Kragujevac Journal of Mathematics
Volume 48(3) (2024), Pages 441–451.

A NEW FIXED POINT RESULT IN GRAPHICAL $b_v(s)$-METRIC SPACE WITH APPLICATION TO DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, motivated by [13, 15], first we give a notion of graphical $b_v(s)$-metric space, which is a graphical version of $b_v(s)$-metric space. Utilizing the graphical Banach contraction mapping we prove fixed point results in graphical $b_v(s)$-metric space. Appropriate examples are also presented to support our results. In the end, the main result ensures the existence of a solution for an ordinary differential equation along with its boundary conditions by using the fixed point result in graphical $b_v(s)$-metric space.

1. Introduction and Preliminaries

After the most renowned Banach contraction principle stated by Banach [3], many authors have provided more general and innovative contraction mappings on a complete metric space and established fixed point results, see [7,8,10,12]. On the contrary, some authors generalize the concept of metric space by introducing more general conditions instead of triangular inequality. Couple of them are given below.

Definition 1.1 ([2, 11])**.** Let $s \geq 1$ be a given real number and X be a non-empty set. A b-metric on X is a mapping $\rho : X \times X \to [0, +\infty)$ such that for all $a, b, c \in X$ it satisfies the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$;
(iii) $\rho(a, c) \leq s[\rho(a, b) + \rho(b, c)]$.

Then (X, ρ) is called a b-metric space with coefficient s.

Key words and phrases. Graph, fixed point, graphical $b_v(s)$ metric, graphic Banach contraction.
2020 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54H25, 05C12.
DOI 10.46793/KgJMat2403.441B
Received: September 20, 2020.
Accepted: May 10, 2021.

Definition 1.2 ([6])**.** Let a mapping $\rho : X \times X \to [0, +\infty)$ defined on a non-empty set X satisfy:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) $\rho(a, b) \leq \rho(a, c) + \rho(c, d) + \rho(d, b)$ for all $a, b \in X$ and all distinct points $c, d \in X \backslash \{a, b\}$.

Then ρ is called a rectangular metric on X and (X, ρ) is called a rectangular metric space.

Definition 1.3 ([6])**.** Let a mapping $\rho : X \times X \to [0, +\infty)$ be defined on a non-empty set X. For $v \in \mathbb{N}$, (X, ρ) is said to be a v-generalized metric space, if the following hold:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$, for all $a, b \in X$;
(iii) $\rho(a, b) \leq \rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)$ for all $a, u_1, u_2, \ldots, u_v, b \in X$ such that $a, u_1, u_2, \ldots, u_v, b$ are all different.

For more details on b-metric spaces and their generalizations we refer to [1]. Recently, Mitrović and Radenović [13] introduced the concept of $b_v(s)$-metric space as follows.

Definition 1.4. Let a mapping $\rho : X \times X \to [0, +\infty)$ be defined on a non-empty set X. For $v \in \mathbb{N}$, (X, ρ) is said to be a $b_v(s)$-metric space, if the following hold:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) there exists a real number $s \geq 1$ such that

$$\rho(a, b) \leq s[\rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)],$$

for distinct points $a, u_1, u_2, \ldots, u_v, b \in X$.

Let X be a non-empty set and $\Delta = \{(x, x) : x \in X\}$. A graph $\mathbb{G}$ is an ordered pair $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$, where $\mathbb{V}(\mathbb{G})$ is the set of vertices of graph $\mathbb{G}$ and $\mathbb{E}(\mathbb{G}) \subseteq \mathbb{V}(\mathbb{G}) \times \mathbb{V}(\mathbb{G})$ is the set of edges of graph $\mathbb{G}$. In this paper, we will use the concept of graph structure on metric space (namely, a graphical metric space) that has been introduced by Shukla et al. [15], in which the non-empty set X is associated with a graph $\mathbb{G}$ by considering the set X as the set of vertices, i.e., $\mathbb{V}(\mathbb{G}) = X$ and allowing that the set of edges $\mathbb{E}(\mathbb{G})$ contains the set Δ of all loops (which are edges that join a vertex to itself), i.e., $\Delta \subset \mathbb{E}(\mathbb{G})$. The sequence of vertices $\{t_i\}_{i=0}^{l}$ such that $t_0 = a$, $t_l = b$ and $(t_{i-1}, t_i) \in \mathbb{E}(\mathbb{G})$ for $j = 1, 2, \ldots, l$, represents the directed path from a to b of length l in the graph $\mathbb{G}$, in short it is written as $(aPb)_{\mathbb{G}}^{l}$. If the vertex $c \in X$ lies on the path from a to b, then we use notation $c \in (aPb)_{\mathbb{G}}$. A connected graph $\mathbb{G}$ states that there is a path between each pair vertices of the graph. A sequence $\{x_n\}$ is said to be $\mathbb{G}$-termwise connected, if for each $n \in \mathbb{N}$ there is a path from x_n to x_{n+1} in the graph $\mathbb{G}$. For $l \in \mathbb{N}$, let $[a]_{\mathbb{G}}^{l}$ be the set defined by

$$[a]_{\mathbb{G}}^{l} = \{b \in X : (aPb)_{\mathbb{G}}^{l}\}.$$

A connected component of the graph $\mathbb{G}$ is a connected sub-graph $\mathbb{G}_1$ of a graph $\mathbb{G}$ such that there is no path between vertices of $\mathbb{G}_1$ and vertices of $\mathbb{G}\backslash\mathbb{G}_1$. For more information we refer reader to [4, 5, 9, 16, 17].

Definition 1.5 ([15])**.** Let X be a non-empty set and $\mathbb{G}$ be a graph associated with X. A graphical metric on X is a mapping $\rho : X \times X \to [0, +\infty)$ satisfying the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) for all $a, b, c \in X$ such that $(aPb)_{\mathbb{G}}$ and $c \in (aPb)_{\mathbb{G}}$ implies $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$,

and the pair (X, ρ) is called graphical metric space.

In this paper, we introduce the concept of graphical $b_v(s)$-metric space, which is graphical version of $b_v(s)$-metric space. Graphical $b_v(s)$-metric space is a generalization of $b_v(s)$-metric space and graphical metric space. In the rest of the paper, all the graphs are directed unless otherwise stated and the set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

Definition 1.6. Let $\mathbb{G}$ be a graph associated with a non-empty set X. For $v \in \mathbb{N}$, a mapping $\rho : X \times X \to [0, +\infty)$ is said to be a graphical $b_v(s)$-metric, if it satisfies the following:

(i) $\rho(a, b) = 0$ if and only if $a = b$;
(ii) $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$;
(iii) for distinct $u_1, u_2, \ldots, u_v \in (aPb)_{\mathbb{G}}$ and a real number $s \geq 1$ holds

$$\rho(a, b) \leq s[\rho(a, u_1) + \rho(u_1, u_2) + \cdots + \rho(u_v, b)],$$

and the pair (X, ρ) is called graphical $b_v(s)$-metric space.

By Definition 1.1–1.6 and [9, 15, 17] it is easy to verify that the following hold.

(i) Graphical $b_1(1)$-metric space is graphical metric space.
(ii) Graphical $b_1(s)$-metric space is b-metric space with coefficient s.
(iii) Graphical $b_2(1)$-metric space is graphical rectangular metric space.
(iv) Graphical $b_2(s)$-metric space is graphical rectangular b-metric space with coefficient s.

Remark 1.1. Every $b_v(s)$-metric space (X, ρ) is a graphical $b_v(s)$-metric space endowed with a graph $\mathbb{G}$ having $\mathbb{E}(\mathbb{G}) = X \times X$, but converse need not be true.

Example 1.1. *Let $X = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ and let $\mathbb{G} = G_1 \cup G_2 \cup G_3$ be an undirected graph, where G_1, G_2 and G_3 are connected components with:*

$$\begin{aligned} \mathbb{V}(G_1) &= \{v_1, v_2\}, & \mathbb{E}(G_1) &= \{e_1\}, \\ \mathbb{V}(G_2) &= \{v_3, v_4, v_5, v_6, v_7, v_8, v_9\}, & \mathbb{E}(G_2) &= \{e_2, e_3, e_4, e_5, e_6, e_7\}, \\ \mathbb{V}(G_3) &= \{v_{10}, v_{11}\}, & \mathbb{E}(G_3) &= \{e_8\}. \end{aligned}$$

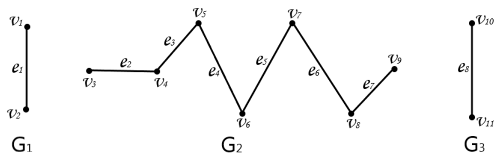

FIGURE 1. $\mathbb{G} = G_1 \cup G_2 \cup G_3$

Let $\rho : X \times X \to [0, +\infty)$ be a mapping defined in a following way:

$$\rho(v_i, v_j) = \begin{cases} 0, & \text{if } v_i = v_j, \\ l_{v_i v_j}, & \text{if } v_i, v_j \in G_k,\ k = \{1, 2, 3\}, \\ 1, & \text{otherwise}, \end{cases}$$

where $l_{v_i v_j}$ denote the length of the shortest path from v_i to v_j. Then (X, ρ) is graphical $b_4(1)$-metric space but not $b_4(1)$-metric space.

Definition 1.7. *Let (X, ρ) be a graphical $b_v(s)$-metric space endowed with a graph $\mathbb{G}$ and let $\{y_n\}$ be a sequence of elements in X. Then $\{y_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ exists $m \in \mathbb{N}$ such that $\rho(y_k, y_l) < \epsilon$, for all $k, l \geq m$, i.e., $\lim_{k,l \to +\infty} \rho(y_k, y_l) = 0$. The sequence $\{y_n\}$ converges to $z \in X$, if for each $\epsilon > 0$, exists $m \in \mathbb{N}$ such that $\rho(y_k, z) < \epsilon$ for all $k \geq m$, i.e., $\lim_{k \to +\infty} \rho(y_k, z) = 0$.*

Definition 1.8. *If every $\mathbb{G}$-termwise connected (briefly, $\mathbb{G}$-TWC) Cauchy sequence in a graphical $b_v(s)$-metric space X endowed with a graph $\mathbb{G}$ converges in X, then X is said to be $\mathbb{G}$-complete.*

2. MAIN RESULTS

First, we provide a definition of Banach contraction mapping in graphical $b_v(s)$ metric space.

Definition 2.1. Let $\mathbb{G}$ be a graph associated with graphical $b_v(s)$-metric space (X, ρ). A graphic Banach contraction (GBC) on X is a mapping $F : X \to X$ such that:

(GBC-1) $(Fa, Fb) \in \mathbb{E}(\mathbb{G})$ whenever $(a, b) \in \mathbb{E}(\mathbb{G})$;

(GBC-2) for all $(a, b) \in \mathbb{E}(\mathbb{G})$, there exists $\eta \in [0, 1)$, such that $\rho(Fa, Fb) \leq \eta \rho(a, b)$.

Remark 2.1. *Every Banach contraction on a non-empty set X is a graphic Banach contraction on X after considering the set of edges is equal to $X \times X$. But, converse is not always true (see Example* 2.1*).*

Definition 2.2. *A graph $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ is said to have property $(\mathcal{P})$ if for each convergent $\mathbb{G}$-TWC F-Picard sequence $\{x_n\}$ there exists $m \in \mathbb{N}$ and a limit ξ of $\{x_n\}$ in X, such that $(x_k, \xi) \in \mathbb{E}(\mathbb{G})$ or $(\xi, x_k) \in \mathbb{E}(\mathbb{G})$ for all $k > m$.*

Theorem 2.1. *Let (X, ρ) be a $\mathbb{G}$-complete graphical $b_v(s)$-metric space and let $F : X \to X$ be an injective GBC on X. Suppose that the following conditions are satisfied.*

(i) *There exists* $a_0 \in X$ *with* $F^k a_0 \in [a_0]^{l_k}_{\mathbb{G}}$ *for* $k = 1, 2, \ldots, v$, *where* $l_k = m_k v + 1$ *and* $m_k \in \mathbb{N}^0$.

(ii) $\mathbb{G}$ *has the property* $(\mathcal{P})$.

Then, for initial term $a_0 \in X$ *the* F*-Picard sequence* $\{a_n\}$ *is* $\mathbb{G}$*-TWC and converges to both* ξ^* *and* $F\xi^*$ *in* X.

Proof. For $k = 1, 2, \ldots, v$, let $a_0 \in X$ be such that $F^k a_0 \in [a_0]^{l_k}_{\mathbb{G}}$, where $l_k = m_k v + 1$ and $m_k \in \mathbb{N}^0$. Then there exists a path $\{e^k_j\}^{l_k}_{j=0}$ such that

$$a_0 = e^k_0\ , \quad F^k a_0 = e^k_{l_k} \quad \text{and} \quad (e^k_{j-1}, e^k_j) \in \mathbb{E}(\mathbb{G}), \quad \text{for all } j = 1, 2, \ldots, l_k.$$

Since $(e^k_{j-1}, e^k_j) \in \mathbb{E}(\mathbb{G})$, by $(\mathrm{GBC}-1)$ we have

$$(Fe^k_{j-1}, Fe^k_j) \in \mathbb{E}(\mathbb{G}), \quad \text{for } j = 1, 2, \ldots, l_k.$$

Therefore, $\{Fe^k_j\}^{l_k}_{j=0}$ is a path from $Fe^k_0 = Fa_0 = a_1$ to $Fe^k_{l_k} = F^2 a_0 = a_2$ of length l_k. Similarly, for all $n \in \mathbb{N}$, $\{F^n e^1_j\}^{l_1}_{j=0}$ is a path from $F^n e^1_0 = F^n a_0 = a_n$ to $F^n e^1_{l_1} = F^n F a_0 = a_{n+1}$ of length l_1. Thus, $\{a_n\}$ is $\mathbb{G}$-TWC sequence.

Therefore, for each $k = 1, 2, \ldots, v$, and $j = 1, 2, \ldots, l_k$, we have $(F^n e^k_{j-1}, F^n e^k_j) \in \mathbb{E}(\mathbb{G})$, for all $n \in \mathbb{N}$.

By $(GBC-2)$, for $j = 1, 2, \ldots, l_k$, we have

$$\rho\left(F^n e^k_{j-1}, F^n e^k_j\right) \leq \eta \rho\left(F^{n-1} e^k_{j-1}, F^{n-1} e^k_j\right) \leq \cdots \leq \eta^n \rho\left(e^k_{j-1}, e^k_j\right). \tag{2.1}$$

By condition (iii) of Definition 1.6, for $k = 1, 2, \ldots, v$ we have

$$\begin{aligned}
\rho(a_0, a_k) =& \rho(e^k_0, e^k_{l_k}) \\
\leq& s[\rho(e^k_0, e^k_1) + \rho(e^k_1, e^k_2) + \cdots + \rho(e^k_{v-1}, e^k_v)] \\
&+ s^2[\rho(e^k_v, e^k_{v+1}) + \rho(e^k_{v+1}, e^k_{v+2}) + \cdots + \rho(e^k_{2v-1}, e^k_{2v})] \\
&\vdots \\
&+ s^{m_k}[\rho(e^k_{(m_k-1)v}, e^k_{(m_k-1)v+1}) + \cdots + \rho(e^k_{l_k-1}, e^k_{l_k})] \\
=& D_{l_k}.
\end{aligned} \tag{2.2}$$

On the same way, using the inequalities (2.1) and (2.2), for $k = 1, 2, \ldots, v$, we have

$$\begin{aligned}
\rho(a_n, a_{n+k}) =& \rho(F^n a_0, F^n a_k) = \rho(F^n e^k_0, F^n e^k_{l_k}) \\
\leq& s[\rho(F^n e^k_0, F^n e^k_1) + \rho(F^n e^k_1, F^n e^k_2) + \cdots + \rho(F^n e^k_{v-1}, F^n e^k_v)] \\
&+ s^2[\rho(F^n e^k_v, F^n e^k_{v+1}) + \rho(F^n e^k_{v+1}, F^n e^k_{v+2}) + \cdots + \rho(F^n e^k_{2v-1}, F^n e^k_{2v})] \\
&\vdots \\
&+ s^{m_k}[\rho(F^n e^k_{(m_k-1)v}, F^n e^k_{(m_k-1)v+1}) + \cdots + \rho(F^n e^k_{l_k-1}, F^n e^k_{l_k})] \\
\leq& \eta^n D_{l_k}.
\end{aligned} \tag{2.3}$$

Now, from equation (2.3) one can prove Cauchy-ness of the sequence $\{x_n\}$, i.e., for all $p \geq 1$, $\rho(x_n, x_{n+p}) \to 0$ as $n \to +\infty$.

Therefore, $\mathbb{G}$-completeness of X implies $a_n \to \xi^*$ for some $\xi^* \in X$. Thanks to Property ($\mathcal{P}$), that ensures that there exists $k \in \mathbb{N}$ such that $(a_n, \xi^*) \in \mathbb{E}(\mathbb{G})$ or $(\xi^*, a_n) \in \mathbb{E}(\mathbb{G})$ for all $n > k$.

Assume that, for all $n > k$, $(a_n, \xi^*) \in \mathbb{E}(\mathbb{G})$, then by $(BGC - 2)$,

$$\rho(Fa_n, F\xi^*) \leq \eta\rho(a_n, \xi^*), \quad \text{for all } n > k.$$

This implies

$$\rho(Fa_n, F\xi^*) \to 0, \quad \text{as} \quad n \to +\infty,$$

i.e., $a_{n+1} \to F\xi^*$. So, $F\xi^*$ is also a limit of $\{a_n\}$.

Analogously, we can prove the case $(\xi^*, a_n) \in \mathbb{E}(\mathbb{G})$ for all $n > k$. □

Remark 2.2. In $b_v(s)$-metric space a sequence may converges to more than one limit, and hence this result also holds in graphical $b_v(s)$-metric space. To remove this difficulty some authors use Housdorff-ness condition on such metric space.

Definition 2.3. Let a graph $\mathbb{G}$ is associated with a graphical $b_v(s)$-metric space (X, ρ) and a mapping F is a graphic Banach contraction on X. The quadruplet $(X, \rho, \mathbb{G}, F)$ has the property S^*, if each $\mathbb{G}$-TWC F-Picard sequence $\{a_n\}$ in X has a unique limit.

Theorem 2.2. *Let the conditions of Theorem* 2.1 *hold along with that the quadruple* $(X, \rho, \mathbb{G}, F)$ *has the Property* S^**, then* F *has a fixed point.*

Proof. From the proof of Theorem 2.1 and the Property S^*, we have $F\xi^* = \xi^*$. □

Theorem 2.3. *Let the conditions of Theorem* 2.2 *hold and suposse that for all* $\xi^*, \zeta^* \in \text{Fix(F)}$ *there exists a path* $(\xi^* P \zeta^*)^t_{\mathbb{G}}$ *between* ξ^* *and* ζ^* *of length* t*, where* $t = 1$ *or* $t = mv + 1$ *for* $m \in \mathbb{N}^0$*. Then* F *has a unique fixed point.*

Proof. Let suposse that for all $\xi^*, \zeta^* \in \text{Fix(F)}$ there exists a path $(\xi^* P \zeta^*)^t_{\mathbb{G}}$ between ξ^* and ζ^* of length t.

Case I. If $t = 1$, then $(\xi^*, \zeta^*) \in \mathbb{E}(\mathbb{G})$, which implies $(F\xi^*, F\zeta^*) \in \mathbb{E}(\mathbb{G})$ by condition (GBC-1).

Now, by (GBC-2), we have $\rho(F\xi^*, F\zeta^*) \leq \eta[\rho(\xi^*, \zeta^*)]$, which implies $\rho(\xi^*, \zeta^*) \leq \eta[\rho(\xi^*, \zeta^*)]$. This contradicts the fact $\eta < 1$. Hence, $\xi^* = \zeta^*$.

Case II. Let $t = mv + 1$, where $m \in \mathbb{N}^0$ and let $\{e_i\}_{i=0}^t$ be the path from ξ^* to ζ^*, such that $e_0 = \xi^*$ and $e_k = \zeta^*$. Then

$$\begin{aligned}
\rho(\xi^*, \zeta^*) =& \rho(F^n\xi^*, F^n\zeta^*) \\
\leq& s[\rho(F^n e_0, F^n e_1) + \rho(F^n e_1, F^n e_2) + \cdots + \rho(F^n e_{v-1}, F^n e_v)] \\
&+ s^2[\rho(F^n e_v, F^n e_{v+1}) + \rho(F^n e_{v+1}, F^n e_{v+2}) + \cdots + \rho(F^n e_{2v-1}, F^n e_{2v})] \\
&\vdots \\
&+ s^m[\rho(F^n e_{(m-1)v}, F^n e_{(m-1)v+1}) + \cdots + \rho(F^n e_{t-1}, F^n e_t)] \\
\to& 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

This implies $\xi^* = \zeta^*$. □

Example 2.1. Let $A = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and a set $X = \{0, 1\} \cup A$ is associated with graph $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ such that $\mathbb{V}(\mathbb{G}) = X$, $\mathbb{E}(\mathbb{G}) = \Delta \cup \{(0, \frac{1}{2^n}) : n \in \mathbb{N}\} \cup \{(\frac{1}{2^n}, \frac{1}{2^m}) \in X \times X : n, m \in \mathbb{N}, n < m\}$. Let a symmetric function $\rho : X \times X \to [0, +\infty)$ such that

$$\rho(a, b) = \begin{cases} 0, & \text{if } a = b, \\ b, & \text{if } a = 0,\ b \in A, \\ a, & \text{if } a \in A,\ b = 1, \\ M, & \text{if } a, b \in A, \\ 1, & \text{if } a = 0,\ b = 1, \end{cases}$$

where $M = \max\{a, b\}$. Then (X, ρ) is $\mathbb{G}$-complete graphical $b_4(1)$-metric space. Now, $F : X \to X$ be a function defined as:

$$Fx = \begin{cases} \frac{x^5}{2}, & \text{if } x \in [0, 1), \\ 1, & \text{otherwise.} \end{cases}$$

Then, the mapping F satisfies all the conditions of Theorem 2.2 and having contraction constant $\eta = \frac{1}{2^5}$. Hence, 0 is the unique fixed point.

Remark 2.3. In Example 2.1, we observed that:

(i) the mapping F is graphic Banach contraction on X but not a Banach contraction;

(ii) (X, ρ) is not a $b_4(1)$-metric space.

3. An Application to Differential Equation

In this section, inspired by [14], we establish the existence of solution for the following second order ordinary differential equation:

$$-\frac{d^2y}{dt^2} = h(t, y(t)) \tag{3.1}$$

having boundary conditions $y(0) = y(1) = 0$, where $h : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

Let $X = C([0, 1], \mathbb{R})$ be the set of all real-valued continuous functions defined on $[0, 1]$. Let's define a set K as:

$$K = \left\{ y \in X : \inf_{t \in [0,1]} y(t) > 0 \text{ and } y(t) \leq 1,\ t \in [0, 1] \right\}.$$

Now, to define a graph structure $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ on X, lets consider $\mathbb{V}(\mathbb{G}) = X$ and

$$\begin{aligned} \mathbb{E}(\mathbb{G}) &= \Delta \cup \{(a, b) \in K \times K : a(t) \leq b(t) \text{ for all } t \in [0, 1]\} \\ &= \{(a, a) : a \in X\} \cup \{(a, b) \in K \times K : a(t) \leq b(t) \text{ for all } t \in [0, 1]\}. \end{aligned}$$

Define the mapping $\rho : X \times X \to \mathbb{R}^+$ as:

$$\rho(a, b) = \sup_{0 \leq t \leq 1} |a(t) - b(t)|,$$

for all $a, b \in X$. Then, (X, ρ) is the $\mathbb{G}$-complete graphical $b_3(1)$-metric space. The problem defined in (3.1) with given boundary condition is equivalent to the following Fredholm integral equation:

$$y(t) = \int_0^1 H(t,s)h(s,y(s))ds, \tag{3.2}$$

where

$$H(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Consider an injective mapping $F : X \to X$ defined as:

$$Fy(t) = \int_0^1 H(t,s)h(s,y(s))ds.$$

Then the fixed point of F is a solution of integral equation (3.2).

Theorem 3.1. *Suppose the following assumptions hold.*

(i) $\psi \in C([0,1], \mathbb{R})$ *is the lower solution of equation* (3.2)*, i.e.,*

$$\psi(t) \le \int_0^1 H(t,s)h(s,\psi(s))ds.$$

(ii) *A function* $h(t,\cdot) : \mathbb{R} \to \mathbb{R}^+$ *is increasing on* $[0,1]$*. In addition,* $h(t,1) = t$ *and* $\inf_{t\in[0,1]} H(t,s) > 0$.

(iii) *For every* $t \in [0,1]$*, holds*

$$|h(s,x(s)) - h(s,y(s))| \le |x(s) - y(s)|.$$

Then the existence of solution for equation (3.2) *provides a solution for* (3.1).

Proof. Clearly, the mapping $F : X \to X$ is well defined. Now, to prove F is GBC on X, we consider $(a,b) \in \mathbb{E}(\mathbb{G})$, i.e., $a, b \in K$ and $a(t) \le b(t)$ for all $t \in [0,1]$. Now, the following holds

$$Fa(t) = \int_0^1 H(t,s)h(s,a(s))ds \le \int_0^1 H(t,s)h(s,1)ds = \int_0^1 H(t,s)sds = \frac{4}{27\sqrt{3}} \le 1,$$

and from condition (ii), it is obvious that $\inf_{t\in[0,1]} Fa(t) > 0$. This implies $Fa(t) \in K$. Similarly, we can prove this for $b(t) \in K$.

Since, $h(t,\cdot) : \mathbb{R} \to \mathbb{R}^+$ is increasing on [0,1], we have

$$Fa(t) = \int_0^1 H(t,s)h(s,a(s))ds \le \int_0^1 H(t,s)h(s,b(s))ds = Fb(t).$$

It gives that $(Fa(t), Fb(t)) \in \mathbb{E}(\mathbb{G})$.

Now, for each $t \in [0,1]$, we have

$$\begin{aligned}
|Fa(t) - Fb(t)| &= \left| \int_0^1 H(t,s)h(s,a(s))ds - \int_0^1 H(t,s)h(s,b(s))ds \right| \\
&\leq \int_0^1 H(t,s)\Big|h(s,a(s)) - h(s,b(s))\Big|ds \\
&\leq \int_0^1 H(t,s)|a(s) - b(s)|ds \\
&\leq \sup_{t\in[0,1]} \int_0^1 H(t,s)|a(s) - b(s)|ds \\
&\leq \sup_{t\in[0,1]} |a(t) - b(t)| \int_0^1 H(t,s)ds \\
&\leq \frac{1}{8} \sup_{t\in[0,1]} |a(t) - b(t)| \\
&\leq \frac{1}{8}\rho(a,b).
\end{aligned}$$

This implies, $\rho(Fa, Fb) \leq \frac{1}{8}\rho(a,b)$. Note that for all $t \in [0,1]$, $\int_0^1 H(t,s)ds = \frac{t}{2} - \frac{t^2}{2}$ which implies that, $\sup_{t\in[0,1]} \int_0^1 H(t,s)ds = \frac{1}{8}$. Thus, F is GBC on X. From the condition (i), there exists $\psi(t) \in X$ such that $F^k\psi(t) \in [\psi(t)]_{\mathbb{G}}^1$, for each $k = 1,2,3$. It is easy to see that, the condition (I) of the Theorem 2.3 and Property S^* are satisfied. Therefore, Theorem 2.3 guarantees that F has an unique fixed point and hence the integral equation (3.2) has solution in X that ensures the existence of the solution of differential equation (3.1). □

4. Conclusion

In this paper, we initiated the concept of $b_v(s)$-metric space equipped with graph structure. Also, graphic Banach contraction is defined and it is proved that every Banach contraction is a graphic Banach contraction but converse need not be true. Fixed point results are established in aforementioned space. The presented results are validated by suitable examples. In the end, obtained results are utilized to solve ordinary differential equation which show the importance of our work.

Future Scope. One can establish the fixed point results in a graphical $b_v(s)$-metric space by using different contraction mappings, like Meir-Keeler, Kannan, Reich. Moreover, topology of the aforementioned space will helps to know the properties and whole structure of the space.

Acknowledgements. The authors would like to thank the anonymous referee(s) and the editor for their careful reading and useful suggestions on the manuscript.

Research supported by Council of Scientific&Industrial Research, Ministry of Science and Technnology, Republic of India, CSIR Sanction Reference-25(0215)/13/EMR-II; Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174013.

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