\mathfrak{K} > 0$ such that

$$\left\| f\left(t, u_t + \overline{v}_t, \int_0^t g(t, \tau, u_\tau + \overline{v}_\tau)d\tau\right) \right\| \leq \mathfrak{K}, \quad v \in D.$$

For convenience, we divide the proof in the following steps.

Step 1. The map $\tilde{Q}$ is continuous map on D.

Let $\{v^{(n)}\}$ be a sequence in D such that $\{v^{(n)}\} \to v \in D$ as $n \to \infty$. For $t \in (-\infty, 0]$ we get

$$\|\tilde{Q}v^{(n)}(t) - \tilde{Q}v(t)\| = 0.$$

Also, from (A1) for $t \in \mathcal{J}$ and as $n \to \infty$, we have

(i) $\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau \to \int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau$;
(ii) $F\left(s,u_s+\overline{v}_s^{(n)},\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau\right) \to F\left(s,u_s+\overline{v}_s,\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau\right)$.

Now, by applying Lebesgue Dominated Convergence Theorem for $t\in\mathcal{J}$, we have

$$\begin{aligned}\|\tilde{Q}v^{(n)}(t)-\tilde{Q}v(t)\| \leq &\int_0^t \left\|\mathcal{T}_{\beta,\gamma_j}(t-s)\right\| \Bigg\| F\left(s,u_s+\overline{v}_s^{(n)},\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau\right)\\ &\qquad - F\left(s,u_s+\overline{v}_s,\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau\right)\Bigg\| ds\\ \to&0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus map $\tilde{Q}$ is continuous on D.

Step 2. $\tilde{Q}$ is a increasing monotonic operator.

Consider x, $y\in D$ with $x\leq y$ then $x(t)\leq y(t)$ for $t\in\mathcal{J}$. Therefore, x_t, y_t belong to the ordered Banach space $\mathbb{X}_0$ such that $x_t\leq y_t$ for $t\in\mathcal{J}$. Using (A2) and positivity of $\mathcal{S}_{\beta,\gamma_j}(t)$, we obtain

$$\tilde{Q}x\leq\tilde{Q}y. \tag{3.3}$$

Now, we show that $v^{(0)}\leq\tilde{Q}v^{(0)}$ and $\tilde{Q}w^{(0)}\leq w^{(0)}$. For this, let

$$g(t)={}^cD^{1+\beta}v^{(0)}(t)+\sum_{j=1}^n\alpha_j{}^cD^{\gamma_j}v^{(0)}(t)-Av^{(0)}(t)$$

subject to the conditions $v^{(0)}(0)=y_0$, $v'^{(0)}(0)=y_1$.

Then by definition of lower solution, we obtain $g(t)\leq F\left(t,y_t,\int_0^t h(t,s,y_s)ds\right)$ for $t\in\mathcal{J}$. Since $v^{(0)}(t)$ is a lower solution of (1.1), we get

$$\begin{aligned}v^{(0)}(t)=&\mathcal{S}_{\beta,\gamma_j}(t)y_0+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)y_1+\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)y_0ds\\ &+\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)g(s)ds\\ \leq&\mathcal{S}_{\beta,\gamma_j}(t)\phi(0)+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)\chi+\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds\\ &+\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s,u_s+v_s^{(0)},\int_0^s h(s,\tau,u_\tau+v_\tau^{(0)})d\tau\right)ds\\ \leq&\tilde{Q}v^{(0)}(t),\quad t\in\mathcal{J},\end{aligned}$$

and also $v^{(0)}(t)\leq\phi(t)$, $v'^{(0)}(0)\leq\chi$. Therefore, $v^{(0)}(t)\leq\tilde{Q}v^{(0)}(t)$ for all $t\in(-\infty,T]$. Similarly, we can show that $w^{(0)}(t)\geq\tilde{Q}w^{(0)}(t)$ for all $t\in(-\infty,T]$. Thus, $\tilde{Q}$ is a increasing monotonic operator.

Step 3. $\tilde{Q}$ is an equicontinuous operator.

For any $v\in D$ and t_1, $t_2\in(-\infty,0]$ such that $t_1<t_2$, we have

$$\|\tilde{Q}v(t_2)-\tilde{Q}v(t_1)\|=0.$$

Further for $v \in D$ and $t_1, t_2 \in \mathfrak{J}$ such that $t_1 < t_2$, we have

$$\begin{aligned}
\|\tilde{Q}v(t_2) - \tilde{Q}v(t_1)\| \leq & \|\mathcal{S}_{\beta,\gamma_j}(t_2)\phi(0) - \mathcal{S}_{\beta,\gamma_j}(t_1)\phi(0)\| \\
& + \|(g_1 * \mathcal{S}_{\beta,\gamma_j})(t_2) - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t_1)\|\|\chi\| \\
& + \sum_{j=1}^{n} \alpha_j S_0 \bigg\| \int_0^{t_2} \frac{(t_2 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \\
& - \int_0^{t_1} \frac{(t_1 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \bigg\| \|\phi(0)\| \\
& + \int_0^{t_1} \|\mathcal{T}_{\beta,\gamma_j}(t_2 - s) - \mathcal{T}_{\beta,\gamma_j}(t_1 - s)\| \\
& \times \left\| F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau) d\tau\right) \right\| ds \\
& + \int_{t_1}^{t_2} \|\mathcal{T}_{\beta,\gamma_j}(t_2 - s)\| \left\| F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau) d\tau\right) \right\| ds. \\
= & \sum_{i=1}^{5} J_i.
\end{aligned}$$

We have

$$\begin{aligned}
J_2 &= \|(g_1 * \mathcal{S}_{\beta,\gamma_j}(t_2) - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t_1)\|\|\chi\| \\
&= \left\| \int_0^{t_2} g_1(t_2 - \tau)\mathcal{S}_{\beta,\gamma_j}(\tau) d\tau - \int_0^{t_1} g_1(t_1 - \tau)\mathcal{S}_{\beta,\gamma_j}(\tau) d\tau \right\| \|\chi\| \\
&\leq \int_{t_1}^{t_2} \|\mathcal{S}_{\beta,\gamma_j}(\tau)\| d\tau \|\chi\| \\
&\leq S_0 \|\chi\| (t_2 - t_1) \\
&\to 0 \quad \text{as} \quad t_1 \to t_2
\end{aligned}$$

and

$$\begin{aligned}
J_3 \leq & \sum_{j=1}^{n} \alpha_j S_0 \left\| \int_0^{t_2} \frac{(t_2 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds - \int_0^{t_1} \frac{(t_1 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \right\| \|\phi(0)\| \\
\leq & \sum_{j=1}^{n} \alpha_j S_0 \left| \frac{t_2^{1+\beta-\gamma_j} - t_1^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \right| \|\phi(0)\| \\
\to & 0 \quad \text{as} \quad t_1 \to t_2.
\end{aligned}$$

From the expressions J_2 and J_3, we can easily deduce that $J_4 \to 0$ and $J_5 \to 0$ as $t_1 \to t_2$ independently of $u \in D$. Therefore, $\|\tilde{Q}v(t_2) - \tilde{Q}v(t_1)\| \to 0$ as $t_1 \to t_2$ independently of $u \in D$. Hence, $Q(D)$ is equicontinuous on $\mathfrak{J}$.

Step 4. Now, we will show $\mu(\{\tilde{Q}v^{(n)}\}_{n=1}^{\infty}) = 0$.

Define the sequences

$$v^{(n)} = \tilde{Q}v^{(n-1)}, \quad w^{(n)} = Qw^{(n-1)}, \quad n = 1, 2, \ldots \tag{3.4}$$

It follows from monotonicity of $\tilde{Q}$ that

$$v^{(0)} \leq v^{(1)} \leq \cdots \leq v^{(n)} \leq \cdots \leq w^{(n)} \leq \cdots \leq w^{(1)} \leq w^{(0)}. \tag{3.5}$$

Next, we will show that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ convergent uniformly in $\mathcal{J}$.

We set $\mathbb{B} = \{v^{(n)} : n \in \mathbb{N}\}$ and $\mathbb{B}_0 = \{v^{(n-1)} : n \in \mathbb{N}\}$. Using normality of cone $\mathbb{P}_0$, we obtain that $\mathbb{B}$ and $\mathbb{B}_0$ are bounded. Since $\mathbb{B}_0 = \mathbb{B} \cup \{v^{(0)}\}$, it follows that $\mu(\mathbb{B}_0(t)) = \mu(\mathbb{B}(t))$ for $t \in (-\infty, T]$. Let

$$\varphi(t) := \mu(\mathbb{B}_0(t)) = \mu(\mathbb{B}(t)), \quad t \in (-\infty, T].$$

Since $\mathbb{B} = \tilde{Q}(\mathbb{B}_0)$, we have

$$\mu(\mathbb{B}(t)) = \mu(\tilde{Q}(\mathbb{B}_0)(t)).$$

For $t \in (-\infty, 0]$, $\varphi(t) := \mu(\tilde{Q}(\mathbb{B}_0)(t)) = 0$. For $t \in \mathcal{J}$, we have

$$\begin{aligned}
\varphi(t) =& \mu(\tilde{Q}(\mathbb{B}_0)(t) \\
\leq & 2\mu(\tilde{Q}\{v^{(n-1)}(t)\}) \\
\leq & 2\mu\Bigg[\mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds \\
& + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\Bigg(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\Bigg)ds\Bigg] \\
\leq & 2\mu\Bigg[\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\Bigg(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\Bigg)ds\Bigg] \\
\leq & \frac{4S_0}{\Gamma(1+\beta)}\Bigg[\int_0^t (t-s)^\beta \mu\Bigg\{F\Bigg(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\Bigg)\Bigg\}ds\Bigg] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}\Bigg[\int_0^t (t-s)^\beta \Bigg\{\sup_{-\infty<r\leq 0} \mu(\overline{v}^{n-1}(s+r)) \\
& + \mu\Bigg(\int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\Bigg)\Bigg\}ds\Bigg] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}\Bigg[\int_0^t (t-s)^\beta \Bigg\{\sup_{0\leq z\leq s} \mu(\overline{v}^{n-1}(z)) \\
& + 2\int_0^s \xi(s,\tau) \sup_{-\infty<r\leq 0} \mu(\overline{v}^{n-1}(\tau+r))d\tau\Bigg)\Bigg\}ds\Bigg] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}(1+2\xi^*) \int_0^t (t-s)^\beta \sup_{0\leq z\leq s} \mu(\overline{v}^{n-1}(z))ds \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}(1+2\xi^*) \int_0^t (t-s)^\beta \varphi(s)ds.
\end{aligned}$$

Now, by the Gronwall's inequality, $\varphi(t) \equiv 0$ on $\mathcal{J}$. So $\mu\{v^{(n)} : n \in \mathbb{N}\} = 0$. This implies that the set $\{v^{(n)} : n \in \mathbb{N}\}$ is relatively compact in D. So, we conclude that the sequence $\{v^{(n)}\}$ admits a convergent subsequence in D. Further by (3.5), we observe that $\{v^{(n)}\}$ itself is convergent sequence in $\mathbb{X}$. So, there exists $v^* \in \mathbb{X}$ satisfying $v^{(n)} \to v^*$ as $n \to \infty$. By (3.2) and (3.4), we have

(3.6)
$$v^{(n)}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) & \\ \times F\left(s, u_s + \overline{v}_s^{(n-1)}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{(n-1)})d\tau\right) ds, & t \in \mathcal{J}. \end{cases}$$

As $n \to \infty$, then applying Lebesgue Dominated Convergence Theorem, we have

(3.7)
$$v^*(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) & \\ \times F\Bigg(s, u_s + \overline{v}_s^*, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^*)d\tau\Bigg) ds, & t \in \mathcal{J}. \end{cases}$$

Then $v^* \in \mathcal{C}(\mathcal{J}, \mathbb{X})$ and $v^* = \tilde{Q}v^*$. Thus v^* is a fixed point of $\tilde{Q}$ and hence v^* will be the solution of (3.2). Similarly, there exists $w^* \in \mathcal{C}(\mathcal{J}, \mathbb{X})$ in such a way $w^{(n)} \to w^*$ as $n \to \infty$ and $w^* = \tilde{Q}w^*$. If $v \in D$ be a fixed point of $\tilde{Q}$ then by (3.3), we get $v^{(1)} \le Qv^{(0)} \le Qv = v \le Qw^{(0)} \le Qw^{(1)}$. Now, by induction principle $v^{(n)} \le v \le w^{(n)}$. In view of (3.5) and as $n \to \infty$, we obtain $v^{(0)} \le v^* \le v \le w^* \le w^{(0)}$. Hence, w^* and v^* are the maximal and minimal mild solutions of the system (1.1) in D, respectively. □

Corollary 3.1. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive regular cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \le w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$ is a positive operator and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. By regularity of the cone $\mathbb{P}_0$, we have that any ordered-bounded and ordered-monotonic sequence in $\mathbb{X}_0$ is convergent. Let $\{y^n\}$ be an increasing or decreasing sequence in D. Then using assumption $(A2)$, $F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)$ is ordered-bounded and ordered-monotonic sequence in $\mathbb{X}_0$ and hence $\{F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)\}$ is convergent. Therefore, $\mu(\{F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)\}) = 0$. Hence, assumption (A3) holds. Now, by Theorem 3.1, we conclude the assertion. □

Corollary 3.2. *Let $\mathbb{X}_0$ be a weakly sequentially complete ordered Banach space with a positive normal cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \le w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$ is a positive operator and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Since, in a weakly sequentially complete and ordered Banach space, the normal cone $\mathbb{P}_0$ is regular. Therefore using Corollary 3.1, we can conclude the assertion. □

Corollary 3.3. *We assume that $\mathbb{X}_0$ is a reflexive and ordered Banach space space with positive normal cone $\mathbb{P}_0$. Also, consider that the system* (1.1) *admits lower and upper solutions $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is positive and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Since, in a reflexive and ordered Banach space, the normal cone $\mathbb{P}_0$ is regular. Now, by Corollary 3.1, we conclude the assertion. ☐

Next, we will show the uniqueness of the mild solution for the system (1.1). For this we consider the following assumption.

(A4) The functions $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ and $F : \mathcal{J} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$ are such that
 (i) h is continuous and there exists an integrable function $\psi : \Delta \to [0, T]$ such that

$$h(t, s, u_2) - h(t, s, u_1) \leq \psi(t, s)[u_2(r) - u_1(r)],$$

 for any $(t, s) \in \Delta$ and $v_t^{(0)} \leq u_1 \leq u_2 \leq w_t^{(0)}$, $r \in (-\infty, 0]$;
 (ii) F is continuous and there exists $\kappa \geq 0$ such that

$$F(t, u_2, v_2) - F(t, u_1, v_1) \leq \kappa[(u_2(r) - u_1(r)) + (v_2 - v_1)], \quad r \in (-\infty, 0],$$

 for any $t \in \mathcal{J}$, $u_1, u_2 \in \mathfrak{B}$ with $v_t^{(0)} \leq u_1 \leq u_2 \leq w_t^{(0)}$ and $v_1, v_2 \in \mathbb{X}$ with $\int_0^t g(t, s, v_s^{(0)})ds \leq v_1 \leq v_2 \leq \int_0^t g(t, s, w_s^{(0)})ds$.

Theorem 3.2. *Let $\mathbb{X}_0$ be an ordered Banach space with normal positive cone $\mathbb{P}_0$ with normal constant N_0. Assume that $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is positive, the system* (1.1) *has upper and lower solutions $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$ and assumptions* (A2) *and* (A4) *hold. Then the system* (1.1) *has a unique mild solution in $[v^{(0)}, w^{(0)}]$.*

Theorem 3.3. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive normal cone $\mathbb{P}_0$ with normal constant N_0. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is a positive operator and the assumptions* (A2)-(A4) *are satisfied. Then the system* (1.1) *admits a unique mild solution in $[v^{(0)}, w^{(0)}]$.*

Proof. Let $\{x_n\} \in [v_t^{(0)}, w_t^{(0)}]$ and $\{y_n\} \in [v^{(0)}, w^{(0)}]$ be two monotonic increasing sequences. For $m, n = 1, 2, \ldots$, with $m > n$, for some $r_1, r_2 \in (-\infty, 0]$ using (A4), we have

$$\theta \leq h(t, s, x_m) - g(t, s, x_n) \leq \xi(t, s)[x_m(r_1) - x_n(r_1)]$$

and

$$\theta \leq F(t, x_m, y_m) - F(t, x_n, y_n) \leq \kappa[(x_m(r_2) - x_n(r_2)) + (y_m - y_n)].$$

Using the normality of positive cone $\mathbb{P}_0$, we get

$$\|h(t, s, x_m) - h(t, s, x_n)\| \leq N_0\xi(t, s)\|x_m(r_1) - x_n(r_1)\|$$

and

$$\|F(t, x_m, y_m) - F(t, x_n, y_n)\| \leq N_0\kappa \|(x_m(r_2) - x_n(r_2)) + (y_m - y_n)\|.$$

Using the property of measure of noncompactness, we have

$$\mu(\{h(t, s, x_m)\}) \leq N_0\xi(t, s) \sup_{-\infty \leqslant r \leqslant 0} \mu(\{x_m(r)\})$$

and

$$\mu(\{F(t, x_m, y_m)\}) \leq N_0\kappa \left[\sup_{-\infty \leqslant r \leqslant 0} \mu(\{x_m(r)\}) + \mu(\{y_m\}) \right].$$

Now, we observed that (A4) implies (A1) and (A3). Therefore, by Theorem 3.1, minimal and maximal mild solutions v^* and w^* exist for the system (1.1) on D, respectively.

By (3.2), for any $t \in (-\infty, 0]$, we have

$$\theta \leq w^*(t) - v^*(t) = \tilde{Q}w^*(t) - \tilde{Q}v^*(t) = 0.$$

Using the normality of positive cone $\mathbb{P}_0$, we get $\|v^*(t) - w^*(t)\| \leq 0$, i.e., $v^*(t) = w^*(t)$ for all $t \in (-\infty, 0]$.

To abbreviate the writing, we set $K_0 := \sup_{0 \leq t \leq T} K_1(t)$. Now using (A4) and the positivity of operator $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t \geq 0}$, for any $t \in \mathcal{J}$, we have

$$\begin{aligned}
\|v^*(t) - w^*(t)\| =& \|\tilde{Q}v^*(t) - \tilde{Q}w^*(t)\| \\
\leq& N_0 \Bigg\| \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) \Bigg[F\Bigg(s, u_s + \overline{v}_s^*, \int_0^s h(s, \tau, u_\tau + \overline{v}_\tau^*) d\tau\Bigg) \\
&- F\Bigg(s, u_s + \overline{w}_s^*, \int_0^s h(s, \tau, u_\tau + \overline{w}_\tau^*) d\tau\Bigg)\Bigg] ds \Bigg\| \\
\leq& N_0\kappa \Bigg[\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\| \Bigg(\|\overline{v}_s^* - \overline{w}_s^*\|_{\mathfrak{B}} \\
&+ \Bigg\| \int_0^s h(s, \tau, u_\tau + \overline{v}_\tau^*) d\tau - \int_0^s h(s, \tau, u_\tau + \overline{w}_\tau^*) d\tau \Bigg\| \Bigg) ds \Bigg] \\
\leq& N_0\kappa \Bigg[\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\| \Bigg(\|\overline{v}_s^* - \overline{w}_s^*\|_{\mathfrak{B}} + \int_0^s \xi(s,\tau) \|\overline{v}_\tau^* - \overline{w}_\tau^*\|_{\mathfrak{B}} d\tau \Bigg) ds \Bigg] \\
\leq& \frac{N_0 S_0 K_0 \kappa}{\Gamma(1+\beta)} \Bigg[\int_0^t (t-s)^\beta \Bigg\{ \sup_{-\infty \leq r \leq 0} \|\overline{v}_s^*(r) - \overline{w}_s^*(r)\| \\
&+ \int_0^s \xi(s,\tau) \sup_{-\infty \leq r \leq 0} \|\overline{v}_\tau^*(r) - \overline{w}_\tau^*(r)\| d\tau \Bigg\} ds \Bigg] \\
\leq& \frac{N_0 S_0 K_0 \kappa}{\Gamma(1+\beta)} \Bigg[\int_0^t (t-s)^\beta \Bigg\{ \sup_{0 \leq z \leq s} \|\overline{v}^*(z) - \overline{w}^*(z)\| \\
&+ \xi^* \sup_{0 \leq z \leq s} \|\overline{v}^*(z) - \overline{w}^*(z)\| \Bigg\} ds \Bigg]
\end{aligned}$$

$$\leq \frac{N_0 S_0 K_0 \kappa}{\Gamma(1+\beta)}(1+\xi^*)\int_0^t (t-s)^{\beta}\|v^*(s)-w^*(s)\|ds.$$

Now, by Lemma 2.1, we get $v^*(t)=w^*(t)$ for all $t\in[0,T]$. So, $v^*(t)=w^*(t)$ for all $t\in(-\infty,T]$. Hence, $v^*(t)=w^*(t)=z^*(t)$(say) for all $t\in(-\infty,T]$ is the unique solution of (3.2). So, we get $y(t)=u(t)+z^*(t)$ is the unique mild solution of the system (1.1). □

4. Example

The fractional order diffusion wave equations have great applications in various fields of science and engineering. These equations represent propagation of mechanical waves through viscoelastic media, charge transport in amorphous semiconductors [15,20,30], and may be used in thermodynamics and shear in fluids, the flow of fluid through fissured rocks [1]. In particular, the fractional delay diffusion wave equations describe the driver reaction time, time taken for a signal traveling to the controlled object, time consume by body to produce red blood cells and cell division time in the dynamics of viral persistence or exhaustion.

Let $\beta,\gamma_j>0$, $j=1,2,3,\dots,n$ be given, satisfying $0<\beta\leq\gamma_n\leq\cdots\leq\gamma_1\leq 1$. Consider the following system

$$(4.1)\quad \begin{cases} {}^cD^{1+\beta}u(t,\nu)+\sum_{j=1}^{n}\alpha_j{}^cD^{\gamma_j}u(t,\nu)=\Delta u(t,\nu)+L\Big(\frac{|u_t(\theta,\nu)|}{|1+u_t(\theta,\nu)|} \\ \qquad\qquad +\int_0^t(t-s)^{-1/2}s^{-1/2}\int_{-\infty}^0\xi(\theta)u_t(\theta,\nu)d\theta ds\Big), \\ u(\theta,\nu)=u_0(\theta,\nu),\quad \theta\in(-\infty,0],\quad \frac{\partial u(t,\nu)}{\partial t}|_{t=0}=z_0, \end{cases}$$

where $\mathbb{X}=L^2([0,1],\mathbb{R})$, $t\in\mathcal{J}=[0,1]$, $T>0$, $\nu\in[0,1]$, $L\geq 0$, $x_t(\theta,\nu)=x(t+\theta,\nu)$, $t\in\mathcal{J}$, $\xi:(-\infty,0]\to\mathbb{R}^+$, $u_0:(-\infty,0]\times[0,1]\to\mathbb{R}$ and Δ is the Laplace operator with maximal domain $\{v\in\mathbb{X}:v\in H^2([0,1],\mathbb{R})\}$. Let $\mathbb{P}=\{v\in\mathbb{X}:v(\nu)\geqslant 0 \text{ a.e. } \nu\in[0,1]\}$. Then the cone $\mathbb{P}$ is normal in Banach space $\mathbb{X}$ with normal constant $N=1$.

Using the theory of cosine families, we can see that Laplacian Δ generates a bounded cosine function $\{C(t)\}_{t\geq 0}$ on the space $L^2([0,1],\mathbb{R})$. Moreover, by Theorem 2.1 the operator Δ in system (4.1) generates a bounded $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$-resolvent family. Let us assume $S_0=\sup_{t\in[0,1]}\|\mathcal{S}_{\beta,\gamma_j}(t)\|$.

For $t\in[0,1]$, $\nu\in[0,1]$ and $\theta\in(-\infty,0]$, we set $z_0=\chi$ and

$$\begin{aligned} y(t)=&u(t,\nu),\\ \phi(\theta)=&u_0(\theta,\nu),\\ h(t,s,y_s)=&(t-s)^{-1/2}s^{-1/2}\int_{-\infty}^0\xi(\theta)u_t(\theta,\nu)d\theta,\\ F(t,y_t,\int_0^t h(t,s,y_s)ds)=&L\Bigg[\frac{|u_t(\theta,\nu)|}{1+|u_t(\theta,\nu)|}+\int_0^t h(t,s,y_s)ds\Bigg]. \end{aligned}$$

Now, we observe that the system (4.1) has a abstract form of system (1.1). Let $v(t) = 0$ for $t \in [0,1]$. Then $F\left(t, v_t, \int_0^t h(t,s,v_s)ds\right) = 0$ for $t \in [0,1]$ and $\phi(t) \geq v(t)$ for $t \in (-\infty, 0]$. Let us suppose that there exists a function $w(t) \geq 0$ such that
(4.2)
$$\begin{cases} {}^cD^{1+\beta}w(t) + \sum_{j=1}^n \alpha_j {}^cD^{\gamma_j}w(t) \geq Aw(t) + F\left(t, w_t, \int_0^t h(t,s,w_s)ds\right), & t \in (0,T], \\ w(t) \geq \phi(t) \in \mathfrak{B}, \quad t \in (-\infty, 0], \quad w'(0) \geq \chi. \end{cases}$$
Thus the system (1.1) admits lower and upper solutions v, w such that $v \leq w$.

Let $\vartheta > 0$ be a constant and
$$\mathfrak{B} = \left\{ y \in \mathcal{C}((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\vartheta\theta}y(\theta) \text{ exists in } \mathbb{R} \right\}.$$
The norm of $\mathfrak{B}$ is given by $\|y\|_{\mathfrak{B}} = \sup_{-\infty<\theta\leq 0} e^{\vartheta\theta}|y(\theta)|$. Let $y : (-\infty, 0] \to \mathbb{R}$ such that $y_0 \in \mathfrak{B}$. Then
$$\lim_{\theta\to-\infty} e^{\vartheta\theta}y_t(\theta) = \lim_{\theta\to-\infty} e^{\vartheta\theta}y(t+\theta) = \lim_{\theta\to-\infty} e^{\vartheta(\theta-t)}y(\theta) = e^{-\vartheta t}\lim_{\theta\to-\infty} e^{\vartheta\theta}y_0(\theta) < \infty.$$
Hence, $y_t \in \mathfrak{B}$. Finally, we will show that
$$\|y_t\|_{\mathfrak{B}} \leq K_1(t)\sup_{s\in[0,t]}|y(s)| + K_2(t)\|y_0\|_{\mathfrak{B}},$$
where $K_1 = K_2 = 1$ and $K = 1$. We have $|y_t(\theta)| = |y(t+\theta)|$. If $t+\theta \leq 0$, we obtain
$$|y_t(\theta)| \leq \sup_{s\in(-\infty,0]}|y(s)|.$$
If $t+\theta \geq 0$, then we get
$$|y_t(\theta)| \leq \sup_{s\in[0,t]}|y(s)|.$$
Thus, for all $(t+\theta) \in [0,1]$ we have
$$|y_t(\theta)| \leq \sup_{s\in(-\infty,0]}|y(s)| + \sup_{s\in[0,t]}|y(s)|.$$
Then
$$\|y_t\|_{\mathfrak{B}} \leq \|y_0\|_{\mathfrak{B}} + \sup_{s\in[0,t]}|y(s)|.$$
One can easily check that $\mathfrak{B}$ is a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{B}}$ and hence conclude that $\mathfrak{B}$ is a phase space. Clearly, the functions f and h satisfies the assumptions (A1) and (A2). For $t \in [0,1]$, $\varphi_1, \varphi_2 \in \mathfrak{B}$ with $0 \leq \varphi_1 \leq \varphi_2$ and $v_1, v_2 \in X$, we have
$$0 \leq h(t,s,\varphi_2) - h(t,s,\varphi_1) \leq (t-s)^{-1/2}s^{-1/2}\int_{-\infty}^0 \xi(\theta)(\varphi_2(\theta) - \varphi_2(\theta)d\theta$$
and
$$0 \leq F(t,\varphi_2,v_2) - F(t,\varphi_1,v_1) \leq L\left[\frac{|\varphi_2(\theta)|}{1+|\varphi_2(\theta)|} - \frac{|\varphi_1(\theta)|}{1+|\varphi_1(\theta)|} + v_2 - v_1\right].$$

Using normality of cone $\mathbb{P}$, we have

$$\begin{aligned}\|h(t,s,\varphi_2)-h(t,s,\varphi_1)\| \leq &(t-s)^{-1/2}s^{-1/2}\int_{-\infty}^{0}|\xi(\theta)|\|\varphi_2(\theta)-\varphi_1(\theta)\|d\theta,\\ \|F(t,\varphi_2,v_2)-F(t,\varphi_1,v_1)\| \leq &L[\|\varphi_2(\theta)-\varphi_1(\theta)\|+\|v_2-v_1\|].\end{aligned}$$

Now, by the property of measure of noncompactness for $U \subset \mathcal{C}((-\infty,0],\mathbb{X})$ and $V \subset \mathbb{X}$, we have

$$\begin{aligned}\mu(h(t,s,U)) \leq &\xi(t,s)\sup_{-\infty\leq\theta\leq 0}\mu(U(\theta)),\\ \mu(f(t,U,V)) \leq &L[\sup_{-\infty<\theta\leq 0}\mu(U(\theta))+\mu(V)],\end{aligned}$$

where $\xi(t,s) = (t-s)^{-1/2}s^{-1/2}\int_{-\infty}^{0}|\xi(\theta)|d\theta$. Let $\xi^* = \sup_{t,s\in(-\infty,1]}\xi(t,s)$. Thus, assumptions (A3) and (A4) are fulfilled. Now by the Theorem 3.1, the system (4.1) admits extrimal mild solutions lying between the lower solution 0 and the upper solution w. Further, by Theorem 3.3 the system (4.1) admits unique mild solution.

5. Conclusion

The monotone iterative technique has been employed to establish the existence and uniqueness of mild solution for a class of multi-term time-fractional delay differential system in an ordered Banach space. Assuming the existence of the lower and upper solutions of the system (1.1), a new set of sufficient conditions has been obtained in which the nonlinear functions satisfy some monotonic properties. One can extend this idea to establish the existence results for multi-term time-fractional differential system with impulsive conditions.

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MORE GENERALIZATIONS OF UNION SOFT HYPERIDEALS OF ORDERED SEMIHYPERGROUPS

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Abstract. In this paper, we introduce the notions of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals of ordered semihypergroups. Some basic operations are investigated and some related properties are also studied. We present characterizations of ordered semihypergroups in terms of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals. We prove that every (M,N)-union soft hyperideal is an (M,N)-union soft interior hyperideal but the converse is not true which is shown with help of an example. However we show that the notions of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals coincide in a regular as well as in intra-regular ordered semihypergroups. Moreover we introduce the notion of (M,N)-union soft simple ordered semihypergroups. Finally, we characterize (M,N)-union soft simple ordered semihypergroups by means of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals.

1. Introduction

There are many examples in chemistry where the sum of two elements is a set of elements. In this case we have a hyperstructure. Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [8] at the 8^{th} Congress of Scandinavian Mathematicians. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an

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element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many researchers have studied different aspects of semihypergroups (see [9–15, 18]).

The uncertainty appeared in economics, engineering, environmental science, medical science and social science and so many other applied sciences is too complicated to be solved by traditional mathematical framework. Molodstov [6], introduced soft set theory and it has received much attention since its inception. Soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breadth of the discipline of informations sciences with intelligent systems, approximate reasoning, expert and decision support systems, self-adaptation and self-organizational systems, information and knowledge, modeling and computing with words. Soft set theory has been regarded as a new mathematical tool for dealing with uncertainties and it has seen a wide-ranging applications in the mean of algebraic structures such as groups [1], semirings [2], ordered semigroups [4], hemirings [5, 7], and so on. Feng et al. discussed soft relations in semigroups (see [3]) and explored decomposition of fuzzy soft sets with finite value spaces. Khan et al. [17], applied soft set theory to ordered semihypergroups and introduced the notions of uni-soft subsemihypergroups and uni-soft left (resp. right) hyperideals.

In this paper, we study the concepts of union soft interior hyperideals, (M, N)-union soft hyperideals and (M, N)-union soft interior hyperideals in ordered semihypergroups and present some related examples of these concepts. We show that (M, N)-union soft hyperideals and (M, N)-union soft interior hyperideals coincide in regular ordered semihypergroups and intra-regular ordered semihypergroups. We characterize ordered semihypergroups in terms of (M, N)-union soft hyperideals and (M, N)-union soft interior hyperideals. We introduce the concept of (M, N)-union soft simple ordered semihypergroups. Moreover we characterize (M, N)-union soft simple ordered semihypergroups in terms of (M, N)-union soft hyperideals and (M, N)-union soft interior hyperideals.

2. Preliminaries

By an ordered semihypergroup we mean a structure $(S, \circ, \leq)$ in which the following conditions are satisfied:

(1) $(S, \circ)$ is a semihypergroup;

(2) $(S, \leq)$ is a poset;

(3) for all $a, b, x \in S$ $a \leq b$ implies $x \circ a \leq x \circ b$ and $a \circ x \leq b \circ x$.

For $A \subseteq S$, we denote $(A] := \{t \in S : t \leq h \text{ for some } h \in A\}$. For $A, B \subseteq S$, we have $A \circ B := \bigcup\{a \circ b : a \in A, b \in B\}$.

A nonempty subset A of an ordered semihypergroup S is called a subsemihypergroup of S if $A^2 \subseteq A$.

A nonempty subset A of S is called a left (resp. right) hyperideal of S if it satisfies the following conditions:

(1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$);

(2) if $a \in A, b \in S$ and $b \leq a$, implying $b \in A$.

By a two sided hyperideal or simply a hyperideal of S we mean a nonempty subset of S which is both a left hyperideal and a right hyperideal of S.

A subsemihypergroup A of S is called an interior hyperideal of S if it satisfies the following conditions:

(1) $S \circ A \circ S \subseteq A$;

(2) if $a \in A, b \in S$ and $b \leq a$, implying $b \in A$.

An ordered semihypergroup $(S, \circ, \leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq a \circ x \circ a$.

An ordered semihypergroup S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq x \circ a \circ a \circ y$.

3. Soft Sets

In what follows, we take $E = S$ as the set of parameters, which is an ordered semihypergroup, unless otherwise specified.

From now on, U is an initial universe set, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C, \ldots \subseteq E$.

Definition 3.1 (see [6])**.** A *soft set* f_A over U is defined as

$$f_A : E \to P(U) \quad \text{such that} \quad f_A(x) = \emptyset \quad \text{if } x \notin A.$$

Hence, f_A is also called an *approximation function.*

A soft set f_A over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}.$$

It is clear that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 3.2 (see [6])**.** Let $f_A, f_B \in S(U)$. Then f_A is called a *soft subset* of f_B, denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

Definition 3.3 (see [6])**.** Two soft sets f_A and f_B are said to be equal soft sets if $f_A \tilde{\subseteq} f_B$ and $f_B \tilde{\subseteq} f_A$ and is denoted by $f_A \tilde{=} f_B$.

Definition 3.4. (see [6]). Let $f_A, f_B \in S(U)$. Then the *soft union* of f_A and f_B, denoted by $f_A \tilde{\cup} f_B = f_{A \cup B}$, is defined by $(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

Definition 3.5 (see [6])**.** Let $f_A, f_B \in S(U)$. Then the *soft intersection* of f_A and f_B, denoted by $f_A \tilde{\cap} f_B = f_{A \cap B}$, is defined by $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

For $x \in S$, we define $A_x = \{(y, z) \in S \times S \mid x \leq y \circ z\}$.

Definition 3.6 (see [17])**.** Let f_A and g_B be two soft sets of an ordered semihypergroup S over U. Then, the uni-soft product, denoted by $f_A\tilde{\diamond}g_B$, is defined by

$$f_A\tilde{\diamond}g_B : S \to P(U), \quad x \mapsto \left(f_A\tilde{\diamond}g_B\right)(x) = \begin{cases} \bigcap\limits_{(y,z)\in A_x} \{f_A(y) \cup g_B(z)\}, & \text{if } A_x \neq \emptyset, \\ U, & \text{if } A_x = \emptyset, \end{cases}$$

for all $x \in S$.

Definition 3.7 (see [17])**.** Let $A \subseteq S$. Then the soft characteristic function

$$\chi_A^c : S \to P(U)$$

is defined by

$$\chi_A(x) := \begin{cases} U, & \text{if } x \in A, \\ \emptyset, & \text{if } x \notin A. \end{cases}$$

For the characteristic soft set χ_A over U, the soft set χ_A^c over U given as follows:

$$\chi_A^c(x) := \begin{cases} \emptyset, & \text{if } x \in A, \\ U, & \text{if } x \notin A. \end{cases}$$

For an ordered semihypergroup, the soft sets "$\emptyset_S$" of S over U is defined as follows:

$$\emptyset_S : S \mapsto P(U), \quad x \mapsto \emptyset_S(x) = \emptyset.$$

Definition 3.8 (see [17])**.** Let f_A be a soft set of an ordered semihypergroup S over U a subset δ such that $\delta \in P(U)$. The δ-exclusive set of f_A is denoted by $e_A(f_A, \delta)$ and defined to be the set

$$e_A(f_A, \delta) = \{x \in S \mid f_A(x) \subseteq \delta\}.$$

Definition 3.9 (see [17]))**.** A soft set f_A of an ordered semihypergroup S over U is called *a union soft subsemihypergroup of* S over U if

$$(\forall x, y \in S) \bigcup_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(x) \cup f_A(y).$$

Definition 3.10 (see [17])**.** Let f_A be a soft set of an ordered semihypergroup S over U. Then f_A is called a union soft left (resp. right) hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x, y \in S) \bigcup\limits_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(y) \left(\text{resp. } \bigcup\limits_{\alpha \in x \circ y} f_A(\alpha) \subseteq f_A(x)\right)$;

(2) $(\forall x, y \in S)\ x \leq y \Rightarrow f_A(x) \subseteq f_A(y)$.

A soft set f_A of an ordered semihypergroup S over U is called a *union soft hyperideal* of S over U if it is both a union soft left hyperideal and a union soft right hyperideal of S over U.

Definition 3.11. A union soft subsemihypergroup f_A of an ordered semihypergroup S over U is called a union soft interior hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x, y, a \in S) \bigcup_{\alpha \in x \circ a \circ y} f_A(\alpha) \subseteq f_A(a)$;

(2) $(\forall x, y \in S)\ x \leq y \Rightarrow f_A(x) \subseteq f_A(y)$.

Example 3.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	e_1	e_2	e_3	e_4
e_1	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$
e_2	$\{e_1\}$	$\{e_1\}$	$\{e_1, e_4\}$	$\{e_1\}$
e_3	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$
e_4	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$	$\{e_1\}$

,

$$\leq := \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_4, e_4), (e_1, e_4)\}.$$

Suppose $U = \{1, 2, 3\}$ and $A = \{e_2, e_3, e_4\}$. Let us define $f_A(e_1) = \emptyset$, $f_A(e_2) = \{1\}$, $f_A(e_3) = \{1, 2, 3\}$ and $f_A(e_4) = \{2, 3\}$. Then f_A is a union soft interior hyperideal of S over U.

4. (M, N)-Union Soft Hyperideals

In this section, we introduce the notions of (M, N)-union soft hyperideal of ordered semihypergroups and investigate some related properties. From now on, $\emptyset \subseteq M \subset N \subseteq U$.

For any soft sets f_A and g_B, we define an order relation $\widetilde{\supseteq}_{[M,N]}$ by putting

$$f_A \widetilde{\supseteq}_{[M,N]} g_B \Leftrightarrow (f_A(x) \cup M) \cap N \widetilde{\supseteq} (g_B(x) \cup M) \cap N,$$

for all $x \in S$.

In case $f_A \widetilde{\supseteq}_{[M,N]} g_B$ and $g_B \widetilde{\supseteq}_{[M,N]} f_A$ then $f_A =_{[M,N]} g_B$.

Theorem 4.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then the set*

$$\left(S(U), \widetilde{\diamond}, \widetilde{\supseteq}_{[M,N]}\right)$$

forms an ordered semihypergroup.

Proof. Obviously, the operation "$\widetilde{\diamond}$" is well-defined.

Let f_A, g_B, and $h_C \in S(U)$ and x be any element of S. If $A_x = \emptyset$, then, clearly, $((((f_A \widetilde{\diamond} g_B) \widetilde{\diamond} h_C)(x)) \cup M) \cap N = (((f_A \widetilde{\diamond} (g_B \widetilde{\diamond} h_C))(x)) \cup M) \cap N$. Let $A_x \neq \emptyset$, then we have

$$\begin{aligned}
&((((f_A \widetilde{\diamond} g_B) \widetilde{\diamond} h_C)(x)) \cup M) \cap N \\
&= \left(\left(\bigcap_{x \leq y \circ z} \{(f_A \widetilde{\diamond} g_B)(y) \cup h_C(z)\}\right) \cup M\right) \cap N \\
&= \left(\left(\bigcap_{x \leq y \circ z} \left\{\bigcap_{y \leq u \circ v} \{f_A(u) \cup g_B(v)\} \cup h_C(z)\right\}\right) \cup M\right) \cap N
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(\bigcap_{x \leq (u \circ v) \circ z} \{ f_A(u) \cup g_B(v) \cup h_C(z) \} \right) \cup M \right) \cap N \\
&= \left(\left(\bigcap_{x \leq u \circ (v \circ z)} \{ f_A(u) \cup (g_B(v) \cup h_C(z)) \} \right) \cup M \right) \cap N \\
&\supseteq \left(\left(\bigcap_{x \leq u \circ (v \circ z)} \left\{ f_A(u) \cup \left\{ \bigcap_{y \leq v \circ z} (g_B(v) \cup h_C(z)) \right\} \right\} \right) \cup M \right) \cap N \\
&= \left(\left(\bigcap_{x \leq u \circ (v \circ z)} \{ f_A(u) \cup (g_B \widetilde{\diamond} h_C)(v \circ z) \} \right) \cup M \right) \cap N \\
&= (((f_A \widetilde{\diamond} (g_B \widetilde{\diamond} h_C))(x)) \cup M) \cap N.
\end{aligned}
$$

It follows that $((f_A \widetilde{\diamond} g_B) \widetilde{\diamond} h_C) \widetilde{\supseteq}_{[M,N]} (f_A \widetilde{\diamond} (g_B \widetilde{\diamond} h_C))$. Similarly, we can prove that $(f_A \widetilde{\diamond} (g_B \widetilde{\diamond} h_C)) \widetilde{\supseteq}_{[M,N]} ((f_A \widetilde{\diamond} g_B) \widetilde{\diamond} h_C)$. Thus we have proved that $((f_A \widetilde{\diamond} g_B) \widetilde{\diamond} h_C) =_{[M,N]} (f_A \widetilde{\diamond} (g_B \widetilde{\diamond} h_C))$.

Assume that $f_A \widetilde{\supseteq}_{[M,N]} g_B$ and let $A_x = \emptyset$. Then obviously, $(f_A \widetilde{\diamond} h_C) \widetilde{\supseteq}_{[M,N]} (g_B \widetilde{\diamond} h_C)$ and $(h_C \widetilde{\diamond} f_A) \widetilde{\supseteq}_{[M,N]} (h_C \widetilde{\diamond} g_B)$. If $A_x \neq \emptyset$, then

$$
\begin{aligned}
(((f_A \widetilde{\diamond} h_C)(x)) \cup M) \cap N &= \left(\left(\bigcap_{(y,z) \in A_x} \{ f_A(y) \cup h_C(z) \} \right) \cup M \right) \cap N \\
&= \left(\left(\bigcap_{(y,z) \in A_x} \{ f_A(y) \cup h_C(z) \cup M \} \right) \cup M \right) \cap N \\
&\supseteq \left(\left(\bigcap_{(y,z) \in A_x} \{ g_B(y) \cup h_C(z) \cap N \} \right) \cup M \right) \cap N \\
&= \left(\bigcap_{(y,z) \in A_x} \{ g_B(y) \cup h_C(z) \cap N \} \right) \cup (M \cap N) \\
&= \left(\left(\bigcap_{(y,z) \in A_x} \{ g_B(y) \cup h_C(z) \} \right) \cap N \right) \cup M \\
&= \left(\left(\bigcap_{(y,z) \in A_x} \{ g_B(y) \cup h_C(z) \} \right) \cup M \right) \cap N \\
&= (g_B \widetilde{\diamond} h_C)(x).
\end{aligned}
$$

In a similar way, we can show that $(h_C \widetilde{\diamond} f_A) \widetilde{\supseteq}_{[M,N]} (h_C \widetilde{\diamond} g_B)$. Thus, $\left(S(U), \widetilde{\diamond}, \widetilde{\supseteq}_{[M,N]} \right)$ is an ordered semihypergroup.

Definition 4.1. A soft set f_A of an ordered semihypergroup S over U is called an (M,N)-*union subsemihypergroup of* S over U if

$$(\forall x,y\in S)\left(\bigcup_{\alpha\in x\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(x)\cap f_A(y)\cup M.$$

Example 4.1. Let $(S,\circ,\leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	p	q	r	s
p	$\{p\}$	$\{p\}$	$\{p\}$	$\{p\}$
q	$\{p\}$	$\{p\}$	$\{p\}$	$\{p\}$
r	$\{p\}$	$\{p\}$	$\{p,q\}$	$\{p\}$
s	$\{p\}$	$\{p\}$	$\{p,q\}$	$\{p,q\}$

,

$$\leq := \{(p,p),(q,q),(r,r),(s,s),(p,q)\}.$$

Suppose $U=\{1,2,3\}$, $A=\{q,r,s\}$, $M=\{2\}$ and $N=\{1,2\}$. Let us define $f_A(p)=\emptyset$, $f_A(q)=\{2\}$, $f_A(r)=\{1,2,3\}$ and $f_A(s)=\{2,3\}$. Then f_A is an (M,N)-union soft subsemihypergroup of S over U.

Theorem 4.2. *A non-empty subset A of an ordered semihypergroup $(S,\circ,\leq)$ is a subsemihypergroup of S if and only if the soft set f_A, defined by*

$$f_A(x)=\begin{cases}\delta_1, & \text{if } x\in A,\\ \delta_2, & \text{if } x\notin A,\end{cases}$$

is an (M,N)-union soft subsemihypergroup of S over U, where $\delta_1,\delta_2\subseteq U$ such that $M\subseteq\delta_1\subseteq\delta_2\subseteq N\subseteq U$.

Proof. Suppose A is a subsemihypergroup of S. Suppose $x,y\in S$. If $x,y\in A$, then $x\circ y\subseteq A$. We have to show that $\bigcup_{\beta\in x\circ y} f_A(\beta)\cap N\subseteq f_A(x)\cap f_A(y)\cup M$. Let $\beta\in x\circ y\subseteq A$. Then $f_A(\beta)=\delta_1$. Also $f_A(x)=\delta_1=f_A(y)$. So $f_A(\beta)=\delta_1=f_A(x)\cup f_A(y)$. Hence $\bigcup_{\beta\in x\circ y} f_A(\beta)\cap N=\delta_1\cap N=\delta_1=f_A(x)\cup f_A(y)\cup M$. If x or y is not in A, then $x\circ y\subseteq A$ or $x\circ y\nsubseteq A$. If $x\circ y\subseteq A$, then for $\beta\in x\circ y\subseteq A$, we have $f_A(\beta)\cap N=\delta_1\cap N=\delta_1$. If $x\circ y\nsubseteq A$, then for $\beta\in x\circ y\nsubseteq A$, we have $f_A(\beta)\cap N=\delta_2\cap N=\delta_2$. But $f_A(x)\cup f_A(y)\cup M=\delta_2\cup M=\delta_2$. Thus, $\bigcup_{\beta\in x\circ y} f_A(\beta)\cap N\subseteq f_A(x)\cup f_A(y)\cup M$.

Conversely, assume that f_A is an (M,N)-union soft subsemihypergroup of S over U. Let $x,y\in A$. Then $f_A(x)=\delta_1=f_A(y)$. By our supposition $\bigcup_{\beta\in x\circ y} f_A(\beta)\cap N\subseteq f_A(x)\cup f_A(y)\cup M=\delta_1\cup M=\delta_1$. But $M\subseteq\delta_1\subseteq\delta_2\subseteq N$. So, $f_A(\beta)\subseteq\delta_1$ for every $\beta\in x\circ y$. Thus, $\beta\in A$. This implies that $x\circ y\subseteq A$. Hence, A is subsemihypergroup of S. $\square$

Theorem 4.3. *If f_A and g_B are two (M,N)-union soft subsemihypergroup of S over U, then their union $f_A\cup g_B$ is an (M,N)-union soft subsemihypergroup of S over U.*

Proof. Let $x, y \in S$. Since f_A and g_B are two (M, N)-union soft subsemihypergroup of S over U. Then for every $\alpha \in x \circ y$, we have

$$\begin{aligned}
(f_A \cup g_B)(\alpha) \cap N &= (f_A(\alpha) \cup g_B(\alpha)) \cap N \\
&= (f_A(\alpha) \cap N) \cup (g_B(\alpha) \cap N) \\
&\subseteq (f_A(x) \cup f_A(y) \cup M) \cup (g_B(x) \cup g_B(y) \cup M) \\
&= ((f_A(x) \cup g_B(x)) \cup (f_A(y) \cup g_B(y))) \cup M \\
&= (f_A \cup g_B)(x) \cup (f_A \cup g_B)(y) \cup M.
\end{aligned}$$

Hence, $\bigcup\limits_{\alpha \in x \circ y} (f_A \cup g_B)(\alpha) \cap N \subseteq (f_A \cup g_B)(x) \cup (f_A \cup g_B)(y) \cup M$. Therefore, $f_A \cup g_B$ is an (M, N)-union soft subsemihypergroup of S over U. $\square$

Definition 4.2. A soft set f_A of an ordered semihypergroup S over U is called an (M, N)-union soft left (resp. right) hyperideal of S over U if it satisfies the following conditions:

(1) $\left(\bigcup\limits_{\alpha \in x \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(y) \cup M$ $\left(\text{resp. } \bigcup\limits_{\alpha \in x \circ y} f_A(\alpha)) \cap N \subseteq f_A(x) \cup M\right)$;

(2) $x \leq y \Rightarrow f_A(x) \cap N \subseteq f_A(y) \cup M$,

for all $x, y \in S$.

A soft set f_A of an ordered semihypergroup S over U is called an (M, N)-*union soft* hyperideal of S over U if it is both an (M, N)-union soft left hyperideal and an (M, N)-union soft right hyperideal of S over U.

Example 4.2. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$
4	$\{1\}$	$\{1\}$	$\{1, 2\}$	$\{1, 2, 3\}$

,

$$\leq := \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4), (3,4)\}.$$

Suppose $U = \{h_1, h_2, h_3\}$, $A = \{1, 3, 4\}$, $M = \{h_1\}$ and $N = \{h_1, h_3\}$. Let us define $f_A(1) = \emptyset$, $f_A(2) = \{h_1\}$, $f_A(3) = \{h_1, h_2\}$ and $f_A(4) = \{h_1, h_2, h_3\}$. Then f_A is an (M, N)-union soft hyperideal of S over U.

Theorem 4.4. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) hyperideal of S if and only if the soft set χ_A^c of A is an (M, N)-union soft left (resp. right) hyperideal of S over U.*

Proof. Suppose that A is a left hyperideal of S. Let $x, y \in S$. Then

$$\left(\bigcup_{\alpha \in x \circ y} \chi_A^c(\alpha)\right) \cap N \subseteq \chi_A^c(y) \cup M.$$

Indeed, if $y \notin A$ then $\chi_A^c(y) = U$. Since $\chi_A^c(x) \subseteq U$ for all $x \in S$ and $\emptyset \subseteq M \subset N \subseteq U$, we have

$$\left(\bigcup_{\alpha \in x \circ y} \chi_A^c(\alpha)\right) \cap N \subseteq U = \chi_A^c(y) \cup M.$$

Let $y \in A$. Since A is a left hyperideal of S and $x \in S$, we have $x \circ y \subseteq S \circ A \subseteq A$. Thus, in this case $\chi_A^c(\alpha) = \emptyset$ for any $\alpha \in x \circ y$. Hence,

$$\left(\bigcup_{\alpha \in x \circ y} \chi_A^c(\alpha)\right) \cap N = \emptyset \subseteq \chi_A^c(y) \cup M.$$

Let now $x, y \in S$, $x \leq y$. Then $\chi_A^c(x) \cap N \subseteq \chi_A^c(y) \cup M$. In fact, if $y \in A$, then $\chi_A^c(y) = \emptyset$. Since $S \ni x \leq y \in A$, by hypothesis we have $x \in A$, then $\chi_A^c(x) = \emptyset$. Thus $\chi_A^c(x) \cap N = \emptyset \subseteq M = \chi_A^c(y) \cup M$. If $y \notin A$, then $\chi_A^c(y) = U$. Since $x \in S$, $\emptyset \subseteq M \subset N \subseteq U$, we have $\chi_A^c(x) \cap N \subseteq U = \chi_A^c(y) \cup M$. Consequently, χ_A^c is an (M, N)-union soft left hyperideal of S over U.

Conversely, let A be a non-empty subset of S such that χ_A^c is an (M, N)-union soft left hyperideal of S over U. We claim that $S \circ A \subseteq A$. To prove our claim, let $x \in S$ and $y \in A$. By hypothesis,

$$\left(\bigcup_{\alpha \in x \circ y} \chi_A^c(\alpha)\right) \cap N \subseteq \chi_A^c(y) \cup M = \emptyset \cup M = M.$$

Thus, by $\emptyset \subseteq M \subset N \subseteq U$, $\bigcup_{\alpha \in x \circ y} \chi_A^c(\alpha) \cap N \subseteq M$. Hence for any $\alpha \in x \circ y$, $\chi_A^c(\alpha) = \emptyset$, i.e., $\alpha \in A$. It thus follows that $S \circ A \subseteq A$. Furthermore, let $x \in A$, $S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $\chi_A^c(y) = \emptyset$. By $x \in A$, we have $\chi_A^c(x) = \emptyset$. Since χ_A^c is an (M, N)-union soft left hyperideal of S over U and $y \leq x$, we have $\chi_A^c(y) \cap N \subseteq \chi_A^c(x) \cup M = \emptyset \cup M = M$. Notice that $\emptyset \subseteq M \subset N \subseteq U$, we conclude that $\chi_A^c(y) = \emptyset$. Therefore, A is a left hyperideal of S.

Similarly we can show that χ_A^c is an (M, N)-union soft right hyperideal of S over U, if and only if A is a right hyperideal of S. □

Corollary 4.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. Then A is a hyperideal of S if and only if the soft set χ_A^c of A is an (M, N)-union soft hyperideal of S over U.*

Theorem 4.5. *Let f_A be a soft set of an ordered semihypergroup S over U and $\delta \in P(U)$. Then f_A is an (M, N)-union soft hyperideal of S over U if and only if the nonempty δ-exclusive set $e_A(f_A, \delta)$ of f_A is a hyperideal of S and $M \subset \delta \subseteq N$.*

Proof. Assume that f_A is an (M, N)-union soft hyperideal of S over U. Let $x \in e_A(f_A, \delta)$ for $M \subset \delta \subseteq N$ and $y \in S$. Then $f_A(x) \subseteq \delta$. It follows from Definition 4.2, that

$$\left(\bigcup_{\alpha \in x \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(x) \cup M \subseteq \delta \cup M = \delta$$

and

$$\left(\bigcup_{\alpha\in y\circ x} f_A(\alpha)\right)\cap N\subseteq f_A(x)\cup M\subseteq \delta\cup M=\delta.$$

Notice that $\delta\subseteq N$ we can deduce that $\bigcup_{\alpha\in x\circ y} f_A(\alpha)\subseteq\delta$ and $\bigcup_{\alpha\in y\circ x} f_A(\alpha)\subseteq\delta$. Thus it can be easily shown that $x\circ y\subseteq e_A(f_A,\delta)$ and $y\circ x\subseteq e_A(f_A,\delta)$. Furthermore, let $x\in e_A(f_A,\delta)$, $S\ni y\leq x$. Then $y\in e_A(f_A,\delta)$. Indeed, since $x\in e_A(f_A,\delta)$, $f_A(x)\subseteq\delta$ and f_A is an (M,N)-union soft hyperideal of S over U, we have $f_A(y)\cap N\subseteq f_A(x)\cup M\subseteq\delta\cup M=\delta$. By $\delta\subset N$, we have $f_A(y)\subseteq\delta$, i.e., $y\in e_A(f_A,\delta)$. Therefore, $e_A(f_A,\delta)$ is a hyperideal of S.

Conversely, let $e_A(f_A,\delta)\neq\emptyset$ be a hyperideal of S for all $M\subset\delta\subseteq N$. If there exist $x_1,y_1\in S$ such that

$$\left(\bigcup_{\alpha\in x_1\circ y_1} f_A(\alpha)\right)\cap N\supset f_A(y_1)\cup M,$$

then there exists $M\subset\delta\subseteq N$ such that

$$\left(\bigcup_{\alpha\in x_1\circ y_1} f_A(\alpha)\right)\cap N\supset\delta\supseteq f_A(y_1)\cup M$$

and we have $f_A(y_1)\subseteq\delta$ and $\bigcup_{\alpha\in x_1\circ y_1} f_A(\alpha)\supset\delta$. Thus, $y_1\in e_A(f_A,\delta)$ and $x_1\circ y_1\nsubseteq e_A(f_A,\delta)$, which is a contradiction. Hence,

$$\left(\bigcup_{\alpha\in x\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(y)\cup M,$$

for all $x,y\in S$. Moreover if $x\leq y$ then $f_A(x)\cap N\subseteq f_A(y)\cup M$. Indeed, if there exist $x_1,y_1\in S$ such that $x_1\leq y_1$ and $f_A(x_1)\cap N\supset f_A(y_1)\cup M$ then there exists $M\subset\delta\subseteq N$ such that $f_A(x_1)\cap N\supset\delta\supseteq f_A(y_1)\cup M$ and we have $f_A(y_1)\subseteq\delta$ and $f_A(x_1)\supset\delta$. Then $y_1\in e_A(f_A,\delta)$ and $x_1\notin e_A(f_A,\delta)$. This is a contradiction that $e_A(f_A,\delta)$ is a hyperideal of S. Therefore f_A is an (M,N)-union soft left hyperideal of S over U. In a similar way we can show that f_A is an (M,N)-union soft right hyperideal of S over U and thus f_A is an (M,N)-union soft hyperideal of S over U. □

Theorem 4.6. *Let $(S,\circ,\leq)$ be an ordered semihypergroup and f_A be a soft set of S over U. Then f_A is an (M,N)-union soft left hyperideal of S over U if and only if f_A satisfies the following conditions:*

(1) $\emptyset_S\tilde{\diamond}f_A\tilde{\supseteq}_{[M,N]}f_A$;

(2) $(\forall x,y\in S)\ x\leq y\Rightarrow f_A(x)\cap N\subseteq f_A(y)\cup M$.

Proof. Suppose that f_A is an (M,N)-union soft left hyperideal of S over U. Then by Definition 4.2, condition (2) holds. To prove the condition (1) holds, it is enough to prove that $(\emptyset_S\tilde{\diamond}f_A)(x)\cup M\supseteq f_A(x)\cap N$ for any $x\in S$. Indeed, let $x\in S$. If $A_x=\emptyset$, then $(\emptyset_S\tilde{\diamond}f_A)(x)\cup M\supseteq f_A(x)\cap N$. Let $A_x\neq\emptyset$. Then there exist $y,z\in S$ such that

$x \leq y \circ z$ and there exists $v \in y \circ z$ such that $x \leq v$. Since f_A is an (M, N)-union soft left hyperideal of S over U, we have for any $x \leq y \circ z$. Thus,

$$
\begin{aligned}
((\emptyset_S \tilde{\diamond} f_A)(x) \cup M) \cap N &= \left(\left(\bigcap_{(y,z) \in A_x} \{\emptyset_S(y) \cup f_A(z)\} \right) \cup M \right) \cap N \\
&= \left(\left(\bigcap_{(y,z) \in A_x} \{\emptyset \cup f_A(z) \cup M\} \right) \cup M \right) \cap N \\
&= \left(\left(\bigcap_{(y,z) \in A_x} \{f_A(z) \cup M\} \right) \cup M \right) \cap N \\
&\supseteq \left(\left(\bigcap_{(y,z) \in A_x} \{f_A(x) \cap N\} \right) \cup M \right) \cap N \\
&= [\{f_A(x) \cap N\} \cup M] \cap N \\
&= (f_A(x) \cap N) \cup (M \cap N) \\
&= (f_A(x) \cap N) \cup M \\
&= (f_A(x) \cup M) \cap N.
\end{aligned}
$$

Thus, $\emptyset_S \tilde{\diamond} f_A \tilde{\supseteq}_{[M,N]} f_A$ for all $x \in S$.

Conversely, assume that the conditions (1) and (2) hold. Let $y, z \in S$. Then we can prove that $\bigcup_{x \in y \alpha z} f_A(x) \cap N \subseteq f_A(z) \cup M$ for any $x \in y \circ z$. In fact, since $x \in y \circ z$, $x \leq x$, we have $x \leq y \circ z$. Thus by hypothesis, we have

$$
\begin{aligned}
f_A(x) \cap N &\subseteq (f_A(x) \cap N) \cup M \\
&\subseteq ((\emptyset_S \tilde{\diamond} f_A)(x) \cap N) \cup M \\
&= \left(\left(\bigcap_{(p,q) \in A_x} \{\emptyset_S(p) \cup f_A(q)\} \right) \cap N \right) \cup M \\
&\subseteq (\{\emptyset_S(y) \cup f_A(z)\} \cap N) \cup M \\
&= (\{\emptyset \cup f_A(z)\} \cap N) \cup M \\
&= (f_A(z) \cap N) \cup M \\
&= (f_A(z) \cup M) \cap (N \cup M) \\
&= (f_A(z) \cup M) \cap N \\
&\subseteq f_A(z) \cup M.
\end{aligned}
$$

Hence, $\bigcup_{x \in y \alpha z} f_A(x) \cap N \subseteq f_A(z) \cup M$ for any $x \in y \circ z$. Hence, f_A is an (M, N)-union soft left hyperideal of S over U $\square$

Similarly we can prove the following theorem.

Theorem 4.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and f_A be a soft set of S over U. Then f_A is an (M, N)-union soft right hyperideal of S over U if and only if f_A satisfies the following conditions:*
(1) $f_A \tilde{\diamond} \emptyset_S \tilde{\supseteq}_{[M,N]} f_A$;
(2) $(\forall x, y \in S)\ x \leq y \Rightarrow f_A(x) \cap N \subseteq f_A(y) \cup M$.

5. (M, N)-Union Soft Interior Hyperideals

In this section, we introduce the notion of (M, N)-union soft interior hyperideal of ordered semihypergroups and will study some related properties.

Definition 5.1. Let f_A be a soft set of an ordered semihypergroup S over U. Then f_A is called an (M, N)-union soft interior hyperideal of S over U if it satisfies the following conditions:

(1) $(\forall x, y \in S) \left(\bigcup_{\alpha \in x \circ y} f_A(\alpha) \right) \cap N \subseteq f_A(x) \cup f_A(y) \cup M$;
(2) $(\forall x, a, y \in S) \left(\bigcup_{\alpha \in x \circ a \circ y} f_A(\alpha) \right) \cap N \subseteq f_A(a) \cup M$;
(3) $(\forall x, y \in S)\, x \leq y \Rightarrow f_A(x) \cap N \subseteq f_A(y) \cup M$.

Example 5.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d	e
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$

,

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e)\}.$$

Let $U = \{1,2,3\}$, $A = \{c,d,e\}$, $M = \{2\}$ and $N = \{1,2\}$. The soft set f_A is defined by

$$f_A = \begin{cases} \emptyset, & \text{if } x \in \{a,b\}, \\ U, & \text{if } x \in \{c,d,e\}. \end{cases}$$

Then f_A is an (M, N)-union soft interior hyperideal of S over U.

Theorem 5.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and A be a nonempty subset of S. Then A is an interior hyperideal of S if and only if the soft set χ_A^c of A is an (M, N)-union soft interior hyperideal of S over U.*

Proof. Suppose that A is an interior hyperideal of S. Let x, y and a be any elements of S. Then $\left(\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\chi_A^c(a)\cup M$. Indeed, if $a\in A$, then $\chi_A^c(a)=\emptyset$. Since A is an interior hyperideal of S, we have $\alpha\in x\circ a\circ y\subseteq S\circ A\circ S\subseteq A$ we have $\chi_A^c(\alpha)=\emptyset$ and $\emptyset\subseteq M\subset N\subseteq U$. Thus, $\left(\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\right)\cap N=\emptyset\subseteq\chi_A^c(a)\cup M$. If $a\notin A$, then $\chi_A^c(a)=U$. Since $\chi_A^c(x)\subseteq U$ for all $x\in S$, thus, $\left(\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq U=\chi_A^c(a)\cup M$. Let $x,y\in S$ with $x\leq y$. Then $\chi_A^c(x)\cap N\subseteq\chi_A^c(y)\cup M$. Indeed, if $y\notin A$, then $\chi_A^c(y)=U$ and $\emptyset\subseteq M\subset N\subseteq U$ so $\chi_A^c(x)\cap N\subseteq U=\chi_A^c(y)\cup M$. If $y\in A$ then $\chi_A^c(y)=\emptyset$. Since $x\leq y$ and A is an interior hyperideal of S, we have $x\in A$ and thus $\chi_A^c(x)\cap N=\emptyset\subseteq\chi_A^c(y)\cup M$. Since A is an interior hyperideal of S., we have, A is a subsemihypergroup of S. Let $x,y\in S$. Then we have $\left(\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\chi_A^c(x)\cup\chi_A^c(y)\cup M$. Indeed, if $x\circ y\nsubseteq A$, then there exists $\alpha\in x\circ y$ such that $\alpha\notin A$, and we have $\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)=U$. Besides that $x\circ y\nsubseteq A$ implies that $x\notin A$ or $y\notin A$. Then $\chi_A^c(x)=U$ or $\chi_A^c(y)=U$ and hence $\left(\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq U=\chi_A^c(x)\cup\chi_A^c(y)\cup M$. Let $x\circ y\subseteq A$. Then $\chi_A^c(\alpha)=\emptyset$ for any $\alpha\in x\circ y$. It implies that $\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)=\emptyset$. Since we have $\chi_A^c(x)\supseteq\emptyset$ for any $x\in A$, it follows, $\left(\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\right)\cap N=\emptyset\subseteq\chi_A^c(x)\cup\chi_A^c(y)\cup M$. Therefore, χ_A^c is an (M,N)-union soft interior hyperideal of S over U.

Conversely, let $\emptyset\neq A\subseteq S$ such that χ_A^c is an (M,N)-union soft interior hyperideal of S over U. We claim that $A\circ A\subseteq A$. To prove the claim, let $x,y\in A$. By hypothesis, $\left(\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\chi_A^c(x)\cup\chi_A^c(y)\cup M$, which implies that $\left(\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\emptyset\cap\emptyset\cup M=M$. Thus by $\emptyset\subseteq M\subset N\subseteq U$, $\bigcup_{\alpha\in x\circ y}\chi_A^c(\alpha)\cap N\subseteq M$. Thus for any $\alpha\in x\circ y$, $\chi_A^c(\alpha)=\emptyset$ implies that $\alpha\in A$. It thus follows that $A\circ A\subseteq A$. Let $\alpha\in S\circ A\circ S$, then there exist $x,y\in S$ and $a\in A$ such that $\alpha\in x\circ a\circ y$. Since $\left(\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\chi_A^c(a)\cup M$, and $a\in A$ we have $\chi_A^c(a)=\emptyset$. Hence for each $\alpha\in S\circ A\circ S$, we have $\left(\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\right)\cap N\subseteq\emptyset\cup M=M$. Thus, by $\emptyset\subseteq M\subset N\subseteq U$, $\bigcup_{\alpha\in x\circ a\circ y}\chi_A^c(\alpha)\cap N\subseteq M$. Thus, for any $\alpha\in x\circ a\circ y$, $\chi_A^c(\alpha)=\emptyset$

implies that $\alpha \in A$. Thus $S \circ A \circ S \subseteq A$. Furthermore, let $x \in A$, $S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $\chi_A^c(y) = \emptyset$. By $x \in A$ we have $\chi_A^c(x) = \emptyset$. Since χ_A^c is an (MN)-union soft interior hyperideal of S over U and $y \leq x$, we have $\chi_A^c(y) \cap N \subseteq \chi_A^c(x) \cup M = \emptyset \cup M = M$. Notice that $\emptyset \subseteq M \subset N \subseteq U$, we conclude that $\chi_A^c(y) = \emptyset$. Hence $y \in A$. Therefore A is a interior hyperideal of S. □

Theorem 5.2. *Let f_A be a soft set of an ordered semihypergroup S over U and $\delta \in P(U)$. Then f_A is an (M,N)-union soft interior hyperideal of S over U if and only if each nonempty δ-exclusive set $e_A(f_A,\delta)$ of f_A is an interior hyperideal of S and $M \subset \delta \subseteq N$.*

Proof. Assume that f_A is an (M,N)-union soft interior hyperideal of S over U. Let $M \subset \delta \subseteq N$ and $e_A(f_A,\delta) \neq \emptyset$. Let $x, y \in e_A(f_A,\delta)$. Then $f_A(x) \subseteq \delta$ and $f_A(y) \subseteq \delta$. By hypothesis, we have $\left(\bigcup_{\alpha \in x \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(x) \cup f_A(y) \cup M \subseteq \delta \cup \delta \cup M = \delta$. Since $M \subset \delta \subseteq N$, we can write as $\bigcup_{\alpha \in x \circ y} f_A(\alpha) \subseteq \delta$. Thus for any $\alpha \in x \circ y$, we have $f_A(\alpha) \subseteq \delta$, implies that $\alpha \in e_A(f_A,\delta)$. It follows that $x \circ y \subseteq e_A(f_A,\delta)$. Hence $e_A(f_A,\delta)$ is a subsemihypergroup of S. Let $y \in e_A(f_A,\delta)$ and $x, z \in S$. Then $f_A(y) \subseteq \delta$. Since f_A is an (M,N)-union soft interior hyperideal of S over U. Thus, $\left(\bigcup_{w \in x \circ y \circ z} f_A(w)\right) \cap N \subseteq f_A(y) \cup M \subseteq \delta \cup M = \delta$. Since $\emptyset \subseteq M \subset \delta \subseteq N \subseteq U$, we can write as $\bigcup_{w \in x \circ y \circ z} f_A(w) \subseteq \delta$. Hence, $f_A(w) \subseteq \delta$ for any $w \in x \circ y \circ z$ implies that $w \in e_A(f_A,\delta)$. Thus, $S \circ e_A(f_A,\delta) \circ S \subseteq e_A(f_A,\delta)$. Furthermore, let $x \in e_A(f_A,\delta)$, $S \ni y \leq x$. Then $y \in e_A(f_A,\delta)$. Indeed, since $x \in e_A(f_A,\delta)$, $f_A(x) \subseteq \delta$ and f_A is an (M,N)-union soft interior hyperideal of S over U, we have $f_A(y) \cap N \subseteq f_A(x) \cup M \subseteq \delta \cup M = \delta$. By $M \subset \delta \subseteq N$, we have $f_A(y) \subseteq \delta$, i.e., $y \in e_A(f_A,\delta)$. Therefore, $e_A(f_A,\delta)$ is an interior hyperideal of S.

Conversely, suppose that $e_A(f_A,\delta) \neq \emptyset$ is an interior hyperideal of S for all $M \subset \delta \subseteq N$. If there exist $x_1, y_1 \in S$ such that $\left(\bigcup_{\alpha \in x_1 \circ y_1} f_A(\alpha)\right) \cap N \supset f_A(x_1) \cup f_A(y_1) \cup M$, then there exists $M \subset \delta \subseteq N$ such that $\left(\bigcup_{\alpha \in x_1 \circ y_1} f_A(\alpha)\right) \cap N \supset \delta \supseteq f_A(x_1) \cup f_A(y_1) \cup M$, and we have $f_A(x_1) \subseteq \delta$, $f_A(y_1) \subseteq \delta$ and $\bigcup_{\alpha \in x_1 \circ y_1} f_A(\alpha) \supset \delta$ which implies that $x_1, y_1 \in e_A(f_A,\delta)$ and $x_1 \circ y_1 \nsubseteq e_A(f_A,\delta)$. It contradicts the fact that $e_A(f_A,\delta)$ is an interior hyperideal of S. Consequently, $\left(\bigcup_{\alpha \in x \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(x) \cup f_A(y) \cup M$ for all $x, y \in S$. Next we show that $\left(\bigcup_{\alpha \in x \circ a \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(a) \cup M$ for all $x, a, y \in S$. If there exist

x_1, a_1, y_1 such that $\left(\bigcup_{\alpha\in x_1\circ a_1\circ y_1} f_A(\alpha)\right)\cap N \supset f_A(a_1)\cup M$, and $M\subset\delta\subseteq N$ such that $\left(\bigcup_{\alpha\in x_1\circ a_1\circ y_1} f_A(\alpha)\right)\cap N \supset \delta \supseteq f_A(a_1)\cup M$, so $f_A(a_1)\subseteq\delta$ and $\bigcup_{\alpha\in x_1\circ a_1\circ y_1} f_A(\alpha)\supset\delta$ then $a_1\in e_A(f_A,\delta)$ and $x_1\circ a_1\circ y_1 \nsubseteq e_A(f_A,\delta)$. This is a contradiction that $e_A(f_A,\delta)$ is an interior hyperideal of S. Moreover if $x\le y$, then $f_A(x)\cap N\subseteq f_A(y)\cup M$. Indeed, if there exist $x_1, y_1\in S$ such that $x_1\le y_1$ and $f_A(x_1)\cap N\supset f_A(y_1)\cup M$, then there exists $M\subset\delta\subseteq N$ such that $f_A(x_1)\cap N\supset\delta\supseteq f_A(y_1)\cup M$ and we have $f_A(y_1)\subseteq\delta$ and $f_A(x_1)\supset\delta$. Then $y_1\in e_A(f_A,\delta)$ and $x_1\notin e_A(f_A,\delta)$. This is a contradiction that $e_A(f_A,\delta)$ is an interior hyperideal of S. Thus if $x\le y$ then $f_A(x)\cap N\subseteq f_A(y)\cup M$. □

Theorem 5.3. *Let $(S,\circ,\le)$ be an ordered semihypergroup and f_A be an (M,N)-union soft hyperideal of S over U. Then f_A is an (M,N)-union soft interior hyperideal of S over U.*

Proof. Suppose that f_A is an (M,N)-union soft hyperideal of S over U. Let $x,y\in S$. Then by hypothesis $\left(\bigcup_{\alpha\in x\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(x)\cup M\subseteq f_A(x)\cup f_A(y)\cup M$. Let $x,a,y\in S$. Since f_A is an (M,N)-union soft hyperideal of S over U, then for any $\alpha\in x\circ a\circ y$, and $\emptyset\subseteq M\subset N\subseteq U$ we have

$$\begin{aligned}
\left(\bigcup_{\alpha\in x\circ a\circ y} f_A(\alpha)\right)\cap N &= \left(\left(\bigcup_{\alpha\in x\circ a\circ y} f_A(\alpha)\right)\cap N\right)\cap N\\
&= \left(\left(\bigcup_{\substack{\alpha\in x\circ\beta\\ \beta\in a\circ y}} f_A(\alpha)\right)\cap N\right)\cap N\\
&\subseteq (f_A(\beta)\cup M)\cap N\\
&= (f_A(\beta)\cap N)\cup(N\cap M) = (f_A(\beta)\cap N)\cup M\\
&\subseteq \left(\left(\bigcup_{\beta\in a\circ y} f_A(\beta)\right)\cap N\right)\cup M\\
&\subseteq (f_A(a)\cup M)\cup M\\
&= f_A(a)\cup M.
\end{aligned}$$

Thus,

$$\left(\bigcup_{\alpha\in x\circ a\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(a)\cup M.$$

Therefore, f_A is an (M,N)-union soft interior hyperideal of S over U. □

The converse of above theorem is not true in general. We can illustrate it by the following example.

Example 5.2. Let $(S,\circ,\leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	v_1	v_2	v_3	v_4
v_1	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$
v_2	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$
v_3	$\{v_1\}$	$\{v_1\}$	$\{v_1,v_2\}$	$\{v_1,v_2\}$
v_4	$\{v_1\}$	$\{v_1\}$	$\{v_1,v_2\}$	$\{v_1\}$

,

$\leq:=\{(v_1,v_1),(v_2,v_2),(v_3,v_3),(v_4,v_4),(v_1,v_2),(v_1,v_3),(v_1,v_4),(v_4,v_2),(v_4,v_3)\}$.

Suppose $U=\{x,y,z\}$, $A=\{v_2,v_3\}$, $M=\{y\}$ and $N=\{y,z\}$. Let us define $f_A(v_1)=\emptyset$, $f_A(v_2)=\{x,z\}$, $f_A(v_3)=\{x,y,z\}$ and $f_A(v_4)=\emptyset$. Then f_A is an (M,N)-union soft interior hyperideal of S over U. This is not an (M,N)-union soft left hyperideal as

$$\bigcup_{\alpha\in v_3\circ v_4=\{v_1,v_2\}} f_A(\alpha)\cap N=f_A(v_1)\cup f_A(v_2)\cap N=\{z\}\nsubseteq\emptyset\cup\{y\}=\{y\}=f_A(v_4)\cup M.$$

Theorem 5.4. *Let $(S,\circ,\leq)$ be a regular ordered semihypergroup and f_A is an (M,N)-union soft interior hyperideal of S over U. Then f_A is an (M,N)-union soft hyperideal of S over U.*

Proof. Let $x,y\in S$. Since f_A is an (M,N)-union soft interior hyperideal of S over U, then $\left(\bigcup_{\alpha\in x\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(x)\cup M$. Indeed, since S is regular and $x\in S$, then there exists $z\in S$ such that $x\leq x\circ z\circ x$. Then we have $x\circ y\leq(x\circ z\circ x)\circ y=(x\circ z)\circ(x\circ y)$. So, there exist $\alpha\in x\circ y$, $v\in x\circ z$ and $\beta\in v\circ x\circ y$ such that $\alpha\leq\beta$. So $f_A(\alpha)\cap N\subseteq f_A(\beta)\cup M$. Since f_A is an (M,N)-union soft interior hyperideal of S over U, and $\emptyset\subseteq M\subset N\subseteq U$, we have

$$\begin{aligned}
f_A(\alpha)\cap N =&(f_A(\alpha)\cap N)\cap N\\
\subseteq&(f_A(\beta)\cup M)\cap N\\
=&(f_A(\beta)\cap N)\cup(N\cap M)=(f_A(\beta)\cap N)\cup M\\
\subseteq&\left(\left(\bigcup_{\beta\in v\circ x\circ y} f_A(\beta)\right)\cap N\right)\cup M\subseteq(f_A(x)\cup M)\cup M\\
=&f_A(x)\cup M.
\end{aligned}$$

Thus,

$$\left(\bigcup_{\alpha\in x\circ y} f_A(\alpha)\right)\cap N\subseteq f_A(x)\cup M.$$

Therefore f_A is an (M,N)-union soft right hyperideal of S over U. In a similar way we prove that f_A is an (M,N)-union soft left hyperideal of S over U. □

By Theorem 5.3 and 5.4 we have the following.

Theorem 5.5. *In regular ordered semihypergroups the concepts of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals coincide.*

Theorem 5.6. *Let $(S,\circ,\leq)$ be an intra-regular ordered semihypergroup and f_A is an (M,N)-union soft interior hyperideal of S over U. Then f_A is an (M,N)-union soft hyperideal of S over U.*

Proof. Let $a,b\in S$. Then $\left(\bigcup\limits_{u\in a\circ b} f_A(u)\right)\cap N\subseteq f_A(a)\cup M$. Indeed, since S is intra-regular and $a\in S$, there exist $x,y\in S$ such that $a\leq x\circ a\circ a\circ y$. Then $a\circ b\leq (x\circ a\circ a\circ y)\circ b = x\circ a\circ(a\circ y\circ b)$. So there exist $u\in a\circ b$, $v\in a\circ y\circ b$ and $\alpha\in x\circ a\circ v$ such that $u\leq\alpha$. So $f_A(u)\cap N\subseteq f_A(\alpha)\cup M$. Since f_A is an (M,N)-union soft interior hyperideal of S over U, we have

$$\begin{aligned}
f_A(u)\cap N =&(f_A(u)\cap N)\cap N\\
\subseteq&(f_A(\alpha)\cup M)\cap N\\
=&(f_A(\alpha)\cap N)\cup(N\cap M) = (f_A(\alpha)\cap N)\cup M\\
\subseteq&\left(\left(\bigcup_{\alpha\in x\circ a\circ v} f_A(\alpha)\right)\cap N\right)\cup M\subseteq (f_A(a)\cup M)\cup M\\
=&f_A(a)\cup M.
\end{aligned}$$

Thus,

$$\left(\bigcup_{u\in a\circ b} f_A(u)\right)\cap N\subseteq f_A(a)\cup M.$$

Hence, f_A is an (M,N)-union soft right hyperideal of S over U. Similarly we can prove that f_A is an (M,N)-union soft left hyperideal of S over U. Therefore, f_A is an (M,N)-union soft hyperideal of S over U. □

By Theorem 5.3 and 5.6, we have the following.

Theorem 5.7. *In intra-regular ordered semihypergroups the concepts of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals coincide.*

6. Characterizations of (M,N)-Union Soft Simple Ordered Semihypergroups in Terms of (M,N)-Union Soft Hyperideals and (M,N)-Union Soft Interior Hyperideals

In this section, we introduce the concept of (M,N)-union soft simple ordered semihypergroups and characterize this type of ordered semihypergroups in terms of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals.

Definition 6.1 (see [16])**.** An ordered semihypergroup $(S,\circ,\leq)$ is called simple if it has no a proper hyperideal, that is for any hyperideal $A\neq\emptyset$ of S we have $A=S$.

Lemma 6.1 (see [16]). *An ordered semihypergroup $(S, \circ, \leq)$ is a simple ordered semihypergroup if and only if for every $a \in S$, $(S \circ a \circ S] = S$.*

Definition 6.2. An ordered semihypergroup $(S, \circ, \leq)$ is called (M, N)-union soft simple if for any (M, N)-union soft hyperideal f_A of S over U, we have $f_A(a) \cap N \subseteq f_A(b) \cup M$ for all $a, b \in S$.

Theorem 6.1. *Let be $(S, \circ, \leq)$ an ordered semihypergroup. Then S is (M, N)-union soft simple if and only if for any (M, N)-union soft hyperideal f_A of S over U, we have $e_A(f_A, \delta) = S$ for all $\emptyset \subseteq M \subset \delta \subseteq N \subseteq U$ if $e_A(f_A, \delta) \neq \emptyset$.*

Proof. Suppose that S is an (M, N)-union soft simple ordered semihypergroup and f_A is an (M, N)-union soft hyperideal of S over U. Let $M \subset \delta \subseteq N$ be such that $e_A(f_A, \delta) \neq \emptyset$. We need to prove that $x \in e_A(f_A, \delta)$ for all $x \in S$. Since $e_A(f_A, \delta) \neq \emptyset$, we can suppose that there exits $y \in e_A(f_A, \delta)$, i.e., $f_A(y) \subseteq \delta$. Hence $f_A(x) \cap N \subseteq f_A(y) \cup M \subseteq \delta \cup M = \delta$. Since $M \subset \delta$, we can conclude that $f_A(x) \subseteq \delta$, which implies that $x \in e_A(f_A, \delta)$.

Conversely, for any (M, N)-union soft hyperideal f_A of S over U, suppose that $e_A(f_A, \delta) = S$ for all $\emptyset \subseteq M \subset \delta \subseteq N \subseteq U$ if $e_A(f_A, \delta) \neq \emptyset$. We claim that $f_A(a) \cap N \subseteq f_A(b) \cup M$ for all $a, b \in S$. If there exist $x, y \in S$ such that $f_A(x) \cap N \supset f_A(y) \cup M$, then we have $f_A(x) \cap N \supset \delta \supseteq f_A(y) \cup M$ for some $M \subset \delta \subseteq N$. Thus, $f_A(x) \supset \delta$, i.e., $x \notin e_A(f_A, \delta) = S$, which is a contradiction. Therefore $f_A(a) \cap N \subseteq f_A(b) \cup M$ holds for all $a, b \in S$. Thus, S is (M, N)-union soft simple. $\square$

Let $(S, \circ, \leq)$ be an ordered semihypergroup and $a \in S$, and f_A be a soft set of S over U we denote by I_a the subset of S defines as follows:

$$I_a = \{b \in S \mid f_A(b) \cap N \subseteq f_A(a) \cup M\}.$$

Clearly $I_a \neq \emptyset$, since $a \in I_a$.

Theorem 6.2. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and f_A is an (M, N)-union soft left hyperideals of S over U. Then the set I_a is a left hyperideal of S for every $a \in S$.*

Proof. Suppose that f_A is an (M, N)-union soft left hyperideals of S over U. Let $b \in I_a$ and $s \in S$. Then $s \circ b \subseteq I_a$. Indeed, since f_A is an (M, N)-union soft left hyperideal of S over U and $b, s \in S$, we have $\left(\bigcup_{\alpha \in s o b} f_A(\alpha)\right) \cap N \subseteq f_A(b) \cup M$. Since $b \in I_a$, we have $f_A(b) \cap N \subseteq f_A(a) \cup M$. Thus,

$$\begin{aligned} f_A(\alpha) \cap N =& (f_A(\alpha) \cap N) \cap N \\ \subseteq& \left(\left(\bigcup_{\alpha \in s o b} f_A(\alpha)\right) \cap N\right) \cap N \subseteq (f_A(b) \cup M) \cap N \\ =& (f_A(b) \cap N) \cup (M \cap N) \end{aligned}$$

$$\begin{aligned}&\subseteq (f_A(a) \cup M) \cup M\\&= f_A(a) \cup M.\end{aligned}$$

Thus, $\alpha \in I_a$ and hence $s \circ b \subseteq I_a$. Let $b \in I_a$ and $S \ni s \leq b$. Then $s \in I_a$. Indeed, since f_A is an (M, N)-union soft left hyperideals of S over U, $b, s \in S$ and $s \leq b$, we have $f_A(s) \cap N \subseteq f_A(b) \cup M$. Since $b \in I_a$, we have $f_A(b) \cap N \subseteq f_A(a) \cup M$. Then $f_A(s) \cap N \subseteq f_A(a) \cup M$, so $s \in I_a$. □

In a similar way we prove the following.

Theorem 6.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and f_A is an (M, N)-union soft right hyperideals of S over U. Then the set I_a is a right hyperideal of S for every $a \in S$.*

By Theorem 6.2 and 6.3 we have the following.

Theorem 6.4. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and f_A is an (M, N)-union soft hyperideals of S over U. Then the set I_a is a hyperideal of S for every $a \in S$.*

Theorem 6.5. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then S is simple if and only if it is (M, N)-union soft simple.*

Proof. Assume that S is a simple ordered semihypergroup. Let f_A is an (M, N)-union soft hyperideal of S over U and $a, b \in S$. By Theorem 6.4, we obtain I_a is a hyperideal of S. Since S is simple, $I_a = S$. Then $b \in I_a$, that is $f_A(b) \cap N \subseteq f_A(a) \cup M$. Therefore, S is (M, N)-union soft simple.

Conversely, suppose that S is (M, N)-union soft simple. Let I be a hyperideal of S. By Corollary 4.1, we obtain the characteristic function χ_I^c is an (M, N)-union soft hyperideal of S over U. We claim that $I = S$. To prove our claim, let $x \in S$. Since S is (M, N)-union soft simple, $\chi_I^c(x) \cap N \subseteq \chi_I^c(y) \cup M$ for all $y \in S$. Since $I \neq \emptyset$, let $a \in I$. Then $\chi_I^c(x) \cap N \subseteq \chi_I^c(a) \cup M = \emptyset \cup M = M$. So, $\chi_I^c(x) \cap N \subseteq M$. Since $M \subset N$, we conclude that $\chi_I^c(x) = \emptyset$, i.e., $x \in I$. Thus, we have shown that $S \subseteq I$, and so, $S = I$. Hence, S is simple. □

Theorem 6.6. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then S is a simple if and only if for every (M, N)-union soft interior hyperideal f_A of S over U, we have $f_A(a) \cap N \subseteq f_A(b) \cup M$ for all $a, b \in S$.*

Proof. Suppose that S is a simple ordered semihypergroup. Let f_A be an (M, N)-union soft interior hyperideal of S over U and $a, b \in S$. By Lemma 6.1, we have $S = (S \circ b \circ S]$. Thus by $a \in S$, we have $a \in (S \circ b \circ S]$. Then there exist $x, y \in S$ such that $a \leq x \circ b \circ y$. Then $a \leq \alpha$ for some $\alpha \in x \circ b \circ y$. Since f_A is an (M, N)-union soft interior hyperideal of S over U, we have $f_A(a) \cap N \subseteq f_A(\alpha) \cup M$. Also since $\left(\bigcup_{\alpha \in x \circ b \circ y} f_A(\alpha)\right) \cap N \subseteq f_A(b) \cup M$. Thus,

$$f_A(a) \cap N = (f_A(a) \cap N) \cap N$$

$$
\begin{aligned}
&\subseteq\left(f_A(\alpha)\cup M\right)\cap N\\
&=\left(f_A(\alpha)\cap N\right)\cup\left(M\cap N\right)=\left(f_A(\alpha)\cap N\right)\cup M\\
&\subseteq\left(\left(\bigcup_{\alpha\in x\circ b\circ y}f_A(\alpha)\right)\cap N\right)\cup M\subseteq\left(f_A(b)\cup M\right)\cup M\\
&=f_A(b)\cup M.
\end{aligned}
$$

Conversely, assume that for every (M,N)-union soft interior hyperideal f_A of S over U, we have $f_A(a)\cap N\subseteq f_A(b)\cup M$ for all $a,b\in S$. Let f_A be any (M,N)-union soft hyperideal of S over U. Then by Theorem 5.3, f_A is an (M,N)-union soft interior hyperideal of S over U. Hence S is (M,N)-union soft simple by Definition 6.2. It thus follows from Theorem 6.5 that S is a simple ordered semihypergroup. □

As a consequence of Lemma 6.1, Theorem 6.5, and Theorem 6.6, we present characterizations of a simple ordered semihypergroup as the following theorem.

Theorem 6.7. *Let $(S,\circ,\leq)$ be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is a simple ordered semihypergroup;*

(2) *$S=(S\circ a\circ S]$ for every $a\in S$;*

(3) *S is (M,N)-union soft simple;*

(4) *for every (M,N)-union soft interior hyperideal of S over U, we have $f_A(a)\cap N\subseteq f_A(b)\cup M$ for all $a,b\in S$.*

7. Conclusion

Ideal theory play a vital role in hyperstructures, in this paper, we introduced the notions of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals of ordered semihypergroups and studied them. When $M=\emptyset$ and $N=U$, we meet union soft hyperideals and union soft interior hyperideals. From this view, we say that (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals are more general concepts than ordinary union soft ones. Moreover we introduced the notion of (M,N)-union soft simple ordered semihypergroup. We characterized (M,N)-union soft simple ordered semihypergroups by means of (M,N)-union soft hyperideals and (M,N)-union soft interior hyperideals. Hopefully that the obtained new characterizations of ordered semihypergroup in terms of (M,N)-union soft hyperideals will be very useful for future study of ordered semihypergroups. In future we will define other (M,N)-union soft hyperideals of ordered semihypergroups and will study their applications.

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POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH LIDSTONE LIKE BOUNDARY CONDITIONS

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Abstract. We consider a higher order fractional boundary value problem with Lidstone like boundary conditions, where the nonlinearity is an L^1-Carathèodory function. We first consider the lower order problem. Then, by using a convolution to construct the Green's function for the higher order problem, we are able to apply a recent fixed point theorem to show the existence of positive solutions of the boundary value problem.

1. Introduction

Let $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$ and $1 \leq \beta \leq n-1$. We study existence and nonexistence of solutions of the fractional differential equation

$$D^{\alpha}_{0+}u + f(t,u) = 0, \quad t \in (0,1), \tag{1.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-2, \quad D^{\beta}_{0+}u(1) = 0, \tag{1.2}$$

where D^{α}_{0+} and D^{β}_{0+} are the standard Riemann-Liouville derivatives. Here $f : (0,1) \times [0,\infty) \to [0,\infty)$ is an L^1-Carathèodory function, i.e., f satisfies the following properties:

(a) $f(\cdot, u)$ is a measurable function for all $u \geq 0$;
(b) $f(t, \cdot)$ is continuous for a.e. $t \in (0,1)$ and

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(c) for all $r > 0$ there exists a $\psi_r \in L^1[0,1]$ such that $|f(t,u)| \leq \psi_r(t)$ for a.e. $t \in (0,1)$ and for all $|u| \leq r$.

We then consider a higher order problem with boundary conditions inspired by Lidstone boundary conditions. Let $m \in \mathbb{N}$, $m \geq 3$, $n \in \mathbb{N}$, $2n-1+m < \gamma \leq 2n+m$, $1 \leq \beta \leq n-1$ and consider the boundary value problem

$$D_{0+}^{\gamma} u(t) + (-1)^n g(t,u) = 0, \quad 0 < t < 1, \tag{1.3}$$

satisfying the boundary conditions

$$\begin{aligned} u^{(i)}(0) = 0, \quad i = 0,1,\ldots,m-2, \quad D_{0+}^{\beta} u(1) = 0, \\ D_{0+}^{\gamma-2l} u(0) = D_{0+}^{\gamma-2l} u(1) = 0, \quad l = 1,\ldots,n-1, \end{aligned} \tag{1.4}$$

where $g : (0,1) \times [0,\infty) \to [0,\infty)$ is an L^1-Carathèodory function. To construct the Green's function for this problem, we use a convolution. The Green's function for the higher order problem therefore inherits properties of the Green's function corresponding to (1.1), (1.2) and similar arguments can be made to show the existence of positive solutions of the boundary value problem.

Fixed point theory has been used extensively to study the existence of positive solutions of fractional boundary value problems [2, 7, 8, 10–12, 20, 23, 25] and singular fractional boundary value problems [1, 9, 14, 16, 18, 21, 22, 24, 26] where the nonlinearity may be singular at $t = 0$ or $t = 1$. Of particular interest to this work is the recent paper by Benmezaï, Chentout and Henderson [3], where the authors prove a new fixed point theorem using strongly positive-like operators and then apply their fixed point theorem to a fractional boundary value problem. The use of convolution to construct Green's functions for higher order problems can be found first in [6]. In [15], the authors used convolution to study positive solutions of some different higher order fractional boundary value problems.

2. Preliminaries

We start with the definition of the Riemann-Liouville fractional integral and fractional derivative.

Definition 2.1. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0+}^{\nu} u$, is defined as

$$I_{0+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s) ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = D^n I_{0+}^{n-\alpha} u(t),$$

provided the right-hand side exists. We refer to [4,13,17,19] for a more in depth study of fractional calculus and fractional differential equations.

Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$ and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Cones generate a natural partial ordering on a Banach space. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \preceq v$ if $v - u \in \mathcal{P}$, $u \prec v$ if $v - u \in \mathcal{P}$, $u \neq v$, and $u \npreceq v$ if $v - u \notin \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are continuous mappings, $M \preceq N$ if for all $u \in \mathcal{P}$, $Mu \preceq Nu$. The relations $N \prec M$ and $N \npreceq M$ are defined similarly. The notation $\succeq$, $\succ$ and $\nsucceq$ define the reverse situations.

Definition 2.2. An operator $L \in L_C(\mathcal{B})$, where $L_C(\mathcal{B})$ is the set of all linear compact self-mappings of B, is said to be positive if $L : \mathcal{P} \to \mathcal{P}$ and strongly positive if $\mathcal{P}^\circ \neq \emptyset$ and $L : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$.

Definition 2.3. Let $L \in L_C(\mathcal{B})$ be positive. L is said to be lower bounded if $\inf\{\|Lu\| : u \in \mathcal{P} \cap \partial B(0,1)\} > 0$.

For all positive operators $L \in L_C(\mathcal{B})$, define the subsets

$$\Lambda_L = \{\lambda \geq 0 : \text{ there exists } u \succ 0_{\mathcal{B}} \text{ such that } Lu \succeq \lambda u\}$$

and

$$\Gamma_L = \{\lambda \geq 0 : \text{ there exists } u \succ 0_{\mathcal{B}} \text{ such that } Lu \preceq \lambda u\}.$$

The proof of the following lemma can be found in [3].

Lemma 2.1. *Let $L \in L_C(\mathcal{B})$ be strongly positive. Then*

$$r(L) = \sup \Lambda_L = \inf \Gamma_L.$$

Definition 2.4. A positive operator $L \in L_C(\mathcal{B})$ is said to be a strong positive-like operator if $r(L) = \sup \Lambda_L = \inf \Gamma_L > 0$.

The following two theorems are the model for which our main result is based. The proofs can be found in the work of Benmezai, Chentout, and Henderson [3]. The first deals with nonexistence of positive fixed points and the second with existence of positive fixed points.

Theorem 2.1. *Let $T : \mathcal{P} \to \mathcal{P}$ be a continuous mapping and let $L \in L_C(B)$ be a strongly positive-like operator. If either*

$$r(L) > 1 \quad \text{and} \quad Tu \succeq Lu, \quad \text{for all } u \in \mathcal{P},$$

or

$$r(L) < 1 \quad \text{and} \quad Tu \preceq Lu, \quad \text{for all } u \in \mathcal{P},$$

then T has no fixed points in $\mathcal{P}$.

Theorem 2.2. *Let $T : \mathcal{P} \to \mathcal{P}$ be a completely continuous mapping and assume that there exist two strongly positive-like operators $L_1, L_2 \in L_c(\mathcal{B})$ and two functions $F_1, F_2 : \mathcal{P} \to \mathcal{P}$ such that L_1 is lower bounded on $\mathcal{P}$, $r(L_2) < 1 < r(L_1)$, and for all $u \in \mathcal{P}$*

$$L_1u - F_1u \preceq Tu \preceq L_2u + F_2u.$$

If either

$$F_1u = o(\|u\|) \quad as \quad u \to \infty \quad and \quad F_2u = o(\|u\|) \quad as \quad u \to 0,$$

or

$$F_1u = o(\|u\|) \quad as \quad u \to 0 \quad and \quad F_2u = o(\|u\|) \quad as \quad u \to \infty,$$

then T has a fixed point in $\mathcal{P}$.

3. Eigenvalue Criteria

Let $E = C[0,1]$ be the Banach space of continuous functions with the usual supremum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the Banach space X as

$$X = \left\{ u \in C[0,1] : \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} \text{ exists} \right\}$$

endowed with the norm

$$\|u\|_X = \sup_{t\in[0,1]} \left| \frac{u(t)}{t^{\alpha-1}} \right|.$$

Fix $\delta \in (0,1)$. Define the cones

$$\begin{aligned} E^+ =&\{u \in E : u(t) \geq 0 \text{ for all } t \in [0,1]\},\\ \mathcal{P} =&\{u \in E^+ : u(t) \geq \delta^{\alpha-1}\|u\|_0 \text{ for all } t \in [\delta,1]\} \end{aligned}$$

and

$$X^+ = \{u \in X : u(t) \geq 0 \text{ for all } t \in [0,1]\}.$$

Define the sets

$$\mathbb{L}^1_+ = \{m \in \mathbb{L}^1(0,1) : m(t) \geq 0 \text{ a.e. } t \in [0,1]\}$$

and

$$\mathbb{L}^1_{++} = \{m \in \mathbb{L}^1_+ : m > 0 \text{ on a subset of positive measure}\}.$$

We also introduce the subset $S \subset X$ by

$$S = \left\{ u \in X : u(t) > 0 \text{ for all } t \in (0,1] \text{ and } \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} > 0 \right\}.$$

The following theorem is given in [3].

Lemma 3.1. *S is open in X.*

The Green's function for $-D^{\alpha}_{0+}u = 0$ satisfying the boundary conditions (1.2) is given by (see, for example, [5])

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{3.1}$$

Therefore, u is a solution of (1.1), (1.2) if and only if

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds, \quad 0 \le t \le 1.$$

Define $v(t,s)$ by

$$v(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{\left(1-\frac{s}{t}\right)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1, \\ \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases}$$

Notice $G(t,s) = t^{\alpha-1}v(t,s)$. The following lemma gives sign properties of G and v.

The proof of (1)–(3) of the following lemma can be found in [15]. The proof of (4) is trivial.

Lemma 3.2. *Let G be defined as in* (3.1).

(1) $G(t,s) \in C([0,1] \times [0,1))$ *with* $G(t,s) > 0$ *for* $(t,s) \in (0,1] \times (0,1)$.
(2) $t^{\alpha-1}G(1,s) \le G(t,s) \le G(1,s)$ *for* $(t,s) \in [0,1] \times [0,1)$.
(3) $G(t,s) \ge \delta^{\alpha-1}G(1,s)$ *for all* $t \in [\delta,1]$ *and all* $s \in [0,1)$.
(4) $v(0,s) > 0$ *for all* $s \in [0,1)$.

Let $m \in \mathbb{L}^1_{++}$. Define $L_m : E \to E$ by

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds.$$

For $u \in X$, define $L^X_x : X \to E$ by $L^X_m u = L_m u$.

Lemma 3.3. *For $m \in \mathbb{L}^1_{++}$, the operator L_m is compact and positive. Moreover, $L_m : E^+ \to \mathcal{P}$.*

Proof. The proof that L_m is compact is standard. Let $u \in E^+$. Then $u(t) \ge 0$ for $t \in [0,1]$. Since $m > 0$ for a.e. $t \in [0,1]$, then by Lemma 3.2 (1),

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds \ge 0.$$

So $L_m u \in E^+$ and $L_m : E^+ \to E^+$. Furthermore, Lemma 3.2 (3) gives that

$$\|L_m u\| = |L_m u(1)|_0$$

and

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds \ge \delta^{\alpha-1}\int_0^1 G(1,s)m(s)u(s)ds = \delta^{\alpha-1}\|L_m u\|.$$

So $L_m u \in \mathcal{P}$ and $L_m : E^+ \to \mathcal{P}$. $\square$

Lemma 3.4. *For $m \in \mathbb{L}^1_{++}$, L_m is a strongly positive-like operator which is lower bounded on the cone $\mathcal{P}$.*

Proof. We start by proving that for $m \in \mathbb{L}^1_+[0,1] \cap C[0,1]$, L_m^X is a strongly positive operator. Using the Arzelà-Ascoli theorem, similar to the argument in [3], we have that L_m^X compact. Next, let $u \in X^+ \setminus \{0\}$. For all $t \in (0,1]$, by Lemma 3.2,

$$L_m^X u(t) = \int_0^1 G(t,s)m(s)u(s)ds > 0.$$

Also,

$$\lim_{t\to 0} \frac{L_m^X u(t)}{t^{\alpha-1}} = \int_0^1 v(0,s)m(s)u(s)ds > 0.$$

So $L_m^X : X \setminus \{0\} \to S \subset X^{+\circ}$. So L_m^X is strongly positive, and by Lemma 2.1,

$$r(L_m^X) = \sup \Lambda_{L_m^X} = \inf \Gamma_{L_m^X}.$$

Since L_m^X is an embedding of the operator L_m into X, $\Lambda_{L_m^X} \subset \Lambda_{L_m}$ and $\Gamma_{L_m^X} \subset \Gamma_{L_m}$. Next, let $\lambda \geq 0$ and $u \in E^+ \setminus \{0\}$ be such that $L_m u \succeq \lambda u$. Then, from an argument similar to that above, $U = L_m u \in X^+ \setminus \{0\}$. Now

$$L_m^X U = L_m^X (L_m u) = L_m (L_m u) \succeq \lambda L_m U.$$

So, $\lambda \in \Lambda_{L_m^X}$, and $\Lambda_{L_m^X} = \Lambda_{L_m}$. Similarly, $\Gamma_{L_m^X} = \Gamma_{L_m}$. So,

$$r(L_m) = \sup \Lambda_{L_m} = \inf \Gamma_{L_m}.$$

So, L_m is a strongly positive-like operator.

Finally, for $u \in \mathcal{P}$,

$$\|L_m u\| = L_m u(1) = \int_0^1 G(1,s)m(s)u(s)ds \geq \delta^{\alpha-1} \int_0^1 G(1,s)m(s)\delta^{\alpha-1}ds\|u\|.$$

So L_m is lower bounded on the cone $\mathcal{P}$. □

4. Existence and Nonexistence Results

Define the operator $T : E^+ \to E$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Notice that u is a solution of the boundary value problem (1.1), (1.2) if and only if u is a fixed point of T.

We have the following lemma.

Lemma 4.1. *$T : E^+ \to E$ is compact and $T : E^+ \to \mathcal{P}$.*

Proof. The fact that T is compact is a standard application of the Arzela-Ascoli theorem. Next, let $u \in E^+$. Then by Lemmma 3.2 (1) and (3),

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq 0,$$

and, since $\|Tu\| = Tu(1)$,

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \delta^{\alpha-1} \int_0^1 G(1,s)f(s,u(s))ds = \delta^{\alpha-1}\|u\|.$$

So, $T : E^+ \to \mathcal{P}$. □

Let $m \in \mathbb{L}^1_{++}$. Consider the linear boundary value problem

$$D^{\alpha}_{0^+}u(t) + \mu m(t)u(t) = 0, \quad \text{a.e. } t \in (0,1), \tag{4.1}$$

satisfying the boundary conditions (1.2), where μ is a real parameter.

Lemma 4.2. *For all $m \in \mathbb{L}^1_{++}$, (4.1), (1.2) admit a unique positive eigenvalue $\mu_\alpha(m)$.*

Proof. Now (μ, u) is a solution of (4.1), (1.2) if and only if $L_m u = \mu^{-1}u$. Lemma 3.4 gives that $\mu^{-1} = r(L_m)$ is the unique positive eigenvalue of L_m. Thus, $\mu_\alpha(m) = 1/r(L_m)$ is the unique positive eigenvalue of (4.1), (1.2). □

Theorem 4.1. *Assume that there exists $m \in \mathbb{L}^1_+$ such that one of the following hypotheses is satisfied:*

$$\mu_\alpha(m) < 1 \quad \textit{and} \quad f(t,u) \geq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{4.2}$$

$$\mu_\alpha(m) > 1 \quad \textit{and} \quad f(t,u) \leq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{4.3}$$

Then (1.1), (1.2) has no positive solutions.

Proof. Let $u \in \mathcal{P}$, and suppose (4.2) holds. Then $f(t,u) \geq m(t)u$, which implies $Tu \succeq L_m u$. But L_m is a strongly positive-like operator with $r(L_m) = 1/\mu_\alpha(m) > 1$. Theorem 2.1 is therefore satisfied and T has no positive fixed points. A similar argument can be made if (4.3) holds. □

Theorem 4.2. *Assume that there exist $m_1, m_2 \in \mathbb{L}^1_{++}$, $q_1, q_2 \in \mathbb{L}^1_+$, and two functions $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ such that $\mu_\alpha(m_1) < 1 < \mu_\alpha(m_2)$ and for all $u \geq 0$ and a.e. $t \in (0,1)$,*

$$m_1(t)u - q_1(t)\phi_1(u) \leq f(t,u) \leq m_2(t)u + q_2(t)\phi_2(u). \tag{4.4}$$

If either

- (H1) *$\phi_1(u) = o(\|u\|)$ as $u \to \infty$, $\phi_2(u) = o(\|u\|)$ as $u \to 0$, ϕ_1 is nondecreasing, and ϕ_2 is nondecreasing near 0 or*
- (H2) *$\phi_1(u) = o(\|u\|)$ as $u \to 0$, $\phi_2(u) = o(\|u\|)$ as $u \to \infty$, ϕ_1 is nondecreasing near 0, and ϕ_2 is nondecreasing,*

then (1.1), (1.2) has at least one positive solution.

Proof. For $i = 1, 2$, let $F_i : \mathcal{P} \to \mathcal{P}$ be defined by

$$F_i u(t) = \int_0^1 G(t,s)\phi_i(u(s))ds.$$

From (4.4), we have that for all $u \in \mathcal{P}$,

$$L_{m_1}u - F_1 u \preceq Tu \preceq L_{m_2}u + F_2 u,$$

with

$$r(L_{m_2}) = \frac{1}{\mu_\alpha(m_2)} < 1 < r(L_{m_1}) = \frac{1}{\mu_\alpha(m_1)}.$$

Suppose (H1) holds. Then, we have,

$$\frac{\|F_i u\|_\infty}{\|u\|_\infty} = \sup_{t\in[0,1]} \frac{F_i u(t)}{\|u\|_\infty} \leq \int_0^1 G(1,s) q_i(s) \frac{\phi_i(u(s))}{\|u\|_\infty} ds \leq \int_0^1 G(1,s) q_i(s) ds,$$

which progresses to our conclusion,

$$F_1 u = o(\|u\|) \quad \text{as} \quad u \to \infty \quad \text{and} \quad F_2 u = o(\|u\|) \quad \text{as} \quad u \to 0.$$

We therefore have from Theorem 2.2 that T has a fixed point, which finally is a positive solution to (1.1), (1.2). The case for (H2) is similar. □

5. An Extension to a Higher Order Problem

In this section, we consider the fractional boundary value problem (1.3), (1.4), motivated by the two-point Lidstone boundary value problem for ordinary differential equations. Define $G_0(t,s) = G(t,s)$ from (3.1) to be the Green's function for $-D^\alpha_{0+} u = 0$, $u^{(i)}(0) = 0$, $i = 0, 1, \dots, m-2$, $D^\beta_{0+} u(1) = 0$. Denote by $G_n(t,s)$ the Green's function for the BVP $-D^\gamma_{0+} u = 0$, (1.4).

The construction for $G_n(t,s)$ is similar to the construction in [6] and is given here for completeness. Define $G_k(t,s)$ by

$$G_k(t,s) = -\int_0^1 G_{k-1}(t,r) G_{conj}(r,s) dr, \tag{5.1}$$

$k = 2, \dots, n-1$, where

$$G_{conj}(t,s) = \begin{cases} t(1-s), & 0 \leq t < s \leq 1, \\ s(1-t), & 0 \leq s < t \leq 1, \end{cases} \tag{5.2}$$

is the Green's function for $-u'' = 0$, $u(0) = u(1) = 0$. Thus the Green's function $G_n(t,s)$ for (1.3), (1.4) is of the form

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r) G_{conj}(r,s) dr,$$

where $G_{n-1}(t,s)$ is the Green's function for

$$\begin{aligned} &D^{\gamma-2}_{0+} u(t) + h(t) = 0, \quad 0 < t < 1, \\ &u^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2, \quad D^\beta_{0+} u(1) = 0, \\ &D^{\gamma-2l}_{0+} u(0) = D^{\gamma-2l}_{0+} u(1) = 0, \quad l = 1, \dots, n-2. \end{aligned}$$

To see this, for the base case, first consider the linear differential equation

$$D^{\alpha+2}_{0+} u(t) + h(t) = 0,$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2, \quad D^\beta_{0+} u(1) = 0,$$

$$D_{0^+}^{\gamma-2(n-1)}u(0) = 0, \quad D_{0^+}^{\gamma-2(n-1)}u(1) = 0.$$

Make the change of variable $v(t) = D_{0^+}^{\alpha+2-2}u(t)$. Then $D^2v(t) = D^2D_{0^+}^{\alpha+2}u(t) = D_{0+}^{\alpha}u(t) = -h(t)$. Since $\alpha = \gamma - 2n + 2$, $v(0) = D_{0+}^{\alpha}u(0) = 0$ and $v(1) = D_{0+}^{\alpha}u(1) = 0$. Thus v satisfies the Dirichlet boundary value problem

$$\begin{aligned} &v'' + h(t) = 0, \quad 0 < t < 1, \\ &v(0) = 0, \quad v(1) = 0. \end{aligned}$$

Also, u now satisfies a lower order boundary value problem,

$$\begin{aligned} &D_{0+}^{\alpha}u(t) = v(t), \quad 0 < t < 1, \\ &u^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2, \quad D_{0+}^{\beta}u(1) = 0, \end{aligned}$$

and so,

$$\begin{aligned} u(t) &= \int_0^1 G_0(t,s)(-v(s))ds \\ &= \int_0^1 \left(-\int_0^1 G_0(t,s)G_{conj}(s,r)ds\right) h(r)dr \\ &= \int_0^1 G_1(t,s)h(s)ds, \end{aligned}$$

where $G_1(t,s) = -\int_0^1 G_0(t,r)G_{conj}(r,s)dr$.

For the inductive step, consider

$$D_{0^+}^{\gamma}u(t) + k(t) = 0,$$

satisfying (1.4). The argument here is similar to above. Make the change of variable $v(t) = D_{0^+}^{\gamma-2}u(t)$. Thus $D^2v(t) = D^2D_{0^+}^{\gamma-2}u(t) = D_{0+}^{\gamma}u(t) = -k(t)$. Since $v(0) = D_{0^+}^{\gamma-2}u(0) = 0$ and $v(1) = D_{0^+}^{\gamma-2}u(1) = 0$, then v satisfies the Dirichlet boundary value problem

$$\begin{aligned} &v'' + k(t) = 0, \quad 0 < t < 1, \\ &v(0) = 0, \quad v(1) = 0. \end{aligned}$$

Here u now satisfies a lower order boundary value problem,

$$\begin{aligned} &D_{0+}^{\gamma-2}u(t) = v(t), \quad 0 < t < 1, \\ &u^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2, \quad D_{0+}^{\beta}u(1) = 0, \\ &D_{0^+}^{\gamma-2l}u(0) = 0, \quad D_{0^+}^{\gamma-2l}u(1) = 0, \quad l = 2, \dots, k, \end{aligned}$$

and by the induction hypothesis,

$$\begin{aligned} u(t) &= \int_0^1 G_{n-1}(t,s)(-v(s))ds \\ &= \int_0^1 \left(-\int_0^1 G_{n-1}(t,s)G_{conj}(s,r)ds\right) k(r)dr \end{aligned}$$

$$= \int_0^1 G_n(t,s)k(s)ds,$$

where $G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{conj}(r,s)dr$.

Define $v_n(t,s)$ so that $t^{\alpha-1}v_n(t,s) = G_n(t,s)$. The following lemma follows from Lemma 3.2.

Lemma 5.1. *Let G_n be defined inductively as above.*

(1) $G_n(t,s) \in C([0,1]\times[0,1))$ *with* $(-1)^n G_n(t,s) > 0$ *for* $(t,s) \in (0,1]\times(0,1)$.
(2) $t^{\alpha-1}(-1)^n G(1,s) \le (-1)^n G(t,s) \le (-1)^n G(1,s)$ *for* $(t,s) \in [0,1]\times[0,1)$.
(3) $(-1)^n G(t,s) \ge (-1)^n \delta^{\alpha-1} G(1,s)$ *for all* $t \in [\delta,1]$ *and all* $s \in [0,1)$.
(4) $(-1)^n v(0,s) \ge 0$ *for all* $s \in [0,1)$.

Proof. We start by showing (1) holds. For the base case, consider that $G_0(t,s) = G(t,s)$ from Lemma 3.2 which does belong to $C([0,1]\times[0,1])$ and is positive. Since

$$G_1(t,s) = -\int_0^1 G_0(t,r)G_{conj}(r,s)ds,$$

and $G_{conj}(r,s) \in C([0,1]\times[0,1])$ and $G_{conj}(t,s) > 0$, it follows that $G_1(t,s) \in C([0,1]\times[0,1])$ and $-G_1(t,s) > 0$. For the inductive step, assume $G_{n-1}(t,s) \in C([0,1]\times[0,1))$ and $(-1)^{n-1}G_{n-1}(t,s) > 0$ for $(t,s) \in (0,1]\times(0,1)$. Then by definition

$$(-1)^n G_n(t,s) = -\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{conj}(r,s)dr,$$

we see that since $G_{conj}(t,s) \in C([0,1]\times[0,1])$ and $G_{conj}(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$, then $(-1)^n G_n(t,s) > 0$ for $(t,s) \in (0,1]\times(0,1)$ and $G_n(t,s) \in C([0,1]\times[0,1))$.

For (2), similar to the first item, the base case follows from Lemma 3.2. Since for $G_0(t,s) = G(t,s)$, we have

$$t^{\alpha-1}G_0(1,s) \le G_0(t,s) \le G_0(1,s),$$

and by the definition of $G_1(t,s)$ we have

$$\begin{aligned}
-t^{\alpha-1}G_1(1,s) &= \int_0^1 t^{\alpha-1}G_0(1,r)G_{conj}(r,s)dr \\
&\le \int_0^1 G_0(t,r)G_{conj}(r,s)dr \\
&= -G_1(t,s) \\
&\le \int_0^1 G_0(1,r)G_{conj}(r,s)dr \\
&= -G_1(1,s).
\end{aligned}$$

For the inductive step, in a similar fashion, assume

$$t^{\alpha-1}(-1)^{n-1}G_{n-1}(1,s) \le (-1)^{n-1}G_{n-1}(t,s) \le (-1)^{n-1}G_{n-1}(1,s).$$

Then by the definition of $G_n(t,s)$, we have

$$\begin{aligned}
t^{\alpha-1}(-1)^n G_n(1,s) &= (-1)^{n+1} t^{\alpha-1}\int_0^1 G_{n-1}(1,r)G_{conj}(r,s)dr \\
&\le (-1)^{n+1}\int_0^1 G_{n-1}(t,r)G_{conj}(r,s)dr \\
&= (-1)^n G_n(t,s) \\
&\le (-1)^{n+1}\int_0^1 G_{n-1}(1,r)G_{conj}(r,s)dr \\
&= (-1)^n G_n(1,s).
\end{aligned}$$

Notice that (3) is a direct result of (2), and a proof of (4) can similarly be obtained using induction. □

Define the sets

$$\mathbb{L}^1_{n+} = \{m \in \mathbb{L}^1(0,1) : (-1)^n m(t) \ge 0 \text{ a.e. } t \in [0,1]\}$$

and

$$\mathbb{L}^1_{n++} = \{m \in \mathbb{L}^1_{n+} : (-1)^n m(t) > 0 \text{ on a subset of positive measure}\}.$$

Let $m \in \mathbb{L}^1_{n++}$. Define $L_{nm} : E \to E$ by

$$L_{nm}u(t) = \int_0^1 G_n(t,s)m(s)u(s)ds.$$

Define $L^X_{nm} : X \to E$ by, for $u \in X$, $L^X_{nm}u = L_{nm}u$.

Lemma 5.2. *For $m \in \mathbb{L}^1_{n+}$, the operator L_{nm} is compact and positive. Moreover, $L_{nm} : E^+ \to \mathcal{P}$.*

Proof. Let $u \in E^+$. Then

$$\begin{aligned}
L_{nm}u(t) &= \int_0^1 G_n(t,s)m(s)u(s)ds \\
&= \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds > 0,
\end{aligned}$$

and, since $\|L_{nm}u\| = |L_{nm}u(1)|_0$,

$$\begin{aligned}
L_{nm}u(t) &= \int_0^1 G_n(t,s)m(s)u(s)ds \\
&= \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds \\
&\ge \delta^{\alpha-1}\int_0^1 (-1)^n G_n(1,s)|m(s)|u(s)ds \\
&= \delta^{\alpha-1}\|L_{nm}u\|,
\end{aligned}$$

concluding the proof. □

Lemma 5.3. *For $m \in \mathbb{L}^1_{n++}$, L_{nm} is a strongly positive-like operator which is lower bounded on the cone $\mathcal{P}$.*

Proof. As in the proof of Lemma 3.4, if we can show $L_{nm}^X : X^+ \setminus \{0\} \to S \subset X^{+\circ}$, the result follows. Let $u \in X^+$. First, notice for $t \in (0,1]$,

$$L_{nm}^X u(t) = \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds > 0.$$

Again, since m and $v_n(0,s)$ have the same sign,

$$\lim_{t\to 0} \frac{L_{mn}^X u(t)}{t^{\alpha-1}} = \int_0^1 (-1)^n v_n(0,s)|m(s)|u(s)ds > 0.$$

So $L_m^X : X \setminus \{0\} \to S \subset X^{+\circ}$, and the result follows. □

Define the operator $T_n : E^+ \to E$ by

$$T_n u(t) = \int_0^1 G_n(t,s)(-1)^n g(s,u(s))ds.$$

Notice that u is a solution of the boundary value problem (1.3), (1.4) if and only if u is a fixed point of T_n.

The following lemma is a direct result of the Arzelà-Ascoli theorem and Lemma 5.1.

Lemma 5.4. *$T_n : E^+ \to E$ is compact and $T_n : E^+ \to \mathcal{P}$.*

The proofs of the main results are similar to the proofs from Section 4 and are therefore omitted.

Let $m \in \mathbb{L}^1_{n++}$. Consider the linear boundary value problem

$$D^\gamma_{0^+} u(t) + \mu m(t)u(t) = 0, \quad \text{a.e. } t \in (0,1), \tag{5.3}$$

satisfying the boundary conditions (1.4), where μ is a real parameter.

Lemma 5.5. *For all $m \in \mathbb{L}^1_{n++}$,* (5.3), (1.4) *admits a unique positive eigenvalue $\mu_\alpha(m)$.*

Theorem 5.1. *Assume that there exists $m \in \mathbb{L}^1_{n+}$ such that one of the following hypotheses is satisfied.*

$$\mu_\alpha(m) < 1 \quad \textit{and} \quad (-1)^n g(t,u) \geq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{5.4}$$

$$\mu_\alpha(m) > 1 \quad \textit{and} \quad (-1)^n g(t,u) \leq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{5.5}$$

then (1.3), (1.4) *has no positive solutions.*

Theorem 5.2. *Assume that there exist $m_1, m_2 \in \mathbb{L}^1_{n++}$, $q_1, q_2 \in \mathbb{L}^1_{n+}$, and two functions $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ such that $\mu_\alpha(m_1) < 1 < \mu_\alpha(m_2)$ and, for all $u \geq 0$ and a.e. $t \in (0,1)$*

$$m_1(t)u - q_1(t)\phi_1(u) \leq (-1)^n g(t,u) \leq m_2(t)u + q_2(t)\phi_2(u). \tag{5.6}$$

If either

(H1) $\phi_1(u) = o(\|u\|)$ *as* $u \to \infty$, $\phi_2(u) = o(\|u\|)$ *as* $u \to 0$, ϕ_1 *is nondecreasing, and* ϕ_2 *is nondecreasing near* 0 *or*

(H2) $\phi_1(u) = o(\|u\|)$ *as* $u \to 0$, $\phi_2(u) = o(\|u\|)$ *as* $u \to \infty$, ϕ_1 *is nondecreasing near* 0*, and* ϕ_2 *is nondecreasing,*

then (1.3)*,* (1.4) *has at least one positive solution.*

We conclude the paper by remarking that the hypotheses of Theorems 4.2 and 5.2 are similar to the hypotheses of the main theorem in [3]. Therefore, the examples of nonlinearities provided in that work could be easily modified for the problems given in this paper.

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