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ON THE ZAGREB INDEX OF TOURNAMENTS

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Abstract. A tournament is an orientation of a complete simple graph. The score of a vertex in a tournament is the out degree of the vertex. The Zagreb index of a tournament is defined as the sum of the squares of the scores of its vertices. In this paper, we obtain various lower and upper bounds for the Zagreb index of a tournament.

1. Introduction

A tournament is an orientation of a complete simple graph. Let T be a tournament with order n and having vertex set $\{v_1, v_2, \dots, v_n\}$. The score of a vertex v_i, $1 \leq i \leq n$, denoted by s_{v_i} (or simply by s_i), is defined as the out degree of v_i. Clearly, $0 \leq s_i \leq n-1$ for all i, $1 \leq i \leq n$. The sequence $[s_1, s_2, \dots, s_n]$ in non-decreasing order is called the score sequence of the tournament T. A regular tournament on n (odd) vertices is a tournament in which score of every vertex is $\frac{n-1}{2}$. Many of the important properties of tournaments were first investigated by Landau [5] (1953) in order to model dominance relations in flocks of chickens. Current applications of tournaments include the study of voting theory and social choice theory among other things. Other undefined notations and terminology can be seen in [8].

The following result [5], also called Landau's theorem, gives a necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

Theorem 1.1 (Landau [5]). *A sequence $[s_1, s_2, \dots, s_n]$ of non-negative integers in non-decreasing order is a score sequence of some tournament if and only if*

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$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}, \quad \textit{for } 1 \leq k \leq n, \tag{1.1}$$

with equality when $k = n$.

Several results for the scores in a tournament can be seen in [3, 6, 7, 9, 13]. Also, stronger inequalities for scores in tournaments can be found in [2]. Further the extension of scores to oriented graphs and digraphs can be seen in [10–12].

For any two distinct vertices u and v of a tournament T, we have one of the following possibilities:

(i) there is an arc directed from u to v which is denoted by $u(1-0)v$;
(ii) there is an arc directed from v to u which is denoted by $u(0-1)v$.

One of the oldest graph invariants is the well-known Zagreb index first introduced by Gutman and Trinajstić [4], where they examined the dependence of total π-electron energy on molecular structure. Some recent work can be seen in [1]. The (first) Zagreb index $M_1(G)$ of a graph G is defined as the sum of the squares of the degrees of the vertices of G and the second Zagreb index $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices. These two topological indices (M_1 and M_2) reflect the extent of branching of the molecular carbon-atom skeleton. Determining the extremal values or bounds of these two topological indices of graphs, as well as characterizing the corresponding extremal graphs, has attracted the attention of many researchers. Analogous to this, we define the Zagreb index $M(T)$ of a tournament T as the sum of the scores of the vertices of T. That is, $M(T) = \sum_{i=1}^{n} s_i^2$.

The rest of the paper is organized as follows. In Section 2, we obtain the lower bounds for the Zagreb index $M(T)$ of a tournament T. In Section 3, we compute the upper bounds for $M(T)$.

2. Lower Bounds for the Zagreb Index $M(T)$

The following result gives the best general lower bound for $M(T)$.

Theorem 2.1. *If $[s_1, s_2, \dots, s_n]$ is the score sequence of a tournament T, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n}{2}\Big\{2m(n-m-2) + n - 1\Big\}, \quad \textit{where } m = \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{2.1}$$

with equality if and only if $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. Let v_i and v_j be two vertices of the tournament T with their respective scores as s_i and s_j such that $s_i \geq s_j$. Also, assume that $M(T) = \sum_{r=1}^{n} {s_r}^2$ is minimum.

We claim that $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. To prove the claim, we assume to the contrary that $s_i - s_j > 1$ for some i, j, $1 \leq i, j \leq n$. Then there exists a vertex v_k with score s_k such that $v_i(1-0)v_k$ and $v_k(1-0)v_j$. Now, reversing the orientation of these arcs to $v_i(0-1)v_k$ and $v_k(0-1)v_j$ respectively, we get a new tournament T_1

with the score sequence $[t_1, t_2, \ldots, t_n]$, where $t_i = s_i - 1$, $t_j = s_j + 1$, $t_r = s_r$ for all r, $1 \le r \le n$ with $r \neq i, j$.

Thus,

$$\begin{aligned}\sum_{r=1}^{n} {t_r}^2 &= \sum_{r=1}^{j-1} {t_r}^2 + t_j^2 + \sum_{r=j+1}^{i-1} {t_r}^2 + t_i^2 + \sum_{r=i+1}^{n} {t_r}^2\\ &= \sum_{r=1}^{j-1} {s_r}^2 + (s_j+1)^2 + \sum_{r=j+1}^{i-1} {s_r}^2 + (s_i-1)^2 + \sum_{r=i+1}^{n} {s_r}^2\\ &= \sum_{r=1}^{n} {s_r}^2 - 2(s_i - s_j - 1).\end{aligned}$$

As $s_i - s_j > 1$, so we obtain

$$\sum_{r=1}^{n} {t_r}^2 < \sum_{r=1}^{n} {s_r}^2,$$

which is a contradiction, since $M(T) = \sum_{r=1}^{n} {s_r}^2$ is minimum. Hence, $s_i - s_j \le 1$ for all i, j, $1 \le i, j \le n$. This means that some of the vertices of T have score m and the remaining vertices (if any) have score $m+1$. If x vertices of T have score m and y vertices have score $m+1$, then

$$x + y = n \tag{2.2}$$

and by (1.1), we have

$$mx + (m+1)y = \frac{n(n-1)}{2}. \tag{2.3}$$

Solving (2.2) and (2.3), we get $x = \frac{n}{2}(2m - n + 3)$ and $y = \frac{n}{2}(n - 2m - 1)$. Therefore,

$$\begin{aligned}\min M(T) &= \min \sum_{i=1}^{n} {s_i}^2 = \min\{s_1^2 + s_2^2 + \cdots + s_n^2\}\\ &= \underbrace{m^2 + m^2 + \cdots + m^2}_{\frac{n}{2}(2m-n+3)-\text{times}} + \underbrace{(m+1)^2 + (m+1)^2 + \cdots + (m+1)^2}_{\frac{n}{2}(n-2m-1)-\text{times}}\\ &= \frac{n}{2}(2m-n+3)m^2 + \frac{n}{2}(n-2m-1)(m+1)^2\\ &= \frac{n}{2}\{2m(n-m-2) + n - 1\}.\end{aligned}$$

That is,

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \ge \frac{n}{2}\{2m(n-m-2) + n - 1\}.$$

Now, assume that equality holds in (2.1). Since $M(T)$ is minimal, so some of the vertices of T have score m and the remaining vertices (if any) have score $m+1$, where $m = \lfloor \frac{n-1}{2} \rfloor$. Therefore, $s_i - s_j \le 1$ for all i, j, $1 \le i, j \le n$.

Conversely, assume that $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. Then as above, we have

$$\begin{aligned}
M(T) &= \sum_{i=1}^{n} {s_i}^2 = s_1^2 + s_2^2 + \cdots + s_n^2 \\
&= \underbrace{m^2 + m^2 + \cdots + m^2}_{\frac{n}{2}(2m-n+3)-\text{times}} + \underbrace{(m+1)^2 + (m+1)^2 + \cdots + (m+1)^2}_{\frac{n}{2}(n-2m-1)-\text{times}} \\
&= \frac{n}{2}(2m - n + 3)m^2 + \frac{n}{2}(n - 2m - 1)(m+1)^2 \\
&= \frac{n}{2}\{2m(n - m - 2) + n - 1\}.
\end{aligned}$$

Therefore equality holds in (2.1). ☐

Theorem 2.2. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T and $m = \lfloor \frac{n-2}{2} \rfloor$ and $x = \frac{n-1}{2}(n - 2m - 2)$. Then the following hold.*

(i) *For $s_n > x$, we have*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n-1}{2}\{(2m+1)(n-m) - m\} + s_n^2 - x(2m+1).$$

(ii) *For $s_n \leq x$, we have*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n-1}{2}\{(2m+1)(n-m) - m\} + s_n^2 + 2x - s_n(2m+3).$$

Proof. Let v_n be the vertex of the tournament T with score s_n. Deleting the vertex v_n, we obtain a new tournament $T_1 = T - \{v_n\}$ with score sequence $[t_1, t_2, \ldots, t_{n-1}]$. By Theorem 2.1, the minimum value of $M(T)$ is attained in terms of n if and only if $s_i - s_j \leq 1$ for all i, j, $1 \leq i, j \leq n$. Using this result, we conclude that the value of $\sum_{i=1}^{n-1} {t_i}^2$ (in terms of the number of vertices) will be minimum if the value of $M(T)$ (in terms of n and s_n) is minimum. So, we have to find the minimum value of $M(T)$ in terms of n and s_n. For this, first we find the minimum value of $\sum_{i=1}^{n-1} {t_i}^2$ in terms of the number of vertices.

As the tournament T_1 has $n-1$ vertices, therefore, by using Theorem 2.1, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} {t_i}^2 &\geq \frac{n-1}{2}\{2m(n - 1 - m - 2) + (n-1) - 1\} \\
&= \frac{n-1}{2}\{2m(n - m - 3) + (n-2)\},
\end{aligned}$$

where $m = \lfloor \frac{(n-1)-1}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor$ and $t_i - t_j \leq 1$ for all i, j, $1 \leq i, j \leq n-1$.

If x vertices of T_1 have score $m+1$ and y vertices have score m, then we have

$$x + y = n - 1. \tag{2.4}$$

Also, by (1.1), we have

$$(m+1)x + my = \frac{(n-1)(n-2)}{2}. \tag{2.5}$$

Solving (2.4) and (2.5) for x, we have $x = \frac{n-1}{2}(n-2m-2)$. So, T_1 has $x = \frac{n-1}{2}(n-2m-2)$ vertices of score $m+1$ and $n-1-x$ vertices of score m.

Now, we add the vertex v_n of score s_n and join it to the other vertices of the tournament T_1 by arcs, such that $M(T) = \sum_{i=1}^{n} {s_i}^2$ is minimum. This can be done as follows. Let $v_n(1-0)u$ to as many vertices u of score $m+1$ as possible and then $v_n(1-0)v$ to the remaining vertices v of score m till the score s_n is exhausted. Note that other arcs are directed towards v_n in order to complete the tournament. Now, we consider the following two cases.

Case (i). When $s_n > x$, then

$$\min M(T) = \min \sum_{i=1}^{n} {s_i}^2 = \min \sum_{i=1}^{n-1} {t_i}^2 + s_n^2 + (n-1-x)(2m+1),$$

that is,

$$\begin{aligned} M(T) &\geq \frac{n-1}{2}\{2m(n-m-3)+n-2\} + s_n^2 + (n-1)(2m+1) - x(2m+1) \\ &= \frac{n-1}{2}\{(2m+1)(n-m)-m\} + s_n^2 - x(2m+1), \end{aligned}$$

where $m = \lfloor \frac{n-2}{2} \rfloor$ and $x = \frac{n-1}{2}(n-2m-2)$.

Case (ii). When $s_n \leq x$, then

$$\begin{aligned} \min M(T) &= \min \sum_{i=1}^{n} {s_i}^2 \\ &= \min \sum_{i=1}^{n-1} {t_i}^2 + s_n^2 + (n-1-x)(2m+1) + (x-s_n)\{2(m+1)+1\}, \end{aligned}$$

that is,

$$\begin{aligned} M(T) \geq& \frac{n-1}{2}\{2m(n-m-3)+n-2\} + s_n^2 + (n-1)(2m+1) - x(2m+1) \\ &+ (x-s_n)(2m+3) \\ =& \frac{n-1}{2}\{(2m+1)(n-m)-m\} + s_n^2 + 2x - s_n(2m+3), \end{aligned}$$

where $m = \lfloor \frac{n-2}{2} \rfloor$ and $x = \frac{n-1}{2}(n-2m-2)$. □

Remark 2.1. The lower bounds given by Theorems 2.1 and 2.2 are best possible, since these bounds hold for every score sequence $[s_1, s_2, \ldots, s_n]$ of a tournament. In particular, these hold for a regular tournament on n (odd) vertices having score sequence $\left[\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2}\right]$. Clearly $\sum_{i=1}^{n} s_i^2$ is minimum and so the equality in Theorems 2.1 and 2.2 hold for regular tournaments.

Theorem 2.3. *If* $[s_1, s_2, \ldots, s_n]$ *is the score sequence of a tournament* T*, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq s_1^2 + s_n^2 + \frac{1}{n-2}\left\{\frac{n(n-1)}{2} - s_1 - s_2\right\}^2,$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$.

Proof. Consider $s_2, s_3, \ldots, s_{n-1}$ as the weights assigned to the scores $s_2, s_3, \ldots, s_{n-1}$, respectively. Since the arithmetic mean is greater than or equal to the harmonic mean, therefore

$$\frac{\sum\limits_{i=2}^{n-1} s_i s_i}{\sum\limits_{i=2}^{n-1} s_i} \geq \frac{\sum\limits_{i=2}^{n-1} s_i}{\sum\limits_{i=2}^{n-1} \frac{s_i}{s_i}},$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$. That is,

$$\sum_{i=2}^{n-1} {s_i}^2 \geq \frac{1}{n-2}\left(\sum_{i=2}^{n-1} s_i\right)^2,$$

with equality if and only if $s_2 = s_3 = \cdots = s_{n-1}$. After simplification, it is easy to see that

$$\sum_{i=1}^{n} {s_i}^2 - s_1^2 - s_n^2 \geq \frac{1}{n-2}\left(\sum_{i=1}^{n} s_i - s_1 - s_n\right)^2.$$

By using (1.1), we have

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq s_1^2 + s_n^2 + \frac{1}{n-2}\left\{\frac{n(n-1)}{2} - s_1 - s_n\right\}^2,$$

equality holds if and only if $s_2 = s_3 = \cdots = s_{n-1}$. □

Theorem 2.4. *If* $[s_1, s_2, \ldots, s_n]$ *is the score sequence of a tournament* T*, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \geq \frac{n}{4}(n-1)^2,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$.

Proof. Applying the Cauchy-Schwartz inequality, we have

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i \cdot 1 \leq \left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}},$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. This is equivalent to

$$\sum_{i=1}^{n} s_i \leq \left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} n^{\frac{1}{2}},$$

which after simplification gives

$$\left(\sum_{i=1}^{n} {s_i}^2\right)^{\frac{1}{2}} \geq \frac{1}{n^{\frac{1}{2}}} \sum_{i=1}^{n} s_i,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. Now, by using (1.1), we have

$$\sum_{i=1}^{n} {s_i}^2 \geq \frac{1}{n}\left\{\frac{n(n-1)}{2}\right\}^2 = \frac{n}{4}(n-1)^2,$$

where equality occurs if and only if $s_1 = s_2 = \cdots = s_n$. Thus,

$$M(T) \geq \frac{n}{4}(n-1)^2,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$. □

3. Upper Bounds for the Zagreb Index $M(T)$

In this section, we obtain the upper bounds for the Zagreb index $M(T)$. In a tournament, we denote with N_i^+ the out-neighbor set of the vertex v_i.

Theorem 3.1. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament and $M(T) = \sum\limits_{i=1}^{n} {s_i}^2$ be maximum. Then*

- **(a)** $N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$ *if and only if* $s_i = s_j$*;*
- **(b)** $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$ *if and only if* $s_i > s_j$*, and*
- **(c)** $s_i < s_j$ *if* $v_i \in N_k^+$ *and* $v_j \in (N_k^+)^c - \{v_k\}$*, where* s_i *and* s_j *are the scores of the two vertices* v_i *and* v_j *respectively.*

Proof. **(a)** Let $s_i = s_j$. Assume to the contrary that $N_i^+ - \{v_j\} \neq N_j^+ - \{v_i\}$. Since $s_i = s_j$, therefore there exist at least two vertex v_p and v_q with their respective scores s_p and s_q such that $v_i(1-0)v_p$, $v_p(1-0)v_j$, $v_j(1-0)v_q$ and $v_q(1-0)v_i$. Now, we consider two cases.

Case (i). When $s_p \geq s_q$. By changing the arcs $v_i(1-0)v_p$ and $v_q(1-0)v_i$ to $v_i(0-1)v_p$ and $v_q(0-1)v_i$ respectively, we get a new score sequence $[t_1, t_2, \ldots, t_n]$, where $t_p = s_p + 1$, $t_q = s_q - 1$ and $t_r = s_r$ for all r, $1 \leq r \leq n$ with $r \neq p, q$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} {t_i}^2 &= \sum_{\substack{i=1\\ i\neq p,q}}^{n} {t_i}^2 + t_p^2 + t_q^2 = \sum_{\substack{i=1\\ i\neq p,q}}^{n} {s_i}^2 + (s_p+1)^2 + (s_q-1)^2\\ &= \sum_{i=1}^{n} {s_i}^2 + 2(s_p - s_q + 1) > \sum_{i=1}^{n} {s_i}^2,\end{aligned}$$

since $s_p \geq s_q$, which is a contradiction, since $M(T) = \sum_{i=1}^{n} {s_i}^2$ was assumed to be maximum.

Case (ii). When $s_p < s_q$. By changing the arcs $v_p(1-0)v_j$ and $v_j(1-0)v_q$ to $v_p(0-1)v_j$ and $v_j(0-1)v_q$, respectively and proceeding as in case (i), we arrive at a contradiction. Hence, $N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$.

Conversely, if $N_i^+ - \{v_j\} = N_j^+ - \{v_i\}$, then $s_i = s_j$.

(b) Let $s_i > s_j$. Assume to the contrary that $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$ is not true. Then there exists a vertex $v_p \in N_j^+ - \{v_i\}$, but $v_p \notin N_i^+ - \{v_j\}$. This means that $v_j(1-0)v_p$ and $v_p(1-0)v_i$, and by changing these arcs to $v_j(0-1)v_p$ and $v_p(0-1)v_i$ respectively, we get a new score sequence $[t_1, t_2, \dots, t_n]$, where $t_i = s_i + 1,\ t_j = s_j - 1$ and $t_r = s_r$ for all r, $1 \le r \le n$ with $r \neq i, j$. Then

$$\begin{aligned}\sum_{r=1}^{n} {t_r}^2 &= \sum_{\substack{r=1\\ r\neq i,j}}^{n} {t_r}^2 + t_i^2 + t_j^2 = \sum_{\substack{r=1\\ r\neq i,j}}^{n} {s_r}^2 + (s_i+1)^2 + (s_j-1)^2\\ &= \sum_{r=1}^{n} {s_r}^2 + 2(s_i - s_j + 1) > \sum_{r=1}^{n} {s_r}^2,\end{aligned}$$

since $s_i > s_j$, which is a contradiction, since $M(T)$ was assumed to be maximum. Hence, $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$.

Conversely, if $N_i^+ - \{v_j\} \supsetneq N_j^+ - \{v_i\}$, then $s_i > s_j$.

(c) Assume to the contrary that $s_i \geq s_j$. Then, by using parts **(a)** and **(b)**, we have $N_i^+ - \{v_j\} \supseteq N_j^+ - \{v_i\}$. Since $v_i \in N_k^+$ and $v_j \in (N_k^+)^c - \{v_k\}$, so $v_k(1-0)v_i$ and $v_j(1-0)v_k$. Therefore,

$$\{v_k\} \subseteq N_j^+ - \{v_i\} \subseteq N_i^+ - \{v_j\},$$

that is, $v_k \in N_i^+ - \{v_j\}$. Thus, we obtain $v_i(1-0)v_k$, which is a contradiction. Hence, the result follows. ☐

Lemma 3.1. *Let $[s_1, s_2, \dots, s_n]$ be the score sequence of a tournament and let m_i be the average of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 = \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i.$$

Proof. Since

$$s_i m_i = s_i \frac{1}{s_i} \sum_{j=1}^{n} \{s_j :\ v_i(1-0)v_j\} = \sum_{j=1}^{n} \{s_j : v_i(1-0)v_j\},$$

therefore, by using (1.1), we have

$$\begin{aligned}\sum_{i=1}^{n} s_i m_i &= \sum_{i=1}^{n}\sum_{j=1}^{n} \{s_j :\ v_i(1-0)v_j\} = \sum_{j=1}^{n}\sum_{i=1}^{n} \{s_j :\ v_i(1-0)v_j\}\\ &= \sum_{j=1}^{n} s_j(n-1-s_j) = (n-1)\sum_{j=1}^{n} s_j - \sum_{j=1}^{n} s_j^2\\ &= (n-1)\frac{n(n-1)}{2} - \sum_{j=1}^{n} s_j^2.\end{aligned}$$

Hence,

$$M(T) = \sum_{i=1}^{n} s_i^2 = \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i.$$

□

Theorem 3.2. *If* $[s_1, s_2, \ldots, s_n]$ *is the score sequence of a tournament* T*, then*

$$M(T) = \sum_{i=1}^{n} {s_i}^2 \leq \frac{n(n-1)}{2} s_n, \tag{3.1}$$

with equality if and only if the tournament is regular.

Proof. Let m_i be the average of the scores of the vertices v_j such that $v_i(1-0)v_j$. Then, by using (1.1), we have

$$\begin{aligned} s_i m_i =& s_i \frac{1}{s_i} \sum_{j=1}^{n} \{s_j : v_i(1-0)v_j\} \geq \sum_{j=1}^{n} s_j - s_i - (n-1-s_i)s_n \\ =& \frac{n(n-1)}{2} - s_i - (n-1-s_i)s_n, \end{aligned} \tag{3.2}$$

with equality if and only if $s_i = \frac{n-1}{2}$ for all i, $1 \leq i \leq n$.

Now, by Lemma 3.1, (3.2) and (1.1), we have

$$\begin{aligned} \sum_{i=1}^{n} s_i^2 &= \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i \leq \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} \left(\frac{n(n-1)}{2} - s_i - (n-1-s_i)s_n \right) \\ &= \frac{n(n-1)^2}{2} - \frac{n^2(n-1)}{2} + \sum_{i=1}^{n} s_i + n(n-1)s_n - s_n \sum_{i=1}^{n} s_i \\ &= \frac{n(n-1)^2}{2} - \frac{n^2(n-1)}{2} + \frac{n(n-1)}{2} + n(n-1)s_n - s_n \frac{n(n-1)}{2} \\ &= \frac{n(n-1)}{2} s_n. \end{aligned}$$

Therefore,

$$M(T) \leq \frac{n(n-1)}{2} s_n.$$

Now suppose that equality holds in (3.1). Then, $s_i = \frac{n-1}{2}$ for all i, $1 \leq i \leq n$, that is, the tournament is regular.

Conversely, suppose that the tournament is regular. Then, it can be easily checked that equality holds in (3.1). □

Theorem 3.3. *Let* $[s_1, s_2, \ldots, s_n]$ *be the score sequence of a tournament with vertex set* V *and let* m_i *be the average of the scores of the vertices* v_j *such that* $v_i(1-0)v_j$*. Then*

$$M(T) = \sum_{j=1}^{n} s_j^2 \leq \frac{n(n-1)}{4} \Big(n - 1 + \max\{s_j - m_j : v_j \in V\} \Big), \tag{3.3}$$

with equality if and only if $\max\{s_j - m_j : v_j \in V\} = n - 1 - 2m_i$*, where* $1 \leq i, j \leq n$*, with* $i \neq j$*.*

Proof. Applying Lemma 3.1, we have

$$\begin{aligned}
2\sum_{j=1}^{n} s_j^2 &= \sum_{j=1}^{n} s_j^2 + \sum_{j=1}^{n} s_j^2 = \sum_{j=1}^{n} s_j^2 + \frac{n(n-1)^2}{2} - \sum_{j=1}^{n} s_j m_j \\
&= \frac{n(n-1)^2}{2} + \sum_{j=1}^{n} s_j(s_j - m_j) \leq \frac{n(n-1)^2}{2} + \max\{s_j - m_j : v_j \in V\} \sum_{j=1}^{n} s_j \\
&= \frac{n(n-1)}{2}\Big(n - 1 + \max\{s_j - m_j : v_j \in V\}\Big).
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} s_j^2 \leq \frac{n(n-1)}{4}\Big(n - 1 + \max\{s_j - m_j : v_j \in V\}\Big)$$

Equality holds in (3.3) if and only if

$$\sum_{j=1}^{n} s_j^2 = \frac{n(n-1)^2}{4} + \frac{n(n-1)}{4}p, \tag{3.4}$$

where $p = \max\{s_j - m_j : v_j \in V\}$. By Lemma 3.1, (3.4) is equivalent to

$$\frac{n(n-1)^2}{2} - \sum_{j=1}^{n} s_j m_j = \frac{n(n-1)^2}{4} + \frac{n(n-1)}{4}p,$$

which after simplification gives

$$\frac{n(n-1)}{2}\Big(\frac{p - n + 1}{2}\Big) + \sum_{j=1}^{n} s_j m_j = 0.$$

By (1.1), this implies that

$$\sum_{j=1}^{n} s_j\Big(\frac{p - n + 1}{2}\Big) + \sum_{j=1}^{n} s_j m_j = 0,$$

that is,

$$\sum_{j=1}^{n} s_j\Big(\frac{p - n + 1}{2} + m_j\Big) = 0.$$

Finally, after simplification, we have

$$\sum_{j=1}^{n} s_j(p - n + 1 + 2m_j) = 0. \tag{3.5}$$

Now, assume that equality holds in (3.3). Then (3.5) holds. Since each term in this summation is non-negative and sum is equal to zero, therefore for each v_i either

$s_i = 0$ or $\max\{s_j - m_j : v_j \in V\} = p = n-1-2m_i$. But $s_i = 0$ is not possible for each v_i in any tournament (except the tournament with only one vertex), therefore

$$\max\{s_j - m_j : v_j \in V\} = n-1-2m_i.$$

Conversely, assume that $\max\{s_j - m_j : v_j \in V\} = n-1-2m_i$. Then, by Lemma 3.1, we have

$$\begin{aligned}\frac{n(n-1)}{4}\Big(n-1+\max\{s_j-m_j:\ v_j\in V\}\Big) &= \frac{n(n-1)^2}{4} + \frac{n(n-1)}{4}(n-1-2m_i)\\ &= \frac{n(n-1)^2}{4} + \frac{n(n-1)^2}{4} - \frac{n(n-1)}{2}m_i\\ &= \frac{n(n-1)^2}{2} - \sum_{i=1}^{n} s_i m_i = \sum_{i=1}^{n} s_i^2.\end{aligned}$$

Therefore, equality holds in (3.3). □

Theorem 3.4. *If $[s_1, s_2, \ldots, s_n]$ is the score sequence of a tournament T, then*

$$M(T) = \sum_{i=1}^{n} s_i^2 \le \frac{n(n-1)}{2}(s_1+s_n) - ns_1s_n, \tag{3.6}$$

with equality if and only if the tournament has only two types of scores s_1 and s_n.

Proof. By using (1.1), we have

$$\begin{aligned}M(T) = \sum_{i=1}^{n} s_i^2 &= \sum_{i=1}^{n}(s_i^2 - s_is_1 + s_is_1) = \sum_{i=1}^{n}\{s_i(s_i-s_1)+s_is_1\}\\ &\le \sum_{i=1}^{n}\{s_n(s_i-s_1)+s_is_1\} = \sum_{i=1}^{n}(s_ns_i - s_ns_1 + s_is_1)\\ &= \sum_{i=1}^{n}(s_n+s_1)s_i - \sum_{i=1}^{n} s_ns_1 = \frac{n(n-1)}{2}(s_1+s_n) - ns_1s_n.\end{aligned}$$

Equality holds if and only if

$$\sum_{i=1}^{n}\{s_i(s_i-s_1)\} = \sum_{i=1}^{n}\{s_n(s_i-s_1)\}$$

or

$$\sum_{i=1}^{n}\{s_n(s_i-1) - s_i(s_i-1)\} = 0$$

or

$$\sum_{i=1}^{n}\{(s_n-s_i)(s_i-s_1)\} = 0. \tag{3.7}$$

Now, assume that equality holds in (3.6). Then equality holds in (3.7). Since each term in this summation is non-negative and sum is equal to zero, therefore either $s_i = s_1$ or $s_i = s_n$ for $i = 1, 2, \ldots, n$. So the tournament has only two types of scores s_1 and s_n.

Conversely, suppose that the tournament has only two types of scores s_1 and s_n. Then $\sum_{i=1}^{n}\{(s_n - s_i)(s_i - s_1)\} = 0$. Hence, the equality holds. □

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