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IDENTITIES WITH MULTIPLICATIVE GENERALIZED (α, α)-DERIVATIONS OF SEMIPRIME RINGS

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Abstract. Let R be a semiprime ring and α be an automorphism of R. A mapping $F : R \to R$ (not necessarily additive) is called multiplicative generalized (α, α)-derivation if there exists a unique (α, α)-derivation d of R such that $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$ for all $x, y \in R$. In the present paper, we intend to study several algebraic identities involving multiplicative generalized (α, α)-derivations on appropriate subsets of semiprime rings and collect the information about the commutative structure of these rings.

1. Introduction

Troughout this paper, R denotes an associative semiprime ring with center $Z(R)$. A ring R is said to be prime if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies $a = 0$. It is straight forward to observe that every prime ring is semiprime but the converse is not true in general, e.g., $\mathbb{Z} \times \mathbb{Z}$, which is a semiprime ring but not prime. For a fixed integer $n \geq 1$, a ring is said to be n-torsion free if $nx = 0$ for all $x \in R$ implies $x = 0$. For any $x, y \in R$, we denote the commutator $xy - yx$ and the anti-commutator $xy + yx$ by the symbols $[x, y]$ and $(x \circ y)$, respectively. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The very first example of a derivation is the differential operator Δ on $C[0, 1]$, the ring of the real valued differentiable functions on $[0, 1]$. The notion of derivation has been generalized in many directions. Brešar [6] introduced the notion of generalized derivation, which is an additive mapping

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$F : R \to R$ satisfying the relation $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is the associated derivation of R. An additive mapping $H : R \to R$ such that $H(xy) = H(x)y$ for all $x, y \in R$ is called the left multiplier of R. Clearly, with $d = 0$, every left multiplier is a generalized derivation and with $F = d$, every derivation is a generalized derivation. Let $\alpha : R \to R$ be an automorphism of R. Then an additive mapping $\delta : R \to R$ is said to be an (α, α)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. Note that every $(1_R, 1_R)$-derivation is the ordinary derivation of R, where 1_R stands for the identity mapping of R. Thus one can now think of the notion of generalized (α, α)-derivation, which is a unified notion of both generalized derivation and (α, α)-derivation. Accordingly, an additive mapping $\xi : R \to R$ is said to be a generalized (α, α)-derivation if there exists a unique (α, α)-derivation δ of R such that $\xi(xy) = \xi(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. If we drop the assumption of additivity of ξ, then it is called multiplicative generalized (α, α)-derivation associated with (α, α)-derivation δ, e.g. let S be any ring and $R = \{ae_{12} + be_{13} + ce_{23} : \forall a, b, c \in S\}$ be a subring of $M_3(S)$, the ring of 3×3 matrices over S. Define a mapping $F : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto (bc)e_{13}$, $d : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13}$ and $\alpha : R \to R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13} - ce_{23}$. Then α is an automorphism of R and F is a multiplicative generalized (α, α)-derivation associated with (α, α)-derivation δ.

The study of commutative structure of prime rings with derivations has been initiated long back by Posner [15]. Precisely, Posner proved that: *If R is a prime ring and d is a derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either R is commutative or $d = 0$.* Since then this result has been extended in many directions by a number of algebraists. In 1987, Bell and Martindale [4] extended this result to the class of semiprime rings and proved that: if a semiprime ring R admits a derivation d which is nonzero and centralizing on a left ideal U of R, then R contains a nonzero central ideal. In the same line of investigation, Daif [7] examined the commutativity of semiprime rings admitting derivations that satisfy the identities: (i) $d([x, y]) - [xy] = 0$, (ii) $d([x, y]) + [x, y] = 0$. Further, Fošner et al. [12] studied a more general situation and proved that: if R is a 2-torsion free semiprime ring and d is a derivation of R such that $[[d(x), x], d(x)] = 0$ for all $x \in R$, then D maps R into $Z(R)$. Dhara [8] discussed the generalized derivations of semiprime rings that act as homomorphisms and anti-homomorphism on appropriate subsets of the ring. More specifically, he proved the following.

1. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(x)F(y)$ for all $x, y \in I$, then $d(I) = (0)$, F is commuting left multiplier mapping on I.
2. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = (0)$, R contains a nonzero central ideal.

Moreover, Dhara and Mozumder [10] generalized these results by studying the identities $F(xy) \pm F(x)F(y) \in Z(R)$ and $F(xy) \pm F(y)F(x) \in Z(R)$ on (semi)prime ring R, where F is multiplicative generalized derivation of R. Very recently, Tiwari and Sharma [17] studied the following identities: (i) $G(xy) \pm F(x)F(y) \in Z(R)$; (ii) $G(xy) \pm F(x)F(y) \pm \alpha(yx) = 0$; (iii) $G(xy) \pm F(x)F(y) \pm \alpha(xy) \in Z(R)$; (iv) $G(xy) \pm F(x)F(y) \pm \alpha([x,y]) = 0$; (v) $G(xy) \pm F(x)F(y) \pm \alpha(x \circ y) = 0$ on Lie ideals of prime rings, where F and G are generalized (α,α)-derivations and α is an automorphism.

On the other hand, Atteya [2] proved that a semiprime ring R that admits a generalized derivation F, contains a nonzero central ideal if any one of the following identity holds true: (i) $F(xy) \pm xy \in Z(R)$; (ii) $F(xy) \pm yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) + yx \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R. In 2013, Dhara et al. [9] studied the following identities: (i) $[d(x), F(y)] = \pm[x,y]$; (ii) $[d(x), F(y)] = \pm(x \circ y)$; (iii) $d(x) \circ F(y) = \pm(x \circ y)$; (iv) $d(x) \circ F(y) = \pm[x,y]$ on ideals of semiprime rings, where F is generalized derivation with associated derivation d. Then it seems more interesting to consider such identities with multiplicative derivations. In this direction, Kumar and Sandhu [14] investigated the following identities: (i) $F([x,y]) \pm xy = 0$; (ii) $F([x,y]) \pm yx = 0$; (iii) $F(x \circ y) \pm xy = 0$; (iv) $F(x \circ y) \pm yx = 0$; (v) $d(x)F(y) \pm xy = 0$; (vi) $d(x)F(y) \pm yx = 0$; (vii) $[F(x), y] \pm x \circ G(y) = 0$; (viii) $F(x) \circ y \pm x \circ G(y) = 0$ on semiprime R, where F and G are multiplicative (generalized)-derivations with the associated mappings d and g respectively. In 2017, Tiwari et al. [16] examined the structure of semiprime rings involving two multiplicative (generalized)-derivations that satisfy a list of algebraic identities on appropriate subsets of the ring.

In light of the above discussion, in this paper, our aim is to study certain identities with multiplicative generalized (α,α)-derivations of semiprime rings. More precisely, we characterize the following situations: (i) $G(xy) \pm F(x)F(y) = 0$; (ii) $G(xy) \pm F(x)F(y) \pm xy = 0$; (iii) $F(x)y + yG(x) = 0$; (iv) $F(xy) \pm G(xy) = 0$; (v) $F(xy) \pm G(yx) = 0$; (vi) $\alpha(x) \circ F(y) \pm G(yx) = 0$; (vii) $[\alpha(x), F(y)] \pm G(yx) = 0$; (viii) $\alpha(x) \circ F(y) \pm \alpha([x,y]) = 0$; (ix) $[\alpha(x), F(y)] \pm \alpha(x \circ y) = 0$; (x) $[F(x), d(y)] \pm x \circ \alpha(y) = 0$; (xi) $d(x) \circ d(y) \pm F(xy) = 0$; (xii) $[d(x), d(y)] \pm F(xy) = 0$; (xiii) $[d(x), F(y)] \pm F([x,y]) = 0$; (xiv) $d(x) \circ F(y) \pm F(x \circ y) = 0$, for all x, y in an appropriate subset of R, where F and G are multiplicative generalized (α,α)-derivations with associated (α,α)-derivations d and g, respectively.

2. Preliminary Results

Lemma 2.1 (Brauer's trick [11])**.** *A group G cannot be the union of two of its proper subgroups.*

Lemma 2.2. ([1, Lemma 2.1]). *If R is a semiprime ring and I is an ideal of R, then I is a semiprime ring.*

Lemma 2.3. ([4, Theorem 3]). *Let R be a semiprime ring and U be a nonzero left ideal of R. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Lemma 2.4. ([13, Lemma 1.1.5]). *Let R be a semiprime ring and ρ be a right ideal of R. Then $Z(\rho) \subset Z(R)$.*

Lemma 2.5. ([3, Lemma 3.1]). *Let R be a 2-torsion free semiprime ring and U be a nonzero left ideal of R. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in U$, then $axb = 0 = bxa$ for all $x \in U$.*

Lemma 2.6. ([12, Theorem 1]). *Let R be a 2-torsion free semiprime ring and $D : R \to R$ be a derivation satisfying the relation $[[D(x), x], D(x)] = 0$ for all $x \in R$. In this case D maps R into $Z(R)$.*

Lemma 2.7. ([5, Lemma 1]). *Let R be a semiprime ring, I be a nonzero ideal of R and $a \in I$ and $b \in R$. If $aIb = (0)$, then $ab = ba = 0$.*

Lemma 2.8. *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. If R admits an (α, α)-derivation d such that $d(I) \subseteq Z(R)$, then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$[r, d(x)] = 0, \quad \text{for all } x \in I,\ r \in R. \tag{2.1}$$

Replacing x by xs in (2.1), where $s \in R$, we get

$$d(x)[r, \alpha(s)] + [r, \alpha(x)]d(s) + \alpha(x)[r, d(s)] = 0. \tag{2.2}$$

Replacing x by sx in (2.2), we find

$$\begin{aligned} &d(s)\alpha(x)[r, \alpha(s)] + \alpha(s)d(x)[r, \alpha(s)] + \alpha(s)[r, \alpha(x)]d(s) \\ &\quad + [r, \alpha(s)]\alpha(x)d(s) + \alpha(s)\alpha(x)[r, d(s)] = 0. \end{aligned} \tag{2.3}$$

Utilization of equation (2.2) in (2.3) gives that

$$d(s)\alpha(x)[r, \alpha(s)] + [r, \alpha(s)]\alpha(x)d(s) = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

Applying Lemma 2.5, we have

$$d(s)\alpha(x)[r, \alpha(s)] = 0. \tag{2.4}$$

Taking $x\alpha^{-1}(p)$ instead of x in (2.4), we get

$$d(s)\alpha(x)p[r, \alpha(s)] = 0, \quad \text{for all } x \in I,\ r, s, p \in R.$$

We now can easily arrive at $d(s)[\alpha(x), \alpha(s)]pd(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s, p \in R$. It forces that $d(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Linearizing on s, we get

$$d(r)[\alpha(x), \alpha(s)] + d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R. \tag{2.5}$$

Substituting sx in place of s in (2.5), we get

$$d(r)[\alpha(x), \alpha(s)]\alpha(x) + d(s)\alpha(x)[\alpha(x), \alpha(r)] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0. \tag{2.6}$$

Right multiply (2.5) by $\alpha(x)$ and then subtract from (2.6) to obtain

$$d(s)[\alpha(x), [\alpha(x), \alpha(r)]] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

In particular, taking $r = x$ in (2.5), we get $d(x)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Using this in the above relation, we get

$$d(s)[\alpha(x), [\alpha(x), \alpha(r)]] = 0, \quad \text{for all } x \in I,\ r, s \in R. \tag{2.7}$$

Replacing r by rp in (2.7), where $p \in R$, we find

$$\begin{aligned} &d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] \\ &+ d(s)[\alpha(x), [\alpha(x), \alpha(r)]]\alpha(p) = 0. \end{aligned}$$

Using (2.7), we get

$$d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

In view of (2.7), it can be re-written as

$$-\alpha(s)d(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Again using (2.7), we get $2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0$ for all $x \in I$ and $r, s, p \in R$. Since R is 2-torsion free, we obtain

$$d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Replacing p by qp, where $q \in R$, we get

$$d(s)[\alpha(x), \alpha(r)]\alpha(q)[\alpha(x), \alpha(p)] = 0.$$

Replacing q by $q\alpha^{-1}(d(s))$ and put $p = r$ in the above relation, we get

$$d(s)[\alpha(x), \alpha(r)]Rd(s)[\alpha(x), \alpha(r)] = (0), \quad \text{for all } x \in I,\ r, s \in R.$$

Since R is semiprime, we get

$$d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s \in R.$$

Replacing s by su in the above relation

$$d(s)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s, u \in R. \tag{2.8}$$

Replacing u by xu in (2.8), we get

$$d(s)\alpha(x)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I,\ r, s, u \in R. \tag{2.9}$$

Left multiply (2.8) by $\alpha(x)$ and then subtract from (2.9), we obtain

$$[d(s), \alpha(x)]\alpha(u)[\alpha(x), \alpha(r)] = 0.$$

In particular, put $r = \alpha^{-1}(d(s))$, we get

$$[\alpha(x), d(s)]\alpha(u)[\alpha(x), d(s)] = 0, \quad \text{for all } x \in I,\ u, s \in R.$$

It implies that $[\alpha(x), d(s)] = 0$ for all $x \in I$ and $s \in R$. That means $d(R) \subseteq Z(I')$, where $I' = \alpha(I)$. And hence by Lemma 2.4, we are done. □

3. Main Results

Theorem 3.1. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $G(xy) \pm F(x)F(y) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal and d and g maps R into $Z(R)$.*

Proof. Assume that

$$G(xy) + F(x)F(y) = 0, \quad \text{for all } x, y \in I. \tag{3.1}$$

Replacing y by yr in (3.1), where $r \in R$, we find

$$\begin{aligned} 0 &= G(xyr) + F(x)F(yr) \\ &= G(xy)\alpha(r) + \alpha(xy)g(r) + F(x)F(y)\alpha(r) + F(x)\alpha(y)d(r). \end{aligned}$$

By using (3.1), we get

$$0 = \alpha(xy)g(r) + F(x)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.2}$$

Replacing x by xs in (3.2), where $s \in R$, we get

$$0 = \alpha(xsy)g(r) + F(x)\alpha(s)\alpha(y)d(r) + \alpha(x)d(s)\alpha(y)d(r). \tag{3.3}$$

Putting sy instead of y in (3.2), where $s \in R$, we get

$$0 = \alpha(xsy)g(r) + F(x)\alpha(sy)d(r). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$0 = \alpha(x)d(s)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r, s \in R.$$

Substituting wy for y and then putting $y = x$ and $s = r$, where $w \in R$, we have

$$0 = \alpha(x)d(r)\alpha(w)\alpha(x)d(r), \quad \text{for all } x \in I,\ r, w \in R.$$

Since R is semiprime ring, we find that

$$0 = \alpha(x)d(r), \quad \text{for all } x \in I,\ r \in R. \tag{3.5}$$

Replacing x by xy in (3.5), where $y \in I$, we have

$$0 = \alpha(x)\alpha(y)d(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.6}$$

Because I is ideal of R, $\alpha(I)$ is ideal of R. In view of Lemma 2.7, we have $\alpha(x)d(r) = 0$ and $d(r)\alpha(x) = 0$ for all $x \in I, r \in R$. That is

$$0 = [\alpha(x), d(r)], \quad \text{for all } x \in I,\ r \in R. \tag{3.7}$$

Moreover, a particular case of (3.7) implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}d$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R. Further, from equation (3.7), we have $d(r) \in Z(I)$ for all $r \in R$. In view of Lemma 2.4, we conclude that d maps R into $Z(R)$.

Using (3.4) in (3.5), we obtain $0=\alpha(x)\alpha(s)\alpha(y)g(r)$ for all $x,y\in I$ and $r,s\in R$. Taking $\alpha^{-1}(g(r))s$ instead of s in the last expression, we get in particular

$$(0)=\alpha(x)g(r)R\alpha(x)g(r),\quad \text{for all } x\in I,\ r\in R.$$

It yields that

$$0=\alpha(x)g(r),\quad \text{for all } x\in I,\ r\in R.$$

Since this expression is same as (3.5) but with g instead of d. Therefore, the same technique implies g maps R into $Z(R)$, as desired.

Using similar approach, we conclude that the same result holds for $G(xy)-F(x)F(y)=0$ for all $x,y\in I$. □

Theorem 3.2. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $G(xy)\pm F(x)F(y)\pm xy=0$ for all $x,y\in I$ then R contains a nonzero central ideal and, d and g maps R into $Z(R)$.*

Proof. Assume that

$$G(xy)+F(x)F(y)+xy=0,\quad \text{for all } x,y\in I. \tag{3.8}$$

Replacing y by yr in (3.8), where $r\in R$, we have

$$0=G(xy)\alpha(r)+\alpha(xy)g(r)+F(x)F(y)\alpha(r)+F(x)\alpha(y)d(r)+xyr. \tag{3.9}$$

Right multiplying (3.8) by $\alpha(r)$, we get

$$0=G(xy)\alpha(r)+F(x)F(y)\alpha(r)+xy\alpha(r),\quad \text{for all } x,y\in I,\ r\in R. \tag{3.10}$$

Subtracting (3.10) from (3.9), we obtain

$$0=\alpha(xy)g(r)+F(x)\alpha(y)d(r)+xyr-xy\alpha(r),\quad \text{for all } x,y\in I,\ r\in R. \tag{3.11}$$

Replacing x by xs in (3.11), where $s\in R$, we have

$$0=\alpha(xsy)g(r)+F(x)\alpha(s)\alpha(y)d(r)+\alpha(x)d(s)\alpha(y)d(r)+xsyr-xsy\alpha(r). \tag{3.12}$$

Again replacing y by sy in (3.11), where $s\in R$, we get

$$0=\alpha(xsy)g(r)+F(x)\alpha(sy)d(r)+xsyr-xsy\alpha(r). \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$0=\alpha(x)d(s)\alpha(y)d(r),\quad \text{for all } x,y\in I,\ r,s\in R. \tag{3.14}$$

Substituting wx for y and r for s in (3.14), where $w\in R$, we have

$$0=\alpha(x)d(r)\alpha(w)\alpha(x)d(r),\quad \text{for all } x\in I,\ r,w\in R.$$

It follows that

$$0=\alpha(x)d(r),\quad \text{for all } x\in I,\ r\in R.$$

This expression also appeared as equation (3.5) in Theorem 3.1, so the result is followed in the same way.

Now replacing x by ux in (3.9), where $u \in R$, we get

$$0 = \alpha(u)\alpha(xy)g(r) + uxyr - uxy\alpha(r).$$

Combining with the above relation, it implies that $(\alpha(u) - u)\alpha(xy)g(r) = 0$ for all $x, y \in I$ and $u, r \in R$. Using (3.11), it gives

$$(\alpha(u) - u)xy(\alpha(r) - r) = 0, \quad \text{for all } x, y \in I, \ r, u \in R.$$

It implies that $x\alpha(r) = xr$ for all $x \in I$ and $r \in R$. Now, using this in (3.11), we obtain $0 = \alpha(xy)g(r)$ for all $x, y \in I$ and $r \in R$. That is, $\alpha(x)\alpha(y)g(r) = 0$. Since α is an automorphism, replacing x by $x\alpha^{-1}(g(r))$ in the last relation to get $\alpha(x)g(r)\alpha(y)g(r) = 0$. Further it implies that $\alpha(x)g(r)R\alpha(y)g(r) = (0)$ for all $x, y \in I$ and $r \in R$. In particular, $\alpha(x)g(r)R\alpha(x)g(r) = (0)$ for all $x \in I$ and $r \in R$. Hence, we get $0 = \alpha(x)g(r)$ for all $x \in I$ and $r \in R$. By repeating the similar argument as above, we get our conclusion.

Using similar approach, we conclude that the same result holds for $G(xy) - F(x)F(y) - xy = 0$, $G(xy) - F(x)F(y) + xy = 0$ and $G(xy) + F(x)F(y) - xy = 0$ for all $x, y \in I$. □

Theorem 3.3. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism on R such that $\alpha(I) = I$. Let F and G be multiplicative generalized (α, α)-derivations of R associated with nonzero (α, α)-derivations d and g respectively. If $F(x)y + yG(x) = 0$ for all $x, y \in I$, then F maps I into $Z(R)$ and G maps I into $Z(R)$. Moreover, R contains a nonzero central ideal and $d + g$ maps R into $Z(R)$.*

Proof. Assume that

$$F(x)y + yG(x) = 0, \quad \text{for all } x, y \in I. \tag{3.15}$$

Substituting xz in place of x in (3.15), where $z \in I$, we obtain

$$0 = F(x)\alpha(z)y + \alpha(x)d(z)y + yG(x)\alpha(z) + y\alpha(x)g(z). \tag{3.16}$$

Replacing y by $\alpha(z)y$ in (3.15), we have

$$0 = F(x)\alpha(z)y + \alpha(z)yG(x), \quad \text{for all } x, y \in I, \ z \in R. \tag{3.17}$$

Subtracting (3.17) from (3.16) and using (3.15), we get

$$0 = -F(x)y\alpha(z) + \alpha(z)F(x)y + \alpha(x)d(z)y + y\alpha(x)g(z). \tag{3.18}$$

Combining (3.16) and (3.18), we get

$$0 = [\alpha(z), F(x)]\, y, \quad \text{for all } x, y, z \in I.$$

Since I is semiprime, we have

$$0 = [\alpha(z), F(x)]\,, \quad \text{for all } x, z \in I.$$

That is $F(I) \subset Z(I)$. In view of Lemma 2.4, we find $F(I) \subset Z(R)$. Now right multiplying (3.15) by $\alpha(z)$, where $z \in I$, we obtain

$$0 = F(x)y\alpha(z) + yG(x)\alpha(z), \quad \text{for all } x, y, z \in I. \tag{3.19}$$

Replacing y with $y\alpha(z)$ in (3.15), we have

$$0 = F(x)y\alpha(z) + y\alpha(z)G(x), \quad \text{for all } x, y, z \in I. \tag{3.20}$$

Application (3.19) and (3.20) yields that

$$0 = y\left[\alpha(z), G(x)\right], \quad \text{for all } x, y, z \in I.$$

By semiprimeness of I, we get

$$0 = \left[\alpha(z), G(x)\right], \quad \text{for all } x, y, z \in I.$$

That is $G(I) \subset Z(I)$. In view of Lemma 2.4, we find $G(I) \subset Z(R)$. In view of $F(I) \subset Z(I)$ and $G(I) \subset Z(I)$, our hypothesis yields

$$0 = y(F + G)(x), \quad \text{for all } x, y \in I. \tag{3.21}$$

Replacing x by xr in (3.21), where $r \in R$, we find

$$0 = y(F + G)(x)\alpha(r) + y\alpha(x)(d + g)(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.22}$$

Application of (3.21) in (3.22), we have

$$0 = y\alpha(x)(d + g)(r), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.23}$$

In view of Lemma 2.7 and the fact that $\alpha(I) = I$, we have $y(d + g)(r) = 0$ and $(d + g)(r)y = 0$ for all $x \in I, r \in R$. That is

$$0 = [y, (d + g)(r)], \quad \text{for all } y \in I,\ r \in R. \tag{3.24}$$

Moreover, a particular case of (3.24) implies that $[\alpha(y), (d + g)(y)] = 0$ for all $y \in I$. That is $[y, \varphi(y)] = 0$ for all $y \in I$, where $\varphi = \alpha^{-1}(d + g)$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R. Applying Lemma 2.4 on relation (3.24), we find that $(d + g)(R) \subseteq Z(R)$. □

Theorem 3.4. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations associated with nonzero (α,α)-derivations d and g respectively. If $F(xy) \pm G(xy) = 0$ for all $x, y \in I$, then $d \pm g$ maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. By the given hypothesis, we have

$$F(xy) \pm G(xy) = 0 = (F \pm G)(xy), \quad \text{for all } x, y \in I.$$

Since sum of two multiplicative generalized (α,α)-derivations is a multiplicative generalized (α,α)-derivation, we take H in place of $F \pm G$, therefore our condition becomes $H(xy) = 0$ for all $x, y \in I$. Which is a particular case of our Theorem 3.1 (with $F = 0$). Hence, we are done. □

Theorem 3.5. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $F(xy) \pm G(yx) = 0$ for all $x, y \in I$, then d and g maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$F(xy) + G(yx) = 0, \quad \text{for all } x, y \in I. \tag{3.25}$$

Replacing x by xr in (3.25), where $r \in R$, we have

$$0 = F(xry) + G(yxr), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.26}$$

Again replace y with ry in (3.25), we find

$$0 = F(xry) + G(ryx), \quad \text{for all } x, y \in I,\ r \in R. \tag{3.27}$$

Combining (3.26) and (3.27), we get

$$G(yxr) = G(ryx), \quad \text{for all } x, y \in I,\ r \in R.$$

Putting yz instead of y in (3.25), where $z \in I$ and using (3.25), we get

$$0 = -G(yx)\alpha(z) + G(yz)\alpha(x) + \alpha(xy)d(z) + \alpha(yz)g(x), \quad \text{for all } x, y, z \in I. \tag{3.28}$$

Substituting xs in place of x in (3.28), where $s \in R$, we obtain

$$-G(yxs)\alpha(z) + G(yz)\alpha(xs) + \alpha(xsy)d(z) + \alpha(yz)g(xs) = 0. \tag{3.29}$$

Taking sy instead of y in (3.28), where $s \in R$, we have

$$-G(syx)\alpha(z) + G(syz)\alpha(x) + \alpha(xsy)d(z) + \alpha(syz)g(x) = 0. \tag{3.30}$$

Combining (3.29) and (3.30) and using $G(yxs) = G(syx)$, we get

$$0 = G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(syz)\alpha(x) - \alpha(syz)g(x).$$

Using $G(syz) = G(yzs)$ in last expression, we find

$$G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(yzs)\alpha(x) - \alpha(syz)g(x) = 0, \quad \text{for all } x, y, z \in I,\ s \in R.$$

That is

$$G(yz)\alpha([x,s]) + \alpha(yz)(g(x)\alpha(s) - g(s)\alpha(x)) + \alpha(yzx)g(s) - \alpha(syz)g(x) = 0, \tag{3.31}$$

for all $x, y, z \in I$, $s \in R$. Replacing s by x in (3.31), we find

$$0 = \alpha([yz, x])g(x), \quad \text{for all } x, y, z \in I. \tag{3.32}$$

Putting wy instead of y in (3.32), where $w \in I$, we have $0 = \alpha([w,x])\alpha(y)\alpha(z)g(x)$ for all $x, y, z, w \in I$. It is implies that $(0) = \alpha([w,x])\alpha(I)\alpha(z)g(x)$ for all $x, y, z \in I$. Because I is an ideal of R, $\alpha(I)$ is an ideal of R. In view of Lemma 2.7, we have

$$0 = \alpha([w,x])\alpha(z)g(x), \quad \text{for all } x, y, z \in I. \tag{3.33}$$

That is, $[I,x]\,I\,((\alpha^{-1}g)(x)) = (0)$ for all $x \in I$. Since I is a semiprime ring in itself, it must contains a family P of prime ideals such that $\cap P_\lambda = (0)$. Let P_{λ_1} be a typical member of this family and $x \in I$; by (3.33), we have find

$$[I,x] \subset P_{\lambda_1} \quad \text{or} \quad \left(\left(\alpha^{-1}g\right)(x)\right) \subset P_{\lambda_1}.$$

Let $A = \{x \in I : [I,x] \subset P_{\lambda_1}\}$ and $B = \{x \in I : ((\alpha^{-1}g)(x)) \subset P_{\lambda_1}\}$. Note that A and B are additive subgroups of I such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[I,I] \subset P_{\lambda_1} \quad \text{or} \quad \left(\left(\alpha^{-1}g\right)(I)\right) \subset P_{\lambda_1}.$$

Together with these both cases, we have $[I,I]((\alpha^{-1}g)(I)) = (0)$. That is

$$\alpha([x,y])g(z) = 0, \quad \text{for all } x,y,z \in I. \tag{3.34}$$

Replacing y by ry in (3.34), where $r \in R$, we have $\alpha([x,r])\alpha(y)g(z) = 0$. That is

$$[\alpha(x),r]\,\alpha(y)g(z) = 0, \quad \text{for all } x,y,z \in I,\ r \in R. \tag{3.35}$$

Right multiplying (3.35) by $\alpha(x)$, we get

$$[\alpha(x),r]\,\alpha(y)g(z)\alpha(x) = 0. \tag{3.36}$$

Substituting yx for y in (3.35), we have

$$[\alpha(x),r]\,\alpha(y)\alpha(x)g(z) = 0. \tag{3.37}$$

Combining (3.36) and (3.37), we obtain $[\alpha(x),r]\,\alpha(y)\,[\alpha(x),g(z)] = 0$ for all $x,y,z \in I, r \in R$. In particular, we have $[\alpha(x),g(z)]\,\alpha(I)\,[\alpha(x),g(z)] = (0)$ for all $x,z \in I$. Since $\alpha(I)$ is semiprime ring, we obtain

$$[\alpha(x),g(z)] = 0, \quad \text{for all } x,z \in I. \tag{3.38}$$

That is $g(I) \subset Z(I^{'})$, where $I^{'} = \alpha(I)$. By Lemma 2.4, $g(I) \subset Z(R)$. In view of Lemma 2.8, we get $g(R) \subset Z(R)$. Moreover, a particular case of (3.38) implies that $[x,\varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}g$ is an ordinary derivation of R. By Lemma 2.3, R contains a nonzero central ideal of R.

Now from (3.25), by replacing y by yr, where $r \in R$, we get

$$F(xyr) + G(yrx) = 0, \quad \text{for all } x,y \in I,\ r \in R. \tag{3.39}$$

And replacing x by rx in (3.25), we obtain

$$F(rxy) + G(yrx), \quad \text{for all } x,y \in I,\ r \in R. \tag{3.40}$$

Combining (3.39) and (3.40), we get

$$F(xyr) = F(rxy), \quad \text{for all } x,y \in I,\ r \in R.$$

Interchanging the role of x and y in this expression, we obtain

$$F(yxr) = F(ryx), \quad \text{for all } x,y \in I,\ r \in R.$$

This relation has already existed in the above proof for G. Hence by repeating the same arguments, we get that d maps R into $Z(R)$. □

Theorem 3.6. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α,α)-derivations of R associated with nonzero (α,α)-derivations d and g respectively. If $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x,y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x, y \in I$. That is

$$\alpha(x)F(y) + F(y)\alpha(x) \pm G(yx) = 0. \tag{3.41}$$

Replacing y by yx in (3.41), we find

$$\alpha(x)F(y)\alpha(x) + \alpha(x)\alpha(y)d(x) + F(y)\alpha(x)\alpha(x) + \alpha(y)d(x)\alpha(x) \pm G(yx)\alpha(x) \pm \alpha(yx)g(x) = 0.$$

Expression (3.41) reduces it to

$$\alpha(x)\alpha(y)d(x) + \alpha(y)d(x)\alpha(x) \pm \alpha(yx)g(x) = 0. \tag{3.42}$$

Taking zy in place of y in (3.42), where $z \in I$, we get

$$\alpha(x)\alpha(z)\alpha(y)d(x) + \alpha(z)\alpha(y)d(x)\alpha(x) \pm \alpha(z)\alpha(yx)g(x) = 0. \tag{3.43}$$

By using (3.42), we find $[\alpha(x), \alpha(z)]\alpha(y)d(x) = 0$ for all $x, y, z \in I$. By using Lemma 2.7, it yields that $[\alpha(x), \alpha(y)]d(x) = 0$ for all $x, y \in I$. Replacing y by $\alpha^{-1}(r)y$, where $r \in R$, we get

$$[\alpha(x), r]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I,\ r \in R.$$

It implies that

$$[\alpha(x), R]R\alpha(I)d(x) = (0). \tag{3.44}$$

Since R contains a family S of prime ideals such that $\cap P_\lambda = (0)$. Let P be a typical member of this family and $x \in I$, by (3.44), we find

$$[\alpha(x), R] \subset P \quad \text{or} \quad \alpha(I)d(x) \subset P.$$

Let $A = \{x \in I : [\alpha(x), R] \subset P\}$ and $B = \{x \in I : \alpha(I)d(x) \subset P\}$. Note that A and B are the additive subgroups of R such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[\alpha(I), R] \subset P \quad \text{or} \quad \alpha(I)d(I) \subset P.$$

Together with these both cases, we have $[\alpha(I), R]d(I) = 0$. That is

$$[\alpha(x), r]d(y) = 0, \quad \text{for all } x, y \in I,\ r \in R. \tag{3.45}$$

Replacing y by yt, in (3.45), where $t \in I$, we find

$$[\alpha(x), r]\alpha(y)d(t) = 0. \tag{3.46}$$

This expression is same as (3.35) of Theorem 3.5 with d instead of g. With the similar arguments, we get our conclusion. □

By the same implications as in Theorem 3.6 with necessary modifications, we can get the following result.

Theorem 3.7. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F and G be multiplicative generalized (α, α)-derivations of R associated with nonzero (α, α)-derivations d and g, respectively. If $[\alpha(x), F(y)] \pm G(yx) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.8. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $\alpha(x)\circ F(y)\pm\alpha([x,y])=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$\alpha(x)\circ F(y)\pm\alpha([x,y])=0,\quad \text{for all } x,y\in I. \tag{3.47}$$

Replacing y by yx in (3.47), we find

$$(\alpha(x)\circ F(y))\alpha(x)+(\alpha(x)\circ\alpha(y)d(x))\pm\alpha([x,y])\alpha(x)=0,\quad \text{for all } x,y\in I.$$

Equation (3.47) reduces it to

$$(\alpha(x)\circ\alpha(y)d(x))=0. \tag{3.48}$$

Replacing y by zy in (3.48), where $z\in I$, and using it, we get $[\alpha(x),\alpha(z)]\alpha(y)d(x)=0$ for all $x,y,z\in I$. This expression also appeared in Theorem 3.6, hence the conclusion follows in the similar manner. □

By the same implications as in Theorem 3.8 with necessary modifications, we can get the following result.

Theorem 3.9. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[\alpha(x),F(y)]\pm\alpha(x\circ y)=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.10. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[F(x),d(y)]\pm(x\circ\alpha(y))=0$ for all $x,y\in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $[F(x),d(y)]-(x\circ\alpha(y))=0$ for all $x,y\in I$. That is

$$F(x)d(y)-d(y)F(x)-x\alpha(y)-\alpha(y)x=0. \tag{3.49}$$

Replacing y by yz in (3.49), where $z\in I$, we find

$$\begin{aligned}&F(x)d(y)\alpha(z)+F(x)\alpha(y)d(z)-d(y)\alpha(z)F(x)\\&-\alpha(y)d(z)F(x)-x\alpha(y)\alpha(z)-\alpha(y)\alpha(z)x=0.\end{aligned}$$

Right multiplying (3.49) by $\alpha(z)$ and then comparing with the above expression to obtain

$$[F(x),\alpha(y)d(z)]+d(y)[F(x),\alpha(z)]-\alpha(y)[\alpha(z),x]=0,\quad \text{for all } x,y,z\in I. \tag{3.50}$$

Taking zy instead of y in (3.50), we have

$$\begin{aligned}&\alpha(z)[F(x),\alpha(y)d(z)]+[F(x),\alpha(z)]\alpha(y)d(z)+d(z)\alpha(y)[F(x),\alpha(z)]\\&+\alpha(z)d(y)[F(x),\alpha(z)]-\alpha(z)\alpha(y)[\alpha(z),x]=0,\quad \text{for all } x,y,z\in I.\end{aligned}$$

Application of (3.50) yields

$$[F(x), \alpha(z)]\alpha(y)d(z) + d(z)\alpha(y)[F(x), \alpha(z)] = 0, \quad \text{for all } x, y, z \in I.$$

In view of Lemma 2.5, it follows that

$$[F(x), \alpha(z)]\alpha(y)d(z) = 0 = d(z)\alpha(y)[F(x), \alpha(z)], \quad \text{for all } x, y, z \in I.$$

Let us consider the expression

$$[F(x), \alpha(z)]\alpha(y)d(z) = 0. \tag{3.51}$$

Replacing x by xz in (3.51), we get

$$[F(x), \alpha(z)]\alpha(z)\alpha(y)d(z) + [\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0. \tag{3.52}$$

Substituting zy in place of y in (3.51) and then subtracting it from (3.52) in order to find

$$[\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0, \quad \text{for all } x, y, z \in I. \tag{3.53}$$

Replacing x by wx in (3.53), where $w \in R$, and using it to obtain

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)\alpha(y)d(z) = 0, \quad \text{for all } x, y, z, w \in I.$$

In particular, taking $y = r[w, z]x$ in above expression, where $r \in R$, we get

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)R[\alpha(w), \alpha(z)]\alpha(x)d(z) = (0), \quad \text{for all } x, z, w \in I.$$

Since R is semiprime ring, it implies that $[\alpha(w), \alpha(z)]\alpha(x)d(z) = 0$ for all $x, z, w \in I$. This expression also appeared in Theorem 3.6, so the result is followed in the same way.

Using similar approach we conclude that the same result holds for $[F(x), d(y)] + (x \circ \alpha(y)) = 0$ for all $x, y \in I$. □

Theorem 3.11. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R. Let F be a multiplicative generalized (α, α)-derivation of R associated with a nonzero (α, α)-derivation d. If $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. We assume that $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$. That is

$$d(x)d(y) + d(y)d(x) \pm F(xy) = 0. \tag{3.54}$$

Replacing y by yz in (3.54) and using it, we get

$$d(x)\alpha(y)d(z) + \alpha(y)d(z)d(x) + d(y)[\alpha(z), d(x)] \pm \alpha(xy)d(z) = 0. \tag{3.55}$$

Taking xy in place of y in (3.55) to get

$$\begin{aligned} &d(x)\alpha(x)\alpha(y)d(z) + \alpha(x)\alpha(y)d(z)d(x) + d(x)\alpha(y)[\alpha(z), d(x)] \\ &+ \alpha(x)d(y)[\alpha(z), d(x)] \pm \alpha(x)\alpha(xy)d(z) = 0, \quad \text{for all } x, y, z \in R. \end{aligned} \tag{3.56}$$

Application of (3.55) in (3.56) yields

$$[d(x), \alpha(x)]\alpha(y)d(z) + d(x)\alpha(y)[\alpha(z), d(x)] = 0, \quad \text{for all } x, y, z \in R.$$

In particular for $x = z$, we get

$$(3.57) \qquad [d(x), \alpha(x)]\alpha(y)d(x) = d(x)\alpha(y)[d(x), \alpha(x)], \quad \text{for all } x, y \in R.$$

Replacing y by $y\alpha^{-1}(d(x))$ in (3.57) and using it, we have

$$d(x)\alpha(y)[d(x), [d(x), \alpha(x)]] = 0, \quad \text{for all } x, y \in R.$$

It implies that

$$[d(x), [d(x), \alpha(x)]]\alpha(y)[d(x), [d(x), \alpha(x)]] = 0, \quad \text{for all } x, y \in R.$$

Using semiprimeness of R, we get $[d(x), [d(x), \alpha(x)]] = 0$ for all $x \in R$. That is equivalent to $[\varphi(x), [\varphi(x), x]] = 0$ for all $x \in R$, where $\varphi = \alpha^{-1}d$, which is an ordinary derivation of R. Invoking Lemma 2.6, we get φ maps R into $Z(R)$, i.e.,

$$[\varphi(x), y] = 0, \quad \text{for all } x, y \in R.$$

In particular, we have $[\varphi(x), x] = 0$ for all $x \in R$, and hence R contains a nonzero central ideal by Lemma 2.3. □

Theorem 3.12. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. Let F be a multiplicative generalized (α,α)-derivation of R associated with a nonzero (α,α)-derivation d. If $[d(x), d(y)] \pm F(xy) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. Following the same arguments as in Theorem 3.11, instead of equation (3.57), we have

$$[d(x), \alpha(x)]\alpha(y)d(x) + d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

By Lemma 2.5, we find

$$(3.58) \qquad d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Taking xy in place of y in (3.58), we get

$$(3.59) \qquad d(x)\alpha(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Left multiply (3.58) by $\alpha(x)$ to obtain

$$(3.60) \qquad \alpha(x)d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Comparing (3.59) and (3.60), we obtain $[d(x), \alpha(x)]\alpha(y)[d(x), \alpha(x)] = 0$ for all $x, y \in I$. By semiprimeness of $\alpha(I)$, we find $[\alpha(x), d(x)] = 0$ for all $x \in I$. It implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}d$, which is an ordinary derivation. Hence, in light of Lemma 2.3, we are done. □

Now onwards, we consider that F is a two sided multiplicative generalized (α,α)-derivation associated with (α,α)-derivation d, i.e., F satisfies the following conditions:

$$F(xy) = F(x)\alpha(y) + \alpha(x)d(y) = d(x)\alpha(y) + \alpha(x)F(y), \quad \text{for all } x, y \in R.$$

Theorem 3.13. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R. If $[d(x), F(y)] \pm F([x, y]) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. Assume that

$$[d(x), F(y)] \pm F([x, y]) = 0, \quad \text{for all } x, y \in I. \tag{3.61}$$

Replacing y by yx in (3.61) and using it, we get

$$F(y)[d(x), \alpha(x)] + [d(x), \alpha(y)]d(x) \pm \alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I. \tag{3.62}$$

Replacing y by xy in (3.62), we have

$$\begin{aligned}&\alpha(x)F(y)[d(x), \alpha(x)] + d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)[d(x), \alpha(y)]d(x)\\ &+ [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I.\end{aligned}$$

Using (3.62), we get

$$d(x)\alpha(y)[d(x), \alpha(x)] + [d(x), \alpha(x)]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I.$$

This expression also appeared in Theorem 3.12, so the result is followed in the same way. □

Theorem 3.14. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R. If $d(x) \circ F(y) \pm F(x \circ y) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$(d(x) \circ F(y)) \pm F(x \circ y) = 0, \quad \text{for all } x, y \in I. \tag{3.63}$$

Replacing y by yx in (3.63) and using it, we get

$$-F(y)[d(x), \alpha(x)] + (d(x) \circ \alpha(y))d(x) \pm \alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I. \tag{3.64}$$

Replacing y by xy in (3.64), we find

$$\begin{aligned}&-\alpha(x)F(y)[d(x), \alpha(x)] - d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)(d(x) \circ \alpha(y))d(x)\\ &+ [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I.\end{aligned}$$

Using (3.64), we obtain

$$d(x)\alpha(y)[d(x), \alpha(x)] = [d(x), \alpha(x)]\alpha(y)d(x), \quad \text{for all } x, y \in I.$$

This expression also appeared as equation (3.57) in Theorem 3.11, so the result is followed in the same way. □

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