ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 48, Number 3, 2024

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Kragujevac : InterPrint). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2403

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Email: krag_j_math@kg.ac.rs
Website: http://kjm.pmf.kg.ac.rs

Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: InterPrint, Kragujevac, Serbia
From 2021 the journal appears in one volume and six issues per annum.

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Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 331–343.

BEST PROXIMITY POINT THEOREMS IN NON-ARCHIMEDEAN MENGER PROBABILISTIC SPACES

ARIFE AYSUN KARAASLAN[1] AND VATAN KARAKAYA[2]

Abstract. In this work, we prove best proximity point theorems for γ-contractions with conditions the weak P-property in non-Archimedean Menger probabilistic metric spaces. We give the notion of γ- proximal contractions of first and second type in non-Archimedean Menger probabilistic metric spaces and also we establish best proximity point theorems for these proximal contractions. Lastly, we complete our study by giving examples that support our results.

1. Introduction

The concept of the probabilistic metric spaces were introduced by Menger [15]. When x and y are two elements of a probabilistic metric space, the idea of distance between these points is changed with function $F_{x,y}(t)$. $F_{x,y}(t)$ is a distribution function that is explained as probability that the distance between x and y is less than t. In fact, studies in these spaces improved with Schweizer and Sklar' s leading works [20]. The probabilistic interpretation of Banach contraction principle is demonstrated by Sehgal and Bharucha-Reid in [22]. Some studies about probabilistic metric spaces are given in list [7, 8, 12, 16–18].

On the other hand, best proximity point was started by Fan [9]. For more details, references are listed in [1, 3, 4, 11, 13, 14, 19, 24]. Sezen introduced γ-contraction and γ-weak contraction in non-Archimedean fuzzy metric spaces [23]. In this paper, we prove some best proximity point theorems for γ-contractions in a non-Archimedean Menger probabilistic metric space.

Key words and phrases. Fixed point, best proximity point, γ-contraction, non-Archimedean Menger probabilistic metric space

2010 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54E70.

DOI 10.46793/KgJMat2403.331K

Received: August 11, 2020.

Accepted: April 21, 2021.

2. Preliminaries

Definition 2.1 ([20])**.** A triangular norm (shorter $\Delta -$ norm/ $t -$ norm) is a binary operation Δ which is defined on the closed interval [0,1],

$$\Delta : [0,1] \times [0,1] \to [0,1]$$

that satisfies the following requirements:

(Δ_1) $\Delta(a_1, 1) = a_1,\ \Delta(0,0) = 0$;
(Δ_2) $\Delta(a_1, a_2) = \Delta(a_2, a_1)$;
(Δ_3) $\Delta(a_3, a_4) \geq \Delta(a_1, a_2)$ for $a_3 \geq a_1,\ a_4 \geq a_2$;
(Δ_4) for all $a_1, a_2, a_3 \in [0,1]$, $\Delta(\Delta(a_1, a_2), a_3) = \Delta(a_1, \Delta(a_2, a_3))$.

Principal examples of $\Delta -$ norms are:

(i) $\Delta_M(a_1, a_2) = \min(a_1, a_2)$;
(ii) $\Delta_P(a_1, a_2) = a_1.a_2$;
(iii) $\Delta_L(a_1, a_2) = \max(a_1 + a_2 - 1, 0)$;
(iv) $\Delta_D(a_1, a_2) = \begin{cases} \min(a_1, a_2), & \text{if } \max(a_1, a_2) = 1, \\ 0, & \text{otherwise.} \end{cases}$

Definition 2.2 ([20])**.** Let F be a function defined from $\mathbb{R}$ to $\mathbb{R}^+$. If it is nondecreasing, left-continuous with

$$\inf \{F(t) : t \in \mathbb{R}\} = 0 \quad \text{and} \quad \sup \{F(t) : t \in \mathbb{R}\} = 1,$$

then F is called a distribution function. In addition, if $F(0) = 0$, then F is called a distance distribution function. L^+ indicate the set of all distance distribution functions and H is a special example of distance distribution function (also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.3 ([20])**.** Let X is a nonempty set and F is a mapping defined from $X \times X$ into L^+. The value of F at the point (x, y) is denoted by $F_{x,y}$. If the following conditions hold, (X, F) ordered pair is called a probabilistic metric space:

(PM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM-3) $F_{x,y}(t) = 1, F_{y,z}(s) = 1,$ then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X,\ t, s \geq 0$.

Every metric space (X, d) can always be realized as a probabilistic metric space by taking into account that $F : X \times X \to L^+$ defined as

$$F_{x,y}(t) = H(t - d(x, y)), \quad \text{for all } x, y \in X.$$

Definition 2.4 ([20])**.** Let (X, F) be a probabilistic metric space and Δ is a $t-$norm that provides the following inequality,

$$F_{x,z}(t + s) \geq \Delta \left(F_{x,y}(t), F_{y,z}(s)\right), \quad \text{for all } x, y, z \in X \text{ and } t, s \geq 0.$$

Then, triplet (X, F, Δ) is named as a Menger probabilistic metric space.

Definition 2.5 ([20])**.** Let (X, F, Δ) be a Menger space.

(i) A sequence (x_n) is called a convergent sequence to $x \in X$ if for every $t > 0$ and $0 < \varepsilon < 1$, there exists $n_0 = n_0(t, \varepsilon) \in \mathbb{N}$ such that $F_{x_n,x}(t) > 1 - \lambda$ for all $n \geq \mathbb{N}$.

(ii) A sequence (x_n) in X is called Cauchy sequence if for every $t > 0$ and $0 < \varepsilon < 1$, there exists $n_0 = n_0(t, \varepsilon) \in \mathbb{N}$ such that $F_{x_n,x_m}(t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A Menger space is said to be complete , if each Cauchy sequence in X is convergent to a point in X.

Definition 2.6 ([5])**.** A probabilistic metric space (X, F) is called non-Archimedean probabilistic metric space if $F_{x,y}(t) = 1, F_{y,z}(s) = 1$, then $F_{x,z}(\max\{t, s\}) = 1$ for every $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.7 ([5,6])**.** A Menger probabilistic metric space (X, F, Δ) is called non-Archimedean if $F_{x,z}(\max\{t, s\}) = \Delta\left(F_{x,y}(t), F_{y,z}(s)\right)$ for all $x, y, z \in X$ and $t, s \geq 0$.

Note. We observe that (X, F, Δ) is non-Archimedean if and only if

$$F_{x,z}(t) \geq \Delta\left(F_{x,y}(t), F_{y,z}(t)\right), \quad \text{for all } x, y, z \in X \text{ and } t \geq 0.$$

Definition 2.8 ([19])**.** Let (X, F, Δ) be a Menger probabilistic metric space and A, B be two nonempty subsets of this space. A mapping $T : A \to B$ satisfies the following equality

$$F_{x,Tx}(t) = F_{A,B}(t), \quad \text{for } t > 0.$$

Then x in A is said to be a best proximity point of T.

Definition 2.9 ([3])**.** Let (X, F, Δ) be a Menger probabilistic metric space and A, B two nonempty subsets of this space. A set A is said to be approximatively compact with respect to a set B if every sequence (x_n) in A satisfies the condition that $F_{y,x_n}(t) \to F_{y,A}(t)$ for some $y \in B$ and for each $t > 0$ has a convergent subsequence.

Definition 2.10. Let $\gamma : [0, 1) \to \mathbb{R}$ be a function that has the following properties:

(a) strictly increasing;

(b) continuous mapping;

(c) for each sequence (α_n) of positive numbers, $\lim_{n\to\infty} \alpha_n = 1$ if and only if $\lim_{n\to\infty} \gamma(\alpha_n) = +\infty$.

Also, Γ represents the family of all γ functions.

Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space. A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0, 1)$ such that for all $x, y \in X$ and $\gamma \in \Gamma$

$$F_{Tx,Ty}(t) < 1 \Rightarrow \gamma(F_{Tx,Ty}(t)) \geq \gamma(F_{x,y}(t)) + \delta. \tag{2.1}$$

3. Main Results

In this section, we present some definitions and some best proximity point results in non-Archimedean Menger probabilistic metric spaces. Let A and B two nonempty subsets of a Menger probabilistic metric space (X, F, Δ). We will use the following notations:

$$\begin{aligned} F_{A,B}(t) =& \sup\{F_{x,y}(t) : x \in A, y \in B\}. \\ A_0(t) =& \{x \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } y \in B\}, \\ B_0(t) =& \{y \in B : F_{x,y}(t) = F_{A,B}(t) \text{ for some } x \in A\}. \end{aligned}$$

Now, let us give our main results.

Definition 3.1. Let (A, B) be a pair of nonempty subsets of a non-Archimedean Menger probabilistic metric space X with $A_0(t) \neq 0$. Then the pair (A, B) is said to have the weak P-property if and only if

$$F_{x_1,y_1}(t) = F_{A,B}(t), \quad F_{x_2,y_2}(t) = F_{A,B}(t) \quad \Rightarrow \quad F_{x_1,x_2}(t) \geq F_{y_1,y_2}(t),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 3.1. Let $X = \mathbb{R} \times \mathbb{R}$ and d defined as the standard metric $d(x, y) = |x - y|$ for all $x \in X$, $\Delta(a, b) = \min(a, b)$ and the distribution function defined as

$$F_{x,y}(t) = \frac{t}{t + d(x, y)}, \quad \text{for all } t > 0.$$

(X, F, Δ) is a non-Archimedean Menger probabilistic metric space. Let $A = \{(0, 0)\}$, $B = \{(1, 0), (-1, 0)\}$. From here, $d(A, B) = 1$ and $F_{A,B}(t) = \frac{t}{t+d(A,B)} = \frac{t}{t+1}$. Now we consider

$$F_{x_1,y_1}(t) = F_{A,B}(t), \quad F_{x_2,y_2}(t) = F_{A,B}(t).$$

We get $(x_1, y_1) = ((0, 0), (1, 0))$ and $(x_2, y_2) = ((0, 0), (-1, 0))$, $F_{x_1,x_2}(t) = F_{(0,0),(0,0)}(t) = 1$ and $F_{y_1,y_2}(t) = F_{(1,0),(-1,0)}(t) = \frac{t}{t+2}$ implies $F_{x_1,x_2}(t) > F_{y_1,y_2}(t)$. Thus, (A, B) is said to have the weak P-property.

Definition 3.2. Let A, B be nonempty subsets of a non-Archimedean Menger probabilisitc metric space (X, F, Δ). The mapping $g : A \to A$ is said to be a probabilistic isometry if

$$F_{gx_1,gx_2}(t) = F_{x_1,x_2}(t),$$

for all $x_1, x_2 \in A$.

Definition 3.3. Let A, B be nonempty subsets of a non-Archimedean Menger probabilistic metric space (X, F, Δ). Given $S : A \to B$ and a probabilistic isometry $g : A \to A$, the mapping S is said to preserve probabilistic distance with respect to g if

$$F_{Sgx_1,Sgx_2}(t) = F_{Sx_1,Sx_2}(t),$$

for all $x_1, x_2 \in A$.

Example 3.2. Let $X = [0,1] \times \mathbb{R}$ and d defined as the standart metric $d(x,y) = |x-y|$ for all $x \in X$ and the distribution function defined as

$$F_{x,y}(t) = \frac{t}{t + d(x,y)}, \quad \text{for all } t > 0.$$

Let $A = \{(0,x) : x \in \mathbb{R}\}$. $g : A \to A$ is defined as $g(0,x) = (0,-x)$. $F_{x,y}(t) = \frac{t}{t+d(x,y)} = F_{gx,gy}(t)$, where $x = (0,x_1), y = (0,y_1) \in A$. This indicates that g is a probabilistic isometry.

Theorem 3.1. *A and B be nonempty, closed subsets of a complete non-Archimedean Menger probabilistic metric space (X, F, Δ) such that $A_0(t)$ is nonempty. Let $T : A \to B$ be a γ-contraction such that $T(A_0(t)) \subseteq B_0(t)$. Suppose that the pair (A,B) has the weak P-property. Then T has a unique x^* in A such that $F_{x^*,Tx^*}(t) = F_{A,B}(t)$.*

Proof. Let start by choosing an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, we can find $x_1 \in A_0(t)$ such that $F_{x_1,Tx_0}(t) = F_{A,B}(t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{x_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(x_n) \in A_0(t)$ satisfying for all $n \in \mathbb{N}$,

$$F_{x_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.1}$$

(A,B) satisfies the weak P-property, from (3.1) we obtain

$$F_{x_n,x_{n+1}}(t) \geq F_{Tx_{n-1},Tx_n}(t), \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Now we will prove that the sequence (x_n) is convergent in $A_0(t)$. If there exists $n_0 \in \mathbb{N}$ such that $F_{Tx_{n_0-1},Tx_{n_0}}(t) = 1$, then by (3.2) we get $F_{x_{n_0},x_{n_0+1}}(t) = 1$ which implies $x_{n_0} = x_{n_0+1}$. Hence, we get

$$Tx_{n_0} = Tx_{n_0+1} \Rightarrow F_{Tx_{n_0},Tx_{n_0+1}}(t) = 1. \tag{3.3}$$

From (3.2) and (3.3), we have that

$$F_{x_{n_0+2},x_{n_0+1}}(t) \geq F_{Tx_{n_0+1},Tx_{n_0}}(t) = 1 \Rightarrow x_{n_0+2} = x_{n_0+1}.$$

Therefore, for all $n \geq n_0$, $x_n = x_{n_0}$ and (x_n) is convergent in $A_0(t)$. Also, we get

$$F_{x_{n_0},Tx_{n_0}}(t) = F_{x_{n_0+1},Tx_{n_0}}(t) = F_{A,B}(t).$$

From this equality we can say that x_{n_0} is a probabilistic best proximity point of T and the proof is finished. For this reason, we suppose that, for all $n \in \mathbb{N}$, $F_{Tx_{n-1},Tx_n}(t) \neq 1$. From the definition of γ-contraction and (3.2), we have

$$\begin{aligned} \gamma(F_{x_n,x_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ &\geq \gamma(F_{x_{n-2},x_{n-1}}(t)) + 2\delta \\ &\;\vdots \\ &\geq \gamma(F_{x_0,x_1}(t)) + n\delta. \end{aligned} \tag{3.4}$$

Letting $n \to \infty$, from (3.4) we have

$$\lim_{n\to\infty} \gamma(F_{x_n,x_{n+1}}(t)) = +\infty.$$

Using the property of γ function we have,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1. \tag{3.5}$$

We shall show that (x_n) is a Cauchy sequence. Suppose that (x_n) is not a Cauchy sequence. Then there exist $\varepsilon \in (0,1)$ and $t_0 > 0$ and two sequences $m(j), n(j)$ of positive integers such that $m(j) > n(j) + 1$ and

$$F_{x_{m(j)},x_{n(j)}}(t_0) < 1-\varepsilon \quad \text{and} \quad F_{x_{m(j)-1},x_{n(j)}}(t_0) \geq 1-\varepsilon. \tag{3.6}$$

So, for all $j \in \mathbb{N}$ we get

$$\begin{aligned} 1-\varepsilon >&F_{x_{m(j)},x_{n(j)}}(t_0) \\ \geq&\Delta(F_{x_{m(j)},x_{m(j)-1}}(t_0), F_{x_{m(j)-1},x_{n(j)}}(t_0)) \\ \geq&\Delta(F_{x_{m(j)},x_{m(j)-1}}(t_0), (1-\varepsilon)). \end{aligned} \tag{3.7}$$

By taking $j \to \infty$ in (3.7) and using (3.5) we have,

$$\lim_{j\to\infty} F_{x_{m(j)},x_{n(j)}}(t_0) = 1-\varepsilon. \tag{3.8}$$

From the property of t-norm

$$\begin{aligned} F_{x_{m(j)+1},x_{n(j)+1}}(t_0) \geq&\Delta(F_{x_{m(j)+1},x_{m(j)}}(t_0), F_{x_{m(j)},x_{n(j)+1}}(t_0)) \\ \geq&\Delta(F_{x_{m(j)+1},x_{m(j)}}(t_0), \Delta(F_{x_{m(j)},x_{n(j)}}(t_0), F_{x_{n(j)},x_{n(j)+1}}(t_0))). \end{aligned}$$

On letting limit as $j \to \infty$ in previous inequality, we obtain

$$\lim_{j\to\infty} F_{x_{m(j)+1},x_{n(j)+1}}(t_0) = 1-\varepsilon. \tag{3.9}$$

By applying inequality in (2.1) with $x = x_{m(j)}$ and $y = x_{n(j)}$,

$$\gamma(F_{x_{m(j)+1},x_{n(j)+1}}(t)) \geq \gamma(F_{x_{m(j)},x_{n(j)}}(t)) + \delta. \tag{3.10}$$

Taking the limit as $j \to \infty$ in (3.10), using definition of γ-contraction, from (3.8) and (3.9), we obtain

$$\gamma(1-\varepsilon) \geq \gamma(1-\varepsilon) + \delta.$$

This is a contraction. Therefore, (x_n) is a Cauchy sequence in X. We know that (X, F, Δ) is complete and $A_0(t)$ is a closed subset of this space, there exists $x^* \in A_0(t)$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

From the continuity of T, we have $Tx_n \to Tx^*$ and $F_{x_{n+1},Tx_n}(t) = F_{x^*,Tx^*}(t)$. From (3.1), $F_{x^*,Tx^*}(t) = F_{A,B}(t)$. This shows that x^* is a probabilistic best proximity point of T. Now, we show that uniqueness of the best proximity point of T. Suppose that x_1 and x_2 are two best proximity points of T. For $x_1, x_2 \in A$, $x_1 \neq x_2$ and

$F_{x_1,Tx_1}(t) = F_{x_2,Tx_2}(t) = F_{A,B}(t)$. Since (A,B) has the weak P-property, we can write $F_{x_1,x_2}(t) \geq F_{Tx_1,Tx_2}(t)$. T is a γ-contraction and $x_1 \neq x_2$ implies $F_{x_1,x_2}(t) \neq 1$,

$$\gamma(F_{x_1,x_2}(t)) \geq \gamma(F_{Tx_1,Tx_2}(t)) \geq \gamma(F_{x_1,x_2}(t)) + \delta > \gamma(F_{x_1,x_2}(t)),$$

which is a contradiction. Hence, T has a unique best proximity point. □

Example 3.3. Let $X = \mathbb{R} \times [0,1]$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2. Let $A = \{(x,0) : \text{for all } x \in \mathbb{R}\}$, $B = \{(y,1) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A,B) = 1$ and $F_{A,B}(t) = \frac{t}{t+1}$. $\gamma : [0,1) \to \mathbb{R}$ defined as $\gamma = \frac{1}{1-x}$, for all $x \in X$. Let $T : A \to B$ and $T(x,0) = \left(\frac{x}{6}, 1\right)$. Then, $T(A_0(t)) = B_0(t)$. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

We have $(a_1, x_1) = \left(\left(\frac{-b_1}{6}, 0\right), (-b_1, 0)\right)$ or $(a_2, x_2) = \left(\left(\frac{-b_2}{6}, 0\right), (-b_2, 0)\right)$. Then using γ-contraction, we have

$$\begin{aligned} \gamma(F_{a_1,a_2}(t)) &= \gamma\left(F_{\left(-\frac{b_1}{6},0\right),\left(-\frac{b_2}{6},0\right)}(t)\right) = \gamma\left(\frac{t}{t + \frac{|b_1 - b_2|}{6}}\right) \\ &= \frac{1}{1 - \frac{t}{t + \frac{|b_1-b_2|}{6}}} > \frac{1}{1 - \frac{t}{t+|b_1-b_2|}} = \gamma\left(\frac{t}{t + |b_1 - b_2|}\right) \\ &= \gamma(F_{x_1,x_2}(t)). \end{aligned} \tag{3.11}$$

From (3.11), $\gamma(F_{a_1,a_2}(t)) > \gamma(F_{x_1,x_2}(t))$. So, we can find a $\delta \in (0,1)$ such that $\gamma(F_{a_1,a_2}(t)) \geq F_{x_1,x_2}(t)) + \delta$. Then T is a γ-contraction and $(0,0)$ is a unique best proximity point of T.

Corollary 3.1. *Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and $A_0(t)$ is a nonempty closed subset of X. Let $T : A \to A$ be a γ-contraction. Then there exists a unique x^* in A.*

Definition 3.4. Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and A, B be two nonempty subsets of this space such that $A_0(t)$ is nonempty. A mapping $T : A \to B$ is said to be a γ-proximal contraction of first type if there exists a $\delta \in (0,1)$ for all $u_1, u_2, x_1, x_2 \in X$ such that

$$\begin{aligned} &F_{u_1,Tx_1}(t) = F_{A,B}(t), \quad F_{u_2,Tx_2}(t) = F_{A,B}(t), \quad F_{u_1,u_2}(t), F_{x_1,x_2}(t) < 1, \\ &\Rightarrow \gamma(F_{u_1,u_2}(t)) \geq \gamma(F_{x_1,x_2}(t)) + \delta. \end{aligned} \tag{3.12}$$

Definition 3.5. Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and A, B be two nonempty subsets of this space such that $A_0(t)$ is nonempty. A mapping $T : A \to B$ is said to be a γ-proximal contraction of second type if there exists a $\delta \in (0,1)$ for all $u_1, u_2, x_1, x_2 \in X$ such that

$$\begin{aligned} &F_{u_1,Tx_1}(t) = F_{A,B}(t), \quad F_{u_2,Tx_2}(t) = F_{A,B}(t), \quad F_{Tu_1,Tu_2}(t), F_{Tx_1,Tx_2}(t) < 1, \\ &\Rightarrow \gamma(F_{Tu_1,Tu_2}(t)) \geq \gamma(F_{Tx_1,Tx_2}(t)) + \delta. \end{aligned} \tag{3.13}$$

Theorem 3.2. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) $T(A_0(t)) \subseteq B_0(t)$;
(2) $T : A \to B$ *is a continuous* $\gamma -$ *proximal contraction of first type;*
(3) g *is an isometry;*
(4) $A_0(t) \subseteq g(A_0(t))$.

Then there exist a unique element $x \in A$ such that $F_{gx,Tx}(t) = F_{A,B}(t)$.

Proof. We will start the proof by choosing an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $F_{gx_1,Tx_0}(t) = F_{A,B}(t)$. Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{gx_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(x_n) \in A_0(t)$ satisfying for all $n \in \mathbb{N}$,

$$F_{gx_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.14}$$

Now we will prove that the sequence (x_n) is convergent in $A_0(t)$. If there exists $n_0 \in \mathbb{N}$ such that $F_{gx_{n_0},gx_{n_0+1}}(t) = 1$, then it is clear that the sequence (x_n) is convergent. So, let for all $n \in \mathbb{N}$, $F_{gx_{n_0},gx_{n_0+1}}(t) \neq 1$. From the hypothesis of the theorem, T is a γ-proximal contraction of first type

$$\begin{aligned} \gamma(F_{gx_n,gx_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ \gamma(F_{x_n,x_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ &\vdots \\ &\geq \gamma(F_{x_0,x_1}(t)) + n\delta. \end{aligned} \tag{3.15}$$

Letting $n \to \infty$, in previous inequality we have $\lim\limits_{n\to\infty} \gamma(F_{x_n,x_{n+1}}(t)) = +\infty$. If we continue with the same way that used in proof of Theorem 3.1, we can say (x_n) is a Cauchy sequence. Since complete non-Archimedean Menger probabilistic metric space (X, F, Δ) has closed subsets, there exist $x \in A_0(t)$ such that $\lim\limits_{n\to\infty} x_n = x$. Applying limit when $n \to \infty$ in (3.14), we have

$$F_{gx,Tx}(t) = F_{A,B}(t).$$

To show the uniqueness, we will suppose the contrary. Let $x^* \in A_0(t)$ and it satisfy the equality $F_{gx^*,Tx^*}(t) = F_{A,B}(t)$ such that $x \neq x^*$. Hence, $F_{x,x^*}(t) \neq 1$. Since g is an isometry and T is a γ-proximal contraction of the first kind, it follows that

$$\gamma(F_{x,x^*}(t)) = \gamma(F_{gx,gx^*}(t)) \geq \gamma(F_{x,x^*}(t)) + \delta > \gamma(F_{x,x^*}(t)),$$

which is a contradiction. Consequently, $x = x^*$. □

Example 3.4. Let $X = [-2, 2] \times \mathbb{R}$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2.

Let $A = \{(-2, x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(2, y) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 4$ and $F_{A,B}(t) = \frac{t}{t+4}$. $\gamma : [0, 1) \to \mathbb{R}$ defined as $\gamma(x) = \frac{1}{1-x^2}$, for all $x \in X$. Let $T : A \to B$ and $g : A \to A$, these are defined as $T(-2, x) = \left(2, \frac{x}{2}\right)$ and $g(-2, x) = (-2, -x)$. Then, $T(A_0(t)) = B_0(t)$, $A_0(t) = g(A_0(t))$ and g is a isometry. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

We have $(a_1, x_1) = \left(\left(-2, \frac{b_1}{2}\right), (-2, b_1)\right)$ or $(a_2, x_2) = \left(\left(-2, \frac{b_2}{2}\right), (-2, b_2)\right)$. We must show that, T is a γ-proximal contraction of first type

$$\begin{aligned}\gamma(F_{a_1,a_2}(t)) &= \gamma\left(F_{\left(-2,\frac{b_1}{2}\right),\left(-2,\frac{b_2}{2}\right)}(t)\right) = \gamma\left(\frac{t}{t + \frac{|b_1-b_2|}{2}}\right) \\ &= \frac{1}{1 - \left(\frac{t}{t+\frac{|b_1-b_2|}{2}}\right)^2} > \frac{1}{1 - \left(\frac{t}{t+|b_1-b_2|}\right)^2} = \gamma\left(\frac{t}{t + |b_1 - b_2|}\right) \\ &= \gamma(F_{x_1,x_2}(t)).\end{aligned} \tag{3.16}$$

From (3.16), we have $\gamma(F_{a_1,a_2}(t)) > \gamma F_{x_1,x_2}(t))$. So, we can find a $\delta \in (0, 1)$ such that $\gamma(F_{a_1,a_2}(t)) \geq F_{x_1,x_2}(t)) + \delta$. Then T is a γ-proximal contraction of first type and $(-2, 0)$ is a unique best proximity point of T.

If we assume that g is the identity mapping, we can give the following result.

Corollary 3.2. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Let $T : A \to B$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*
(2) *$T : A \to B$ is a continuous $\gamma -$ proximal contraction of first type.*

Then T has a unique best proximity point in A.

Theorem 3.3. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Suppose that A is approximatively compact with respect to B. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*
(2) *$T : A \to B$ is a continuous $\gamma -$ proximal contraction of second type;*
(3) *g is an isometry;*
(4) *$A_0(t) \subseteq g(A_0(t))$;*
(5) *T preserves probabilistic distance with respect to g.*

Then there exists a unique element $x \in A$ such that $F_{gx,Tx}(t) = F_{A,B}(t)$.

Proof. Let start by choosing an element Tx_0 in $T(A_0(t))$. Using the hypothesis, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $F_{gx_1,Tx_0}(t) =$

$F_{A,B}(t)$. Further, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{gx_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(Tx_n) \in B$ satisfying for all $n \in \mathbb{N}$

$$F_{gx_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.17}$$

Now we will prove that the sequence (Tx_n) is convergent in B. If there exists $n_0 \in \mathbb{N}$ such that $F_{Tgx_{n_0},Tgx_{n_0+1}}(t) = 1$, then it is clear that the sequence (Tx_n) is convergent. So, let for all $n \in \mathbb{N}$, $F_{Tgx_{n_0},Tgx_{n_0+1}}(t) \neq 1$. From the hypothesis of the theorem, T is a γ-proximal contraction of second type

$$\begin{aligned} \gamma(F_{Tgx_n,Tgx_{n+1}}(t)) \geq& \gamma(F_{Tx_{n-1},Tx_n}(t)) + \delta \\ \gamma(F_{Tx_n,Tx_{n+1}}(t)) \geq& \gamma(F_{Tx_{n-1},Tx_n}(t)) + \delta \\ &\vdots \\ \geq& \gamma(F_{Tx_0,Tx_1}(t)) + n\delta. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$, in previous inequality we have $\lim\limits_{n\to\infty} \gamma(F_{Tx_n,Tx_{n+1}}(t)) = +\infty$. If we continue same way that used in proof of Theorem 3.1, we can say that (Tx_n) is a Cauchy sequence in B. In theorem hyphothesis, complete non-Archimedean Menger probabilistic metric space (X, F, Δ) has closed subsets, there exists $y \in B$ such that $\lim\limits_{n\to\infty} Tx_n = y$. Using the triangle inequality

$$\begin{aligned} F_{y,A}(t) \geq F_{y,gx_n}(t) \geq& \Delta(F_{y,Tx_{n-1}}(t), F_{Tx_{n-1},gx_n}(t)) \\ =& \Delta(F_{y,Tx_{n-1}}(t), F_{A,B}(t)) \\ \geq& \Delta(F_{y,Tx_{n-1}}(t), F_{y,A}(t)). \end{aligned} \tag{3.19}$$

In (3.19), if we take the limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} F_{y,gx_n}(t) = F_{y,A}(t)$. Due to the fact that A is approximatively compact with respect to B, there exists a subsequence (gx_{n_k}) of (gx_n) such that converges to some $w \in A$.

Hence, $F_{w,y}(t) = \lim\limits_{k\to\infty} F_{gx_{n_k},Tgx_{n_k-1}}(t) = F_{y,A}(t)$. It implies that $w \in A_0(t)$. $A_0(t) \subseteq g(A_0(t))$, there exists $x \in A_0(t)$ such that $w = gx$. As we know, $\lim\limits_{n\to\infty} gx_{n_k} = gx$ and g is an isometry, we have $\lim\limits_{n\to\infty} x_{n_k} = x$. (Tx_n) converges to y and the continuity of T, we can write $\lim\limits_{n\to\infty} Tx_{n_k} = Tx = y$. As a result that, $F_{gx,Tx}(t) = \lim\limits_{n\to\infty} F_{gx_{n_k},Tgx_{n_k}} = F_{A,B}(t)$. The uniqueness can be shown using the same way in Theorem 3.1. $\square$

Example 3.5. Let $X = \mathbb{R} \times [0,1]$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2. Let $A = \{(x,0) : \text{for all } x \in \mathbb{R}\}$, $B = \{(y,1) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A,B) = 1$ and $F_{A,B}(t) = \frac{t}{t+1}$. $\gamma : [0,1) \to \mathbb{R}$ defined as $\gamma(x) = \frac{1}{\sqrt{1-x}}$, for all $x \in X$. Let $T : A \to B$ and $g : A \to A$, these are defined as $T(x,0) = \left(\frac{x}{3}, 1\right)$ and $g(x,0) = (-x,0)$. Then, $T(A_0(t)) = B_0(t)$, $A_0(t) = g(A_0(t))$ and g is an isometry. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

Also, $F_{Tgx_1,Tgx_2}(t) = F_{Tx_1,Tx_2}(t)$ and this says that T preserves isometric distance with respect to g. We have $(a_1, x_1) = \left(\left(\frac{b_1}{3}, 0\right), (b_1, 0)\right)$ or $(a_2, x_2) = \left(\left(\frac{b_2}{3}, 0\right), (b_2, 0)\right)$. We must show that, T is a γ-proximal contraction of second type

$$\begin{aligned}
\gamma(F_{Ta_1,Ta_2}(t)) &= \gamma\left(F_{\left(\frac{b_1}{9},1\right),\left(\frac{b_2}{9},1\right)}(t)\right) = \gamma\left(\frac{t}{t + \frac{|b_1-b_2|}{9}}\right) \\
&= \frac{1}{\sqrt{1 - \left(\frac{t}{t+\frac{|b_1-b_2|}{9}}\right)}} > \frac{1}{\sqrt{1 - \left(\frac{t}{t+\frac{|b_1-b_2|}{3}}\right)}} = \gamma\left(\frac{t}{t + \frac{|b_1-b_2|}{3}}\right) \qquad (3.20) \\
&= \gamma(F_{Tx_1,Tx_2}(t)).
\end{aligned}$$

From (3.20), we have $\gamma(F_{Ta_1,Ta_2}(t)) > \gamma(F_{Tx_1,Tx_2}(t))$. So, we can find a $\delta \in (0, 1)$ such that $\gamma(F_{Ta_1,Ta_2}(t)) \geq F_{Tx_1,Tx_2}(t)) + \delta$. Then T is a γ-contraction of second type and $(0, 0)$ is a unique best proximity point of T.

If we assume that g is the identity mapping, we can give the following result.

Corollary 3.3. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Assume that A is approximately compact with respect to B. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*

(2) *$T : A \to B$ is a continuous $\gamma -$ proximal contraction of second type.*

Then, T has a unique probabilistic best proximity point in A.

4. Conclusion

The purpose of this paper is to give best proximity point theorems for γ-contractions and also γ-proximal contractions of first and second type. These are proved and supported with examples.

Acknowledgements. First author was supported by the Scientifc and Technological Research Council of Turkey (TUBITAK) Programme, 2211-A.

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[1]Department of Mathematics,
Faculty of Arts and Science,
Işik University
Email address: karaaslan.aysun@gmail.com

[2]Department of Mathematical Engineering,
Faculty of Chemistry-Metallurgical,
Yildiz Technical University
Email address: vkkaya@yildiz.edu.tr

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 345–364.

THREE-WEIGHT AND FIVE-WEIGHT LINEAR CODES OVER FINITE FIELDS

PAVAN KUMAR[1] AND NOOR MOHAMMAD KHAN[1]

Abstract. Recently, linear codes constructed from defining sets have been studied extensively. For an odd prime p, let Tr_e^m be the trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$, where e is a divisor of m. In this paper, for the defining set $D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\} = \{d_1, d_2, \dots, d_n\}$ (say), we define a p^e-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \{c_x = \left(\mathrm{Tr}_e^m(xd_1), \mathrm{Tr}_e^m(xd_2), \dots, \mathrm{Tr}_e^m(xd_n)\right) : x \in \mathbb{F}_{p^m}\}$$

and present three-weight and five-weight linear codes with their weight distributions. We show that each nonzero codeword of $\mathcal{C}_D$ is minimal for $\frac{m}{e} \geq 5$ and, thus, such codes are applicable in secret sharing schemes.

1. Introduction

Throughout this paper, let p be an odd prime, and let $\mathbb{F}_{p^m}$ be the finite field with p^m elements for any positive integer m. Denote by $\mathbb{F}_{p^m}^* = \mathbb{F}_{p^m} \setminus \{0\}$ the multiplicative group of $\mathbb{F}_{p^m}$.

An (n, M) code over $\mathbb{F}_{p^e}$, where $e \mid m$ and $\frac{m}{e} > 2$, is a subset of $\mathbb{F}_{p^e}^n$ of size M. Since linear codes are easier to describe, encode and decode than nonlinear codes, they have been an interesting topic in both theory and practice for many years. A linear code $\mathcal{C}$ over $\mathbb{F}_{p^e}$ is a subspace of $\mathbb{F}_{p^e}^n$. An $[n, k, d]$ linear code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_{p^e}^n$ with minimum Hamming-distance d. The vectors in a linear code $\mathcal{C}$ are known as *codewords*. The number of nonzero coordinates in $c \in \mathcal{C}$ is called the Hamming-weight $\mathrm{wt}(c)$ of a codeword c. Let A_i denote the number of codewords with Hamming weight

Key words and phrases. Linear code, weight distribution, Gauss sum, cyclotomic number, secret sharing.

2010 *Mathematics Subject Classification.* Primary: 11T71. Secondary: 94B05.

DOI 10.46793/KgJMat2403.345K

Received: March 22, 2020.

Accepted: April 23, 2021.

i in a linear code $\mathcal{C}$ of length n. The weight enumerator of $\mathcal{C}$ is defined by

$$1 + A_1 z + A_2 z^2 + \cdots + A_n z^n,$$

where $(1, A_1, \ldots, A_n)$ is called the *weight distribution* of $\mathcal{C}$. Throughout the paper, $\#\{\cdot\}$ denotes the cardinality of the set. If $\#\{i : A_i \neq 0, 1 \leq i \leq n\} = t$, then the code $\mathcal{C}$ is said to be t-weight code. Several classes of linear codes with various weights have been constructed in [3, 5, 6, 8, 19], and a lot of literature is present on the weight distributions of some special linear codes [1, 2, 14, 15].

Let $D = \{d_1, d_2, \ldots, d_n\} \subseteq F_{p^m}$. A linear code $\mathcal{C}_D$ of length n over $\mathbb{F}_p$ is defined by

$$\mathcal{C}_D = \{\Big(\mathrm{Tr}_1^m(xd_1), \mathrm{Tr}_1^m(xd_2), \ldots, \mathrm{Tr}_1^m(xd_n)\Big) : x \in \mathbb{F}_{p^m}\},$$

where Tr_1^m denotes the absolute trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_p$. The set D is known as the defining set of this code $\mathcal{C}_D$. Ding et al. introduced this construction (see [6, 7]), and many others used it to obtain linear codes with few weights [8, 17]. In [3, 6, 11, 14, 17, 19], the authors constructed the code $\mathcal{C}_D$ over $\mathbb{F}_p$ with few weights by considering certain defining sets with absolute trace function. In particular, the authors, in [11], give linear codes over $\mathbb{F}_p$ by employing Gauss sums and Pless Power Moments [10, page 260].

In this paper, we use Gauss sums and cyclotomic numbers to find linear codes over $\mathbb{F}_{p^e}$ by considering a new defining set obtained by replacing Tr by Tr_e^m in the defining set D given in [11]. Let m, s and e are positive integers with $s > 2$ and $m = es$. Now we define the trace function Tr_e^m from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$ as follows:

$$\mathrm{Tr}_e^m(x) = \sum_{k=0}^{s-1} x^{p^{ke}}.$$

Now, set

$$D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\} = \{d_1, d_2, \ldots, d_n\},$$

$$C_D = \{\boldsymbol{c}_x = (\mathrm{Tr}_e^m(xd_1), \mathrm{Tr}_e^m(xd_2), \ldots, \mathrm{Tr}_e^m(xd_n)) : x \in \mathbb{F}_{p^m}\}. \tag{1.1}$$

Then we present the weight distribution of the proposed linear code $\mathcal{C}_D$ of (1.1) in the Section 4.

2. Preliminaries

We begin with some preliminaries by introducing the concept of cyclotomic numbers. Let a be a primitive element of $\mathbb{F}_{p^m}$ and $p^m = Nh + 1$ for two positive integers $N > 1$, $h > 1$. The *cyclotomic classes* of order N in $\mathbb{F}_{p^m}$ are the cosets $\mathcal{C}_i^{(N,p^m)} = a^i\langle a^N\rangle$ for $i = 0, 1 \ldots, N-1$, where $\langle a^N\rangle$ denotes the subgroup of $\mathbb{F}_{p^m}^*$ generated by a^N. It is obvious that $\#\mathcal{C}_i^{(N,p^m)} = h$. For fixed i and j, we define the *cyclotomic number* $(i,j)^{(N,p^m)}$ to be the number of solutions of the equation

$$x_i + 1 = x_j, \quad x_i \in \mathcal{C}_i^{(N,p^m)}, x_j \in \mathcal{C}_j^{(N,p^m)},$$

where $1 = a^0$ is the multiplicative identity of $\mathbb{F}_{p^m}$. That is, $(i,j)^{(N,p^m)}$ is the number of ordered pairs (s,t) such that

$$a^{Ns+i} + 1 = a^{Nt+j}, \quad 0 \le s,t \le h-1.$$

Now, we present some notions and results about group characters and Gauss sums for later use (see [12] for details).

An additive character χ of $\mathbb{F}_{p^m}$ is a mapping from $\mathbb{F}_{p^m}$ into the multiplicative group of complex numbers of absolute value 1 with $\chi\ (g_1 + g_2) = \chi\ (g_1)\chi(g_2)$ for all $g_1, g_2 \in \mathbb{F}_{p^m}$. By ([12], Theorem 5.7), for any $b \in \mathbb{F}_{p^m}$,

$$\chi_b(x) = \zeta_p^{\mathrm{Tr}_1^m(bx)}, \quad \text{for all } x \in \mathbb{F}_{p^m}, \tag{2.1}$$

defines an additive character of $\mathbb{F}_{p^m}$, where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$, and every additive character can be obtained in this way. An additive character defined by $\chi_0(x) = 1$ for all $x \in \mathbb{F}_{p^m}$ is called the trivial character while all other characters are called nontrivial characters. The character χ_1 in (2.1) is called the canonical additive character of $\mathbb{F}_{p^m}$.

The orthogonal property of additive characters of $\mathbb{F}_{p^m}$ can be found in ([12], Theorem 5.4) and is given as

$$\sum_{x \in \mathbb{F}_{p^m}} \chi(x) = \begin{cases} p^m, & \text{if } \chi \text{ trivial}, \\ 0, & \text{if } \chi \text{ non-trivial}. \end{cases} \tag{2.2}$$

Characters of the multiplicative group $\mathbb{F}_{p^m}^*$ of $\mathbb{F}_{p^m}$ are called multiplicative character of $\mathbb{F}_{p^m}$. By [12, Theorem 5.8], for each $j = 0, 1, \ldots, p^m - 2$, the function ψ_j with

$$\psi_j(g^k) = e^{\frac{2\pi\sqrt{-1}jk}{p^m-1}}, \quad \text{for } k = 0, 1, \ldots, p^m - 2$$

defines a multiplicative character of $\mathbb{F}_{p^m}$, where g is a generator of $\mathbb{F}_{p^m}^*$. For $j = \frac{p^m-1}{2}$, we have the quadratic character $\eta = \psi_{\frac{p^m-1}{2}}$ defined by

$$\eta(g^k) = \begin{cases} -1, & \text{if } 2 \nmid k, \\ 1, & \text{if } 2 \mid k. \end{cases}$$

Moreover, we extend this quadratic character by letting $\eta(0) = 0$.

The quadratic Gauss sum $G = G(\eta, \chi_1)$ over $\mathbb{F}_{p^m}$ is defined by

$$G(\eta, \chi_1) = \sum_{x \in \mathbb{F}_{p^m}^*} \eta(x)\chi_1(x).$$

Now, let $\overline{\eta}$ and $\overline{\chi}_1$ denote the quadratic and canonical character of $\mathbb{F}_{p^e}$ respectively. Then we define the quadratic Gauss sum $\overline{G} = G(\overline{\eta}, \overline{\chi}_1)$ over $\mathbb{F}_{p^e}$ by

$$G(\overline{\eta}, \overline{\chi}_1) = \sum_{x \in \mathbb{F}_{p^e}^*} \overline{\eta}(x)\overline{\chi}_1(x).$$

The explicit values of quadratic Gauss sums are given by the following lemma.

Lemma 2.1. ([12, Theorem 5.15]). *Let the symbols be the same as before. Then*

$$G(\eta,\chi_1)=(-1)^{m-1}\sqrt{-1}^{\frac{(p-1)^2m}{4}}\sqrt{p^m},\quad G(\overline{\eta},\overline{\chi}_1)=(-1)^{e-1}\sqrt{-1}^{\frac{(p-1)^2e}{4}}\sqrt{p^e}.$$

Lemma 2.2. ([13, Lemma 2]). *Let the symbols be the same as before. Then the following hold.*

1. *If $s\geq 2$ is even, then $\eta(y)=1$ for each $y\in\mathbb{F}_{p^e}^*$;*
2. *If s is odd, then $\eta(y)=\overline{\eta}(y)$ for each $y\in\mathbb{F}_{p^e}^*$.*

Lemma 2.3. ([16]). *When $N=2$, the cyclotomic numbers are given by*

1. *h even: $(0,0)^{(2,p^m)}=\frac{h-2}{2}$, $(0,1)^{(2,p^m)}=(1,0)^{(2,p^m)}=(1,1)^{(2,p^m)}=\frac{h}{2}$;*
2. *h odd: $(0,0)^{(2,p^m)}=(1,0)^{(2,p^m)}=(1,1)^{(2,p^m)}=\frac{h-1}{2}$, $(0,1)^{(2,p^m)}=\frac{h+1}{2}$.*

Lemma 2.4. ([12, Theorem 5.33]). *Let χ be a non-trivial additive character of $\mathbb{F}_{p^m}$, and let $f(x)=a_2x^2+a_1x+a_0\in\mathbb{F}_{p^m}[x]$ with $a_2\neq 0$. Then*

$$\sum_{x\in\mathbb{F}_{p^m}}\chi(f(x))=\chi(a_0-a_1^2(4a_2)^{-1})\eta(a_2)G(\eta,\chi).$$

Lemma 2.5. ([12, Theorem 2.26]). *Let Tr_1^m and Tr_1^e be absolute trace functions over $\mathbb{F}_{p^m}$ and $\mathbb{F}_{p^e}$ respectively, and let Tr_e^m be the trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$. Then*

$$\mathrm{Tr}_1^m(x)=\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(x)),$$

for all $x\in\mathbb{F}_{p^m}$.

3. Basic Results

In this section, we provide some important results to establish our main results.

Lemma 3.1. *For each $\lambda\in\mathbb{F}_{p^e}$, set $S_\lambda=\#\{x\in\mathbb{F}_{p^m}:\mathrm{Tr}_e^m(x^2)=\lambda\}$. If s is odd, then*

$$S_\lambda=\begin{cases}p^{m-e}+p^{-e}\overline{\eta}(-1)\overline{\eta}(\lambda)G\overline{G}, & \text{if } \lambda\neq 0,\\ p^{m-e}, & \text{if } \lambda=0.\end{cases}$$

Proof. For each $\lambda\in\mathbb{F}_{p^e}$, we have

$$\begin{aligned}S_\lambda&=\frac{1}{p^e}\sum_{x\in\mathbb{F}_{p^m}}\left(\sum_{y\in\mathbb{F}_{p^e}}\zeta_p^{\mathrm{Tr}_1^e(y(\mathrm{Tr}_e^m(x^2)-\lambda))}\right)\\&=\frac{1}{p^e}\sum_{x\in\mathbb{F}_{p^m}}\left(1+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_e^m(yx^2)-\mathrm{Tr}_1^e(\lambda y)}\right)\\&=p^{m-e}+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{-\mathrm{Tr}_1^e(\lambda y)}\sum_{x\in\mathbb{F}_{p^m}}\chi_1(yx^2)\\&=p^{m-e}+G\sum_{y\in\mathbb{F}_{p^e}^*}\overline{\chi}_1(-\lambda y)\eta(y).\end{aligned}$$

This completes the proof. ∎

Lemma 3.2. *For $\lambda, \mu \in \mathbb{F}_{p^e}$, define*

$$N(\lambda,\mu) = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x^2) = \lambda \text{ and } \mathrm{Tr}_e^m(x) = \mu\}.$$

Then the following assertions hold.

1. *If $2 \mid s$ and $p \mid s$, then*
$$N(\lambda,\mu) = \begin{cases} p^{m-2e} + p^{-e}(p^e-1)G, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ p^{m-2e}, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ p^{m-2e} - p^{-e}G, & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$
2. *If $2 \mid s$ and $p \nmid s$, then*
$$N(\lambda,\mu) = \begin{cases} p^{m-2e}, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ p^{m-2e} + p^{-e}G, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0, \\ p^{m-2e} + \overline{\eta}(\mu^2 - s\lambda)p^{-e}G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0. \end{cases}$$
3. *If $2 \nmid s$ and $p \mid s$, then*
$$N(\lambda,\mu) = \begin{cases} p^{m-2e}, & \text{if } \lambda = 0, \\ p^{m-2e} + \overline{\eta}(-\lambda)p^{-e}G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$
4. *If $2 \nmid s$ and $p \nmid s$, then*
$$N(\lambda,\mu) = \begin{cases} p^{m-2e} + \overline{\eta}(-s)p^{-2e}(p^e-1)G\overline{G}, & \text{if } \mu^2 - s\lambda = 0, \\ p^{m-2e} - \overline{\eta}(-s)p^{-2e}G\overline{G}, & \text{if } \mu^2 - s\lambda \neq 0. \end{cases}$$

Proof. By the properties of additive character and Lemma 2.4, we have

$$\begin{aligned} N(\lambda,\mu) &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \Bigg(\sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(y(\mathrm{Tr}_e^m(x^2)-\lambda))} \Bigg) \Bigg(\sum_{z \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(z(\mathrm{Tr}_e^m(x)-\mu))} \Bigg) \\ &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \Bigg(1 + \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)} \Bigg) \Bigg(1 + \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)} \Bigg) \\ &= p^{m-2e} + p^{-2e}(S_1 + S_2 + S_3), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} S_1 &= \sum_{x \in \mathbb{F}_{p^m}} \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)} = \sum_{z \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-z\mu) \sum_{x \in \mathbb{F}_{p^m}} \chi_1(zx) = 0, \\ S_2 &= \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)} = \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-y\lambda) \sum_{x \in \mathbb{F}_{p^m}} \chi_1(yx^2), \\ S_3 &= \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)} \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)} \end{aligned}$$

$$= \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-y\lambda) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-z\mu) \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2+zx).$$

By Lemma 2.4, it is easy to prove that

$$S_2 = \begin{cases} G(p^e-1), & \text{if } \lambda = 0 \text{ and } 2 \mid s, \\ 0, & \text{if } \lambda = 0 \text{ and } 2 \nmid s, \\ -G, & \text{if } \lambda \neq 0 \text{ and } 2 \mid s, \\ \overline{\eta}(-\lambda)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } 2 \nmid s. \end{cases}$$

By Lemma 2.4, we have

$$\begin{aligned} S_3 &= \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-y\lambda) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-z\mu) \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2+zx) \\ &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\eta(y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right), \end{aligned}$$

and there are the following cases to consider.

Case I. Suppose that $2 \mid s$ and $p \mid s$. Then

$$S_3 = G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\mu z) = \begin{cases} G(p^e-1)^2, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ -G(p^e-1), & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ -G(p^e-1), & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ G, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$

Case II. We consider that $2 \mid s$ and $p \nmid s$. Then, from Lemma 2.4, we have

$$\begin{aligned} S_3 &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right) \\ &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(\frac{\mu^2 - s\lambda}{s} y\right) \overline{\eta}\left(-\frac{s}{4y}\right) \overline{G} - G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \\ &= \begin{cases} -G(p^e-1), & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ G, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0, \\ \left(\overline{\eta}(\mu^2 - s\lambda)p^e + 1\right)G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0. \end{cases} \end{aligned}$$

Case III. Assume that $2 \nmid s$ and $p \mid s$. Then

$$S_3 = G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\mu z) = \begin{cases} 0, & \text{if } \lambda = 0, \\ \overline{\eta}(-\lambda)(p^e-1)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ -\overline{\eta}(-\lambda)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$

Case IV. Suppose that $2 \nmid s$ and $p \nmid s$. Then, by Lemma 2.4, we have

$$\begin{aligned}
S_3 &= G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \sum_{z \in \mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right) \\
&= G\overline{G} \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y)\overline{\chi}_1\left(\frac{\mu^2 y}{s}\right)\overline{\eta}\left(-\frac{s}{4y}\right) - G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \\
&= \overline{\eta}(-s)G\overline{G} \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1\left(\frac{\mu^2 - s\lambda}{s} y\right) - G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \\
&= \begin{cases}
\overline{\eta}(-s)(p^e - 1)G\overline{G}, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\
-\overline{\eta}(-s)G\overline{G}, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\
\Big((p^e - 1)\overline{\eta}(-s) - \overline{\eta}(-\lambda)\Big)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0, \\
-\Big(\overline{\eta}(-s) + \overline{\eta}(-\lambda)\Big)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0.
\end{cases}
\end{aligned}$$

Combining (3.1) and the values of S_1, S_2 and S_3, we get the complete proof. □

Lemma 3.3. *Let the symbols be the same as before, and let*

$$\Omega_1 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2 + x))}.$$

Then

$$\Omega_1 = \begin{cases}
(p^e - 1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\
-G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\
0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\
\overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s.
\end{cases}$$

Proof. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
\Omega_1 &= \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \chi_1(yx^2 + yx) = G \sum_{y \in \mathbb{F}_{p^e}^*} \chi_1\left(-\frac{y}{4}\right)\eta(y) \\
&= G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^m(-\frac{y}{4})} = G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{y}{4}\mathrm{Tr}_e^m(1))} \\
&= \begin{cases}
G \sum\limits_{y \in \mathbb{F}_{p^e}^*} \eta(y), & \text{if } p \mid s, \\
G \sum\limits_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } p \nmid s.
\end{cases}
\end{aligned}$$

$$= \begin{cases} G \sum\limits_{y\in\mathbb{F}_{p^e}^*} 1, & \text{if } 2 \mid s \text{ and } p \mid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \overline{\eta}(y), & \text{if } 2 \nmid s \text{ and } p \mid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \overline{\eta}(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } 2 \nmid s \text{ and } p \nmid s, \end{cases}$$

$$= \begin{cases} (p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ -G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ 0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s, \end{cases}$$

as required. □

Lemma 3.4. *For $b \in \mathbb{F}_{p^m}^*$ and $c \in \mathbb{F}_{p^e}^*$, let*

$$\Omega_3 = \sum_{z\in\mathbb{F}_{p^e}^*} \sum_{y\in\mathbb{F}_{p^e}^*} \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2 + yx + bzx).$$

Then we have the following statemets.

1. *If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then*

$$\Omega_3 = \begin{cases} \overline{\eta}(-1)G\overline{G}^2 - G(p^e-1), & \text{if } 2 \mid s \text{ and } p \mid s, \\ G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 = s\mathrm{Tr}_e^m(b^2), \\ \overline{\eta}\big(s\mathrm{Tr}_e^m(b^2) - (\mathrm{Tr}_e^m(b)+2c)^2\big)G\overline{G}^2 + G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 \neq s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}\big(-\mathrm{Tr}_e^m(b^2)\big)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}\big(-\mathrm{Tr}_e^m(b^2)\big)G\overline{G}(p^e-1) - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 = s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}\big(-\mathrm{Tr}_e^m(b^2)\big)G\overline{G} - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

2. *If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) = 0$, then*

$$\Omega_3 = \begin{cases} -(p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ \overline{\eta}(s\mathrm{Tr}_e^m(b^2))G\overline{G}^2 + G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ \overline{\eta}(-\mathrm{Tr}_e^m(b^2))(p^e-1)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ -\Big(\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) + \overline{\eta}(-s)\Big)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s. \end{cases}$$

3. *If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then*

$$\Omega_3 = \begin{cases} -(p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ 0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ -\overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s. \end{cases}$$

4. *If* $\mathrm{Tr}_e^m(b^2)=0$ *and* $\mathrm{Tr}_e^m(b)=0$*, then*

$$\Omega_3=\begin{cases}(p^e-1)^2G, & \text{if } 2\mid s \text{ and } p\mid s,\\ -(p^e-1)G, & \text{if } 2\mid s \text{ and } p\nmid s,\\ 0, & \text{if } 2\nmid s \text{ and } p\mid s,\\ \overline{\eta}(-s)(p^e-1)G\overline{G}, & \text{if } 2\nmid s \text{ and } p\nmid s.\end{cases}$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}\Omega_3&=\sum_{z\in\mathbb{F}_{p^e}^*}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\chi_1(yx^2+yx+bzx)\\
&=G\sum_{z\in\mathbb{F}_{p^e}^*}\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{(y+bz)^2}{4y}\right)\\
&=G\sum_{z\in\mathbb{F}_{p^e}^*}\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(\frac{-y^2-2byz-b^2z^2}{4y}\right)\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\sum_{z\in\mathbb{F}_{p^e}^*}\chi_1\left(-\frac{bz}{2}-\frac{b^2z^2}{4y}\right)\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^m(-\frac{bz}{2}-\frac{b^2z^2}{4y})}\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e\left(-\frac{z}{2}\mathrm{Tr}_e^m(b)-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2)\right)}.\end{aligned}$$

Note that, in the first part, $\mathrm{Tr}_e^m(b^2)\neq 0$ and $\mathrm{Tr}_e^m(b)\neq 0$. Therefore,

$$\begin{aligned}\Omega_3&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1(-\frac{y}{4})\sum_{z\in\mathbb{F}_{p^e}^*}\overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2)-\frac{z}{2}\mathrm{Tr}_e^m(b)\right)\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\left(\sum_{z\in\mathbb{F}_{p^e}}\overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2)-\frac{z}{2}\mathrm{Tr}_e^m(b)\right)-1\right)\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\sum_{z\in\mathbb{F}_{p^e}}\overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2)-\frac{z}{2}\mathrm{Tr}_e^m(b)\right)-G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\\
&=G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\overline{\chi}_1\left(\frac{y\left(\mathrm{Tr}_e^m(b)\right)^2}{4\mathrm{Tr}_e^m(b^2)}\right)\overline{\eta}\left(-y\mathrm{Tr}_e^m(b^2)\right)\overline{G}-G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\\
&=\overline{\eta}\left(-\mathrm{Tr}_e^m(b^2)\right)G\overline{G}\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\overline{\chi}_1\left(\frac{\left(\mathrm{Tr}_e^m(b)\right)^2-s\mathrm{Tr}_e^m(b^2)}{4\mathrm{Tr}_e^m(b^2)}y\right)\overline{\eta}(y)-G\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\chi_1\left(-\frac{y}{4}\right)\end{aligned}$$

$$= \begin{cases} \overline{\eta}(-1)G\overline{G}^2 - G(p^e-1), & \text{if } 2 \mid s \text{ and } p \mid s, \\ G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 = s\mathrm{Tr}_e^m(b^2), \\ \overline{\eta}(s\mathrm{Tr}_e^m(b^2) - (\mathrm{Tr}_e^m(b))^2)G\overline{G}^2 + G, & \text{if } 2 \mid s, p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 \neq s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}(p^e-1) - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 = s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G} - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s, p \nmid s \text{ and } \big(\mathrm{Tr}_e^m(b)\big)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

This completes the proof of the first part.

Following the similar arguments used in the first part, one can easily prove the remaining parts. $\square$

Lemma 3.5. *For $\mu \in \mathbb{F}_{p^e}^*$, let $V = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x) = \mu$ and $\big(\mathrm{Tr}_e^m(x)\big)^2 = s\mathrm{Tr}_e^m(x^2)\}$. Then, for $p \nmid s$, we have*

$$V = \begin{cases} p^{m-2e}, & \text{if } 2 \mid s, \\ p^{m-2e} + \overline{\eta}(-s)p^{-2e}(p^e-1)G\overline{G}, & \text{if } 2 \nmid s. \end{cases}$$

Proof. We can rewrite V as

$$V = \#\left\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x) = \mu \text{ and } \mathrm{Tr}_e^m(x^2) = \frac{\mu^2}{s}\right\}.$$

Then, by definition, we have

$$\begin{aligned} V &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(\sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e\left(y(\mathrm{Tr}_e^m(x^2) - \frac{\mu^2}{s})\right)} \right) \left(\sum_{z \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e\left(z(\mathrm{Tr}_e^m(x) - \mu)\right)} \right) \\ &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(1 + \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2) - \frac{y\mu^2}{s})} \right) \left(1 + \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(zx) - z\mu)} \right) \\ &= p^{m-2e} + p^{-2e}(N_1 + N_2 + N_3), \end{aligned}$$

where

$$N_1 = \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(zx) - \mathrm{Tr}_1^e(z\mu)} = 0, \quad N_2 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2) - \mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)},$$

$$N_3 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2 + zx) - \mathrm{Tr}_1^e\left(\frac{y\mu^2}{s} + z\mu\right)}.$$

Now, by Lemma 2.4, we obtain

$$N_2 = \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)} \chi(0)\eta(y)G$$

$$= \begin{cases} G \sum\limits_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)}, & \text{if } 2 \mid s, \\ \overline{\eta}(-s) G \sum\limits_{y \in \mathbb{F}_{p^e}^*} \overline{\eta}\left(-\frac{y\mu^2}{s}\right) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right), & \text{if } 2 \nmid s, \end{cases}$$

$$= \begin{cases} -G, & \text{if } 2 \mid s, \\ \overline{\eta}(-s) G\overline{G}, & \text{if } 2 \nmid s, \end{cases}$$

and

$$\begin{aligned} N_3 &= \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s} + z\mu\right)} \sum_{x \in \mathbb{F}_{p^m}} \chi_1(yx^2 + zx) \\ &= \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s} + z\mu\right)} \chi_1\left(-\frac{z^2}{4y}\right) \eta(y) G \\ &= G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \sum_{z \in \mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - z\mu\right) \\ &= G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \sum_{z \in \mathbb{F}_{p^e}} \overline{\chi}_1\left(-\frac{sz^2}{4y} - z\mu\right) - G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \overline{\chi}_1\left(\frac{\mu^2}{s} y\right) \overline{\eta}\left(-\frac{s}{4y}\right) \overline{G} - G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= \overline{\eta}(-s) G\overline{G} \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\eta}\big(y\big) - G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y) \overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= \begin{cases} G, & \text{if } 2 \mid s, \\ \overline{\eta}(-s)(p^e - 1) G\overline{G} - \overline{\eta}(-s) G\overline{G}, & \text{if } 2 \nmid s. \end{cases} \end{aligned}$$

Also

$$V = p^{m-2e} + p^{-2e}(N_1 + N_2 + N_3).$$

Thus, we get the desired result. $\square$

Lemma 3.6. *Suppose that $\lambda, \mu \in \mathbb{F}_{p^e}^*$. For $i \in \{1, -1\}$, let K_i denote the number of pairs (λ, μ) such that $\overline{\eta}(\mu^2 - s\lambda) = i$. Then we have*

$$K_1 = \frac{1}{2}(p^e - 1)(p^e - 3), \quad K_{-1} = \frac{1}{2}(p^e - 1)^2.$$

Proof. We can rewrite $\mu^2 - s\lambda \neq 0$ as

$$\frac{s\lambda}{\mu^2 - s\lambda} + 1 = \frac{\mu^2}{\mu^2 - s\lambda}. \tag{3.2}$$

Set $p^e = 2h+1$. Now, for any fixed $\overline{\mu}^2 - s\overline{\lambda}$ such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = 1$, the number of the pairs (λ, μ^2) satisfying (3.2) is equal to

$$(0,0)^{(2,p^e)} + (1,0)^{(2,p^e)} = h-1 \quad \text{(by Lemma 2.2)}.$$

Similarly, for a fixed $\overline{\mu}^2 - s\overline{\lambda}$ such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = -1$, the number of pairs (λ, μ^2) satisfying (4.1) is equal to

$$(0,1)^{(2,p^e)} + (1,1)^{(2,p^e)} = h \quad \text{(from Lemma 2.2)}.$$

Consequently, the number of the pairs (λ, μ) such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = 1$ (resp. $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = -1$) is $2(h-1)$ (resp. $2h$). We conclude that $K_1 = (p^e-1)(h-1)$ (resp. $K_{-1} = (p^e-1)h$), and hence the result follows. □

Lemma 3.7. *Suppose that $\lambda, \mu \in \mathbb{F}_{p^e}^*$ and $\mu^2 - s\lambda \neq 0$. For $i \in \{1, -1\}$, let ψ_i denote the number of the pairs (λ, μ) such that $\overline{\eta}(-\lambda) = i$. Then we have*

$$\psi_1 = \begin{cases} \frac{1}{2}(p^e-1)(p^e-3), & \textit{if } \overline{\eta}(-s) = 1, \\ \frac{1}{2}(p^e-1)^2, & \textit{if } \overline{\eta}(-s) = -1, \end{cases}$$

and

$$\psi_{-1} = \begin{cases} \frac{1}{2}(p^e-1)^2, & \textit{if } \overline{\eta}(-s) = 1, \\ \frac{1}{2}(p^e-1)(p^e-3), & \textit{if } \overline{\eta}(-s) = -1. \end{cases}$$

Proof. The proof of the lemma is similar to the proof of Lemma 3.6 and is omitted here. □

4. Main Results

Our task in this section is to prove some lemmas needed to obtain a class of 3-weight and 5-weight linear codes over $\mathbb{F}_{p^e}$.

Now, let D be the defining set defined by

$$D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\}.$$

Assume that $l_0 = |D| + 1$. Then

$$l_0 = \frac{1}{p^e} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))} = p^{m-e} + \frac{1}{p^e} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))}.$$

Define $N_b = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x^2+x) = 0 \text{ and } \mathrm{Tr}_e^m(bx) = 0\}$. Let $\mathrm{wt}(c_b)$ denote the Hamming-weight of the codeword $\boldsymbol{c}_b$ of the code $\mathcal{C}_D$. It is easy to verify that

$$\mathrm{wt}(c_b) = l_0 - N_b. \tag{4.1}$$

For $b \in \mathbb{F}_{p^m}^*$, we have

$$N_b = p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(\sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))} \right) \left(\sum_{z \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(z \mathrm{Tr}_e^m(bx))} \right)$$

$$
\begin{aligned}
&= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(1 + \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))} \right) \left(1 + \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(z \mathrm{Tr}_e^m(bx))} \right) \\
&= p^{m-2e} + p^{-2e} \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2+yx))} + p^{-2e} \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(zbx))} \\
&\quad + p^{-2e} \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2+yx+bzx))}
\end{aligned}
$$

$$
= p^{m-2e} + p^{-2e} \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2+yx)} + p^{-2e} \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2+yx+bzx)}. \tag{4.2}
$$

In this section, we calculate l_0, N_b and give the proofs of the main results.

4.1. **The first case of three-weight linear codes.** In this subsection, we consider that $2 \mid s$ and $p \mid s$. In order to determine the weight distribution of $\mathcal{C}_D$ of (1.1), we need the following lemma.

Lemma 4.1. *Let $b \in \mathbb{F}_{p^m}^*$ and the symbols be the same as before. Then*

$$
N_b = \begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0, \\ & \text{or } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} + p^{-e}(p^e-1)G, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} + p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0. \end{cases}
$$

Proof. The proof of the lemma directly follows from (4.2), Lemmas 3.3 and 3.4. □

Theorem 4.1. *Let s be even and $p \mid s$. Then the code $\mathcal{C}_D$ of (1.1) is a $[p^{m-e} - 1 + p^{-e}(p^e-1)G, s]$ linear code with the weight distribution given in Table 1, where $G = -(-1)^{\frac{m(p-1)^2}{8}} p^{\frac{m}{2}}$.*

TABLE 1. The weight ditribution of the codes in Theorem 4.1

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e}$	$p^{m-2e} - 1 + p^{-e}(p^e-1)G$
$(p^e-1)p^{m-2e} + p^{-e}(p^e-1)G$	$2(p^e-1)p^{m-2e} - p^{-e}(p^e-1)G$
$(p^e-1)p^{m-2e} + p^{-e}(p^e-2)G$	$(p^e-1)^2 p^{m-2e}$

Proof. By Lemma 3.3, we have

$$
l_0 = p^{m-e} + p^{-e}(p^e-1)G.
$$

Combining (4.1) and Lemma 4.1, we have the following distinct cases.

Case I. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$ or $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) = 0$, then we obtain

$$
\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-e}(p^e-1)G.
$$

By Lemma 3.2, $\mathrm{wt}(c_b) = (p^e-1)p^{m-2e} + p^{-e}(p^e-1)G$ occurs $2(p^e-1)p^{m-2e} - p^{-e}(p^e-1)G$ times.

Case II. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) = 0$, then we have $\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e}$. By Lemma 3.2, the frequency is $p^{m-2e} - 1 + p^{-e}(p^e-1)G$.

Case III. If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then we have

$$\mathrm{wt}(c_b) = (p^e-1)p^{m-2e} + p^{-e}(p^e-2)G.$$

From Lemma 3.2, the frequency is $(p^e-1)^2 p^{m-2e}$. Hence, the result is established. $\square$

Example 4.1. Let $(p, m, s, e) = (3, 12, 6, 2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[58400, 6, 51840]$ and the weight enumerator $1 + 105624z^{51840} + 419904z^{51921} + 5912z^{52488}$.

4.2. **The second case of three-weight linear codes.** In this subsection, suppose $2 \mid s$ and $p \nmid s$. By (4.2), Lemmas 3.3 and 3.4, it is easy to get the following lemma.

Lemma 4.2. *Let $b \in \mathbb{F}_{p^m}^*$. Then*

$$N_b = \begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0,\\ p^{m-2e} + \overline{\eta}(-s\mathrm{Tr}_e^m(b^2))p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) = 0,\\ p^{m-2e} - p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) = 0,\\ p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0\\ & \text{and } \Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2),\\ p^{m-2e} + \overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2 - s\mathrm{Tr}_e^m(b^2)\Big)p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0\\ & \text{and } \Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq (s\mathrm{Tr}_e^m(b^2)). \end{cases}$$

Theorem 4.2. *Let $2 \mid s$ and $p \nmid s$. Then the code $\mathcal{C}_D$ of* (1.1) *is a $[p^{m-e} - p^{-e}G - 1, s]$ linear code with the weight distribution given in Table* 2*, where $G = -(-1)^{\frac{m(p-1)^2}{8}} p^{\frac{m}{2}}$.*

TABLE 2. The weight distribution for the codes in Theorem 4.2

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e} - p^{-e}G$	$(p^e-1)(2p^{m-2e} + p^{-e}G)$
$(p^e-1)p^{m-2e}$	$\frac{1}{2}(p^e-1)(p^{m-e} - G) + p^{m-2e} - 1$
$(p^e-1)p^{m-2e} - 2p^{-e}G$	$\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} + p^{-e}G)$

Proof. If $2 \mid s$ and $p \nmid s$, then by Lemma 3.3, we have

$$l_0 = p^{m-e} - p^{-e}G.$$

By (4.2) and Lemma 4.2, we have following distinct cases to consider.

Case I. If $\mathrm{Tr}_e^m(b^2)=0$ and $\mathrm{Tr}_e^m(b)\neq 0$ or $\mathrm{Tr}_e^m(b^2)\neq 0$ and $\Big(\mathrm{Tr}_e^m(b)\Big)^2=s\mathrm{Tr}_e^m(b^2)$, then we can acquire

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-p^{-e}G.$$

By Lemmas 3.2 and 3.5, the frequency is $(p^e-1)(2p^{m-2e}+p^{-e}G)$.

Case II. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(-s\mathrm{Tr}_e^m(b^2))=1$ or $\mathrm{Tr}_e^m(b^2)\neq 0, \mathrm{Tr}_e^m(b)\neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2\neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2-s\mathrm{Tr}_e^m(b^2)\Big)=1$, then we have

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-2p^{-e}G.$$

From Lemmas 3.2 and 3.5, the frequency is $\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e}+p^{-e}G)$.

Case III. If $\mathrm{Tr}_e^m(b^2)=0$ and $\mathrm{Tr}_e^m(b)=0$ or $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(-s\mathrm{Tr}_e^m(b^2))=-1$ or $\mathrm{Tr}_e^m(b^2)\neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2\neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2-s\mathrm{Tr}_e^m(b^2)\Big)=-1$, then

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}.$$

It follows from Lemmas 3.2 and 3.5 that $\mathrm{wt}(c_b)=(p^e-1)p^{m-2e}-2p^{-e}G$ occurs $\frac{1}{2}(p^e-1)(p^{m-e}-G)+p^{m-2e}-1$ times. Thus, the proof is completed. □

Example 4.2. Let $(p,m,s,e)=(3,8,4,2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[737,4,648]$ and the weight enumerator $1+3320z^{648}+1224z^{657}+2016z^{666}$.

4.3. **The first case of 5-weight linear codes.** In this subsection, we assume that $2\nmid s$ and $p\mid s$. By (4.2), Lemma 3.3 and Lemma 3.4, we get the following lemma.

Lemma 4.3. *For $b\in\mathbb{F}_{p^m}^*$ and $\mathrm{Tr}_e^m(b^2)\neq 0$, we have*

$$N_b=\begin{cases} p^{m-2e}-p^{-2e}\overline{\eta}(-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b)\neq 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big)=1,\\ p^{m-2e}+p^{-2e}\overline{\eta}(-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b)\neq 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big)=-1,\\ p^{m-2e}+p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b)=0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big)=1,\\ p^{m-2e}-p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b)=0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big)=-1.\end{cases}$$

Moreover, if $\mathrm{Tr}_e^m(b^2)=0$, then $N_b=p^{m-2e}$.

Theorem 4.3. *Let $2\nmid s$ and $p\mid s$. Then the linear code $\mathcal{C}_D$ of* (1.1) *has parameters $[p^{m-e}-1,s]$ and weight distribution in Table* 3, *where $G\overline{G}=(-1)^{m+e-2}(-1)^{\frac{(p-1)^2(m+e)}{8}}p^{\frac{(m+e)}{2}}$.*

Proof. Note that $2\nmid s$ and $p\mid s$. By Lemma 3.3, we have $l_0=p^{m-e}$, which gives the length of the code $\mathcal{C}_D$. It follows from (4.1) and Lemma 4.3 that $\mathrm{wt}(c_b)$ has five distinct values under following cases.

Case I. If $\mathrm{Tr}_e^m(b^2)=0$, then we have $\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}$. By Lemma 3.1, the frequency of such codewords is $p^{m-e}-1$.

Case II. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)\neq 0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=1$, then we can acquire

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)G\overline{G}.$$

From Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)^2p^{m-2e}$.

Case III. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)\neq 0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=-1$, then we can obtain

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)G\overline{G}.$$

It follows from Lemma 3.2 that the frequency is $\frac{1}{2}(p^e-1)^2p^{m-2e}$.

Case IV. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=1$, then we can obtain

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}.$$

By Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)(p^{m-2e}+p^{-e}\overline{\eta}(-1)G\overline{G})$.

Case V. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=-1$, then we have

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}.$$

From Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)(p^{m-2e}-p^{-e}\overline{\eta}(-1)G\overline{G})$. Hence, the result is established. $\square$

Example 4.3. Let $(p,m,s,e)=(3,6,3,2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[80,3,64]$ and the weight enumerator $1+72z^{64}+288z^{71}+80z^{72}+288z^{73}$. By Table 3, $\mathcal{C}_D$ in Theorem 4.3 is a four-weight linear code if and only if $p=s=3$.

TABLE 3. The weight distribution of the codes in Theorem 4.3

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e}$	$p^{m-e}-1$
$(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)G\overline{G}$	$\frac{1}{2}(p^e-1)^2p^{m-2e}$
$(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)G\overline{G}$	$\frac{1}{2}(p^e-1)^2p^{m-2e}$
$(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-2e}+p^{-e}\overline{\eta}(-1)G\overline{G})$
$(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-2e}-p^{-e}\overline{\eta}(-1)G\overline{G})$

Example 4.4. Let $(p,m,s,e)=(5,10,5,2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[5^8-1,5,24\times 5^6-600]$ and the weight enumerator $1+A_{w_1}z^{w_1}+A_{w_2}z^{w_2}+A_{w_3}z^{w_3}+A_{w_4}z^{w_4}+A_{w_5}z^{w_5}$, where the values of A_{w_i} and w_i for $1\leq i\leq 5$, are given in Table 4.

TABLE 4. The weight distribution of the code in Theorem 4.3 for $(p,m,s,e)=(5,10,5,2)$

Weight	Frequency
$w_1=24\times 5^6-600$	$A_{w_1}=12(5^6+5^4)$
$w_2=24\times 5^6-25$	$A_{w_2}=12\times 24\times 5^6$
$w_3=24\times 5^6$	$A_{w_3}=5^8-1$
$w_4=24\times 5^6+25$	$A_{w_4}=12\times 24\times 5^6$
$w_5=24\times 5^6+600$	$A_{w_5}=12(5^6-5^4)$

4.4. **The second case of five-weight linear codes.** In this subsection, suppose $2 \nmid s$ and $p \nmid s$. The auxiliary result that we need is the following.

Lemma 4.4. *Let $b \in \mathbb{F}_{p^m}^*$ and the symbols be the same as before. Then*

$$N_b = \begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0, \\ p^{m-2e} + p^{-e}\overline{\eta}(-s)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} - p^{-2e}\overline{\eta}\Big(-\mathrm{Tr}_e^m(b^2)\Big)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} + p^{-2e}\overline{\eta}(-s)(p^e-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0, \mathrm{Tr}_e^m(b) \neq 0 \text{ and} \\ & \Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2), \\ p^{m-2e} - p^{-2e}\overline{\eta}\Big(-\mathrm{Tr}_e^m(b^2)\Big)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0 \text{ and} \\ & \Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

Proof. The proof of the lemma follows from (4.2), Lemmas 3.3 and 3.4. □

Theorem 4.4. *Let s be odd with $p \nmid s$. Then the linear code C_D of* (1.1) *has parameters $[p^{m-e} + p^{-e}\overline{\eta}(-s)G\overline{G} - 1, s]$ and weight distribution in Tables* 5 *and* 6*, where $G\overline{G} = (-1)^{m+e-2}(-1)^{\frac{(p-1)^2(m+e)}{8}}p^{\frac{(m+e)}{2}}$.*

Proof. Firstly, we assume that $\overline{\eta}(-s) = 1$. For $2 \nmid s$ and $p \nmid s$, by Lemma 3.3, we have

$$l_0 = p^{m-e} + p^{-e}G\overline{G}.$$

It follows from (4.1) and Lemma 4.4 that $\mathrm{wt}(c_b)$ has five distinct values under following cases.

Case I. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then we have

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} + p^{-e}G\overline{G}.$$

By Lemma 3.2, the frequency is $(p^e - 1)(p^{m-2e} - p^{-2e}G\overline{G})$.

Case II. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) = 0$, then $\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e}$. From Lemma 3.2, the frequency is $p^{m-2e} + p^{-2e}(p^e - 1)G\overline{G} - 1$.

Case III. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$ and $\Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2)$, then we can obtain

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} + p^{-2e}G\overline{G}.$$

It follows from Lemmas 3.2 and 3.5 that the frequency of such codewords is $(p^e - 1)p^{m-2e} + p^{-2e}(p^e - 1)^2G\overline{G}$.

Case IV. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = 1$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = 1$, then we can obtain

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} + p^{-2e}(p^e + 1)G\overline{G}.$$

By Lemmas 3.2 and 3.7, the frequency is $\frac{1}{2}(p^e - 1)(p^{m-e} - p^{-e}G\overline{G})$.

Case V. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = -1$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$, $\left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = -1$, then we have

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-2e}(p^e-1)G\overline{G}.$$

From Lemmas 3.2 and 3.7, the frequency is $\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} - p^{-2e}G\overline{G})$, which completes the Table 5. Similarly, we can complete the Table 6 by taking $\overline{\eta}(-s) = -1$. □

Table 5. The weight distribution of the codes in Theorem 4.4 with $\overline{\eta}(-s) = 1$

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e} + p^{-e}G\overline{G}$	$(p^e-1)(p^{m-2e} - p^{-2e}G\overline{G})$
$(p^e-1)p^{m-2e}$	$p^{m-2e} + p^{-2e}(p^e-1)G\overline{G} - 1$
$(p^e-1)p^{m-2e} + p^{-2e}G\overline{G}$	$(p^e-1)p^{m-2e} + p^{-2e}(p^e-1)^2G\overline{G}$
$(p^e-1)p^{m-2e} + p^{-2e}(p^e+1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-e} - p^{-e}G\overline{G})$
$(p^e-1)p^{m-2e} + p^{-2e}(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} - p^{-2e}G\overline{G})$

Table 6. The weight distribution of the codes in Theorem 4.4 with $\overline{\eta}(-s) = -1$

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e} - p^{-e}G\overline{G}$	$(p^e-1)(p^{m-2e} + p^{-2e}G\overline{G})$
$(p^e-1)p^{m-2e}$	$p^{m-2e} - p^{-2e}(p^e-1)G\overline{G} - 1$
$(p^e-1)p^{m-2e} - p^{-2e}G\overline{G}$	$(p^e-1)p^{m-2e} - p^{-2e}(p^e-1)^2G\overline{G}$
$(p^e-1)p^{m-2e} - p^{-2e}(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-e} + p^{-e}G\overline{G})$
$(p^e-1)p^{m-2e} - p^{-2e}(p^e+1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} + p^{-2e}G\overline{G})$

Example 4.5. Let $(p, m, s, e) = (5, 6, 3, 2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[649, 3, 600]$ and the weight enumerator as $1 + 48z^{600} + 1176z^{601} + 6624z^{624} + 576z^{625} + 7200z^{626}$.

5. Concluding Remarks

In this paper, we have presented a class of three-weight and five-weight linear codes. A number of three-weight and five-weight codes were discussed in [1,3,4,6,9,14,19,20].

Let w_0 and w_∞ denote the minimum and maximum non-zero weight of a linear code $\mathcal{C}_D$, respectively. The linear code $\mathcal{C}_D$ with $\frac{w_0}{w_\infty} > \frac{(p^e-1)}{p^e}$ can be used to construct a secret sharing scheme with interesting access structures (see [18]).

For the linear code $\mathcal{C}_D$ in Theorem 4.1, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-2e}{2}}}.$$

Let $\frac{m}{e} > 4$. Then by simple computation, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-2e}{2}}} > \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.2, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - 2p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + 2p^{\frac{m-2e}{2}}}.$$

Then it can easily be checked that

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + 2p^{\frac{m-2e}{2}}} > \frac{(p^e-1)p^{m-2e} - 2p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}, \quad \text{for } \frac{m}{e} > 4.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.3, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-3e}{2}}} > \frac{(p^e-1)}{p^e}, \quad \text{for } \frac{m}{e} \geq 5.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.4, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e+1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e+1)p^{\frac{m-3e}{2}}}.$$

Let $\frac{m}{e} \geq 5$. Then by simple calculation, we can show that

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e+1)p^{\frac{m-3e}{2}}} > \frac{(p^e-1)p^{m-2e} - (p^e+1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}.$$

Consequently, one can easily see that the codewords of the linear code $\mathcal{C}_D$ are minimal for $\frac{m}{e} \geq 5$. These linear codes can be used in secret sharing schemes.

Acknowledgements. The present research is supported by University Grants Commission, New Delhi, India, under JRF in Science, Humanities & Social Sciences scheme, with Grant number 11-04-2016-413564.

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[1]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
Email address: pavan4957@gmail.com
Email address: nm_khan123@yahoo.co.in

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 48(3) (2024), PAGES 365–382.

IDENTITIES WITH MULTIPLICATIVE GENERALIZED (α, α)-DERIVATIONS OF SEMIPRIME RINGS

GURNINDER SINGH SANDHU[1], AYŞE AYRAN[2], AND NEŞET AYDIN[2]

ABSTRACT. Let R be a semiprime ring and α be an automorphism of R. A mapping $F : R \to R$ (not necessarily additive) is called multiplicative generalized (α, α)-derivation if there exists a unique (α, α)-derivation d of R such that $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$ for all $x, y \in R$. In the present paper, we intend to study several algebraic identities involving multiplicative generalized (α, α)-derivations on appropriate subsets of semiprime rings and collect the information about the commutative structure of these rings.

1. INTRODUCTION

Troughout this paper, R denotes an associative semiprime ring with center $Z(R)$. A ring R is said to be prime if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies $a = 0$. It is straight forward to observe that every prime ring is semiprime but the converse is not true in general, e.g., $\mathbb{Z} \times \mathbb{Z}$, which is a semiprime ring but not prime. For a fixed integer $n \geq 1$, a ring is said to be n-torsion free if $nx = 0$ for all $x \in R$ implies $x = 0$. For any $x, y \in R$, we denote the commutator $xy - yx$ and the anti-commutator $xy + yx$ by the symbols $[x, y]$ and $(x \circ y)$, respectively. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The very first example of a derivation is the differential operator Δ on $C[0, 1]$, the ring of the real valued differentiable functions on $[0, 1]$. The notion of derivation has been generalized in many directions. Brešar [6] introduced the notion of generalized derivation, which is an additive mapping

Key words and phrases. Semiprime ring, multiplicative generalized (α, α)-derivation, (α, α)-derivation, automorphism.
2010 *Mathematics Subject Classification.* Primary: 16W25. Secondary: 16N60, 16U80.
DOI 10.46793/KgJMat2403.365S
Received: January 02, 2021.
Accepted: April 24, 2021.

$F : R \to R$ satisfying the relation $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is the associated derivation of R. An additive mapping $H : R \to R$ such that $H(xy) = H(x)y$ for all $x, y \in R$ is called the left multiplier of R. Clearly, with $d = 0$, every left multiplier is a generalized derivation and with $F = d$, every derivation is a generalized derivation. Let $\alpha : R \to R$ be an automorphism of R. Then an additive mapping $\delta : R \to R$ is said to be an (α, α)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. Note that every $(1_R, 1_R)$-derivation is the ordinary derivation of R, where 1_R stands for the identity mapping of R. Thus one can now think of the notion of generalized (α, α)-derivation, which is a unified notion of both generalized derivation and (α, α)-derivation. Accordingly, an additive mapping $\xi : R \to R$ is said to be a generalized (α, α)-derivation if there exists a unique (α, α)-derivation δ of R such that $\xi(xy) = \xi(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. If we drop the assumption of additivity of ξ, then it is called multiplicative generalized (α, α)-derivation associated with (α, α)-derivation δ, e.g. let S be any ring and $R = \{ae_{12}+be_{13}+ce_{23} : \forall a, b, c \in S\}$ be a subring of $M_3(S)$, the ring of 3×3 matrices over S. Define a mapping $F : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto (bc)e_{13}$, $d : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13}$ and $\alpha : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13} - ce_{23}$. Then α is an automorphism of R and F is a multiplicative generalized (α, α)-derivation associated with (α, α)-derivation δ.

The study of commutative structure of prime rings with derivations has been initiated long back by Posner [15]. Precisely, Posner proved that: *If R is a prime ring and d is a derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either R is commutative or $d = 0$.* Since then this result has been extended in many directions by a number of algebraists. In 1987, Bell and Martindale [4] extended this result to the class of semiprime rings and proved that: if a semiprime ring R admits a derivation d which is nonzero and centralizing on a left ideal U of R, then R contains a nonzero central ideal. In the same line of investigation, Daif [7] examined the commutativity of semiprime rings admitting derivations that satisfy the identities: (i) $d([x, y]) - [xy] = 0$, (ii) $d([x, y]) + [x, y] = 0$. Further, Fošner et al. [12] studied a more general situation and proved that: if R is a 2-torsion free semiprime ring and d is a derivation of R such that $[[d(x), x], d(x)] = 0$ for all $x \in R$, then D maps R into $Z(R)$. Dhara [8] discussed the generalized derivations of semiprime rings that act as homomorphisms and anti-homomorphism on appropriate subsets of the ring. More specifically, he proved the following.

1. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(x)F(y)$ for all $x, y \in I$, then $d(I) = (0)$, F is commuting left multiplier mapping on I.
2. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = (0)$, R contains a nonzero central ideal.

Moreover, Dhara and Mozumder [10] generalized these results by studying the identities $F(xy) \pm F(x)F(y) \in Z(R)$ and $F(xy) \pm F(y)F(x) \in Z(R)$ on (semi)prime ring R, where F is multiplicative generalized derivation of R. Very recently, Tiwari and Sharma [17] studied the following identities: (i) $G(xy) \pm F(x)F(y) \in Z(R)$; (ii) $G(xy) \pm F(x)F(y) \pm \alpha(yx) = 0$; (iii) $G(xy) \pm F(x)F(y) \pm \alpha(xy) \in Z(R)$; (iv) $G(xy) \pm F(x)F(y) \pm \alpha([x, y]) = 0$; (v) $G(xy) \pm F(x)F(y) \pm \alpha(x \circ y) = 0$ on Lie ideals of prime rings, where F and G are generalized (α,α)-derivations and α is an automorphism.

On the other hand, Atteya [2] proved that a semiprime ring R that admits a generalized derivation F, contains a nonzero central ideal if any one of the following identity holds true: (i) $F(xy) \pm xy \in Z(R)$; (ii) $F(xy) \pm yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) + yx \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R. In 2013, Dhara et al. [9] studied the following identities: (i) $[d(x), F(y)] = \pm[x, y]$; (ii) $[d(x), F(y)] = \pm(x \circ y)$; (iii) $d(x) \circ F(y) = \pm(x \circ y)$; (iv) $d(x) \circ F(y) = \pm[x, y]$ on ideals of semiprime rings, where F is generalized derivation with associated derivation d. Then it seems more interesting to consider such identities with multiplicative derivations. In this direction, Kumar and Sandhu [14] investigated the following identities: (i) $F([x, y]) \pm xy = 0$; (ii) $F([x, y]) \pm yx = 0$; (iii) $F(x \circ y) \pm xy = 0$; (iv) $F(x \circ y) \pm yx = 0$; (v) $d(x)F(y) \pm xy = 0$; (vi) $d(x)F(y) \pm yx = 0$; (vii) $[F(x), y] \pm x \circ G(y) = 0$; (viii) $F(x) \circ y \pm x \circ G(y) = 0$ on semiprime R, where F and G are multiplicative (generalized)-derivations with the associated mappings d and g respectively. In 2017, Tiwari et al. [16] examined the structure of semiprime rings involving two multiplicative (generalized)-derivations that satisfy a list of algebraic identities on appropriate subsets of the ring.

In light of the above discussion, in this paper, our aim is to study certain identities with multiplicative generalized (α,α)-derivations of semiprime rings. More precisely, we characterize the following situations: (i) $G(xy) \pm F(x)F(y) = 0$; (ii) $G(xy) \pm F(x)F(y) \pm xy = 0$; (iii) $F(x)y + yG(x) = 0$; (iv) $F(xy) \pm G(xy) = 0$; (v) $F(xy) \pm G(yx) = 0$; (vi) $\alpha(x) \circ F(y) \pm G(yx) = 0$; (vii) $[\alpha(x), F(y)] \pm G(yx) = 0$; (viii) $\alpha(x) \circ F(y) \pm \alpha([x, y]) = 0$; (ix) $[\alpha(x), F(y)] \pm \alpha(x \circ y) = 0$; (x) $[F(x), d(y)] \pm x \circ \alpha(y) = 0$; (xi) $d(x) \circ d(y) \pm F(xy) = 0$; (xii) $[d(x), d(y)] \pm F(xy) = 0$; (xiii) $[d(x), F(y)] \pm F([x, y]) = 0$; (xiv) $d(x) \circ F(y) \pm F(x \circ y) = 0$, for all x, y in an appropriate subset of R, where F and G are multiplicative generalized (α,α)-derivations with associated (α,α)-derivations d and g, respectively.

2. Preliminary Results

Lemma 2.1 (Brauer's trick [11])**.** *A group G cannot be the union of two of its proper subgroups.*

Lemma 2.2. ([1, Lemma 2.1]). *If R is a semiprime ring and I is an ideal of R, then I is a semiprime ring.*

Lemma 2.3. ([4, Theorem 3]). *Let R be a semiprime ring and U be a nonzero left ideal of R. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Lemma 2.4. ([13, Lemma 1.1.5]). *Let R be a semiprime ring and ρ be a right ideal of R. Then $Z(\rho) \subset Z(R)$.*

Lemma 2.5. ([3, Lemma 3.1]). *Let R be a 2-torsion free semiprime ring and U be a nonzero left ideal of R. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in U$, then $axb = 0 = bxa$ for all $x \in U$.*

Lemma 2.6. ([12, Theorem 1]). *Let R be a 2-torsion free semiprime ring and $D : R \to R$ be a derivation satisfying the relation $[[D(x), x], D(x)] = 0$ for all $x \in R$. In this case D maps R into $Z(R)$.*

Lemma 2.7. ([5, Lemma 1]). *Let R be a semiprime ring, I be a nonzero ideal of R and $a \in I$ and $b \in R$. If $aIb = (0)$, then $ab = ba = 0$.*

Lemma 2.8. *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. If R admits an (α, α)-derivation d such that $d(I) \subseteq Z(R)$, then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$[r, d(x)] = 0, \quad \text{for all } x \in I,\ r \in R. \tag{2.1}$$

Replacing x by xs in (2.1), where $s \in R$, we get

$$d(x)[r, \alpha(s)] + [r, \alpha(x)]d(s) + \alpha(x)[r, d(s)] = 0. \tag{2.2}$$

Replacing x by sx in (2.2), we find

$$\begin{aligned} &d(s)\alpha(x)[r, \alpha(s)] + \alpha(s)d(x)[r, \alpha(s)] + \alpha(s)[r, \alpha(x)]d(s) \\ &\quad + [r, \alpha(s)]\alpha(x)d(s) + \alpha(s)\alpha(x)[r, d(s)] = 0. \end{aligned} \tag{2.3}$$

Utilization of equation (2.2) in (2.3) gives that

$$d(s)\alpha(x)[r, \alpha(s)] + [r, \alpha(s)]\alpha(x)d(s) = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

Applying Lemma 2.5, we have

$$d(s)\alpha(x)[r, \alpha(s)] = 0. \tag{2.4}$$

Taking $x\alpha^{-1}(p)$ instead of x in (2.4), we get

$$d(s)\alpha(x)p[r, \alpha(s)] = 0, \quad \text{for all } x \in I,\ r, s, p \in R.$$

We now can easily arrive at $d(s)[\alpha(x), \alpha(s)]pd(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s, p \in R$. It forces that $d(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Linearizing on s, we get

$$d(r)[\alpha(x), \alpha(s)] + d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R. \tag{2.5}$$

Substituting sx in place of s in (2.5), we get

$$d(r)[\alpha(x), \alpha(s)]\alpha(x) + d(s)\alpha(x)[\alpha(x), \alpha(r)] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0. \tag{2.6}$$

Right multiply (2.5) by $\alpha(x)$ and then subtract from (2.6) to obtain

$$d(s)[\alpha(x), [\alpha(x), \alpha(r)]] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

In particular, taking $r = x$ in (2.5), we get $d(x)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Using this in the above relation, we get

$$d(s)[\alpha(x), [\alpha(x), \alpha(r)]] = 0, \quad \text{for all } x \in I,\ r, s \in R. \tag{2.7}$$

Replacing r by rp in (2.7), where $p \in R$, we find

$$\begin{aligned} &d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] \\ &+ d(s)[\alpha(x), [\alpha(x), \alpha(r)]]\alpha(p) = 0. \end{aligned}$$

Using (2.7), we get

$$d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

In view of (2.7), it can be re-written as

$$-\alpha(s)d(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Again using (2.7), we get $2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0$ for all $x \in I$ and $r, s, p \in R$. Since R is 2-torsion free, we obtain

$$d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Replacing p by qp, where $q \in R$, we get

$$d(s)[\alpha(x), \alpha(r)]\alpha(q)[\alpha(x), \alpha(p)] = 0.$$

Replacing q by $q\alpha^{-1}(d(s))$ and put $p = r$ in the above relation, we get

$$d(s)[\alpha(x), \alpha(r)]Rd(s)[\alpha(x), \alpha(r)] = (0), \quad \text{for all } x \in I,\ r, s \in R.$$

Since R is semiprime, we get

$$d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

Replacing s by su in the above relation

$$d(s)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s, u \in R. \tag{2.8}$$

Replacing u by xu in (2.8), we get

$$d(s)\alpha(x)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s, u \in R. \tag{2.9}$$

Left multiply (2.8) by $\alpha(x)$ and then subtract from (2.9), we obtain

$$[d(s), \alpha(x)]\alpha(u)[\alpha(x), \alpha(r)] = 0.$$

In particular, put $r = \alpha^{-1}(d(s))$, we get

$$[\alpha(x), d(s)]\alpha(u)[\alpha(x), d(s)] = 0, \quad \text{for all } x \in I,\ u, s \in R.$$

It implies that $[\alpha(x), d(s)] = 0$ for all $x \in I$ and $s \in R$. That means $d(R) \subseteq Z(I^{'})$, where $I^{'} = \alpha(I)$. And hence by Lemma 2.4, we are done. $\square$

3. Main Results

Theorem 3.1. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $G(xy)\pm F(x)F(y) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal and d and g maps R into $Z(R)$.*

Proof. Assume that

$$G(xy) + F(x)F(y) = 0, \quad \text{for all } x, y \in I. \tag{3.1}$$

Replacing y by yr in (3.1), where $r \in R$, we find

$$\begin{aligned} 0 &= G(xyr) + F(x)F(yr) \\ &= G(xy)\alpha(r) + \alpha(xy)g(r) + F(x)F(y)\alpha(r) + F(x)\alpha(y)d(r). \end{aligned}$$

By using (3.1), we get

$$0 = \alpha(xy)g(r) + F(x)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.2}$$

Replacing x by xs in (3.2), where $s \in R$, we get

$$0 = \alpha(xsy)g(r) + F(x)\alpha(s)\alpha(y)d(r) + \alpha(x)d(s)\alpha(y)d(r). \tag{3.3}$$

Putting sy instead of y in (3.2), where $s \in R$, we get

$$0 = \alpha(xsy)g(r) + F(x)\alpha(sy)d(r). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$0 = \alpha(x)d(s)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r, s \in R.$$

Substituting wy for y and then putting $y = x$ and $s = r$, where $w \in R$, we have

$$0 = \alpha(x)d(r)\alpha(w)\alpha(x)d(r), \quad \text{for all } x \in I,\ r, w \in R.$$

Since R is semiprime ring, we find that

$$0 = \alpha(x)d(r), \quad \text{for all } x \in I,\ r \in R. \tag{3.5}$$

Replacing x by xy in (3.5), where $y \in I$, we have

$$0 = \alpha(x)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.6}$$

Because I is ideal of R, $\alpha(I)$ is ideal of R. In view of Lemma 2.7, we have $\alpha(x)d(r) = 0$ and $d(r)\alpha(x) = 0$ for all $x \in I, r \in R$. That is

$$0 = [\alpha(x), d(r)], \quad \text{for all } x \in I,\ r \in R. \tag{3.7}$$

Moreover, a particular case of (3.7) implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}d$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R. Further, from equation (3.7), we have $d(r) \in Z(I)$ for all $r \in R$. In view of Lemma 2.4, we conclude that d maps R into $Z(R)$.

Using (3.4) in (3.5), we obtain $0 = \alpha(x)\alpha(s)\alpha(y)g(r)$ for all $x, y \in I$ and $r, s \in R$. Taking $\alpha^{-1}(g(r))s$ instead of s in the last expression, we get in particular

$$(0) = \alpha(x)g(r)R\alpha(x)g(r), \quad \text{for all } x \in I,\, r \in R.$$

It yields that

$$0 = \alpha(x)g(r), \quad \text{for all } x \in I,\, r \in R.$$

Since this expression is same as (3.5) but with g instead of d. Therefore, the same technique implies g maps R into $Z(R)$, as desired.

Using similar approach, we conclude that the same result holds for $G(xy) - F(x)F(y) = 0$ for all $x, y \in I$. □

Theorem 3.2. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α, α)-derivations of R associated with nonzero (α, α)-derivations d and g respectively. If $G(xy) \pm F(x)F(y) \pm xy = 0$ for all $x, y \in I$ then R contains a nonzero central ideal and, d and g maps R into $Z(R)$.*

Proof. Assume that

$$G(xy) + F(x)F(y) + xy = 0, \quad \text{for all } x, y \in I. \tag{3.8}$$

Replacing y by yr in (3.8), where $r \in R$, we have

$$0 = G(xy)\alpha(r) + \alpha(xy)g(r) + F(x)F(y)\alpha(r) + F(x)\alpha(y)d(r) + xyr. \tag{3.9}$$

Right multiplying (3.8) by $\alpha(r)$, we get

$$0 = G(xy)\alpha(r) + F(x)F(y)\alpha(r) + xy\alpha(r), \quad \text{for all } x, y \in I,\, r \in R. \tag{3.10}$$

Subtracting (3.10) from (3.9), we obtain

$$0 = \alpha(xy)g(r) + F(x)\alpha(y)d(r) + xyr - xy\alpha(r), \quad \text{for all } x, y \in I,\, r \in R. \tag{3.11}$$

Replacing x by xs in (3.11), where $s \in R$, we have

$$0 = \alpha(xsy)g(r) + F(x)\alpha(s)\alpha(y)d(r) + \alpha(x)d(s)\alpha(y)d(r) + xsyr - xsy\alpha(r). \tag{3.12}$$

Again replacing y by sy in (3.11), where $s \in R$, we get

$$0 = \alpha(xsy)g(r) + F(x)\alpha(sy)d(r) + xsyr - xsy\alpha(r). \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$0 = \alpha(x)d(s)\alpha(y)d(r), \quad \text{for all } x, y \in I,\, r, s \in R. \tag{3.14}$$

Substituting wx for y and r for s in (3.14), where $w \in R$, we have

$$0 = \alpha(x)d(r)\alpha(w)\alpha(x)d(r), \quad \text{for all } x \in I,\, r, w \in R.$$

It follows that

$$0 = \alpha(x)d(r), \quad \text{for all } x \in I,\, r \in R.$$

This expression also appeared as equation (3.5) in Theorem 3.1, so the result is followed in the same way.

Now replacing x by ux in (3.9), where $u \in R$, we get

$$0 = \alpha(u)\alpha(xy)g(r) + uxyr - uxy\alpha(r).$$

Combining with the above relation, it implies that $(\alpha(u) - u)\alpha(xy)g(r) = 0$ for all $x, y \in I$ and $u, r \in R$. Using (3.11), it gives

$$(\alpha(u) - u)xy(\alpha(r) - r) = 0, \quad \text{for all } x, y \in I, \; r, u \in R.$$

It implies that $x\alpha(r) = xr$ for all $x \in I$ and $r \in R$. Now, using this in (3.11), we obtain $0 = \alpha(xy)g(r)$ for all $x, y \in I$ and $r \in R$. That is, $\alpha(x)\alpha(y)g(r) = 0$. Since α is an automorphism, replacing x by $x\alpha^{-1}(g(r))$ in the last relation to get $\alpha(x)g(r)\alpha(y)g(r) = 0$. Further it implies that $\alpha(x)g(r)R\alpha(y)g(r) = (0)$ for all $x, y \in I$ and $r \in R$. In particular, $\alpha(x)g(r)R\alpha(x)g(r) = (0)$ for all $x \in I$ and $r \in R$. Hence, we get $0 = \alpha(x)g(r)$ for all $x \in I$ and $r \in R$. By repeating the similar argument as above, we get our conclusion.

Using similar approach, we conclude that the same result holds for $G(xy) - F(x)F(y) - xy = 0$, $G(xy) - F(x)F(y) + xy = 0$ and $G(xy) + F(x)F(y) - xy = 0$ for all $x, y \in I$. □

Theorem 3.3. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism on R such that $\alpha(I) = I$. Let F and G be multiplicative generalized (α, α)-derivations of R associated with nonzero (α, α)-derivations d and g respectively. If $F(x)y + yG(x) = 0$ for all $x, y \in I$, then F maps I into $Z(R)$ and G maps I into $Z(R)$. Moreover, R contains a nonzero central ideal and $d + g$ maps R into $Z(R)$.*

Proof. Assume that

$$F(x)y + yG(x) = 0, \quad \text{for all } x, y \in I. \tag{3.15}$$

Substituting xz in place of x in (3.15), where $z \in I$, we obtain

$$0 = F(x)\alpha(z)y + \alpha(x)d(z)y + yG(x)\alpha(z) + y\alpha(x)g(z). \tag{3.16}$$

Replacing y by $\alpha(z)y$ in (3.15), we have

$$0 = F(x)\alpha(z)y + \alpha(z)yG(x), \quad \text{for all } x, y \in I, \; z \in R. \tag{3.17}$$

Subtracting (3.17) from (3.16) and using (3.15), we get

$$0 = -F(x)y\alpha(z) + \alpha(z)F(x)y + \alpha(x)d(z)y + y\alpha(x)g(z). \tag{3.18}$$

Combining (3.16) and (3.18), we get

$$0 = [\alpha(z), F(x)]\, y, \quad \text{for all } x, y, z \in I.$$

Since I is semiprime, we have

$$0 = [\alpha(z), F(x)]\,, \quad \text{for all } x, z \in I.$$

That is $F(I) \subset Z(I)$. In view of Lemma 2.4, we find $F(I) \subset Z(R)$. Now right multiplying (3.15) by $\alpha(z)$, where $z \in I$, we obtain

$$0 = F(x)y\alpha(z) + yG(x)\alpha(z), \quad \text{for all } x, y, z \in I. \tag{3.19}$$

Replacing y with $y\alpha(z)$ in (3.15), we have

$$0 = F(x)y\alpha(z) + y\alpha(z)G(x), \quad \text{for all } x, y, z \in I. \tag{3.20}$$

Application (3.19) and (3.20) yields that

$$0 = y\left[\alpha(z), G(x)\right], \quad \text{for all } x, y, z \in I.$$

By semiprimeness of I, we get

$$0 = \left[\alpha(z), G(x)\right], \quad \text{for all } x, y, z \in I.$$

That is $G(I) \subset Z(I)$. In view of Lemma 2.4, we find $G(I) \subset Z(R)$. In view of $F(I) \subset Z(I)$ and $G(I) \subset Z(I)$, our hypothesis yields

$$0 = y(F+G)(x), \quad \text{for all } x, y \in I. \tag{3.21}$$

Replacing x by xr in (3.21), where $r \in R$, we find

$$0 = y(F+G)(x)\alpha(r) + y\alpha(x)(d+g)(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.22}$$

Application of (3.21) in (3.22), we have

$$0 = y\alpha(x)(d+g)(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.23}$$

In view of Lemma 2.7 and the fact that $\alpha(I) = I$, we have $y(d+g)(r) = 0$ and $(d+g)(r)y = 0$ for all $x \in I, r \in R$. That is

$$0 = [y, (d+g)(r)], \quad \text{for all } y \in I,\ r \in R. \tag{3.24}$$

Moreover, a particular case of (3.24) implies that $[\alpha(y), (d+g)(y)] = 0$ for all $y \in I$. That is $[y, \varphi(y)] = 0$ for all $y \in I$, where $\varphi = \alpha^{-1}(d+g)$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R. Applying Lemma 2.4 on relation (3.24), we find that $(d+g)(R) \subseteq Z(R)$. $\square$

Theorem 3.4. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations associated with nonzero (α,α)-derivations d and g respectively. If $F(xy) \pm G(xy) = 0$ for all $x, y \in I$, then $d \pm g$ maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. By the given hypothesis, we have

$$F(xy) \pm G(xy) = 0 = (F \pm G)(xy), \quad \text{for all } x, y \in I.$$

Since sum of two multiplicative generalized (α,α)-derivations is a multiplicative generalized (α,α)-derivation, we take H in place of $F \pm G$, therefore our condition becomes $H(xy) = 0$ for all $x, y \in I$. Which is a particular case of our Theorem 3.1 (with $F = 0$). Hence, we are done. $\square$

Theorem 3.5. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $F(xy) \pm G(yx) = 0$ for all $x, y \in I$, then d and g maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$F(xy) + G(yx) = 0, \quad \text{for all } x, y \in I. \tag{3.25}$$

Replacing x by xr in (3.25), where $r \in R$, we have

$$0 = F(xry) + G(yxr), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.26}$$

Again replace y with ry in (3.25), we find

$$0 = F(xry) + G(ryx), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.27}$$

Combining (3.26) and (3.27), we get

$$G(yxr) = G(ryx), \quad \text{for all } x, y \in I,\ r \in R.$$

Putting yz instead of y in (3.25), where $z \in I$ and using (3.25), we get

$$0 = -G(yx)\alpha(z) + G(yz)\alpha(x) + \alpha(xy)d(z) + \alpha(yz)g(x), \quad \text{for all } x, y, z \in I. \tag{3.28}$$

Substituting xs in place of x in (3.28), where $s \in R$, we obtain

$$-G(yxs)\alpha(z) + G(yz)\alpha(xs) + \alpha(xsy)d(z) + \alpha(yz)g(xs) = 0. \tag{3.29}$$

Taking sy instead of y in (3.28), where $s \in R$, we have

$$-G(syx)\alpha(z) + G(syz)\alpha(x) + \alpha(xsy)d(z) + \alpha(syz)g(x) = 0. \tag{3.30}$$

Combining (3.29) and (3.30) and using $G(yxs) = G(syx)$, we get

$$0 = G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(syz)\alpha(x) - \alpha(syz)g(x).$$

Using $G(syz) = G(yzs)$ in last expression, we find

$$G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(yzs)\alpha(x) - \alpha(syz)g(x) = 0, \quad \text{for all } x, y, z \in I,\ s \in R.$$

That is

$$G(yz)\alpha([x,s]) + \alpha(yz)(g(x)\alpha(s) - g(s)\alpha(x)) + \alpha(yzx)g(s) - \alpha(syz)g(x) = 0, \tag{3.31}$$

for all $x, y, z \in I$, $s \in R$. Replacing s by x in (3.31), we find

$$0 = \alpha([yz, x])g(x), \quad \text{for all } x, y, z \in I. \tag{3.32}$$

Putting wy instead of y in (3.32), where $w \in I$, we have $0 = \alpha([w,x])\alpha(y)\alpha(z)g(x)$ for all $x, y, z, w \in I$. It is implies that $(0) = \alpha([w,x])\alpha(I)\alpha(z)g(x)$ for all $x, y, z \in I$. Because I is an ideal of R, $\alpha(I)$ is an ideal of R. In view of Lemma 2.7, we have

$$0 = \alpha([w,x])\alpha(z)g(x), \quad \text{for all } x, y, z \in I. \tag{3.33}$$

That is, $[I,x]\, I\, ((\alpha^{-1}g)(x)) = (0)$ for all $x \in I$. Since I is a semiprime ring in itself, it must contains a family P of prime ideals such that $\cap P_\lambda = (0)$. Let P_{λ_1} be a typical member of this family and $x \in I$; by (3.33), we have find

$$[I,x] \subset P_{\lambda_1} \quad \text{or} \quad \left(\left(\alpha^{-1}g\right)(x)\right) \subset P_{\lambda_1}.$$

Let $A = \{x \in I : [I,x] \subset P_{\lambda_1}\}$ and $B = \{x \in I : ((\alpha^{-1}g)(x)) \subset P_{\lambda_1}\}$. Note that A and B are additive subgroups of I such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[I,I] \subset P_{\lambda_1} \quad \text{or} \quad \left(\left(\alpha^{-1}g\right)(I)\right) \subset P_{\lambda_1}.$$

Together with these both cases, we have $[I,I]\,((\alpha^{-1}g)(I)) = (0)$. That is

$$\alpha([x,y])g(z) = 0, \quad \text{for all } x,y,z \in I. \tag{3.34}$$

Replacing y by ry in (3.34), where $r \in R$, we have $\alpha([x,r])\alpha(y)g(z) = 0$. That is

$$[\alpha(x),r]\,\alpha(y)g(z) = 0, \quad \text{for all } x,y,z \in I,\ r \in R. \tag{3.35}$$

Right multiplying (3.35) by $\alpha(x)$, we get

$$[\alpha(x),r]\,\alpha(y)g(z)\alpha(x) = 0. \tag{3.36}$$

Substituting yx for y in (3.35), we have

$$[\alpha(x),r]\,\alpha(y)\alpha(x)g(z) = 0. \tag{3.37}$$

Combining (3.36) and (3.37), we obtain $[\alpha(x),r]\,\alpha(y)\,[\alpha(x),g(z)] = 0$ for all $x,y,z \in I, r \in R$. In particular, we have $[\alpha(x),g(z)]\,\alpha(I)\,[\alpha(x),g(z)] = (0)$ for all $x,z \in I$. Since $\alpha(I)$ is semiprime ring, we obtain

$$[\alpha(x),g(z)] = 0, \quad \text{for all } x,z \in I. \tag{3.38}$$

That is $g(I) \subset Z(I')$, where $I' = \alpha(I)$. By Lemma 2.4, $g(I) \subset Z(R)$. In view of Lemma 2.8, we get $g(R) \subset Z(R)$. Moreover, a particular case of (3.38) implies that $[x,\varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}g$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R.

Now from (3.25), by replacing y by yr, where $r \in R$, we get

$$F(xyr) + G(yrx) = 0, \quad \text{for all } x,y \in I,\ r \in R. \tag{3.39}$$

And replacing x by rx in (3.25), we obtain

$$F(rxy) + G(yrx), \quad \text{for all } x,y \in I,\ r \in R. \tag{3.40}$$

Combining (3.39) and (3.40), we get

$$F(xyr) = F(rxy), \quad \text{for all } x,y \in I,\ r \in R.$$

Interchanging the role of x and y in this expression, we obtain

$$F(yxr) = F(ryx), \quad \text{for all } x,y \in I,\ r \in R.$$

This relation has already existed in the above proof for G. Hence by repeating the same arguments, we get that d maps R into $Z(R)$. □

Theorem 3.6. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x,y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x, y \in I$. That is

$$\alpha(x)F(y) + F(y)\alpha(x) \pm G(yx) = 0. \tag{3.41}$$

Replacing y by yx in (3.41), we find

$$\begin{aligned}&\alpha(x)F(y)\alpha(x) + \alpha(x)\alpha(y)d(x) + F(y)\alpha(x)\alpha(x)\\ &+ \alpha(y)d(x)\alpha(x) \pm G(yx)\alpha(x) \pm \alpha(yx)g(x) = 0.\end{aligned}$$

Expression (3.41) reduces it to

$$\alpha(x)\alpha(y)d(x) + \alpha(y)d(x)\alpha(x) \pm \alpha(yx)g(x) = 0. \tag{3.42}$$

Taking zy in place of y in (3.42), where $z \in I$, we get

$$\alpha(x)\alpha(z)\alpha(y)d(x) + \alpha(z)\alpha(y)d(x)\alpha(x) \pm \alpha(z)\alpha(yx)g(x) = 0. \tag{3.43}$$

By using (3.42), we find $[\alpha(x), \alpha(z)]\alpha(y)d(x) = 0$ for all $x, y, z \in I$. By using Lemma 2.7, it yields that $[\alpha(x), \alpha(y)]d(x) = 0$ for all $x, y \in I$. Replacing y by $\alpha^{-1}(r)y$, where $r \in R$, we get

$$[\alpha(x), r]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I,\ r \in R.$$

It implies that

$$[\alpha(x), R]R\alpha(I)d(x) = (0). \tag{3.44}$$

Since R contains a family S of prime ideals such that $\cap P_\lambda = (0)$. Let P be a typical member of this family and $x \in I$, by (3.44), we find

$$[\alpha(x), R] \subset P \quad \text{or} \quad \alpha(I)d(x) \subset P.$$

Let $A = \{x \in I : [\alpha(x), R] \subset P\}$ and $B = \{x \in I : \alpha(I)d(x) \subset P\}$. Note that A and B are the additive subgroups of R such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[\alpha(I), R] \subset P \quad \text{or} \quad \alpha(I)d(I) \subset P.$$

Together with these both cases, we have $[\alpha(I), R]d(I) = 0$. That is

$$[\alpha(x), r]d(y) = 0, \quad \text{for all } x, y \in I,\ r \in R. \tag{3.45}$$

Replacing y by yt, in (3.45), where $t \in I$, we find

$$[\alpha(x), r]\alpha(y)d(t) = 0. \tag{3.46}$$

This expression is same as (3.35) of Theorem 3.5 with d instead of g. With the similar arguments, we get our conclusion. □

By the same implications as in Theorem 3.6 with necessary modifications, we can get the following result.

Theorem 3.7. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α, α)-derivations of R associated with nonzero (α, α)-derivations d and g, respectively. If $[\alpha(x), F(y)] \pm G(yx) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.8. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $\alpha(x)\circ F(y)\pm\alpha([x,y])=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$\alpha(x)\circ F(y)\pm\alpha([x,y])=0,\quad \text{for all } x,y\in I. \tag{3.47}$$

Replacing y by yx in (3.47), we find

$$(\alpha(x)\circ F(y))\alpha(x)+(\alpha(x)\circ\alpha(y)d(x))\pm\alpha([x,y])\alpha(x)=0,\quad \text{for all } x,y\in I.$$

Equation (3.47) reduces it to

$$(\alpha(x)\circ\alpha(y)d(x))=0. \tag{3.48}$$

Replacing y by zy in (3.48), where $z\in I$, and using it, we get $[\alpha(x),\alpha(z)]\alpha(y)d(x)=0$ for all $x,y,z\in I$. This expression also appeared in Theorem 3.6, hence the conclusion follows in the similar manner. □

By the same implications as in Theorem 3.8 with necessary modifications, we can get the following result.

Theorem 3.9. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[\alpha(x),F(y)]\pm\alpha(x\circ y)=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.10. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[F(x),d(y)]\pm(x\circ\alpha(y))=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $[F(x),d(y)]-(x\circ\alpha(y))=0$ for all $x,y\in I$. That is

$$F(x)d(y)-d(y)F(x)-x\alpha(y)-\alpha(y)x=0. \tag{3.49}$$

Replacing y by yz in (3.49), where $z\in I$, we find

$$\begin{aligned}&F(x)d(y)\alpha(z)+F(x)\alpha(y)d(z)-d(y)\alpha(z)F(x)\\&-\alpha(y)d(z)F(x)-x\alpha(y)\alpha(z)-\alpha(y)\alpha(z)x=0.\end{aligned}$$

Right multiplying (3.49) by $\alpha(z)$ and then comparing with the above expression to obtain

$$[F(x),\alpha(y)d(z)]+d(y)[F(x),\alpha(z)]-\alpha(y)[\alpha(z),x]=0,\quad \text{for all } x,y,z\in I. \tag{3.50}$$

Taking zy instead of y in (3.50), we have

$$\begin{aligned}&\alpha(z)[F(x),\alpha(y)d(z)]+[F(x),\alpha(z)]\alpha(y)d(z)+d(z)\alpha(y)[F(x),\alpha(z)]\\&+\alpha(z)d(y)[F(x),\alpha(z)]-\alpha(z)\alpha(y)[\alpha(z),x]=0,\quad \text{for all } x,y,z\in I.\end{aligned}$$

Application of (3.50) yields

$$[F(x), \alpha(z)]\alpha(y)d(z) + d(z)\alpha(y)[F(x), \alpha(z)] = 0, \quad \text{for all } x, y, z \in I.$$

In view of Lemma 2.5, it follows that

$$[F(x), \alpha(z)]\alpha(y)d(z) = 0 = d(z)\alpha(y)[F(x), \alpha(z)], \quad \text{for all } x, y, z \in I.$$

Let us consider the expression

$$[F(x), \alpha(z)]\alpha(y)d(z) = 0. \tag{3.51}$$

Replacing x by xz in (3.51), we get

$$[F(x), \alpha(z)]\alpha(z)\alpha(y)d(z) + [\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0. \tag{3.52}$$

Substituting zy in place of y in (3.51) and then subtracting it from (3.52) in order to find

$$[\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0, \quad \text{for all } x, y, z \in I. \tag{3.53}$$

Replacing x by wx in (3.53), where $w \in R$, and using it to obtain

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)\alpha(y)d(z) = 0, \quad \text{for all } x, y, z, w \in I.$$

In particular, taking $y = r[w, z]x$ in above expression, where $r \in R$, we get

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)R[\alpha(w), \alpha(z)]\alpha(x)d(z) = (0), \quad \text{for all } x, z, w \in I.$$

Since R is semiprime ring, it implies that $[\alpha(w), \alpha(z)]\alpha(x)d(z) = 0$ for all $x, z, w \in I$. This expression also appeared in Theorem 3.6, so the result is followed in the same way.

Using similar approach we conclude that the same result holds for $[F(x), d(y)] + (x \circ \alpha(y)) = 0$ for all $x, y \in I$. ☐

Theorem 3.11. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R. Let F be a multiplicative generalized (α, α)-derivation of R associated with a nonzero (α, α)-derivation d. If $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. We assume that $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$. That is

$$d(x)d(y) + d(y)d(x) \pm F(xy) = 0. \tag{3.54}$$

Replacing y by yz in (3.54) and using it, we get

$$d(x)\alpha(y)d(z) + \alpha(y)d(z)d(x) + d(y)[\alpha(z), d(x)] \pm \alpha(xy)d(z) = 0. \tag{3.55}$$

Taking xy in place of y in (3.55) to get

$$\begin{aligned} &d(x)\alpha(x)\alpha(y)d(z) + \alpha(x)\alpha(y)d(z)d(x) + d(x)\alpha(y)[\alpha(z), d(x)] \\ &+ \alpha(x)d(y)[\alpha(z), d(x)] \pm \alpha(x)\alpha(xy)d(z) = 0, \quad \text{for all } x, y, z \in R. \end{aligned} \tag{3.56}$$

Application of (3.55) in (3.56) yields

$$[d(x), \alpha(x)]\alpha(y)d(z) + d(x)\alpha(y)[\alpha(z), d(x)] = 0, \quad \text{for all } x, y, z \in R.$$

In particular for $x=z$, we get

$$[d(x),\alpha(x)]\alpha(y)d(x)=d(x)\alpha(y)[d(x),\alpha(x)],\quad \text{for all } x,y\in R. \tag{3.57}$$

Replacing y by $y\alpha^{-1}(d(x))$ in (3.57) and using it, we have

$$d(x)\alpha(y)[d(x),[d(x),\alpha(x)]]=0,\quad \text{for all } x,y\in R.$$

It implies that

$$[d(x),[d(x),\alpha(x)]]\alpha(y)[d(x),[d(x),\alpha(x)]]=0,\quad \text{for all } x,y\in R.$$

Using semiprimeness of R, we get $[d(x),[d(x),\alpha(x)]]=0$ for all $x\in R$. That is equivalent to $[\varphi(x),[\varphi(x),x]]=0$ for all $x\in R$, where $\varphi=\alpha^{-1}d$, which is an ordinary derivation of R. Invoking Lemma 2.6, we get φ maps R into $Z(R)$, i.e.,

$$[\varphi(x),y]=0,\quad \text{for all } x,y\in R.$$

In particular, we have $[\varphi(x),x]=0$ for all $x\in R$, and hence R contains a nonzero central ideal by Lemma 2.3. □

Theorem 3.12. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[d(x),d(y)]\pm F(xy)=0$ for all $x,y\in I$, then R contains a nonzero central ideal.*

Proof. Following the same arguments as in Theorem 3.11, instead of equation (3.57), we have

$$[d(x),\alpha(x)]\alpha(y)d(x)+d(x)\alpha(y)[d(x),\alpha(x)]=0,\quad \text{for all } x,y\in I.$$

By Lemma 2.5, we find

$$d(x)\alpha(y)[d(x),\alpha(x)]=0,\quad \text{for all } x,y\in I. \tag{3.58}$$

Taking xy in place of y in (3.58), we get

$$d(x)\alpha(x)\alpha(y)[d(x),\alpha(x)]=0,\quad \text{for all } x,y\in I. \tag{3.59}$$

Left multiply (3.58) by $\alpha(x)$ to obtain

$$\alpha(x)d(x)\alpha(y)[d(x),\alpha(x)]=0,\quad \text{for all } x,y\in I. \tag{3.60}$$

Comparing (3.59) and (3.60), we obtain $[d(x),\alpha(x)]\alpha(y)[d(x),\alpha(x)]=0$ for all $x,y\in I$. By semiprimeness of $\alpha(I)$, we find $[\alpha(x),d(x)]=0$ for all $x\in I$. It implies that $[x,\varphi(x)]=0$ for all $x\in I$, where $\varphi=\alpha^{-1}d$, which is an ordinary derivation. Hence, in light of Lemma 2.3, we are done. □

Now onwards, we consider that F is a two sided multiplicative generalized (α,α)-derivation associated with (α,α)-derivation d, i.e., F satisfies the following conditions:

$$F(xy)=F(x)\alpha(y)+\alpha(x)d(y)=d(x)\alpha(y)+\alpha(x)F(y),\quad \text{for all } x,y\in R.$$

Theorem 3.13. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. If $[d(x), F(y)] \pm F([x, y]) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. Assume that

$$[d(x), F(y)] \pm F([x, y]) = 0, \quad \text{for all } x, y \in I. \tag{3.61}$$

Replacing y by yx in (3.61) and using it, we get

$$F(y)[d(x), \alpha(x)] + [d(x), \alpha(y)]d(x) \pm \alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I. \tag{3.62}$$

Replacing y by xy in (3.62), we have

$$\begin{aligned}&\alpha(x)F(y)[d(x), \alpha(x)] + d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)[d(x), \alpha(y)]d(x)\\ &+ [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I.\end{aligned}$$

Using (3.62), we get

$$d(x)\alpha(y)[d(x), \alpha(x)] + [d(x), \alpha(x)]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I.$$

This expression also appeared in Theorem 3.12, so the result is followed in the same way. □

Theorem 3.14. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R. If $d(x) \circ F(y) \pm F(x \circ y) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$(d(x) \circ F(y)) \pm F(x \circ y) = 0, \quad \text{for all } x, y \in I. \tag{3.63}$$

Replacing y by yx in (3.63) and using it, we get

$$-F(y)[d(x), \alpha(x)] + (d(x) \circ \alpha(y))d(x) \pm \alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I. \tag{3.64}$$

Replacing y by xy in (3.64), we find

$$\begin{aligned}&-\alpha(x)F(y)[d(x), \alpha(x)] - d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)(d(x) \circ \alpha(y))d(x)\\ &+ [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I.\end{aligned}$$

Using (3.64), we obtain

$$d(x)\alpha(y)[d(x), \alpha(x)] = [d(x), \alpha(x)]\alpha(y)d(x), \quad \text{for all } x, y \in I.$$

This expression also appeared as equation (3.57) in Theorem 3.11, so the result is followed in the same way. □

Acknowledgements. We would like to thank the anonymous referee for his/her comments and suggestions that lead many improvements in the article.

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[1]Department of Mathematics,
Patel Memorial National College,
Rajpura 140401, India
Email address: gurninder_rs@pbi.ac.in

[2]Department of Mathematics,
Çanakkale Onsek*I*z Mart University,
Çanakkale 17020, Turkey
Email address: ayseayran@stu.comu.edu.tr
Email address: neseta@comu.edu.tr

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 383–392.

NEW MIXED RECURRENCE RELATIONS OF TWO-VARIABLE ORTHOGONAL POLYNOMIALS VIA DIFFERENTIAL OPERATORS

MOSAED M. MAKKY[1] AND MOHAMMAD SHADAB[2]

Abstract. In this paper, we derive new recurrence relations for two-variable orthogonal polynomials for example Jacobi polynomial, Bateman's polynomial and Legendre polynomial via two different differential operators $\Xi = \left(\frac{\partial}{\partial z} + \sqrt{w}\frac{\partial}{\partial w}\right)$ and $\Delta = \left(\frac{1}{w}\frac{\partial}{\partial z} + \frac{1}{z}\frac{\partial}{\partial w}\right)$. We also derive some special cases of our main results.

1. Introduction and Preliminaries

In recent decades, the study of the multi-variable orthogonal polynomials has been substantially developed by many authors [3,5,15]. The properties of the multi-variable orthogonal polynomials have been analyzed by different approaches. The analytical properties of two-variable orthogonal polynomials like generating functions, recurrence relations, partial differential equations, and orthogonality have remained the main attraction of the topic due to its wide range of applications in different research areas [1, 4, 7, 10, 16].

Some new classes of two-variables analogues of the Jacobi polynomials have been introduced from Jacobi weights by Koornwinder [9]. These all classes are introduced by means of two different partial differential operators D_1 and D_2, where D_1 has order two, and D_2 may have any arbitrary order. Koornwinder constructed bases of orthogonal polynomials in two-variables by using a tool given by Agahanov [2].

In 2017, M. Marriaga et al. [11] derived some new recurrence relations involving two-variable orthogonal polynomials in a different way. In 2019, G. V. Milovanović et

Key words and phrases. Jacobi polynomials, Legendre polynomials, Bateman's polynomials, differential operators.

2020 *Mathematics Subject Classification.* Primary: 33C45, 33C47. Secondary: 11B37.

DOI 10.46793/KgJMat2403.383M

Received: September 21, 2020.

Accepted: April 28, 2021.

al. [12] presented the study of various recurrence relations, generating functions and series expansion formulas for two families of orthogonal polynomials in two-variables. Motivated by these two studies, we present here some recurrence relations of two-variables orthogonal polynomials via differential operators.

The generalized hypergeometric function [14, p. 42–43] can be defined as

$$ {}_pF_q\left[\begin{array}{l}\alpha_1,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\cdot\frac{z^n}{n!}, $$

with certain convergence conditions given in [14, p. 43].

The Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ [13, p. 22, (1)], is defined by

$$ (\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1, & \nu=0,\lambda\in\mathbb{C}\setminus\{0\},\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \nu=n\in\mathbb{N},\lambda\in\mathbb{C},\end{cases} $$

being understood *conventionally* that $(0)_0=1$ and assumed *tacitly* that the Γ quotient exists.

The classical Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n, $n=0,1,2,\dots$, [13, p. 254 (1)] is defined as

$$ P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}{}_2F_1\left(-n,1+\alpha+\beta+n;1+\alpha;\frac{1-x}{2}\right), $$
$$ \operatorname{Re}(\alpha)>-1,\ \operatorname{Re}(\beta)>-1,\ x\in(-1,1). $$

The generating function of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n [13, p. 270, (2)] is defined by

$$ \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)\,t^n=F_4\left(1+\beta,1+\alpha;1+\alpha,1+\beta;\frac{1}{2}t(x-1),\frac{1}{2}t(x+1)\right), $$

where

$$ F_4\left(1+\beta,1+\alpha;1+\alpha,1+\beta;\frac{1}{2}t(x-1),\frac{1}{2}t(x+1)\right), $$

is an Appell polynomial [14, p. 53, (7)].

An elementary generating function of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [13, p. 271, (6)] can be presented in the form

$$ \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n=\rho^{-1}\left(\frac{2}{1+t+\rho}\right)^{\beta}\left(\frac{2}{1-t+\rho}\right)^{\alpha} $$

or

$$ \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n=2^{\alpha+\beta}\rho^{-1}(1+t+\rho)^{-\beta}(1-t+\rho)^{-\alpha}, $$

where $\rho=(1-2xt+t^2)^{\frac{1}{2}}$ and on setting $\alpha=\beta=0$, the Jacobi polynomial reduce to the Legendre polynomial.

Recently, R. Khan et al. [8] introduced generalization of two-variable Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x,y)=\sum_{k=0}^{n}\frac{(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n}\left(\frac{x-\sqrt{y}}{2}\right)^k, \tag{1.1}$$
$$n=0,1,2,\ldots,\ \mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ x,y\in(-1,1),$$

which can be presented in the alternate form

$$P_n^{(\alpha,\beta)}(x,y)=\sum_{n,k=0}^{\infty}\frac{(1+\alpha)_n\,(1+\beta)_n}{k!(n-k)!\,(1+\alpha)_k\,(1+\beta)_{n-k}}\left(\frac{x-\sqrt{y}}{2}\right)^k\left(\frac{x+\sqrt{y}}{2}\right)^{n-k}$$

and

$$P_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n}{n!}\left(\frac{x+\sqrt{y}}{2}\right)^n{}_2F_1\left(-n,-\beta-n;1+\alpha;\frac{x-\sqrt{y}}{x+\sqrt{y}}\right)$$

or

$$P_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n}{n!}{}_2F_1\left(-n,1+\alpha+\beta+n;1+\alpha;\frac{\sqrt{y}-x}{2}\right).$$

The generating functions of generalized Jacobi polynomial of two-variables $P_n^{(\alpha,\beta)}(x,y)$ [8] can be presented as follows

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x,y)\,t^n=\mu^{-1}\left(\frac{2}{1+\sqrt{y}t+\mu}\right)^{\beta}\left(\frac{2}{1-\sqrt{y}t+\mu}\right)^{\alpha}$$

or

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x,y)t^n=2^{\alpha+\beta}\mu^{-1}\left(1+\sqrt{y}t+\mu\right)^{-\beta}\left(1-\sqrt{y}t+\mu\right)^{-\alpha},$$

where $\mu=(1-2xt+y\,t^2)^{\frac{1}{2}}$.

In another way, the generating function of generalized Jacobi polynomials of two variables $P_n^{(\alpha,\beta)}(x,y)$ [8] can be presented as follows

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x,y)t^n=F_4\left(1+\beta,1+\alpha;1+\alpha,1+\beta;\frac{1}{2}t(x-\sqrt{y}),\frac{1}{2}t(x+\sqrt{y})\right),$$

which can be written in the form

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x,y)\,t^n=\sum_{n,k=0}^{\infty}\frac{(1+\alpha)_{n+k}\,(1+\beta)_{n+k}\,\frac{1}{2}\left(x-\sqrt{y}\right)^k\frac{1}{2}\left(x+\sqrt{y}\right)^n t^n}{k!n!\,(1+\alpha)_k\,(1+\beta)_n}.$$

Bateman's generating function for $P_n^{(\alpha,\beta)}(x,y)$ [8] can be presented as follows

$$\mathrm{B}_n^{(\alpha,\beta)}(x,y;t)=\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(x-\sqrt{y}\right)^n t^n}{n!\,(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(x+\sqrt{y}\right)^n t^n}{n!\,(1+\beta)_n}\right],$$
$$\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ |x|<1,\ |y|<1,$$

where

$$\mathrm{B}_n^{(\alpha,\beta)}(x,y;t)=\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n\,(1+\beta)_n}.$$

The generalized Jacobi polynomial of two-variables $P_n^{(\alpha,\beta)}(x,y)$ reduces to the Legendre polynomial of two variables $P_n(x,y)$ for $\alpha=\beta=0$ in (1.1)

$$P_n(x,y)=\sum_{k=0}^{n}\frac{(n+k)!}{(k!)^2\,(n-k)!}\left(\frac{x-\sqrt{y}}{2}\right)^k,$$

and its generating function can be given by

$$\sum_{n=0}^{\infty}P_n(x,y)\,t^n=\left(1-2xt+yt^2\right)^{-\frac{1}{2}}.$$

Also, Khan and Abukhammash [6] defined the Legendre Polynomials of two-variables $P_n(x,y)$ as

$$P_n(x,y)=\sum_{k=0}^{[n/2]}\frac{(-y)^k\left(\frac{1}{2}\right)_{n-k}(2x)^{n-2k}}{k!(n-k)!}$$

and the generating function for $P_n(x,y)$ is given by

$$\sum_{k=0}^{n}P_n(x,y)\,t^n=\left(1-2xt+y\,t^2\right)^{\frac{1}{2}}.$$

2. Recurrence Relations for Jacobi Polynomials

In this section, we will study the action of the following differential operator

$$\Xi=\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right) \tag{2.1}$$

on complex bivariate Jacobi polynomial $P_n^{(\alpha,\beta)}(z,w)$ (2.2) to obtain the desired results.

Now, we present complex bivariate Jacobi polynomial by replacing $x,y\in\mathbb{R}$ by $z,w\in\mathbb{C}$ such that

$$P_n^{(\alpha,\beta)}(z,w)=\sum_{k=0}^{n}\frac{(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n}\left(\frac{z-\sqrt{w}}{2}\right)^k, \tag{2.2}$$
$$\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ |z|<1,\ |w|<1.$$

Following conjugate relations will be use frequently in the paper.

$$\begin{aligned}
(1+\alpha+\beta)_{n+k+1}&=(1+\alpha+\beta)\,(2+\alpha+\beta)\,[1+(1+\alpha)+(1+\beta)]_{(n-1)+k},\\
(1+\alpha)_{k+1}&=(1+\alpha)\,(1+(1+\alpha))_k,\\
(1+\alpha)_n&=(1+\alpha)\,(1+(1+\alpha))_{n-1},\\
(1+\alpha+\beta)_{n+1}&=(1+\alpha+\beta)_n\,(1+\alpha+\beta+n).
\end{aligned}$$

Theorem 2.1. *Following recurrence relation for the Jacobi Polynomial* $P_n^{(\alpha,\beta)}(z,w)$, *it holds true*

$$\begin{aligned}&\frac{\partial}{\partial z}\,P_n^{(\alpha,\beta)}(z,w)+\sqrt{w}\frac{\partial}{\partial w}\,P_n^{(\alpha,\beta)}(z,w)\\&-\frac{(1+\alpha+\beta+n)}{4}\left(\frac{z-\sqrt{w}}{2}\right)\,P_{n-1}^{(1+\alpha),(1+\beta)}(z,w)=0,\\&\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ |z|<1,\ |w|<1.\end{aligned}$$

Proof. On applying the operator (2.1) in (2.2), we get

$$\begin{aligned}&\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right)P_n^{(\alpha,\beta)}(z,w)\\&=\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right)\sum_{k=0}^{n}\frac{(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n}\left(\frac{z-\sqrt{w}}{2}\right)^k\\&=\sum_{k=0}^{n}\frac{k\,(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}\left(\frac{1}{2}-\frac{1}{4}\right)\\&=\frac{1}{4}\sum_{k=0}^{n}\frac{k\,(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}.\end{aligned}$$

Now, on replacing $k\to k+1$ and simplifications, we get

$$\begin{aligned}&\frac{1}{4}\sum_{k=0}^{n}\frac{(1+\alpha)_n\,(1+\alpha+\beta)_{n+k+1}}{k!\,[n-(k+1)]!\,(1+\alpha)_{k+1}\,(1+\alpha+\beta)_n}\left(\frac{z-\sqrt{w}}{2}\right)^k\\&=\frac{1}{4}\sum_{k=0}^{n}\frac{(1+\alpha)\,(1+(1+\alpha))_{n-1}\,(1+\alpha+\beta)\,(2+\alpha+\beta)}{k!\,((n-1)-k)!\,(1+\alpha)\,(1+(1+\alpha))_k}\\&\quad\times\frac{(1+(1+\alpha)+(1+\beta))_{(n-1)+k}\,(1+\alpha+\beta+n)}{(1+(1+\alpha)+(1+\beta))_{(n-1)}\,(1+\alpha+\beta)\,(2+\alpha+\beta)}\left(\frac{z-\sqrt{w}}{2}\right)^k\\&=\frac{1+(\alpha+\beta+n)}{4}\left(\frac{z-\sqrt{w}}{2}\right)\\&\quad\times\sum_{k=0}^{n}\frac{(1+(1+\alpha))_{n-1}\,[1+(1+\alpha)+(1+\beta)]_{(n-1)+k}}{k!\,((n-1)-k)!\,(1+(1+\alpha))_k\,[1+(1+\alpha)+(1+\beta)]_{n-1}}\left(\frac{z-\sqrt{w}}{2}\right)^{k-1}\\&=\frac{(1+\alpha+\beta+n)}{4}\left(\frac{z-\sqrt{w}}{2}\right)\,P_{n-1}^{(1+\alpha),(1+\beta)}(z,w).\end{aligned}$$

Therefore, we get the desired result. □

Corollary 2.1. *Following recurrence relation for the Jacobi Polynomial* $P_n^{(\alpha,\beta)}(z,1)$, *it holds true*

$$\frac{\partial}{\partial z}\,P_n^{(\alpha,\beta)}(z,1)-\frac{(1+\alpha+\beta+n)}{2}\left(\frac{z-1}{2}\right)\,P_{n-1}^{(1+\alpha),(1+\beta)}(z,1)=0,$$

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, $|z| < 1$.

Proof. First, put $w = 1$ in (2.2) we consider the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(z,1) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\alpha)_k (1+\alpha+\beta)_n} \left(\frac{z-1}{2}\right)^k.$$

Taking differential operator $\Xi_z = \left(\frac{\partial}{\partial z}\right)$ then following the same process used in the above theorem leads to the desired result. □

Corollary 2.2. *Following recurrence relation for the Jacobi Polynomial $P_n^{(\alpha,\beta)}(1,w)$, it holds true*

$$\left(\sqrt{w}\frac{\partial}{\partial w}\right) P_n^{(\alpha,\beta)}(1,w) + \frac{(1+\alpha+\beta+n)}{4}\left(\frac{1-\sqrt{w}}{2}\right) P_{n-1}^{(1+\alpha),(1+\beta)}(1,w) = 0,$$

$\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, $|w| < 1$.

Proof. By puting $z = 1$ in (2.2) we get

$$P_n^{(\alpha,\beta)}(1,w) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\alpha)_k (1+\alpha+\beta)_n} \left(\frac{1-\sqrt{w}}{2}\right)^k.$$

Taking differential operator $\Xi_w = \left(\sqrt{w}\frac{\partial}{\partial w}\right)$ then following the same process used in the above theorem leads to the desired result. □

3. Recurrence Relations for Bateman's Generating Function

Now, we present complex bivariate Bateman's generating function by replacing $x, y \in \mathbb{R}$ by $z, w \in \mathbb{C}$ such that

$$\mathrm{B}_n^{(\alpha,\beta)}(z,w;t) = \left[\sum_{n=0}^{\infty} \frac{\frac{1}{2}(z-\sqrt{w})^n t^n}{n!(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty} \frac{\frac{1}{2}(z+\sqrt{w})^n t^n}{n!(1+\beta)_n}\right] \tag{3.1}$$

and

$$\mathrm{B}_n^{(\alpha,\beta)}(z,w;t) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(z,w)t^n}{(1+\alpha)_n (1+\beta)_n}, \quad \mathrm{Re}(\alpha) > -1, \mathrm{Re}(\beta) > -1, |z| < 1, |w| < 1.$$

We can also write the conjugate relationships for the purpose to use these relations in this section

$$(1+\alpha)_{n+1} = (1+\alpha)(1+(1+\alpha))_n,$$
$$(1+\beta)_{n+1} = (1+\beta)(1+(1+\beta))_n.$$

Theorem 3.1. *Following recurrence relation for the Bateman's generating function $\mathrm{B}_n^{(\alpha,\beta)}(z,w;t)$, it holds true*

$$\frac{\partial}{\partial z}\mathrm{B}_n^{(\alpha,\beta)}(z,w;t) + \sqrt{w}\frac{\partial}{\partial w}\mathrm{B}_n^{(\alpha,\beta)}(z,w;t) - \frac{t}{2(1+\alpha)}\mathrm{B}_n^{[(1+\alpha),\beta]}(z,w;t)$$
$$- \frac{3t}{2(1+\beta)}\mathrm{B}_n^{[\alpha,(1+\beta)]}(z,w;t) = 0, \quad \mathrm{Re}(\alpha) > -1, \mathrm{Re}(\beta) > -1, |z| < 1, |w| < 1.$$

Proof. Using the differential operator Ξ for the Bateman's generating function $\mathrm{B}_n^{(\alpha,\beta)}(z,w;t)$, we see that

$$\begin{aligned}
&\Xi\ \mathrm{B}_n^{(\alpha,\beta)}(z,w;t)\\
=&\left(\frac{\partial}{\partial z}+\sqrt{w}\frac{\partial}{\partial w}\right)\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z-\sqrt{w}\right)^n t^n}{n!\,(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z+\sqrt{w}\right)^n t^n}{n!\,(1+\beta)_n}\right]\\
=&\frac{t}{2\,(1+\alpha)}\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z-\sqrt{w}\right)^n t^n}{n!\,[1+(1+\alpha)]_n}\right]\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z+\sqrt{w}\right)^n t^n}{n!\,(1+\beta)_n}\right]\\
&+\frac{3\,t}{2\,(1+\beta)}\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z-\sqrt{w}\right)^n t^n}{n!\,(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z+\sqrt{w}\right)^n t^n}{n!\,[1+(1+\beta)]_n}\right]\\
=&\frac{t}{2\,(1+\alpha)}\mathrm{B}_n^{[(1+\alpha),\beta]}(z,w;t)-\frac{3\,t}{2\,(1+\beta)}\mathrm{B}_n^{[\alpha,(1+\beta)]}(z,w;t).
\end{aligned}$$

Therefore, we get the desired result. □

Corollary 3.1. *Following recurrence relation for the Bateman's generating function* $\mathrm{B}_n^{(\alpha,\beta)}(z,1)$*, it holds true*

$$\begin{aligned}
&\frac{\partial}{\partial z}\ \mathrm{B}_n^{(\alpha,\beta)}(z,1;t)-\frac{t}{(1+\alpha)}\mathrm{B}_n^{[(1+\alpha),\beta]}(z,1;t)-\frac{t}{(1+\beta)}\mathrm{B}_n^{[\alpha,(1+\beta)]}(z,1;t)=0,\\
&\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ |z|<1.
\end{aligned}$$

Proof. First, substitute w=1 in the Bateman's generating function (3.1), we have

$$\mathrm{B}_n^{(\alpha,\beta)}(z,1;t)=\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z-1\right)^n t^n}{n!\,(1+\alpha)_n}\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(z+1\right)^n t^n}{n!\,(1+\beta)_n}.$$

Taking differential operator $\Xi_z=\left(\frac{\partial}{\partial z}\right)$ then following the same process used in the above theorem leads to the desired result. □

Corollary 3.2. *Following recurrence relation for the Bateman's generating function* $\mathrm{B}_n^{(\alpha,\beta)}(1,w)$*, it holds true*

$$\begin{aligned}
&\sqrt{w}\frac{\partial}{\partial w}\mathrm{B}_n^{(\alpha,\beta)}(1,w;t)+\frac{t}{2(1+\alpha)}\mathrm{B}_n^{[(1+\alpha),\beta]}(1,w;t)-\frac{t}{2(1+\beta)}\mathrm{B}_n^{[\alpha,(1+\beta)]}(1,w;t)=0,\\
&\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}\,(\beta)>-1,\ |w|<1.
\end{aligned}$$

Proof. Put $z=1$ in the Bateman's generating function (3.1), we get

$$\mathrm{B}_n^{(\alpha,\beta)}(1,w;t)=\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(1-\sqrt{w}\right)^n t^n}{n!\,(1+\alpha)_n}\right]\left[\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(1+\sqrt{w}\right)^n t^n}{n!\,(1+\beta)_n}\right].$$

Taking differential operator $\Xi_w=\left(\sqrt{w}\frac{\partial}{\partial w}\right)$ then following the same process used in the above theorem leads to the desired result. □

4. Recurrence Relations for Legendre Polynomials

In this sections, we will study the action of the following differential operator

$$\Delta = \left(\frac{1}{w}\frac{\partial}{\partial z} + \frac{1}{z}\frac{\partial}{\partial w}\right),$$

on complex bivariate Legendre polynomial $P_n(z,w)$ (2.2) to obtain the desired results.

Now, we present complex bivariate Legendre polynomial by replacing $x, y \in \mathbb{R}$ by $z, w \in \mathbb{C}$ such that

$$P_n(z,w) = \sum_{k=0}^{[n/2]} \frac{(-w)^k \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k}}{k!(n-k)!}, \tag{4.1}$$

where $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, $|z| < 1$, $|w| < 1$.

Theorem 4.1. *Following recurrence relation for the Legendre polynomials $P_n(z,w)$, it holds true*

$$\frac{1}{w}\frac{\partial}{\partial z}P_n(z,w) + \left(\frac{1}{z}\frac{\partial}{\partial w}\right)P_n(z,w) - \left(\frac{n}{zw}\right)P_n(z,w) + \left(\frac{1}{2z^2}\right)P_{n-1}(z,w) = 0,$$

$\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, $|z| < 1$, $|w| < 1$.

Proof. For Legendre polynomials (4.1) of two variables $P_n(z,w)$ we see that

$$\Delta\ P_n(z,w) = \left(\frac{1}{w}\frac{\partial}{\partial z} + \frac{1}{z}\frac{\partial}{\partial w}\right)\sum_{k=0}^{[n/2]} \frac{(-w)^k \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k}}{k!(n-k)!}.$$

After applying the differential operator, we get

$$\begin{aligned}
&-2\sum_{k=0} \frac{(-w)^{k-1} \left(\frac{1}{2}\right)_{n-k} (n-2k)\,(2z)^{n-2k-1}}{k!(n-k)!} - 2\sum_{k=0}^{[n/2]} \frac{k\,(-w)^{k-1} \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k-1}}{k!(n-k)!}\\
=&-2\sum_{k=0}^{[n/2]} \frac{(n-k)\,(-w)^{k-1} \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k-1}}{k!(n-k)!}.\\
=&-2n\sum_{k=0}^{[n/2]} \frac{(-w)^{k-1} \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k-1}}{k!(n-k)!} - 2\sum_{k=0}^{[n/2]} \frac{k\,(-w)^{k-1} \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k-1}}{k!(n-k)!}\\
=&\left(\frac{n}{zw}\right)\sum_{k=0}^{[n/2]} \frac{(-w)^{k} \left(\frac{1}{2}\right)_{n-k} (2z)^{n-2k}}{k!(n-k)!}\\
&-2\left(\frac{1}{2z}\right)^2\sum_{k=0}^{[n/2]} \frac{(-w)^{k} \left(\frac{1}{2}\right)_{(n-1)-k} (2z)^{[(n-1)-2(k-1)]}}{k!\,[(n-1)-k)]!}.\\
=&\left(\frac{n}{zw}\right)P_n(z,w) - \left(\frac{1}{2z^2}\right)P_{n-1}(z,w).
\end{aligned}$$

Now, on some simplification, we get our desired result. $\square$

Corollary 4.1. *Following recurrence relation for the Legendre polynomials $P_n(z,1)$, it holds true*

$$\frac{1}{w}\frac{\partial}{\partial z}P_n(z,1) \;-\left(\frac{n}{z}\right)P_n(z,1) \;+\left(\frac{1}{2z^2}\right)P_{n-1}(z,1)=0,$$
$$\operatorname{Re}(\alpha)>-1,\ \operatorname{Re}(\beta)>-1,\ |z|<1.$$

Proof. First, substitute $w=1$ in equation (4.1) we get

$$P_n(z,1)=\sum_{k=0}^{[n/2]}\frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2z)^{n-2k}}{k!(n-k)!}.$$

Taking differential operator $\Delta_z=\left(\frac{1}{w}\frac{\partial}{\partial z}\right)$ then following the same process used in the above theorem leads to the desired result. □

Corollary 4.2. *Following recurrence relation for the Legendre polynomials $P_n(1,w)$, it holds true*

$$\frac{1}{z}\frac{\partial}{\partial w}P_n(1,w) \;-\left(\frac{n}{w}\right)P_n(1,w) \;+\left(\frac{1}{2}\right)P_{n-1}(1,w)=0,$$
$$\operatorname{Re}(\alpha)>-1,\ \operatorname{Re}(\beta)>-1,\ |w|<1.$$

Proof. Put $z=1$ in equation (4.1) we get

$$P_n(1,w)=\sum_{k=0}^{[n/2]}\frac{(-w)^k\left(\frac{1}{2}\right)_{n-k}(2)^{n-2k}}{k!(n-k)!}.$$

Taking differential operator $\Delta_w=\left(\frac{1}{z}\frac{\partial}{\partial w}\right)$ then following the same process used in the above theorem leads to the desired result. □

Acknowledgements. The authors are highly thankful to the editor and anonymous referees for their efforts which could shape the manuscript in the present form.

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[1]Departement of Mathematics,
Faculty of Science, South Valley University,
(Qena-Egypt)
Email address: mosaed_makky11@yahoo.com, mosaed_makky@sci.svu.edu.eg

[2]Department of Natural and Applied Sciences,
School of Science and Technology, Glocal University,
Saharanpur 247121, India
Email address: shadabmohd786@gmail.com

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 393–405.

GEOMETRIC INEQUALITIES FOR STATISTICAL SUBMANIFOLDS IN COSYMPLECTIC STATISTICAL MANIFOLDS

MUSTAFA KAZAZ[1], MOHD ASLAM[2], AND MOHD AQUIB[3]

Abstract. In this paper, we obtain two important geometric inequalities namely, Euler's inequality and Chen's inequality for statistical submanifolds in cosymplectic statistical manifolds with constant curvature, and discuss the equality case of the inequalities. We also give some applications of the inequalities obtained.

1. Introduction

Since Lauritzen introduced the notion of statistical manifolds in 1987 [18], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry [2,17,28].

The notion of statistical submanifold was introduced in 1989 by Vos [27]. Though, it has made very little progress due to the hardness to find classical differential geometric approaches for study of statistical submanifolds. However, in the recent years many research has been published in the area and it remains a hot topic for the researchers [4,6,13,14,23,25,26].

On the other hand, in 1993, B.-Y. Chen [8] established the simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds know as the theory of Chen invariants, which is one of the most interesting research area of differential geometry.

Since then different geometers obtained the similar inequalities for different submanifolds and ambient spaces due to its rich geometric importance [5,9,10,12,16,20,21,24]. In [22], A. Mihai and I. Mihai established a Chen-Ricci inequality with respect to a

Key words and phrases. Chen's inequality, statistical manifolds, Cosymplectic statistical manifolds.
2010 *Mathematics Subject Classification.* Primary: 53B05. Secondary: 53B20, 53C40.
DOI 10.46793/KgJMat2403.393K
Received: December 15, 2020.
Accepted: May 02, 2021.

sectional curvature of the ambient Hessian manifold. In [3], Aquib obtained Chen's inequality for totally real statistical submanifolds of quaternion Kaehler-like statistical space forms. Recently, Chen et al. [11] obtained Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. In [19], F. Malek and H. Akbari, obtained bounds for Casorati curvatures of submanifolds in Cosymplectic statistical space forms.

In the present article, motivated by the problems proposed in [7] we derive Chen's inequality for statistical manifolds in Cosymplectic statistical manifold with constant curvature and investigate the equality case of the inequality. We also give some applications of the inequalities we derived.

2. Preliminaries

Let $(\overline{M}, g)$ be a Riemannian manifold and $\overline{\nabla}$ and $\overline{\nabla}^*$ be torsion-free affine connections on $\overline{M}$ such that

$$Zg(X, Y) = g(\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}^*_Z Y),$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Then Riemannian manifold $(\overline{M}, g)$ is called a statistical manifold. It is denoted by $(\overline{M}, g, \overline{\nabla}, \overline{\nabla}^*)$. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections. The pair $(\overline{\nabla}, g)$ is said to be a statistical structure.

If $(\overline{\nabla}, g)$ is a statistical structure on $\overline{M}$, then $(\overline{\nabla}^*, g)$ is also statistical structure on $\overline{M}$.

For the dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ we have

$$2\overline{\nabla}^\circ = \overline{\nabla} + \overline{\nabla}^*, \tag{2.1}$$

where $\overline{\nabla}^\circ$ is Levi-Civita connection for g.

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold and let M be an $(m+1)$-dimensional submanifolds of $\overline{M}$. Then, the Gauss formulae are [27]

$$\begin{cases} \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \\ \overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \end{cases}$$

where h and h^* are symmetric, bilinear, imbedding curvature tensors of M in $\overline{M}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

Let $\overline{R}$ and $\overline{R}^*$ be Riemannian curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Then [27]

$$\begin{aligned} g(\overline{R}(X, Y)Z, W) =& g(R(X, Y)Z, W) + g(h(X, Z), h^*(Y, W)) \\ & - g(h^*(X, W), h(Y, Z)) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} g(\overline{R}^*(X, Y)Z, W) =& g(R^*(X, Y)Z, W) + g(h^*(X, Z), h(Y, W)) \\ & - g(h(X, W), h^*(Y, Z)), \end{aligned} \tag{2.3}$$

where

$$g(\overline{R}^*(X,Y)Z,W) = -g(Z,\overline{R}(X,Y)W).$$

Let us denote the normal bundle of M by $TM^\perp$. The linear transformations A_N and A_N^* are defined by

$$\begin{cases} g(A_N X,Y) = g(h(X,Y),N), \\ g(A_N^* X,Y) = g(h^*(X,Y),N), \end{cases}$$

for any $N \in \Gamma(TM^\perp)$ and $X,Y \in \Gamma(TM)$. The corresponding Weingarten formulae are [27]

$$\begin{cases} \overline{\nabla}_X N = -A_N^* X + \nabla_X^\perp N, \\ \overline{\nabla}_X^* N = -A_N X + \nabla_X^{*\perp} N, \end{cases}$$

where $N \in \Gamma(TM^\perp)$, $X \in \Gamma(TM)$ and $\nabla_X^\perp$ and $\nabla_X^{*\perp}$ are Riemannian dual connections with respect to the induced metric on $\Gamma(TM^\perp)$.

For a statistical manifod $(\overline{M},\overline{g},\overline{\nabla})$, the difference (1,2)-tensor K of the torsion free affine connection $\overline{\nabla}$ and levi-Civita connection $\overline{\nabla}^\circ$ is defined as (see [15])

$$K_X Y = K(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_X^\circ Y.$$

K is a difference tensor field on $\overline{M}$, that is, $K_X Y = K_Y X$ and

$$\overline{g}(K_X Y, Z) = \overline{g}(Y, K_X Z).$$

Now, we consider a cosymplectic statistical structure on a cosymplectic manifold and define cosymplectic statistical manifold and cosymplectic statistical space form.

Definition 2.1 ([19])**.** $(\overline{\nabla},\overline{g},\phi,\xi,\eta)$ is called cosymplectic statistical structure on $\overline{M}$ if $(\overline{\nabla},\overline{g})$ is a statistical structure and $\phi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$, $\phi(\xi) = 0$, $\overline{g}(\phi X,Y) = -\overline{g}(X,\phi Y)$, $\overline{\nabla}_X^\circ \phi = 0$. That means $(\overline{g},\phi,\xi,\eta)$ is a cosymplectic structure on $\overline{M}$, and the formula $K_X\phi Y + \phi K_X Y = 0$ holds for any $X,Y \in \Gamma(T\overline{M})$.

Let $(\overline{M},\overline{\nabla},\overline{g})$ be a statistical manifold. The tensor field $\overline{R}(X,Y,Z,W)$ is not skew-symmetric relative to Z and W. Then, the sectional curvature on $\overline{M}$ can not be defined by the standard definition. In [15] Furuhata and Hasegawa have defined the statistical curvature tensor field $\overline{S}$ for a statistical manifold $(\overline{M},\overline{\nabla},\overline{g})$ as follows:

$$\overline{S}(X,Y)Z = \frac{1}{2}\{\overline{R}(X,Y)Z + \overline{R}^*(X,Y)Z\}. \tag{2.4}$$

Definition 2.2 ([19])**.** $(\overline{M},\overline{\nabla},\overline{g},\phi,\xi)$ be cosymplectic statistical manifold and c a real constant. The cosymplectic statistical structure is said to be of constant ϕ-sectional curvature c if

$$\begin{aligned} \overline{S}(X,Y)Z =& \frac{c}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(X,\phi Z)\phi Y \\ & - \overline{g}(Y,\phi Z)\phi X + 2\overline{g}(X,\phi Y)\phi Z + \eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)\xi - \overline{g}(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.5}$$

holds for any $X, Y, Z \in \Gamma(T\overline{M})$.

Let ξ be tangent to the submanifolds M and let $\{e_1, \dots, e_{m+1} = \xi\}$ and $\{e_{m+2}, \dots, e_{2n+1}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on M. Then, the mean curvature vector fields H, H^*, H° are given by

$$H = \frac{1}{m+1} \sum_{i=1}^{m+1} h(e_i, e_i),$$
$$H^* = \frac{1}{m+1} \sum_{i=1}^{m+1} h^*(e_i, e_i)$$

and

$$H^\circ = \frac{1}{m+1} \sum_{i=1}^{m+1} h^\circ(e_i, e_i).$$

We also set

$$\|h\|^2 = \sum_{i,j=1}^{m+1} g(h(e_i, e_j), h(e_i, e_j)),$$
$$\|h^*\|^2 = \sum_{i,j=1}^{m+1} g(h^*(e_i, e_j), h^*(e_i, e_j))$$

and

$$\|h^\circ\|^2 = \sum_{i,j=1}^{m+1} g(h^\circ(e_i, e_j), h^\circ(e_i, e_j)).$$

The second fundamental form h°(resp. h, or h^*) has several geometric properties due to which we got following different classes of the submanifolds.

- A submanifold is said to be totally geodesic submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$ or $\overline{\nabla}^*$), if the second fundamental form h° (resp. h or h^*) vanishes identically, that is $h^\circ = 0$ (resp. $h = 0$ or $h^* = 0$).
- A submanifold is said to be minimal submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$ or $\overline{\nabla}^*$), if the mean curvature vector H° (resp. H or H^*) vanishes identically, that is $H^\circ = 0$ (resp. $H = 0$ or $H^* = 0$).

Let $K(\pi)$ denotes the sectional curvature of a Riemannian manifold M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \dots, e_{m+1}\}$ be an orthonormal basis of T_pM and $\{e_{m+2}, \dots, e_{2n+1}\}$ be an orthonormal basis of $T_p^\perp M$ at any $p \in M$, then

$$\tau(p) = \sum_{1 \le i < j \le m+1} K(e_i \wedge e_j),$$

where τ is the scalar curvature. The normalized scalar curvature ρ is defined as

$$2\tau = m(m+1)\rho.$$

We also put

$$h_{ij}^\gamma = g(h(e_i, e_j), e_\gamma), \quad h_{ij}^{*\gamma} = g(h^*(e_i, e_j), e_\gamma),$$

where $i, j \in \{1, \ldots, m+1\}$, $\gamma \in \{m+2, \ldots, 2n+1\}$.

The square norm P at $p \in M$ is defined as

$$||P||^2 = \sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j).$$

Lemma 2.1 ([11]). *For $m+1 \geq 3$, $a_1, a_2, \ldots, a_{m+1}$, $m+1$ real numbers*

$$\sum_{1 \leq i < j \leq m+1} a_i a_j - a_1 a_2 \leq \frac{m-1}{2m} \left(\sum_{i=1}^{m+1} a_i \right)^2.$$

Moreover, equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_{m+1}$.

3. Euler's Inequality for Cosymplectic Manifold

In this section we will prove the Euler's inequality for statistical submanifolds of Cosymplectic manifold. To be precise we will prove the following.

Theorem 3.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

$$2\tau \geq \frac{c}{4}[m(m-1) + 3||P||^2] - ||h^\circ||^2 + (m+1)^2 g(H, H^*).$$

Further, equality case of the inequality holds if and only if $h = h^*$.

Proof. From (2.2), (2.3) and (2.5), we have

$$\begin{aligned}
g(S(X,Y)Z,W) =& g(\overline{S}(X,Y)Z,W) - \frac{1}{2}[g(h(X,Z),h^*(Y,W)) \\
& - g(h^*(X,W),h(Y,Z)) + (h^*(X,Z),h(Y,W)) \\
& - g(h(X,W),h^*(Y,Z))] \\
=& \frac{c}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
& + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
& + 2g(X,\phi Y)g(\phi Z,W) + \eta(X)\eta(Z)g(Y,W) \\
& - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)g(\xi,W) \\
& - g(Y,Z)\eta(X)g(\xi,W)] \\
& - \frac{1}{2}[g(h(X,Z),h^*(Y,W)) - g(h^*(X,W),h(Y,Z)) \\
& + g(h^*(X,Z),h(Y,W)) - g(h(X,W),h^*(Y,Z))].
\end{aligned}$$

Put $X = W = e_i$, $Y = Z = e_j$, we have

$$\begin{aligned}
g(S(e_i,e_j)e_j,e_i) =& \frac{c}{4}\{g(e_j,e_j)g(e_i,e_i) - g(e_i,e_j)g(e_j,e_i) \\
& + g(e_i,\phi e_j)g(\phi e_j,e_i) - g(e_j,\phi e_j)g(\phi e_i,e_i) \\
& + 2g(e_i,\phi e_j)g(\phi e_j,e_i) + \eta(e_i)\eta(e_j)g(e_j,e_i)
\end{aligned}$$

$$
\begin{aligned}
&- \eta(e_j)\eta(e_j)g(e_i, e_i) + g(e_i, e_j)\eta(e_j)g(\xi, e_i) \\
&- g(e_j, e_j)\eta(e_i)g(\xi, e_i)\} \\
&- \frac{1}{2}\Big[g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big) \\
&+ g\Big(h^*(e_i, e_j), h(e_j, e_i)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big].
\end{aligned}
$$

Taking summation, we derive

$$
\begin{aligned}
2\tau =& \frac{c}{4}\Big[(m+1)^2 - (m+1) \\
&+ 3\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j) + 1 - (m+1) + 1 - (m+1)\Big] \\
&- \frac{1}{2}\sum_{i,j=1}^{m+1} \Big[g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big) \\
&+ g\Big(h^*(e_i, e_j), h(e_j, e_i)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big] \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] - \frac{1}{2}\sum_{i,j=1}^{m+1} \Big[2g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) \\
&- g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big] \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] \\
&- \frac{1}{2}\Big[2g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - 2(m+1)^2 g(H, H^*) \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] - g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) + (m+1)^2 g(H, H^*) \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*) - \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ij}^{\alpha} h_{ij}^{*\alpha} \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*) \\
&- \frac{1}{4}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} \Big[(h_{ij}^{\alpha} + h_{ij}^{*\alpha})^2 - (h_{ij}^{\alpha} - h_{ij}^{*\alpha})^2\Big] \\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*) - ||h^{\circ}||^2 \\
&+ \frac{1}{4}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} \Big[(h_{ij}^{\alpha} - h_{ij}^{*\alpha})^2 \\
\geq& \frac{c}{4}\Big[m(m-1) - ||h^{\circ}||^2 + 3||P||^2\Big] + (m+1)^2 g(H, H^*),
\end{aligned}
$$

which is the required result. $\square$

An immediate consequence of the Theorem 3.1 is the following result.

Corollary 3.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

Angle between H and H^*	*Inequalities*
θ	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2 + (m+1)^2\|H\|\|H^*\|\cos\theta$
0°	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2 + (m+1)^2\|H\|\|H^*\|$
90°	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2$

4. Chen's Inequality for Cosymplectic Statistical Manifolds

This section is devoted to the main result of the article. Here, we obtain Chen's inequality for statistical submanifolds of Cosymplectic statistical manifolds with constant ϕ-sectional curvature.

Theorem 4.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

$$\begin{aligned}\tau - K(\pi) \geq & 2\tau_\circ - k_\circ(\pi) + \frac{c}{4}[(1+m-m^2) - 3||P||^2 \\ & + 3\Theta(\pi) - \Phi(\pi)] - \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2),\end{aligned}$$

where $\Theta(\pi) = g^2(\phi e_1, e_2)$, $\Phi(\pi) = \eta^2(e_1) + \eta^2(e_2)$, $\pi = e_1 \wedge e_2$. Moreover, the equality holds if and only if

$$\begin{aligned} & h^\alpha_{11} + h^\alpha_{22} = h^\alpha_{33} = \cdots = h^\alpha_{m+1m+1}, \\ & h^{*\alpha}_{11} + h^{*\alpha}_{22} = h^{*\alpha}_{33} = \cdots = h^{*\alpha}_{m+1m+1}, \\ & h^\alpha_{ij} = h^{*\alpha}_{ij}, \quad 1 \leq i \neq j \leq m+1,\ (i,j) \neq (1,2),(2,1),\ \alpha \in \{m+2,\ldots,2n+1\}. \end{aligned} \tag{4.1}$$

Proof. From (2.2), (2.3) and (2.4), we have

$$\begin{aligned} 2g(\overline{S}(X,Y)Z,W) = & 2g(S(X,Y)Z,W) + g(h(X,Z),h^*(Y,W)) \\ & - g(h^*(X,W),h(Y,Z)) + g(h^*(X,Z),h(Y,W)) \\ & - g(h(X,W),h^*(Y,Z)). \end{aligned} \tag{4.2}$$

Using (2.5) in (4.2) and putting $X = W = e_i$, $Y = Z = e_j$, we get

$$\begin{aligned}2g(S(e_i,e_j)e_j,e_i) =&\frac{c}{2}[g(e_j,e_j)g(e_i,e_j) - g(e_i,e_j)g(e_j,e_i)\\&+ g(e_i,\phi e_j)g(\phi e_j,e_i) - g(e_j,\phi e_j)g(\phi e_i,e_j)\\&+ 2g(e_i,\phi e_j)g(\phi e_j,e_i) + \eta(e_i)\eta(e_j)g(e_j,e_i)\\&- \eta(e_j)\eta(e_j)g(e_i,e_i) + g(e_i,e_j)\eta(e_j)g(\xi,e_i)\\&- g(e_j,e_j)\eta(e_i)g(\xi,e_i)] - g(h(e_i,e_j),h^*(e_j,e_i))\\&+ g(h^*(e_i,e_i),h(e_j,e_j)) - g(h^*(e_i,e_j),h(e_j,e_i))\\&+ g(h(e_i,e_i),h^*(e_j,e_j)).\end{aligned}$$

Applying summation over $i, j = 1, 2, \ldots, m+1$, we obtain

$$\begin{aligned}\sum_{1\le i<j\le m+1} 2g(S(e_i,e_j)e_j,e_i) =&\frac{c}{2}[(m+1)^2 - (m+1)\\&+ 3\sum_{i,j=1}^{m+1} g^2(\phi e_j,e_i) + 1 - (m+1) + 1 - (m+1)]\\&- g(h(e_i,e_j),h^*(e_j,e_i)) + g(h^*(e_i,e_i),h(e_j,e_j))\\&- g(h^*(e_i,e_j),h(e_j,e_i)) + g(h(e_i,e_i),h^*(e_j,e_j))\\=&\frac{c}{2}[m^2 - m + 3||P||^2]\\&- g(h(e_i,e_j),h^*(e_j,e_i)) + g(h^*(e_i,e_i),h(e_j,e_j))\\&- g(h^*(e_i,e_j),h(e_j,e_i)) + g(h(e_i,e_i),h^*(e_j,e_j)),\end{aligned}$$

which implies

$$\begin{aligned}\tau =&\frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h^{*\alpha}_{ii}h^{\alpha}_{jj} + h^{\alpha}_{ii}h^{*\alpha}_{jj} - 2h^{\alpha}_{ij}h^{*\alpha}_{ij})\\=&\frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1)\\&+ \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[(h^{\alpha}_{ii} + h^{*\alpha}_{ii})(h^{\alpha}_{jj} + h^{*\alpha}_{jj}) - h^{\alpha}_{ii}h^{\alpha}_{jj} - h^{*\alpha}_{ii}h^{*\alpha}_{jj}\\&- (h^{\alpha}_{ij} + h^{*\alpha}_{ij})^2 + (h^{\alpha}_{ij})^2 + (h^{*\alpha}_{ij})^2]\\=&\frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1)\\&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\left\{2[h^{\circ\alpha}_{ii}h^{\circ\alpha}_{jj} - (h^{\circ\alpha}_{ij})^2] - \frac{1}{2}[h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2] - \frac{1}{2}[h^{*\alpha}_{ii}h^{*\alpha}_{jj} - (h^{*\alpha}_{ij})^2]\right\}.\end{aligned} \tag{4.3}$$

Further, with respect to Levi-Civita connection, we have

$$\tau_\circ = \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) + \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})^2]. \tag{4.4}$$

We also have

$$\begin{aligned}
K(\pi) =& \frac{1}{2}[g(R(e_1,e_2)e_2,e_1) + g(R^*(e_1,e_2)e_2,e_1)]\\
=& \frac{1}{2}[g(\overline{R}(e_1,e_2)e_2,e_1) + g(\overline{R}^*(e_1,e_2)e_2,e_1)\\
& - 2g(h^*(e_1,e_2),h(e_2,e_1)) + 2g(h^*(e_1,e_1),h(e_2,e_2))]\\
=& g(\overline{S}(e_1,e_2)e_2,e_1) + \sum_{\alpha=m+2}^{2n+1}\left[\frac{1}{2}h_{11}^{*\alpha}h_{22}^{\alpha} + \frac{1}{2}h_{11}^{\alpha}h_{22}^{*\alpha} - h_{12}^{*\alpha}h_{12}^{\alpha}\right]\\
=& g(\overline{S}(e_1,e_2)e_2,e_1) + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}[(h_{11}^{\alpha}+h_{11}^{*\alpha})(h_{22}^{\alpha}+h_{22}^{*\alpha})\\
& - h_{11}^{\alpha}h_{22}^{\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{\alpha}+h_{12}^{*\alpha})^2 + (h_{12}^{\alpha})^2 + (h_{12}^{*\alpha})^2]\\
=& g(\overline{S}(e_1,e_2)e_2,e_1) + \sum_{\alpha=m+2}^{2n+1}\Big[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\}\\
& - \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}\Big].
\end{aligned} \tag{4.5}$$

On the other hand

$$\begin{aligned}
g(\overline{S}(e_1,e_2)e_2,e_1) =& \frac{c}{4}[g(e_2,e_2)g(e_1,e_1) - g(e_1,e_2)g(e_2,e_1) + g(e_1,\phi e_2)g(\phi e_2,e_1)\\
& - g(e_2,\phi e_2)g(\phi e_1,e_1) + 2g(e_1,\phi e_2)g(\phi e_2,e_1)\\
& + \eta(e_1)\eta(e_2)g(e_2,e_1) - \eta(e_2)\eta(e_2)g(e_1,e_1)\\
& + g(e_1,e_2)\eta(e_2)g(\xi,e_1) - g(e_2,e_2)\eta(e_1)g(\xi,e_1)]\\
=& \frac{c}{4}[1 + 3g^2(e_1,\phi e_2) - \eta(e_1)^2 - \eta(e_2)],\\
=& \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)].
\end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned}
K(\pi) =& \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] + \sum_{\alpha=m+2}^{2n+1}[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\}\\
& - \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}].
\end{aligned} \tag{4.7}$$

Also, with respect to Levi-Civita connection, we get

$$K_\circ(\pi) = \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] + \sum_{\alpha=m+2}^{2n+1}\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\}. \tag{4.8}$$

Substract (4.7) from (4.3), we get

$$\begin{aligned}
\tau - K(\pi) =& \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) - \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] \\
&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\Bigg\{2[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})^2] - \frac{1}{2}[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] \\
&- \frac{1}{2}[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2]\Bigg\} - \sum_{\alpha=m+2}^{2n+1}\Big[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\} \\
&- \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}\Big] \\
=& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} 2h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\circ\alpha})^2 - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ii}^{\alpha}h_{jj}^{\alpha} \\
&+ \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\alpha})^2 - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ii}^{*\alpha}h_{jj}^{*\alpha} + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{*\alpha})^2 \\
&- 2\sum_{\alpha=m+2}^{2n+1} h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} + 2\sum_{\alpha=m+2}^{2n+1}(h_{12}^{\circ\alpha})^2 + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1} h_{11}^{\alpha}h_{22}^{\alpha} \\
&- \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}(h_{12}^{\alpha})^2 + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1} h_{11}^{*\alpha}h_{22}^{*\alpha} - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}(h_{12}^{*\alpha}) \\
\geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{11}^{\alpha}h_{22}^{\alpha}) - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ii}^{*\alpha}h_{jj}^{*\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha}) \\
&+ 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[(h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha}) - h_{11}^{\circ\alpha}h_{22}^{\circ\alpha}] - 2\left[\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\circ\alpha})^2 - (h_{12}^{\circ\alpha})^2\right] \\
\geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) \\
&+ 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})] - 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})].
\end{aligned}$$

Using (4.4) and (4.8), we have

$$\begin{aligned}
\tau - K(\pi) \geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) + 2\left[\tau_\circ - \frac{3}{4}c||P||^2\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{c}{4}m(m-1)-2\left[-\frac{c}{4}(1+3\theta(\pi)-\Phi(\pi)+K_{\pi}\right]\\
=&\frac{3}{4}||P||^{2}+\frac{c}{4}(m^{2}-m)-\frac{c}{4}-\frac{3}{4}c\theta(\pi)+\frac{c}{4}\Phi(\pi)\\
&-\frac{(m+1)^{2}(m-1)}{4m}(||H||^{2}+||H^{*}||^{2})+2\tau_{\circ}-\frac{3}{2}c||P||^{2}\\
&-\frac{c}{2}(m^{2}-m)+\frac{c}{2}+\frac{3}{2}c\theta(\pi)-\frac{c}{2}\Phi(\pi)-2K_{\circ}(\pi)\\
\geq&\frac{c}{4}-\frac{3}{4}c||P||^{2}-\frac{c}{4}(m^{2}-m)+\frac{3}{4}c\theta(\pi)-\frac{c}{4}\Phi(\pi)\\
&-\frac{(m+1)^{2}(m-1)}{4m}(||H||^{2}+||H^{*}||^{2})+2\tau_{\circ}-K_{\circ}(\pi)\\
\geq&\frac{c}{4}[(1+m-m^{2})-3||P||^{2}+3\theta(\pi)-\Phi(\pi)]\\
&-\frac{(m+1)^{2}(m-1)}{4m}(||H||^{2}+||H^{*}||^{2})+2\tau_{\circ}-K_{\circ}(\pi),
\end{aligned}
$$

which is the required inequality. Moreover, equality holds if and only if it satisfies (4.1). □

From the above theorem we have the following non-existence result of minimal statistical submanifolds in cosymplectic statistical manifold.

Corollary 4.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$ such that*

$$(\tau-K(\pi))-(2\tau_{\circ}-K_{\circ}(\pi))<\frac{c}{4}[(1+m-m^{2})-3||P||^{2}+3\theta(\pi)-\Phi(\pi)],$$

then M can not be minimally immersed in $\overline{M}(c)$ with respect to $\overline{\nabla}$ and $\overline{\nabla}^{*}$ symulteneously.

Acknowledgements. The authors would like to express their sincere gratitude to the referees for useful remarks, which definitely improved the manuscript. The authors are also very thankful to Prof. Mohammad Hasan Shahid for his valuable suggestions.

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[1]Department of Mathematics,
Faculty of Arts and Sciences,
Manisa Celal Bayar University,
45140, Manisa, Turkey
Email address: mustafa.kazaz@cbu.edu.tr

[2]Department of Mathematics,
Faculty of Natural Sciences,
Jamia Millia Islamia,
New Delhi-110025, India
Email address: kidwaiaslam@gmail.com

[3]Department of Mathematics,
Sri Venkateswara College,
University of Delhi,
New Delhi-110021, India
Email address: aquib80@gmail.com

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 407–421.

EXPLORING THE ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SEMIGROUPS THROUGH THEIR PRIME m-BI IDEALS

MOHAMMAD MUNIR[1], NASREEN KAUSAR[2], K. ANITHA[3], GUNASEELAN MANI[4], AND TEHREEM[5]

Abstract. We introduce the concepts of the prime m-bi ideal and their associated types in the semigroups. Different characterizations of the semigroups using these m-bi ideals are presented. The forms of the topologies induced by the prime and strongly prime m-bi ideals in the semigroups are also explored. The result shows that either both the conditions of m-regularity and m-intraregularity or existence of pairwise comaximal m-bi ideals in a semigroup is necessary for strongly prime m-bi ideals to induce a topology; whereas the existence of pairwise comaximal m-bi ideals is necessary for the prime m-bi ideals to induce topology on the semigroups. We concluded that the prime m-bi ideals are as important to study the semigroups as the prime bi ideals.

1. Introduction and Preliminaries

A short introduction to our work and the important concepts are described in this section.

1.1. **Introduction.** A non-empty set M together with a given associative binary operation $\cdot$ is called a semigroup. A semigroup primarily need not to possess the additive identity 0 or absorbing *zero* [15], or the multiplicative identity e as against many other algebraic structures which do possess these two or one of these elements. Moreover, different powers of semigroups through a positive integer so-called *index* also produce sub-structures like subsemigroups and ideals which have different forms,

Key words and phrases. prime m-bi ideals, m-regularity, comaximal ideals, topology.
2020 *Mathematics Subject Classification.* Primary: 20M17. Secondary: 22A15.
DOI 10.46793/KgJMat2403.407M
Received: February 09, 2020.
Accepted: May 04, 2021.

and behave differently. In order to explore these properties, the author of this article generalized the bi ideals in the semirings and semigruops respectively in articles [9] and [10] through the *indices*. As a follow-up of these generalizations, it was felt that the major classes of the m-bi ideals like, the prime, strongly prime, semiprime, maximal and minimal, irreducible and strongly irreducible should also be studied in a similar way so that the hidden properties of the semigroups should also be uncovered. To meet this end, we started this work to explore the algebraic and consequently topological properties of these ideals in the semigroups. Our work is aimed at mainly semigroups without *zero*. We divide this article into six sections.

In Subsection 1.2, we present the preliminary concepts from the literature consisting of books on semigroups especially [5] and [2], and the research articles in the field of semigroup theory. We demonstrate the major results of our work on the prime and associated m-bi ideals in semigroups in Section 2. Section 3 deals with the characterization of semigroups through maximal m-bi ideals. In Section 4, we characterize the m-regular and m-intraregular semirings using the prime, semiprime and strongly m-bi ideals. We present the forms of the topologies formed by these m-bi ideals in Section 5. The conclusion of the whole work is given in Section 6.

1.2. **Preliminaries.** A nonempty subset K of M is called its *subsemirgoup* if K itself is a semigroup under the operation $\cdot$ of M. The subsemigroup K of M becomes a left (right) ideal of M if the condition $MK \subseteq K$ ($KM \subseteq K$) is imposed on M [13]. If K is a left as well as a right ideal, then it is called an ideal (or a two-sided ideal) of M. The intersection (if it is nonempty) and sum of two (ideals, left-ideals, right-rights) of a semgroup is (an ideal, a left-ideal, a right-ideal). Right and left ideals generalize to the quasi ideals. A quasi ideal Q of semigroup M is a subsemigroup if $MQ \cap QM \subseteq Q$. A further generalization of quasi ideals gives introduction to bi ideals. A subsemigroup B of M is called a *bi ideal* of M if $BMB \subseteq B$. Every bi ideals is a quasi ideal, however every quasi ideal may or may not be a bi ideal. An *m-bi ideal* B of M is a subsemigroup of M such that $BM^mB \subseteq B$, where $m \geq 1$ is a positive integer. Every *bi ideal* B of M is a 1-bi ideal of M, but every m-bi ideals is not a bi ideal. An m-bi ideal is called principal m-bi ideal if it is generated by a single element. If $a \in M$, the m-bi ideal generated by M is $\langle a\rangle_{m-b} = \{a\} \cup \{a^2\} \cup aM^m a$. A semigroup having no nontrivial *two-sided* ideals is called a simple semigroup [13].

2. Prime m-Bi Ideals

In this section, we develop the definitions of prime m-bi and their associated types, and characterize the semigroups using their properties.

Definition 2.1. An m-bi ideal B of a semigroup M is known as a prime m-bi ideal (strongly prime m-bi ideal) if the proposition "$B_1B_2 \subseteq B$" ("$B_1B_2 \cap B_2B_1 \subseteq B$") infers either "$B_1 \subseteq B$" or "$B_2 \subseteq B$" for any two m-bi ideals B_1 and B_2 of M.

Definition 2.2. An m-bi ideal B of a semigroup M is known as semiprime m-bi ideal if the proposition "$B_1^2 \subseteq B_1$" implies "$B_1 \subseteq B$" for any m-bi ideal B_1 of M.

This is very obvious that each strongly prime m-bi ideal of M is its prime m-bi ideal and its each prime m-bi-ideal is its semiprime m-bi ideal; but the converses do not hold. The prime (strongly prime) bi ideals as defined in [15] are the 1-bi ideals, but the prime (strongly prime) m-bi-ideal of M are not its prime (strongly prime) bi ideals. Similarly, semiprime bi ideals are the semiprime 1-bi ideals; but the converse does not follow.

Example 2.1. The semigroup M itself is always a prime, a semiprime and a strongly m-bi ideal of M. Moreover, M can have these type of ideals different from M.

Example 2.2. Consider the semigroup $M = \{\alpha, \beta, \gamma, \delta\}$ with the binary operation $\cdot$ given in the following table:

$\cdot$	α	β	γ	δ
α	α	α	α	α
β	α	β	α	α
γ	α	α	γ	α
δ	α	α	α	α

.

Taking $m = 2$, we get $M^2 = \{\alpha, \beta, \gamma\}$. Then the 2-bi ideals in M are $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$, $\{\alpha, \beta, \gamma\}$ and $\{\alpha, \beta, \gamma, \delta\}$. The prime 2-bi ideals are $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$ and $\{\alpha, \beta, \gamma, \delta\}$ and so are semiprime 2-bi-ideals of M. These are also strongly prime except the prime 2-bi ideal, $\{\alpha\}$, which is not strongly prime 2-bi ideal because $\{\alpha, \beta\}\{\alpha, \gamma\} \cap \{\alpha, \gamma\}\{\alpha, \beta\} = \{\alpha, \beta\} \cap \{\alpha, \gamma\} = \{\alpha\}$, but none of $\{\alpha, \beta\}$ and $\{\alpha, \gamma\}$ is contained in $\{\alpha\}$. More comments on the ideal viz., $\{\alpha, \beta, \gamma\}$ will be given in Section 3, Remark 3.1.

The following two successive examples demonstrate when the m-bi ideals and the subsets of a semigroup are also the prime and semiprime m-bi ideals.

Example 2.3. In a right zero semigroup M with the cardinality $|M| > 1$, we have $yx = x$ for all $x, y \in M$. So, for an arbitrary $x \in M$, $xx = x$, i.e., x is idempotent, $M^2 = M$. Consequently, $M^m = M$ for any integer $m \geq 1$. If S is a subset of M, then $SMS = MS = S$, i.e., S is bi ideal of M. Moreover, since $SM^mS = S$, every bi ideal is an m-bi ideal. That is, every subset is m-bi ideal of M.

In this case, all m-bi ideals of M coincide with the prime m-bi ideal, and so with the semiprime m-bi ideals. This is because for m-bi ideals B_1, B_2, we have $B_1B_2 = B_2$. On the other hand, if B is any m-bi ideal of M such that $|M - B| \geq 2$, then B is not strongly prime since for distinct $a, b \in M - B$, $(B \cup \{a\})(B \cup \{b\}) \cap (B \cup \{b\})(B \cup \{a\}) = (B \cup \{a\}) \cap (B \cup \{b\}) = B$, but none of $(B \cup \{a\})$, $(B \cup \{b\})$ is contained in B.

Example 2.4. Let M be a Kronecker delta semigroup, that is, M has a zero 0 and

$$xy = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

and assume that $|M| > 2$. Since $x^2 = x$ for all $x \in M$, so consequently, $M^m = M$. Clearly, every subset B of M is its m-bi ideals because $B^2 = B$ and $BM^mB = BMB = B$. Moreover, if $B_1^2 \subseteq B$, then since and $B_1^2 = B_1$ for any subsets B_1, B of M, so $B_1 \subseteq B$ imply that all subsets of M are all semiprime m-bi ideals of M. If B is an m-bi ideal of M such that $|M - B| > 2$, then B is not a prime m-bi ideal of M since for distinct $a, b \in M - B$, $(B \cup \{a\})(B \cup \{b\}) = (B \cup \{a\}) \cap (B \cup \{b\}) = B$, neither $(B \cup \{a\})$ nor $(B \cup \{b\})$ is contained in B. In particular case, $\{0\}$ is a semiprime m-bi ideal of M which is not a prime m-bi ideal.

Definition 2.3. An m-bi ideal B of a semigroup M is known as an irreducible (strongly irreducible) m-bi ideal if the proposition "$B_1 \cap B_2 = B$" ("$B_1 \cap B_2 \subseteq B$") infers either "$B_1 = B$" or "$B_2 = B$" (either "$B_1 \subseteq B$" or "$B_2 \subseteq B$"), for any two m-bi ideals B_1 and B_2 of M.

In a semigroup, strongly irreducible m-bi ideal irreducible m-bi ideal, but the converse is not true. This is evident by the following example.

Example 2.5. For the semigroup, $M = \{\pi, \rho, \sigma, \tau, \phi, \psi, \omega\}$ with binary operation $\cdot$ defined in the Table 1. We take $m = 2$, $M^2 = \{\pi, \rho, \sigma, \tau, \phi, \psi\}$. The m-bi ideals

TABLE 1.

.	π	ρ	σ	τ	ϕ	ψ	ω
π	π	π	π	π	π	π	π
ρ	π	ρ	ρ	ρ	ρ	ρ	ρ
σ	π	ρ	σ	τ	ρ	ρ	ρ
τ	π	ρ	ρ	ρ	σ	τ	ρ
ϕ	π	ρ	ϕ	ψ	ρ	ρ	ρ
ψ	π	ρ	ρ	ρ	ϕ	ψ	ρ
ω	π	ρ	ρ	ρ	ρ	ρ	ρ

in M are $\{\pi\}$, $\{\pi, \rho\}$, $\{\pi, \rho, \sigma\}$, $\{\pi, \rho, \tau\}$, $\{\pi, \rho, \phi\}$, $\{\pi, \rho, \psi\}$, $\{\pi, \rho, \sigma, \phi\}$, $\{\pi, \rho, \tau, \psi\}$, $\{\pi, \rho, \sigma, \tau\}$, $\{\pi, \rho, \phi, \psi\}$ and M itself. The irreducible m-bi ideals are $\{\pi\}$, $\{\pi, \rho, \sigma, \phi\}$, $\{\pi, \rho, \tau, \psi\}$, $\{\pi, \rho, \sigma, \tau\}$, $\{\pi, \rho, \phi, \psi\}$ and M. Strongly irreducible m-bi ideal is $\{\pi\}$.

Remark 2.1. (a) The intersection of any collection of *prime* (*strongly prime*) m-bi ideals in a semigroup M is generally not a *prime* (*strongly prime*) m-bi ideal. M be the semigroup of non-zero integers under ordinary multiplication. Let B_1 be the 2-bi ideal of M divisible by 2 and B_2 is the 3-bi ideal of M divisible by 3. Both of them are prime m-bi ideals of M as 2 and 3 are prime integers. Now $B_1 \cap B_2$ consists of non-zero integers divisible by 6, and $B_1B_2 = B_1 \cap B_2$, but neither $B_1 \cap B_1 \cap B_2$ nor $B_2 \cap B_1 \cap B_2$ implying that $B_1 \cap B_2 \subseteq B_1 \cap B_2$ is not prime.

(b) The intersection of any collection of *semiprime* m-bi ideals in M is a *semiprime* m-bi ideal.

This is to be reminded that each prime m-bi ideal of a semigroup is semiprime m-bi ideal. The following proposition describes the conditions when the semiprime m-bi ideal is a prime m-bi ideal in a semigroup.

Proposition 2.1. *If a semiprime m-bi ideal B of a semigroup M is strongly irreducible, then B is a strongly prime m-bi-ideal.*

Proof. We consider B_1, B_2 as two m-bi ideals of M with the additional assumption that

$$B_1B_2 \cap B_2B_1 \subseteq B. \tag{2.1}$$

Then, after a little simplification, we get,

$$(B_1 \cap B_2)^2 \subseteq B_1B_2 \cap B_2B_1. \tag{2.2}$$

Combining (2) and (2.2) by the transitive property of inclusion, we get, $(B_1 \cap B_2)^2 \subseteq B$, which gives $B_1 \cap B_2 \subseteq B$, because B is a semiprime. Moreover, since B is strongly irreducible m-bi ideal of M, so we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$, making B a strongly prime m-bi ideal of M. ☐

Proposition 2.2. *For any m-bi B of a semigroup M, such that $c \in M$ and $c \notin B$, there exists an irreducible m-bi ideal I such that $B \subseteq I$ and $c \notin I$.*

Proof. Take $\mathcal{B} = \{B : B$ is an m-bi ideal of M so that $c \in M$ and $c \notin B\}$. Then $\mathcal{B} \neq \emptyset$, because $B \in \mathcal{B}$. $\mathcal{B}$ is clearly a partially ordered set under the binary operation of *inclusion* of m-bi ideals in $\mathcal{B}$. If $\mathcal{S}$ is any totally ordered subset of $\mathcal{B}$, then $S = \bigcup_{S_\alpha \in \mathcal{S}, \alpha \in \wedge} S_\alpha$ is an m-bi ideal of M containing B. So we can find a maximal m-bi ideal, J, in $\mathcal{B}$ [6]. To show that J is an irreducible, we suppose $J = J_1 \cap J_2$ for two m-bi ideals J_1 and J_2 of M. If, on contrary, both J_1 and J_2 contain J properly, then $c \in J_1$ and $c \in J_2$. Hence $c \in J_1 \cap J_2 = J$, which contradicts the hypothesis that $c \notin J$. Thus, $J = J_1$ or $J = J_2$; implying that J is an irreducible m-bi ideal. ☐

The last theorem in this section characterizes the semigroups in which each m-bi ideal is irreducible and strongly irreducible.

Theorem 2.1. *The following statements are equivalent for a given semigroup M.*

(a) *The set $\mathcal{B}$ of all m-bi ideals of M is a totally ordered set under binary operation of inclusion of sets.*
(b) *Each m-bi ideal of M is strongly irreducible m-bi ideal.*
(c) *Each m-bi ideal of M is irreducible m-bi ideal.*

Proof. (a) $\Rightarrow$ (b) Suppose B is an m-bi ideal of M and for B_1, B_2 to be any two m-bi ideals of M, the statement $B_1 \cap B_2 \subseteq B$ holds. Since $\mathcal{B}$ is totally ordered set under inclusion, so $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This implies, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. So, from our previous articulation $B_1 \cap B_2 \subseteq B$, we derive either $B_1 \subseteq B$ or $B_2 \subseteq B$, making B a strongly irreducible m-bi ideal of M.

(b) $\Rightarrow$ (c) Straightforward as *strongly irreducible m-bi ideal of M is irreducible m-bi ideal.*

(c) $\Rightarrow$ (a) For any two m-bi ideals of M, namely B_1 and B_2, we can compose the statement $B_1 \cap B_2 = B_1 \cap B_2$. Since each m-bi ideal of M is irreducible m-bi, $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, which further implies, $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. That is, B_1 and B_2 are comparable making the collection of m-bi ideals of M a totally ordered set. $\square$

3. Maximal m-Bi Ideals

Maximal ideals of semigroups, like all other algebraic structures, are an important category of ideals used to characterize semigroups in a different way along with the prime, strong prime and semiprime ideals. In the following section, we define the maximal m-bi ideals and characterize semigroups through their properties.

Definition 3.1. An m-bi ideal S of a semigroup M is called its maximal m-bi ideal if $S \subset M$ (M contains S properly) and there exists no m-bi ideal S_1 of M to give $S \subset S_1 \subset M$ [14].

In Example 2.2 of Section 2, $\mathcal{M} = \{\alpha, \beta, \gamma\}$ is the maximal m-bi ideals of the semigroup, $M = \{\alpha, \beta, \gamma, \delta\}$.

Theorem 3.1. *Every maximal m-bi ideal S of a semigroup M is a prime m-bi ideal of M, if $M = M^2$.*

Proof. Suppose $M - S = P$, where P is the set complement of the ideal S with respect to the semigroup M. Then, $M = (S \cup P)^2 = S^2 \cup SP \cup PS \cup P^2 \subset S \cup P^2$. That is, $M \subset S \cup P^2$. This gives that $M \cap P \subset S \cap P \cup P^2$. But $S \cap P \neq \emptyset$, therefore, we get $P \subset P^2$. Assume $B_1 B_2 \subset S$ for two m-bi ideals B_1 and B_2 of M. Suppose on contrary that neither B_1 nor B_2 is contained in S. Since $B_1 \nsubseteq S$ and S is maximal, we have $B_1 \cup S = M$, hence $P \subset B_1$. Analogously, $P \subset B_2$. Thus, $P^2 \subset B_1 B_2$, hence $P \subset B_1 B_2$, which is contradiction to $B_1 B_2 \subset S$. $\square$

Remark 3.1. If $M \neq M^2$, then every maximal m-bi ideal of M is not prime m-bi ideal. This is evident in Example 2.2. The maximal m-bi ideal $\{\alpha, \beta, \gamma\}$ is not prime because $\{\alpha, \beta, \gamma, \delta\}\{\alpha, \beta, \gamma, \delta\} \subseteq \{\alpha, \beta, \gamma\}$, but $\{\alpha, \beta, \gamma, \delta\} \nsubseteq \{\alpha, \beta, \gamma\}$, so $\{\alpha, \beta, \gamma\}$ is not prime.

Intersection of maximal ideals performs an important role in characterizing semigroups [3]. The following theorems deal with the sets of maximal m-bi ideals, their intersections and their complement sets in the semigroups. These help us in defining the topologies on the semigroups, and tell when the prime m-bi ideals are maximal m-bi ideals. Š. Schwarz proved these theorem in [14] for ideals in semigroups, we prove them for the maximal m-bi ideals in semigroups.

Theorem 3.2. *Let $\{S_\alpha : \alpha \in \wedge\}$ be the set of different maximal m-bi ideals of a semigroup M. Suppose $|\wedge| \geq 2$ and denote $Q_\alpha = M^m - S_\alpha$ and $S = \bigcap_{\alpha\in\wedge} S_\alpha$, we have the following.*

(a) $Q_\alpha \cap Q_\beta = \emptyset$ *for* $\alpha \neq \beta$.
(b) $M^m = \left(\bigcup_{\alpha\in\wedge} Q_\alpha\right) \cup S$.
(c) *For every* $\nu \neq \alpha$, *we have* $Q_\alpha \subset S_\nu$.
(d) *If* J *is an* m-*bi ideal of* M *and* $J \cap Q_\alpha \neq \emptyset$, *then* $Q_\alpha \subset J$.
(e) *For* $\alpha \neq \beta$, *we have* $Q_\alpha Q_\beta^m Q_\alpha \subset S$, *that is* S *is not empty.*

Proof. The case $|\wedge| = 1$ is obvious.

(a) We have $S_\alpha \cup S_\beta = M^m$ for $\alpha \neq \beta$. Thus, $Q_\alpha \cap Q_\beta = (M^m - S_\alpha) \cap (M^m - S_\beta) = M^m - (S_\alpha \cup S_\beta) = \emptyset$.

(b) Since $S = \bigcap_{\alpha\in\wedge} S_\alpha = \bigcap_{\alpha\in\wedge}(M^m - Q_\alpha) = M^m - \bigcup_{\alpha\in\wedge} Q_\alpha$. Thus, $M^m = (\bigcup_{\alpha\in\wedge} Q_\alpha) \cup S$.

(c) For $\nu \neq \alpha$, we have $Q_\alpha = M^m \cap Q_\alpha = (S_\nu \cup Q_\nu) \cap Q_\alpha = S_\nu \cap Q_\alpha$. Thus, $Q_\alpha \subset S_\nu$.

(d) Since $J \cap Q_\alpha \neq \emptyset$ and J is an m-bi ideal of M whereas S_α is the maximal m-bi ideal, therefore union set $S_\alpha \cup J$ is an m-bi ideal of M greater than S_α. So, $S_\alpha \cup J = M^m$. Since $S_\alpha \cap Q_\alpha = \emptyset$, we have $Q_\alpha \cap S_\alpha \cup J = Q_\alpha \cap M^m$, i.e., $Q_\alpha \cap (S_\alpha \cup J) = Q_\alpha \cap M^m$, which gives that $(Q_\alpha \cap S_\alpha) \cup (Q_\alpha \cap S_\alpha) = Q_\alpha$, and $\emptyset \cup (Q_\alpha \cap S_\alpha) = Q_\alpha$ i.e., $(Q_\alpha \cap J) = Q_\alpha$, which gives that $Q_\alpha \subset J$.

(e) Suppose on contrary that there exist $u_\alpha, u_\delta \in Q_\alpha$ and $u_\beta \in Q_\beta$ such that $u_\alpha u_\beta u_\delta = u_\gamma$ and $u_\gamma \notin S$. Using (ii), we can find Q_γ such that $u_\gamma \in Q_\gamma$. If $Q_\gamma \neq Q_\alpha$. Then $Q_\alpha \subset M^m - Q_\gamma = S_\gamma$. That is, $Q_\alpha \subset S_\gamma$ and similarly, $Q_\delta \subset S_\gamma$. This gives, $Q_\alpha Q_\beta^m Q_\alpha \subset S_\gamma Q_\beta^m S_\gamma \subset S_\gamma M^m S_\gamma \subset S_\gamma$, hence, $u_\gamma \in S_\gamma$, which is a contradiction to $u_\gamma \in Q_\gamma = M^m \backslash S_\gamma$. Suppose now, $Q_\gamma = Q_\beta$. Then, $Q_\beta \subset M^m - Q_\gamma = S_\gamma$ and $Q_\alpha Q_\beta^m Q_\alpha \subset S_\alpha M^m S_\alpha \subset S_\alpha$, hence $u_\gamma \in S_\alpha = M^m - Q_\alpha$, which is a contradiction to $u_\gamma \in Q_\gamma$. Thus, $Q_\alpha Q_\beta^m Q_\alpha \subset S$ and S is not empty. □

Theorem 3.3. *Let M be a semigroup containing maximal m-bi ideals and let S be the intersection of all maximal m-bi ideals of M. Then every prime m-bi ideal of M containing S and different from M is a maximal m-bi ideal of M.*

Proof. Let U be a prime m-bi ideal of M containing S and $U \neq M$. Then Theorem 3.2, part (iv),

$$U = M^m - \left(\bigcup_{\nu\in\Lambda} Q_\nu\right) = \bigcap_{\nu\in\Lambda}(M^m - Q_\nu) = \bigcap_{\nu\in\Lambda} S_\nu,$$

where $\Lambda \subseteq \wedge$ and $\Lambda \neq \emptyset$. If $|\Lambda| = 1$, we have $U = S_\nu$, i.e. U is a maximal m-bi ideal of M and the theorem is proved. We shall show that $|\Lambda| \geq 2$ is not possible. Suppose on contrary that $|\Lambda| \geq 2$. Let $\beta \in \Lambda$ and denote $H = \bigcup_{\nu\in\Lambda, \nu\neq\beta} S_\nu$. Then we have $U = H \cap S_\beta$. Since both H and S_β are m-bi ideals, their product is also m-bi ideal, and so $HS_\beta \subset H \cap S_\beta = U$. Since U is prime m-bi ideal, so either $H \subset U$ or $S_\beta \subset U$. We discuss these two possibilities separately.

(a) Let $H \subset U$. Since $U \subset H$ also, so $U = H$. Further $H = U = H \cap S_\beta$ implies $H \subseteq S_\beta$, by Theorem 3.2, part (iii), we have $Q_\beta \subseteq \bigcup_{\nu \in \Lambda, \nu \neq \beta} S_\nu = H$. Hence $Q_\beta \subset S_\beta$, a contradiction with $Q_\beta \cap S_\beta = \emptyset$.

(b) Let $S_\beta \subset U$. Since also $U \subset S_\beta$, so $U = S_\beta$. Now $U = S_\beta = H \cap S_\beta$ would imply $S_\beta \subset H$. Since S_β is maximal and H is a proper subset of M, so $H = S_\beta$. The relation $Q_\beta \subset H = S_\beta$ gives an another contradiction.

These two cases complete the proof of the theorem. □

Theorem 3.4. *If M is a semigroup containing at least one maximal m-bi ideal, then a prime m-bi ideal U different from M is a maximal m-bi ideal of M if and only if $S \subset U$, where $S = \bigcap_{\alpha \in \wedge} S_\alpha$.*

Proof. If U is a maximal ideal, then clearly $S \subset U$. Conversely, if $S \subset U$, then by Theorem 3.3, U is a maximal ideal of M. □

Definition 3.2. An m-bi ideal N different from $\{0\}$ (if $0 \in M$) of a semigroup M is known as its minimal m-bi ideal if there does exist any other proper m-bi ideal in M which is contained in N properly.

In Example 2.2, $\mathcal{N} = \{\alpha\}$ is a minimal m-bi ideal of M. Detailed studies of the maximal and minimal m-bi ideals of a semigroup will be given in a future article on chains of the m-bi ideals in a semigroup.

4. Charterizing m-Regular and m-Intraregular Semigroups

In this section, we describe the m-regular and m-intraregular semigroups using the properties of their m-bi ideals, prime, semiprime and strongly prime m-bi ideals.

Definition 4.1. An element a of a semigroup M is called m-regular if $axa = a$ for some $x \in M^m$. A semigroup M is called m-regular if every element of M is m-regular. M is m-regular if $a \in aM^m a$ for all $a \in M$ [11].

Definition 4.2. An element a of a semigroup M is called m-intraregular if $ya^2z = a$ for some elements $y, z \in M^m$. Semigroup M is called m-intraregular if every element of M is m-intraregular [11].

Theorem 4.1. *For a semigroup M, the given conditions are equivalent.*

(a) *M is m-regular & m-intraregular.*
(b) *$B^2 = B$ for all m-bi ideal B in M.*
(c) *$B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for all m-bi ideals B_1, B_2 in M.*
(d) *Every m-bi ideal of M is semiprime.*
(e) *For any proper m-bi ideal B of M, if $B = \bigcap_{\alpha \in \wedge} \{B_\alpha : B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B\}$.*

Proof. (a) ⇒ (b) Let M be m-regular and m-intraregular. Trivially, $B^2 \subseteq B$. For the converse, let $b \in B$. So, $b \in M$, using (a), we have $b = bsb$ and $b = ub^2w$, for some $s \in M^m$, $y, w \in M^m$. So, $b = bsb = bsbsb = bs(sb^2u)sb = (bssb)(busb)$. As $b \in B$, therefore, $b(ss)b \in BM^mB \subseteq B$. Also, $b(us)b \in BM^mB \subseteq B$. So, $b = (bssb)(busb) \in BB = B^2$. We get $B \subseteq B^2$.

(b) $\Rightarrow$ (c) Since $(B_1 \cap B_2)$ being intersection of two m-bi ideal is again an m-bi ideals, by the truth of part (b), we have $B_1 \cap B_2 = (B_1 \cap B_2)^2$. After a short simplification, we

$$B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1. \tag{4.1}$$

Since $B_1B_2 \cap B_2B_1$ being the intersection of the products B_1B_2 and B_2B_1 of m-bi ideals of M is again an m-bi ideal [10]. So, by (b), we obtain $B_1B_2 \cap B_2B_1 = (B_1B_2 \cap B_2B_1)^2 \subseteq B_1B_2B_2B_1 \subseteq B_1M^mB_1 \subseteq B_1$. Analogously, $B_1B_2 \cap B_2B_1 \subseteq B_2$. Thus,

$$B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2. \tag{4.2}$$

Consequently, by (4.1) and (4.2), we get $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$.

(c) $\Rightarrow$ (d) In order to show that each m-bi ideal B of M is semiprime, we take another arbitrary m-bi C of M and assume that $C^2 \subseteq B$. By the truth of (c), we get $C = C \cap C = CC \cap CC = C^2$. This gives that $C \subseteq B$. Thus, every m-bi ideal of M is semiprime.

(d) $\Rightarrow$ (e) For a proper m-bi ideal B of M, let $\mathcal{B} = \bigcap\limits_{\alpha \in \wedge} \{B_\alpha :\ B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B\}$. Clearly, $B \subseteq \mathcal{B}$. We claim that $\mathcal{B} \subseteq B$, because if not, let $c \in \mathcal{B}$ and $c \notin B$. Then, Proposition 2.2 says that there exists an irreducible m-bi ideal in M say B_γ, for some $\gamma \in \wedge$, such that $B_\gamma \supset B$ and $c \notin B_\gamma$. By our assumption, every m-bi ideal is semiprime, and so each B_γ is irreducible semiprime m-bi ideal. But $c \notin B_\gamma$ creates contradiction to the assumption that $c \in B_\gamma$ for all $\gamma \in \wedge$. Hence, our claim is valid that $\mathcal{B} \subseteq B$. This completes the proof of the theorem.

(e) $\Rightarrow$ (b) Assuming the validity of (v), we have to show that each m-bi ideals B of M is idempotent i.e., $B^2 = B$. Clearly, $B^2 \subseteq B$. We again claim that $B \subseteq B^2$, because if not, let $c \in B$ such that $c \notin B^2$. Two possibilities arise. Firstly, when B^2 is contained in M properly, then, by (e), $B^2 = \bigcap\limits_{\alpha \in \wedge} \{B_\alpha :\ B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B^2\}$. That is, $B^2 \subseteq B_\alpha$ for all α. But B_α is semiprime, so $B \subseteq \bigcap\limits_{\alpha \in \wedge} B_\alpha = B^2$, i.e., $B \subseteq B^2$. Lastly, $B^2 = B$. Secondly, when B^2 is not a proper m-bi ideal of M, then $B^2 = M$, so B is idempotent, i.e., $B^2 = B$. □

Proposition 4.1. *An m-bi ideal B of an m-regular and m-intraregular semigroup M is strongly irreducible if and only if B is strongly prime.*

Proof. Suppose that B is strongly irreducible m-bi ideal of M, then from (4.1) of Theorem 4.1, for any two m-bi ideals B_1 and B_2 of M, $B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B(say)$. But by our hypothesis "B is strongly irreducible", we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$, resulting B into a strongly prime m-bi ideal of M.

Conversely, suppose that B is strongly prime, then (4.2) of Theorem 4.1 leads to $B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2 \subseteq B$ (say). This produces $B_1 \subseteq B$ or $B_2 \subseteq B$ because of our hypothesis. Thus, B becomes a strongly irreducible m-bi ideal. □

The following theorem characterizes the semigroups in which all m-bi ideals are strongly prime.

Theorem 4.2. *Each m-bi ideal of a semigroup M is strongly prime if and only if M is m-regular, m-intraregular and all the m-bi ideals of M become a totally ordered set with respect to inclusion.*

Proof. Assume that all m-bi ideal of M are strongly prime, so are also semiprime. By Theorem 4.1, M is m-regular and m-intraregular, and $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$ for two m-bi ideals B_1 and B_2 of M. It stays to show that the collection of all m-bi ideals of M is totally ordered with regards to the inclusion of m-bi ideals. Since $B_1 \cap B_2$ being an m-bi ideal of M is strongly prime, so the result, $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$ gives either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. Eventually, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus, the collection of m-bi ideals of M is totally ordered set.

Conversely, assume that M is m-regular, m-intraregular and the collection of m-bi ideals of M is totally ordered under the set inclusion. Let B be any m-bi ideal of M. We want to show B is strongly prime. Suppose B_1, B_2 be any two m-bi ideals of M with the property that $B_1B_2 \cap B_2B_1 \subseteq B$. By Theorem 4.1, $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$. So, $B_1 \cap B_2 \subseteq B$. By our assumption, either $B_1 \subseteq B_2$ or $B_2 \subset B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Along these lines, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Subsequently B is strongly prime. □

Theorem 4.3. *If the collection of all m-bi ideals in a semigroup M becomes a totally ordered set under the inclusion, then M is m-regular and m-intraregular if and only if each m-bi ideal of M is prime.*

Proof. Assume M is m-regular and m-intraregular and that B, B_1 and B_2 be any three m-bi ideals of M with the assumption that $B_1B_2 \subseteq B$. This to be noted, by theorem 4.1, B is semiprime. Since the collection of m-bi ideals of M is totally ordered, so by the definition of the total order on B_1 and B_2, one gets $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Without loss of generality, let $B_1 \subseteq B_2$, this produces $B_1^2 \subseteq B_1B_2 \subseteq B$. At this point, $B_1 \subseteq B$ because B is semiprime. Thus, B is a prime m-bi ideal.

For the converse, suppose that every m-bi ideal of M is prime. It is given that the collection of m-bi standards of M is totally ordered, therefore the prime m-bi ideals coincide with the semiprime m-bi. Eventually, by Theorem 4.2, M is both m-standard and m-intraregular. □

5. Topologies of m-Bi Ideals

Topology studies the set structures that are aimed to generalize the geometrical properties of the objects [1]. The ideals in all algebraic structures form topological spaces called the structure space [8]. Such structure spaces are compact and T_1. All algebraic structures have different families of ideals, so these families represent the topologies intrinsically [4]. A natural way in which to study this situation is the semigroups. Detailed procedures of defining the topologies in semigroups and other algebraic structures are given in [8] and [4]. In the beginning of this section, we define the *comaximal* m-bi ideals in semigroups, which will be used later in the section.

Definition 5.1. We say that two m-bi ideals A and B in a semigroup M intersect *transversally* or said to be *comaximal* m-bi ideals ([7, 12]), if $A \cap B = AB$.

We shall use the following notations in our onward work.

Notation 5.1. Let $\mathcal{B}$ is set of all the m-bi ideals of M, we define for each $B \in \mathcal{B}$, the collections.

(a) $\mathcal{P}$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_B = \{I \in \mathcal{P} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{P}) = \{\mathcal{K}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

(b) $\mathcal{S}$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_B = \{I \in \mathcal{S} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{S}) = \{\mathcal{C}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

(c) $\mathcal{H}$ be the family of all properly containing semiprime m-bi ideals in M, $\mathcal{Y}_B = \{I \in \mathcal{H} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{H}) = \{\mathcal{Y}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

Notation 5.2. Let $\mathcal{L}$ be the set of all m-left ideals of M, we define for each $L \in \mathcal{L}$, the collections.

(a) $\mathcal{P}_L$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_L = \{I \in \mathcal{P} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{P}_L) = \{\mathcal{K}_L : L \text{ is } m\text{-left ideal in } M\}$.

(b) $\mathcal{S}$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_L = \{I \in \mathcal{S} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{S}_L) = \{\mathcal{C}_L : L \text{ is } m\text{-left ideal in } M\}$.

(c) $\mathcal{H}$ be the family of all properly containing semiprime m-bi ideals of M, $\mathcal{Y}_L = \{I \in \mathcal{H} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{H}_L) = \{\mathcal{Y}_L : L \text{ is } m\text{-left ideal in } M\}$.

Notation 5.3. Let $\mathcal{R}$ be the set of all m-right ideals of M, we define for each $R \in \mathcal{R}$, the collections.

(a) $\mathcal{P}_R$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_R = \{I \in \mathcal{P} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{P}_R) = \{\mathcal{K}_R : R \text{ is } m\text{-right ideal in } M\}$.

(b) $\mathcal{S}_R$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_R = \{I \in \mathcal{S} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{S}_R) = \{\mathcal{C}_R : R \text{ is } m\text{-right ideal in } M\}$.

(c) $\mathcal{H}_R$ be the collection of all properly containing semiprime m-bi ideals of M, $\mathcal{Y}_R = \{I \in \mathcal{H} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{H}_R) = \{\mathcal{Y}_R : R \text{ is } m\text{-right ideal in } M\}$.

Theorem 5.1. *If, in the semigroup M containing* 0*, the m-bi ideals are pairwise comaximal in the sense of Definition* 5.1*, then the $\mathcal{T}(\mathcal{P})$ forms a topology on the set $\mathcal{P}$.*

Proof. We show that $\mathcal{T}(\mathcal{P})$ satisfies all the three *axioms* of a topology.

(a) Since $\{0\}$ is an m-bi ideal of M, $\mathcal{K}_{\{0\}} = \{I \in \mathcal{P} : \{0\} \nsubseteq I\} = \emptyset$ because 0 belongs to every m-bi ideal of M. So, $\emptyset \in \mathcal{T}(\mathcal{P})$. Since M is also an m-bi ideal of M

whereas $\mathcal{P}$ is the family of all properly containing strongly prime m-bi ideals of M, so $\mathcal{K}_M = \{I \in \mathcal{P} : M \nsubseteq I\} = \mathcal{P}$. Therefore, $\mathcal{P} \in \mathcal{T}(\mathcal{P})$.

(b) Let $\{\mathcal{K}_{B_\alpha} : \alpha \in I\}$ be an arbitrary collection from $\mathcal{T}(\mathcal{P})$. Then

$$\underset{\alpha\in I}{\cup}\mathcal{K}_{B_\alpha} = \{I \in \mathcal{P} : B_\alpha \nsubseteq I \text{ for some } \alpha \in I\} = \left\{ I \in \mathcal{P} : \overset{\wedge}{\bigcup_{\alpha\in I}} B_\alpha \nsubseteq I \right\} = \mathcal{K}_{\overset{\wedge}{\bigcup\limits_{\alpha\in I}} B_\alpha},$$

where $\overset{\wedge}{\bigcup\limits_{\alpha\in I}} B_\alpha$ is the m-bi ideal of M generated by $\underset{\alpha\in I}{\cup} B_\alpha$. Therefore, $\underset{\alpha\in I}{\cup}\mathcal{K}_{B_\alpha} \in \mathcal{T}(\mathcal{P})$.

(c) Let $\mathcal{K}_{B_1}$ and $\mathcal{K}_{B_2} \in \mathcal{T}(\mathcal{P})$. If $I \in \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$, then $I \in \mathcal{P}$ and neither $B_1 \nsubseteq I$ nor $B_2 \nsubseteq I$. Let $B_1 \cap B_2 \subseteq I$. Since B_1 and B_2 are two *comaximal*, $B_1 \cap B_2 = B_1B_2$. So, $B_1B_2 \subseteq I$. But I is strongly prime m-bi ideal, so either $B_1 \subseteq I$ or $B_2 \subset I$. This is contradiction. Therefore, I does not contain $B_1 \cap B_2$, and so $I \in \mathcal{K}_{B_1\cap B_2}$. Thus $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2} \subseteq \mathcal{K}_{B_1\cap B_2}$. On the other hand, $I \in \mathcal{K}_{B_1\cap B_2}$ gives that $I \in \mathcal{P}$, but $B_1 \cap B_2 \nsubseteq I$. That is, $B_1 \nsubseteq I$ and $B_2 \nsubseteq I$. Thus $I \in \mathcal{K}_{B_1}$, $I \in \mathcal{K}_{B_2}$. Therefore, $I \in \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$. Hence $\mathcal{K}_{B_1\cap B_2} \subseteq \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$. Thus, $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}\mathcal{K}_{B_1\cap B_2}$. So, $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2} \in \mathcal{T}(\mathcal{P})$. This shows that $\mathcal{T}(\mathcal{P})$ is a topology on $\mathcal{P}$. □

Corollary 5.1. *The collection $\mathcal{T}_L(\mathcal{P}_L)$ as defined in Notations 5.2 for the m-left ideals forms topology.*

Corollary 5.2. *The collection $\mathcal{T}_R(\mathcal{P}_R)$ as defined in Notations 5.3 for the m-right ideals forms a topology.*

Theorem 5.2. *If M is an m-regular and m-intraregular semigroup containing 0, then $\mathcal{T}(\mathcal{S})$ forms a topology on the set $\mathcal{S}$.*

Proof. We show that $\mathcal{T}(\mathcal{S})$ satisfies all the three *axioms* of a topology.

(a) Similar to (a) of proof of Theorem 5.1.

(b) Similar to (b) of proof of Theorem 5.1.

(c) Let $\mathcal{C}_{B_1}$ and $\mathcal{C}_{B_2} \in \mathcal{T}(\mathcal{S})$. If $I \in \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$, then $I \in \mathcal{S}$ and $B_1 \nsubseteq I$, $B_2 \nsubseteq I$. Suppose $B_1 \cap B_2 \subseteq I$. Since M is both m-regular and m-intraregular, $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$. Hence, $B_1B_2 \cap B_2B_1 \subseteq I$. This implies either $B_1 \subseteq I$ or $B_2 \subset I$; a contradiction. Therefore, $B_1 \cap B_2 \nsubseteq I$, and so $I \in \mathcal{C}_{B_1\cap B_2}$. Thus $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2} \subseteq \mathcal{C}_{B_1\cap B_2}$. If $I \in \mathcal{C}_{B_1\cap B_2}$, then we have $I \in \mathcal{S}$ and $B_1 \cap B_2 \nsubseteq I$. This implies that $B_1 \nsubseteq I$ and $B_2 \nsubseteq I$. Thus, $I \in \mathcal{C}_{B_1}$ and $I \in \mathcal{C}_{B_2}$, and therefore $I \in \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$. Hence $\mathcal{C}_{B_1\cap B_2} \subseteq \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$. Consequently, $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}\mathcal{C}_{B_1\cap B_2}$. So, $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2} \in \mathcal{T}(\mathcal{S})$.

This shows that $\mathcal{T}(\mathcal{S})$ is a topology on $\mathcal{S}$. □

Corollary 5.3. *The collection $\mathcal{T}_L(\mathcal{S}_L)$ as defined in Notations 5.2 for the m-left ideals forms topology.*

Corollary 5.4. *The collection $\mathcal{T}_R(\mathcal{S}_R)$ as defined in Notations 5.2 for the m-right ideals forms a topology.*

Remark 5.1. (a) In Theorem 5.2 and Theorem 5.1, and their associated corollaries, we took semigroups with 0 in order to present more explicit form of topologies; however, semigroups without 0 also form the topologies.

(b) The m-regularity and m-intraregularity alone on the semigroup M are not enough to prove Theorem 5.1. Theorem 5.2 can also be proved if M possesses the pairwise *comaximal* m-bi ideals, even if in the absence of m-regularity and m-intraregularity property. The imposition of both these two conditions on M does not admit the collection $\mathcal{T}(\mathcal{Y})$ defined in Notation 5.1, (iii) to form a topology on $\mathcal{H}$.

(c) We can compare the strength of these two conditions with that of the pairwise *comaximal* m-bi ideals of M in the domain of the topological spaces. This signifies that the pairwise *comaximal* property is stronger than the m-regularity and m-intraregularity.

(d) The collection $\mathcal{T}_L(\mathcal{H}_L)$ as defined in Notations 5.2 for the m-left ideals does not admit topology.

(e) The collection $\mathcal{T}_R(\mathcal{H}_R)$ as defined in Notations 5.2 for the m-right ideals does not admit a topology.

6. Conclusions

The main conclusions of the article are summarized in the following lines.

(a) The concepts of the prime, semiprime and strongly prime m-bi in the semigroups were introduced. With the help of the examples, we showed that the m-bi ideals have different properties than the bi ideals in semigroups.

(b) We also presented the concept of the maximal, minimal, irreducible and strongly irreducible m-bi ideals and gave the important characterizations of m-regular and m-intraregular semigroups using these m-bi ideals.

(c) We showed that the prime m-bi ideals form topology when the pairwise *comaximal* property is satisfied. The strongly prime m-bi ideals form the topology separately when the semigroups is m-regular and m-intraregular. And also, when the pairwise comaximal property holds in it. However, the semiprime m-bi ideals do not admit the topology even if both the properties are satisfied by the semigroups.

(d) In the future, we can extend this work to explore more topologies of these m-bi ideals. We can explore metric topologies on these ideals. We can extend the work on the chains of the m-bi deals and characterize the semigroups through the maximal m-bi ideals, and other classes of prime m-bi ideals. The work can be extended to other algebraic structures, especially semirings. The idea of the m-bi ideal is also of importance to explore the properties of the simple semigroups.

Acknowledgements. The authors thank the editor and anonymous referees for their useful suggestions which have improved the article.

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[1]Department of Mathematics,
Government Postgraduate College No. 1,
Abbottabad, Pakistan
Email address: dr.mohammadmunir@gpgc-atd.edu.pk

[2]Department of Mathematics and Statistics,
University of Agriculture,
Faisalabad, Pakistan
Email address: kausar.nasreen57@gmail.com

[3]SRM Institute of Science and Technology,
Ramapuram Chennai, Tamilnadu, India
Email address: anithamaths2019@gmail.com

[4]Department of Mathematics,
Sri Sankara Arts and Science College,
Enathur, Kanchipuram- 631 561, Tamilnadu, India
Email address: mathsguna@yahoo.com

[5]Department of Mathematics,
Air University(AU),
Islamabad, Pakistan
Email address: tehreem@mail.au.edu.pk

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 423–432.

L-FUZZY HOLLOW MODULES AND L-FUZZY MULTIPLICATION MODULES

SHRIRAM K. NIMBHORKAR[1] AND JYOTI A. KHUBCHANDANI[2]

Abstract. In this paper, we give some characterizations of L-fuzzy hollow modules and of L-fuzzy multiplication modules.

1. Introduction

The concept of a fuzzy set, which is a generalization of a crisp set, was introduced by Zadeh [13]. Rosenfeld [12] used this concept to develop the theory of fuzzy subgroups. Naegoita and Ralescu [9] applied this concept to modules and defined a fuzzy submodule of a module.

Barnad [3] introduced the concept of a multiplication module. An R-module M is called a *multiplication module* if every submodule of M is of the form IM, for some ideal I of R. Also, Elbast and Smith [4] have studied multiplication modules.

Lee and Park [6] studied fuzzy prime submodules of a fuzzy multiplication module. Recently, Atani [2] introduced and investigated L-fuzzy multiplication modules over a commutative ring with nonzero identity. He has proved a relation between a multiplication module and an L-fuzzy multiplication module.

In this paper we introduce a notion of a hollow fuzzy module and prove some results. Our notion is different from that of Rahman [11]. We prove some results on L-fuzzy multiplication modules. We also show that an L-hollow fuzzy module is an L-fuzzy multiplication module.

Key words and phrases. L-Fuzzy hollow module, L-fuzzy multiplication module, L-fuzzy Noetherian module.

2010 *Mathematics Subject Classification.* Primary: 08A72. Secondary: 03E72.

DOI 10.46793/KgJMat2403.423N

Received: September 17, 2020.

Accepted: May 07, 2021.

2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We recall some definitions and results from Moderson and Malik [8] which will be used in this paper.

Definition 2.1. ([8, Definition 1.1.1]). A fuzzy subset of an R-module M is a mapping $\mu : M \to [0,1]$. We denote the set of all fuzzy subsets of M by $[0,1]^M$.

If μ is a mapping from M to L, where L is a complete Heyting algebra, then μ is called an L-subset of M. We denote the set of all L-subsets of R by L^R and the set of all L-subsets of M by L^M.

Definition 2.2. ([8, Definition 1.1.3]). If $N \subseteq M$ and $\alpha \in [0,1]^M$, then α_N is defined as

$$\alpha_N(x) = \begin{cases} \alpha, & \text{if } x \in N, \\ 0, & \text{otherwise.} \end{cases}$$

If $N = \{x\}$, then α_x is often called a fuzzy point and is denoted by χ_α. If $\alpha = 1$, then 1_N is known as the characteristic function of N and is denoted by χ_N.

If $\mu, \sigma \in [0,1]^M$, then for $x, y, z \in M$, we define

(i) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$;
(ii) $(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;
(iii) $(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$;
(iv) $(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$.

Definition 2.3. ([8, Definition 4.1.6]). Let $\zeta \in L^R$ and $\mu \in L^M$. Define $\zeta \cdot \mu$ as

$$(\zeta \cdot \mu)(x) = \vee\{\zeta(r) \wedge \mu(y) \mid r \in R, y \in M, ry = x\}, \quad \text{for all } x \in M.$$

Definition 2.4. ([8, Definition 3.1.7]). Suppose that $\mu \in L^R$ satisfies the following conditions:

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$.

Then μ is called an L-ideal of R.

We denote the set of all L-ideals of R by $LI(R)$.

Definition 2.5. ([8, Definition 4.1.8]). Let M be a module over a ring R and L be a complete Heyting algebra. An L subset μ in M is called an L-submodule of M, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu(\theta) = 1$;
(ii) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(iii) $\mu(rx) \geq \mu(x)$.

Definition 2.6. ([8, Definition 4.5.1]). For $\mu, \nu \in L^M$ and $\zeta \in L^R$, define the residual quotients $\mu : \nu \in L^R$ and $\mu : \zeta \in L^M$ as follows:

$$\mu : \nu = \cup\{\eta \mid \eta \in L^R,\, \eta \cdot \nu \subseteq \mu\},$$
$$\mu : \zeta = \cup\{\xi \mid \xi \in L^M,\, \zeta \cdot \xi \subseteq \mu\}.$$

Theorem 2.1. ([8, Theorem 4.5.3]). *Let μ, $\nu \in L^M$ and $\zeta \in L^R$. Then*
(1) $(\mu : \nu)\nu \subseteq \mu$*;*
(2) $\zeta \cdot \nu \subseteq \mu$ *if and only if* $\zeta \subseteq (\mu : \nu)$ *if and only if* $\nu \subseteq \mu : \zeta$*.*

Definition 2.7 ([8])**.** Let $c \in L\backslash\{1\}$. Then
(i) c is called a prime element of L if $a \wedge b \leq c$, implies that $a \leq c$ or $b \leq c$ for all a, $b \in L$;
(ii) c is called a maximal element if there does not exist $a \in L\backslash\{1\}$ such that $c < a < 1$.

Remark 2.1 ([8]). If $\mu, \nu \in LI(R)$, then $(\mu \circ \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in R, yz = x\}$. We write $\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$.

Definition 2.8. ([8, Definition 3.5.1]). Let $\xi \in LI(R)$. Then ξ is called a prime L-ideal of R if ξ is non-constant and $\mu \circ \nu \subseteq \xi$, $\mu, \nu \in LI(R)$ implies either $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

Definition 2.9. ([8, Definition 3.6.1]). Let $\xi \in LI(R)$ and let ρ_ξ be the family of all prime L-ideals μ of R such that $\xi \subseteq \mu$. The L-radical of ξ, denoted by $\sqrt{\xi}$, is defined by

$$\sqrt{\xi} = \begin{cases} \cap\{\mu \mid \mu \in \rho_\xi\}, & \text{if } \rho_\xi \neq \phi, \\ 1_R, & \text{if } \rho_\xi = \phi. \end{cases}$$

Definition 2.10. ([8, Definition 3.7.1]). Let $\xi \in LI(R)$. Then ξ is called a primary L-ideal of R if ξ is nonconstant and for any $\mu, \nu \in LI(R)$, $\mu \circ \nu \subseteq \xi$ implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$.

Theorem 2.2. ([8, Theorem 3.5.3]). *If ξ is a prime L-ideal of R, then ξ_* is a prime ideal of R.*

Theorem 2.3. ([8, Theorem 3.5.5]). *Let $\xi \in L^R$. Then ξ is a prime L-ideal of R if and only if $\xi(0) = 1$, ξ_* is a prime ideal of R, $\xi(R) = \{1, c\}$, where c is a prime element in L.*

Definition 2.11 ([5])**.** A ring R is called regular if, for each element $x \in R$, there exists $y \in R$ such that $xyx = x$.

Definition 2.12. A dense chain in a lattice L is a non-empty sublattice C such that, for all ordered pairs $x < y$ with x, $y \in C$, there exists some $z \in C$ such that $x < y < z$.

Theorem 2.4 ([8])**.** *Let R be a ring with identity, L be a dense chain and ξ be a primary L-ideal of R. Then $\sqrt{\xi}$ is a prime L-ideal of R.*

Theorem 2.5. ([7, Theorem 3.10]). *Let R be a ring with* 1 *and A be a nonconstant fuzzy left* (*right*) *ideal of R. Then there exists a fuzzy maximal left* (*right*) *ideal B of R such that $A \subseteq B$.*

Definition 2.13. ([5, Definition 4.3.2]). A fuzzy ideal μ of a ring R is called fuzzy semiprime if, for any fuzzy ideal ζ of R, the condition $\zeta^n \subseteq \mu$ implies that $\zeta \subseteq \mu$, where $n \in \mathbb{Z}_+$.

Theorem 2.6. ([5, Theorem 4.4.3]). *A commutative ring with unity is regular if and only if each of its fuzzy ideal is fuzzy semiprime.*

Definition 2.14 ([2])**.** Let M be a module over a commutative ring R. M is called an L-fuzzy multiplication module provided for each L-fuzzy submodule μ of M, there exists $\zeta \in LI(R)$ with $\zeta(0_R) = 1$ such that $\mu = \zeta\chi_M$.

One can easily show that if $\mu = \zeta\chi_M$ for some $\zeta \in LI(R)$ with $\zeta(0_R) = 1$, then $\mu = (\mu : \chi_M)\chi_M$.

Theorem 2.7. ([2, Theorem 10]). *Let M be an R-module. Then M is a multiplication module if and only if M is an L-fuzzy multiplication module.*

Theorem 2.8. ([1, Theorem 2]). *Let P be a primary ideal of R and M a faithful multiplication R-module. Let $a \in R$, $x \in M$ satisfy $ax \in PM$. Then $a \in \sqrt{P}$ or $x \in PM$.*

Definition 2.15. ([10, Definition 4.1]). Let M be a module over a ring R and $\mu \in L(M)$. Then μ is said to be a small L-submodule of M, if for any $\nu \in L(M)$ satisfying $\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$.

Definition 2.16. ([11, Definition 2.10]). A fuzzy submodule $\mu(\neq \chi_\theta)$ of a module M is said to be fuzzy indecomposable if there do not exist fuzzy submodules σ, γ of M with $\sigma \neq \chi_\theta$, $\gamma \neq \chi_\theta$ and $\sigma \neq \mu$, $\gamma \neq \mu$ such that $\mu = \sigma \oplus \gamma$.

Theorem 2.9. ([10, Theorem 5.2]). *Let $\mu \in L^M$. Then μ is a maximal L-submodule of M if and only if μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M and α is a maximal element of $L - \{1\}$.*

Definition 2.17. ([11, Definition 3.1]). A fuzzy submodule ν with $\nu_* \neq \{\theta\}$ of M is said to be a fuzzy hollow submodule if for every fuzzy submodule μ of ν with $\mu_* \neq \nu_*$, μ is a fuzzy small submodule of ν. We say that an R-module $M \neq \{\theta\}$ is fuzzy hollow module if for every $\sigma \in F(M)$ with $\sigma_* \neq M$ implies $\sigma \ll_f M$.

Theorem 2.10. ([11, Theorem 3.6]). *Every fuzzy hollow submodule is indecomposable.*

Theorem 2.11. ([2, Theorem 14]). *Let M be a non-zero L-fuzzy multiplication R-module. Then every L-fuzzy submodule $\mu \neq \chi_M$ of M is contained in a generalized maximal L-fuzzy submodule of M.*

Proposition 2.1. ([2, Proposition 18]). *Suppose that M is a faithful L-fuzzy multiplication R-module. Let ζ be an L-fuzzy prime ideal of R. If η is an L-fuzzy ideal of R such that $\eta\chi_M \subseteq \zeta\chi_M$ and $\zeta\chi_M \neq \chi_M$, then $\eta \subseteq \zeta$. In particular, $(\zeta\chi_M : \chi_M) = \zeta$.*

Notations:

$fspec(R)$: the set of all prime L-submodules of R;

$Max_L(M)$: the set of all maximal L-submodules of M;

$JLR(M)$: the intersection of all maximal L-submodules of M is known as Jacobson L-radical of M.

Definition 2.18 ([2])**.** An R-module M is called an L-fuzzy Noetherian module, if every ascending chain of L-fuzzy submodules is stationary.

Definition 2.19. A module M is called L-local if M has exactly one maximal L-submodule.

Definition 2.20. A module M is called L-serial if any two L-submodules of M are comparable with respect to inclusion.

3. L-Fuzzy Hollow Modules and L-Fuzzy Multiplication Modules

In this section we introduce a slightly different notion of L-fuzzy hollow modules. Also, we obtain some properties of the same and L-fuzzy multiplication module.

Definition 3.1. Let M be a module over a commutative ring R. M is called an L-fuzzy hollow module if either $Max_L(M) = \chi_\theta$ or for each maximal L-fuzzy submodule μ of M and for each L-fuzzy submodule σ of M, the equality $\mu + \sigma = \chi_M$ implies that $\sigma = \chi_M$.

Theorem 3.1. *Let M be a non-zero module. Then the following statements are equivalent.*

(1) *M is an L-fuzzy hollow module and $Max_L(M) \neq \chi_\theta$.*

(2) *M is a cyclic and an L-local module.*

(3) *M is a finitely generated L-local module.*

Proof. (1) $\Rightarrow$ (2) Let μ be a maximal L-submodule of M and for $m \in M$, $\chi_{\{m\}}$ be an L-submodule of M such that $\chi_{\{m\}} \nsubseteq \mu$. Since, $\mu + \chi_{\{m\}} = \chi_M$, and as M is a L-fuzzy hollow module we have $\chi_M = \chi_{\{m\}}$. Hence, M has only one maximal L-submodule.

Also, as $\chi_M = \langle \chi_{\{m\}} \rangle = \chi_{Rm}$ implies that, $M = Rm$. Hence, M is cyclic.

(2) $\Rightarrow$ (3) It is obivous.

(3) $\Rightarrow$ (1) Let μ be a maximal L-submodule of M and σ be an L-fuzzy submodule of M. If $\mu + \sigma = \chi_M$ and $\sigma \neq \chi_M$, then by Zorn's lemma there exists a maximal L-submodule δ of M containing σ. Since, M is an L-local module, $\delta = \mu$ and so $\chi_M = \mu + \sigma = \mu$, a contradiction. Thus, $\sigma = \chi_M$. □

Theorem 3.2. *Let M be an R-module and μ be an L-fuzzy submodule of M. Then the following statements are equivalent.*

(1) *μ is a serial submodule.*

(2) *μ is an L-fuzzy hollow submodule.*

(3) *μ is fuzzy indecomposable.*

Proof. (1) $\Rightarrow$ (2) Suppose that $Max_L(\mu) \neq \chi_\theta$ and $\mu_1, \mu_2 \in L(M)$ be such that $\mu_1 + \mu_2 = \mu$, where μ_1 is a maximal L-submodule of μ and μ_2 is an L-submodule of μ. Since, μ_1, μ_2 are L-submodules of μ and μ is a serial submodule either $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

If $\mu_1 \subseteq \mu_2$, then $\mu = \mu_1 + \mu_2 = \mu_2$. If $\mu_2 \subseteq \mu_1$, then $\mu = \mu_1 + \mu_2 = \mu_1$, which is not possible as μ_1 is a maximal L-submodule of μ. Thus, μ is an L-fuzzy hollow submodule of M.

(2) $\Rightarrow$ (3) Follows from Theorem 2.10.

(3) $\Rightarrow$ (1) Let μ_1, μ_2 be L-fuzzy submodules of μ with $\mu_1 \neq \chi_\theta$, $\mu_2 \neq \chi_\theta$, $\mu_1 \neq \mu$, $\mu_2 \neq \mu$ and $\mu_1 \nsubseteq \mu_2$. As μ is fuzzy indecomposable, μ_1, μ_2 does not satisfy $\mu_1 + \mu_2 = \mu$ and $\mu_1 \cap \mu_2 = \chi_\theta$. Then, $\mu_2 \subseteq \mu_1$, thus μ is a serial submodule. ☐

Lemma 3.1. *Let M be an L-fuzzy multiplication module and μ be an L-fuzzy submodule of M. Then the following are equivalent.*

(1) $\mu \subseteq JLR(M)$.

(2) μ *is an L-small submodule in M.*

Proof. (1) $\Rightarrow$ (2) Let σ be an L-fuzzy submodule of M such that $\chi_M = \mu + \sigma$. If $\sigma \neq \chi_M$, then by Theorem 2.11, there exists a maximal L-submodule δ of M such that $\sigma \subseteq \delta$. But, $\mu \subseteq JLR(M) \subseteq \delta$ implies that $\mu + \sigma \subseteq \delta \neq \chi_M$. Thus, $\sigma = \chi_M$ implies that μ is an L-small submodule in M.

(2) $\Rightarrow$ (1) Assume that μ is an L-small submodule of M. Suppose that $\mu \nsubseteq JLR(M)$. Then there exists a maximal L-submodule β of M such that $\mu \nsubseteq \beta$. Thus, $\mu + \beta = \chi_M$. But $\beta \neq \chi_M$, a contradiction. Hence, $\mu \subseteq \beta$. ☐

Theorem 3.3. *If M is an L-fuzzy hollow module, then M is an L-fuzzy multiplication module.*

Proof. As M is an L-fuzzy hollow module, by Theorem 3.1, M is cyclic. But, we know that every cyclic module is a multiplication module. Thus, by Theorem 2.7, M is an L-fuzzy multiplication module. ☐

We give an example of an L-fuzzy multiplication module by using Theorem 3.3.

Example 3.1. Let $L = \{0, 0.25, 0.5, 0.75, 1\}$. Then L is a complete Heyting algebra together with the operations minimium (meet), maximium (join) and $\leq$ (partial ordering), then 0.75 is a maximal element of $L - \{1\}$.

Consider, $M = \mathbb{Z}_{27} = \{0, 1, 2, \dots, 26\}$ under addition modulo 27, then M is a module over the ring $\mathbb{Z}$. Let $A = \{0, 3, 6, \dots, 24\}$.

Define, $\mu \in [0,1]^M$ as follows:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0.75, & \text{otherwise.} \end{cases}$$

Then $\mu_* = \{0, 3, 6, \dots, 24\} = A$, which is a maximal submodule of $\mathbb{Z}_{27}$. Also, $\mu = \chi_{\mu_*} \cup 0.75_M$, where 0.75 is a maximal element of $L - \{1\}$. So, by Theorem 2.9, μ is a maximal L-submodule of $\mathbb{Z}_{27}$. Infact, μ is the only maximal L-submodule of $\mathbb{Z}_{27}$.

Let $B = \{0, 9, 18\}$ and define $\nu \in [0,1]^M$ as follows,

$$\nu(x) = \begin{cases} 1, & \text{if } x \in B, \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha < 0.75$. Then clearly μ, ν are the only fuzzy submodules of M. Also, here $\nu \neq \chi_M$ implies that $\mu + \nu \neq \chi_M$. This shows that M is an L-fuzzy hollow module and by Theorem 3.3, M is an L-fuzzy multiplication module.

Corollary 3.1. *For $\xi_1, \xi_2 \in L^R$ with $\xi_1 \subseteq \xi_2$, then $\xi_1 \cdot \chi_M \subseteq \xi_2 \cdot \chi_M$ and thus $(\xi_1\chi_M : \chi_M) \subseteq (\xi_2\chi_M : \chi_M)$.*

Proof. We have

$$\begin{aligned}
(\xi_1 \cdot \chi_M)(x) &= \bigvee\{\xi_1(r) \wedge \chi_M(y) \mid r \in R, y \in M \ \wedge \ ry = x\} \\
&= \bigvee\{\xi_1(r) \mid r \in R, x \in rM\} \\
&\leq \bigvee\{\xi_2(r) \mid r \in R, x \in rM\} \\
&\leq \bigvee\{\xi_2(r) \wedge \chi_M(y) \mid r \in R, y \in M \ \wedge \ ry = x\} \\
&= (\xi_2 \cdot \chi_M)(x).
\end{aligned}$$

Hence, $\xi_1 \cdot \chi_M \subseteq \xi_2 \cdot \chi_M$, for all $x \in M$.

Again we have

$$\begin{aligned}
(\xi_1\chi_M : \chi_M) &= \bigvee\{\eta \mid \eta \in L^R, \eta \cdot \chi_M \subseteq \xi_1 \cdot \chi_M\} \\
&\leq \bigvee\{\eta \mid \eta \in L^R, \eta \cdot \chi_M \subseteq \xi_2 \cdot \chi_M\} \\
&\leq (\xi_2\chi_M : \chi_M).
\end{aligned}$$

Hence, $(\xi_1\chi_M : \chi_M) \subseteq (\xi_2\chi_M : \chi_M)$. □

Theorem 3.4. *Let M be an L-fuzzy multilpication module. Then μ is a maximal L-fuzzy submodule of M if and only if there exists a maximal ideal ξ of $LI(R)$ such that $\mu = \xi\chi_M \neq \chi_M$.*

Proof. By Theorem 2.11, if ξ is a maximal L-fuzzy ideal of R and $\chi_M \neq \xi\chi_M$, then $\xi\chi_M$ is a maximal L-submodule of M.

Conversely, assume that μ is a maximal L-submodule of M. Then there exists an L-ideal ν of $LI(R)$ such that $\mu = \nu\chi_M$. Suppose that ν is not a maximal L-ideal of R. Then $\nu \subseteq \beta$ for some $\beta \in LI(R)$ and so $\nu\chi_M \subseteq \beta\chi_M$ implies that $\mu \subseteq \beta\chi_M$. This implies μ is not a maximal L-submodule of M, a contradiction. Thus, ν is a maximal L-fuzzy ideal of R. □

Theorem 3.5. *Let M be a faithful L-fuzzy Noetherian R-module. Then R satisfies the ascending chain condition on L-prime ideals.*

Proof. Let $\xi_1 \subseteq \xi_2 \subseteq \xi_3 \subseteq \cdots$ be an ascending chain of L-prime ideals of R. Then by Corollary 3.1, $\xi_1\chi_M \subseteq \xi_2\chi_M \subseteq \xi_3\chi_M \subseteq \cdots$. But as M is an L-fuzzy Noetherian R-module, there exists some $n \in \mathbb{N}$ such that $\xi_n\chi_M = \xi_{n+1}\chi_M = \cdots$. Hence, by Proposition 2.1, $\xi_1 \subseteq \xi_2 \subseteq \xi_3 \subseteq \cdots \subseteq \xi_n$. $\square$

Theorem 3.6. *Let R be regular ring with unity which satisfies ascending chain condition on fuzzy semiprime ideals and M be an L-fuzzy multiplication module. Then M is an L-fuzzy Noetherian module.*

Proof. Let $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots$ be an ascending chain of L-fuzzy submodules of M. Then by Corollary 3.1, $(\mu_1 : \chi_M) \subseteq (\mu_2 : \chi_M) \subseteq (\mu_3 : \chi_M) \subseteq \cdots$ is an ascending chain of ideals of R. By Theorem 2.6, $(\mu_1 : \chi_M) \subseteq (\mu_2 : \chi_M) \subseteq (\mu_3 : \chi_M) \subseteq \cdots$ is an ascending chain of fuzzy semiprime ideals of R. By assumption there exists positive integer t such that $(\mu_t : \chi_M) = (\mu_{t+s} : \chi_M)$, for every positive integer s. Hence, $\mu_t = (\mu_t : \chi_M)\chi_M = (\mu_{t+s} : \chi_M)\chi_M = \mu_{t+s}$ gives $\mu_t = \mu_{t+s}$ for every s and so the chain is stationary. Hence, M is an L-fuzzy Noetherian module. $\square$

Theorem 3.7. *Let M be an faithful L-fuzzy multiplication module. Then for every L-fuzzy submodule μ of M, if $\mu\chi_M \subseteq \xi\chi_M$, where $\xi \in fspec(R)$, then $\mu \subseteq \xi$.*

Proof. Given, $\mu\chi_M \subseteq \xi\chi_M$. As, $\mu \subseteq (\mu\chi_M : \chi_M) \subseteq (\xi\chi_M : \chi_M) = \xi$ by Proposition 2.1. Hence, $\mu \subseteq \xi$. $\square$

Theorem 3.8. *Let R be a ring and M be an L-fuzzy multiplication R-module. Then $\xi\chi_M \neq \chi_M$ for any proper fuzzy ideal ξ of R.*

Proof. As ξ is a proper fuzzy ideal of R, by Theorem 2.5, there exists a maximal fuzzy ideal η of R such that $\xi \subseteq \eta$. Let μ be a proper L-fuzzy submodule of M. As M is an L-fuzzy multiplication module, by Theorem 2.11, μ is contained in a generalized maximal L-fuzzy submodule of M say ν. Then, ν is a maximal L-fuzzy submodule of M. Hence, by Theorem 3.4, $\nu = \eta\chi_M \neq \chi_M$. But as $\xi \subseteq \eta$ implies that $\xi\chi_M \subseteq \eta\chi_M \neq \chi_M$ and so $\xi\chi_M \neq \chi_M$. $\square$

Theorem 3.9. *Let L be a dense chain and M be a faithful L-fuzzy multiplication R-module. Let μ be a primary L-fuzzy ideal of R, $a, b \in L$ and $r_a x_b \in \mu\chi_M$ for some $r \in R$ and $x \in M$. Then $r_a \in \mu$ or $x_b \in \mu\chi_M$.*

Proof. As μ is a primary L-fuzzy ideal of R and L is a dense chain, then by Theorem 2.4 $\sqrt{\mu}$ is prime L-fuzzy ideal of R. Now, by Theorem 2.3, for each $r \in R$, there exist a prime ideal P of R and a prime element $c \in L$ such that

$$\sqrt{\mu(r)} = \begin{cases} 1, & \text{if } r \in P, \\ c, & \text{otherwise.} \end{cases}$$

(I) As $r_a x_b \in \mu\chi_M$, it follows that $\mu\chi_M(rx) \geq a \wedge b$.

But, (II)

$$\begin{aligned}\mu\chi_M(rx) &= \vee\{\mu(s) \wedge \chi_M(y) \mid s \in R, y \in M, rx = sy\}\\ &= \vee\{\mu(s) \mid s \in R, rx \in sM\}.\end{aligned}$$

Let $A = \{s \in P \mid rx \in sM\}$.

Case(I). If $A = \emptyset$, then there does not exist $s \in P$ such that $rx \in sM$. Hence, from (I) $\mu\chi_M(rx) = c \geq a \wedge b$. As c is a prime element of L, either $c \geq a$ or $c \geq b$.

(i) Suppose that $c \geq a$. As $\mu(r) \in \{1, c\}$, we have $\sqrt{\mu(r)} \geq a$ and so $r_a \in \sqrt{\mu}$.

(ii) If $c \geq b$, then similarly from (II) $\mu\chi_M(x) = \vee\{\mu(s') : s' \in R, x \in s'M\}$. So, $\mu\chi_M(x) \in \{1, c\}$. Therefore, $\mu\chi_M(x) \geq b$ and so $x_b \in \mu\chi_M$.

Case (II). If $A \neq \emptyset$, then there exists $s' \in P$ such that $rx \in s'M$. Therefore, using (I) we have $\mu\chi_M(rx) = \vee\{\mu(s) \mid s \in R, rx \in sM\} = 1$ and $rx \in s'M \subseteq PM$. Now, by using Theorem 2.7 and Theorem 2.8, we get either $r \in P$ or $x \in PM$.

(i) If $r \in P$, then $\sqrt{\mu(r)} = 1 \geq a$ implies that $ra \in \sqrt{\mu}$.

(ii) If $x \in PM$, then $x = r_1x_1 + \cdots + r_nx_n$ for some $r_i \in P$ and $x_i \in M$ such that $i = 1, 2, \ldots, n$. Hence, $\mu\chi_M(x) = \mu\chi_M(\Sigma r_i x_i) \geq \mu\chi_M(r_1x_1) \wedge \cdots \wedge \mu\chi_M(r_nx_n) = 1 \geq b$ and so, $x_b \in \mu\chi_M$. □

Corollary 3.2. *Assume that M is a faithful L-fuzzy multiplication R-module and μ is a primary L-fuzzy ideal of R such that $\chi_M \neq \mu\chi_M$. Then $\mu\chi_M$ is a primary L-fuzzy submodule of M.*

Proof. Let μ be a primary L-fuzzy ideal of R and M be a faithful L-fuzzy multiplication R-module. If $r_a x_b \in \mu\chi_M$, for $r \in R$ and $x \in M$, then by Theorem 3.9, $r_a \in \sqrt{\mu} \subseteq \sqrt{(\mu\chi_M : \chi_M)}$ or $x_b \in \mu\chi_M$. Thus, $\mu\chi_M$ is a primary L-fuzzy submodules of M. □

4. Conclusion

In this article, we have defined an L-fuzzy hollow submodule in a different way and some of its properties are investigated. Also, some theorems on L-fuzzy multiplication modules are proved. Thus, this concept of an L-fuzzy multiplication module can be extended to an L-fuzzy fully invariant multiplication modules.

Acknowledgements. The authors are thankful to the referee for helpful suggestions, which improved the paper.

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[1]Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad 431004, India
Email address: sknimbhorkar@gmail.com

[2]Department of Engineering, Sciences and Humanities,
Vishwakarma Institute of Technology,
Pune, 411406, India
Email address: khubchandani_jyoti@yahoo.com

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 433–439.

FAULT-TOLERANT METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS

ALI AHMAD[1], MUHAMMAD A. ASIM[1], AND MARTIN BAČA[2]

Abstract. Metric dimension and fault-tolerant metric dimension of any graph G is subject to size of resolving set. It has become more important in modern GPS and sensors based world as resolving set ensures that in case of semi outage system is still scalable using redundant interfaces. Metric dimension of several interesting classes of graphs have been investigated like Cayley digraphs, Cartesian product of graphs, wheel graphs, convex polytopes and certain networks for categorical product of graphs. In this paper we used the phenomena of barycentric subdivision of graph and proved that fault-tolerant metric dimension of barycentric subdivision of Cayley graph is constant.

1. Introduction

Concept of metric dimension in graph theory was first introduced by Slater [18], Harary and Melter [10] in mid 70's. In a connected graph G, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. *Metric dimension* of any graph G can be defined as $S \subseteq V(G)$ with minimum cardinality where all other vertices of G are uniquely determined by their distances to the vertices in S. A vertex x resolves two vertices u and v if $d(x, u) \neq d(x, v)$, hence minimum cardinality of a resolving set of G is called the metric dimension and is denoted by $\beta(G)$. Similarly a resolving set R is said to be fault-tolerant, if $R\backslash\{x\}$ is also a resolving set for every $x \in R$ that is why *fault-tolerant metric dimension* is the minimum cardinality of a fault-tolerant resolving set of G. The fault-tolerant metric

Key words and phrases. Metric dimension, fault-tolerant metric dimension, barycentric subdivision, Cayley graph.

2010 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C76.

DOI 10.46793/KgJMat2403.433A

Received: January 15, 2020.

Accepted: May 10, 2021.

dimension of graph G is denoted by $\beta'(G)$. A fault-tolerant resolving set of order $\beta'(G)$ is also called a fault-tolerant metric basis of G.

Lot of work has been done in the area of metric dimension and has used in different domains of scientific research. Work of Slater on fault-tolerant metric dimension of graphs carried out in different dimensions like resolvability of crystal structures, network analysis, chemical structures of Methylene, mathematical formalization of woven structures and most significant in Fast-Cluster for removing redundant sequences. Concept of metric dimension using radio navigation by considering the vertices as sonar or loran station ruled in last three decades but now its obsolete and is replaced by GPS and sensor and ad-hoc networks. Through fault-tolerant resolving, a system can continue operating somehow even in case of any failure in one or more of its components. In case of semi outage that leads to graceful degradation of service, system tries to act as scalable system by discovering redundant network interfaces. Fault-tolerant metric dimension can support physical connectivity and link discovery in distributed network based systems.

Metric dimension of several interesting classes of graphs have been investigated: Johnson and Kneser graph [2], Grassmann graphs [3], Cayley digraphs [7] and Cartesian product of graphs [5]. Siddiqui et al. [17] investigated the metric dimension of some infinite families of wheel-related graphs. Kratica et al. [16] studied the metric dimension problem of convex polytopes. Imran et al. [13] studied further the metric dimension of convex polytopes generated by wheel-related graphs. Ahmad et al. [1] studied the metric dimension of Cayley graph of certain finite groups. Vetrik et al. [19] studied the metric dimension problem for certain networks which can be obtained as the categorical product of graphs. In [4], it has been shown that metric dimension of a graphs is not necessarily a finite natural number. They proved that some infinite graphs have infinite metric dimension. The computational complexity of these problems is studied in [8]. Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor memory pairs, each of which is called a processing vertex. Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips.

By inserting a new vertex at any edge to split it into two equi-halves this phenomena is known as edge subdivision. If edge subdivision is applied on multiple edges then it is called graph subdivision, whereas if all edges are subdivided then it is called barycentric subdivision of graph. Gross and Yellen [9] explained nice properties that barycentric subdivided graph will be bipartite, loopless and any loopless graph will be simple as well. Gary and Johnson [8] put an argument that problem of determining $\beta(G) < k$ is NP-Complete problem. In this paper we determined that fault-tolerant metric dimension of barycentric subdivision of Cayley graph is constant and four vertices are sufficient to resolve all the vertices of graph.

2. Results

Let P_n be a path of n vertices, Chartrand et al. [6] determined the metric dimension in the following theorem.

Theorem 2.1 ([6]). *A connected graph G has metric dimension 1 if and only if $G \cong P_n$.*

By considering the two endpoints of the path, the fault-tolerant metric basis obtained. It is easy to observe that $\beta(P_n) = 1$ and $\beta'(P_n) = 2$, for path P_n, $n \geq 2$. From this result and the definition of the fault-tolerant metric dimension the following inequality holds

$$\beta'(G) \geq \beta(G) + 1.$$

Javaid et al. [14] proved in the following theorem that the difference between metric dimension and fault-tolerant of a graph can be arbitrary large.

Theorem 2.2 ([14]). *For every positive integer n, there exists a graph such that $\beta'(G) - \beta(G) \geq n$.*

Let SG be a semigroup, and let H be a nonempty subset of SG. The Cayley graph $Cay(SG, H)$ of SG relative to H is defined as the graph with vertex set SG and edge set $E(SG)$ consisting of those ordered pairs (x, y) such that $hx = y$ for some $h \in H$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $Cay(SG, H)$ of a group SG is symmetric or undirected if and only if $H = H^{-1}$.

The Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$, $n \geq 3$, $m \geq 2$, is a graph which can be obtained as the Cartesian product $P_m \square C_n$ of a path on m vertices with a cycle on n vertices. The vertex set and the edge set of $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$ are defined as: $V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = \{(a_s, b_t) : 1 \leq s \leq n, 1 \leq t \leq m\}$ and $E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = \{(a_s, b_t)(a_{s+1}, b_t) : 1 \leq s \leq n, 1 \leq t \leq m\} \cup \{(a_s, b_t)(a_s, b_{t+1}) : 1 \leq s \leq n, 1 \leq t \leq m-1\}$. We have $|V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))| = mn$, $|E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))| = (2m-1)n$, where $|V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))|$, $|E(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))|$ denote the number of vertices, edges of the Cayley graphs $Cay(Z_n \oplus Z_m)$, respectively.

The metric dimension of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has been determined in [5] while the metric dimension of Cayley graphs $Cay(\mathbb{Z}_n : H)$ for all $n \geq 7$ and $H = \{\pm 1, \pm 3\}$ has been determined in [15].

The barycentric subdivision graph $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ can be obtained by adding a new vertex (c_s, d_t) between (a_s, b_t) and (a_{s+1}, b_t) and adding a new vertex (u_s, v_t) between (a_s, b_t) and (a_s, b_{t+1}). Clearly, $S(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ has $3nm - n$ vertices and $4nm - 2n$ edges.

The metric dimension of $P_m \square C_n$ has been determined in [5] and Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ is actually the Cartesian product of $P_2 \square C_n$. In the next theorem, we prove that the fault-tolerant metric dimension of barycentric subdivision $BS(Cay(Z_n \oplus Z_m))$ is constant and only four vertices appropriately chosen suffice to resolve all the vertices of the $BS(Cay(Z_n \oplus Z_m))$ for $n \geq 6$ and $m \geq 2$.

Theorem 2.3. *Let $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ be the barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$. Then the fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is 4 for $n \geq 6$ and $m \geq 2$.*

Proof. Theorem will be proved for equality using double inequality.

Case 1: $n \equiv 0 \pmod 2$. Let

$$R = \{(a_1, b_1), (a_2, b_1), (a_{\frac{n}{2}+1}, b_1), (a_n, b_1)\} \subseteq V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)))$$

that shows R is a fault-tolerant resolving set for this case. With respect to R a representation for the vertices of BS(Cay($\mathbb{Z}_n \oplus \mathbb{Z}_m$)) is as follows.

For $1 \leq t \leq m$,

$$\gamma((a_s, b_t)|R) = \begin{cases} (2t-2, 2t, 2t+n-2, 2t), & \text{for } s = 1, \\ (2t, 2t-2, 2t+n-4, 2t+2), & \text{for } s = 2, \\ (2t+2s-2, 2t+2s-6, 2t-2s+n, 2t+2s-2), & \text{for } 3 \leq s \leq \frac{n}{2}, \\ (2t+n-2, 2t+n-4, 2t-2, 2t+n-4), & \text{for } s = \frac{n}{2}+1, \\ (2t-2s+2n, 2t-2s+2n+2, 2t+2s-n-4, & \\ 2t-2s+2n-2), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m$,

$$\gamma((c_s, d_t)|R) = \begin{cases} (2t-1, 2t-1, 2t+n-3, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-1, 2t+2s-1), & \text{for } 2 \leq s \leq \frac{n}{2}-1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-1, 2t+2s-3), & \text{for } s = \frac{n}{2}, \\ (2t+n-3, 2t+n-3, 2t-1, 2t+n-5), & \text{for } s = \frac{n}{2}+1, \\ 2t-2s+2n-1, 2t-2s+2n+1, 2t+2s-n-3, & \\ 2t-2s+2n-3), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m-1$,

$$\gamma((u_s, v_t)|R) = \begin{cases} (2t-1, 2t+1, 2t+n-1, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n+1, 2t+2s-1), & \text{for } 2 \leq s \leq \frac{n}{2}, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n+1, 2t+2s-5), & \text{for } s = \frac{n}{2}+1, \\ (2t-2s+2n+1, 2t-2s+2n+3, 2t+2s-n-3, & \\ 2t-2s+2n-1), & \text{for } \frac{n}{2}+2 \leq s \leq n. \end{cases}$$

These vertex representation are distinct, so R is the fault-tolerant resolving set of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$. Therefore fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is less than equal to 4 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \leq 4$.

Imran [12] showed that metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$ is 3, for $m = 2$ and Ahmad et al. [1] proved that metric dimension of

$BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)) = 3$ for $m \geq 3$, therefore the fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ must be greater than 3 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \geq 4$. Hence proved that fault-tolerant metric dimension is $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) = 4$ for $n \geq 6$ and $m \geq 2$.

Case 2: $n \equiv 1 \pmod 2$. Let

$$R = \{(a_1, b_1), (a_2, b_1), (a_{\lceil \frac{n}{2} \rceil}, b_1), (a_n, b_1)\} \subseteq V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)))$$

that shows R is a fault-tolerant resolving set for this case. With respect to R a representation for the vertices of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is as follows.

For $1 \leq t \leq m$,

$$\gamma((a_s, b_t)|R) = \begin{cases} (2t-2, 2t, 2t+n-3, 2t), & \text{for } s = 1, \\ (2t, 2t-2, 2t+n-5, 2t+2), & \text{for } s = 2, \\ (2t+2s-4, 2t+2s-6, 2t-2s+n-1, & \\ 2t+2s-2), & \text{for } 3 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-3, 2t+n-5, 2t-2, 2t+n-3), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t+n-3, 2t+n-3, 2t, 2t+n-5), & \text{for } s = \lceil \frac{n}{2} \rceil + 1, \\ (2t-2s+2n, 2t-2s+2n+2, & \\ 2t+2s-n-3, 2t-2s+2n-2), & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq s \leq n. \end{cases}$$

For $1 \leq t \leq m$,

$$\gamma((c_s, d_t)|R) = \begin{cases} (2t-1, 2t-1, 2t+n-4, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n-2, & \\ 2t+2s-1), & \text{for } 2 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-2, 2t+n-4, 2t-1, 2t+n-4), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t-2s+2n-1, 2t-2s+2n+1, & \\ 2t+2s-n-2, 2t-2s+2n-3), & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq s \leq n-1, \\ (2t-1, 2t+1, 2t+n-2, 2t-1), & \text{for } s = n. \end{cases}$$

For $1 \leq t \leq m-1$,

$$\gamma((u_s, v_t)|R) = \begin{cases} (2t-1, 2t+1, 2t+n-2, 2t+1), & \text{for } s = 1, \\ (2t+2s-3, 2t+2s-5, 2t-2s+n, 2t+2s-1), & \text{for } 2 \leq s \leq \lceil \frac{n}{2} \rceil - 1, \\ (2t+n-2, 2t+n-4, 2t-1, 2t+n-2), & \text{for } s = \lceil \frac{n}{2} \rceil, \\ (2t+n-2, 2t+n-2, 2t+1, 2t+n-4), & \text{for } s = \lceil \frac{n}{2} \rceil + 1, \\ (2t-2s+2n+1, 2t-2s+2n+3, & \\ 2t+2s-n-2, 2t-2s+2n-1), & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq s \leq n. \end{cases}$$

These vertex representations are distinct, so R is the fault-tolerant resolving set of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$. Therefore, fault-tolerant metric dimension of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))$ is less than or equal to 4 that means $\beta'(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m))) \leq 4$.

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Acknowledgements. The research for this article was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-19-0153 and by VEGA 1/0233/18.

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[1]College of Computer Science & Information Technology,
Jazan University,
Jazan, Saudi Arabia
Email address: `ahmadsms@gmail.com`
Email address: `mr.ahsan.asim@gmail.com`

[2]Department of Applied Mathematics and Informatics,
Technical University,
042 00 Košice, Slovak Republic
Email address: `martin.baca@tuke.sk`

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 48(3) (2024), PAGES 441–451.

A NEW FIXED POINT RESULT IN GRAPHICAL $b_v(s)$-METRIC SPACE WITH APPLICATION TO DIFFERENTIAL EQUATIONS

PRAVIN BARADOL[1], DHANANJAY GOPAL[1], AND NADA DAMLJANOVIĆ[2]

ABSTRACT. In the present paper, motivated by [13, 15], first we give a notion of graphical $b_v(s)$-metric space, which is a graphical version of $b_v(s)$-metric space. Utilizing the graphical Banach contraction mapping we prove fixed point results in graphical $b_v(s)$-metric space. Appropriate examples are also presented to support our results. In the end, the main result ensures the existence of a solution for an ordinary differential equation along with its boundary conditions by using the fixed point result in graphical $b_v(s)$-metric space.

1. INTRODUCTION AND PRELIMINARIES

After the most renowned Banach contraction principle stated by Banach [3], many authors have provided more general and innovative contraction mappings on a complete metric space and established fixed point results, see [7,8,10,12]. On the contrary, some authors generalize the concept of metric space by introducing more general conditions instead of triangular inequality. Couple of them are given below.

Definition 1.1 ([2, 11])**.** Let $s \geq 1$ be a given real number and X be a non-empty set. A b-metric on X is a mapping $\rho : X \times X \to [0, +\infty)$ such that for all $a, b, c \in X$ it satisfies the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$;
(iii) $\rho(a, c) \leq s[\rho(a, b) + \rho(b, c)]$.

Then (X, ρ) is called a b-metric space with coefficient s.

Key words and phrases. Graph, fixed point, graphical $b_v(s)$ metric, graphic Banach contraction.
2020 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54H25, 05C12.
DOI 10.46793/KgJMat2403.441B
Received: September 20, 2020.
Accepted: May 10, 2021.

Definition 1.2 ([6])**.** Let a mapping $\rho : X \times X \to [0, +\infty)$ defined on a non-empty set X satisfy:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) $\rho(a, b) \leq \rho(a, c) + \rho(c, d) + \rho(d, b)$ for all $a, b \in X$ and all distinct points $c, d \in X \backslash \{a, b\}$.

Then ρ is called a rectangular metric on X and (X, ρ) is called a rectangular metric space.

Definition 1.3 ([6])**.** Let a mapping $\rho : X \times X \to [0, +\infty)$ be defined on a non-empty set X. For $v \in \mathbb{N}$, (X, ρ) is said to be a v-generalized metric space, if the following hold:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$, for all $a, b \in X$;
(iii) $\rho(a, b) \leq \rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)$ for all $a, u_1, u_2, \ldots, u_v, b \in X$ such that $a, u_1, u_2, \ldots, u_v, b$ are all different.

For more details on b-metric spaces and their generalizations we refer to [1]. Recently, Mitrović and Radenović [13] introduced the concept of $b_v(s)$-metric space as follows.

Definition 1.4. Let a mapping $\rho : X \times X \to [0, +\infty)$ be defined on a non-empty set X. For $v \in \mathbb{N}$, (X, ρ) is said to be a $b_v(s)$-metric space, if the following hold:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) there exists a real number $s \geq 1$ such that
$$\rho(a, b) \leq s[\rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)],$$
for distinct points $a, u_1, u_2, \ldots, u_v, b \in X$.

Let X be a non-empty set and $\Delta = \{(x, x) : x \in X\}$. A graph $\mathbb{G}$ is an ordered pair $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$, where $\mathbb{V}(\mathbb{G})$ is the set of vertices of graph $\mathbb{G}$ and $\mathbb{E}(\mathbb{G}) \subseteq \mathbb{V}(\mathbb{G}) \times \mathbb{V}(\mathbb{G})$ is the set of edges of graph $\mathbb{G}$. In this paper, we will use the concept of graph structure on metric space (namely, a graphical metric space) that has been introduced by Shukla et al. [15], in which the non-empty set X is associated with a graph $\mathbb{G}$ by considering the set X as the set of vertices, i.e., $\mathbb{V}(\mathbb{G}) = X$ and allowing that the set of edges $\mathbb{E}(\mathbb{G})$ contains the set Δ of all loops (which are edges that join a vertex to itself), i.e., $\Delta \subset \mathbb{E}(\mathbb{G})$. The sequence of vertices $\{t_i\}_{i=0}^{l}$ such that $t_0 = a$, $t_l = b$ and $(t_{i-1}, t_i) \in \mathbb{E}(\mathbb{G})$ for $j = 1, 2, \ldots, l$, represents the directed path from a to b of length l in the graph $\mathbb{G}$, in short it is written as $(aPb)^l_{\mathbb{G}}$. If the vertex $c \in X$ lies on the path from a to b, then we use notation $c \in (aPb)_{\mathbb{G}}$. A connected graph $\mathbb{G}$ states that there is a path between each pair vertices of the graph. A sequence $\{x_n\}$ is said to be $\mathbb{G}$-termwise connected, if for each $n \in \mathbb{N}$ there is a path from x_n to x_{n+1} in the graph $\mathbb{G}$. For $l \in \mathbb{N}$, let $[a]^l_{\mathbb{G}}$ be the set defined by
$$[a]^l_{\mathbb{G}} = \{b \in X : (aPb)^l_{\mathbb{G}}\}.$$

A connected component of the graph $\mathbb{G}$ is a connected sub-graph $\mathbb{G}_1$ of a graph $\mathbb{G}$ such that there is no path between vertices of $\mathbb{G}_1$ and vertices of $\mathbb{G}\backslash\mathbb{G}_1$. For more information we refer reader to [4, 5, 9, 16, 17].

Definition 1.5 ([15])**.** Let X be a non-empty set and $\mathbb{G}$ be a graph associated with X. A graphical metric on X is a mapping $\rho : X \times X \to [0, +\infty)$ satisfying the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) for all $a, b, c \in X$ such that $(aPb)_{\mathbb{G}}$ and $c \in (aPb)_{\mathbb{G}}$ implies $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$,

and the pair (X, ρ) is called graphical metric space.

In this paper, we introduce the concept of graphical $b_v(s)$-metric space, which is graphical version of $b_v(s)$-metric space. Graphical $b_v(s)$-metric space is a generalization of $b_v(s)$-metric space and graphical metric space. In the rest of the paper, all the graphs are directed unless otherwise stated and the set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

Definition 1.6. Let $\mathbb{G}$ be a graph associated with a non-empty set X. For $v \in \mathbb{N}$, a mapping $\rho : X \times X \to [0, +\infty)$ is said to be a graphical $b_v(s)$-metric, if it satisfies the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) for distinct $u_1, u_2, \ldots, u_v \in (aPb)_{\mathbb{G}}$ and a real number $s \geq 1$ holds

$$\rho(a, b) \leq s[\rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)],$$

and the pair (X, ρ) is called graphical $b_v(s)$-metric space.

By Definition 1.1–1.6 and [9, 15, 17] it is easy to verify that the following hold.

(i) Graphical $b_1(1)$-metric space is graphical metric space.
(ii) Graphical $b_1(s)$-metric space is b-metric space with coefficient s.
(iii) Graphical $b_2(1)$-metric space is graphical rectangular metric space.
(iv) Graphical $b_2(s)$-metric space is graphical rectangular b-metric space with coefficient s.

Remark 1.1. Every $b_v(s)$-metric space (X, ρ) is a graphical $b_v(s)$-metric space endowed with a graph $\mathbb{G}$ having $\mathbb{E}(\mathbb{G}) = X \times X$, but converse need not be true.

Example 1.1. *Let* $X = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ *and let* $\mathbb{G} = G_1 \cup G_2 \cup G_3$ *be an undirected graph, where* G_1, G_2 *and* G_3 *are connected components with:*

$$\mathbb{V}(G_1) = \{v_1, v_2\}, \qquad \mathbb{E}(G_1) = \{e_1\},$$
$$\mathbb{V}(G_2) = \{v_3, v_4, v_5, v_6, v_7, v_8, v_9\}, \qquad \mathbb{E}(G_2) = \{e_2, e_3, e_4, e_5, e_6, e_7\},$$
$$\mathbb{V}(G_3) = \{v_{10}, v_{11}\}, \qquad \mathbb{E}(G_3) = \{e_8\}.$$

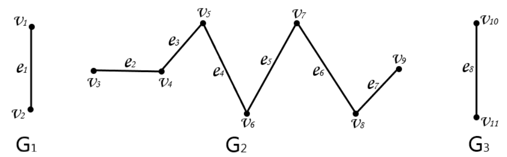

FIGURE 1. $\mathbb{G} = G_1 \cup G_2 \cup G_3$

Let $\rho : X \times X \to [0, +\infty)$ be a mapping defined in a following way:

$$\rho(v_i, v_j) = \begin{cases} 0, & \text{if } v_i = v_j, \\ l_{v_i v_j}, & \text{if } v_i, v_j \in G_k,\ k = \{1, 2, 3\}, \\ 1, & \text{otherwise}, \end{cases}$$

where $l_{v_i v_j}$ denote the length of the shortest path from v_i to v_j. Then (X, ρ) is graphical $b_4(1)$-metric space but not $b_4(1)$-metric space.

Definition 1.7. *Let (X, ρ) be a graphical $b_v(s)$-metric space endowed with a graph $\mathbb{G}$ and let $\{y_n\}$ be a sequence of elements in X. Then $\{y_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ exists $m \in \mathbb{N}$ such that $\rho(y_k, y_l) < \epsilon$, for all $k, l \geq m$, i.e., $\lim_{k,l \to +\infty} \rho(y_k, y_l) = 0$. The sequence $\{y_n\}$ converges to $z \in X$, if for each $\epsilon > 0$, exists $m \in \mathbb{N}$ such that $\rho(y_k, z) < \epsilon$ for all $k \geq m$, i.e., $\lim_{k \to +\infty} \rho(y_k, z) = 0$.*

Definition 1.8. *If every $\mathbb{G}$-termwise connected (briefly, $\mathbb{G}$-TWC) Cauchy sequence in a graphical $b_v(s)$-metric space X endowed with a graph $\mathbb{G}$ converges in X, then X is said to be $\mathbb{G}$-complete.*

2. MAIN RESULTS

First, we provide a definition of Banach contraction mapping in graphical $b_v(s)$ metric space.

Definition 2.1. Let $\mathbb{G}$ be a graph associated with graphical $b_v(s)$-metric space (X, ρ). A graphic Banach contraction (GBC) on X is a mapping $F : X \to X$ such that:

(GBC-1) $(Fa, Fb) \in \mathbb{E}(\mathbb{G})$ whenever $(a, b) \in \mathbb{E}(\mathbb{G})$;

(GBC-2) for all $(a, b) \in \mathbb{E}(\mathbb{G})$, there exists $\eta \in [0, 1)$, such that $\rho(Fa, Fb) \leq \eta \rho(a, b)$.

Remark 2.1. *Every Banach contraction on a non-empty set X is a graphic Banach contraction on X after considering the set of edges is equal to $X \times X$. But, converse is not always true (see Example* 2.1*).*

Definition 2.2. *A graph $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ is said to have property $(\mathcal{P})$ if for each convergent $\mathbb{G}$-TWC F-Picard sequence $\{x_n\}$ there exists $m \in \mathbb{N}$ and a limit ξ of $\{x_n\}$ in X, such that $(x_k, \xi) \in \mathbb{E}(\mathbb{G})$ or $(\xi, x_k) \in \mathbb{E}(\mathbb{G})$ for all $k > m$.*

Theorem 2.1. *Let (X, ρ) be a $\mathbb{G}$-complete graphical $b_v(s)$-metric space and let $F : X \to X$ be an injective GBC on X. Suppose that the following conditions are satisfied.*

(i) *There exists* $a_0 \in X$ *with* $F^k a_0 \in [a_0]_{\mathbb{G}}^{l_k}$ *for* $k = 1, 2, \ldots, v$*, where* $l_k = m_k v + 1$ *and* $m_k \in \mathbb{N}^0$*.*

(ii) $\mathbb{G}$ *has the property* $(\mathcal{P})$*.*

Then, for initial term $a_0 \in X$ *the* F*-Picard sequence* $\{a_n\}$ *is* $\mathbb{G}$*-TWC and converges to both* ξ^* *and* $F\xi^*$ *in* X*.*

Proof. For $k = 1, 2, \ldots, v$, let $a_0 \in X$ be such that $F^k a_0 \in [a_0]_{\mathbb{G}}^{l_k}$, where $l_k = m_k v + 1$ and $m_k \in \mathbb{N}^0$. Then there exists a path $\{e_j^k\}_{j=0}^{l_k}$ such that

$$a_0 = e_0^k\ , \quad F^k a_0 = e_{l_k}^k \quad \text{and} \quad (e_{j-1}^k, e_j^k) \in \mathbb{E}(\mathbb{G}), \quad \text{for all } j = 1, 2, \ldots, l_k.$$

Since $(e_{j-1}^k, e_j^k) \in \mathbb{E}(\mathbb{G})$, by $(\mathrm{GBC}-1)$ we have

$$(Fe_{j-1}^k, Fe_j^k) \in \mathbb{E}(\mathbb{G}), \quad \text{for } j = 1, 2, \ldots, l_k.$$

Therefore, $\{Fe_j^k\}_{j=0}^{l_k}$ is a path from $Fe_0^k = Fa_0 = a_1$ to $Fe_{l_k}^k = F^2 a_0 = a_2$ of length l_k. Similarly, for all $n \in \mathbb{N}$, $\{F^n e_j^1\}_{j=0}^{l_1}$ is a path from $F^n e_0^1 = F^n a_0 = a_n$ to $F^n e_{l_1}^1 = F^n F a_0 = a_{n+1}$ of length l_1. Thus, $\{a_n\}$ is $\mathbb{G}$-TWC sequence.

Therefore, for each $k = 1, 2, \ldots, v$, and $j = 1, 2, \ldots, l_k$, we have $(F^n e_{j-1}^k, F^n e_j^k) \in \mathbb{E}(\mathbb{G})$, for all $n \in \mathbb{N}$.

By $(GBC - 2)$, for $j = 1, 2, \ldots, l_k$, we have

$$\rho\left(F^n e_{j-1}^k, F^n e_j^k\right) \le \eta \rho\left(F^{n-1} e_{j-1}^k, F^{n-1} e_j^k\right) \le \cdots \le \eta^n \rho\left(e_{j-1}^k, e_j^k\right). \tag{2.1}$$

By condition (iii) of Definition 1.6, for $k = 1, 2, \ldots, v$ we have

$$\begin{aligned} \rho(a_0, a_k) =& \rho(e_0^k, e_{l_k}^k) \\ \le& s[\rho(e_0^k, e_1^k) + \rho(e_1^k, e_2^k) + \cdots + \rho(e_{v-1}^k, e_v^k)] \\ &+ s^2[\rho(e_v^k, e_{v+1}^k) + \rho(e_{v+1}^k, e_{v+2}^k) + \cdots + \rho(e_{2v-1}^k, e_{2v}^k)] \\ &\vdots \\ &+ s^{m_k}[\rho(e_{(m_k-1)v}^k, e_{(m_k-1)v+1}^k) + \cdots + \rho(e_{l_k-1}^k, e_{l_k}^k)] \\ =& D_{l_k}. \end{aligned} \tag{2.2}$$

On the same way, using the inequalities (2.1) and (2.2), for $k = 1, 2, \ldots, v$, we have

$$\begin{aligned} \rho(a_n, a_{n+k}) =& \rho(F^n a_0, F^n a_k) = \rho(F^n e_0^k, F^n e_{l_k}^k) \\ \le& s[\rho(F^n e_0^k, F^n e_1^k) + \rho(F^n e_1^k, F^n e_2^k) + \cdots + \rho(F^n e_{v-1}^k, F^n e_v^k)] \\ &+ s^2[\rho(F^n e_v^k, F^n e_{v+1}^k) + \rho(F^n e_{v+1}^k, F^n e_{v+2}^k) + \cdots + \rho(F^n e_{2v-1}^k, F^n e_{2v}^k)] \\ &\vdots \\ &+ s^{m_k}[\rho(F^n e_{(m_k-1)v}^k, F^n e_{(m_k-1)v+1}^k) + \cdots + \rho(F^n e_{l_k-1}^k, F^n e_{l_k}^k)] \\ \le& \eta^n D_{l_k}. \end{aligned} \tag{2.3}$$

Now, from equation (2.3) one can prove Cauchy-ness of the sequence $\{x_n\}$, i.e., for all $p \ge 1$, $\rho(x_n, x_{n+p}) \to 0$ as $n \to +\infty$.

Therefore, $\mathbb{G}$-completeness of X implies $a_n \to \xi^*$ for some $\xi^* \in X$. Thanks to Property ($\mathcal{P}$), that ensures that there exists $k \in \mathbb{N}$ such that $(a_n, \xi^*) \in \mathbb{E}(\mathbb{G})$ or $(\xi^*, a_n) \in \mathbb{E}(\mathbb{G})$ for all $n > k$.

Assume that, for all $n > k$, $(a_n, \xi^*) \in \mathbb{E}(\mathbb{G})$, then by $(BGC-2)$,

$$\rho(Fa_n, F\xi^*) \leq \eta\rho(a_n, \xi^*), \quad \text{for all } n > k.$$

This implies

$$\rho(Fa_n, F\xi^*) \to 0, \quad \text{as} \quad n \to +\infty,$$

i.e., $a_{n+1} \to F\xi^*$. So, $F\xi^*$ is also a limit of $\{a_n\}$.

Analogously, we can prove the case $(\xi^*, a_n) \in \mathbb{E}(\mathbb{G})$ for all $n > k$. □

Remark 2.2. In $b_v(s)$-metric space a sequence may converges to more than one limit, and hence this result also holds in graphical $b_v(s)$-metric space. To remove this difficulty some authors use Housdorff-ness condition on such metric space.

Definition 2.3. Let a graph $\mathbb{G}$ is associated with a graphical $b_v(s)$-metric space (X, ρ) and a mapping F is a graphic Banach contraction on X. The quadruplet $(X, \rho, \mathbb{G}, F)$ has the property S^*, if each $\mathbb{G}$-TWC F-Picard sequence $\{a_n\}$ in X has a unique limit.

Theorem 2.2. *Let the conditions of Theorem* 2.1 *hold along with that the quadruple* $(X, \rho, \mathbb{G}, F)$ *has the Property* S^**, then* F *has a fixed point.*

Proof. From the proof of Theorem 2.1 and the Property S^*, we have $F\xi^* = \xi^*$. □

Theorem 2.3. *Let the conditions of Theorem* 2.2 *hold and suposse that for all* $\xi^*, \zeta^* \in \text{Fix(F)}$ *there exists a path* $(\xi^* P \zeta^*)^t_{\mathbb{G}}$ *between* ξ^* *and* ζ^* *of length* t*, where* $t = 1$ *or* $t = mv + 1$ *for* $m \in \mathbb{N}^0$*. Then* F *has a unique fixed point.*

Proof. Let suposse that for all $\xi^*, \zeta^* \in \text{Fix(F)}$ there exists a path $(\xi^* P \zeta^*)^t_{\mathbb{G}}$ between ξ^* and ζ^* of length t.

Case I. If $t = 1$, then $(\xi^*, \zeta^*) \in \mathbb{E}(\mathbb{G})$, which implies $(F\xi^*, F\zeta^*) \in \mathbb{E}(\mathbb{G})$ by condition (GBC-1).

Now, by (GBC-2), we have $\rho(F\xi^*, F\zeta^*) \leq \eta[\rho(\xi^*, \zeta^*)]$, which implies $\rho(\xi^*, \zeta^*) \leq \eta[\rho(\xi^*, \zeta^*)]$. This contradicts the fact $\eta < 1$. Hence, $\xi^* = \zeta^*$.

Case II. Let $t = mv + 1$, where $m \in \mathbb{N}^0$ and let $\{e_i\}^t_{i=0}$ be the path from ξ^* to ζ^*, such that $e_0 = \xi^*$ and $e_k = \zeta^*$. Then

$$\begin{aligned}
\rho(\xi^*, \zeta^*) =&\rho(F^n\xi^*, F^n\zeta^*)\\
\leq& s[\rho(F^n e_0, F^n e_1) + \rho(F^n e_1, F^n e_2) + \cdots + \rho(F^n e_{v-1}, F^n e_v)]\\
&+ s^2[\rho(F^n e_v, F^n e_{v+1}) + \rho(F^n e_{v+1}, F^n e_{v+2}) + \cdots + \rho(F^n e_{2v-1}, F^n e_{2v})]\\
&\vdots\\
&+ s^m[\rho(F^n e_{(m-1)v}, F^n e_{(m-1)v+1}) + \cdots + \rho(F^n e_{t-1}, F^n e_t)]\\
\to& 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

This implies $\xi^* = \zeta^*$. □

Example 2.1. Let $A = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and a set $X = \{0,1\} \cup A$ is associated with graph $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ such that $\mathbb{V}(\mathbb{G}) = X$, $\mathbb{E}(\mathbb{G}) = \Delta \cup \{(0, \frac{1}{2^n}) : n \in \mathbb{N}\} \cup \{(\frac{1}{2^n}, \frac{1}{2^m}) \in X \times X : n, m \in \mathbb{N}, n < m\}$. Let a symmetric function $\rho : X \times X \to [0, +\infty)$ such that

$$\rho(a,b) = \begin{cases} 0, & \text{if } a = b, \\ b, & \text{if } a = 0,\ b \in A, \\ a, & \text{if } a \in A,\ b = 1, \\ M, & \text{if } a, b \in A, \\ 1, & \text{if } a = 0,\ b = 1, \end{cases}$$

where $M = \max\{a, b\}$. Then (X, ρ) is $\mathbb{G}$-complete graphical $b_4(1)$-metric space. Now, $F : X \to X$ be a function defined as:

$$Fx = \begin{cases} \frac{x^5}{2}, & \text{if } x \in [0,1), \\ 1, & \text{otherwise.} \end{cases}$$

Then, the mapping F satisfies all the conditions of Theorem 2.2 and having contraction constant $\eta = \frac{1}{2^5}$. Hence, 0 is the unique fixed point.

Remark 2.3. In Example 2.1, we observed that:

(i) the mapping F is graphic Banach contraction on X but not a Banach contraction;
(ii) (X, ρ) is not a $b_4(1)$-metric space.

3. An Application to Differential Equation

In this section, inspired by [14], we establish the existence of solution for the following second order ordinary differential equation:

$$-\frac{d^2y}{dt^2} = h(t, y(t)) \tag{3.1}$$

having boundary conditions $y(0) = y(1) = 0$, where $h : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

Let $X = C([0,1], \mathbb{R})$ be the set of all real-valued continuous functions defined on $[0,1]$. Let's define a set K as:

$$K = \left\{ y \in X : \inf_{t \in [0,1]} y(t) > 0 \text{ and } y(t) \le 1,\ t \in [0,1] \right\}.$$

Now, to define a graph structure $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ on X, lets consider $\mathbb{V}(\mathbb{G}) = X$ and

$$\begin{aligned} \mathbb{E}(\mathbb{G}) &= \Delta \cup \{(a,b) \in K \times K : a(t) \le b(t) \text{ for all } t \in [0,1]\} \\ &= \{(a,a) : a \in X\} \cup \{(a,b) \in K \times K : a(t) \le b(t) \text{ for all } t \in [0,1]\}. \end{aligned}$$

Define the mapping $\rho : X \times X \to \mathbb{R}^+$ as:

$$\rho(a,b) = \sup_{0 \le t \le 1} |a(t) - b(t)|,$$

for all $a, b \in X$. Then, (X, ρ) is the $\mathbb{G}$-complete graphical $b_3(1)$-metric space. The problem defined in (3.1) with given boundary condition is equivalent to the following Fredholm integral equation:

$$y(t) = \int_0^1 H(t,s)h(s,y(s))ds, \tag{3.2}$$

where

$$H(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Consider an injective mapping $F : X \to X$ defined as:

$$Fy(t) = \int_0^1 H(t,s)h(s,y(s))ds.$$

Then the fixed point of F is a solution of integral equation (3.2).

Theorem 3.1. *Suppose the following assumptions hold.*

(i) $\psi \in C([0,1], \mathbb{R})$ *is the lower solution of equation* (3.2)*, i.e.,*

$$\psi(t) \leq \int_0^1 H(t,s)h(s,\psi(s))ds.$$

(ii) *A function* $h(t,\cdot) : \mathbb{R} \to \mathbb{R}^+$ *is increasing on* $[0,1]$*. In addition,* $h(t,1) = t$ *and* $\inf_{t\in[0,1]} H(t,s) > 0$.

(iii) *For every* $t \in [0,1]$*, holds*

$$|h(s,x(s)) - h(s,y(s))| \leq |x(s) - y(s)|.$$

Then the existence of solution for equation (3.2) *provides a solution for* (3.1).

Proof. Clearly, the mapping $F : X \to X$ is well defined. Now, to prove F is GBC on X, we consider $(a,b) \in \mathbb{E}(\mathbb{G})$, i.e., $a, b \in K$ and $a(t) \leq b(t)$ for all $t \in [0,1]$. Now, the following holds

$$Fa(t) = \int_0^1 H(t,s)h(s,a(s))ds \leq \int_0^1 H(t,s)h(s,1)ds = \int_0^1 H(t,s)sds = \frac{4}{27\sqrt{3}} \leq 1,$$

and from condition (ii), it is obvious that $\inf_{t\in[0,1]} Fa(t) > 0$. This implies $Fa(t) \in K$. Similarly, we can prove this for $b(t) \in K$.

Since, $h(t,\cdot) : \mathbb{R} \to \mathbb{R}^+$ is increasing on [0,1], we have

$$Fa(t) = \int_0^1 H(t,s)h(s,a(s))ds \leq \int_0^1 H(t,s)h(s,b(s))ds = Fb(t).$$

It gives that $(Fa(t), Fb(t)) \in \mathbb{E}(\mathbb{G})$.

Now, for each $t \in [0,1]$, we have

$$\begin{aligned}
|Fa(t) - Fb(t)| &= \left| \int_0^1 H(t,s)h(s,a(s))ds - \int_0^1 H(t,s)h(s,b(s))ds \right| \\
&\leq \int_0^1 H(t,s)\Big|h(s,a(s)) - h(s,b(s))\Big|ds \\
&\leq \int_0^1 H(t,s)|a(s) - b(s)|ds \\
&\leq \sup_{t\in[0,1]} \int_0^1 H(t,s)|a(s) - b(s)|ds \\
&\leq \sup_{t\in[0,1]} |a(t) - b(t)| \int_0^1 H(t,s)ds \\
&\leq \frac{1}{8} \sup_{t\in[0,1]} |a(t) - b(t)| \\
&\leq \frac{1}{8}\rho(a,b).
\end{aligned}$$

This implies, $\rho(Fa, Fb) \leq \frac{1}{8}\rho(a,b)$. Note that for all $t \in [0,1]$, $\int_0^1 H(t,s)ds = \frac{t}{2} - \frac{t^2}{2}$ which implies that, $\sup_{t\in[0,1]} \int_0^1 H(t,s)ds = \frac{1}{8}$. Thus, F is GBC on X. From the condition (i), there exists $\psi(t) \in X$ such that $F^k\psi(t) \in [\psi(t)]^1_{\mathbb{G}}$, for each $k = 1,2,3$. It is easy to see that, the condition (I) of the Theorem 2.3 and Property S^* are satisfied. Therefore, Theorem 2.3 guarantees that F has an unique fixed point and hence the integral equation (3.2) has solution in X that ensures the existence of the solution of differential equation (3.1). □

4. Conclusion

In this paper, we initiated the concept of $b_v(s)$-metric space equipped with graph structure. Also, graphic Banach contraction is defined and it is proved that every Banach contraction is a graphic Banach contraction but converse need not be true. Fixed point results are established in aforementioned space. The presented results are validated by suitable examples. In the end, obtained results are utilized to solve ordinary differential equation which show the importance of our work.

Future Scope. One can establish the fixed point results in a graphical $b_v(s)$-metric space by using different contraction mappings, like Meir-Keeler, Kannan, Reich. Moreover, topology of the aforementioned space will helps to know the properties and whole structure of the space.

Acknowledgements. The authors would like to thank the anonymous referee(s) and the editor for their careful reading and useful suggestions on the manuscript.

Research supported by Council of Scientific&Industrial Research, Ministry of Science and Technnology, Republic of India, CSIR Sanction Reference-25(0215)/13/EMR-II; Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174013.

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[1]S.V. National Institute of Technology,
Ichchhanath, Surat, Gujarat,
India
Email address: bardolpr@gmail.com
Email address: gopal.dhananjay@rediffmail.com, dg@ashd.svnit.ac.in

[2]Faculty of Technical Sciences Čačak,
University of Kragujevac,
Serbia
Email address: nada.damljanovic@ftn.kg.ac.rs, nada.damljanovic@gmail.com

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 453–471.

WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF ALMOST CONTACT MANIFOLDS

ION MIHAI[1], SIRAJ UDDIN[2], AND ADELA MIHAI[3,4]

Abstract. B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds. By using the notion of pointwise slant submanifolds, we investigate the geometry of pointwise semi-slant submanifolds and their warped products in Sasakian and cosymplectic manifolds. We prove that there exist no proper pointwise semi-slant warped product submanifold other than contact CR-warped products in Sasakian manifolds. We give non-trivial examples of such submanifolds in cosymplectic manifolds and obtain several fundamental results, including a characterization for warped product pointwise semi-slant submanifolds.

1. Introduction

In [7], B.-Y. Chen introduced the notion of slant submanifolds of almost Hermitian manifolds as a natural generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. Afterwards, the geometry of slant submanifolds became an active topic of research in differential geometry. Later, A. Lotta [19] has extended this study for almost contact metric manifolds. J. L. Cabrerizo et al. investigated slant submanifolds of a Sasakian manifold [6]. N. Papaghiuc introduced in [21] a class of submanifolds, called semi-slant submanifolds of almost Hermitian manifolds, which are the generalizations of slant and CR-submanifolds. Later on, Cabrerizo et al. [5] extended this idea for semi-slant submanifolds of contact metric manifolds and provided many examples of such submanifolds.

Next, as an extension of slant submanifolds of an almost Hermitian manifold, F. Etayo [15] introduced the notion of pointwise slant submanifolds of almost Hermitian

Key words and phrases. Warped products, pointwise slant submanifolds, pointwise semi-slant submanifolds, Sasakian manifolds, cosymplectic manifold.

2010 *Mathematics Subject Classification.* Primary: 53C15, 53C40, 53C42. Secondary: 53B25.

DOI 10.46793/KgJMat2403.453M

Received: March 5, 2021.

Accepted: May 31, 2021.

manifolds. In 2012, B.-Y. Chen and O. J. Garay [13] studied pointwise slant submanifolds of almost Hermitian manifolds. They have obtained several fundamental results, in particular, a characterization of these submanifolds. K. S. Park [22] has extended this study for almost contact metric manifolds. In his definition of pointwise slant submanifolds of almost contact metric manifolds he did not mention whether the structure vector field ξ is either tangent or normal to the submanifold. B. Sahin studied pointwise semi-slant submanifolds and warped product pointwise semi-slant submanifolds by using the notion of pointwise slant submanifolds [25]. In [31] the authors modified the definition of pointwise slant submanifolds of an almost contact metric manifold such that the structure vector field ξ is tangent to the submanifold. We have obtained a simple characterization for such submanifolds and studied warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In 1969, R. L. Bishop and B. O'Neill [3] introduced and studied warped product manifolds. 30 years later, around the beginning of this century, B.-Y. Chen initiated in [8,9] the study of warped product CR-submanifolds of Kaehler manifolds. Chen's work in this line of research motivated many geometers to study the geometry of warped product submanifolds by using his idea for different structures on manifolds (see, for instance [2,16,20,26]). For a detailed survey on warped product submanifolds we refer to Chen's books [10,12] and his survey article [11] as well.

In [23], B. Sahin showed that there exists no proper warped product semi-slant submanifold of Kaehler manifolds. Then, he introduced the notion of warped product hemi-slant submanifolds of Kaehler manifolds [24]. In 2013, he defined and studied warped product pointwise semi-slant submanifolds and showed that there exists a non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kaehler manifold $\tilde{M}$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively [25]. For almost contact metric manifolds, we have seen in [18] and [1] that there are no proper warped product semi-slant submanifolds in cosymplectic and Sasakian manifolds. Then, we have considered warped product pseudo-slant submanifolds (warped product hemi-slant submanifolds [24], in the same sense of almost Hermitian manifolds) of cosymplectic manifolds [28] and Sasakian manifolds [29].

Recently, K. S. Park [22] studied warped product pointwise semi-slant submanifolds of almost contact metric manifolds. He proved that there do not exist warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ in $\tilde{M}$, where $\tilde{M}$ is either a cosymplectic manifold, a Sasakian manifold or a Kenmotsu manifold such that M_θ and M_T are proper pointwise slant and invariant submanifolds of $\tilde{M}$, respectively. Then he provided many examples and obtained several results for warped products by reversing these two factors, including sharp estimations for the squared norm of the second fundamental form in terms of the warping functions. Later, we also extended this idea in [31] to warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In this paper, we study warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$ of Sasakian and cosymplectic manifolds.

The present paper is organized as follows: In Section 2 we give basic definitions and formulae needed for this paper. Section 3 is devoted to the study of pointwise semi-slant submanifolds of almost contact metric manifolds. We define pointwise semi-slant submanifolds and in the definition of pointwise semi-slant submanifolds we assume that the structure vector field ξ is always tangent to the submanifold. We give two non-trivial examples of such submanifolds for the justification of our definition and a result which is useful to the next section. In Section 4 we study warped product pointwise semi-slant submanifolds of Sasakian and cosymplectic manifolds. We prove that there is no proper pointwise semi-slant warped product $M = M_T \times_f M_\theta$ other than contact CR-warped product in Sasakian manifolds, but if we assume the ambient space is cosymplectic then there exists a non-trivial class of such warped products. We obtain several new results which are generalizations of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In Section 5 we provide nontrivial examples of Riemannian product and warped product pointwise semi-slant submanifolds in Euclidean spaces.

2. Preliminaries

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional manifold $\tilde{M}$ is defined by a $(1,1)$ tensor field φ, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying the following conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \eta(\xi) = 1,$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map [4]. There always exists a Riemannian metric g on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any $X, Y \in \Gamma(T\tilde{M})$, the Lie algebra of vector fields on $\tilde{M}$. The metric g is called a *compatible metric* and the manifold $\tilde{M}$ together with the structure (φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.1), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. If ξ is a Killing vector field with respect to g, then the contact metric structure is called a *K-contact structure*. An almost contact metric manifold is called *almost cosymplectic* if $d\eta = 0$ and $d\varphi = 0$, according to D. E. Blair ([4]). In particular, a normal almost cosymplectic manifold is called *cosymplectic* and satisfies

$$\widetilde{\nabla}\varphi = 0, \quad \widetilde{\nabla}\xi = 0. \tag{2.2}$$

A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of φ, the Sasakian condition can be expressed by

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From the formula (2.3), it follows that

$$\tilde{\nabla}_X \xi = -\varphi X, \tag{2.4}$$

for any $X \in \Gamma(T\tilde{M})$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M; the Gauss and Weingarten formulae are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_N is the shape operator of M with respect to the normal vector N. Moreover, $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_N and h are related by [32]

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

For any X tangent to M, we write

$$\varphi X = PX + FX, \tag{2.8}$$

where PX and FX are the tangential and normal components of φX, respectively. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$\varphi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of φN, respectively. Moreover, from (2.1) and (2.8), we have

$$g(PX, Y) = -g(X, PY), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$.

Throughout this paper, we assume that the structure vector field ξ is tangent to M, otherwise M is a C-totally real submanifold [19]. Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *slant* [6] if for each non-zero vector X tangent to M at $p \in M$ such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and on the choice of $X \in T_pM - \langle \xi_p \rangle$.

A slant submanifold is said to be *proper slant* if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$. We note that if $\theta = 0$ then the submanifold is an invariant submanifold and if $\theta = \frac{\pi}{2}$ then it is an anti-invariant submanifold (equivalently, a slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant).

As a natural extension of slant submanifolds, F. Etayo [15] introduced pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. Later on, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and obtained many interesting results [13]. In a similar way, K. S. Park [22] defined and studied pointwise slant submanifols of almost contact metric manifolds. His definition of pointwise slant submanifolds of almost contact metric manifold is similar to the pointwise slant submanifolds of almost Hermitian manifolds, therefore we have modified his definition by considering the structure vector field ξ is tangent to the submanifold and studied pointwise slant submanifolds of almost contact metric manifolds in [31].

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *pointwise slant* if for any nonzero vector X tangent to M at $p \in M$, such that X is not propotional to ξ_p, the angle $\theta(X)$ between φX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in T_p^*M$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold.

We note that every slant submanifold is a pointwise slant submanifold, but the converse may not be true. We also note that a pointwise slant submanifold is *invariant* (respectively, *anti-invariant*) if for each point $p \in M$, the slant function $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$). A pointwise slant submanifold is slant if and only if the slant function θ is constant on M. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor θ is constant.

In [31], the authors have obtained the following characterization theorem.

Theorem 2.1 ([31])**.** *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is pointwise slant if and only if*

$$P^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{2.11}$$

for some real valued function θ defined on the tangent bundle TM of M.

The following relations are immediate consequences of Theorem 2.1.

Let M be a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$. Then we have

$$g(PX, PY) = \cos^2\theta\,[g(X,Y) - \eta(X)\eta(Y)],$$

$$g(FX, FY) = \sin^2\theta\,[g(X,Y) - \eta(X)\eta(Y)],$$

for any $X, Y \in \Gamma(TM)$.

The next useful relation for a pointwise slant submanifold of an almost contact metric manifold was obtained in [31]

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi)\,, \quad fFX = -FPX, \tag{2.12}$$

for any $X \in \Gamma(TM)$.

3. Pointwise Semi-Slant Submanifolds

In [25], B. Sahin defined and studied pointwise semi-slant submanifolds of Kaehler manifolds. In this section, we define and study pointwise semi-slant submanifolds of almost contact metric manifolds.

Definition 3.1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a *pointwise semi-slant submanifold* if there exists a pair of orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ on M such that

(i) the tangent bundle TM admits the orthogonal direct decomposition $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle$;
(ii) the distribution $\mathfrak{D}$ is invariant under φ, i.e., $\varphi(\mathfrak{D}) = \mathfrak{D}$;
(iii) the distribution $\mathfrak{D}^\theta$ is pointwise slant with slant function θ.

Note that the normal bundle $T^\perp M$ of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = F\mathfrak{D}^\theta \oplus \nu, \quad F\mathfrak{D}^\theta \perp \nu,$$

where ν is an invariant normal subbundle of $T^\perp M$ under φ.

We denote the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by m_1 and m_2, respectively. Then the following hold.

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is an invariant submanifold.
(iii) If $m_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(iv) If $m_1 \neq 0$ and $\theta = \frac{\pi}{2}$, then M is a contact CR-submanifold.
(v) If θ is constant on M, then M is a semi-slant submanifold with slant angle θ.

We also note that a pointwise semi-slant submanifold is *proper* if neither $m_1, m_2 = 0$ nor $\theta = 0, \frac{\pi}{2}$ and θ should not be a constant.

We provide the following non-trivial examples of pointwise semi-slant submanifolds of almost contact metric manifolds.

Example 3.1. Let $(\mathbb{R}^7, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1,\, y_1,\, x_2,\, y_2,\, x_3,\, y_3,\, z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 3,$$

where $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g is the standard Euclidean metric on $\mathbb{R}^7$. Then (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Consider a submanifold M of $\mathbb{R}^7$ defined by

$$\psi(u, v, w, t, z) = (u + v,\ -u + v,\ t\cos w,\ t\sin w,\ w\cos t,\ w\sin t,\ z),$$

such that w, t $(w \neq t)$ are non vanishing real valued functions on M. Then the tangent space TM is spanned by the following vector fields

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1},\\
X_3 &= -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3},\\
X_4 &= \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - w\sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_3},\\
X_5 &= \frac{\partial}{\partial z}.
\end{aligned}$$

Thus, we observe that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}((t-w)/\sqrt{(t^2+1)(w^2+1)})$. Hence, M is a pointwise semi-slant submanifold of $\mathbf{R}^7$ such that $\xi = \frac{\partial}{\partial z}$ is tangent to M.

Example 3.2. Consider a submanifold of $\mathbb{R}^7$ with almost contact structure φ given in Example 3.1. If the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^7$ is given by

$$\psi(u_1, u_2, u_3, u_4, t) = (u_1, (u_3^2 + u_4^2)/2, \cos u_4, -u_2,\, (u_3^2 - u_4^2)/2, \sin u_4,\, t), \quad u_4 \neq 0,$$

then the tangent space TM is spanned by X_1, X_2, X_3, X_4 and X_5 where

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1}, \;\; X_2 = -\frac{\partial}{\partial y_1}, \;\; X_3 = u_3\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial y_2},\\
X_4 &= u_4\frac{\partial}{\partial x_2} - u_4\frac{\partial}{\partial y_2} - \sin u_4\frac{\partial}{\partial x_3} + \cos u_4\frac{\partial}{\partial y_3},\\
X_5 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Then, M is a pointwise semi-slant submanifold such that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}(\sqrt{2}\,u_4/\sqrt{1+2u_4^2})$.

Now, we obtain the following useful results for semi-slant submanifolds of a Sasakian (or cosymplectic) manifold.

Lemma 3.1. *Let M be a pointwise semi-slant submanifold of a Sasakian* (*or cosymplectic*) *manifold $\tilde{M}$. Then we have*

(i) $\sin^2\theta\, g(\nabla_X Y, Z) = g(h(X, \varphi Y), FZ) - g(h(X,Y), FPZ)$;
(ii) $\sin^2\theta\, g(\nabla_Z W, X) = g(h(X,Z), FPW) - g(h(\varphi X, Z), FW)$,

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first and second parts of the lemma can be proved in a similar way. For any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$ we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z).$$

From the covariant derivative formula of φ, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

Then from (2.3), (2.8) and the orthogonality of the two distributions, we find

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X \varphi Y, PZ) + g(\tilde{\nabla}_X \varphi Y, FZ) \\ &= -g(\tilde{\nabla}_X PZ, \varphi Y) + g(h(X, \varphi Y), FZ) \\ &= g(\varphi\tilde{\nabla}_X PZ, Y) + g(h(X, \varphi Y), FZ). \end{aligned}$$

Again, from the covariant derivative formula of φ, we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi PZ, Y) - g((\tilde{\nabla}_X \varphi)PZ, Y) + g(h(X, \varphi Y), FZ).$$

Using (2.3), (2.8) and the orthogonality of vector fields, we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X P^2 Z, Y) + g(\tilde{\nabla}_X FPZ, Y) + g(h(X, \varphi Y), FZ).$$

Then, from (2.11) and (2.6), we have

$$\begin{aligned} g(\nabla_X Y, Z) = &- \cos^2\theta\, g(\tilde{\nabla}_X Z, Y) + \sin 2\theta\, X(\theta)\, g(Y, Z) - g(h(X, Y), FPZ) \\ &+ g(h(X, \varphi Y), FZ). \end{aligned}$$

From the orthogonality of the two distributions the above equation takes the form

$$g(\nabla_X Y, Z) = \cos^2\theta\, g(\tilde{\nabla}_X Y, Z) - g(h(X, Y), FPZ) + g(h(X, \varphi Y), FZ).$$

Hence, (i) follows from the above relation. In a similar way we can prove (ii). □

4. Warped Product Pointwise Semi-Slant Submanifolds

In [3], R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian metric

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y),$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant.

Let X be a vector field tangent to M_1 and Z be an another vector field on M_2; then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{4.1}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold, then the base manifold M_1 is totally geodesic in M and the fiber M_2 is totally umbilical in M [3, 8].

By analogy to CR-warped products, which were introduced by B.-Y. Chen in [8], we define the warped product pointwise semi-slant submanifolds as follows.

Definition 4.1. A warped product of an invariant submanifold M_T and a pointwise slant submanifold M_θ of an almost contact metric manifold $\tilde{M}$ is called a *warped product pointwise semi-slant submanifold*.

A warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is called *proper* if M_θ is a proper pointwise slant submanifold and M_T is an invariant submanifold of $\tilde{M}$ and the function f on M is not constant.

The non-existence of warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ of Kaehler and Sasakian manifolds is proved in [25] and [22]. On the other hand, there exist non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Kaehler manifolds [25] and contact metric manifolds [22].

In this section, we study the warped product pointwise semi-slant submanifold of the form $M = M_T \times_f M_\theta$. Notice that a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product contact CR-submanifold if the slant function $\theta = \frac{\pi}{2}$. Similarly, the warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product semi-slant submanifold if θ is constant on M, i.e., M_θ is a proper slant submanifold.

Remark 4.1. On a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ of a Sasakian (or cosymplectic) manifold $\tilde{M}$, we consider the structure vector field ξ tangent to M; then either $\xi \in \Gamma(TM_T)$ or $\xi \in \Gamma(TM_\theta)$.

When ξ is tangent to M_θ, then it is easy to check that warped product is trivial (see [26] and [18]), therefore we always consider $\xi \in \Gamma(TM_T)$.

First, we prove the following non-existence result of pointwise semi-slant warped products.

Theorem 4.1. *There do not exist any proper pointwise semi-slant warped product submanifolds of Sasakian manifolds other than contact CR-warped products.*

Proof. Let $M = M_\theta \times_f M_T$ be a pointwise semi-slant warped product submanifold. Then, in a similar way of Theorem 11 and Theorem 12 of [22], we find that M is a Riemannian product of M_T and M_θ.

On the other hand, if $M = M_T \times_f M_\theta$ and ξ is tangent to M_θ, then by Remark 4.1, M is again a Riemannian product of M_T and M_θ. Furthermore, if ξ is tangent to M_T, then from (2.3) and (2.4), we have

$$\nabla_Z \xi + h(Z, \xi) = -PZ - FZ, \tag{4.2}$$

for any $Z \in \Gamma(TM_\theta)$. Then, equating the tangential component of (4.2) and using (4.1), we obtain

$$\xi(\ln f)Z = -PZ. \tag{4.3}$$

Taking the inner product with $W \in \Gamma(TM_\theta)$, we find

$$\xi(\ln f)g(Z,W) = -g(PZ,W). \tag{4.4}$$

By interchanging Z and W, we get

$$\xi(\ln f)g(Z,W) = g(PZ,W). \tag{4.5}$$

From (4.4) and (4.5), we find $\xi(\ln f) = 0$. Then, from (4.3), we get $PZ = 0$, which means that $\theta = \frac{\pi}{2}$. Hence, M is a contact CR-warped product, which proves the theorem completely. □

Next we find that if we replace the ambient manifold Sasakian to cosymplectic, then there exists a non-trivial class of pointwise semi-slant warped products.

Lemma 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then we have*

$$g(h(X,W),FPZ) - g(h(X,PZ),FW) = \sin 2\theta\, X(\theta)\, g(Z,W),$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(\tilde{\nabla}_X Z, W) = X(\ln f)\, g(Z,W). \tag{4.6}$$

On the other hand, we can obtain

$$g(\tilde{\nabla}_X Z, W) = g(\varphi\tilde{\nabla}_X Z, \varphi W).$$

Using the covariant derivative formula of φ, we get

$$g(\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X \varphi Z, \varphi W).$$

Then, from (2.5), (2.8), (4.1) and the orthogonality of vector fields, we find

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& g(\tilde{\nabla}_X PZ, PW) + g(\tilde{\nabla}_X PZ, FW) + g(\tilde{\nabla}_X FZ, \varphi W) \\ =& X(\ln f)\, g(PZ,PW) + g(h(X,PZ),FW) - g(\varphi\tilde{\nabla}_X FZ, W) \\ =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) - g(\tilde{\nabla}_X \varphi FZ, W). \end{aligned}$$

From (2.9) and (2.12), we derive

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) + \sin^2\theta\, g(\tilde{\nabla}_X Z, W) \\ & + \sin 2\theta\, X(\theta)\, g(Z,W) + g(\tilde{\nabla}_X FPZ, W). \end{aligned} \tag{4.7}$$

Hence, the result follows from (4.6) and (4.7) by using (2.6), (2.7) and (4.1). □

Lemma 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

(i) $\xi(\ln f) = 0$;
(ii) $g(h(X,Y), FZ) = 0$;
(iii) $g(h(X,Z), FW) = X(\ln f)\, g(PZ, W) - \varphi X(\ln f)\, g(Z, W)$,

for any $X, Y \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. From (2.2), (2.5) and (2.8), we have $\nabla_Z \xi + h(Z,\xi) = 0$ for any $Z \in \Gamma(TM_\theta)$, which implies that $\xi(\ln f) = 0$ by using (4.1). (ii) is proved in [22] (see relation (100) in [22]). Now, for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(h(X,Z), FW) = g(\tilde{\nabla}_Z X, FW) = g(\tilde{\nabla}_Z X, \varphi W) - g(\tilde{\nabla}_Z X, PW).$$

Using the covariant derivative formula of the Riemannain connection and (4.1), we get

$$g(h(X,Z), FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - X(\ln f)\, g(Z, PW).$$

Then from (2.2), (2.5) and (4.1), we derive

$$g(h(X,Z), FW) = -\varphi X(\ln f)\, g(Z,W) - X(\ln f)\, g(Z, PW),$$

which is third part of the lemma. Hence, the proof is complete. □

Interchanging X and φX, for any $X \in \Gamma(TM_T)$ in Lemma 4.2 (iii), we obtain relation

$$g(h(\varphi X, Z), FW) = X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z, PW), \tag{4.8}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Similarly, interchanging Z and PZ, for any $Z \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), we obtain

$$g(h(X, PZ), FW) = \varphi X(\ln f)\, g(Z, PW) - \cos^2\theta\, X(\ln f)\, g(Z,W), \tag{4.9}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Similarly, if we interchange W and PW, for any $W \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), then we derive

$$g(h(X,Z), FPW) = \cos^2\theta\, X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z, PW), \tag{4.10}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Lemma 4.3. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then we have*

$$g(A_{FW}\varphi X, Z) - g(A_{FPW} X, Z) = \sin^2\theta\, X(\ln f)\, g(Z,W), \tag{4.11}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Subtracting (4.10) from (4.8), we get (4.11). □

A warped product submanifold $M = M_1 \times_f M_2$ is *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$.

From Lemma 4.3, we obtain the following result.

Theorem 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is warped product of invariant submanifolds or the warping function f is constant on M.*

Proof. From (4.11) and the mixed totally geodesic condition, we have

$$\sin^2\theta\, X(\ln f)\, g(Z, W) = 0.$$

Since g is a Riemannian metric, then either $\sin^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is warped product of invariant submanifolds or f is constant on M. Thus, the proof is complete. ☐

Lemma 4.4. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then we have*

$$g(A_{FPZ}W, X) - g(A_{FW}PZ, X) = 2\cos^2\theta\, X(\ln f)\, g(Z, W), \tag{4.12}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchanging Z and W in (4.10) and using (2.10), we get

$$g(h(X, W), FPZ) = \cos^2\theta\, X(\ln f)\, g(Z, W) + \varphi X(\ln f)\, g(Z, PW), \tag{4.13}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$. Subtracting (4.9) from (4.13), we find (4.12). ☐

Also, with the help of Lemma 4.4, we find the following result.

Theorem 4.3. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is a contact CR-warped product of the form $M_T \times_f M_\perp$ or the warping function f is constant on M.*

Proof. From (4.12) and the mixed totally geodesic condition, we have

$$\cos^2\theta\, X(\ln f)\, g(Z, W) = 0.$$

Since g is a Riemannian metric, then either $\cos^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is a contact CR-warped product or f is constant on M, which ends the proof. ☐

From Theorem 4.2 and Theorem 4.3, we conclude the following.

Corollary 4.1. *There does not exist any mixed totally geodesic proper warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ of a cosymplectic manifold.*

Also, from Lemma 4.1 and Lemma 4.4, we have the following result.

Theorem 4.4. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a pointwise slant submanifold of $\tilde{M}$. Then, either M is a contact CR-warped product of the form $M = M_T \times_f M_\perp$ or $\nabla(\ln f) = \tan\theta\,\nabla\theta$, for any $X \in \Gamma(TM_T)$, where $M_\perp$ is an anti-invariant submanifold and ∇f is the gradient of f.*

Proof. From Lemma 4.1 and Lemma 4.4, we have

$$\cos^2\theta\{X(\ln f) - \tan\theta\, X(\theta)\}\, g(Z, W) = 0.$$

Since g is a Riemannian metric, therefore we conclude that either $\cos^2\theta = 0$ or $X(\ln f) - \tan\theta\, X(\theta) = 0$. Consequently, either $\theta = \frac{\pi}{2}$ or $X(\ln f) = \tan\theta\, X(\theta)$, which proves the theorem completely. ☐

As an application of Theorem 4.4, we have the following consequence.

Remark 4.2. If we consider that the slant function θ is constant, i.e., M_θ is a proper slant submanifold in Theorem 4.4, then $Z(\ln f) = 0$, i.e., there are no warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$ in cosymplectic manifolds. Hence, Theorem 4.1 of [18] is a special case of Theorem 4.4.

In order to give a characterization result for pointwise semi-slant submanifolds of a cosymplectic manifold, we recall the following well-known result of Hiepko [17].

Theorem 4.5 (Hiepko's Theorem). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

By using Theorem 4.5, we prove the following theorem.

Theorem 4.6. *Let M be a pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally a non-trivial warped product submanifold of the form $M_T \times_f M_\theta$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$ if and only if*

$$A_{FW}\varphi X - A_{FPW}X = \sin^2\theta\, X(\mu)W, \quad \textit{for all } X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle),\ W \in \Gamma(\mathfrak{D}^\theta), \tag{4.14}$$

for some smooth function μ on M satisfying $Z(\mu) = 0$ for any $Z \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. Then for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, from Lemma 4.2 (ii) we have

$$g(A_{FW}X, Y) = 0. \tag{4.15}$$

Interchanging X and φX in (4.15), we get $g(A_{FW}\varphi X, Y) = 0$, which means that $A_{FW}\varphi X$ has no component in TM_T. Similarly, if we interchange W and PW in (4.15) then, we get $g(A_{FPW}X, Y) = 0$, i.e., $A_{FPW}X$ has no component in TM_T, too. Therefore, $A_{FW}\varphi X - A_{FPW}X$ lies in TM_θ, which together with Lemma 4.3, give (4.14).

Conversely, if M is a pointwise semi-slant submanifold such that (4.14) holds, then from Lemma 3.1 (i), we have

$$g(\nabla_X Y, W) = \csc^2\theta\, g(A_{FW}\varphi Y - A_{FPW}Y, X),$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathfrak{D}^\theta)$. From (4.14), we arrive at

$$g(\nabla_X Y, W) = Y(\mu) g(X, W) = 0,$$

which means that the leaves of the distribution $\mathfrak{D} \oplus \langle\xi\rangle$ are totally geodesic in M. Also, from Lemma 3.1 (ii), we have

$$g(\nabla_Z W, X) = \csc^2\theta\, g(A_{FPW}X - A_{FW}\varphi X, Z), \tag{4.16}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$. Interchanging Z and W, we derive

$$g(\nabla_W Z, X) = \csc^2\theta\, g(A_{FPZ}X - A_{FZ}\varphi X, W). \tag{4.17}$$

Subtracting (4.17) from (4.16), we get

$$\sin^2\theta\, g([Z, W], X) = g(A_{FZ}\varphi X - A_{FPZ}X, W) - g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.14), we get

$$\sin^2\theta\, g([Z, W], X) = X(\mu)\, g(Z, W) - X(\mu)\, g(W, Z) = 0.$$

Since M is proper pointwise semi-slant, then $\sin^2\theta \neq 0$, thus we conclude that the pointwise slant distribution $\mathfrak{D}^\theta$ is integrable.

Let M_θ be a leaf of $\mathfrak{D}^\theta$ and h^θ is the second fundamental form of M_θ in M. Then from (4.17), we have

$$g(h^\theta(Z, W), X) = g(\nabla_Z W, X) = -\csc^2\theta\, g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.14), we find

$$g(h^\theta(Z, W), X) = -X(\mu)\, g(Z, W).$$

Then from the definition of the gradient of a function, we arrive at

$$h^\theta(Z, W) = -(\vec{\nabla}\mu)\, g(Z, W).$$

Hence, M_θ is a totally umbilical submanifold of M with the mean curvature vector $H^\theta = -\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ. Since $Z(\mu) = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$, then we can show that $H^\theta = -\vec{\nabla}\mu$ is parallel with respect to the normal connection, say D^n, of M_θ in M (see [24, 25, 28]). Thus, M_θ is a totally umbilical submanifold of M with a non vanishing parallel mean curvature vector $H^\theta = -\vec{\nabla}\mu$, i.e., M_θ is an extrinsic sphere in M. Then from Heipko's Theorem [17], we conclude that M is a warped product manifold of M_T and M_θ, where M_T and M_θ are integral manifolds of $\mathfrak{D} \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively. Thus, the proof is complete. $\square$

As an application of Theorem 4.6, for $\theta = \frac{\pi}{2}$ we obtain the following result as a special case of Theorem 4.6.

Corollary 4.2. ([27, Theorem 4.2]). *A proper CR-submanifold M of a cosymplectic manifold $\tilde{M}$ tangent to the structure vector field ξ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X = -\left(\varphi X(\mu)\right)Z, \quad X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle),\ Z \in \Gamma(\mathfrak{D}^\perp),$$

for some function μ on M satisfying $W\mu = 0$ for all $W \in \Gamma(\mathfrak{D}^\perp)$.

5. Examples

In this section, we provide the following non-trivial examples of Riemannian products and warped product pointwise semi-slant submanifolds in Euclidean spaces.

Example 5.1. Let M be a submanifold of the Euclidean 7-space $\mathbb{R}^7$ with its Cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, t)$ and the almost contact structure

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \quad \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \quad \varphi\Big(\frac{\partial}{\partial t}\Big) = 0, \quad 1 \le i, j \le 3.$$

If M is given by the equations

$$x_1 = u_1, \quad x_2 = u_3\cos u_4, \quad x_3 = u_3^2/2, \quad y_1 = u_2, \quad y_2 = u_3\sin u_4, \quad y_3 = u_4, \quad t = t,$$

for any non-zero function u_3 on M, then tangent space TM of M is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \cos u_4\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3} + \sin u_4\frac{\partial}{\partial y_2},$$

$$X_4 = -u_3\sin u_4\frac{\partial}{\partial x_2} + u_3\sin u_4\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}.$$

Then M is a pointwise semi-slant submanifold with invariant distribution $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ and the pointwise slant distribution $\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{X}_3, \mathrm{X}_4\}$. Clearly, the slant function is $\theta = \cos^{-1}(2u_3/\sqrt{1+u_3^2})$. Moreover, $\mathfrak{D}$ and $\mathfrak{D}^\theta$ are integrable. If M_T and M_θ are integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively, then, $M = M_T \times M_\theta$ is a Riemannian product of M_T and M_θ in $\mathbb{R}^9$.

Example 5.2. Consider the Euclidean 9-space $\mathbb{R}^9$ with its Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, t)$ and the almost contact structure

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \quad \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \quad \varphi\Big(\frac{\partial}{\partial t}\Big) = 0, \quad 1 \le i, j \le 4.$$

Let M be a submanifold of $\mathbb{R}^9$ defined by the following immersion:

$$\psi(u, v, w, s, t) = \Big(u + v, \frac{1}{2}w^2, s\cos w, s\sin w, -u + v, \frac{1}{2}s^2, -w\sin s, w\cos s, t\Big),$$

for any non-zero functions w and s. The tangent space of M is spanned by the following vectors

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1},\\
X_3 &= w\frac{\partial}{\partial x_2} - s\sin w\frac{\partial}{\partial x_3} + s\cos w\frac{\partial}{\partial x_4} - \sin s\frac{\partial}{\partial y_3} + \cos v\frac{\partial}{\partial y_4},\\
X_4 &= \cos w\frac{\partial}{\partial x_3} + \sin w\frac{\partial}{\partial x_4} + s\frac{\partial}{\partial y_2} - w\cos s\frac{\partial}{\partial y_3} - w\sin s\frac{\partial}{\partial y_4},\\
X_5 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Then, M is a pointwise semi-slant submanifold with the structure vector field $\xi = \frac{\partial}{\partial t}$ tangent to M, $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{X}_3, \mathrm{X}_4\}$ is a pointwise slant distribution with slant function

$$\theta = \cos^{-1}\left(\frac{(1-ws)\sin(w-s) - ws}{1+w^2+s^2}\right).$$

It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then M is a Riemannian product of invariant and pointwise slant submanifolds in $\mathbb{R}^9$.

Example 5.3. Let M be a submanifold of $\mathbb{R}^{13}$ given by the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^{13}$ as follows:

$$\begin{aligned}
\psi(u_1, v_1, u_2, v_2, t) =&(u_1 - v_1, u_1\cos(u_2+v_2), u_1\sin(u_2+v_2), v_2, u_1\cos(u_2-v_2),\\
&u_1\sin(u_2-v_2), u_1+v_1, v_1\cos(u_2+v_2), v_1\sin(u_2+v_2), u_2,\\
&v_1\cos(u_2-v_2), v_1\sin(u_2-v_2), t),
\end{aligned}$$

for non-zero functions u_1 and v_1. We use the almost contact structure from Example 5.2. Then, we have

$$\begin{aligned}
X_1 =&\frac{\partial}{\partial x_1} + \cos(u_2+v_2)\frac{\partial}{\partial x_2} + \sin(u_2+v_2)\frac{\partial}{\partial x_3} + \cos(u_2-v_2)\frac{\partial}{\partial x_5}\\
&+ \sin(u_2-v_2)\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_1},\\
X_2 =& -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \cos(u_2+v_2)\frac{\partial}{\partial y_2} + \sin(u_2+v_2)\frac{\partial}{\partial y_3} + \cos(u_2-v_2)\frac{\partial}{\partial y_5}\\
&+ \sin(u_2-v_2)\frac{\partial}{\partial y_6},\\
X_3 =& -u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\,\frac{\partial}{\partial x_3} - u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\
&+ u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2}, +v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial}{\partial y_4}-v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5}+v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\
X_4=&-u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2}+u_1\cos(u_2+v_2)\,\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}+u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\
&-u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6}-v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2}+v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}\\
&+v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5}-v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\
X_5=&\frac{\partial}{\partial t}.
\end{aligned}
$$

By direct computations we find that $\mathfrak{D}=\mathrm{Span}\{\mathrm{X}_1,\mathrm{X}_2\}$ is an invariant distribution and $\mathfrak{D}^\theta=\mathrm{Span}\{\mathrm{X}_3,\mathrm{X}_4\}$ is a pointwise slant distribution with slant function $\theta=\cos^{-1}\left(\frac{1}{1+2u_1^2+2v_1^2}\right)$. Hence, M is a pointwise semi-slant submanifold of $\mathbb{R}^{13}$. It is easy to observe that both distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then the product metric structure of M is given by

$$g=4(du_1^2+dv_1^2)+(1+2u_1^2+2v_1^2)(du_2^2+dv_2^2)=g_{M_T}+f^2g_{M_\theta}.$$

Hence, $M=M_T\times_f M_\theta$ is a warped product submanifold in $\mathbb{R}^{13}$ with warping function $f=\sqrt{1+2u_1^2+2v_1^2}$.

Acknowledgements. The authors thank Professor Bang-Yen Chen and Professor Kwang Soon Park for pointing out an error in an earlier version of this article.

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[1]Faculty of Mathematics,
University of Bucharest,
Str. Academiei 14, 010014 Bucharest, Romania
Email address: imihai@fmi.unibuc.ro

[2]Department of Mathematics, Faculty of Science,
King Abdulaziz University,
21589 Jeddah, Saudi Arabia
Email address: siraj.ch@gmail.com

[3]Department of Mathematics and Computer Science,
Technical University of Civil Engineering Bucharest,
Lacul Tei Bvd. 122-124, 020396 Bucharest, Romania
Email address: adela.mihai@utcb.ro

[4]Interdisciplinary Doctoral School,
Transilvania University of Braşov,
Bd. Eroilor 29, 500036 Braşov, Romania
Email address: adela.mihai@unitbv.ro

Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 473–483.

CERTAIN PROPERTIES ON MEROMORPHIC FUNCTIONS DEFINED BY A NEW LINEAR OPERATOR INVOLVING THE MITTAG-LEFFLER FUNCTION

AQEEL KETAB AL-KHAFAJI[1] AND ABBAS KAREEM WANAS[2]

Abstract. Our paper introduces a new linear operator using the convolution between a Mittag–Leffler Function and basic hypergeometric function. Use of the linear operator creates a new class of meromorphic functions defined in the punctured open unit disk. Consequently, the paper examines different aspects Apps and assets like, extreme points, coefficient inequality, growth and distortion. In conclusion, the work discusses modified Hadamard product and closure theorems.

1. Introduction

Let Σ indicate the class of type functions

$$h(z) = z^{-1} + \sum_{j=1}^{\infty} a_j z^j, \quad j \in \mathbb{N} = \{1, 2, 3, \dots\}, \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

Denote by $\Sigma_i(\delta)$ and $\Sigma^*(\delta)$ the subclasses of Σ that are meromorphically convex function of order δ, and meromorphically starlike of order δ, respectively. Function $h \in \Sigma$ of the type (1.1), is in the class $\Sigma_i(\delta)$, if it meets

$$\operatorname{Re}\left\{-\left(1 + \frac{zh''(z)}{h'(z)}\right)\right\} > \delta, \quad z \in U^*,$$

Key words and phrases. Linear operator, basic hypergeometric function, meromorphic function, Mittag–Leffler function, coefficient inequality, convex linear combination, growth and distortion theorem, extreme points, Hadamard product.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C50.

DOI 10.46793/KgJMat2403.473AK

Received: November 06, 2020.

Accepted: May 31, 2021.

and h is in the class $\Sigma^*(\delta)$, if it meets

$$\operatorname{Re}\left\{-\frac{zh'(z)}{h(z)}\right\} > \delta, \quad z \in U^*.$$

The Hadamard product (or convolution) $h * k$ for two analytic functions h given by (1.1) in U^* and

$$k(z) = z^{-1} + \sum_{j=1}^{\infty} b_j z^j,$$

is define by

$$(h * k)(z) = z^{-1} + \sum_{j=1}^{\infty} a_j b_j z^j.$$

For complex components $q, b_k, a_i,\ b_k \in \mathbb{C}\backslash\{0,-1,-2,\dots\},\ k = 1,\dots,r,\ i = 1,\dots,m,$ the basic hypergeometric function or (q-hypergeometric function) ψ_r^m is defined by:

(1.2)

$$\psi_r^m(a_1,\dots,a_m; b_1,\dots,b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1,q)_j \cdots (a_m,q)_j}{(q,q)_j (b_1,q)_j \cdots (b_r,q)_j} \left[(-1)^j q^{\frac{j(1-j)}{2}}\right]^{1+r-m} z^j,$$

where $q \neq 0$, when $m > r+1$, $m, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $(a,q)_j$ is q-analogue of the Pochhammer symbol $(a)_j$ is defined by

$$(a,q)_j = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{j-1}), & j = 1,2,3,\dots, \\ 1, & j = 0. \end{cases}$$

Initially, the function ψ_r^m given by (1.2), was introduced and referred to by Heine in 1846, as the series of Heine. For readers to refer to further q-theory information can be found in (see [9] and [11]).

Now, for $|q| < 1$, $m = r+1$ and $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, the q-hypergeometric function [25] defined in Equation (1.2), takes the form below

$$\psi_r^m(a_1,\dots,a_m; b_1,\dots,b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1,q)_j \cdots (a_m,q)_j}{(q,q)_j (b_1,q)_j \cdots (b_r,q)_j} z^j$$

that absolutely converges in the open unit disk $\mathbb{U}$.

With regard to the function $\psi_r^m(a_1,\dots,a_m; b_1,\dots,b_r; q, z)$, for meromorphic function $h \in \Sigma$ that includes functions in shape of (1.1) (see work of [1] and [18]), which is shown below, have successfully introduced the q-analogue of the Liu–Srivastava operator

$$\begin{aligned} \mathcal{G}_r^m(a_1,\dots,a_m; b_1,\dots,b_r; q, z)\, h(z) &= z_l^{-1}\psi_r^m(a_1,\dots,a_m; b_1,\dots,b_r; q, z) * h(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^{m}(a_i,q)_{j+1}}{(q,q)_{j+1}\prod_{k=1}^{r}(b_k,q)_{j+1}} a_j z^j. \end{aligned}$$

Before we continue moving on, the Mittag-Leffler function $E_{\delta}(z)$, suggested by Mittag-Leffler (see [16] and [17]) and defined by

$$E_{\alpha}(z)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+1)},\quad z\in\mathbb{U},\alpha\in\mathbb{C},\mathrm{Re}\,(\alpha)>0,$$

where $\Gamma(\delta)$ denotes the Gamma function.

Also, Wiman [26], studied another function $E_{\delta,\mu}(z)$ have numerous similarities of $E_{\delta}(z)$, and given by

$$E_{\alpha,\mu}(z)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+\mu)},\quad z\in\mathbb{U},\ \alpha,\mu\in\mathbb{C},\ \mathrm{Re}\,(\alpha)>0,\ \mathrm{Re}\,(\mu)>0. \tag{1.3}$$

In recent years, there has been growing interest in Mittag-Leffler for application problems including, electric network, fluid flow, probability, statistical distribution theory, etc. (see [2, 4, 8, 12, 15, 19, 22–24] and [27] for more information about this function and its applications). Bansal and Prajapat recently investigated geometric characteristics in [5] for the function $E_{\alpha,\mu}(z)$, like starlikeness, convexity and closed to convex. In addition, certain results were obtained in [21] for the partial sum of the Mettag-Leffler function.

We note that, the function given by (1.3), is not part of class Σ. Therefore, the function $E_{\alpha,\mu}(z)$, is then normalized on the basis of the following:

$$\mathcal{E}_{\alpha,\mu}(z)=\Gamma(\mu)\,z^{-1}E_{\alpha,\mu}(z)=z^{-1}+\sum_{j=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\alpha(j+1)+\mu)}z^{j}. \tag{1.4}$$

Application of the function $\mathcal{E}_{\alpha,\mu}(z)$ defined by (1.4), a new operator $\mathfrak{J}_{\alpha,\mu}:\Sigma\rightarrow\Sigma$, is defined in terms of Hadamard product as follows

$$\begin{aligned}\mathfrak{J}_{\alpha,\mu}h(z)&=\mathcal{E}_{\alpha,\mu}(z)*\mathcal{G}_{r}^{m}(a_{1},\dots,a_{m};b_{1},\dots,b_{r};q,z)\,h(z)\\&=z^{-1}+\sum_{j=1}^{\infty}\Delta_{(j+1,\alpha,\mu)}(a_{m},b_{r})a_{j}z^{j},\end{aligned}$$

where

$$\delta_{(j+1,\alpha,\mu)}(a_{m},b_{r},q)=\frac{\prod_{i=1}^{m}(a_{i},q)_{j+1}}{(q,q)_{j+1}\prod_{k=1}^{r}(b_{k},q)_{j+1}}\left(\frac{\Gamma(\mu)}{\Gamma(\alpha(j+1)+\mu)}\right).$$

Remark 1.1. You can see that when the parameters are defined r, m, α, μ, q, $a_1,\dots,a_m$ and $b_1,\dots,b_r$, it's here noted that the operator defined $\mathfrak{J}_{\alpha,\mu}h(z)$, performs different operators. For further explanation, examples are given.

(a) For $\alpha=0$, $\mu=1$, $a_i=q^{a_i}$, $b_k=q^{b_k}$, $a_i>0$, $b_k>0$, $i=1,\dots,m$, $k=1,\dots,r$, $m=r+1$ and $q\rightarrow 1$, we obtain the operator defined in [14].

(b) For $m=2$, $r=1$, $\alpha=0$, $\mu=1$, $a_2=q$ and $q\rightarrow 1$, we obtain the operator defined in [13].

(c) For $m=1$, $r=0$, $\alpha=0$, $\mu=1$, $a_1=\lambda+1$ and $q\rightarrow 1$, we obtain the operator defined in [10], and it was then generalized through [29].

Some other authors have studied various classes of meromorphic univalent functions, such as, see [3, 6, 7, 20, 28] and [30]). Such works encouraged us to create the new class $\mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d)$ of Σ, that includes the operator $\mathfrak{J}_{\alpha,\mu}h(z)$, and it is presented as follows.

Definition 1.1. For $d \geq 1$, $\tau > 0$, the function $h \in \Sigma$ is in the class $\mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d)$ if it satisfies the inequality

$$\left|\frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)}-1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)}+d}\right|<\tau. \tag{1.5}$$

Denote by Σ^* the subclass of Σ composed of the form functions

$$h(z)=z^{-1}+\sum_{j=1}^{\infty}|a_j|\,z^j. \tag{1.6}$$

Define the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ by

$$\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)=\mathcal{T}^{\tau}_{\alpha,\mu}(a_m, b_r, d)\cap\Sigma^*.$$

2. Main Results

This section introduces work to obtain sufficient conditions for the function h given by (1.6), in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$, it also shows that for functions belonging to this class, this requirement is necessary, as well as growth and distortion bounds, extreme points and linear combinations are submitted for the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$.

Theorem 2.1. *A function h given by* (1.6) *is in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ if and only if*

$$\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m, b_r)\,|a_j|\leq\tau(1+d). \tag{2.1}$$

Proof. Assume that the inequality (1.6) holds true. We have

$$\begin{aligned}&\left|\frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)}-1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)}+d}\right|\\
=&\left|\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'-\mathfrak{J}_{\alpha,\mu}h(z)}{\tau\left[z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'+d\mathfrak{J}_{\alpha,\mu}h(z)\right]}\right|\\
=&\left|\frac{\sum_{j=1}^{\infty}\left[j^2-1\right]\Delta_{(j+1,\alpha,\mu)}(a_m, b_r)\,|a_j|\,z^j}{(1+d)+\sum_{j=1}^{\infty}\left[j^2+d\right]\Delta_{(j+1,\alpha,\mu)}(a_m, b_r)\,|a_j|\,z^j}\right|<\tau,\quad z\in U^*.\end{aligned}$$

So, we have $h\in\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ (by the maximum modulus theorem).

Conversely, let $h \in \mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ where h given by (1.6), then we obtain from inequality (1.5),

$$\begin{aligned}&\left|\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'-\mathfrak{J}_{\alpha,\mu}h(z)}{z^2(\mathfrak{J}_{\alpha,\mu}h(z))''+z(\mathfrak{J}_{\alpha,\mu}h(z))'+d\mathfrak{J}_{\alpha,\mu}h(z)}\right|\\ &=\left|\frac{\sum_{j=1}^{\infty}[j^2-1]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j|\,z^j}{(1+d)+\sum_{j=1}^{\infty}[j^2+d]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j|\,z^j}\right|<\tau, \end{aligned}\tag{2.2}$$

since the last inequality is real for all $z \in U^*$, choose values of z on the real axis. Following explanation, the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j| \le \tau(1+d).$$

Therefore, we get the required inequality (2.1) of Theorem 2.1. □

Corollary 2.1. *If the function h given by* (1.6) *is in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$, then*

$$|a_j| \le \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}, \quad j \ge 1, \tag{2.3}$$

the result is sharp of the function

$$h(z) = z^{-1} + \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j, \quad j \ge 1.$$

Theorem 2.2. *Let $h_o(z) = z^{-1}$ and*

$$h_j(z) = z^{-1} + \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j.$$

Then, $h \in \mathcal{T}^{\tau,*}_{\alpha,\mu}(a_m, b_r, d)$ if and only if it can be expressed form

$$h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j h_j(z), \tag{2.4}$$

where

$$v_j \ge 0 \quad \text{and} \quad \sum_{j=0}^{\infty} v_j = 1.$$

Proof. Using the function h which is defined in (2.4), then

$$h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j \frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\,\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}z^j,$$

and for last function, we get

$$\sum_{j=1}^{\infty}\left[j^{2}(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)$$
$$\times v_j\frac{\tau(1+d)}{\left[j^{2}(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}$$
$$=\sum_{j=1}^{\infty}v_j\tau(1+d)=\tau(1+d)(1-v_o)=\tau(1+d),$$

that is, condition (2.1) is met. Therefore, $h\in\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$.

Conversely, we assume that $h\in\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$, from equation (2.3), we have:

$$|a_j|\leq\frac{\tau(1+d)}{\left[j^{2}(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)},\quad j\geq 1,$$

we set

$$v_i=\frac{\left[j^{2}(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\ |a_j|,\quad j\geq 1,$$

and

$$v_0=1-\sum_{j=1}^{\infty}v_j.$$

That is the result

$$h(z)=\sum_{j=0}^{\infty}v_jf_j.$$

The declaration of Theorem 2.2, is thus complete. $\square$

Theorem 2.3. *If a function h defined by (1.6), is in the class $\mathfrak{T}_{\alpha,\mu}^{\tau,*}(a_m,b_r,d)$, then for $|z|=r$, we have*

$$\frac{1}{r}-\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}r$$
$$\leq|h(z)|\leq\frac{1}{r}+\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}r$$

and

$$\frac{1}{r^{2}}-\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}$$
$$\leq|h'(z)|\leq\frac{1}{r^{2}}+\frac{\tau(1+d)}{\left[(1-\tau)-(1+\tau d)\right]\Delta_{(2,\alpha,\mu)}(a_m,b_r)}.$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}&[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)\sum_{j=1}^{\infty}|a_j|\\ \le&\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\,|a_j|\\ \le&\tau(1+d),\end{aligned}$$

which results

$$\sum_{j=1}^{\infty}|a_j|\le\frac{\tau(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)}.$$

Therefore,

$$|h(z)|\le\frac{1}{|z|}+|z|\sum_{j=1}^{\infty}|a_j|\le\frac{1}{|z|}+\frac{\tau(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)}|z|$$

and

$$|h(z)|\ge\frac{1}{|z|}-|z|\sum_{j=1}^{\infty}|a_j|\ge\frac{1}{|z|}-\frac{\tau(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)}|z|.$$

On the other hand, for (1.6), differentiating both sides with respect to z, we get:

$$|h'(z)|\le\frac{1}{|z|^2}+\sum_{j=1}^{\infty}|a_j|\le\frac{1}{|z|}+\frac{\tau(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)}$$

and

$$|h'(z)|\ge\frac{1}{|z|^2}-\sum_{j=1}^{\infty}|a_j|\ge\frac{1}{|z|}-\frac{\tau(1+d)}{[(1-\tau)-(1+\tau d)]\,\Delta_{(2,\alpha,\mu)}(a_m,b_r)}.$$

Define the functions h_i, $i=1,2$, by

$$h_i(z)=z^{-1}+\sum_{j=1}^{\infty}|a_{j,i}|\,z^j,\quad z\in\ U^*. \tag{2.5}$$ □

Theorem 2.4. *Let the functions h_i, $i=1,2$, which are defined in* (2.5), *be in the class $\mathfrak{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$. Then for $0\le s\le 1$, the function $h(z)=sh_1(z)+(1-s)\,h_2(z)$, in the class $\mathfrak{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$.*

Proof. Using

$$h_i(z)=z^{-1}+\sum_{j=1}^{\infty}|a_{j,i}|\,z^j,\quad i=1,2,$$

we have:

$$h(z)=z^{-1}+\sum_{j=1}^{\infty}\{s\,|a_{j,1}|+(1-s)\,|a_{j,2}|\}\,z^j,\quad 0\le s\le 1.$$

Now, by Theorem 2.1, we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left\{s\left|a_{j,1}\right|+(1-s)\left|a_{j,2}\right|\right\}\\
=&s\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left|a_{j,1}\right|\\
&+(1-s)\sum_{j=1}^{\infty}\left[j^2(1-\tau)-(1+\tau d)\right]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)\left|a_{j,2}\right|\\
\leq& s\tau(1+d)+(1-s)\tau(1+d)=\tau(1+d),
\end{aligned}$$

that demonstrates $h(z)\in\mathfrak{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$. □

Theorem 2.5. *Let the function h_i, $i=1,2$, which are defined in (2.5), be in the class $\mathfrak{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$. Then $h_1*h_2\in\mathfrak{T}^{\delta,*}_{\alpha,\mu}(a_m,b_r,d)$, where*

$$\delta\leq\frac{(j^2-1)\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)+(j^2+d)\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}.$$

Proof. It's enough to find the Littlest δ, such that

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\delta)-(1+\delta d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\delta(1+d)}a_{j,1}a_{j,2}\leq 1.$$

Since $h_i\in\mathfrak{T}^{\tau,*}_{\alpha,\mu}(a_m,b_r,d)$, $i=1,2$, then

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}a_{j,1}a_{j,2}\leq 1.$$

By Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\sqrt{a_{j,1}a_{j,2}}\leq 1. \tag{2.6}$$

We just want to demonstrate that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\frac{[j^2(1-\delta)-(1+\delta d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\delta(1+d)}a_{j,1}a_{j,2}\\
\leq&\sum_{j=1}^{\infty}\frac{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}{\tau(1+d)}\sqrt{a_{j,1}a_{j,2}},
\end{aligned}$$

or equivalent to

$$\sqrt{a_{j,1}a_{j,2}}\leq\frac{[j^2(1-\delta)-(1+\delta d)]\tau}{[j^2(1-\tau)-(1+\tau d)]\delta}.$$

From (2.6), we get

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{\tau(1+d)}{[j^2(1-\tau)-(1+\tau d)]\Delta_{(j+1,\alpha,\mu)}(a_m,b_r)}.$$

Therefore, it is sufficient to show that

$$\frac{\tau\left(1+d\right)}{\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}\leq\frac{\left[j^{2}\left(1-\delta\right)-\left(1+\delta d\right)\right]\tau}{\left[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\right]\delta}.$$

Finally, we have

$$\delta\leq\frac{\left(j^{2}-1\right)\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}{\tau\left(1+d\right)+\left(j^{2}+d\right)\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)}.\qquad\square$$

Theorem 2.6. *If the function h_i, $i=1,2$, given by equation (2.5) is in the class $\mathcal{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$, then $h_1 * h_2 \in \mathcal{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$.*

Proof. Because $h_1 \in \mathcal{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$, by Theorem 2.1, we obtain

$$\sum_{j=1}^{\infty}\Big[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\Big]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j}\right|\leq\tau\left(1+d\right).$$

Since

$$\begin{aligned}&\sum_{j=1}^{\infty}\Big[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\Big]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}a_{j,2}\right|\\&=\sum_{j=1}^{\infty}\Big[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\Big]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}\right|\left|a_{j,2}\right|\\&\leq\sum_{j=1}^{\infty}\Big[j^{2}\left(1-\tau\right)-\left(1+\tau d\right)\Big]\Delta_{\left(j+1,\alpha,\mu\right)}\left(a_{m},b_{r}\right)\left|a_{j,1}\right|\\&\leq1,\end{aligned}$$

we have $h_1 * h_2 \in \mathcal{T}^{\tau,*}_{\alpha,\mu}\left(a_m,b_r,d\right)$. $\square$

Acknowledgements. The authors would like to thank the referee(s) for their helpful comments and suggestions.

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[1]Department of Mathematics,
College of Education for Pure Sciences, The University of Babylon,
51002 Babylon, Iraq
Email address: `aqeel.ketab@uobabylon.edu.iq`

[2]Department of Mathematics,
College of Science, University of Al-Qadisiyah,
Al Diwaniyah, Al-Qadisiyah, Iraq
Email address: `abbas.kareem.w@qu.edu.iq`

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