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ON BOUNDS FOR NORMS OF SINE AND COSINE ALONG A CIRCLE ON THE COMPLEX PLANE

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Dedicated to Dr. Prof. Aliakbar Montazer Haghighi at Prairie View A&M University in USA

Abstract. In the paper, the author presents lower and upper bounds for norms of the sine and cosine functions along a circle on the complex plane.

1. Motivations

This paper is a companion of the formally published article [6].

In the theory of complex functions, the sine and cosine functions $\sin z$ and $\cos z$ on the complex plane $\mathbb{C}$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

respectively, where $z = x + iy$, $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ is the imaginary unit. They have the least positive periodicity 2π, that is,

$$\sin(z + 2k\pi) = \sin z \quad \text{and} \quad \cos(z + 2k\pi) = \cos z,$$

for $k \in \mathbb{Z}$.

When restricting $z = x \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ become $\sin x$ and $\cos x$ and satisfy

$$0 \le |\sin x| \le 1 \quad \text{and} \quad 0 \le |\cos x| \le 1.$$

Key words and phrases. Bound, norm, sine, cosine, double inequality, circle, complex plane, difference, open problem.

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When restricting $z = iy$ for $y \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ reduce to

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y \to \pm i\infty$$

and

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y \to +\infty,$$

as $y \to \pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

In the textbook [9, page 93], Exercise 6 states that, if $z \in \mathbb{C}$ and $|z| \le R$, then

$$|\sin z| \le \cosh R \quad \text{and} \quad |\cos z| \le \cosh R.$$

In [7], a criterion to justify a holomorphic function was discussed.

In [6], the author discussed and computed bounds of the sine and cosine functions $\sin z$ and $\cos z$ along straight lines on the complex plane $\mathbb{C}$. The main results in the paper [6] can be recited as follows.

(a) The complex functions $\sin z$ and $\cos z$ are bounded along straight lines parallel to the real x-axis on the complex plane $\mathbb{C}$:
   (i) along the horizontal straight line $y = \alpha$ on the complex plane $\mathbb{C}$

$$|\sinh \alpha| \le |\sin(x + i\alpha)| \le \cosh \alpha \tag{1.1}$$

   and

$$|\sinh \alpha| \le |\cos(x + i\alpha)| \le \cosh \alpha, \tag{1.2}$$

   where $\alpha \in \mathbb{R}$ is a constant and $x \in \mathbb{R}$;
   (ii) the equalities in the left hand side of (1.1) and in the right hand side of (1.2) hold if and only if $x = k\pi$ for $k \in \mathbb{Z}$;
   (iii) the equalities in the right hand side of (1.1) and in the left hand side of (1.2) hold if and only if $x = k\pi + \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

(b) The complex functions $\sin z$ and $\cos z$ are unbounded along straight lines whose slopes are not horizontal:
   (i) along the sloped straight line $y = \alpha + \beta x$ on the complex plane $\mathbb{C}$

$$|\sin z| \ge |\sinh(\alpha + \beta x)| \quad \text{and} \quad |\cos z| \ge |\sinh(\alpha + \beta x)|,$$

   where $\alpha \in \mathbb{R}$ and $\beta \ne 0$ are constants;
   (ii) along the vertical straight line $x = \gamma$ on the complex plane $\mathbb{C}$

$$|\sin z| \ge |\sinh y| \quad \text{and} \quad |\cos z| \ge |\sinh y|,$$

   where $\gamma \in \mathbb{R}$ is a constant.

In this paper, we present bounds for norms $|\sin(re^{i\theta})|$ and $|\cos(re^{i\theta})|$ of the sine and cosine functions $\sin z$ and $\cos z$ along a circle $C(0, r)$ centered at the origin $z = 0$ of radius $r > 0$ on the complex plane $\mathbb{C}$ in terms of two double inequalities.

2. A Double Inequality for the Norm of Sine Along a Circle

In this section, we present a double inequality for the norm $|\sin(re^{i\theta})|$ of the sine function $\sin z$ along a circle $C(0,r)$ centered at the origin $z=0$ of radius $r>0$ on the complex plane $\mathbb{C}$.

Theorem 2.1. *Let $r>0$ be a constant and let $C(0,r): z=re^{i\theta}$ for $\theta\in[0,2\pi)$ denote a circle centered at the origin $z=0$ of radius r. Then*

$$|\sin r| \le |\sin(re^{i\theta})| \le \sinh r, \quad \theta\in[0,2\pi). \tag{2.1}$$

The left equality is valid if and only if $\theta=0,\pi$, while the right equality is valid if and only if $\theta=\frac{\pi}{2},\frac{3\pi}{2}$.

Proof. The circle $C(0,r)$ can be represented by

$$z=re^{i\theta}, \quad \theta\in[0,2\pi).$$

It is not difficult to see that, for fixed $r>0$, $|\sin(re^{i\theta})|=|\sin r|$ for $\theta=0,\pi$, $|\sin(re^{i\theta})|=\sinh r$ for $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, and $|\sin(re^{i\theta})|$ has a least positive periodicity π with respect to the argument θ.

Straightforward computation yields

(2.2)

$$\begin{aligned}
\sin(re^{i\theta}) &= \sin(r\cos\theta + ir\sin\theta)\\
&= \frac{e^{i(r\cos\theta+ir\sin\theta)} - e^{-i(r\cos\theta+ir\sin\theta)}}{2i}\\
&= \frac{e^{-(r\sin\theta-ir\cos\theta)} - e^{r\sin\theta-ir\cos\theta}}{2i}\\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta)+i\sin(r\cos\theta)] - e^{r\sin\theta}[\cos(r\cos\theta)-i\sin(r\cos\theta)]}{2i}\\
&= \frac{(e^{-r\sin\theta}-e^{r\sin\theta})\cos(r\cos\theta) + i(e^{-r\sin\theta}+e^{r\sin\theta})\sin(r\cos\theta)]}{2i}\\
&= \cosh(r\sin\theta)\sin(r\cos\theta) + i\sinh(r\sin\theta)\cos(r\cos\theta)
\end{aligned}$$

and

$$|\sin(re^{i\theta})| = \sqrt{[\cosh(r\sin\theta)\sin(r\cos\theta)]^2 + [\sinh(r\sin\theta)\cos(r\cos\theta)]^2}\,.$$

In Figure 1, we plot the 3D graph of $|\sin(re^{i\theta})|$ for $r\in[0,5]$ and $\theta\in[0,2\pi)$. In Figure 2, we plot the polarized 3D graph of the norm $|\sin(re^{i\theta})|$ for $r\in[0,4]$ and $\theta\in[0,2\pi)$. In Figure 3, we plot the graph of $|\sin(\pi e^{i\theta})|$ for $\theta\in[0,2\pi)$. These three figures are helpful for analyzing and understanding the behaviour of the sine function $\sin z$ along the circle $C(0,r)$ centered at the origin $z=0$ of radius r.

From Figure 3, we can see that the norm $|\sin(\pi e^{i\theta})|$ has only two maximums at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$, while it has only two minimums at $\theta=0,\pi$ on the interval $[0,2\pi)$.

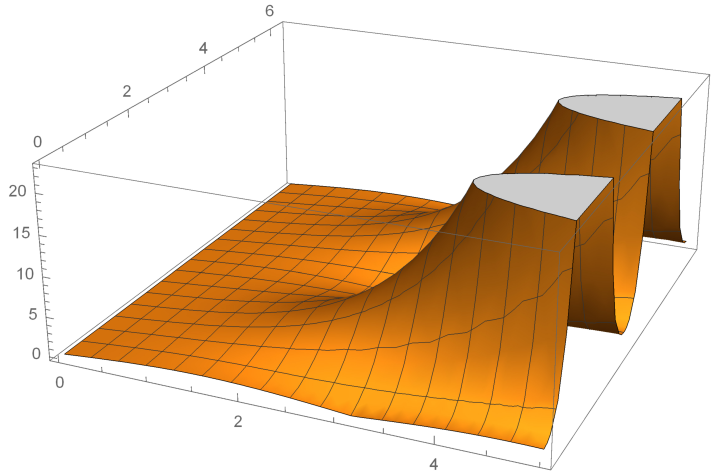

FIGURE 1. The 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$

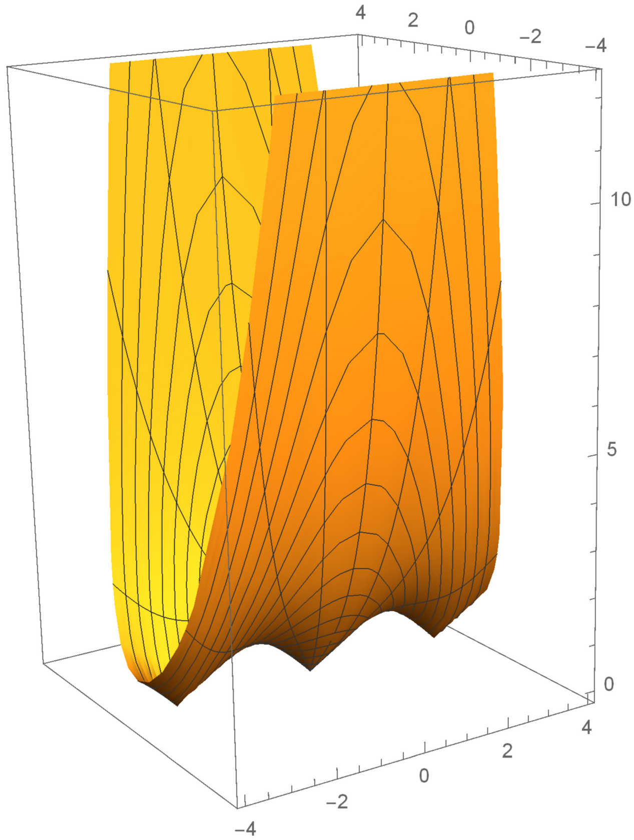

FIGURE 2. The polarized 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0, 4]$ and $\theta \in [0, 2\pi)$

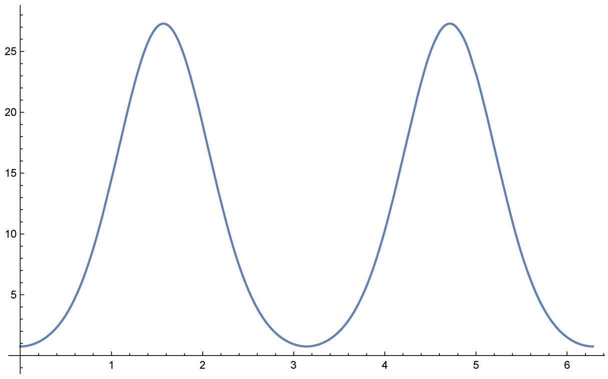

FIGURE 3. The graph of $|\sin(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$

Differentiating the square of $|\sin(re^{i\theta})|$ yields

$$\begin{aligned}\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta} &= r[\cos\theta\sinh(2r\sin\theta) - \sin\theta\sin(2r\cos\theta)] \\ &= r[\sinh(2r\sin\theta) - \tan\theta\sin(2r\cos\theta)]\cos\theta \\ &= r[\cot\theta\sinh(2r\sin\theta) - \sin(2r\cos\theta)]\sin\theta \\ &= r^2\left[\frac{\sinh(2r\sin\theta)}{2r\sin\theta} - \frac{\sin(2r\cos\theta)}{2r\cos\theta}\right]\sin(2\theta).\end{aligned}$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $(k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ for $k = 0, 1, 2, 3$, in order that $\frac{\mathrm{d}\,|\sin(re^{i\theta})|^2}{\mathrm{d}\,\theta} \ne 0$ for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$, it is sufficient to find

$$\frac{\sinh(2r\sin\theta)}{2r\sin\theta} > 1 \tag{2.3}$$

and

$$\frac{\sin(2r\cos\theta)}{2r\cos\theta} < 1, \tag{2.4}$$

for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$. Then, for fixed $r > 0$, the square $|\sin(re^{i\theta})|^2$ and the norm $|\sin(re^{i\theta})|$ have only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while they have only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. At $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, the values of $|\sin(re^{i\theta})|$ are both $\sinh r$; at $\theta = 0, \pi$, the values of $|\sin(re^{i\theta})|$ are both $|\sin r|$.

Considering the odevity of $\sinh t$ and $\sin t$, we see that two inequalities in (2.3) and (2.4) are equivalent to

$$\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} < 1, \tag{2.5}$$

for $t \in (0, \infty)$. The first inequality in (2.5) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 \tag{2.6}$$

in [2, page 270, 3.6.9]. When $t \in (0, \frac{\pi}{2})$, the second inequality in (2.5) follows from the right hand side of the Jordan inequality

$$\frac{\pi}{2} \leq \frac{\sin t}{t} < 1, \quad 0 < |t| \leq \frac{\pi}{2}, \tag{2.7}$$

in [2, Section 2.3] and the papers [1,3,4,8]. When $t > \frac{\pi}{2}$, the second inequality in (2.5) follows from $\sin t \leq 1$ on $(0, \infty)$ and standard argument. The double inequality (2.1) is thus proved. The proof of Theorem 2.1 is complete. □

3. A Double Inequality for the Norm of Cosine Along a Circle

In this section, we present a double inequality for the norm $|\cos(re^{i\theta})|$ of the cosine function $\cos z$ along a circle $C(0, r)$ centered at the origin $z = 0$ of radius $r > 0$ on the complex plane $\mathbb{C}$.

Theorem 3.1. *Let $r > 0$ be a constant and let $C(0, r) : z = re^{i\theta}$ for $\theta \in [0, 2\pi)$ denote a circle centered at the origin $z = 0$ of radius r. Then*

$$|\cos r| \leq |\cos(re^{i\theta})| \leq \cosh r, \quad \theta \in [0, 2\pi). \tag{3.1}$$

The left equality is valid if and only if $\theta = 0, \pi$, while the right equality is valid if and only if $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Proof. It is easy to see that, for fixed $r > 0$, $|\cos(re^{i\theta})| = |\cos r|$ for $\theta = 0, \pi$, $|\cos(re^{i\theta})| = \cosh r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $|\cos(re^{i\theta})|$ has a least positive periodicity π with respect to the argument θ.

Direct calculation yields

(3.2)

$$\begin{aligned}
\cos(re^{i\theta}) &= \cos(r\cos\theta + ir\sin\theta) \\
&= \frac{e^{i(r\cos\theta + ir\sin\theta)} + e^{-i(r\cos\theta + ir\sin\theta)}}{2} \\
&= \frac{e^{-(r\sin\theta - ir\cos\theta)} + e^{r\sin\theta - ir\cos\theta}}{2} \\
&= \frac{e^{-r\sin\theta}[\cos(r\cos\theta) + i\sin(r\cos\theta)] + e^{r\sin\theta}[\cos(r\cos\theta) - i\sin(r\cos\theta)]}{2} \\
&= \frac{(e^{-r\sin\theta} + e^{r\sin\theta})\cos(r\cos\theta) + i(e^{-r\sin\theta} - e^{r\sin\theta})\sin(r\cos\theta)]}{2} \\
&= \cosh(r\sin\theta)\cos(r\cos\theta) - i\sinh(r\sin\theta)\sin(r\cos\theta)
\end{aligned}$$

and

$$|\cos(re^{i\theta})| = \sqrt{[\cosh(r\sin\theta)\cos(r\cos\theta)]^2 + [\sinh(r\sin\theta)\sin(r\cos\theta)]^2}\,.$$

In Figure 4, we plot the 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$. In

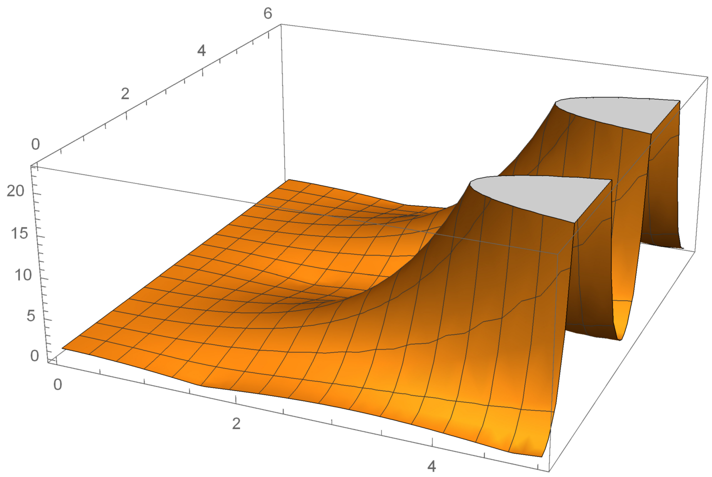

FIGURE 4. The 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,5]$ and $\theta \in [0,2\pi)$

Figure 5, we plot the polarized 3D graph of the norm $|\cos(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$. In Figure 6, we plot the graph of $|\cos(re^{i\theta})|$ for $r = \pi$ and $\theta \in [0,2\pi)$. These three figures are helpful for analyzing and understanding the behaviour of the cosine function $\cos z$ along the circle $C(0,r)$ centered at the origin $z = 0$ of radius r.

From Figure 6, we can see that the norm $|\cos(\pi e^{i\theta})|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0,2\pi)$.

Differentiating the square of $|\cos(re^{i\theta})|$ with respect to θ gives

$$\begin{aligned}
\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta} &= r[\sin\theta\sin(2r\cos\theta) + \cos\theta\sinh(2r\sin\theta)] \\
&= r[\tan\theta\sin(2r\cos\theta) + \sinh(2r\sin\theta)]\cos\theta \\
&= r[\sin(2r\cos\theta) + \cot\theta\sinh(2r\sin\theta)]\sin\theta \\
&= r^2\left[\frac{\sin(2r\cos\theta)}{2r\cos\theta} + \frac{\sinh(2r\sin\theta)}{2r\sin\theta}\right]\sin(2\theta).
\end{aligned}$$

From the first three expressions above, we conclude that the derivative $\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $(k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ for $k = 0,1,2,3$, in order that $\frac{\mathrm{d}\,|\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta} \ne 0$, it is sufficient to show

$$\frac{\sinh(2r\sin\theta)}{2r\sin\theta} > 1 \tag{3.3}$$

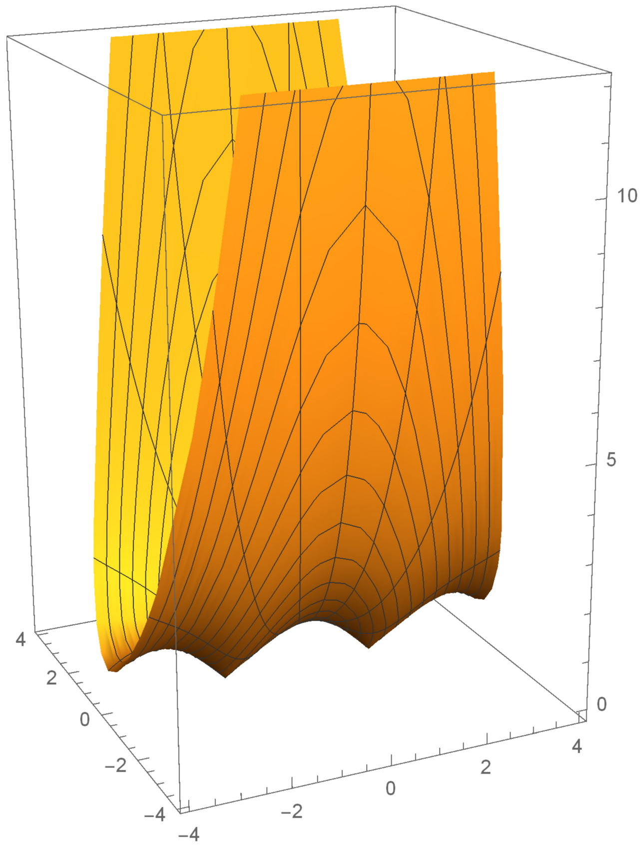

FIGURE 5. The polarized 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0,4]$ and $\theta \in [0,2\pi)$

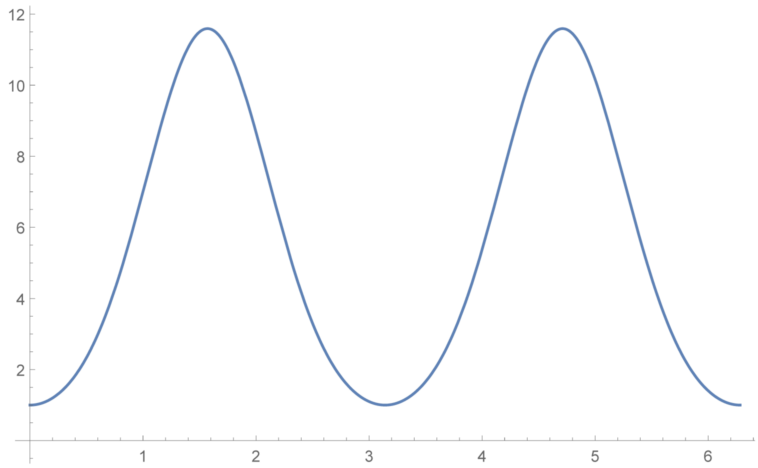

FIGURE 6. The graph of $|\cos(\pi e^{i\theta})|$ for $\theta \in [0,2\pi)$

and

$$\frac{\sin(2r\cos\theta)}{2r\cos\theta} > -1, \tag{3.4}$$

for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$. Then, for fixed $r > 0$, the square $|\cos(re^{i\theta})|^2$ and the norm $|\cos(re^{i\theta})|$ have only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while they have only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. At $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, the values of $|\cos(re^{i\theta})|$ are both $\cosh r$, at $\theta = 0, \pi$ the values of $|\cos(re^{i\theta})|$ are both $|\cos r|$.

Considering odevity of $\sinh t$ and $\sin t$, two inequalities in (3.3) and (3.4) are equivalent to

$$\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} > -1, \tag{3.5}$$

for $t \in (0, \infty)$. The first inequality in (3.5) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality (2.6). When $t \in (0, \frac{\pi}{2})$, the second inequality in (3.5) follows from the left hand side of the Jordan inequality (2.7). When $t > \frac{\pi}{2}$, the second inequality in (3.5) follows from $\sin t \geq -1$ on $(0, \infty)$ and simple argument. The double inequality (3.1) is thus proved. The proof of Theorem 3.1 is complete. □

4. Remarks

In this final section, we list several remarks on our main results in this paper.

Remark 4.1. Comparing Figure 1 and 4, it is not easy to see the difference between $|\sin(re^{i\theta})|$ and $|\cos(re^{i\theta})|$. However, the difference $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r \in [0, 2\pi]$ and $\theta \in [0, 2\pi)$ can be showed by Figure 7.

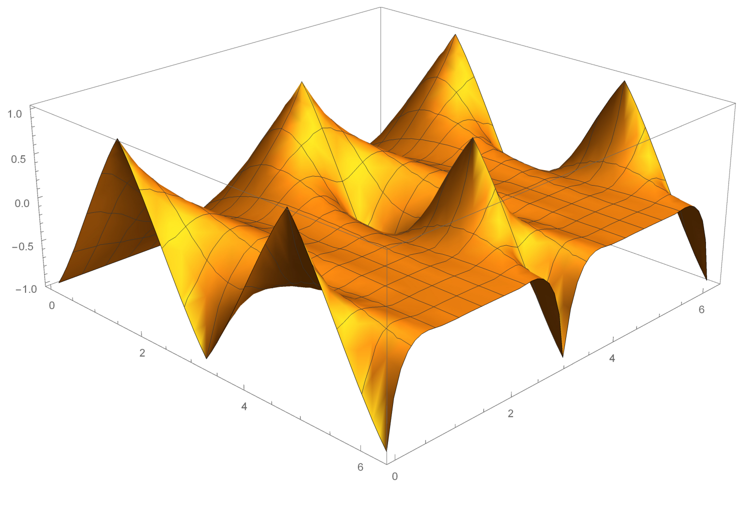

FIGURE 7. The 3D graph of $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r, \theta \in [0, 2\pi)$

Comparing Figure 2 and 5, it is not easy to find the difference between $|\sin(\pi e^{i\theta})|$ and $|\cos(\pi e^{i\theta})|$ yet. However, the difference $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$ can be presented by Figure 8.

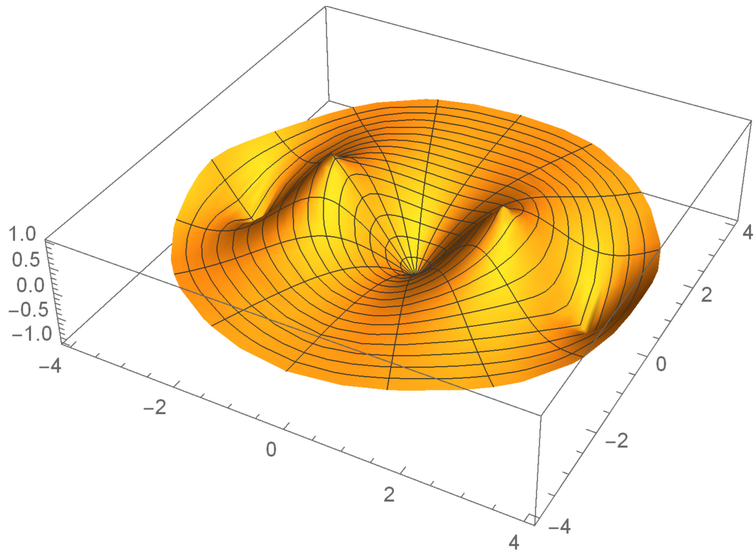

FIGURE 8. The polarized 3D graph of $|\sin(re^{i\theta})| - |\cos(re^{i\theta})|$ for $r \in [0, 4]$ and $\theta \in [0, 2\pi)$

Comparing Figure 3 and 6, it is also not easy to see the difference between $|\sin(\pi e^{i\theta})|$ and $|\cos(\pi e^{i\theta})|$. However, the difference $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$ can be demonstrated by Figure 9.

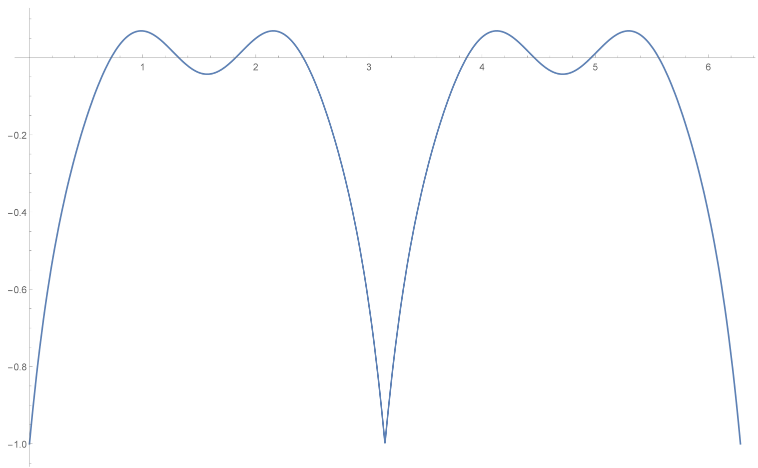

FIGURE 9. The graph of $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$

Remark 4.2. From Figure 7, 8, and 9, we can guess that the double inequality

$$-1 \le |\sin(re^{i\theta})| - |\cos(re^{i\theta})| \le 1 \tag{4.1}$$

is seemingly valid for all $r > 0$ and $\theta \in [0, 2\pi)$. Can one verify, deny, or strengthen this guess?

Remark 4.3. It is standard that

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2=|[\sin(re^{i\theta})-\cos(re^{i\theta})]^2|=|1-\sin(2re^{i\theta})|. \tag{4.2}$$

From (4.2), it follows that

$$|1-|\sin(2re^{i\theta})||\le|\sin(re^{i\theta})-\cos(re^{i\theta})|^2\le1+|\sin(2re^{i\theta})|.$$

Further by virtue of the double inequality (2.1) in Theorem 2.1, we obtain

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2\le1+|\sin(2re^{i\theta})|\le1+\sinh(2r).$$

This means that

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|\le\sqrt{1+\sinh(2r)}, \tag{4.3}$$

for $r>0$ and $\theta\in[0,2\pi)$.

Motivated by the guess expressed in terms of the double inequality (4.1) and by the inequality (4.3), we pose an open problem: what are the nontrivial lower and upper bounds of the norm $|\sin(re^{i\theta})-\cos(re^{i\theta})|$ for $r>0$ and $\theta\in[0,2\pi)$?

Remark 4.4. From (2.2) and (3.2), it follows that

$$\begin{aligned}\sin(re^{i\theta})-\cos(re^{i\theta})&=\cosh(r\sin\theta)[\sin(r\cos\theta)-\cos(r\cos\theta)]\\&\quad+i[\cos(r\cos\theta)+\sin(r\cos\theta)]\sinh(r\sin\theta).\end{aligned}$$

Hence, we have

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|=\sqrt{\sinh^2(r\sin\theta)-\sin(2r\cos\theta)+\cosh^2(r\sin\theta)},$$

which is equivalent to

$$|\sin(re^{i\theta})-\cos(re^{i\theta})|^2=\cosh(2r\sin\theta)-\sin(2r\cos\theta). \tag{4.4}$$

From (4.4), it follows that

$$\begin{aligned}\frac{\mathrm{d}\,|\sin(re^{i\theta})-\cos(re^{i\theta})|^2}{\mathrm{d}\,\theta}&=2r[\sin\theta\cos(2r\cos\theta)+\cos\theta\sinh(2r\sin\theta)]\\&=2r[\cos(2r\cos\theta)+\cot\theta\sinh(2r\sin\theta)]\sin\theta\\&=2r[\tan\theta\cos(2r\cos\theta)+\sinh(2r\sin\theta)]\cos\theta\\&=2r^2\left[\frac{\cos(2r\cos\theta)}{2r\cos\theta}+\frac{\sinh(2r\sin\theta)}{2r\sin\theta}\right]\sin(2\theta),\end{aligned}$$

which is clearly equal to 0 at $\theta=0,\pi$ for all $r>0$. The function $\frac{\sinh t}{t}$ is even and not less than 1 on $(-\infty,\infty)$. The function $\frac{\cos t}{t}$ is odd on $(-\infty,\infty)$. By finding the set of all zeros of the function

$$\frac{\cos t}{t}+\frac{\sinh\sqrt{4r^2-t^2}}{\sqrt{4r^2-t^2}},\quad t\ne0,\ r>0,$$

we can obtain sharp bounds of $|\sin(re^{i\theta})-\cos(re^{i\theta})|$ for $r>0$ and $\theta\in[0,2\pi)$. This is a hint, clue, sketch, or approach to solve the above open problem.

Remark 4.5. To the best of my knowledge, the double inequalities (2.1) and (3.1) in Theorems 2.1 and 3.1 are fundamental and new in the literature.

Remark 4.6. This paper is a revised version of the preprint [5].

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