

**CERTAIN PROPERTIES ON MEROMORPHIC FUNCTIONS
DEFINED BY A NEW LINEAR OPERATOR INVOLVING THE
MITTAG-LEFFLER FUNCTION**

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ABSTRACT. Our paper introduces a new linear operator using the convolution between a Mittag–Leffler Function and basic hypergeometric function. Use of the linear operator creates a new class of meromorphic functions defined in the punctured open unit disk. Consequently, the paper examines different aspects Apps and assets like, extreme points, coefficient inequality, growth and distortion. In conclusion, the work discusses modified Hadamard product and closure theorems.

1. INTRODUCTION

Let Σ indicate the class of type functions

$$(1.1) \quad h(z) = z^{-1} + \sum_{j=1}^{\infty} a_j z^j, \quad j \in \mathbb{N} = \{1, 2, 3, \dots\},$$

which are analytic in the punctured open unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

Denote by $\Sigma_i(\delta)$ and $\Sigma^*(\delta)$ the subclasses of Σ that are meromorphically convex function of order δ , and meromorphically starlike of order δ , respectively. Function $h \in \Sigma$ of the type (1.1), is in the class $\Sigma_i(\delta)$, if it meets

$$\operatorname{Re} \left\{ - \left(1 + \frac{zh''(z)}{h'(z)} \right) \right\} > \delta, \quad z \in U^*,$$

Key words and phrases. Linear operator, basic hypergeometric function, meromorphic function, Mittag–Leffler function, coefficient inequality, convex linear combination, growth and distortion theorem, extreme points, Hadamard product.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C50.

DOI 10.46793/KgJMat2403.473AK

Received: November 06, 2020.

Accepted: May 31, 2021.

and h is in the class $\Sigma^*(\delta)$, if it meets

$$\operatorname{Re} \left\{ -\frac{zh'(z)}{h(z)} \right\} > \delta, \quad z \in U^*.$$

The Hadamard product (or convolution) $h * k$ for two analytic functions h given by (1.1) in U^* and

$$k(z) = z^{-1} + \sum_{j=1}^{\infty} b_j z^j,$$

is define by

$$(h * k)(z) = z^{-1} + \sum_{j=1}^{\infty} a_j b_j z^j.$$

For complex components $q, b_k, a_i, b_k \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $k = 1, \dots, r$, $i = 1, \dots, m$, the basic hypergeometric function or (q -hypergeometric function) ψ_r^m is defined by:

$$(1.2) \quad \psi_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1, q)_j \cdots (a_m, q)_j}{(q, q)_j (b_1, q)_j \cdots (b_r, q)_j} \left[(-1)^j q^{\frac{j(1-j)}{2}} \right]^{1+r-m} z^j,$$

where $q \neq 0$, when $m > r + 1$, $m, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $(a, q)_j$ is q -analogue of the Pochhammer symbol $(a)_j$ is defined by

$$(a, q)_j = \begin{cases} (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{j-1}), & j = 1, 2, 3, \dots, \\ 1, & j = 0. \end{cases}$$

Initially, the function ψ_r^m given by (1.2), was introduced and referred to by Heine in 1846, as the series of Heine. For readers to refer to further q -theory information can be found in (see [9] and [11]).

Now, for $|q| < 1$, $m = r + 1$ and $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, the q -hypergeometric function [25] defined in Equation (1.2), takes the form below

$$\psi_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1, q)_j \cdots (a_m, q)_j}{(q, q)_j (b_1, q)_j \cdots (b_r, q)_j} z^j$$

that absolutely converges in the open unit disk \mathbb{U} .

With regard to the function $\psi_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z)$, for meromorphic function $h \in \Sigma$ that includes functions in shape of (1.1) (see work of [1] and [18]), which is shown below, have successfully introduced the q -analogue of the Liu–Srivastava operator

$$\begin{aligned} \mathcal{G}_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z) h(z) &= z_l^{-1} \psi_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z) * h(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^m (a_i, q)_{j+1}}{(q, q)_{j+1} \prod_{k=1}^r (b_k, q)_{j+1}} a_j z^j. \end{aligned}$$

Before we continue moving on, the Mittag-Leffler function $E_\delta(z)$, suggested by Mittag-Leffler (see [16] and [17]) and defined by

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad z \in \mathbb{U}, \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0,$$

where $\Gamma(\delta)$ denotes the Gamma function.

Also, Wiman [26], studied another function $E_{\delta,\mu}(z)$ have numerous similarities of $E_\delta(z)$, and given by

$$(1.3) \quad E_{\alpha,\mu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \mu)}, \quad z \in \mathbb{U}, \alpha, \mu \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\mu) > 0.$$

In recent years, there has been growing interest in Mittag-Leffler for application problems including, electric network, fluid flow, probability, statistical distribution theory, etc. (see [2, 4, 8, 12, 15, 19, 22–24] and [27] for more information about this function and its applications). Bansal and Prajapat recently investigated geometric characteristics in [5] for the function $E_{\alpha,\mu}(z)$, like starlikeness, convexity and closed to convex. In addition, certain results were obtained in [21] for the partial sum of the Mittag-Leffler function.

We note that, the function given by (1.3), is not part of class Σ . Therefore, the function $E_{\alpha,\mu}(z)$, is then normalized on the basis of the following:

$$(1.4) \quad \mathcal{E}_{\alpha,\mu}(z) = \Gamma(\mu) z^{-1} E_{\alpha,\mu}(z) = z^{-1} + \sum_{j=1}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\alpha(j+1) + \mu)} z^j.$$

Application of the function $\mathcal{E}_{\alpha,\mu}(z)$ defined by (1.4), a new operator $\mathfrak{J}_{\alpha,\mu} : \Sigma \rightarrow \Sigma$, is defined in terms of Hadamard product as follows

$$\begin{aligned} \mathfrak{J}_{\alpha,\mu} h(z) &= \mathcal{E}_{\alpha,\mu}(z) * \mathcal{G}_r^m(a_1, \dots, a_m; b_1, \dots, b_r; q, z) h(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) a_j z^j, \end{aligned}$$

where

$$\delta_{(j+1,\alpha,\mu)}(a_m, b_r, q) = \frac{\prod_{i=1}^m (a_i, q)_{j+1}}{(q, q)_{j+1} \prod_{k=1}^r (b_k, q)_{j+1}} \left(\frac{\Gamma(\mu)}{\Gamma(\alpha(j+1) + \mu)} \right).$$

Remark 1.1. You can see that when the parameters are defined $r, m, \alpha, \mu, q, a_1, \dots, a_m$ and b_1, \dots, b_r , it's here noted that the operator defined $\mathfrak{J}_{\alpha,\mu} h(z)$, performs different operators. For further explanation, examples are given.

- (a) For $\alpha = 0, \mu = 1, a_i = q^{a_i}, b_k = q^{b_k}, a_i > 0, b_k > 0, i = 1, \dots, m, k = 1, \dots, r, m = r + 1$ and $q \rightarrow 1$, we obtain the operator defined in [14].
- (b) For $m = 2, r = 1, \alpha = 0, \mu = 1, a_2 = q$ and $q \rightarrow 1$, we obtain the operator defined in [13].
- (c) For $m = 1, r = 0, \alpha = 0, \mu = 1, a_1 = \lambda + 1$ and $q \rightarrow 1$, we obtain the operator defined in [10], and it was then generalized through [29].

Some other authors have studied various classes of meromorphic univalent functions, such as, see [3, 6, 7, 20, 28] and [30]). Such works encouraged us to create the new class $\mathcal{T}_{\alpha,\mu}^\tau(a_m, b_r, d)$ of Σ , that includes the operator $\mathfrak{J}_{\alpha,\mu}h(z)$, and it is presented as follows.

Definition 1.1. For $d \geq 1, \tau > 0$, the function $h \in \Sigma$ is in the class $\mathcal{T}_{\alpha,\mu}^\tau(a_m, b_r, d)$ if it satisfies the inequality

$$(1.5) \quad \left| \frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} - 1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} + d} \right| < \tau.$$

Denote by Σ^* the subclass of Σ composed of the form functions

$$(1.6) \quad h(z) = z^{-1} + \sum_{j=1}^{\infty} |a_j| z^j.$$

Define the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$ by

$$\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d) = \mathcal{T}_{\alpha,\mu}^\tau(a_m, b_r, d) \cap \Sigma^*.$$

2. MAIN RESULTS

This section introduces work to obtain sufficient conditions for the function h given by (1.6), in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$, it also shows that for functions belonging to this class, this requirement is necessary, as well as growth and distortion bounds, extreme points and linear combinations are submitted for the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$.

Theorem 2.1. *A function h given by (1.6) is in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$ if and only if*

$$(2.1) \quad \sum_{j=1}^{\infty} [j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| \leq \tau(1 + d).$$

Proof. Assume that the inequality (1.6) holds true. We have

$$\begin{aligned} & \left| \frac{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} - 1}{\frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))'}{\mathfrak{J}_{\alpha,\mu}h(z)} + d} \right| \\ &= \left| \frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))' - \mathfrak{J}_{\alpha,\mu}h(z)}{\tau [z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))' + d\mathfrak{J}_{\alpha,\mu}h(z)]} \right| \\ &= \left| \frac{\sum_{j=1}^{\infty} [j^2 - 1] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| z^j}{(1 + d) + \sum_{j=1}^{\infty} [j^2 + d] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| z^j} \right| < \tau, \quad z \in U^*. \end{aligned}$$

So, we have $h \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$ (by the maximum modulus theorem).

Conversely, let $h \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$ where h given by (1.6), then we obtain from inequality (1.5),

$$(2.2) \quad \left| \frac{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))' - \mathfrak{J}_{\alpha,\mu}h(z)}{z^2(\mathfrak{J}_{\alpha,\mu}h(z))'' + z(\mathfrak{J}_{\alpha,\mu}h(z))' + d\mathfrak{J}_{\alpha,\mu}h(z)} \right| = \left| \frac{\sum_{j=1}^{\infty} [j^2 - 1] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| z^j}{(1 + d) + \sum_{j=1}^{\infty} [j^2 + d] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| z^j} \right| < \tau,$$

since the last inequality is real for all $z \in U^*$, choose values of z on the real axis. Following explanation, the denominator in (2.2) and letting $z \rightarrow 1^-$ through real values, we obtain

$$\sum_{j=1}^{\infty} [j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| \leq \tau(1 + d).$$

Therefore, we get the required inequality (2.1) of Theorem 2.1. □

Corollary 2.1. *If the function h given by (1.6) is in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$, then*

$$(2.3) \quad |a_j| \leq \frac{\tau(1 + d)}{[j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}, \quad j \geq 1,$$

the result is sharp of the function

$$h(z) = z^{-1} + \frac{\tau(1 + d)}{[j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)} z^j, \quad j \geq 1.$$

Theorem 2.2. *Let $h_o(z) = z^{-1}$ and*

$$h_j(z) = z^{-1} + \frac{\tau(1 + d)}{[j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)} z^j.$$

Then, $h \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$ if and only if it can be expressed form

$$(2.4) \quad h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j h_j(z),$$

where

$$v_j \geq 0 \quad \text{and} \quad \sum_{j=0}^{\infty} v_j = 1.$$

Proof. Using the function h which is defined in (2.4), then

$$h(z) = z^{-1} + \sum_{j=0}^{\infty} v_j \frac{\tau(1 + d)}{[j^2(1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)} z^j,$$

and for last function, we get

$$\begin{aligned} & \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) \\ & \times v_j \frac{\tau(1+d)}{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)} \\ & = \sum_{j=1}^{\infty} v_j \tau(1+d) = \tau(1+d)(1-v_0) = \tau(1+d), \end{aligned}$$

that is, condition (2.1) is met. Therefore, $h \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$.

Conversely, we assume that $h \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$, from equation (2.3), we have:

$$|a_j| \leq \frac{\tau(1+d)}{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}, \quad j \geq 1,$$

we set

$$v_i = \frac{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\tau(1+d)} |a_j|, \quad j \geq 1,$$

and

$$v_0 = 1 - \sum_{j=1}^{\infty} v_j.$$

That is the result

$$h(z) = \sum_{j=0}^{\infty} v_j f_j.$$

The declaration of Theorem 2.2, is thus complete. □

Theorem 2.3. *If a function h defined by (1.6), is in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$, then for $|z| = r$, we have*

$$\begin{aligned} & \frac{1}{r} - \frac{\tau(1+d)}{[(1-\tau) - (1+\tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)} r \\ & \leq |h(z)| \leq \frac{1}{r} + \frac{\tau(1+d)}{[(1-\tau) - (1+\tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)} r \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{r^2} - \frac{\tau(1+d)}{[(1-\tau) - (1+\tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)} \\ & \leq |h'(z)| \leq \frac{1}{r^2} + \frac{\tau(1+d)}{[(1-\tau) - (1+\tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)}. \end{aligned}$$

Proof. By Theorem 2.1, we have

$$\begin{aligned} & [(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r) \sum_{j=1}^{\infty} |a_j| \\ & \leq \sum_{j=1}^{\infty} [j^2 (1 - \tau) - (1 + \tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_j| \\ & \leq \tau (1 + d), \end{aligned}$$

which results

$$\sum_{j=1}^{\infty} |a_j| \leq \frac{\tau (1 + d)}{[(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)}.$$

Therefore,

$$|h(z)| \leq \frac{1}{|z|} + |z| \sum_{j=1}^{\infty} |a_j| \leq \frac{1}{|z|} + \frac{\tau (1 + d)}{[(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)} |z|$$

and

$$|h(z)| \geq \frac{1}{|z|} - |z| \sum_{j=1}^{\infty} |a_j| \geq \frac{1}{|z|} - \frac{\tau (1 + d)}{[(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)} |z|.$$

On the other hand, for (1.6), differentiating both sides with respect to z , we get:

$$|h'(z)| \leq \frac{1}{|z|^2} + \sum_{j=1}^{\infty} |a_j| \leq \frac{1}{|z|} + \frac{\tau (1 + d)}{[(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)}$$

and

$$|h'(z)| \geq \frac{1}{|z|^2} - \sum_{j=1}^{\infty} |a_j| \geq \frac{1}{|z|} - \frac{\tau (1 + d)}{[(1 - \tau) - (1 + \tau d)] \Delta_{(2,\alpha,\mu)}(a_m, b_r)}.$$

Define the functions $h_i, i = 1, 2$, by

$$(2.5) \quad h_i(z) = z^{-1} + \sum_{j=1}^{\infty} |a_{j,i}| z^j, \quad z \in U^*. \quad \square$$

Theorem 2.4. *Let the functions $h_i, i = 1, 2$, which are defined in (2.5), be in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$. Then for $0 \leq s \leq 1$, the function $h(z) = sh_1(z) + (1 - s)h_2(z)$, in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$.*

Proof. Using

$$h_i(z) = z^{-1} + \sum_{j=1}^{\infty} |a_{j,i}| z^j, \quad i = 1, 2,$$

we have:

$$h(z) = z^{-1} + \sum_{j=1}^{\infty} \{s |a_{j,1}| + (1 - s) |a_{j,2}|\} z^j, \quad 0 \leq s \leq 1.$$

Now, by Theorem 2.1, we obtain

$$\begin{aligned} & \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) \{s|a_{j,1}| + (1-s)|a_{j,2}|\} \\ &= s \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_{j,1}| \\ & \quad + (1-s) \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r) |a_{j,2}| \\ & \leq s\tau(1+d) + (1-s)\tau(1+d) = \tau(1+d), \end{aligned}$$

that demonstrates $h(z) \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$. □

Theorem 2.5. *Let the function $h_i, i = 1, 2$, which are defined in (2.5), be in the class $\mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d)$. Then $h_1 * h_2 \in \mathcal{T}_{\alpha,\mu}^{\delta,*}(a_m, b_r, d)$, where*

$$\delta \leq \frac{(j^2 - 1) \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\tau(1+d) + (j^2 + d) \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}.$$

Proof. It's enough to find the Littlest δ , such that

$$\sum_{j=1}^{\infty} \frac{[j^2(1-\delta) - (1+\delta d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\delta(1+d)} a_{j,1} a_{j,2} \leq 1.$$

Since $h_i \in \mathcal{T}_{\alpha,\mu}^{\tau,*}(a_m, b_r, d), i = 1, 2$, then

$$\sum_{j=1}^{\infty} \frac{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\tau(1+d)} a_{j,1} a_{j,2} \leq 1.$$

By Cauchy-Schwarz inequality, we get

$$(2.6) \quad \sum_{j=1}^{\infty} \frac{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\tau(1+d)} \sqrt{a_{j,1} a_{j,2}} \leq 1.$$

We just want to demonstrate that

$$\begin{aligned} & \sum_{j=1}^{\infty} \frac{[j^2(1-\delta) - (1+\delta d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\delta(1+d)} a_{j,1} a_{j,2} \\ & \leq \sum_{j=1}^{\infty} \frac{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}{\tau(1+d)} \sqrt{a_{j,1} a_{j,2}}, \end{aligned}$$

or equivalent to

$$\sqrt{a_{j,1} a_{j,2}} \leq \frac{[j^2(1-\delta) - (1+\delta d)] \tau}{[j^2(1-\tau) - (1+\tau d)] \delta}.$$

From (2.6), we get

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{\tau(1+d)}{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1,\alpha,\mu)}(a_m, b_r)}.$$

Therefore, it is sufficient to show that

$$\frac{\tau(1+d)}{[j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1, \alpha, \mu)}(a_m, b_r)} \leq \frac{[j^2(1-\delta) - (1+\delta d)] \tau}{[j^2(1-\tau) - (1+\tau d)] \delta}.$$

Finally, we have

$$\delta \leq \frac{(j^2 - 1) \Delta_{(j+1, \alpha, \mu)}(a_m, b_r)}{\tau(1+d) + (j^2 + d) \Delta_{(j+1, \alpha, \mu)}(a_m, b_r)}. \quad \square$$

Theorem 2.6. *If the function h_i , $i = 1, 2$, given by equation (2.5) is in the class $\mathcal{T}_{\alpha, \mu}^{\tau, *}(a_m, b_r, d)$, then $h_1 * h_2 \in \mathcal{T}_{\alpha, \mu}^{\tau, *}(a_m, b_r, d)$.*

Proof. Because $h_1 \in \mathcal{T}_{\alpha, \mu}^{\tau, *}(a_m, b_r, d)$, by Theorem 2.1, we obtain

$$\sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1, \alpha, \mu)}(a_m, b_r) |a_j| \leq \tau(1+d).$$

Since

$$\begin{aligned} & \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1, \alpha, \mu)}(a_m, b_r) |a_{j,1} a_{j,2}| \\ &= \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1, \alpha, \mu)}(a_m, b_r) |a_{j,1}| |a_{j,2}| \\ &\leq \sum_{j=1}^{\infty} [j^2(1-\tau) - (1+\tau d)] \Delta_{(j+1, \alpha, \mu)}(a_m, b_r) |a_{j,1}| \\ &\leq 1, \end{aligned}$$

we have $h_1 * h_2 \in \mathcal{T}_{\alpha, \mu}^{\tau, *}(a_m, b_r, d)$. □

Acknowledgements. The authors would like to thank the referee(s) for their helpful comments and suggestions.

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