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WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF ALMOST CONTACT MANIFOLDS

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Abstract. B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds. By using the notion of pointwise slant submanifolds, we investigate the geometry of pointwise semi-slant submanifolds and their warped products in Sasakian and cosymplectic manifolds. We prove that there exist no proper pointwise semi-slant warped product submanifold other than contact CR-warped products in Sasakian manifolds. We give non-trivial examples of such submanifolds in cosymplectic manifolds and obtain several fundamental results, including a characterization for warped product pointwise semi-slant submanifolds.

1. Introduction

In [7], B.-Y. Chen introduced the notion of slant submanifolds of almost Hermitian manifolds as a natural generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. Afterwards, the geometry of slant submanifolds became an active topic of research in differential geometry. Later, A. Lotta [19] has extended this study for almost contact metric manifolds. J. L. Cabrerizo et al. investigated slant submanifolds of a Sasakian manifold [6]. N. Papaghiuc introduced in [21] a class of submanifolds, called semi-slant submanifolds of almost Hermitian manifolds, which are the generalizations of slant and CR-submanifolds. Later on, Cabrerizo et al. [5] extended this idea for semi-slant submanifolds of contact metric manifolds and provided many examples of such submanifolds.

Next, as an extension of slant submanifolds of an almost Hermitian manifold, F. Etayo [15] introduced the notion of pointwise slant submanifolds of almost Hermitian

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manifolds. In 2012, B.-Y. Chen and O. J. Garay [13] studied pointwise slant submanifolds of almost Hermitian manifolds. They have obtained several fundamental results, in particular, a characterization of these submanifolds. K. S. Park [22] has extended this study for almost contact metric manifolds. In his definition of pointwise slant submanifolds of almost contact metric manifolds he did not mention whether the structure vector field ξ is either tangent or normal to the submanifold. B. Sahin studied pointwise semi-slant submanifolds and warped product pointwise semi-slant submanifolds by using the notion of pointwise slant submanifolds [25]. In [31] the authors modified the definition of pointwise slant submanifolds of an almost contact metric manifold such that the structure vector field ξ is tangent to the submanifold. We have obtained a simple characterization for such submanifolds and studied warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In 1969, R. L. Bishop and B. O'Neill [3] introduced and studied warped product manifolds. 30 years later, around the beginning of this century, B.-Y. Chen initiated in [8, 9] the study of warped product CR-submanifolds of Kaehler manifolds. Chen's work in this line of research motivated many geometers to study the geometry of warped product submanifolds by using his idea for different structures on manifolds (see, for instance [2, 16, 20, 26]). For a detailed survey on warped product submanifolds we refer to Chen's books [10, 12] and his survey article [11] as well.

In [23], B. Sahin showed that there exists no proper warped product semi-slant submanifold of Kaehler manifolds. Then, he introduced the notion of warped product hemi-slant submanifolds of Kaehler manifolds [24]. In 2013, he defined and studied warped product pointwise semi-slant submanifolds and showed that there exists a non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kaehler manifold $\tilde{M}$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively [25]. For almost contact metric manifolds, we have seen in [18] and [1] that there are no proper warped product semi-slant submanifolds in cosymplectic and Sasakian manifolds. Then, we have considered warped product pseudo-slant submanifolds (warped product hemi-slant submanifolds [24], in the same sense of almost Hermitian manifolds) of cosymplectic manifolds [28] and Sasakian manifolds [29].

Recently, K. S. Park [22] studied warped product pointwise semi-slant submanifolds of almost contact metric manifolds. He proved that there do not exist warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ in $\tilde{M}$, where $\tilde{M}$ is either a cosymplectic manifold, a Sasakian manifold or a Kenmotsu manifold such that M_θ and M_T are proper pointwise slant and invariant submanifolds of $\tilde{M}$, respectively. Then he provided many examples and obtained several results for warped products by reversing these two factors, including sharp estimations for the squared norm of the second fundamental form in terms of the warping functions. Later, we also extended this idea in [31] to warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In this paper, we study warped product pointwise semi-slant submanifolds of the form $M_T \times_f M_\theta$ of Sasakian and cosymplectic manifolds.

The present paper is organized as follows: In Section 2 we give basic definitions and formulae needed for this paper. Section 3 is devoted to the study of pointwise semi-slant submanifolds of almost contact metric manifolds. We define pointwise semi-slant submanifolds and in the definition of pointwise semi-slant submanifolds we assume that the structure vector field ξ is always tangent to the submanifold. We give two non-trivial examples of such submanifolds for the justification of our definition and a result which is useful to the next section. In Section 4 we study warped product pointwise semi-slant submanifolds of Sasakian and cosymplectic manifolds. We prove that there is no proper pointwise semi-slant warped product $M = M_T \times_f M_\theta$ other than contact CR-warped product in Sasakian manifolds, but if we assume the ambient space is cosymplectic then there exists a non-trivial class of such warped products. We obtain several new results which are generalizations of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In Section 5 we provide nontrivial examples of Riemannian product and warped product pointwise semi-slant submanifolds in Euclidean spaces.

2. Preliminaries

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional manifold $\tilde{M}$ is defined by a $(1,1)$ tensor field φ, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying the following conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \eta(\xi) = 1,$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map [4]. There always exists a Riemannian metric g on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any $X, Y \in \Gamma(T\tilde{M})$, the Lie algebra of vector fields on $\tilde{M}$. The metric g is called a *compatible metric* and the manifold $\tilde{M}$ together with the structure (φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.1), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. If ξ is a Killing vector field with respect to g, then the contact metric structure is called a *K-contact structure*. An almost contact metric manifold is called *almost cosymplectic* if $d\eta = 0$ and $d\varphi = 0$, according to D. E. Blair ([4]). In particular, a normal almost cosymplectic manifold is called *cosymplectic* and satisfies

$$\widetilde{\nabla}\varphi = 0, \quad \widetilde{\nabla}\xi = 0. \tag{2.2}$$

A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of φ, the Sasakian condition can be expressed by

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From the formula (2.3), it follows that

$$\tilde{\nabla}_X \xi = -\varphi X, \tag{2.4}$$

for any $X \in \Gamma(T\tilde{M})$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^{\perp}M)$ the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M; the Gauss and Weingarten formulae are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of M with respect to the normal vector N. Moreover, $h : TM \times TM \to T^{\perp}M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_N and h are related by [32]

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

For any X tangent to M, we write

$$\varphi X = PX + FX, \tag{2.8}$$

where PX and FX are the tangential and normal components of φX, respectively. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$\varphi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of φN, respectively. Moreover, from (2.1) and (2.8), we have

$$g(PX, Y) = -g(X, PY), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$.

Throughout this paper, we assume that the structure vector field ξ is tangent to M, otherwise M is a C-totally real submanifold [19]. Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *slant* [6] if for each non-zero vector X tangent to M at $p \in M$ such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and on the choice of $X \in T_pM - \langle \xi_p \rangle$.

A slant submanifold is said to be *proper slant* if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$. We note that if $\theta = 0$ then the submanifold is an invariant submanifold and if $\theta = \frac{\pi}{2}$ then it is an anti-invariant submanifold (equivalently, a slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant).

As a natural extension of slant submanifolds, F. Etayo [15] introduced pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. Later on, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and obtained many interesting results [13]. In a similar way, K. S. Park [22] defined and studied pointwise slant submanifols of almost contact metric manifolds. His definition of pointwise slant submanifolds of almost contact metric manifold is similar to the pointwise slant submanifolds of almost Hermitian manifolds, therefore we have modified his definition by considering the structure vector field ξ is tangent to the submanifold and studied pointwise slant submanifolds of almost contact metric manifolds in [31].

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *pointwise slant* if for any nonzero vector X tangent to M at $p \in M$, such that X is not propotional to ξ_p, the angle $\theta(X)$ between φX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in T_p^*M$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold.

We note that every slant submanifold is a pointwise slant submanifold, but the converse may not be true. We also note that a pointwise slant submanifold is *invariant* (respectively, *anti-invariant*) if for each point $p \in M$, the slant function $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$). A pointwise slant submanifold is slant if and only if the slant function θ is constant on M. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor θ is constant.

In [31], the authors have obtained the following characterization theorem.

Theorem 2.1 ([31])**.** *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is pointwise slant if and only if*

$$P^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{2.11}$$

for some real valued function θ defined on the tangent bundle TM of M.

The following relations are immediate consequences of Theorem 2.1.

Let M be a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$. Then we have

$$g(PX, PY) = \cos^2\theta\,[g(X,Y) - \eta(X)\eta(Y)],$$

$$g(FX, FY) = \sin^2\theta\,[g(X,Y) - \eta(X)\eta(Y)],$$

for any $X, Y \in \Gamma(TM)$.

The next useful relation for a pointwise slant submanifold of an almost contact metric manifold was obtained in [31]

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi)\,, \quad fFX = -FPX, \tag{2.12}$$

for any $X \in \Gamma(TM)$.

3. Pointwise Semi-Slant Submanifolds

In [25], B. Sahin defined and studied pointwise semi-slant submanifolds of Kaehler manifolds. In this section, we define and study pointwise semi-slant submanifolds of almost contact metric manifolds.

Definition 3.1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a *pointwise semi-slant submanifold* if there exists a pair of orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ on M such that

(i) the tangent bundle TM admits the orthogonal direct decomposition $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle$;
(ii) the distribution $\mathfrak{D}$ is invariant under φ, i.e., $\varphi(\mathfrak{D}) = \mathfrak{D}$;
(iii) the distribution $\mathfrak{D}^\theta$ is pointwise slant with slant function θ.

Note that the normal bundle $T^\perp M$ of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = F\mathfrak{D}^\theta \oplus \nu, \quad F\mathfrak{D}^\theta \perp \nu,$$

where ν is an invariant normal subbundle of $T^\perp M$ under φ.

We denote the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by m_1 and m_2, respectively. Then the following hold.

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is an invariant submanifold.
(iii) If $m_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(iv) If $m_1 \neq 0$ and $\theta = \frac{\pi}{2}$, then M is a contact CR-submanifold.
(v) If θ is constant on M, then M is a semi-slant submanifold with slant angle θ.

We also note that a pointwise semi-slant submanifold is *proper* if neither $m_1, m_2 = 0$ nor $\theta = 0, \frac{\pi}{2}$ and θ should not be a constant.

We provide the following non-trivial examples of pointwise semi-slant submanifolds of almost contact metric manifolds.

Example 3.1. Let $(\mathbb{R}^7, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1,\, y_1,\, x_2,\, y_2,\, x_3,\, y_3,\, z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 3,$$

where $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g is the standard Euclidean metric on $\mathbb{R}^7$. Then (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Consider a submanifold M of $\mathbb{R}^7$ defined by

$$\psi(u, v, w, t, z) = (u + v,\ -u + v,\ t\cos w,\ t\sin w,\ w\cos t,\ w\sin t,\ z),$$

such that w, t $(w \neq t)$ are non vanishing real valued functions on M. Then the tangent space TM is spanned by the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \\ X_3 &= -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3}, \\ X_4 &= \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - w\sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_3}, \\ X_5 &= \frac{\partial}{\partial z}. \end{aligned}$$

Thus, we observe that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}((t-w)/\sqrt{(t^2+1)(w^2+1)})$. Hence, M is a pointwise semi-slant submanifold of $\mathbf{R}^7$ such that $\xi = \frac{\partial}{\partial z}$ is tangent to M.

Example 3.2. Consider a submanifold of $\mathbb{R}^7$ with almost contact structure φ given in Example 3.1. If the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^7$ is given by

$$\psi(u_1, u_2, u_3, u_4, t) = (u_1, (u_3^2 + u_4^2)/2, \cos u_4, -u_2,\, (u_3^2 - u_4^2)/2, \sin u_4,\, t), \quad u_4 \neq 0,$$

then the tangent space TM is spanned by X_1, X_2, X_3, X_4 and X_5 where

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1}, \;\; X_2 = -\frac{\partial}{\partial y_1}, \;\; X_3 = u_3\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial y_2}, \\ X_4 &= u_4\frac{\partial}{\partial x_2} - u_4\frac{\partial}{\partial y_2} - \sin u_4\frac{\partial}{\partial x_3} + \cos u_4\frac{\partial}{\partial y_3}, \\ X_5 &= \frac{\partial}{\partial t}. \end{aligned}$$

Then, M is a pointwise semi-slant submanifold such that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}(\sqrt{2}\,u_4/\sqrt{1+2u_4^2})$.

Now, we obtain the following useful results for semi-slant submanifolds of a Sasakian (or cosymplectic) manifold.

Lemma 3.1. *Let M be a pointwise semi-slant submanifold of a Sasakian* (*or cosymplectic*) *manifold $\tilde{M}$. Then we have*

(i) $\sin^2\theta\, g(\nabla_X Y, Z) = g(h(X,\varphi Y), FZ) - g(h(X,Y), FPZ)$;
(ii) $\sin^2\theta\, g(\nabla_Z W, X) = g(h(X,Z), FPW) - g(h(\varphi X, Z), FW)$,

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first and second parts of the lemma can be proved in a similar way. For any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$ we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z).$$

From the covariant derivative formula of φ, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

Then from (2.3), (2.8) and the orthogonality of the two distributions, we find

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X \varphi Y, PZ) + g(\tilde{\nabla}_X \varphi Y, FZ) \\ &= -g(\tilde{\nabla}_X PZ, \varphi Y) + g(h(X, \varphi Y), FZ) \\ &= g(\varphi\tilde{\nabla}_X PZ, Y) + g(h(X, \varphi Y), FZ). \end{aligned}$$

Again, from the covariant derivative formula of φ, we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi PZ, Y) - g((\tilde{\nabla}_X \varphi)PZ, Y) + g(h(X, \varphi Y), FZ).$$

Using (2.3), (2.8) and the orthogonality of vector fields, we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X P^2 Z, Y) + g(\tilde{\nabla}_X FPZ, Y) + g(h(X, \varphi Y), FZ).$$

Then, from (2.11) and (2.6), we have

$$\begin{aligned} g(\nabla_X Y, Z) = &- \cos^2\theta\, g(\tilde{\nabla}_X Z, Y) + \sin 2\theta\, X(\theta)\, g(Y, Z) - g(h(X, Y), FPZ) \\ &+ g(h(X, \varphi Y), FZ). \end{aligned}$$

From the orthogonality of the two distributions the above equation takes the form

$$g(\nabla_X Y, Z) = \cos^2\theta\, g(\tilde{\nabla}_X Y, Z) - g(h(X, Y), FPZ) + g(h(X, \varphi Y), FZ).$$

Hence, (i) follows from the above relation. In a similar way we can prove (ii). □

4. Warped Product Pointwise Semi-Slant Submanifolds

In [3], R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian metric

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y),$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant.

Let X be a vector field tangent to M_1 and Z be an another vector field on M_2; then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{4.1}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold, then the base manifold M_1 is totally geodesic in M and the fiber M_2 is totally umbilical in M [3, 8].

By analogy to CR-warped products, which were introduced by B.-Y. Chen in [8], we define the warped product pointwise semi-slant submanifolds as follows.

Definition 4.1. A warped product of an invariant submanifold M_T and a pointwise slant submanifold M_θ of an almost contact metric manifold $\tilde{M}$ is called a *warped product pointwise semi-slant submanifold*.

A warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is called *proper* if M_θ is a proper pointwise slant submanifold and M_T is an invariant submanifold of $\tilde{M}$ and the function f on M is not constant.

The non-existence of warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ of Kaehler and Sasakian manifolds is proved in [25] and [22]. On the other hand, there exist non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Kaehler manifolds [25] and contact metric manifolds [22].

In this section, we study the warped product pointwise semi-slant submanifold of the form $M = M_T \times_f M_\theta$. Notice that a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product contact CR-submanifold if the slant function $\theta = \frac{\pi}{2}$. Similarly, the warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product semi-slant submanifold if θ is constant on M, i.e., M_θ is a proper slant submanifold.

Remark 4.1. On a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ of a Sasakian (or cosymplectic) manifold $\tilde{M}$, we consider the structure vector field ξ tangent to M; then either $\xi \in \Gamma(TM_T)$ or $\xi \in \Gamma(TM_\theta)$.

When ξ is tangent to M_θ, then it is easy to check that warped product is trivial (see [26] and [18]), therefore we always consider $\xi \in \Gamma(TM_T)$.

First, we prove the following non-existence result of pointwise semi-slant warped products.

Theorem 4.1. *There do not exist any proper pointwise semi-slant warped product submanifolds of Sasakian manifolds other than contact CR-warped products.*

Proof. Let $M = M_\theta \times_f M_T$ be a pointwise semi-slant warped product submanifold. Then, in a similar way of Theorem 11 and Theorem 12 of [22], we find that M is a Riemannian product of M_T and M_θ.

On the other hand, if $M = M_T \times_f M_\theta$ and ξ is tangent to M_θ, then by Remark 4.1, M is again a Riemannian product of M_T and M_θ. Furthermore, if ξ is tangent to M_T, then from (2.3) and (2.4), we have

$$\nabla_Z \xi + h(Z, \xi) = -PZ - FZ, \tag{4.2}$$

for any $Z \in \Gamma(TM_\theta)$. Then, equating the tangential component of (4.2) and using (4.1), we obtain

$$\xi(\ln f)Z = -PZ. \tag{4.3}$$

Taking the inner product with $W \in \Gamma(TM_\theta)$, we find

$$\xi(\ln f)g(Z,W) = -g(PZ,W). \tag{4.4}$$

By interchanging Z and W, we get

$$\xi(\ln f)g(Z,W) = g(PZ,W). \tag{4.5}$$

From (4.4) and (4.5), we find $\xi(\ln f) = 0$. Then, from (4.3), we get $PZ = 0$, which means that $\theta = \frac{\pi}{2}$. Hence, M is a contact CR-warped product, which proves the theorem completely. □

Next we find that if we replace the ambient manifold Sasakian to cosymplectic, then there exists a non-trivial class of pointwise semi-slant warped products.

Lemma 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then we have*

$$g(h(X,W), FPZ) - g(h(X,PZ), FW) = \sin 2\theta\, X(\theta)\, g(Z,W),$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(\tilde{\nabla}_X Z, W) = X(\ln f)\, g(Z,W). \tag{4.6}$$

On the other hand, we can obtain

$$g(\tilde{\nabla}_X Z, W) = g(\varphi\tilde{\nabla}_X Z, \varphi W).$$

Using the covariant derivative formula of φ, we get

$$g(\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X \varphi Z, \varphi W).$$

Then, from (2.5), (2.8), (4.1) and the orthogonality of vector fields, we find

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& g(\tilde{\nabla}_X PZ, PW) + g(\tilde{\nabla}_X PZ, FW) + g(\tilde{\nabla}_X FZ, \varphi W) \\ =& X(\ln f)\, g(PZ,PW) + g(h(X,PZ),FW) - g(\varphi\tilde{\nabla}_X FZ, W) \\ =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) - g(\tilde{\nabla}_X \varphi FZ, W). \end{aligned}$$

From (2.9) and (2.12), we derive

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) + \sin^2\theta\, g(\tilde{\nabla}_X Z, W) \\ &+ \sin 2\theta\, X(\theta)\, g(Z,W) + g(\tilde{\nabla}_X FPZ, W). \end{aligned} \tag{4.7}$$

Hence, the result follows from (4.6) and (4.7) by using (2.6), (2.7) and (4.1). □

Lemma 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

(i) $\xi(\ln f) = 0$;

(ii) $g(h(X,Y), FZ) = 0$;

(iii) $g(h(X,Z), FW) = X(\ln f)\, g(PZ,W) - \varphi X(\ln f)\, g(Z,W)$,

for any $X, Y \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. From (2.2), (2.5) and (2.8), we have $\nabla_Z \xi + h(Z,\xi) = 0$ for any $Z \in \Gamma(TM_\theta)$, which implies that $\xi(\ln f) = 0$ by using (4.1). (ii) is proved in [22] (see relation (100) in [22]). Now, for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(h(X,Z), FW) = g(\tilde{\nabla}_Z X, FW) = g(\tilde{\nabla}_Z X, \varphi W) - g(\tilde{\nabla}_Z X, PW).$$

Using the covariant derivative formula of the Riemannain connection and (4.1), we get

$$g(h(X,Z), FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - X(\ln f)\, g(Z, PW).$$

Then from (2.2), (2.5) and (4.1), we derive

$$g(h(X,Z), FW) = -\varphi X(\ln f)\, g(Z,W) - X(\ln f)\, g(Z, PW),$$

which is third part of the lemma. Hence, the proof is complete. □

Interchanging X and φX, for any $X \in \Gamma(TM_T)$ in Lemma 4.2 (iii), we obtain relation

$$g(h(\varphi X, Z), FW) = X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z, PW), \tag{4.8}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Similarly, interchanging Z and PZ, for any $Z \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), we obtain

$$g(h(X, PZ), FW) = \varphi X(\ln f)\, g(Z, PW) - \cos^2\theta\, X(\ln f)\, g(Z,W), \tag{4.9}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Similarly, if we interchange W and PW, for any $W \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), then we derive

$$g(h(X,Z), FPW) = \cos^2\theta\, X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z, PW), \tag{4.10}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Lemma 4.3. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then we have*

$$g(A_{FW}\varphi X, Z) - g(A_{FPW}X, Z) = \sin^2\theta\, X(\ln f)\, g(Z,W), \tag{4.11}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Subtracting (4.10) from (4.8), we get (4.11). □

A warped product submanifold $M = M_1 \times_f M_2$ is *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$.

From Lemma 4.3, we obtain the following result.

Theorem 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is warped product of invariant submanifolds or the warping function f is constant on M.*

Proof. From (4.11) and the mixed totally geodesic condition, we have

$$\sin^2\theta \, X(\ln f)\, g(Z, W) = 0.$$

Since g is a Riemannian metric, then either $\sin^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is warped product of invariant submanifolds or f is constant on M. Thus, the proof is complete. □

Lemma 4.4. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then we have*

$$g(A_{FPZ}W, X) - g(A_{FW}PZ, X) = 2\cos^2\theta \, X(\ln f)\, g(Z, W), \tag{4.12}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchanging Z and W in (4.10) and using (2.10), we get

$$g(h(X, W), FPZ) = \cos^2\theta \, X(\ln f)\, g(Z, W) + \varphi X(\ln f)\, g(Z, PW), \tag{4.13}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$. Subtracting (4.9) from (4.13), we find (4.12). □

Also, with the help of Lemma 4.4, we find the following result.

Theorem 4.3. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is a contact CR-warped product of the form $M_T \times_f M_\perp$ or the warping function f is constant on M.*

Proof. From (4.12) and the mixed totally geodesic condition, we have

$$\cos^2\theta \, X(\ln f)\, g(Z, W) = 0.$$

Since g is a Riemannian metric, then either $\cos^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is a contact CR-warped product or f is constant on M, which ends the proof. □

From Theorem 4.2 and Theorem 4.3, we conclude the following.

Corollary 4.1. *There does not exist any mixed totally geodesic proper warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ of a cosymplectic manifold.*

Also, from Lemma 4.1 and Lemma 4.4, we have the following result.

Theorem 4.4. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a pointwise slant submanifold of $\tilde{M}$. Then, either M is a contact CR-warped product of the form $M = M_T \times_f M_\perp$ or $\nabla(\ln f) = \tan\theta\,\nabla\theta$, for any $X \in \Gamma(TM_T)$, where $M_\perp$ is an anti-invariant submanifold and ∇f is the gradient of f.*

Proof. From Lemma 4.1 and Lemma 4.4, we have

$$\cos^2\theta\{X(\ln f) - \tan\theta\, X(\theta)\}\, g(Z, W) = 0.$$

Since g is a Riemannian metric, therefore we conclude that either $\cos^2\theta = 0$ or $X(\ln f) - \tan\theta\, X(\theta) = 0$. Consequently, either $\theta = \frac{\pi}{2}$ or $X(\ln f) = \tan\theta\, X(\theta)$, which proves the theorem completely. ☐

As an application of Theorem 4.4, we have the following consequence.

Remark 4.2. If we consider that the slant function θ is constant, i.e., M_θ is a proper slant submanifold in Theorem 4.4, then $Z(\ln f) = 0$, i.e., there are no warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$ in cosymplectic manifolds. Hence, Theorem 4.1 of [18] is a special case of Theorem 4.4.

In order to give a characterization result for pointwise semi-slant submanifolds of a cosymplectic manifold, we recall the following well-known result of Hiepko [17].

Theorem 4.5 (Hiepko's Theorem)**.** *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

By using Theorem 4.5, we prove the following theorem.

Theorem 4.6. *Let M be a pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally a non-trivial warped product submanifold of the form $M_T \times_f M_\theta$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$ if and only if*

$$A_{FW}\varphi X - A_{FPW}X = \sin^2\theta\, X(\mu)W, \quad \text{for all } X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle),\ W \in \Gamma(\mathfrak{D}^\theta), \tag{4.14}$$

for some smooth function μ on M satisfying $Z(\mu) = 0$ for any $Z \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a cosymplectic manifold $\tilde{M}$. Then for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, from Lemma 4.2 (ii) we have

$$g(A_{FW}X, Y) = 0. \tag{4.15}$$

Interchanging X and φX in (4.15), we get $g(A_{FW}\varphi X, Y) = 0$, which means that $A_{FW}\varphi X$ has no component in TM_T. Similarly, if we interchange W and PW in (4.15) then, we get $g(A_{FPW}X, Y) = 0$, i.e., $A_{FPW}X$ has no component in TM_T, too. Therefore, $A_{FW}\varphi X - A_{FPW}X$ lies in TM_θ, which together with Lemma 4.3, give (4.14).

Conversely, if M is a pointwise semi-slant submanifold such that (4.14) holds, then from Lemma 3.1 (i), we have

$$g(\nabla_X Y, W) = \csc^2\theta\, g(A_{FW}\varphi Y - A_{FPW}Y, X),$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathfrak{D}^\theta)$. From (4.14), we arrive at

$$g(\nabla_X Y, W) = Y(\mu) g(X, W) = 0,$$

which means that the leaves of the distribution $\mathfrak{D} \oplus \langle\xi\rangle$ are totally geodesic in M. Also, from Lemma 3.1 (ii), we have

$$g(\nabla_Z W, X) = \csc^2\theta\, g(A_{FPW}X - A_{FW}\varphi X, Z), \tag{4.16}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$. Interchanging Z and W, we derive

$$g(\nabla_W Z, X) = \csc^2\theta\, g(A_{FPZ}X - A_{FZ}\varphi X, W). \tag{4.17}$$

Subtracting (4.17) from (4.16), we get

$$\sin^2\theta\, g([Z, W], X) = g(A_{FZ}\varphi X - A_{FPZ}X, W) - g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.14), we get

$$\sin^2\theta\, g([Z, W], X) = X(\mu)\, g(Z, W) - X(\mu)\, g(W, Z) = 0.$$

Since M is proper pointwise semi-slant, then $\sin^2\theta \neq 0$, thus we conclude that the pointwise slant distribution $\mathfrak{D}^\theta$ is integrable.

Let M_θ be a leaf of $\mathfrak{D}^\theta$ and h^θ is the second fundamental form of M_θ in M. Then from (4.17), we have

$$g(h^\theta(Z, W), X) = g(\nabla_Z W, X) = -\csc^2\theta\, g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.14), we find

$$g(h^\theta(Z, W), X) = -X(\mu)\, g(Z, W).$$

Then from the definition of the gradient of a function, we arrive at

$$h^\theta(Z, W) = -(\vec{\nabla}\mu)\, g(Z, W).$$

Hence, M_θ is a totally umbilical submanifold of M with the mean curvature vector $H^\theta = -\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ. Since $Z(\mu) = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$, then we can show that $H^\theta = -\vec{\nabla}\mu$ is parallel with respect to the normal connection, say D^n, of M_θ in M (see [24, 25, 28]). Thus, M_θ is a totally umbilical submanifold of M with a non vanishing parallel mean curvature vector $H^\theta = -\vec{\nabla}\mu$, i.e., M_θ is an extrinsic sphere in M. Then from Heipko's Theorem [17], we conclude that M is a warped product manifold of M_T and M_θ, where M_T and M_θ are integral manifolds of $\mathfrak{D} \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively. Thus, the proof is complete. ☐

As an application of Theorem 4.6, for $\theta = \frac{\pi}{2}$ we obtain the following result as a special case of Theorem 4.6.

Corollary 4.2. ([27, Theorem 4.2]). *A proper CR-submanifold M of a cosymplectic manifold $\tilde{M}$ tangent to the structure vector field ξ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X = -\left(\varphi X(\mu)\right) Z, \quad X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle),\ Z \in \Gamma(\mathfrak{D}^\perp),$$

for some function μ on M satisfying $W\mu = 0$ for all $W \in \Gamma(\mathfrak{D}^\perp)$.

5. Examples

In this section, we provide the following non-trivial examples of Riemannian products and warped product pointwise semi-slant submanifolds in Euclidean spaces.

Example 5.1. Let M be a submanifold of the Euclidean 7-space $\mathbb{R}^7$ with its Cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, t)$ and the almost contact structure

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \quad \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \quad \varphi\Big(\frac{\partial}{\partial t}\Big) = 0, \quad 1 \le i, j \le 3.$$

If M is given by the equations

$$x_1 = u_1, \quad x_2 = u_3 \cos u_4, \quad x_3 = u_3^2/2, \quad y_1 = u_2, \quad y_2 = u_3 \sin u_4, \quad y_3 = u_4, \quad t = t,$$

for any non-zero function u_3 on M, then tangent space TM of M is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \cos u_4 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} + \sin u_4 \frac{\partial}{\partial y_2},$$
$$X_4 = -u_3 \sin u_4 \frac{\partial}{\partial x_2} + u_3 \sin u_4 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}.$$

Then M is a pointwise semi-slant submanifold with invariant distribution $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ and the pointwise slant distribution $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$. Clearly, the slant function is $\theta = \cos^{-1}(2u_3/\sqrt{1+u_3^2})$. Moreover, $\mathfrak{D}$ and $\mathfrak{D}^\theta$ are integrable. If M_T and M_θ are integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively, then, $M = M_T \times M_\theta$ is a Riemannian product of M_T and M_θ in $\mathbb{R}^9$.

Example 5.2. Consider the Euclidean 9-space $\mathbb{R}^9$ with its Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, t)$ and the almost contact structure

$$\varphi\Big(\frac{\partial}{\partial x_i}\Big) = -\frac{\partial}{\partial y_i}, \quad \varphi\Big(\frac{\partial}{\partial y_j}\Big) = \frac{\partial}{\partial x_j}, \quad \varphi\Big(\frac{\partial}{\partial t}\Big) = 0, \quad 1 \le i, j \le 4.$$

Let M be a submanifold of $\mathbb{R}^9$ defined by the following immersion:

$$\psi(u, v, w, s, t) = \Big(u + v, \frac{1}{2}w^2, s \cos w, s \sin w, -u + v, \frac{1}{2}s^2, -w \sin s, w \cos s, t\Big),$$

for any non-zero functions w and s. The tangent space of M is spanned by the following vectors

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1},\\
X_3 &= w\frac{\partial}{\partial x_2} - s\sin w\frac{\partial}{\partial x_3} + s\cos w\frac{\partial}{\partial x_4} - \sin s\frac{\partial}{\partial y_3} + \cos v\frac{\partial}{\partial y_4},\\
X_4 &= \cos w\frac{\partial}{\partial x_3} + \sin w\frac{\partial}{\partial x_4} + s\frac{\partial}{\partial y_2} - w\cos s\,\frac{\partial}{\partial y_3} - w\sin s\frac{\partial}{\partial y_4},\\
X_5 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Then, M is a pointwise semi-slant submanifold with the structure vector field $\xi = \frac{\partial}{\partial t}$ tangent to M, $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{X}_3, \mathrm{X}_4\}$ is a pointwise slant distribution with slant function

$$\theta = \cos^{-1}\left(\frac{(1-ws)\sin(w-s) - ws}{1+w^2+s^2}\right).$$

It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then M is a Riemannian product of invariant and pointwise slant submanifolds in $\mathbb{R}^9$.

Example 5.3. Let M be a submanifold of $\mathbb{R}^{13}$ given by the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^{13}$ as follows:

$$\begin{aligned}
\psi(u_1, v_1, u_2, v_2, t) =&(u_1 - v_1, u_1\cos(u_2+v_2), u_1\sin(u_2+v_2), v_2, u_1\cos(u_2-v_2),\\
&u_1\sin(u_2-v_2), u_1+v_1, v_1\cos(u_2+v_2), v_1\sin(u_2+v_2), u_2,\\
&v_1\cos(u_2-v_2), v_1\sin(u_2-v_2), t),
\end{aligned}$$

for non-zero functions u_1 and v_1. We use the almost contact structure from Example 5.2. Then, we have

$$\begin{aligned}
X_1 =&\frac{\partial}{\partial x_1} + \cos(u_2+v_2)\frac{\partial}{\partial x_2} + \sin(u_2+v_2)\frac{\partial}{\partial x_3} + \cos(u_2-v_2)\frac{\partial}{\partial x_5}\\
&+ \sin(u_2-v_2)\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_1},\\
X_2 =&-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \cos(u_2+v_2)\frac{\partial}{\partial y_2} + \sin(u_2+v_2)\frac{\partial}{\partial y_3} + \cos(u_2-v_2)\frac{\partial}{\partial y_5}\\
&+ \sin(u_2-v_2)\frac{\partial}{\partial y_6},\\
X_3 =&-u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\,\frac{\partial}{\partial x_3} - u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\
&+ u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2}, +v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\partial}{\partial y_4} - v_1 \sin(u_2 - v_2)\frac{\partial}{\partial y_5} + v_1 \cos(u_2 - v_2)\frac{\partial}{\partial y_6},\\
X_4 =& - u_1 \sin(u_2 + v_2)\frac{\partial}{\partial x_2} + u_1 \cos(u_2 + v_2)\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + u_1 \sin(u_2 - v_2)\frac{\partial}{\partial x_5}\\
&- u_1 \cos(u_2 - v_2)\frac{\partial}{\partial x_6} - v_1 \sin(u_2 + v_2)\frac{\partial}{\partial y_2} + v_1 \cos(u_2 + v_2)\frac{\partial}{\partial y_3}\\
&+ v_1 \sin(u_2 - v_2)\frac{\partial}{\partial y_5} - v_1 \cos(u_2 - v_2)\frac{\partial}{\partial y_6},\\
X_5 =& \frac{\partial}{\partial t}.
\end{aligned}
$$

By direct computations we find that $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{\mathrm{X}_3, \mathrm{X}_4\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{1+2u_1^2+2v_1^2}\right)$. Hence, M is a pointwise semi-slant submanifold of $\mathbb{R}^{13}$. It is easy to observe that both distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then the product metric structure of M is given by

$$g = 4(du_1^2 + dv_1^2) + (1 + 2u_1^2 + 2v_1^2)(du_2^2 + dv_2^2) = g_{M_T} + f^2 g_{M_\theta}.$$

Hence, $M = M_T \times_f M_\theta$ is a warped product submanifold in $\mathbb{R}^{13}$ with warping function $f = \sqrt{1 + 2u_1^2 + 2v_1^2}$.

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