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BEST PROXIMITY POINT THEOREMS IN NON-ARCHIMEDEAN MENGER PROBABILISTIC SPACES

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Abstract. In this work, we prove best proximity point theorems for γ-contractions with conditions the weak P-property in non-Archimedean Menger probabilistic metric spaces. We give the notion of γ- proximal contractions of first and second type in non-Archimedean Menger probabilistic metric spaces and also we establish best proximity point theorems for these proximal contractions. Lastly, we complete our study by giving examples that support our results.

1. Introduction

The concept of the probabilistic metric spaces were introduced by Menger [15]. When x and y are two elements of a probabilistic metric space, the idea of distance between these points is changed with function $F_{x,y}(t)$. $F_{x,y}(t)$ is a distribution function that is explained as probability that the distance between x and y is less than t. In fact, studies in these spaces improved with Schweizer and Sklar' s leading works [20]. The probabilistic interpretation of Banach contraction principle is demonstrated by Sehgal and Bharucha-Reid in [22]. Some studies about probabilistic metric spaces are given in list [7, 8, 12, 16–18].

On the other hand, best proximity point was started by Fan [9]. For more details, references are listed in [1, 3, 4, 11, 13, 14, 19, 24]. Sezen introduced γ-contraction and γ-weak contraction in non-Archimedean fuzzy metric spaces [23]. In this paper, we prove some best proximity point theorems for γ-contractions in a non-Archimedean Menger probabilistic metric space.

Key words and phrases. Fixed point, best proximity point, γ-contraction, non-Archimedean Menger probabilistic metric space

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2. Preliminaries

Definition 2.1 ([20])**.** A triangular norm (shorter $\Delta-$ norm/ $t-$ norm) is a binary operation Δ which is defined on the closed interval [0,1],

$$\Delta : [0,1] \times [0,1] \to [0,1]$$

that satisfies the following requirements:

(Δ_1) $\Delta(a_1, 1) = a_1,\ \Delta(0,0) = 0$;
(Δ_2) $\Delta(a_1, a_2) = \Delta(a_2, a_1)$;
(Δ_3) $\Delta(a_3, a_4) \geq \Delta(a_1, a_2)$ for $a_3 \geq a_1,\ a_4 \geq a_2$;
(Δ_4) for all $a_1, a_2, a_3 \in [0,1]\,,\ \Delta(\Delta(a_1, a_2), a_3) = \Delta(a_1, \Delta(a_2, a_3))$.

Principal examples of $\Delta-$ norms are:

(i) $\Delta_M(a_1, a_2) = \min(a_1, a_2)$;
(ii) $\Delta_P(a_1, a_2) = a_1.a_2$;
(iii) $\Delta_L(a_1, a_2) = \max(a_1 + a_2 - 1, 0)$;
(iv) $\Delta_D(a_1, a_2) = \begin{cases} \min(a_1, a_2), & \text{if } \max(a_1, a_2) = 1, \\ 0, & \text{otherwise.} \end{cases}$

Definition 2.2 ([20])**.** Let F be a function defined from $\mathbb{R}$ to $\mathbb{R}^+$. If it is nondecreasing, left-continuous with

$$\inf \{F(t) : t \in \mathbb{R}\} = 0 \quad \text{and} \quad \sup \{F(t) : t \in \mathbb{R}\} = 1,$$

then F is called a distribution function. In addition, if $F(0) = 0$, then F is called a distance distribution function. L^+ indicate the set of all distance distribution functions and H is a special example of distance distribution function (also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.3 ([20])**.** Let X is a nonempty set and F is a mapping defined from $X \times X$ into L^+. The value of F at the point (x, y) is denoted by $F_{x,y}$. If the following conditions hold, (X, F) ordered pair is called a probabilistic metric space:

(PM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM-3) $F_{x,y}(t) = 1, F_{y,z}(s) = 1,$ then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X,\ t, s \geq 0$.

Every metric space (X, d) can always be realized as a probabilistic metric space by taking into account that $F : X \times X \to L^+$ defined as

$$F_{x,y}(t) = H(t - d(x, y)), \quad \text{for all } x, y \in X.$$

Definition 2.4 ([20])**.** Let (X, F) be a probabilistic metric space and Δ is a $t-$norm that provides the following inequality,

$$F_{x,z}(t + s) \geq \Delta\left(F_{x,y}(t), F_{y,z}(s)\right), \quad \text{for all } x, y, z \in X \text{ and } t, s \geq 0.$$

Then, triplet (X, F, Δ) is named as a Menger probabilistic metric space.

Definition 2.5 ([20]). Let (X, F, Δ) be a Menger space.

(i) A sequence (x_n) is called a convergent sequence to $x \in X$ if for every $t > 0$ and $0 < \varepsilon < 1$, there exists $n_0 = n_0(t, \varepsilon) \in \mathbb{N}$ such that $F_{x_n,x}(t) > 1 - \lambda$ for all $n \geq \mathbb{N}$.

(ii) A sequence (x_n) in X is called Cauchy sequence if for every $t > 0$ and $0 < \varepsilon < 1$, there exists $n_0 = n_0(t, \varepsilon) \in \mathbb{N}$ such that $F_{x_n,x_m}(t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A Menger space is said to be complete , if each Cauchy sequence in X is convergent to a point in X.

Definition 2.6 ([5]). A probabilistic metric space (X, F) is called non-Archimedean probabilistic metric space if $F_{x,y}(t) = 1, F_{y,z}(s) = 1$, then $F_{x,z}(\max\{t, s\}) = 1$ for every $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.7 ([5,6]). A Menger probabilistic metric space (X, F, Δ) is called non-Archimedean if $F_{x,z}(\max\{t, s\}) = \Delta\left(F_{x,y}(t), F_{y,z}(s)\right)$ for all $x, y, z \in X$ and $t, s \geq 0$.

Note. We observe that (X, F, Δ) is non-Archimedean if and only if

$$F_{x,z}(t) \geq \Delta\left(F_{x,y}(t), F_{y,z}(t)\right), \quad \text{for all } x, y, z \in X \text{ and } t \geq 0.$$

Definition 2.8 ([19]). Let (X, F, Δ) be a Menger probabilistic metric space and A, B be two nonempty subsets of this space. A mapping $T : A \to B$ satisfies the following equality

$$F_{x,Tx}(t) = F_{A,B}(t), \quad \text{for } t > 0.$$

Then x in A is said to be a best proximity point of T.

Definition 2.9 ([3]). Let (X, F, Δ) be a Menger probabilistic metric space and A, B two nonempty subsets of this space. A set A is said to be approximatively compact with respect to a set B if every sequence (x_n) in A satisfies the condition that $F_{y,x_n}(t) \to F_{y,A}(t)$ for some $y \in B$ and for each $t > 0$ has a convergent subsequence.

Definition 2.10. Let $\gamma : [0, 1) \to \mathbb{R}$ be a function that has the following properties:

(a) strictly increasing;

(b) continuous mapping;

(c) for each sequence (α_n) of positive numbers, $\lim_{n\to\infty} \alpha_n = 1$ if and only if $\lim_{n\to\infty} \gamma(\alpha_n) = +\infty$.

Also, Γ represents the family of all γ functions.

Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space. A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0, 1)$ such that for all $x, y \in X$ and $\gamma \in \Gamma$

$$F_{Tx,Ty}(t) < 1 \Rightarrow \gamma(F_{Tx,Ty}(t)) \geq \gamma(F_{x,y}(t)) + \delta. \tag{2.1}$$

3. Main Results

In this section, we present some definitions and some best proximity point results in non-Archimedean Menger probabilistic metric spaces. Let A and B two nonempty subsets of a Menger probabilistic metric space (X, F, Δ). We will use the following notations:

$$\begin{aligned} F_{A,B}(t) =& \sup\{F_{x,y}(t) : x \in A, y \in B\}.\\ A_0(t) =& \{x \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } y \in B\},\\ B_0(t) =& \{y \in B : F_{x,y}(t) = F_{A,B}(t) \text{ for some } x \in A\}.\end{aligned}$$

Now, let us give our main results.

Definition 3.1. Let (A, B) be a pair of nonempty subsets of a non-Archimedean Menger probabilistic metric space X with $A_0(t) \neq 0$. Then the pair (A, B) is said to have the weak P-property if and only if

$$F_{x_1,y_1}(t) = F_{A,B}(t), \quad F_{x_2,y_2}(t) = F_{A,B}(t) \quad \Rightarrow \quad F_{x_1,x_2}(t) \geq F_{y_1,y_2}(t),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 3.1. Let $X = \mathbb{R} \times \mathbb{R}$ and d defined as the standard metric $d(x, y) = |x - y|$ for all $x \in X$, $\Delta(a, b) = \min(a, b)$ and the distribution function defined as

$$F_{x,y}(t) = \frac{t}{t + d(x, y)}, \quad \text{for all } t > 0.$$

(X, F, Δ) is a non-Archimedean Menger probabilistic metric space. Let $A = \{(0, 0)\}$, $B = \{(1, 0), (-1, 0)\}$. From here, $d(A, B) = 1$ and $F_{A,B}(t) = \frac{t}{t+d(A,B)} = \frac{t}{t+1}$. Now we consider

$$F_{x_1,y_1}(t) = F_{A,B}(t), \quad F_{x_2,y_2}(t) = F_{A,B}(t).$$

We get $(x_1, y_1) = ((0, 0), (1, 0))$ and $(x_2, y_2) = ((0, 0), (-1, 0))$, $F_{x_1,x_2}(t) = F_{(0,0),(0,0)}(t) = 1$ and $F_{y_1,y_2}(t) = F_{(1,0),(-1,0)}(t) = \frac{t}{t+2}$ implies $F_{x_1,x_2}(t) > F_{y_1,y_2}(t)$. Thus, (A, B) is said to have the weak P-property.

Definition 3.2. Let A, B be nonempty subsets of a non-Archimedean Menger probabilisitc metric space (X, F, Δ). The mapping $g : A \to A$ is said to be a probabilistic isometry if

$$F_{gx_1,gx_2}(t) = F_{x_1,x_2}(t),$$

for all $x_1, x_2 \in A$.

Definition 3.3. Let A, B be nonempty subsets of a non-Archimedean Menger probabilistic metric space (X, F, Δ). Given $S : A \to B$ and a probabilistic isometry $g : A \to A$, the mapping S is said to preserve probabilistic distance with respect to g if

$$F_{Sgx_1,Sgx_2}(t) = F_{Sx_1,Sx_2}(t),$$

for all $x_1, x_2 \in A$.

Example 3.2. Let $X = [0,1] \times \mathbb{R}$ and d defined as the standart metric $d(x,y) = |x-y|$ for all $x \in X$ and the distribution function defined as

$$F_{x,y}(t) = \frac{t}{t + d(x,y)}, \quad \text{for all } t > 0.$$

Let $A = \{(0,x) : x \in \mathbb{R}\}$. $g : A \to A$ is defined as $g(0,x) = (0,-x)$. $F_{x,y}(t) = \frac{t}{t+d(x,y)} = F_{gx,gy}(t)$, where $x = (0,x_1), y = (0,y_1) \in A$. This indicates that g is a probabilistic isometry.

Theorem 3.1. *A and B be nonempty, closed subsets of a complete non-Archimedean Menger probabilistic metric space (X,F,Δ) such that $A_0(t)$ is nonempty. Let $T : A \to B$ be a γ-contraction such that $T(A_0(t)) \subseteq B_0(t)$. Suppose that the pair (A,B) has the weak P-property. Then T has a unique x^* in A such that $F_{x^*,Tx^*}(t) = F_{A,B}(t)$.*

Proof. Let start by choosing an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, we can find $x_1 \in A_0(t)$ such that $F_{x_1,Tx_0}(t) = F_{A,B}(t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{x_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(x_n) \in A_0(t)$ satisfying for all $n \in \mathbb{N}$,

$$F_{x_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.1}$$

(A,B) satisfies the weak P-property, from (3.1) we obtain

$$F_{x_n,x_{n+1}}(t) \geq F_{Tx_{n-1},Tx_n}(t), \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Now we will prove that the sequence (x_n) is convergent in $A_0(t)$. If there exists $n_0 \in \mathbb{N}$ such that $F_{Tx_{n_0-1},Tx_{n_0}}(t) = 1$, then by (3.2) we get $F_{x_{n_0},x_{n_0+1}}(t) = 1$ which implies $x_{n_0} = x_{n_0+1}$. Hence, we get

$$Tx_{n_0} = Tx_{n_0+1} \Rightarrow F_{Tx_{n_0},Tx_{n_0+1}}(t) = 1. \tag{3.3}$$

From (3.2) and (3.3), we have that

$$F_{x_{n_0+2},x_{n_0+1}}(t) \geq F_{Tx_{n_0+1},Tx_{n_0}}(t) = 1 \Rightarrow x_{n_0+2} = x_{n_0+1}.$$

Therefore, for all $n \geq n_0$, $x_n = x_{n_0}$ and (x_n) is convergent in $A_0(t)$. Also, we get

$$F_{x_{n_0},Tx_{n_0}}(t) = F_{x_{n_0+1},Tx_{n_0}}(t) = F_{A,B}(t).$$

From this equality we can say that x_{n_0} is a probabilistic best proximity point of T and the proof is finished. For this reason, we suppose that, for all $n \in \mathbb{N}$, $F_{Tx_{n-1},Tx_n}(t) \neq 1$. From the definition of γ-contraction and (3.2), we have

$$\begin{aligned} \gamma(F_{x_n,x_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ &\geq \gamma(F_{x_{n-2},x_{n-1}}(t)) + 2\delta \\ &\vdots \\ &\geq \gamma(F_{x_0,x_1}(t)) + n\delta. \end{aligned} \tag{3.4}$$

Letting $n \to \infty$, from (3.4) we have

$$\lim_{n\to\infty} \gamma(F_{x_n,x_{n+1}}(t)) = +\infty.$$

Using the property of γ function we have,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1. \tag{3.5}$$

We shall show that (x_n) is a Cauchy sequence. Suppose that (x_n) is not a Cauchy sequence. Then there exist $\varepsilon \in (0,1)$ and $t_0 > 0$ and two sequences $m(j), n(j)$ of positive integers such that $m(j) > n(j) + 1$ and

$$F_{x_{m(j)},x_{n(j)}}(t_0) < 1 - \varepsilon \quad \text{and} \quad F_{x_{m(j)-1},x_{n(j)}}(t_0) \geq 1 - \varepsilon. \tag{3.6}$$

So, for all $j \in \mathbb{N}$ we get

$$\begin{aligned} 1 - \varepsilon >& F_{x_{m(j)},x_{n(j)}}(t_0) \\ \geq& \Delta(F_{x_{m(j)},x_{m(j)-1}}(t_0), F_{x_{m(j)-1},x_{n(j)}}(t_0)) \\ \geq& \Delta(F_{x_{m(j)},x_{m(j)-1}}(t_0), (1-\varepsilon)). \end{aligned} \tag{3.7}$$

By taking $j \to \infty$ in (3.7) and using (3.5) we have,

$$\lim_{j\to\infty} F_{x_{m(j)},x_{n(j)}}(t_0) = 1 - \varepsilon. \tag{3.8}$$

From the property of t-norm

$$\begin{aligned} F_{x_{m(j)+1},x_{n(j)+1}}(t_0) \geq& \Delta(F_{x_{m(j)+1},x_{m(j)}}(t_0), F_{x_{m(j)},x_{n(j)+1}}(t_0)) \\ \geq& \Delta(F_{x_{m(j)+1},x_{m(j)}}(t_0), \Delta(F_{x_{m(j)},x_{n(j)}}(t_0), F_{x_{n(j)},x_{n(j)+1}}(t_0))). \end{aligned}$$

On letting limit as $j \to \infty$ in previous inequality, we obtain

$$\lim_{j\to\infty} F_{x_{m(j)+1},x_{n(j)+1}}(t_0) = 1 - \varepsilon. \tag{3.9}$$

By applying inequality in (2.1) with $x = x_{m(j)}$ and $y = x_{n(j)}$,

$$\gamma(F_{x_{m(j)+1},x_{n(j)+1}}(t)) \geq \gamma(F_{x_{m(j)},x_{n(j)}}(t)) + \delta. \tag{3.10}$$

Taking the limit as $j \to \infty$ in (3.10), using definition of γ-contraction, from (3.8) and (3.9), we obtain

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta.$$

This is a contraction. Therefore, (x_n) is a Cauchy sequence in X. We know that (X, F, Δ) is complete and $A_0(t)$ is a closed subset of this space, there exists $x^* \in A_0(t)$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

From the continuity of T, we have $Tx_n \to Tx^*$ and $F_{x_{n+1},Tx_n}(t) = F_{x^*,Tx^*}(t)$. From (3.1), $F_{x^*,Tx^*}(t) = F_{A,B}(t)$. This shows that x^* is a probabilistic best proximity point of T. Now, we show that uniqueness of the best proximity point of T. Suppose that x_1 and x_2 are two best proximity points of T. For $x_1, x_2 \in A$, $x_1 \neq x_2$ and

$F_{x_1,Tx_1}(t) = F_{x_2,Tx_2}(t) = F_{A,B}(t)$. Since (A, B) has the weak P-property, we can write $F_{x_1,x_2}(t) \geq F_{Tx_1,Tx_2}(t)$. T is a γ-contraction and $x_1 \neq x_2$ implies $F_{x_1,x_2}(t) \neq 1$,

$$\gamma(F_{x_1,x_2}(t)) \geq \gamma(F_{Tx_1,Tx_2}(t)) \geq \gamma(F_{x_1,x_2}(t)) + \delta > \gamma(F_{x_1,x_2}(t)),$$

which is a contradiction. Hence, T has a unique best proximity point. $\square$

Example 3.3. Let $X = \mathbb{R} \times [0, 1]$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2. Let $A = \{(x, 0) : \text{for all } x \in \mathbb{R}\}$, $B = \{(y, 1) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $F_{A,B}(t) = \frac{t}{t+1}$. $\gamma : [0, 1) \to \mathbb{R}$ defined as $\gamma = \frac{1}{1-x}$, for all $x \in X$. Let $T : A \to B$ and $T(x, 0) = \left(\frac{x}{6}, 1\right)$. Then, $T(A_0(t)) = B_0(t)$. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

We have $(a_1, x_1) = \left(\left(\frac{-b_1}{6}, 0\right), (-b_1, 0)\right)$ or $(a_2, x_2) = \left(\left(\frac{-b_2}{6}, 0\right), (-b_2, 0)\right)$. Then using γ-contraction, we have

$$\begin{aligned}\gamma(F_{a_1,a_2}(t)) =& \gamma\left(F_{\left(-\frac{b_1}{6},0\right),\left(-\frac{b_2}{6},0\right)}(t)\right) = \gamma\left(\frac{t}{t + \frac{|b_1-b_2|}{6}}\right) \\ =& \frac{1}{1 - \frac{t}{t+\frac{|b_1-b_2|}{6}}} > \frac{1}{1 - \frac{t}{t+|b_1-b_2|}} = \gamma\left(\frac{t}{t + |b_1 - b_2|}\right) \\ =& \gamma(F_{x_1,x_2}(t)).\end{aligned} \tag{3.11}$$

From (3.11), $\gamma(F_{a_1,a_2}(t)) > \gamma(F_{x_1,x_2}(t))$. So, we can find a $\delta \in (0, 1)$ such that $\gamma(F_{a_1,a_2}(t)) \geq F_{x_1,x_2}(t)) + \delta$. Then T is a γ-contraction and $(0, 0)$ is a unique best proximity point of T.

Corollary 3.1. *Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and $A_0(t)$ is a nonempty closed subset of X. Let $T : A \to A$ be a γ-contraction. Then there exists a unique x^* in A.*

Definition 3.4. Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and A, B be two nonempty subsets of this space such that $A_0(t)$ is nonempty. A mapping $T : A \to B$ is said to be a γ-proximal contraction of first type if there exists a $\delta \in (0, 1)$ for all $u_1, u_2, x_1, x_2 \in X$ such that

$$\begin{aligned}&F_{u_1,Tx_1}(t) = F_{A,B}(t), \quad F_{u_2,Tx_2}(t) = F_{A,B}(t), \quad F_{u_1,u_2}(t), F_{x_1,x_2}(t) < 1, \\ &\Rightarrow \gamma(F_{u_1,u_2}(t)) \geq \gamma(F_{x_1,x_2}(t)) + \delta.\end{aligned} \tag{3.12}$$

Definition 3.5. Let (X, F, Δ) be a non-Archimedean Menger probabilistic metric space and A, B be two nonempty subsets of this space such that $A_0(t)$ is nonempty. A mapping $T : A \to B$ is said to be a γ-proximal contraction of second type if there exists a $\delta \in (0, 1)$ for all $u_1, u_2, x_1, x_2 \in X$ such that

$$\begin{aligned}&F_{u_1,Tx_1}(t) = F_{A,B}(t), \quad F_{u_2,Tx_2}(t) = F_{A,B}(t), \quad F_{Tu_1,Tu_2}(t), F_{Tx_1,Tx_2}(t) < 1, \\ &\Rightarrow \gamma(F_{Tu_1,Tu_2}(t)) \geq \gamma(F_{Tx_1,Tx_2}(t)) + \delta.\end{aligned} \tag{3.13}$$

Theorem 3.2. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) $T(A_0(t)) \subseteq B_0(t)$;
(2) $T : A \to B$ *is a continuous* $\gamma -$ *proximal contraction of first type;*
(3) g *is an isometry;*
(4) $A_0(t) \subseteq g(A_0(t))$.

Then there exist a unique element $x \in A$ such that $F_{gx,Tx}(t) = F_{A,B}(t)$.

Proof. We will start the proof by choosing an element x_0 in $A_0(t)$. Since $T\left(A_0(t)\right) \subseteq B_0(t)$ and $A_0(t) \subseteq g\left(A_0(t)\right)$, we can find $x_1 \in A_0(t)$ such that $F_{gx_1,Tx_0}(t) = F_{A,B}(t)$. Since $Tx_1 \in T\left(A_0(t)\right) \subseteq B_0(t)$ and $A_0(t) \subseteq g\left(A_0(t)\right)$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{gx_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(x_n) \in A_0(t)$ satisfying for all $n \in \mathbb{N}$,

$$F_{gx_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.14}$$

Now we will prove that the sequence (x_n) is convergent in $A_0(t)$. If there exists $n_0 \in \mathbb{N}$ such that $F_{gx_{n_0},gx_{n_0+1}}(t) = 1$, then it is clear that the sequence (x_n) is convergent. So, let for all $n \in \mathbb{N}$, $F_{gx_{n_0},gx_{n_0+1}}(t) \neq 1$. From the hypothesis of the theorem, T is a γ-proximal contraction of first type

$$\begin{aligned} \gamma(F_{gx_n,gx_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ \gamma(F_{x_n,x_{n+1}}(t)) &\geq \gamma(F_{x_{n-1},x_n}(t)) + \delta \\ &\vdots \\ &\geq \gamma(F_{x_0,x_1}(t)) + n\delta. \end{aligned} \tag{3.15}$$

Letting $n \to \infty$, in previous inequality we have $\lim\limits_{n\to\infty} \gamma(F_{x_n,x_{n+1}}(t)) = +\infty$. If we continue with the same way that used in proof of Theorem 3.1, we can say (x_n) is a Cauchy sequence. Since complete non-Archimedean Menger probabilistic metric space (X, F, Δ) has closed subsets, there exist $x \in A_0(t)$ such that $\lim\limits_{n\to\infty} x_n = x$. Applying limit when $n \to \infty$ in (3.14), we have

$$F_{gx,Tx}(t) = F_{A,B}(t).$$

To show the uniqueness, we will suppose the contrary. Let $x^* \in A_0(t)$ and it satisfy the equality $F_{gx^*,Tx^*}(t) = F_{A,B}(t)$ such that $x \neq x^*$. Hence, $F_{x,x^*}(t) \neq 1$. Since g is an isometry and T is a γ-proximal contraction of the first kind, it follows that

$$\gamma(F_{x,x^*}(t)) = \gamma(F_{gx,gx^*}(t)) \geq \gamma(F_{x,x^*}(t)) + \delta > \gamma(F_{x,x^*}(t)),$$

which is a contradiction. Consequently, $x = x^*$. □

Example 3.4. Let $X = [-2, 2] \times \mathbb{R}$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2.

Let $A = \{(-2, x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(2, y) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 4$ and $F_{A,B}(t) = \frac{t}{t+4}$. $\gamma : [0, 1) \to \mathbb{R}$ defined as $\gamma(x) = \frac{1}{1-x^2}$, for all $x \in X$. Let $T : A \to B$ and $g : A \to A$, these are defined as $T(-2, x) = \left(2, \frac{x}{2}\right)$ and $g(-2, x) = (-2, -x)$. Then, $T(A_0(t)) = B_0(t)$, $A_0(t) = g(A_0(t))$ and g is a isometry. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

We have $(a_1, x_1) = \left(\left(-2, \frac{b_1}{2}\right), (-2, b_1)\right)$ or $(a_2, x_2) = \left(\left(-2, \frac{b_2}{2}\right), (-2, b_2)\right)$. We must show that, T is a γ-proximal contraction of first type

$$\begin{aligned} \gamma(F_{a_1,a_2}(t)) &= \gamma\left(F_{\left(-2,\frac{b_1}{2}\right),\left(-2,\frac{b_2}{2}\right)}(t)\right) = \gamma\left(\frac{t}{t + \frac{|b_1-b_2|}{2}}\right) \\ &= \frac{1}{1 - \left(\frac{t}{t+\frac{|b_1-b_2|}{2}}\right)^2} > \frac{1}{1 - \left(\frac{t}{t+|b_1-b_2|}\right)^2} = \gamma\left(\frac{t}{t + |b_1 - b_2|}\right) \\ &= \gamma(F_{x_1,x_2}(t)). \end{aligned} \tag{3.16}$$

From (3.16), we have $\gamma(F_{a_1,a_2}(t)) > \gamma F_{x_1,x_2}(t))$. So, we can find a $\delta \in (0, 1)$ such that $\gamma(F_{a_1,a_2}(t)) \geq F_{x_1,x_2}(t)) + \delta$. Then T is a γ-proximal contraction of first type and $(-2, 0)$ is a unique best proximity point of T.

If we assume that g is the identity mapping, we can give the following result.

Corollary 3.2. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Let $T : A \to B$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*
(2) *$T : A \to B$ is a continuous $\gamma-$proximal contraction of first type.*

Then T has a unique best proximity point in A.

Theorem 3.3. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Suppose that A is approximatively compact with respect to B. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*
(2) *$T : A \to B$ is a continuous $\gamma-$proximal contraction of second type;*
(3) *g is an isometry;*
(4) *$A_0(t) \subseteq g(A_0(t))$;*
(5) *T preserves probabilistic distance with respect to g.*

Then there exists a unique element $x \in A$ such that $F_{gx,Tx}(t) = F_{A,B}(t)$.

Proof. Let start by choosing an element Tx_0 in $T(A_0(t))$. Using the hypothesis, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $F_{gx_1,Tx_0}(t) =$

$F_{A,B}(t)$. Further, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $F_{gx_2,Tx_1}(t) = F_{A,B}(t)$. Recursively, we obtain a sequence $(Tx_n) \in B$ satisfying for all $n \in \mathbb{N}$

$$F_{gx_{n+1},Tx_n}(t) = F_{A,B}(t). \tag{3.17}$$

Now we will prove that the sequence (Tx_n) is convergent in B. If there exists $n_0 \in \mathbb{N}$ such that $F_{Tgx_{n_0},Tgx_{n_0+1}}(t) = 1$, then it is clear that the sequence (Tx_n) is convergent. So, let for all $n \in \mathbb{N}$, $F_{Tgx_{n_0},Tgx_{n_0+1}}(t) \neq 1$. From the hypothesis of the theorem, T is a γ-proximal contraction of second type

$$\begin{aligned} \gamma(F_{Tgx_n,Tgx_{n+1}}(t)) \geq& \gamma(F_{Tx_{n-1},Tx_n}(t)) + \delta \\ \gamma(F_{Tx_n,Tx_{n+1}}(t)) \geq& \gamma(F_{Tx_{n-1},Tx_n}(t)) + \delta \\ &\vdots \\ \geq& \gamma(F_{Tx_0,Tx_1}(t)) + n\delta. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$, in previous inequality we have $\lim_{n\to\infty} \gamma(F_{Tx_n,Tx_{n+1}}(t)) = +\infty$. If we continue same way that used in proof of Theorem 3.1, we can say that (Tx_n) is a Cauchy sequence in B. In theorem hyphothesis, complete non-Archimedean Menger probabilistic metric space (X, F, Δ) has closed subsets, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$. Using the triangle inequality

$$\begin{aligned} F_{y,A}(t) \geq F_{y,gx_n}(t) \geq& \Delta(F_{y,Tx_{n-1}}(t), F_{Tx_{n-1},gx_n}(t)) \\ =& \Delta(F_{y,Tx_{n-1}}(t), F_{A,B}(t)) \\ \geq& \Delta(F_{y,Tx_{n-1}}(t), F_{y,A}(t)). \end{aligned} \tag{3.19}$$

In (3.19), if we take the limit as $n \to \infty$, we have $\lim_{n\to\infty} F_{y,gx_n}(t) = F_{y,A}(t)$. Due to the fact that A is approximatively compact with respect to B, there exists a subsequence (gx_{n_k}) of (gx_n) such that converges to some $w \in A$.

Hence, $F_{w,y}(t) = \lim_{k\to\infty} F_{gx_{n_k},Tgx_{n_k-1}}(t) = F_{y,A}(t)$. It implies that $w \in A_0(t)$. $A_0(t) \subseteq g(A_0(t))$, there exists $x \in A_0(t)$ such that $w = gx$. As we know, $\lim_{n\to\infty} gx_{n_k} = gx$ and g is an isometry, we have $\lim_{n\to\infty} x_{n_k} = x$. (Tx_n) converges to y and the continuity of T, we can write $\lim_{n\to\infty} Tx_{n_k} = Tx = y$. As a result that, $F_{gx,Tx}(t) = \lim_{n\to\infty} F_{gx_{n_k},Tgx_{n_k}} = F_{A,B}(t)$. The uniqueness can be shown using the same way in Theorem 3.1. □

Example 3.5. Let $X = \mathbb{R} \times [0, 1]$ and (X, F, Δ) be the non-Archimedean Menger probabilistic metric space given in Example 3.2. Let $A = \{(x, 0) : \text{for all } x \in \mathbb{R}\}$, $B = \{(y, 1) : \text{for all } y \in \mathbb{R}\}$. Then, here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $F_{A,B}(t) = \frac{t}{t+1}$. $\gamma : [0, 1) \to \mathbb{R}$ defined as $\gamma(x) = \frac{1}{\sqrt{1-x}}$, for all $x \in X$. Let $T : A \to B$ and $g : A \to A$, these are defined as $T(x, 0) = \left(\frac{x}{3}, 1\right)$ and $g(x, 0) = (-x, 0)$. Then, $T(A_0(t)) = B_0(t)$, $A_0(t) = g(A_0(t))$ and g is an isometry. Let us consider

$$F_{a_1,Tx_1}(t) = F_{A,B}(t), \quad F_{a_2,Tx_2}(t) = F_{A,B}(t).$$

Also, $F_{Tgx_1,Tgx_2}(t) = F_{Tx_1,Tx_2}(t)$ and this says that T preserves isometric distance with respect to g. We have $(a_1, x_1) = \left(\left(\frac{b_1}{3}, 0\right), (b_1, 0)\right)$ or $(a_2, x_2) = \left(\left(\frac{b_2}{3}, 0\right), (b_2, 0)\right)$. We must show that, T is a γ-proximal contraction of second type

$$\begin{aligned}
\gamma(F_{Ta_1,Ta_2}(t)) &= \gamma\left(F_{\left(\frac{b_1}{9},1\right),\left(\frac{b_2}{9},1\right)}(t)\right) = \gamma\left(\frac{t}{t+\frac{|b_1-b_2|}{9}}\right) \\
&= \frac{1}{\sqrt{1-\left(\frac{t}{t+\frac{|b_1-b_2|}{9}}\right)}} > \frac{1}{\sqrt{1-\left(\frac{t}{t+\frac{|b_1-b_2|}{3}}\right)}} = \gamma\left(\frac{t}{t+\frac{|b_1-b_2|}{3}}\right) \qquad (3.20)\\
&= \gamma(F_{Tx_1,Tx_2}(t)).
\end{aligned}$$

From (3.20), we have $\gamma(F_{Ta_1,Ta_2}(t)) > \gamma(F_{Tx_1,Tx_2}(t))$. So, we can find a $\delta \in (0, 1)$ such that $\gamma(F_{Ta_1,Ta_2}(t)) \geq F_{Tx_1,Tx_2}(t)) + \delta$. Then T is a γ-contraction of second type and $(0, 0)$ is a unique best proximity point of T.

If we assume that g is the identity mapping, we can give the following result.

Corollary 3.3. *Let (X, F, Δ) be a complete non-Archimedean Menger probabilistic metric space and A, B be two nonempty, closed subsets of this space such that $A_0(t)$ is nonempty. Assume that A is approximately compact with respect to B. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(1) *$T(A_0(t)) \subseteq B_0(t)$;*

(2) *$T : A \to B$ is a continuous $\gamma -$ proximal contraction of second type.*

Then, T has a unique probabilistic best proximity point in A.

4. Conclusion

The purpose of this paper is to give best proximity point theorems for γ-contractions and also γ-proximal contractions of first and second type. These are proved and supported with examples.

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References

[1] F. Ali, J. Ali and I. Uddin, *A novel approach for the solution of BVPs via Green's function and fixed point iterative method*, J. Appl. Math. Comput. **60** (2020), 1–15. https://doi.org/10.1007/s12190-020-01431-7

[2] J. Ali, M. Imdad and D. Bahuguna, *Common fixed point theorems in Menger spaces with common property (E.A)*, Comput. Math. Appl. **60**(12) (2010), 3152–3159. https://doi.org/10.1016/j.camwa.2010.10.020

[3] G. A. Anastassiou, Y. J. Cho, R. Saadati and Y. O. Yang, *Common best proximity points for proximally commuting mappings in non-Archimedean pm-spaces*, J. Comput. Anal. Appl. **20**(6) (2016), 1021–1030.

[4] M. Asim, I. Uddin and M. Imdad, *Fixed point results in M_v-metrices with an application*, J. Inequal. Appl. **280** (2019), 1–19. https://doi.org/10.1186/s13660-019-2223-3

[5] S. S. Chang, *On some fixed point theorems in probabilistic metric space and its application*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **63** (1983), 463–474.

[6] S. S. Chang, *On some fixed point theorems for generalized Meir-Keeler type mappings*, Journal of Sichuan University (Natural Science Edition) (1983), 17–20.

[7] L. Ćirić, *Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces*, Nonlinear Anal. **72**(3-4) (2010), 2009–2018. https://doi.org/10.1016/j.na.2009.10.001

[8] P. Das, S. Som, S. Ghosal and V. Karakaya, *A notion of $\alpha\beta$-statistical convergence of order γ in probability*, Kragujevac J. Math. **42**(1) (2018), 51–67.

[9] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. **112** (1969), 234–240.

[10] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[11] M. Imdad, J. Ali and M. Tanveer, *Coincidence and common fixed point theorems for nonlinear contractions in Menger pm spaces*, Chaos Solitons Fractals **42**(5) (2009), 3121–3129. https://doi.org/10.1016/j.chaos.2009.04.017

[12] J. Jachymski, *On probabilistic ϕ-contractions on Menger spaces*, Nonlinear Anal. **73**(7) (2010), 2199–2203. https://doi.org/10.1016/j.na.2010.05.046

[13] M. Jamali and S. M. Vaezpour, *Best proximity point for certain nonlinear contractions in Menger probabilistic metric spaces*, J. Adv. Math. Stud. **9**(2) (2016), 338–347.

[14] M. Jamali and S. M. Vaezpour, *Existence and convergence of best proximity points in new type of contractions in Menger probabilistic metric spaces*, J. Nonlinear Funct. Anal. **2016** (2016), Article ID 36.

[15] K. Menger, *Statistical metrics*, Proc. Amer. Math. Soc. **28**(12) (1942), 535–537. https://doi.org/10.1073/pnas.28.12.535

[16] D. Mihet, *Fixed point theorems in probabilistic metric spaces*, Chaos Solitions Fractals **41**(2) (2009), 1014–1019. https://doi.org/10.1016/j.chaos.2008.04.030

[17] D. Mihet, *Altering distances in probabilistic Menger spaces*, Nonlinear Anal. **71**(7–8) (2009), 2734–2738. https://doi.org/10.1016/j.na.2009.01.107

[18] D. O'Regan and R. Saadati, *Nonlinear contraction theorems in probabilistic spaces*, Appl. Math. Comput. **195**(1) (2008), 86–93. https://doi.org/10.1016/j.amc.2007.04.070

[19] R. Saadati, *Best proximity point theorems for probabilistic proximal cyclic contraction with applications in nonlinear programming*, Fixed Point Theory and Algorithms for Sciences and Engineering **2015** (2015), Article ID 79. https://doi.org/10.1186/s13663-015-0330-5

[20] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North Holland series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

[21] V. M. Sehgal, *Some fixed point theorems in functional analysis and probability*, Ph.D. Thesis, Wayne State University, 1966.

[22] V. M. Sehgal and A. T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Mathematical Systems Theory **6** (1972), 97–102.

[23] M. Sangurlu Sezen, *Fixed point theorems for new type contractive mappings*, J. Funct. Spaces **2019** (2019). https://doi.org/10.1155/2019/2153563

[24] Y. Su and J. Zhang, *Fixed point and best proximity point theorems for contractions in new class of probabilistic metric spaces*, Fixed Point Theory and Algorithms for Sciences and Engineering **2014** (2014), Article ID 170.

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