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WOVEN (WEAVING) FRAMES IN BANACH SPACES

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Abstract. Banach frames are defined by the straightforward generalization of Hilbert space frames. Woven (weaving) frames are the recent generalization of standard frames which appeared in the applications of distributed signal processing. In this paper, we introduce the concepts of woven (weaving) Bessel and frame sequences in Banach spaces and characterize the woven frames in terms of bounded operators. We also give some equivalent conditions for woven X_d-frame in Banach spaces.

1. Introduction

The origin of frame theory can be traced back to the early 1950s with the seminal work of Duffin and Schaeffer [13] in nonharmonic Fourier series. Today, the theory of frames has expanded into an independent and broad field of research with widespread applications to signal processing, image processing, data compression, pattern matching, sampling theory, spherical codes, wavelet analysis, communication and data transmission [4,8,11,18,19]. Inspired by a problem raised in distributed signal processing, Bemrose et al. [1] introduced the concept of weaving frames in separable Hilbert spaces and observed that the weaving frames may be applied in sensor networks which requires distributed processing under different frames. In recent years, a considerable amount of research has been conducted to extend the notion of weaving frames to different settings which include weaving frames in Banach spaces, continuous weaving frames, generalized weaving frames, weaving Riesz bases, weaving fusion frames, weaving controlled frames and weaving vector-valued frames [5,6,20,22,24–26,31–34].

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Frames in Hilbert spaces were extended to Banach spaces by Feichtinger and Gröchenig [15] who introduced the concept of atomic decompositions in Banach spaces. Later on, Gröchenig [17] laid down the foundations for the theory of coherent Banach frames and constructed Banach frames for a wide class of Banach spaces, the so-called coorbit spaces. Keeping in view the fact that the weaving frames have potential applications in wireless sensor networks and other allied areas, we are deeply motivated to extend the concept of woven (weaving) frames to Banach spaces by invoking certain fundamental concepts of operator theory.

This article is organized as follows. Section 2 contains basic definitions and results regarding frames and weaving frames in Hilbert spaces. In Section 3, we introduce the notion of weaving frames in Banach spaces and then generalize the definitions of X_d-frame and p-frame for the woven.

2. Frames and Woven Frames in Hilbert Spaces

In this section, we give a short review of the concept of frames and woven frames in Hilbert spaces and make some preparatory observations. For a complete treatment of frame theory, we recommend the excellent book of Christensen [8], the tutorials of Casazza [2,3] and the memoir of Han and Larson [21]. Throughout this paper, H denotes a separable infinite-dimensional Hilbert space, X, Y, Z the separable Banach spaces with dual X^*, Y^*, Z^*, X_d a Banach sequence space and $\mathbb{I}$ an index set which is finite or countable. Let $\mathbb{N}$ be the set of all positive integers and let $m \in \mathbb{N}$ be fixed. Then for this choice of m, we set $[m] = \{1, 2, \dots, m\}$ and $[m]^c = \mathbb{I} \setminus [m] = \{m+1, m+2, \dots\}$. Let us start with the well-known notion of Hilbert space frames.

2.1. **Discrete frame in Hilbert spaces.**

In this section, we give a short review of the concept of frames in Hilbert spaces, and make some preparatory observations. Let us start with the well known notion of Hilbert space frames.

Definition 2.1. A family of vectors $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ in a Hilbert space H is said to be a frame if there exist constants $0 < A \leq B < \infty$ so that for all $x \in H$

$$A\|x\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2,$$

where A and B are lower and upper frame bounds, respectively. If only B is assumed, then it is called B-Bessel sequence. If $A = B$, it is said to be a tight frame and if $A = B = 1$, it is called a Parseval frame.

If $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ is a Bessel sequence for H, then the synthesis operator of Φ defined as

$$T : l^2(\mathbb{I}) \to H, \quad T\{c_i\} := \sum_{i\in\mathbb{I}} c_i\varphi_i,$$

and the adjoint of T is the analysis operator

$$T^* : H \to l^2(\mathbb{I}), \quad T^*x := \{\langle x, \varphi_i\rangle\}_{i\in\mathbb{I}}.$$

The frame operator $S : H \to H$ is defined by $S := TT^*$

$$Sx = TT^*x = \sum_{i\in\mathbb{I}} \langle x, \varphi_i\rangle\varphi_i, \quad \text{for all } x \in H.$$

The operator S is positive, self-adjoint, invertible and $AI \le S \le BI$. Any $x \in H$ has an expansion

$$x = \sum_{i\in\mathbb{I}} \langle S^{-1}\varphi_i, x\rangle\varphi_i = \sum_{i\in\mathbb{I}} \langle \varphi_i, x\rangle S^{-1}\varphi_i. \tag{2.1}$$

The family $\{S^{-1}\varphi_i\}_{i\in\mathbb{I}}$ is also a frame with bounds B^{-1}, A^{-1} and this frame is called the canonical dual or reciprocal frame of $\{\varphi_i\}_{i\in\mathbb{I}}$.

Definition 2.2. A family of vectors $\Phi = \{\varphi_i\}_{i\in\mathbb{I}}$ in a Hilbert space H is said to be a Riesz sequence if there exist constants $0 \le A \le B < \infty$ so that for all $\{c_i\}_{i\in\mathbb{I}} \in l^2(\mathbb{I})$

$$A\sum_{i\in\mathbb{I}} |c_i|^2 \le \left\| \sum_{i\in\mathbb{I}} c_i\varphi_i \right\|^2 \le B\sum_{i\in\mathbb{I}} |c_i|^2,$$

where A and B are the lower Riesz bound and upper Riesz bound, respectively. If in addition, Φ is complete in H, then it is called as the Riesz basis for H.

2.2. **Woven Frame in Hilbert spaces.** Woven frames in Hilbert spaces were introduced by Bemros et al. [1, 6] in 2015. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. In this subsection, we review the notions of woven and weaving frames in Hilbert spaces and present certain new examples.

Definition 2.3. A family of frames $\{f_{ij}\}_{i\in\mathbb{I}}$ with $j \in [m]$ for a Hilbert space H is said to be woven if there exist universal constants A and B so that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{f_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively. For every $j \in [m]$, the frames $\{f_{ij}\}_{i\in\sigma_j}$ are called weaving frames.

The following proposition shows that every weaving frame has always a universal upper frame bound.

Proposition 2.1. *If each $\phi = \{\varphi_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ is a Bessel sequence for H with bounds B_j for all $j \in [m]$, then every weaving frame is a Bessel sequence with $\sum_{j=1}^m B_j$ as a Bessel bound.*

Proof. For every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$ and every $x \in H$, the inequality

$$\sum_{j=1}^m \sum_{i\in\sigma_j} |\langle x, \varphi_{ij}\rangle|^2 \le \sum_{j=1}^m \sum_{i\in\mathbb{I}} |\langle x, \varphi_{ij}\rangle|^2 \le \|x\|^2 \sum_{j=1}^m B_j,$$

yields the desired bound. □

Example 2.1. There exist two Parseval frames that yield weaving frames with arbitrary weaving bounds. For showing this, assume $\varepsilon > 0$, set $\delta = (1+\varepsilon^2)^{-\frac{1}{2}}$, and let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of $\mathbb{R}^3$. Then the sets $\phi = \{\varphi_i\}_{i=1}^n = \{\delta e_1 + \delta\varepsilon e_1, \delta e_2 + \delta\varepsilon e_2, \delta e_3 + \delta\varepsilon e_3\}$ and $\psi = \{\psi_i\}_{i=1}^n = \{\delta\varepsilon e_2 + \delta e_2, \delta\varepsilon e_1 + \delta e_1, \delta\varepsilon e_3 + \delta e_3\}$, are Parseval frames, which are woven since any choice of σ gives the spaning set. Since they are Parseval, as a consequence of Proposition 2.1, the universal frame bound for every weaving frame can be chosen to be n. For $\sigma = \{2, 4, 6\}$, we have

$$\begin{aligned}
&\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\sigma^c} |\langle x, \psi_i\rangle|^2 \\
=&|\langle x, \delta e_1 + \delta\varepsilon e_1\rangle|^2 + |\langle x, \delta e_2 + \delta\varepsilon e_2\rangle|^2 \\
&+ |\langle x, \delta e_3 + \delta\varepsilon e_3\rangle|^2 + |\langle x, \delta\varepsilon e_2 + \delta e_2\rangle|^2 + |\langle x, \delta\varepsilon e_1 + \delta e_1\rangle|^2 + |\langle x, \delta\varepsilon e_3 + \delta e_3\rangle|^2 \\
=&2\left(\delta^2 + \delta^2\varepsilon^2\right)|\langle x, e_1\rangle|^2 + 2\left(\delta^2\varepsilon^2 + \delta^2\right)|\langle x, e_2\rangle|^2 + 2\left(\delta^2\varepsilon^2 + \delta^2\right)|\langle x, e_3\rangle|^2 \\
=&2\delta^2\left(1+\varepsilon^2\right)\|x\|^2 = \frac{2\varepsilon^2}{1+\varepsilon^2}\|x\|^2,
\end{aligned}$$

which lies between 0, 3 for arbitrary choice of $\varepsilon \in (0, \infty)$.

The following proposition demonstrates that the perturbed frames are obtained as the image of a bounded and invertible operator of a given frame.

Proposition 2.2. *Let $\{\varphi_i\}_{i\in\mathbb{I}}$ be a frame with bounds A, B and V be a bounded operator. If $\|Id - V\|^2 \leq \frac{A}{B}$ and $\|V - V^2\|^2 \leq \frac{A}{B}$, then the frames $\{\varphi_i\}_{i\in\mathbb{I}}$, $\{V\varphi_i\}_{i\in\mathbb{I}}$ and $\{V^2\varphi_i\}_{i\in\mathbb{I}}$ are woven.*

Proof. Note that by Neumann's Theorem V is invertible and thus $\{V\varphi_i\}_{i\in\mathbb{I}}$ and $\{V^2\varphi_i\}_{i\in\mathbb{I}}$ automatically constitute frames. For every partitions $\sigma, \Delta \subset \mathbb{I}$ and every $x \in H$ by using *Minkowski's* inequality:

$$\begin{aligned}
&\left(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, V\varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle x, V^2\varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}} \\
=&\Bigg(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 - \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle V^* x, \varphi_i\rangle|^2 \\
&+ \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^* x, \varphi_i\rangle|^2 - \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^* x, \varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle (V^2)^* x, \varphi_i\right\rangle\right|^2\Bigg)^{\frac{1}{2}} \\
=&\Bigg(\sum_{i\in\sigma} |\langle x, \varphi_i\rangle|^2 + \sum_{i\in\Delta} |\langle x, \varphi_i\rangle|^2 - \sum_{i\in\Delta} |\langle (I - V^*) x, \varphi_i\rangle|^2 + \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} |\langle V^* x, \varphi_i\rangle|^2 \\
&- \sum_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)} \left|\left\langle (V^* - (V^2)^*) x, \varphi_i\right\rangle\right|^2\Bigg)^{\frac{1}{2}}
\end{aligned}$$

$$\geq\left(\sum_{i\in\mathbb{I}}|\langle x,\varphi_i\rangle|^2\right)^{\frac{1}{2}}-\left(\sum_{i\in\Delta}|\langle(I-V^*)x,\varphi_i\rangle|^2\right)^{\frac{1}{2}}+\left(\sum_{i\in\Delta\cup(\mathbb{I}\setminus(\sigma\cup\Delta))}|\langle V^*x,\varphi_i\rangle|^2\right)^{\frac{1}{2}}$$
$$-\left(\left|\left\langle\left(V^*-(V^2)^*\right)x,\varphi_i\right\rangle\right|^2\right)^{\frac{1}{2}}$$
$$\geq\sqrt{A}\,\|x\|-\sqrt{B}\|(I-V^*)x\|+\sqrt{B}\|V^*x\|-\sqrt{B}\|(V^*-(V^2)^*)x\|$$
$$\geq\left(\sqrt{A}-\sqrt{B}\|I-V^*\|+\sqrt{B}\|V^*\|-\sqrt{B}\|V^*\|\|I-V^*\|\right)\|x\|.$$

Thus, $\{\varphi\}_{i\in\sigma}\cup\{V\varphi_i\}_{i\in\Delta}\cup\{V^2\varphi_i\}_{i\in\mathbb{I}\setminus(\sigma\cup\Delta)}$ forms a woven frames having

$$\left(\sqrt{A}-\sqrt{B}\|I-V^*\|+\sqrt{B}\|V^*\|-\sqrt{B}\|V^*\|\|I-V^*\|\right)^2>0.$$

□

3. Woven Frames in Banach Space

3.1. **Frames in Banach Space.** Frames were extended to Banach spaces by Feichtinger and Gröchenig [15] who introduced the notion of atomic decompositions for Banach spaces. Later, Gröchenig [17] introduced a more general concept called Banach frame. Banach frames were further studied in [4]. An analysis of p-frames in general Banach spaces first appeared in [9]. The aim of an atomic decomposition for a space of functions or distributions is to represent every element as a sum of simple functions usually called atoms. If this is possible, some properties of these function spaces, such as duality, interpolation, or operator theory for them, can be understood better by means of the atomic decomposition. Decomposition methods have been used for many important theoretical contributions. A Banach space of scalar valued sequences (often called BK-space) is a linear space of sequences equipped with a norm under which it constitutes a Banach Space (i.e., it is complete in the norm) and for which the coordinate functionals are continuous. In a Banach space of scalar valued sequences, the unit vectors are the elements e_i's defined by $e_i\,(j)=\delta_{ij}$ (δ_{ij} the Kronechker delta).

Definition 3.1. A sequence space X_d is called BK-space, if it is a Banach space and the coordinate functionals $\{a_k\}\to a_k$ are continuous on X_d, that is, the relations $x_n=\{\alpha_j^{(n)}\}$, $x=\{\alpha_j\}\in X_d$, $\lim_{n\to\infty}x_n=x$ imply

$$\lim_{n\to\infty}\alpha_j^{(n)}=\alpha_j,\quad j=1,2,\ldots$$

A BK-space is called solid if whenever $\{a_k\}$ and $\{b_k\}$ are sequences with $\{b_k\}\in X_d$ and $|a_k|\leq|b_k|$ for each $k\in\mathbb{I}$, then it follows that $\{a_k\}\in X_d$ and

$$\|\{a_k\}\|_{X_d}\leq\|\{b_k\}\|_{X_d}.$$

A sequence space X_d is called an AK-space if it is a topological vector space and

$$\{a_k\}=\lim_n\rho_n\left(\{a_k\}\right),\quad\text{for all }\{a_k\}\in X_d,$$

where $\rho_n(\{a_k\}) = (a_1, a_2, \dots, a_n, 0, \dots)$.

If the canonical vectors form a Schauder basis for X_d, then X_d is called a CB-space and its canonical basis is denoted by $\{e_j\}_{i=1}^{\infty}$. If X_d is reflexive and a CB-space, then X_d is called an RCB-space. Also, the dual of X_d is denoted by X_d^*.

Definition 3.2. Let X be a Banach space and X_d be a BK-space. A countable family $\{g_i\}_{i\in\mathbb{I}}$ in the dual X^* is called an X_d-frame for X if

(a) $\{g_i(f)\}_{i\in\mathbb{I}} \in X_d$ for all $f \in X$;

(b) the norms $\|f\|_X$ and $\|\{g_i(f)\}_{i\in\mathbb{I}}\|_{X_d}$ are equivalent, that is, there exist constants $A, B > 0$ such that
$$A\|f\|_X \leq \|\{g_i(f)\}_{i\in\mathbb{I}}\|_{X_d} \leq B\|f\|_X, \quad \text{for all } f \in X.$$
A and B are called X_d-frame bounds.

If at least (a) and the upper condition in (b) are satisfied, $\{g_i\}_{i\in\mathbb{I}}$ is called an X_d-Bessel sequence for X. In case $X_d = \ell^p$, the X_d-frame is called p-frame which is introduced by Christensen and Stoeva [9, 30].

Definition 3.3. A countable family $\{g_i\}_{i\in\mathbb{I}} \subset X^*$ is a p-frame for X, $1 < p < \infty$, if there exist $A, B > 0$ such that
$$A\|f\|_X \leq \left(\sum_{i\in\mathbb{I}} |g_i(f)|^p\right)^{\frac{1}{p}} \leq B\|f\|_X, \quad \text{for all } f \in X.$$
The family $\{g_i\}_{i\in\mathbb{I}}$ is a p-Bessel sequence if at least the upper p-frame condition is satisfied.

Lemma 3.1 ([28])**.** *If X is a Banach space and $\{f_n\} \subset X^*$ is total over X, then X is linearly isometric to the Banach space $X = \{\{f_n(x)\} : x \in X\}$, where the norm is given by $\|\{f_n(x)\}\|_X = \|x\|_X$ for $x \in X$.*

Definition 3.4. Let X be a Banach space and let X_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S : X_d \to X$ be given. The pair $(\{f_n\}, S)$ is called a Banach frame for X with respect to X_d if

(a) $\{f_n(x)\} \in X_d$ for each $x \in X$;

(b) there exist positive constants A and B with $0 < A \leq B < \infty$ such that
$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \quad \text{for all } x \in X; \tag{3.1}$$

(c) S is a bounded linear operator such that $S(\{f_n(x)\}) = x$ for all $x \in X$.

The positive constants A and B are called the lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : X_d \to X$ is called the reconstruction operator (or the pre-frame operator). The inequality (b) is called the frame inequality. The Banach frame $(\{f_n\}, S)$ is called tight if $A = B$ and normalized tight if $A = B = 1$.

Example 3.1. Let $X = l^p$ and $\{e_n\}$ be the sequence of unit vectors in X. Define $\{f_n\} \subset X^*$ by

$$f_n = f_{n+2} = e_n, \quad n \in \mathbb{I}.$$

Then by Lemma 3.1, there exists an associated Banach space $X_d = \{\{f_n(x)\} : x \in X\}$ and a reconstruction operator $S : X_d \to X$ such that $(\{f_n\}, S)$ is a Banach frame for X.

3.2. **Woven in Banach spaces.** As we mentioned earlier, Bemrose, Casazza et al. in [1, 6] proposed weaving frames in a separable Hilbert space. Weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, frames in Hilbert spaces. Improving and extending this notion on Hilbert spaces, we generalize the concept of woven (weaving) on Banach spaces.

Definition 3.5. Let X be a Banach space and X_d be a BK-space. The family of Banach frames $\{g_{ij}\}_{i\in\mathbb{I}}$ for $j \in [m]$ is woven X_d-frame for dual X^* with universal bounds A, B if

(a) $\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]} \in X_d$, $f \in X$;
(b) the norms $\|f\|_X$ and $\|\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]}\|_{X_d}$ are equivalent, that is, there exist constants A, $B > 0$ such that

$$A\|f\|_X \leq \|\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]}\|_{X_d} \leq B\|f\|_X, \quad f \in X.$$

The constants A and B are called woven X_d-frame bounds. If at least (a) and the upper condition in (b) are satisfied, $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is called a woven X_d-Bessel for X.

Definition 3.6. Let X be a Banach space and let X_d be an associated Banach space of scalar valued sequences indexed by $\mathbb{I}$. Let $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ and $S : X_d \to X$ be given. The pair $\Big(\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}, S\Big)$ is called a woven Banach frame for X with respect to X_d if the pair $\Big(\{f_{ij}\}_{i\in\sigma_j,j\in[m]}, S\Big)$ is a Banach frame for each partitions $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$.

The lack of an inner-product in Banach spaces led G. Lumer [23] in 1961 to introducing the theory of semi-inner product spaces. His procedure suggested the existence of a general theory which it seemed should be useful in the study of operator (normed) algebras by providing better insight on known facts, a more adequate language to "classify" special types of operators, as well as new techniques. This notion was further modified by J. R. Giles [16] and other researchers thereon, and the same is presented below.

Definition 3.7 ([16])**.** Let X be a complex (real) vector space. A complex (real) semi-inner product defined on X is a function from $[\cdot,\cdot] : X \times X \to \mathbb{C}$ such that for all $f, g, h \in X$, $\lambda \in \mathbb{C}$ complex (real)

(a) $[\lambda f + g, h] = \lambda\,[f, h] + [g, h]$, $[f, \lambda g] = \overline{\lambda}\,[f, g]$;
(b) $[f, f] \geq 0$ for $f \in X$ and $[f, f] = 0$ implies $f = 0$;
(c) $|\,[f, g]\,|^2 \leq [f, f]\,[g, g]$.

We call X a complex (real) semi-inner product space, abbreviated with S.I.P.S. An S.I.P.S need not satisfy the following properties

(a) $[f,g]=\overline{[g,f]}$;
(b) $[f,g+h]=[f,g]+[f,h]$.

If $[\cdot,\cdot]$ is a S.I.P.S. on X then $\|f\|:=[f,f]^{\frac{1}{2}}$ is a norm on X. Conversely, if X is a normed vector space then it has a S.I.P.S. that induces its norm in this manner which is called the compatible semi-inner product [23]. Let X be a Banach space. We define a duality map $\Phi_X : X \to X^*$ as follows. Given $f \in X$, by the Hahn-Banach theorem, there exists an $f^* \in X^*$ such that $\|f\| = \|f^*\|$ and $f^*(f) = \|f\|^2$. Set $\Phi_X(f) = f^*$, and $\Phi_X(\lambda f) = \overline{\lambda} f^*$, and define Φ_X on the rest of X in the same manner. In general, Φ_X is not unique, linear or continuous. The duality map Φ_X induces a semi-inner product $[\cdot,\cdot]$ if we set $[f,g] = g^*(f)$ [29]. We shall use this definition for $g^*, g \in X$. Note that if X is a Banach space, then the duality map is unique [29]. Recall that a Banach space X is called strictly convex, if for any pair of vectors $f, g \neq 0$ in X, the equation $\|f+g\| = \|f\|_X + \|g\|_X$, implies that there exists a $\lambda > 0$ such that $f = \lambda g$ [12]. In these spaces, the duality mapping from X to X^* is unique and bijective when X is reflexive [12, 14].

In 2011, H. Zhang and J. Zhang [35] introduced frames in Banach space X via S.I.P.S. that is presented in the following definition. The extra condition in Definition 3.5 means that S is a left-inverse of U and thus US is a bounded linear projection of X_d onto the range $R(U)$ of the operator U.

Lemma 3.2 ([10])**.** *If X_d is a CB-space with the canonical unit vectors e_i, $i \in J$, then the space $X_d^{\circledast} := \{\{G(e_i)\}_{i=1}^{\infty} : G \in X_d^*\}$ with the norm $\|\{G(e_i)\}_{i=1}^{\infty}\|_{X_d^{\circledast}} := \|G\|_{X_d^*}$ is a BK-space isometrically isomorphic to X_d^*. Also, every continuous linear functional Ψ on X_d has the form*

$$\Psi(\{c_j\}) = \sum_j c_j d_j,$$

where $\{d_j\} \in X_d^{\circledast}$ is uniquely determined by $d_j = \Psi(e_j)$, $\|\Psi\| = \|\{\Psi(e_i)\}_{i=1}^{\infty}\|_{X_d^{\circledast}}$. When X_d^* is a CB-space then its canonical basis is denoted by $\{e_j^*\}$.

Remark 3.1. It is easy to see that Lemma 3.2 holds in the following more general case: If Y is a Banach space and $\{y_i\}_{i=1}^{\infty}$ is a complete system in Y, then $Y^{\circledast} := \{\{Gy_i\}_{i=1}^{\infty} : G \in Y^*\}$ normed by $\|\{Gy_i\}_{i=1}^{\infty}\|_{Y^{\circledast}} := \|G\|_{Y^*}$ is a BK-space, isometrically isomorphic to Y^*. Thus, the dual of every separable Banach space can be considered as a BK-space, because every separable Banach space has a complete system [28].

In the following theorem, we will see that the Bessel woven condition can be expressed in terms of the synthesis operator T on X_d. As a prerequisite for analysis, synthesis and frame operators of weaving frames, we define the following space.

For $j \in [m]$, let $(X_d)_j := \left\{ \{c_{ij}\}_{i\in\sigma_j} : \sigma_j \subset \mathbb{I}, \|\{c_{ij}\}_{i\in\sigma_j}\|_{X_d} < \infty \right\}$. Define the space

$$\left(\sum_{j\in[m]} \oplus (X_d)_j \right) = \left\{ \{c_{ij}\}_{i\in\mathbb{I}, j\in[m]} : \{c_{ij}\}_{i\in\mathbb{I}} \in (X_d)_j \text{ for all } j \in [m] \right\},$$

with the semi-inner product

$$\left[\{c_{ij}\}_{i\in\mathbb{I}, j\in[m]}, \{c'_{ij}\}_{i\in\mathbb{I}, j\in[m]} \right] = \sum_{i\in\mathbb{I}, j\in[m]} \left| c_{ij} \overline{c'_{ij}} \right|.$$

The following proposition characterizes a woven Bessel in term of a bounded operator.

Theorem 3.1. *Let $\{(X_d)_1, (X_d)_2, \dots\}$ be a sequence of Banach spaces. $(X_d)_i$ and $(X_d^*)_i$'s are BK-spaces. Then,*

$$((X_d)_1 \oplus (X_d)_2 \oplus \cdots)^*_{X_d} = ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}.$$

Proof. We shall establish the result when X_d, X_d^* are BK-space. Assume that

$$C = (\{c_{i1}\}, \{c_{i2}\}, \dots) \in ((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$$

and

$$C^* = (\{c^*_{i1}\}, \{c^*_{i2}\}, \dots) \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}.$$

Then the mapping $C^* \mapsto \varphi_{C^*}$, where

$$\varphi_{c^*}(\{c_{i1}\}, \{c_{i2}\}, \dots) = \sum_{i=1}^{\infty} c^*_{in}(c_{in}),$$

is an isometry from $((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$ onto $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$. Fix $C^* \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$. For each $C = (\{c_{i1}\}, \{c_{i2}\}, \dots)$ in $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$, the mapping $\varphi_{C^*}(\{c_{i1}\}, \{c_{i2}\}, \dots) = \sum_{i=1}^{\infty} c^*_{in}(c_{in})$ defines a continuous linear functional on $((X_d)_1 \oplus (X_d)_2 \oplus \cdots)_{X_d}$ satisfying $\|\varphi_{C^*}\| \le \|C^*\|_{X_d^*}$, since using Lemma 3.2 we have

$$\begin{aligned}
\|\varphi_{C^*}(\{c_{i1}\}, \{c_{i2}\}, \dots)\| &= \left\| \sum c^*_{in}(c_{in}) \right\| \\
&= \sup_{g\in X^*, \|g\|\le 1} \left| g\left(\sum c^*_{in}(c_{in}) \right) \right| \\
&= \sup_{g\in X^*, \|g\|\le 1} \left| G_g\left(\sum c^*_{in}(c_{in}) \right) \right| \\
&\le \sup_{g\in X^*, \|g\|\le 1} \|\{g(c_{in})\}\|_{X_d} \|\{c^*_{in}(c_{in})\}\|_{X_d^*} \\
&\le \|g\| \|\{c^*_{in}(c_{in})\}\|_{X_d^*}.
\end{aligned}$$

Thus,

$$\|\varphi_{C^*}\| \le \|C^*\|_{X_d^*}, \tag{3.2}$$

for all $C^* \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$.

Fix $0 < \varepsilon < 1$. For each n pick some $\{c_{in}\} \in (X_d)$ with $\|c_{in}\| = 1$ and $c^*_{in}(c_{in}) \ge \varepsilon \|c^*_{in}\|$.

Using Lemma 3.2, we have

$$\varepsilon\|c_{in}^*\|_{X_d^*} = \varepsilon \sup_{g\in X^*,\|g\|\le 1} |g(c_{in}^*)| = \varepsilon \sup_{g\in X^*,\|g\|\le 1} |G_g(c_{in}^*)|$$
$$\le \varepsilon\|g\|\|\{c_{in}^*(c_{in})\}\| \le \varepsilon\|g\|\|\sum_{i=1}^{\infty} c_{in}^*(c_{in})\|.$$

This implies that $C^* = (\{c_{i1}^*\}, \{c_{i2}^*\}, \dots) \in ((X_d^*)_1 \oplus (X_d^*)_2 \oplus \cdots)_{X_d^*}$, $\|C^*\|_{X_d^*} \le \|\varphi_{C^*}\|$. Finally, as a consequence of (3.2), we conclude that $C^* \mapsto \varphi_{C^*}$ is an onto linear isometry. □

Proposition 3.1. *Suppose that X_d is a BK-space, for which the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ forms a Schauder basis. Then $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subseteq X^*$ is an X_d^*-Bessel woven for X with universal bound B if and only if the operator*

$$T : \{\ c_{ij}\} \to \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$$

is well defined (hence bounded) from $\sum \oplus X_d$ into X^* and $\|T\| \le B$.

Proof. Let $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ be an X_d^*-Bessel woven for X with universal bound B and let $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ be the canonical unit vector basis of X_d. Define

$$R : X \to \sum \oplus (X_d)^*$$

by

$$R(g) = \{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]}.$$

We have

$$\|R(g)\| = \|\{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]}\| = \sup |f_{ij}(g(f))| = \sup_{g\in X^*,\|g\|=1} |G_g(f_{ij}(g(f)))|$$
$$\le \sup \|G_g\|\|f_{ij}(g(f))\|.$$

Then $\|R\| \le B$, the linear bounded operator $R^* : \sum \oplus (X_d)^{**} \to X^*$ satisfies:

$$R^*(e_{ij})(g) = e_{ij}(R(g)) = f_{ij}(g), \quad \text{for all } g \in X,$$

and thus $R^* e_{ij} = f_{ij}$. Letting $T = R^*|_{\sum \oplus X_d}$, we have

$$\|T\| \le \|R^*\| = \|R\| \le B.$$

Finally, $T\left(\{c_{ij}\}_{i\in\mathbb{I},j\in[m]}\right) = T\left(\sum_{i\in\mathbb{I},j\in[m]} c_{ij}e_{ij}\right) = \sum_{i\in\mathbb{I},j\in[m]} c_{ij}f_{ij}$.

Now suppose that $T : \sum \oplus X_d \longrightarrow X^*$ given by $T(\{c_{ij}\}) = \sum_{i\in\mathbb{I},j\in[m]} c_{ij}f_{ij}$ is well defined and thus bounded by the Banach-Steinhaus theorem. Then $T(e_{ij}) = f_{ij}$ and for every $g \in X$ the operator

$$T^* : X^{**} \to \sum \oplus (X_d)^*, \quad T^*(g)(e_{ij}) = g(T(e_{ij})) = g(f_{ij}),$$

is bounded. That is, $\{f_{ij}(g)\}_{i\in\mathbb{I},j\in[m]} = \{T^*(g)(e_{ij})\}_{i\in\mathbb{I},j\in[m]}$ which is identified with $T^*(g)$ (by Lemma 3.2). So, $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is a X_d^*- Bessel sequence for X with a bound $\|T^*\| = \|T\| \le B$. □

Theorem 3.2. *The family* $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ *is a Bessel woven with Bessel bound* B *if and only if the operator*

$$T : \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \to \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}, \quad \text{for all } \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right),$$

is a well-defined bounded operator from $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ *into* X *and* $\|T\| \leq \sqrt{B}$.

Proof. First assume that $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is a Bessel woven with bound B.

Let $\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ be in $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$. We show that $T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ is well-defined, that is, $\sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}$ is convergent. Consider $n, m \in \mathbb{I}$, $n > m$. Then

$$\begin{aligned}\left\| \sum_{i=1,j\in[m]}^{n} l_{i,j}\varphi_{ij} - \sum_{i=1,j\in[m]}^{m} l_{ij}\varphi_{ij} \right\| &= \left\| \sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij} \right\| \\ &= \sup_{\|g^*\|=1,g\in X} g^* \left(\sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij} \right) = *.\end{aligned}$$

Using the duality mappings Φ_X and its induced semi-inner product $[f, g] = g^*(f)$ we have

$$\begin{aligned} * = \sup_{\|g\|=1} \left| \left[\sum_{i=m+1,j\in[m]}^{n} l_{ij}\varphi_{ij}, g \right] \right| &\leq \sup_{\|g\|=1} \sum_{i=m+1,j\in[m]}^{n} |l_{ij} [\varphi_{ij}, g]| \\ \leq \sup \|\{l_{ij}\}\|_{X_d} \| [\varphi_{ij}, g] \|_{X_d^*} &\leq \sup \|\{l_{ij}\}\|_{X_d} B \|g\|_X. \end{aligned}$$

Since $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$, we know that $\|\{l_{ij}\}_{i=1,j\in[m]}^{n}\|_{X_d}$ is a Cauchy sequence in $\mathbb{C}$, The above calculation shows that $\{\sum_{i=1,j\in[m]}^{n} l_{ij}\varphi_{ij}\}_{n=1}^{\infty}$ is a Cauchy sequence in X, and therefore convergent. Thus, $T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}$ is well-defined. Clearly T is linear, and

$$\left\| T\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \right\| = \sup_{\|f\|=1} \left| \left[T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}, f \right] \right|,$$

that is, $\|T\| \leq \sqrt{B}$.

Conversely, suppose T well-defined and that $\|T\| \leq \sqrt{B}$, for every $f \in X$, we have

$$[T\{l_{ij}\}, f] = \left[\sum l_{ij} f_{ij}, f \right] = [\{l_{ij}\}, \{[f, f_{ij}]\}],$$

therefore

$$T^* f = \{[f, f_{ij}]\}_{i\in\mathbb{I},j\in[m]}$$

and

$$\sum_{i\in\mathbb{I},j\in[m]} |[f, f_{ij}]|^2 = \|T^* f\|^2 \leq \|T^*\|^2 \|f\|^2 \leq \sqrt{B} \|f\|^2.$$

Hence, we conclude that the family $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is Bessel woven. $\square$

Theorem 3.3. *Let the sequence* $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *in* X *be woven for* X, *and the series* $\sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}$ *converges for all* $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$. *Then the operator*

$$T : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X, \quad T\{l_{ij}\}_{i=1,j\in[m]}^{\infty} := \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij},$$

defines a bounded linear operator. The adjoint operator is given by

$$T^* : X^* \to \left(\sum_{j\in[m]} \oplus (X_d)_j\right)^*, \quad T^*\varphi = \{[\varphi, \varphi_{ij}]\}_{i=1,j\in[m]}^{\infty}.$$

Furthermore,

$$\sum_{i=1,j\in[m]}^{\infty} |\,[\varphi, \varphi_{ij}]\,|^2 \leq \|T\|^2\|\varphi\|^2.$$

Proof. Consider the sequence of bounded linear operators

$$T_n : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X, \quad T_n\{l_{ij}\}_{i=1,j\in[m]}^{\infty} := \sum_{i=1,j\in[m]}^{n} l_{ij}\varphi_{ij}.$$

Clearly $T_n \to T$ pointwise as $n \to \infty$, so T is bounded. In order to find the expression for T^*, let $f, \varphi \in X$, $\{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right)$. Then

$$\left[\varphi, T\{l_{ij}\}_{i=1,j\in[m]}^{\infty}\right]_X = \left[\varphi, \sum_{i=1,j\in[m]}^{\infty} l_{ij}\varphi_{ij}\right] = \sum_{i=1,j\in[m]}^{\infty} [\varphi, \varphi_{ij}]\,\overline{l_{ij}}.$$

Alternatively, when $T : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X$ is bounded, then clearly T^* is a bounded operator from X^* to $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)^*$. Therefore, the i-th coordinate function is bounded from X to $\mathbb{C}$; by Riesz representation theorem, T^* has the form

$$T^*\varphi = \{[\varphi, \varphi_{ij}]\}_{i=1,j\in[m]}^{\infty},$$

for some $\{\varphi_{ij}\}_{i\in\mathbb{I},j\in[m]}$ in X. By definition of T^*, we conclude

$$\sum_{i=1,j\in[m]}^{\infty} [\varphi, f_{ij}]\,\overline{l_{ij}} = \sum_{i=1,j\in[m]}^{\infty} [\varphi, \varphi_{ij}]\,\overline{l_{ij}}, \quad \text{for all } \{l_{ij}\}_{i=1,j\in[m]}^{\infty} \in \left(\sum_{j\in[m]} \oplus (X_d)_j\right), f \in X.$$

It follows from here that $f_{ij} = \varphi_{ij}$. The adjoint of a bounded operator T is itself bounded, and $\|T\| = \|T^*\|$. Under the assumption in Theorem 3.2, we have

$$\|T^*\varphi\|^2 \leq \|T\|^2\|\varphi\|^2, \quad \text{for all } \varphi \in X,$$

which leads to

$$\sum_{i=1,j\in[m]}^{\infty} |\,[\varphi, \varphi_{ij}]\,|^2 \leq \|T\|^2\|\varphi\|^2, \quad \text{for all } \varphi \in X. \qquad \square$$

Definition 3.8. Let X be a Banach space and X_d a sequence space. Given a bounded linear operator $S : \left(\sum_{j\in[m]} \oplus (X_d)_j\right) \to X$ and a $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$-woven $\{g_{ij}\} \subset X^*$, we say that $(\{g_{ij}\}, S)$ is a Banach frame for X with respect to $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ if

$$S\left(\{g_{ij}(f)\}\right) = f, \quad \text{for all } f \in X. \tag{3.3}$$

Note that (3.3) can be considered as some kind of generalized reconstruction formula, in the sense that it tells how to come back to $f \in X$ via the coefficients $\{g_{ij}(f)\}$.

The condition, however, does not imply reconstruction via an infinite series, as we will see later. For more information on Banach frames we refer to [7, 17].

The woven X_d-frame condition implies that one can define the following isomorphism

$$U : X \to \left(\sum_{j\in[m]} \oplus (X_d)_j\right), \quad Uf := \{g_{ij}(f)\}, \quad f \in X.$$

The extra condition in Definition 3.8 means that S is a left-inverse of U, and thus US is a bounded linear projection of $\left(\sum_{j\in[m]} \oplus (X_d)_j\right)$ onto the range $R(U)$ of the operator U.

Proposition 3.2. *Suppose that X_d is a BK-space and $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X^*$ is a woven X_d-frame for X. Then, the following conditions are equivalent.*

(a) *$R(U)$ is complemented in X_d.*
(b) *The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : X_d \to X$.*
(c) *There exists a linear bounded operator S, such that $\left(\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}, S\right)$ is a Banach woven for X with respect to X_d.*

Also, the condition

(d) *there exists a family $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X$ such that $\{\sum c_{ij} f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is convergent for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$ and*

$$f = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}, \quad \text{for all } f \in X;$$

implies each of (a)-(c)*.*

If we also assume that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d, (d) *is equivalent to* (a)-(c)*.*

(e) *There exists an X_d^*-Bessel woven $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X \subseteq X^{**}$ for X^* such that*

$$f = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}, \quad \text{for all } f \in X.$$

If the canonical unit vectors form a basis for both X_d and X_d^*, (a)-(e) *is equivalent to*

(f) *there exists an X_d^*-Bessel woven $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]} \subset X \subset X^{**}$ for X^* such that*

$$g = \sum_{i\in\mathbb{I},j\in[m]} g(f_{ij}) g_{ij}, \quad \text{for all } g \in X^*.$$

In each of the cases (e) *and* (f), $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *is actually an* X_d^**-woven for* X^*.

Proof. For convenience, we index $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ and $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$ by the natural numbers. Suppose that X_d is a BK-space. $(a)\to(b)$ is trivial. For the converse, assume (b) and let $V : X_d \to X$ be a linear bounded extension of U^{-1}. Now consider the bounded operator $P : X_d \to R(U)$ defined by $P = UV$. Using the fact that $VU = I$ (on X), we get $P^2 = P$. For every $f \in X$, we have

$$Uf = UVUf = P(Uf) \in R(P).$$

Hence $R(P) = R(U)$, i.e., the range of U equals the range of a bounded projection. Thus, $R(U)$ is complemented (see [27, page 127]). The equivalence $(b) \leftrightarrow (c)$ is clear. We now relate the condition (d) to (a)-(c). First, assume that (d) is satisfied. By assumption, we can define an operator

$$V : X_d \to X, \quad V : \{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \to \sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}.$$

By the Banach-Steinhaus theorem, V is bounded. Let $\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]} \in R(U)$. Furthermore,

$$V(g_{ij}(f)) = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij} = f = U^{-1}Uf = U^{-1}\{g_{ij}(f)\}_{i\in\mathbb{I},j\in[m]},$$

that is, V is an extension of U^{-1}. That is, (b) holds, according to the equivalences proved so far, this means that (a)-(c) holds.

Assume now that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d. Assuming that (b) is satisfied, we show that (d) holds. Let $f_{ij} := Ve_{ij}$. Since V is linear and bounded, for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$, we have

$$\sum_{i=1,j\in[m]}^{n} c_{ij} f_{ij} = V\left(\sum_{i=1,j\in[m]}^{n} c_{ij} e_{ij}\right) \to V(c_{ij}).$$

That is, $\sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$ is convergent. Also, by construction, for all $f \in X$ we have

$$f = VUf = \sum_{i\in\mathbb{I},j\in[m]} g_{ij}(f) f_{ij}. \tag{3.4}$$

Thus, (d) holds as claimed.

Under the assumption that the canonical unit vectors $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ form a basis for X_d, we now prove the equivalence of (d) and (e). First, assume that (d) holds. Due to the equivalence of (b) and (d), we can define $f_{ij} := Le_{ij}$, and (3.4) is available. By Lemma 3.2, for every $g \in X^*$ we have

$$\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]} = \{gV(e_{ij})\}_{i\in\mathbb{I},j\in[m]} \in X_d^*$$

and

$$\|\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]}\|_{X_d^*} = \|gV\| \le \|V\|\,\|g\|_{X^*},$$

hence $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$, considered as a sequence in X^{**}, is an X_d^*-Bessel sequence for X^*. Thus, we have proved the claims in (e). On the other hand, if (e) is valid, then Proposition 3.1 shows that $\sum_{i\in\mathbb{I},j\in[m]} c_{ij} f_{ij}$ is convergent for all $\{c_{ij}\}_{i\in\mathbb{I},j\in[m]} \in X_d$ and hence (d) holds.

Now, assume that the canonical unit vectors form a basis for both X_d and X_d^*; in this case, we want to prove the equivalence of (e) and (f). Let B denote a Bessel bound for the X_d-Bessel sequence $\{g_{ij}\}_{i\in\mathbb{I},j\in[m]}$. Denote the canonical basis for X_d by $\{e_{ij}\}_{i\in\mathbb{I},j\in[m]}$ and the canonical basis for X_d^* by $\{z_{ij}\}_{i\in\mathbb{I},j\in[m]}$. Assume that (e) is valid. Let $g \in X^*$. For given $n \in \mathbb{N}$

$$\begin{aligned}
\left\| g - \sum_{i=1,j\in[m]}^{n} g(f_{ij}) g_{ij} \right\|_{X^*} &= \sup_{f\in X,\|f\|=1} \left| g(f) - \sum_{i=1,j\in[m]}^{n} g(f_{ij}) g_{ij}(f) \right| \\
&= \sup_{f\in X,\|f\|=1} \left| \sum_{i=1,j\in[m]}^{\infty} g(f_{ij}) g_{ij}(f) - \sum_{i=1,j\in[m]}^{n} g(f_{ij}) g_{ij}(f) \right| \\
&= \sup_{f\in X,\|f\|=1} \left| \sum_{i=n+1,j\in[m]}^{\infty} g(f_{ij}) g_{ij}(f) \right| \\
&\leq B \left\| \sum_{i=n+1,j\in[m]}^{\infty} g(f_{ij}) z_{ij} \right\| \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

and hence (f) holds. Assume (f) and let K be an X_d^*-Bessel bound for $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$. For every $g \in X^*$, $\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]}$ belongs to X_d^*, which by Lemma 3.2 is isometrically isomorphic to the space $\{\{G(e_{ij})\}_{i\in\mathbb{I},j\in[m]} | G \in X_d^*\}$, and hence $\{g(f_{ij})\}_{i\in\mathbb{I},j\in[m]}$ can be identified with $\{G_g(e_{ij})\}_{i\in\mathbb{I},j\in[m]}$ for a unique $G_g \in X_d^*$. Then for every $f \in X$

$$\begin{aligned}
\left\| f - \sum_{i=1,j\in[m]}^{n} g_{ij}(f) f_{ij} \right\|_{X} &= \sup_{g\in X^*,\|g\|=1} \left| g(f) - \sum_{i=1,j\in[m]}^{n} g(f_{ij}) g_{ij}(f) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left| \sum_{i=1,j\in[m]}^{\infty} g(f_{ij}) g_{ij}(f) - \sum_{i=1}^{n} g(f_{ij}) g_{ij}(f) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left| \sum_{i=n+1,j\in[m]}^{\infty} g(f_{ij}) g_{ij}(f) \right| \\
&= \sup_{g\in X^*,\|g\|=1} \left\| G_g \left(\sum_{i=n+1,j\in[m]}^{\infty} g_{ij}(f) e_{ij} \right) \right\| \\
&\leq \sup_{g\in X^*,\|g\|=1} \|G_g\| \left\| \sum_{i=n+1,j\in[m]}^{\infty} g_{ij}(f) e_{ij} \right\| \\
&= \sup_{g\in X^*,\|g\|=1} \left\| \{g(f_{ij})\} \sum_{i=n+1,j\in[m]}^{\infty} g_{ij}(f) e_{ij} \right\|
\end{aligned}$$

$$\leq K \left\| \sum_{i=n+1, j\in[m]}^{\infty} g_{ij}(f)\, e_{ij} T \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, (f) is valid. Moreover, by a similar calculations as above, for every $g \in X^*$ we have

$$\|g\| = \sup_{f\in X^*, \|f\|=1} |g(f)| = \sup_{f\in X^*, \|f\|=1} \left| \sum_{i\in\mathbb{I}, j\in[m]} g(f_{ij})\, g_{ij}(f) \right| \leq B \left\| \{g(f_{ij})\}_{i\in\mathbb{I}, j\in[m]} \right\|_{X_d^*},$$

and hence $\{f_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ is a woven X_d^*-frame for X^*. □

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