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ITERATIVE ALGORITHM OF SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEM FOR NEW MAPPINGS

MOHAMMAD FARID[1], SYED SHAKAIB IRFAN[2], AND IQBAL AHMAD[3]

Abstract. In this paper, we developed a new type iterative scheme to approximate a common solution of split monotone variational inclusion, variational inequality and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we proved that the sequence generated by the proposed iterative method converges strongly to a common solution of split monotone variational inclusion, variational inequality and fixed point problems. Furthermore, we give some consequences of the main result. Finally, we discuss a numerical example to demonstrate the applicability of the iterative algorithm. The result presented in this paper unifies and extends some known results in this area.

1. Introduction

Throughout the paper, let C_1 and C_2 be nonempty subsets of real Hilbert spaces H_1 and H_2, respectively.

A mapping $S_1 : C_1 \to C_1$ is said to be nonexpansive if

$$\|S_1x_1 - S_1x_2\| \leq \|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in C_1.$$

Let $\text{Fix}(S_1)$ denotes the fixed points of S_1 that is $\text{Fix}(S_1) = \{x_1 \in C_1 : S_1x_1 = x_1\}$.

The classical scalar nonlinear variational inequality problem (in brief, VIP) is: Find $x_1 \in C_1$ such that

$$\langle Bx_1, x_2 - x_1\rangle \geq 0, \quad \text{for all } x_2 \in C_1, \tag{1.1}$$

Key words and phrases. Iterative method, strong convergence, fixed point problem, split monotone variational inclusion problem, nonexpansive mapping, variational inequality problem.

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where $B : C_1 \to H_1$ is a nonlinear mapping. It was introduced by Hartman and Stampacchia [10].

A mapping $T : H_1 \to H_1$ is said to be

(i) monotone, if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \geq 0, \quad \text{for all } x_1, x_2 \in H_1;$$

(ii) γ-inverse strongly monotone (in brief, ism), if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \ \geq \gamma\|Tx_1 - Tx_2\|^2, \quad \text{for all } x_1, x_2 \in H_1 \text{ and } \gamma > 0;$$

(iii) firmly nonexpansive, if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \ \geq \|Tx_1 - Tx_2\|^2, \quad \text{for all } x_1, x_2 \in H_1;$$

(iv) L-Lipschitz continuous, if

$$\|Tx_1 - Tx_2\| \leq L\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in H_1 \text{ and } L > 0.$$

A set valued mapping $M_1 : H_1 \to 2^{H_1}$ is called monotone if for every $x_1, x_2 \in H_1$, $u_1 \in M_1x_1$ and $u_2 \in M_1x_2$ such that

$$\langle x_1 - x_2, u_1 - u_2\rangle \ \geq 0.$$

And it is maximal if $G(M_1)$, graph of M_1 defined as $G(M_1) = \{(x_1, u_1) : u_1 \in M_1x_1\}$ does not contain properly in the graph of other. Note that, M_1 is maximal if and only if for $(x_1, u_1) \in H_1 \times H_1$, $\langle x_1 - x_2, u_1 - u_2\rangle \ \geq 0$, for all $(x_2, u_2) \in G(M_1)$ implies $u_1 \in M_1x_1$.

An operator $J_{\rho_1}^{M_1} : H_1 \to H_1$ is defined as

$$J_{\rho_1}^{M_1}x_1 = (I + \rho_1 M_1)^{-1}x_1, \quad \text{for all } x_1 \in H_1,$$

known as resolvent operator, where $\rho_1 > 0$ and I stands for identity mapping on H_1.

In this paper, we consider the split monotone variational inclusion problem (in brief, SpMVIP). Find $\tilde{x} \in H_1$ such that

$$0 \in g_1(\tilde{x}) + M_1(\tilde{x}) \tag{1.2}$$

and

$$\tilde{y} = D\tilde{x} \in H_2 \text{ solves } 0 \in g_2(\tilde{y}) + M_2(\tilde{y}), \tag{1.3}$$

where $g_1 : H_1 \to H_1$, $g_2 : H_2 \to H_2$ be inverse strongly monotone mappings, $D : H_1 \to H_2$ be a bounded linear mapping and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings, which is introduced by Moudafi [17]. Let $\Lambda = \{\tilde{x} \in H_1 : \tilde{x} \in \mathrm{Sol(MVIP(1.2))} \text{ and } D\tilde{x} \in \mathrm{Sol(MVIP(1.3))}\}$ denote the solution of SpMVIP (1.2)–(1.3).

The split feasibility, split zero and the split fixed point problems include as a special cases. It studied broadly by various authors and solved real life problems essentially in modelling of inverse problems, sensor networks in computerised tomography and radiation therapy; for details [3, 5, 7].

If $g_1 \equiv 0$ and $g_2 \equiv 0$ then we find a split null point problem (in brief, S_PNPP):Find $\tilde{x} \in H_1$ such that

$$0 \in M_1(\tilde{x}) \tag{1.4}$$

and

$$\tilde{y} = D\tilde{x} \in H_2 \text{ solves } 0 \in M_2(\tilde{y}). \tag{1.5}$$

The iterative algorithm for S_PMVIP (1.2)–(1.3) was introduced and studied by Moudafi [17]:

$$x_0 \in H_1, \quad x_{n+1} = P(x_n + \eta D^*(Q - I)Dx_n), \quad \text{for } \rho > 0,$$

where $P := J_\rho^{M_1}(I - \rho g_1)$, $Q := J_\rho^{M_2}(I - \rho g_2)$, D^* be the adjoint operator of D and $0 < \eta < \frac{1}{\varsigma}$, ς be the spectral radius of D^*D.

The convergence analysis was studied by Byrne et al. [4] of some iterative algorithm for S_PNPP (1.4)–(1.5). Moreover, Kazmi et al. [15] established an iterative method to find a common solution of S_PNPP (1.4)–(1.5) and fixed point problem. For instance, see [1, 12–14, 20–22].

Recently, Qin et al. [19] proposed an algorithm for infinite family of nonexpansive mappings as:

$$x_0 \in C_1,\ x_{n+1} = \mu_n \theta g(x_n) + \eta_n x_n + ((1 - \eta_n)I - \mu_n D)\mathbb{W}_n u_n,$$

where g be a contraction mapping on H_1, D be a strongly positive bounded linear operator, $\mathbb{W}_n$ generated by $S_1, S_2, \ldots$ as:

$$\begin{aligned}
\mathbb{V}_{n,n+1} &:= I,\\
\mathbb{V}_{n,n} &:= \lambda_n S_n \mathbb{V}_{n,n+1} + (1 - \lambda_n)I,\\
\mathbb{V}_{n,n-1} &:= \lambda_{n-1} S_{n-1} \mathbb{V}_{n,n} + (1 - \lambda_{n-1})I,\\
&\vdots\\
\mathbb{V}_{n,m} &:= \lambda_m S_m \mathbb{V}_{n,m+1} + (1 - \lambda_m)I,\\
\mathbb{V}_{n,m-1} &:= \lambda_{m-1} S_{m-1} \mathbb{V}_{n,m} + (1 - \lambda_{m-1})I,\\
&\vdots\\
\mathbb{V}_{n,2} &:= \lambda_2 S_2 \mathbb{V}_{n,3} + (1 - \lambda_3)I,\\
\mathbb{W}_n \equiv \mathbb{V}_{n,1} &:= \lambda_1 S_1 \mathbb{V}_{n,2} + (1 - \lambda_1)I,
\end{aligned} \tag{1.6}$$

where $S_1, S_2, \ldots, \mathbb{W}_n$ are nonexpansive mappings, $\{\lambda_n\} \subset (0, 1]$, for $n \geq 1$. For further work see [8, 11].

Inspirited by Moudafi [17], Byrne et al. [4], Kazmi et al. [14, 15], Qin et al. [19] and by continuing work, we propose and analyze a new type iterative algorithm to find a common solution of split monotone variational inclusion, variational inequality and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we endowed that the sequence generated by the algorithm converges strongly to common solution. Furthermore, we listed some consequences of our established theorem. Finally, we provide a numerical example to demonstrate the applicability of algorithm. We emphasize that the result accounted in manuscript is unifies and extends of various results in this field of study.

2. Preliminaries

This section is devoted to recall few definitions, entailing mathematical tools and helpful results that are required in the sequel.

To each $x_1 \in H_1$, there exists a unique nearest point $P_{C_1}x_1$ to x_1 in C_1 such that

$$\|x_1 - P_{C_1}x_1\| \leq \|x_1 - x_2\|, \quad \text{for all } x_2 \in C_1,$$

where P_{C_1} is a metric projection of H_1 onto C_1. Also, P_{C_1} is nonexpansive and holds

$$\langle x_1 - x_2, P_{C_1}x_1 - P_{C_1}x_2\rangle \geq \|P_{C_1}x_1 - P_{C_1}x_2\|^2, \quad \text{for all } x_1, x_2 \in H_1.$$

Moreover, $P_{C_1}x_1$ is characterized by the fact that $P_{C_1}x_1 \in C_1$ and

$$\langle x_1 - P_{C_1}x_1, x_2 - P_{C_1}x_1\rangle \leq 0, \quad \text{for all } x_2 \in C_1.$$

This implies that

$$\|x_1 - x_2\|^2 \geq \|x_1 - P_{C_1}x_1\|^2 + \|x_2 - P_{C_1}x_1\|^2, \quad \text{for all } x_1 \in H_1, \text{for all } x_2 \in C_1,$$

and

$$\|\mu x_1 + (1-\mu)x_2\|^2 = \mu\|x_1\|^2 + (1-\mu)\|x_2\|^2 - \mu(1-\mu)\|x_1 - x_2\|^2,$$

for all $x_1, x_2 \in H_1$ and $\mu \in [0, 1]$.

Also, on H_1 holds following inequalities.

(a) Opial's condition [18], that is for any $\{x_n\}$ with $x_n \rightharpoonup x_1$ and

$$\liminf_{n\to\infty} \|x_n - x_1\| < \liminf_{n\to\infty} \|x_n - x_2\|,$$

holds, for all $x_2 \in H_1$ with $x_2 \neq x_1$.

(b)

$$\|x_1 + x_2\|^2 \leq \|x_1\|^2 + 2\langle x_2, x_1 + x_2\rangle, \quad \text{for all } x_1, x_2 \in H_1. \tag{2.1}$$

Definition 2.1. ([2]) A mapping $T_1 : H_1 \to H_1$ is called averaged if and only if

$$T_1 = (1-\lambda)I + \lambda S_1,$$

where $\lambda \in (0, 1)$, I be the identity mapping on H_1 and $S_1 : H_1 \to H_1$ be nonexpansive mapping.

Lemma 2.1. ([17])

(i) *If $T_2 = (1-\lambda)T_1 + \lambda S_1$, where $T_1 : H_1 \to H_1$ be averaged, $S_1 : H_1 \to H_1$ be nonexpansive and $0 < \lambda < 1$, then T_2 is averaged.*

(ii) *If T_1 is γ-ism, then βT_1 is $\frac{\gamma}{\beta}$-ism, for $\beta > 0$.*

(iii) *T_1 is averaged if and only if $I - T$ is γ-ism for some $\gamma > \frac{1}{2}$.*

Lemma 2.2. ([17]) *Let $\rho > 0$, f be a γ-ism and M be a maximal monotone mapping. If $\rho \in (0, 2\gamma)$, then $J_\rho^M(I - \rho f)$ is averaged.*

Lemma 2.3. ([17]) *Let $\rho_1, \rho_2 > 0$ and M_1, M_2 be maximal monotone mapping. Then*

$$\tilde{x} \text{ solves (1.2)–(1.3)} \iff \tilde{x} = J_{\rho_1}^{M_1}(I - \rho_1 f_1)\tilde{x} \text{ and } B\tilde{x} = J_{\rho_2}^{M_2}(I - \rho_2 f_2)B\tilde{x}.$$

Lemma 2.4. ([24]) *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in E, a Banach space and let $0 < \mu_n < 1$ with $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$. Consider $v_{n+1} = (1-\mu_n)v_n + \mu_n u_n$, $n \geq 0$, and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$. Then*

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Lemma 2.5. ([16]) *Assume that B is a strongly positive self-adjoint bounded linear operator on H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.6. ([25]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers with*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \alpha_n, \quad n \geq 0,$$

where $\lambda_n \in (0, 1)$ and $\{\alpha_n\}$ in $\mathbb{R}$ with

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n} \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. ([9]) *Let $S_1 : C_1 \to H_1$ be a nonexpansive mapping. If S_1 has a fixed point, then $(I - S_1)$, where I be the identity mapping, be demiclosed that is if $x_n \rightharpoonup x_1 \in H_1$ and $x_n - S_1 x_n \to x_2$, then $(I - S_1)x_1 = x_2$.*

Lemma 2.8. ([23]) *Let $C_1 \neq \emptyset$ be closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. Then $\lim_{i\to\infty} \mathbb{V}_{i,j}\tilde{x}$ exists for all $\tilde{x} \in C_1$ and $j \in \mathbb{N}$.*

Remark 2.1. By Lemma 2.8, define a mapping $\mathbb{W} : C_1 \to C_1$ such that $\mathbb{W}\tilde{x} = \lim_{i\to\infty} \mathbb{W}_i\tilde{x} = \lim_{i\to\infty} \mathbb{V}_{i,1}\tilde{x}$ for all $\tilde{x} \in C_1$, which is called the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ In the whole paper, we consider $0 < \lambda_i < 1$ for all $i \geq 1$.

Lemma 2.9. ([23]) *Let $C_1 \neq \emptyset$ be closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. Then $\mathrm{Fix}(\mathbb{W}) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.*

Lemma 2.10. ([6]) *Let $C_1 \neq \emptyset$ be closed convex subset of H_1. Let $S_1, S_2, \dots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. If K be any bounded bounded subset of C_1, then $\lim_{i\to\infty} \sup_{\tilde{x}\in K} \|\mathbb{W}_i\tilde{x} - \mathbb{W}\tilde{x}\| = 0$.*

3. Main Result

We study the following convergence result for a new type iterative method to find a common solution of $\mathrm{S_P}$MVIP (1.2)–(1.3), VIP (1.1) and fixed point problem.

Theorem 3.1. *Let H_1 and H_2 denote the Hilbert spaces and $C_1 \subset H_1$ be nonempty closed convex subset of H_1. Let $B : H_1 \to H_1$ be a γ-inverse strongly monotone mapping, $D : H_1 \to H_2$ be a bounded linear operator with its adjoint operator D^*, $M_1 : H_1 \to 2^{H_1}$,$M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone operators and $g_1 : H_1 \to H_1$, $g_2 : H_2 \to H_2$ be α_1, α_2-inverse strongly monotone mappings, respectively. Let $f : C_1 \to C_1$ be a contraction mapping with constant $\tau \in (0, 1)$, A be a strongly positive bounded linear self adjoint operator on C_1 with constant $\bar{\theta} > 0$ such that $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$ and $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be an infinite family of nonexpansive mappings such that $\Gamma := \Lambda \cap \mathrm{Sol(VIP(1.1))} \cap (\cap_{i=1}^{\infty}\mathrm{Fix}(S_i)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated as:*

$$\left.\begin{aligned}
&x_1 \in C_1,\\
&z_n = R(I + \xi D^*(S - I)D)x_n,\\
&u_n = P_{C_1}(z_n - \sigma_n B z_n),\\
&v_n = \delta_n u_n + (1 - \delta_n)\mathbb{W}_n u_n,\\
&x_{n+1} = \mu_n \theta f(x_n) + \eta_n x_n + ((1 - \eta_n)I - \mu_n A)v_n, \quad n \geq 1,
\end{aligned}\right\}$$

where $R = J_{\rho_1}^{(g_1,M_1)}(I - \rho_1 g_1)$, $S = J_{\rho_2}^{(g_2,M_2)}(I - \rho_2 g_2)$, $\mathbb{W}_n$ defined in (1.6), $\{\mu_n\}$, $\{\eta_n\}$, $\{\delta_n\} \subset (0, 1)$ and $\xi \in (0, \frac{1}{\epsilon})$, ϵ be the spectral radius of D^*D. Let the control sequences satisfying conditions:

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=0}^{\infty} \mu_n = \infty$;
(ii) $0 < \rho_1 < 2\alpha_1$, $0 < \rho_2 < 2\alpha_2$;
(iii) $0 < \liminf_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \eta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2\gamma$;
(v) $\lim_{n\to\infty} \delta_n = 0$.

Then, the sequence $\{x_n\}$ converges strongly to some $\tilde{x} \in \Gamma$, where $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ which solves:

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x}\rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

Proof. For sake of simplicity, we divide the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Let $\tilde{x} \in \Gamma$ then $\tilde{x} \in \Lambda$ and thus $R\tilde{x} = \tilde{x}$, $S(D\tilde{x}) = D\tilde{x}$ and $P\tilde{x} = \tilde{x}$, where $P = I + \eta D^*(S - I)D$. By Lemma 2.2 and firmly nonexpansive, R and S are averaged.

Also, P is averaged since it is $\frac{\nu}{\epsilon}$-ism for some $\nu > \frac{1}{2}$. From Lemma 2.1 (iii), $I - S$ is ν-ism. Thus, we obtain

$$\begin{aligned}\langle D^*(I-S)Dx_1 - D^*(I-S)Dx_2, x_1 - x_2\rangle =&\langle (I-S)Dx_1 - (I-S)Dx_2, Dx_1 - Dx_2\rangle\\ \geq&\nu\|(I-S)Dx_1 - (I-S)Dx_2\|^2\\ \geq&\frac{\nu}{\epsilon}\|D^*(I-S)Dx_1 - D^*(I-S)Dx_2\|^2.\end{aligned}$$

This implies that $\eta D^*(I-S)D$ is $\frac{\nu}{\xi\epsilon}$-ism. Since $0 < \xi < \frac{1}{\epsilon}$ therefore its complement $(I - \xi D^*(I-S)D)$ is averaged and hence $R(I + \xi D^*(S-I)D) = \mathbb{Z}$(say). Thus, $I + \xi D^*(S-I)D$, R, S and $\mathbb{Z}$ are nonexpansive mappings.

Next, we calculate

$$\begin{aligned}\|z_n - \tilde{x}\|^2 &= \|J_{\rho_1}^{g_1,M_1}(I-\rho_1 g_1)(x_n + \xi D^*(S-I)Dx_n) - J_{\rho_1}^{g_1,M_1}(I-\rho_1 g_1)\tilde{x}\|^2\\ &\leq \|x_n + \xi D^*(S-I)Dx_n - \tilde{x}\|^2\\ &= \|x_n - \tilde{x}\|^2 + \xi^2\|D^*(S-I)Dx_n\|^2 + 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle. \qquad (3.1)\end{aligned}$$

Now,

$$\begin{aligned}\xi^2\|D^*(S-I)Dx_n\|^2 &= \xi^2\langle (S-I)Dx_n, DD^*(S-I)Dx_n\rangle\\ &\leq \epsilon\xi^2\langle (S-I)Dx_n, (S-I)Dx_n\rangle\\ &= \epsilon\xi^2\|(S-I)Dx_n\|^2. \qquad (3.2)\end{aligned}$$

Consider $\Upsilon_n := 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle$ and we estimate

$$\begin{aligned}\Upsilon_n &= 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle\\ &= 2\xi\langle D(x_n - \tilde{x}) + (S-I)Dx_n - (S-I)Dx_n, (S-I)Dx_n\rangle\\ &= 2\xi[\langle S(D(x_n) - D\tilde{x}, (S-I)Dx_n\rangle - \|(S-I)Dx_n\|^2]\\ &\leq 2\xi\left[\frac{1}{2}\|(S-I)Dx_n\|^2 - \|(S-I)Dx_n\|^2\right]\\ &= -\xi\|(S-I)Dx_n\|^2. \qquad (3.3)\end{aligned}$$

From (3.1), (3.2), (3.3), we obtain

$$\|z_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 + \xi(\epsilon\xi - 1)\|(S-I)Dx_n\|^2. \qquad (3.4)$$

Since $0 < \xi < \frac{1}{\epsilon}$, therefore

$$\|z_n - \tilde{x}\| \leq \|x_n - \tilde{x}\|. \qquad (3.5)$$

Using γ-ism and $0 < \sigma_n < 2\gamma$, we have

$$\begin{aligned}\|u_n - \tilde{x}\|^2 &= \|P_{C_1}(z_n - \sigma_n Bz_n) - P_{C_1}(z_n - \sigma_n B\tilde{x})\|^2\\ &\leq \|z_n - \sigma_n Bz_n - (z_n - \sigma_n B\tilde{x})\|^2\\ &= \|(z_n - \tilde{x}) - \sigma_n(Bz_n - B\tilde{x})\|^2\\ &= \|z_n - \tilde{x}\|^2 - 2\sigma_n\langle Bz_n - B\tilde{x}, z_n - \tilde{x}\rangle + \sigma_n^2\|Bz_n - B\tilde{x}\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \|z_n - \tilde{x}\|^2 - 2\sigma_n\gamma\|Bz_n - B\tilde{x}\|^2 + \sigma_n^2\|Bz_n - B\tilde{x}\|^2 \\
&= \|z_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2 \\
&\le \|z_n - \tilde{x}\|^2, \qquad (3.6)
\end{aligned}
$$

this implies

$$\|u_n - \tilde{x}\| \le \|z_n - \tilde{x}\|. \qquad (3.7)$$

By using (3.5) and (3.6), we calculate

$$
\begin{aligned}
\|v_n - \tilde{x}\| &\le \delta_n\|u_n - \tilde{x}\| + (1-\delta_n)\|\mathbb{W}_n u_n - \tilde{x}\| \\
&= \delta_n\|u_n - \tilde{x}\| + (1-\delta_n)\|u_n - \tilde{x}\| \\
&\le \|u_n - \tilde{x}\| \\
&= \|z_n - \tilde{x}\| \\
&= \|x_n - \tilde{x}\|. \qquad (3.8)
\end{aligned}
$$

By using (3.7) and (3.8), we calculate

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \|\mu_n\theta f(x_n) + \eta_n x_n + ((1-\eta_n)I - \mu_n A)v_n - \tilde{x}\| \\
&= \|\mu_n(\theta f(x_n) - A\tilde{x}) + \eta_n(x_n - \tilde{x}) \\
&\quad + ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x})\| \\
&\le \mu_n\|\theta f(x_n) - A\tilde{x}\| + \eta_n\|x_n - \tilde{x}\| \\
&\quad + ((1-\eta_n)I - \mu_n\overline{\theta})\|v_n - \tilde{x}\| \\
&\le \mu_n\|\theta f(x_n) - \theta f(\tilde{x}) + \theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + \eta_n\|x_n - \tilde{x}\| + ((1-\eta_n)I - \mu_n\overline{\theta})\|u_n - \tilde{x}\| \\
&\quad \le \mu_n\theta\|f(x_n) - f(\tilde{x})\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + \eta_n\|x_n - \tilde{x}\| + ((1-\eta_n)I - \mu_n\overline{\theta})\|x_n - \tilde{x}\| \\
&\le \mu_n\theta\tau\|x_n - \tilde{x}\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + (1 - \mu_n\overline{\theta})\|x_n - \tilde{x}\| \\
&\le (1 - \mu_n(\overline{\theta} - \theta\tau))\|x_n - \tilde{x}\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\le \max\left\{\|x_n - \tilde{x}\|, \frac{\|\theta f(\tilde{x}) - A\tilde{x}\|}{\overline{\theta} - \theta\tau}\right\}, \quad n \ge 1.
\end{aligned}
$$

Using induction, we get

$$\|x_{n+1} - \tilde{x}\| \le \max\left\{\|x_1 - \tilde{x}\|, \frac{\|\theta f(\tilde{x}) - A\tilde{x}\|}{\overline{\theta} - \theta\tau}\right\}.$$

Thus, $\{x_n\}$ is bounded. Also, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{\mathbb{W}(x_n)\}$ are bounded due to (3.4), (3.7) and (3.8).

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|x_n - \mathbb{W}_n u_n\| = 0$, $\lim_{n\to\infty}\|v_n - x_n\| = 0$ and $\lim_{n\to\infty}\|v_n - u_n\| = 0$.

Since $R(I+\xi D^*(S-I)D)$ is nonexpansive therefore

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|R(I+\xi D^*(S-I)D)x_{n+1}-R(I+\xi D^*(S-I)D)x_n\|\\ &\leq \|x_{n+1}-x_n\|. \end{aligned}\tag{3.9}$$

Using (3.9), we estimate

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|P_C(I-\sigma_{n+1}B)z_{n+1}-P_C(I-\sigma_nB)z_n\|\\ &\leq \|(I-\sigma_{n+1}B)z_{n+1}-(I-\sigma_nB)z_n\|\\ &= \|(I-\sigma_{n+1}B)z_{n+1}-(I-\sigma_{n+1}B)z_n+(\sigma_n-\sigma_{n+1})Av_n\|\\ &\leq \|z_{n+1}-z_n\|+|\sigma_n-\sigma_{n+1}|\|Bz_n\|\\ &\leq \|x_{n+1}-x_n\|+|\sigma_n-\sigma_{n+1}|\|Bz_n\|\\ &\leq \|x_{n+1}-x_n\|+\mathbb{N}_1|\sigma_n-\sigma_{n+1}|, \end{aligned}\tag{3.10}$$

where $\mathbb{N}_1=\sup_{n\geq 1}\|Bz_n\|$.

For $i\in 1,2,\dots,n$, S_i and $\mathbb{V}_{n,i}$, are nonexpansive therefore from (1.6), we obtain

$$\begin{aligned}\|\mathbb{W}_{n+1}u_n-\mathbb{W}_nu_n\| &= \|\lambda_1S_1\mathbb{V}_{n+1,2}u_n-\lambda_1S_1\mathbb{V}_{n,2}u_n\|\\ &\leq \lambda_1\|\mathbb{V}_{n+1,2}u_n-\mathbb{V}_{n,2}u_n\|\\ &\leq \lambda_1\|\lambda_2S_2\mathbb{V}_{n+1,3}u_n-\lambda_2S_2\mathbb{V}_{n,3}u_n\|\\ &\leq \lambda_1\lambda_2\|\mathbb{V}_{n+1,3}u_n-\mathbb{V}_{n,3}u_n\|\\ &\vdots\\ &\leq \lambda_1\lambda_2\cdots\lambda_n\|\mathbb{V}_{n+1,n+1}u_n-\mathbb{V}_{n,n+1}u_n\|\\ &\leq \mathbb{N}_2\prod_{i=1}^n\lambda_i, \end{aligned}\tag{3.11}$$

where $\mathbb{N}_2\geq 0$ with $\|\mathbb{V}_{n+1,n+1}u_n-\mathbb{V}_{n,n+1}u_n\|\leq\mathbb{N}_2$ for all $n\geq 1$.

Using (3.10) and (3.11), we estimate

$$\begin{aligned}\|v_{n+1}-v_n\| \leq&\|\delta_{n+1}u_{n+1}+(1-\delta_{n+1})\mathbb{W}_{n+1}u_{n+1}-\delta_nu_n-(1-\delta_n)\mathbb{W}_nu_n\|\\ \leq&\|\delta_{n+1}u_{n+1}+(1-\delta_{n+1})\mathbb{W}_{n+1}u_{n+1}-\delta_nu_n-(1-\delta_n)\mathbb{W}_nu_n\\ &+(1-\delta_n)\mathbb{W}_{n+1}u_n-(1-\delta_n)\mathbb{W}_{n+1}u_n\|\\ \leq&(1-\delta_n)\|\mathbb{W}_{n+1}u_n-\mathbb{W}_nu_n\|+\|\mathbb{W}_{n+1}u_{n+1}-\mathbb{W}_{n+1}u_n\|\\ &+\delta_{n+1}\|\mathbb{W}_{n+1}u_{n+1}-u_{n+1}\|+\delta_n\|\mathbb{W}_{n+1}u_n-u_n\|\\ \leq&(1-\delta_n)\mathbb{N}_2\prod_{i=1}^n\lambda_i+\|u_{n+1}-u_n\|+\delta_{n+1}\mathbb{N}_3+\delta_n\mathbb{N}_4\\ \leq&(1-\delta_n)\mathbb{N}_2\prod_{i=1}^n\lambda_i+\|x_{n+1}-x_n\|+\mathbb{N}_1|\sigma_n-\sigma_{n+1}|+\delta_{n+1}\mathbb{N}_3+\delta_n\mathbb{N}_4, \end{aligned}\tag{3.12}$$

where $\mathbb{N}_3 = \sup_{n\geq 1} \|\mathbb{W}_{n+1}u_{n+1} - u_{n+1}\|$ and $\mathbb{N}_4 = \sup_{n\geq 1} \|\mathbb{W}_{n+1}u_n - u_n\|$. Setting $x_{n+1} = (1-\eta_n)s_n + \eta_n x_n$, then we have $s_n = \frac{x_{n+1}-\eta_n x_n}{1-\eta_n}$ and

$$\begin{aligned} s_{n+1} - s_n =& \frac{\mu_{n+1}\theta f(x_{n+1}) + ((1-\eta_{n+1})I - \mu_{n+1}A)v_{n+1}}{1-\eta_{n+1}} \\ &- \frac{\mu_n \theta f(x_n) + ((1-\eta_n)I - \mu_n A)v_n}{1-\eta_n} \\ =& \frac{\mu_{n+1}}{1-\eta_{n+1}}(\theta f(x_{n+1}) - Av_{n+1}) + \frac{\mu_n}{1-\eta_n}(Av_n - \theta f(x_n)) + v_{n+1} - v_n. \end{aligned}$$

Hence,

$$\begin{aligned} \|s_{n+1} - s_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}(\|\theta f(x_{n+1})\| + \|Av_{n+1}\|) \\ &+ \frac{\mu_n}{1-\eta_n}(\|Av_n\| + \|\theta f(x_n)\|) + \|v_{n+1} - v_n\| \\ \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 + \|v_{n+1} - v_n\|, \end{aligned}$$

where $\mathbb{N}_5 = \sup_{n\geq 1}(\|\theta f(x_{n+1})\| + \|Av_{n+1}\|)$ and $\mathbb{N}_6 = \sup_{n\geq 1}(\|Av_n\| + \|\theta f(x_n)\|)$.

Using (3.12) in above inequality

$$\begin{aligned} \|s_{n+1} - s_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 + (1-\delta_n)\mathbb{N}_2 \prod_{i=1}^{n} \lambda_i \\ &+ \|x_{n+1} - x_n\| + \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + \delta_{n+1}\mathbb{N}_3 + \delta_n \mathbb{N}_4, \end{aligned}$$

and thus

$$\begin{aligned} \|s_{n+1} - s_n\| - \|x_{n+1} - x_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 \\ &+ \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + (1-\delta_n)\mathbb{N}_2 \prod_{i=1}^{n} \lambda_i + \delta_{n+1}\mathbb{N}_3 + \delta_n \mathbb{N}_4. \end{aligned}$$

Using the given conditions in above inequality, we have

$$\limsup_{n\to\infty}(\|s_{n+1} - s_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.4, we get

$$\lim_{n\to\infty} \|s_n - x_n\| = 0.$$

As $x_{n+1} = (1-\eta_n)s_n + \eta_n x_n$, therefore

$$\|x_{n+1} - x_n\| = \|(1-\eta_n)(s_n - x_n)\|,$$

which yields

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

Now,

$$\begin{aligned} \|x_n - \mathbb{W}_n u_n\| =& \|x_n - x_{n+1} + x_{n+1} - \mathbb{W}_n u_n\| \\ \leq& \|x_{n+1} - x_n\| + \|\mu_n \theta f(x_n) + \eta_n x_n \end{aligned}$$

$$\begin{aligned}&+((1-\eta_n)I-\mu_n A)v_n-\mathbb{W}_n u_n\|\\ =&\|x_{n+1}-x_n\|+\|\mu_n(\theta f(x_n)-Av_n)\|\\ &+((1-\eta_n)I-\mu_n A)(v_n-\mathbb{W}_n u_n)+\eta_n(x_n-\mathbb{W}_n u_n)\\ \le&\|x_{n+1}-x_n\|+\mu_n\|\theta f(x_n)-Av_n\|+\eta_n\|x_n-\mathbb{W}_n u_n\|. \end{aligned} \tag{3.14}$$

Hence,

$$(1-\eta_n)\|x_n-\mathbb{W}_n u_n\|\le\|x_{n+1}-x_n\|+\mu_n\|\theta f(x_n)-Av_n\|.$$

Using the given conditions and (3.13) in (3.14), we get

$$\lim_{n\to\infty}\|x_n-\mathbb{W}_n u_n\|=0. \tag{3.15}$$

By (3.5) and (3.7), we compute

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2=&\|\mu_n\theta f(x_n)+\eta_n x_n+((1-\eta_n)I-\mu_n A)v_n-\tilde{x}\|^2\\ \le&\|(1-\eta_n)(v_n-\tilde{x})+\eta_n(x_n-\tilde{x})\|^2\\ &+2\langle\mu_n\theta f(x_n)-\mu_n Av_n,x_{n+1}-\tilde{x}\rangle\\ \le&(1-\eta_n)\|v_n-\tilde{x}\|^2+\eta_n\|x_n-\tilde{x}\|^2\\ &-\eta_n(1-\eta_n)\|x_n-v_n\|^2\\ &+2\mu_n\|\theta f(x_n)-Av_n\|\|x_{n+1}-\tilde{x}\|\\ \le&\eta_n\|x_n-\tilde{x}\|^2+(1-\eta_n)\|v_n-\tilde{x}\|^2+2\mu_n\mathbb{N}_7, \end{aligned} \tag{3.16}$$

where $\mathbb{N}_7=\max\{\sup_{n\ge1}\|\theta f(x_n)-Av_n\|,\sup_{n\ge1}\|x_{n+1}-\tilde{x}\|\}$. From (3.4) and (3.8), we get

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2\le&\eta_n\|x_n-\tilde{x}\|^2+(1-\eta_n)\|x_n-\tilde{x}\|^2\\ &+(1-\eta_n)\xi(\epsilon\xi-1)\|(S-I)Dx_n\|^2+2\mu_n\mathbb{N}_7\\ \le&\|x_n-\tilde{x}\|^2+(1-\eta_n)\xi(\epsilon\xi-1)\|(S-I)Dx_n\|^2+2\mu_n\mathbb{N}_7,\end{aligned}$$

which yields

$$(1-\eta_n)\xi(1-\epsilon\xi)\|(S-I)Dx_n\|^2\le\|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2+2\mu_n\mathbb{N}_7.$$

Since $\epsilon(1-\epsilon\xi)>0$, $\lim_{n\to\infty}\mu_n=0$ and $\{x_n\}$, $\{u_n\}$ are bounded, and using (3.13), we have

$$\lim_{n\to\infty}\|(S-I)Dx_n\|=0. \tag{3.17}$$

Next, prove that $\lim_{n\to\infty}\|z_n-x_n\|=0$.

By using firmly nonexpansive of $J_{\rho_1}^{(g_1,M_1)}$, we compute

$$\begin{aligned}\|z_n-\tilde{x}\|^2=&\|J_{\rho_1}^{(g_1,M_1)}(x_n+\xi D^*(S-I)Dx_n)-J_{\rho_1}^{(g_1,M_1)}\tilde{x}\|^2\\ \le&\langle z_n-\tilde{x},x_n+\xi D^*(S-I)Dx_n-\tilde{x}\rangle\\ =&\frac{1}{2}\Big\{\|z_n-\tilde{x}\|^2+\|x_n+\xi D^*(S-I)Dx_n-\tilde{x}\|^2-\|(z_n-\tilde{x})\end{aligned}$$

$$
\begin{aligned}
&- [x_n + \xi D^*(S-I)Dx_n - \tilde{x}]\,\|^2 \Big\} \\
=&\frac{1}{2}\Big\{ \|z_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \|z_n - x_n - \xi D^*(S-I)Dx_n\|^2 \Big\} \\
=&\frac{1}{2}\Big\{ \|z_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \Big[\|z_n - x_n\|^2 + \xi^2 \|D^*(S-I)Dx_n\|^2 \\
&- 2\xi \langle z_n - x_n, D^*(S-I)Dx_n \rangle \Big] \Big\}.
\end{aligned}
$$

Hence, we obtain

$$\|z_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 - \|z_n - x_n\|^2 + 2\xi \|D(z_n - x_n)\| \|(S-I)Dx_n\|.$$

Using (3.7), (3.8) and (3.16) in above inequality, we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq& \eta_n \|x_n - \tilde{x}\|^2 + (1-\eta_n)\|x_n - \tilde{x}\|^2 \\
&- (1-\eta_n)\|z_n - x_n\|^2 \\
&+ 2(1-\eta_n)\xi \|D(z_n - x_n)\| \|(S-I)Dx_n\| + 2\mu_n \mathbb{N}_7 \\
(1-\eta_n)\|z_n - x_n\|^2 \leq& \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&+ 2(1-\eta_n)\xi \|D(z_n - x_n)\| \|(S-I)Dx_n\| + 2\mu_n \mathbb{N}_7 \\
\leq& (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| \\
&+ 2(1-\eta_n)\xi \|D(z_n - x_n)\| \|(S-I)Dx_n\| + 2\mu_n \mathbb{N}_7.
\end{aligned}
$$

By (3.13), (3.17) and the given conditions, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.18}$$

Next, prove that $\lim_{n\to\infty} \|z_n - u_n\| = 0$.

We estimate

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 =& \|(1-\eta_n)(v_n - \tilde{x}) + \eta_n(x_n - \tilde{x}) \\
&+ \mu_n(\theta f(x_n) - Av_n)\|^2 \\
\leq& (1-\eta_n)\|v_n - \tilde{x}\|^2 + \eta_n \|x_n - \tilde{x}\|^2 \\
&+ 2\mu_n \langle \kappa_n, x_{n+1} - \tilde{x} \rangle \\
\leq& (1-\eta_n)\|v_n - \tilde{x}\|^2 + \eta_n \|x_n - \tilde{x}\|^2 + 2\omega^2 \mu_n \\
\leq& (1-\eta_n)\|u_n - \tilde{x}\|^2 + \eta_n \|x_n - \tilde{x}\|^2 + 2\omega^2 \mu_n.
\end{aligned}
$$

In the above inequality we set $\kappa_n = \theta f(x_n) - Av_n$ and let $\omega > 0$ be a suitable constant with $\omega \geq \sup_n\{\|\kappa_n\|, \|x_n - \tilde{x}\|\}$. Thus,

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq& (1-\eta_n)\Big\{ \|P_{C_1}(z_n - \sigma_n B z_n) - P_{C_1}(\tilde{x} - \sigma_n B\tilde{x})\|^2 \Big\} \\
&+ \eta_n \|x_n - \tilde{x}\|^2 + 2\omega^2 \mu_n \\
\leq& (1-\eta_n)\Big\{ \|z_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(1-\eta_n)\Big\{\|x_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2\Big\} \\
&+ \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(1-\eta_n)\sigma_n(\sigma_n - 2\omega)\|Bz_n - B\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n,
\end{aligned} \tag{3.19}
$$

which implies

$$
\begin{aligned}
(1-\eta_n)\sigma_n(2\omega - \sigma_n)\|Bz_n - B\tilde{x}\|^2 \leq&\|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| + 2\omega^2\mu_n.
\end{aligned}
$$

By (3.13) and the given conditions, we get

$$
\lim_{n\to\infty}\|Bz_n - B\tilde{x}\| = 0. \tag{3.20}
$$

From (2.1), we compute

$$
\begin{aligned}
\|u_n - \tilde{x}\|^2 =&\|P_{C_1}(z_n - \sigma_n Bz_n) - P_{C_1}(\tilde{x} - \sigma_n B\tilde{x})\|^2 \\
\leq&\langle u_n - \tilde{x}, (z_n - \sigma_n Bz_n) - (\tilde{x} - \sigma_n B\tilde{x})\rangle \\
\leq&\frac{1}{2}\Big\{\|u_n - \tilde{x}\|^2 + \|(z_n - \sigma_n Bz_n) \\
&- (\tilde{x} - \sigma_n B\tilde{x})\|^2 - \|(u_n - z_n) + \sigma_n(Bz_n - B\tilde{x})\|^2\Big\} \\
\leq&\frac{1}{2}\Big\{\|u_n - \tilde{x}\|^2 + \|z_n - \tilde{x}\|^2 - \|(u_n - z_n) + \sigma_n(Bz_n - B\tilde{x})\|^2\Big\} \\
\leq&\|z_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 - \sigma_n^2\|Bz_n - B\tilde{x}\|^2 \\
&+ 2\sigma_n\langle u_n - z_n, Bu_n - B\tilde{x}\rangle \\
\leq&\|z_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 + 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| \\
\leq&\|x_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 + 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\|.
\end{aligned}
$$

By (3.19), we obtained

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq&(1-\eta_n)\|u_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(1-\eta_n)\Big\{\|x_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 \\
&+ 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\|\Big\} + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1-\eta_n)\|u_n - z_n\|^2 \leq&\|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&+ 2(1-\eta_n)\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| + 2\omega^2\mu_n \\
\leq&(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| \\
&+ 2(1-\eta_n)\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| + 2\omega^2\mu_n.
\end{aligned}
$$

Using (3.13), (3.20) and the given conditions, we get

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.21}$$

From (3.18) and (3.21), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.22}$$

By (3.15) and (3.22), we get

$$\lim_{n\to\infty} \|\mathbb{W}_n u_n - u_n\| = 0. \tag{3.23}$$

Further, using (3.22) and (3.23)

$$\begin{aligned} \|v_n - x_n\|^2 &\leq \|\delta_n u_n + (1-\delta_n)\mathbb{W}_n u_n - x_n\| \\ &\leq \delta_n \|u_n - x_n\| + (1-\delta_n)\|\mathbb{W}_n u_n - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.24}$$

Therefore, by (3.22) and (3.24), we get

$$\lim_{n\to\infty} \|u_n - v_n\| = 0. \tag{3.25}$$

Step 3. We claim that $\tilde{x} \in \Gamma$.

Since $\{x_n\}$ is bounded therefore consider $\tilde{x} \in H_1$ be any weak cluster point of $\{x_n\}$. Hence, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_{n_j} \rightharpoonup \tilde{x}$. By Lemma 2.7 and (3.23), we have $\tilde{x} \in \cap_{i=1}^{\infty} \text{Fix}(S_i)$.

And $z_{n_j} = R[x_{n_j} + \eta D^*(S-I)Dx_{n_j}]$ can write as

$$\frac{(x_{n_j} - z_{n_j}) + D^*(S-I)Dx_{n_j}}{\rho_1} \in M_1 z_{n_j}. \tag{3.26}$$

Taking $j \to \infty$ in (3.26) and by (3.17), (3.18) and the concept of the graph of a maximal monotone mapping and $\frac{1}{\alpha_1}$-Lipschitz continuity of g_1, we get $0 \in M_1\tilde{x} + g_1\tilde{x}$ that is $\tilde{x} \in \text{Sol(MVIP(1.2))}$. Furthermore, since $\{x_n\}$ and $\{z_n\}$ have the same asymptotical behaviour, $Dx_{n_j} \rightharpoonup D\tilde{x}$. As S is nonexpansive, by (3.17) and Lemma 2.7, we get $(I-S)D\tilde{x} = 0$. Hence, by Lemma 2.3, $0 \in g_2(D\tilde{x}) + M_2 D\tilde{x}$ that is $D\tilde{x} \in \text{Sol(MVIP(1.3))}$. Thus, $\tilde{x} \in \Lambda$.

Next, we prove $\tilde{x} \in \text{Sol(VIP(1.1))}$. Since $\lim_{n\to\infty} \|z_n - u_n\| = 0$ and $\lim_{n\to\infty} \|z_n - x_n\| = 0$, there exist subsequences $\{z_{n_i}\}$ and $\{u_{n_i}\}$ of $\{z_n\}$ and $\{u_n\}$, respectively such that $z_{n_i} \rightharpoonup \tilde{x}$ and $u_{n_i} \rightharpoonup \tilde{x}$.

Define the mapping $\mathbb{M}$ as

$$\mathbb{M}(p_1) = \begin{cases} D(p_1) + \mathbb{N}_{C_1}(p_1), & \text{if } p_1 \in C_1, \\ \emptyset, & \text{if } p_1 \notin C_1, \end{cases}$$

where $\mathbb{N}_{C_1}(p_1) := \{p_2 \in H_1 : \langle p_1 - y, p_2 \rangle \geq 0 \text{ for all } y \in C_1\}$ is the normal cone to C_1 at $p_1 \in H_1$. Thus, $\mathbb{M}$ is a maximal monotone and hence $0 \in \mathbb{M}p_1$ mapping if and only if $p_1 \in \text{Sol(VIP(1.1))}$. Let $(p_1, p_2) \in \text{graph}(\mathbb{M})$. Then, we have $p_2 \in \mathbb{M}p_1 =$

$Bp_1 + \mathbb{N}_{C_1}(p_1)$ and hence $p_2 - Bp_1 \in \mathbb{N}_{C_1}(p_1)$. So, we have $\langle p_1 - y, p_2 - Bp_1\rangle \geq 0$, for all $y \in C_1$. On the other hand, from $u_n = P_{C_1}(z_n - \sigma_n Bz_n)$ and $z_1 \in C_1$, we have

$$\langle (z_n - \sigma_n Bz_n) - u_n, u_n - p_1\rangle \geq 0.$$

This implies that

$$\left\langle p_1 - u_n, \frac{u_n - z_n}{\sigma_n} + Bz_n \right\rangle \geq 0.$$

Since $\langle p_1 - y, p_2 - Bp_1\rangle \geq 0$ for all $y \in C_1$ and $u_{n_i} \in C_1$, using monotonicity of B, we have

$$\begin{aligned}
\langle p_1 - u_{n_i}, p_2\rangle \geq & \langle p_1 - u_{n_i}, Bp_1\rangle \\
\geq & \langle p_1 - u_{n_i}, Bp_1\rangle - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}} + Bu_{n_i} \right\rangle \\
= & \langle p_1 - u_{n_i}, Bp_1 - Bu_{n_i}\rangle + \langle p_1 - u_{n_i}, Bu_{n_i} - Bz_{n_i}\rangle \\
& - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}} \right\rangle \\
\geq & \langle p_1 - u_{n_i}, Bu_{n_i} - Bz_{n_i}\rangle - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}} \right\rangle.
\end{aligned}$$

Since B is continuous therefore on taking limit $i \to \infty$, we have $\langle p_1 - \tilde{x}, p_2\rangle \geq 0$. Since $\mathbb{M}$ is maximal monotone, we have $\tilde{x} \in \mathbb{M}^{-1}(0)$ and hence $\tilde{x} \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Thus, $\tilde{x} \in \Gamma$.

Step 4. Finally, we prove that $\limsup_{n\to\infty} \langle (\theta f - A)z, x_n - z\rangle \leq 0$, where $z = P_\Gamma(I - A + \theta f)z$ and $x_n \to \tilde{x}$.

By (3.24), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\theta f - A)z, x_n - z\rangle &= \limsup_{n\to\infty} \langle (\theta f - A)z, v_n - z\rangle \\
&\leq \limsup_{i\to\infty} \langle (\theta f - A)z, v_{n_i} - z\rangle \\
&= \langle (\theta f - A)z, \tilde{x} - z\rangle \\
&\leq 0. \qquad (3.27)
\end{aligned}$$

Using (3.5) and (3.7), we calculate

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 =& \langle \mu_n(\theta f(x_n) - A\tilde{x}) + \eta_n(x_n - \tilde{x}) \\
& + ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\
=& \mu_n\langle \theta f(x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle + \eta_n\langle x_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\
& + \langle ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\
\leq& \mu_n\left(\theta\langle f(x_n) - f(\tilde{x}), x_{n+1} - \tilde{x}\rangle + \langle \theta f(\tilde{x}) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle\right) \\
& + \eta_n\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + \|(1-\eta_n)I - \mu_n A\|\|v_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| \\
\leq& \mu_n\tau\theta\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + \mu_n\langle \theta f(\tilde{x}) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\eta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|+(1-\eta_n-\mu_n\bar{\theta})\|v_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
=&[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2\leq&\frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\left(\|x_n-\tilde{x}\|^2+\|x_{n+1}-\tilde{x}\|^2\right) \\
&+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
\leq&\frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\|x_n-\tilde{x}\|^2+\frac{1}{2}\|x_{n+1}-\tilde{x}\|^2+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2&\leq[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|^2+2\mu_n(\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&=[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|^2+2\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle. \qquad (3.28)
\end{aligned}
$$

Thus, by (3.27), (3.28), Lemma 2.6 and using $\lim_{n\to\infty}\mu_n=0$, we get $x_n\to\tilde{x}$, where $\tilde{x}=P_\Gamma(I+\theta f-A)$. □

Now, we listed following consequences from Theorem 3.1.

Corollary 3.1. *Let H_1 and H_2 denote the Hilbert spaces and $C_1\subset H_1$ be nonempty closed convex subset of H_1. Let $B:H_1\to H_1$ be a γ-inverse strongly monotone mapping, $D:H_1\to H_2$ be a bounded linear operator with its adjoint operator D^*, $M_1:H_1\to 2^{H_1}$, $M_2:H_2\to 2^{H_2}$ be multi-valued maximal monotone operators and $g_1:H_1\to H_1$, $g_2:H_2\to H_2$ be α_1, α_2-inverse strongly monotone mappings, respectively. Let $f:C_1\to C_1$ be a contraction mapping with constant $\tau\in(0,1)$, A be a strongly positive bounded linear self adjoint operator on C_1 with constant $\bar{\theta}>0$ such that $0<\theta<\frac{\bar{\theta}}{\tau}<\theta+\frac{1}{\tau}$ and $S:C_1\to C_1$ be a nonexpansive mapping such that $\Gamma:=\Lambda\cap\mathrm{Sol(VIP(1.1))}\cap\mathrm{Fix}(S))\neq\emptyset$. Let $\{x_n\}$ be the sequence generated as:*

$$
\left.
\begin{aligned}
&x_1\in C_1,\\
&z_n=R(I+\xi D^*(S-I)D)x_n,\\
&u_n=P_{C_1}(z_n-\sigma_nBz_n),\\
&v_n=\delta_nu_n+(1-\delta_n)Su_n,\\
&x_{n+1}=\mu_n\theta f(x_n)+\eta_nx_n+((1-\eta_n)I-\mu_nA)v_n,\quad n\geq1,
\end{aligned}
\right\}
$$

where $R=J_{\rho_1}^{(g_1,M_1)}(I-\rho_1g_1)$, $S=J_{\rho_2}^{(g_2,M_2)}(I-\rho_2g_2)$, $\{\mu_n\},\{\eta_n\},\{\delta_n\}\subset(0,1)$ and $\xi\in(0,\frac{1}{\epsilon})$, ϵ be the spectral radius of D^*D. Let the control sequences satisfying conditions:

(i) $\lim_{n\to\infty}\mu_n=0$, $\sum_{n=0}^{\infty}\mu_n=\infty$;
(ii) $0<\rho_1<2\alpha_1$, $0<\rho_2<2\alpha_2$;
(iii) $0<\liminf_{n\to\infty}\eta_n\leq\limsup_{n\to\infty}\eta_n<1$;
(iv) $0<\liminf_{n\to\infty}\sigma_n\leq\limsup_{n\to\infty}\sigma_n<2\gamma$;
(v) $\lim_{n\to\infty}\delta_n=0$.

Then, the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$, *where* $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ *which solves:*

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

If we consider $\rho_1 = \rho_2$, $g_1 = g_2 = B \equiv 0$ and $\eta_n = 0$ in Theorem 3.1 then we have following corollary.

Corollary 3.2. *Let* H_1 *and* H_2 *denote the Hilbert spaces. Let* $D : H_1 \to H_2$ *be a bounded linear operator with its adjoint operator* D^*, $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ *be multi-valued maximal monotone operators, respectively. Let* $f : H_1 \to H_1$ *be a contraction mapping with constant* $\tau \in (0, 1)$, A *be a strongly positive bounded linear self adjoint operator on* H_1 *with constant* $\bar{\theta} > 0$ *such that* $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$ *and* $\{S_i\}_{i=1}^{\infty} : H_1 \to H_1$ *be an infinite family of nonexpansive mappings such that* $\Gamma := \Lambda \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated as:*

$$\left.\begin{array}{l} x_1 \in H_1, \\ z_n = J_\rho^{M_1}(I + \xi D^*(J_\rho^{M_2} - I)D)x_n, \\ v_n = \delta_n z_n + (1 - \delta_n)\mathbb{W}_\ltimes z_n, \\ x_{n+1} = \mu_n \theta f(x_n) + (I - \mu_n A)v_n, \quad n \geq 1, \end{array}\right\}$$

where $\mathbb{W}_n$ *defined in* (1.6), $\{\mu_n\}$, $\{\delta_n\} \subset (0, 1)$ *and* $\xi \in (0, \frac{1}{\epsilon})$, ϵ *be the spectral radius of* D^*D. *Let the control sequences satisfying conditions:*

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=0}^{\infty} \mu_n = \infty$;
(ii) $\lim_{n\to\infty} \delta_n = 0$.

Then, the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$, *where* $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ *which solves:*

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

4. Numerical Example

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle u, v \rangle = uv$ for all $u, v \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C_1 = [0, \infty)$; let the mappings $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be defined by $g_1(u) = \frac{3}{2}u$ for all $u \in H_1$ and $g_2(v) = v + 3$ for all $v \in H_2$. Let $M_1, M_2 : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_1(u) = \{u - 1\}$ for all $u \in \mathbb{R}$ and $M_2(v) = \{4v\}$ for all $v \in \mathbb{R}$. Let the mapping $D : \mathbb{R} \to \mathbb{R}$ be defined by $D(u) = -\frac{3}{2}u$ for all $u \in \mathbb{R}$. Let the mappings $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be defined by $S_i u = \frac{u+2i}{1+5i}$ for each $i \in \mathbb{N}$, let the mapping $B : H_1 \to \mathbb{R}$ be defined by $Bu = 5u - 2$ for all $u \in H_1$. Let the mapping $f : C_1 \to C_1$ be defined by $f(u) = \frac{u}{5}$ for all $u \in C_1$ and $Au = \frac{u}{2}$ with $\theta = \frac{1}{10}$. Setting $\{\mu_n\} = \{\frac{1}{10n}\}$, $\{\eta_n\} = \{\frac{1}{2n^2}\}$, $\{\sigma_n\} = \frac{1}{4}$, $\{\delta_n\} = \frac{1}{n}$ and $\{\lambda_n\} = \{\frac{1}{3n^2}\}$ for all $n \geq 1$. Let $\mathbb{W}_n$ be the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, which is defined by (1.6). Then, there are sequences $\{x_n\}$, $\{z_n\}$, $\{u_n\}$

and $\{v_n\}$ as: Given x_1,

$$\left.\begin{array}{l} t_n = SDx_n = J_{\rho_2}^{(g_2,M_2)}(I-\rho_2 g_2)Dx_n \\ r_n = x_n + \xi D^*(t_n - Dx_n) \\ z_n = J_{\rho_1}^{(g_1,M_1)} r_n \\ u_n = P_{C_1}(z_n - \sigma_n B z_n), \\ v_n = \delta_n z_n + (1-\delta_n)\mathbb{W}_{\ltimes} z_n, \\ x_{n+1} = \mu_n \theta f(x_n) + \eta_n x_n + ((1-\eta_n)I - \mu_n A)v_n, \end{array}\right\}$$

Then, $\{x_n\}$ converges to $\tilde{x} = \{\frac{2}{5}\} \in \Gamma$.

Proof. Obviously, D is a bounded linear operator on $\mathbb{R}$ with adjoint D^* and $\|D\| = \|D^*\| = \frac{3}{2}$, and hence $\xi \in (0, \frac{4}{9})$. Therefore, we choose $\xi = 0.1$. Further, g_1 and g_2 are 3 and 1-ism, therefore $\rho_1 \subset (0, \frac{4}{3})$ and $\rho_2 \subset (0, 2)$, thus choose $\rho_1 = \frac{1}{3} > 0$ and $\rho_2 = \frac{1}{3} > 0$. For each i, S_i is nonexpansive with $\mathrm{Fix}(S_i) = \left\{\frac{2}{5}\right\}$. Further, B is 5-ism and $\mathrm{Sol(VIP(1.1))} = \{\frac{2}{5}\}$. Furthermore, $\mathrm{Sol(MVIP(1.2))} = \left\{\frac{2}{5}\right\}$ and $\mathrm{Sol(MVIP(1.3))} = \left\{-\frac{3}{5}\right\}$, and thus $\Lambda = \{\frac{2}{5} \in C : \frac{2}{5} \in \mathrm{Sol(MVIP(1.2))} : D(\frac{2}{5}) \in \mathrm{Sol(MVIP(1.3))}\} = \left\{\frac{2}{5}\right\}$. Therefore, $\Gamma := \Lambda \cap \mathrm{Sol(VIP(1.1))} \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$. Thus,

$$t_n = \frac{-6x_n + 9}{14}; \quad r_n = \frac{31x_n - 6t_n}{40}; \quad z_n = \frac{3r_n + 1}{4};$$

$$u_n = P_{C_1}(z_n - \sigma_n B z_n) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 1, \\ \frac{-z_n+2}{4} & \text{otherwise;} \end{cases}$$

$\mathbb{W}_n = u_n;$

Step 1:

$i = 1;$

$$\mathbb{W}_n = \frac{1}{3n^2}\frac{(\mathbb{W}_n + 2i)}{1+5i} + \left(1 - \frac{1}{3n^2}\right)u_n;$$

$i = i + 1;$

if $(i \le N)$ go to Step 1;

$$v_n = \frac{1}{n}u_n + \left(1 - \frac{1}{n}\right)\mathbb{W}_n u_n,$$

$$x_{n+1} = \frac{1}{100n}\frac{x_n}{5} + \frac{1}{2n^2}x_n + \left(1 - \frac{1}{2n^2}\right)v_n - \frac{1}{10n}\frac{v_n}{2},$$

which show that $\{x_n\}$ converges to $\tilde{x} = \frac{2}{5}$ as $n \to +\infty$ and $\lim_{n\to\infty} \|\mathbb{W}_n x - \mathbb{W}x\| = 0$ for each $x \in C_1$. □

5. Figures

Finally, by the software Matlab 7.8.0, we obtain following figures which show that $\{x_n\}$ converges to $\tilde{x} = \frac{2}{5}$ as $n \to +\infty$, and $\lim_{n\to\infty} \|\mathbb{W}_n x - \mathbb{W}x\| = 0$ for each $x \in C_1$.

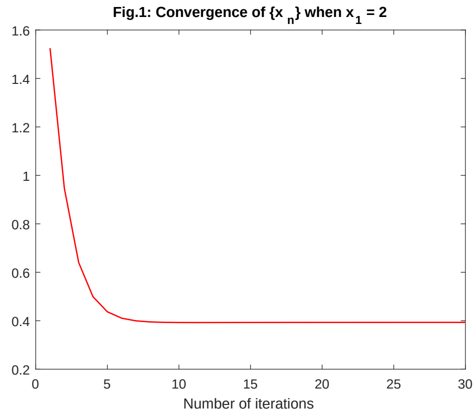

FIGURE 1. Convergence of $\{x_n\}$ when $x_1 = 2$.

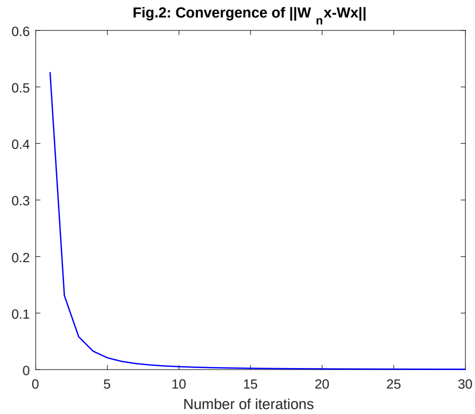

FIGURE 2. Convergence of $\|\mathbb{W}_n x - \mathbb{W}x\|$.

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PRABHAKAR AND HILFER-PRABHAKAR FRACTIONAL DERIVATIVES IN THE SETTING OF Ψ-FRACTIONAL CALCULUS AND ITS APPLICATIONS

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Abstract. The aim of this paper is to study to fractional calculus for class of Ψ function. The present study is designed to study generalized fractional derivatives and find their generalized transforms called Ψ-Laplace transform and Ψ-Sumudu transform. Moreover, find the analytical solutions of some applications in physics the form of generalized fractional derivatives by transform technique.

1. Introduction

In recent years, many researchers investigated generalization of integration and differentiation operators in the field of fractional calculus. In literature several different definitions of fractional integrals and derivatives are available, like Riemann-Liouville integral and derivative Caputo fractional derivative etc. (see [3,18,19]). In [13] defined new fractional derivative called Hilfer fractional derivative which is generalization of Riemann and Caputo fractional derivative. The first investigated generalized Mittage-Leffler function by Prabhakar [17]. The so-called Prabhakar integral is defined in a similar way Riemann-Lioville integral [12,14,17]. Roberto Garra et al. [12] introduced fractional derivative by definition of Hilfer derivative replacing Riemann-Liouville integral operator by Prabhakar integral operator called Hilfer-Prabhakar derivative also defined Prabhakar and Hilfer-Prabhakar derivatives regularized version. Dorrego defined generalization of Prabhakar integral and derivative called k-Prabhakar integral

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and k-Prabhakar derivative [7]. In [5, 6] Dorrego and Cerutti defined the kernel of k-Mittage-Leffler function and generalized derivative called k-fractional Hilfer derivative. Recently Dole et al. [15, 16] defined generalized fractional derivative like k-Hilfer-Prabhakar derivative as well as defined regularized version of k-Prabhakar derivative. Moreover, find Laplace and Sumudu transform to regularized version of k-Prabhakar derivative also k-Hilfer-Prabhakar derivative and its regularized version. In [11] Sausa and Oliviera introduced new fractional derivative in the setting of Ψ-fractional operator called Ψ-Hilfer fractional derivative defined as

$$D^{\mu,\nu,\Psi} = I^{\nu(m-\mu),\Psi}\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^m I^{(1-\nu)(m-\mu),\Psi} f(t).$$

The new generalized integral transform called Ψ-Laplace transform published on an arXiv, by Fahad et al. and obtain for Ψ-Hilfer fractional derivative as follows

$$L_\Psi\Big\{D^{\mu,\nu,\Psi} f(t)\Big\} = s^\mu L_\Psi\Big\{f(t)\Big\} - \sum_{k=0}^{m-1} s^{m(1-\nu)+\mu\nu-k-1}\big(I^{(1-\nu)(m-\mu)-k,\Psi} f(t)\big).$$

In this paper, we define new integral transform called Ψ-Sumudu transform. Define some generalized definitions of fractional derivatives in the setting of Ψ-fractional calculus like Ψ-Prabhakar, Ψ-Hilfer-Prabhakar, Ψ-k-RL-fractional, Ψ-k-Hilfer, Ψ-k-Prabhakar, Ψ-k-Hilfer-Prabhakar fractional integrals and derivatives as well as define all these new fractional derivatives regularized versions. These results are used to obtain the relation between Ψ-Prabhakar fractional derivative and its regularized version and also the relation between Ψ-Hilfer-Prabhakar fractional derivative and its regularized version involving Mittag-Leffler function. Moreover, we obtain Ψ-Laplace transform [9] and Ψ-Sumudu transform to find solutions of fractional differential equations.

2. Preliminaries

Definition 2.1 ([17])**.** Let $n \in \mathbb{N}$, $\alpha, \mu, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\mu) > 0$. The Mittag-Leffler function is defined as

$$E^\gamma_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\mu)}\cdot\frac{z^n}{n!},$$

where $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+(n-1))$ is the Pochhammer symbol.

Definition 2.2 ([5])**.** Let $n \in \mathbb{N}$, $\alpha, \mu, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\mu) > 0$. The k-Mittag-Leffler function is defined as

$$E^\gamma_{k,\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\mu)}\cdot\frac{z^n}{n!},$$

where $(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\cdots(\gamma+(n-1)k)$ is the Pochhammer symbol.

Definition 2.3 ([8, 10])**.** Let μ be a real number such that $\mu > 0$, $-\infty \le a < b \le \infty$, $m = \mu+1$, f be an integrable function defined on $[a, b]$ and $\Psi \in C^1([a, b])$ be increasing function such that $\Psi^{'}(t) \neq 0$ for all t $\in [a,b]$. Then, the Ψ-RL-fractional integral and Ψ-RL fractional derivative of a function f of order μ are defined as

$$I_0^{\mu,\Psi} f(t) = \frac{1}{\Gamma(\mu)} \int_0^\infty (\Psi(t) - \Psi(s))^{\mu-1} \Psi^{'}(s) f(s) ds, \tag{2.1}$$

$$D_0^{\mu,\Psi} = \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m I_0^{m-\mu,\Psi} f(t). \tag{2.2}$$

It is to be noted that for $\Psi(t) \to t$, $I_0^{\mu,\Psi} f(t) \to I_0^{\mu} f(t)$ which is the standard Riemann- Liouville integral. Moreover for $\Psi(t) \to \ln(t)$ the integral defined in equation (2.1) towards Hadamard fractional integral.

Inspired by Caputo's concept [2] of fractional derivative, Almeidea [1] presents the following Caputo version of equation (2.2) and studies some important properties of fractional calculus.

Definition 2.4 ([1])**.** Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-C-fractional derivative of a function f of order μ is defined as

$$^C D_0^{\mu,\Psi} = I_0^{m-\mu,\Psi} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f(t).$$

Definition 2.5 ([11])**.** Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, and $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$D_0^{\mu,\nu,\Psi} = I_0^{\nu(m-\mu,\Psi)} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m I_0^{(1-\nu)(m-\mu),\Psi} f(t).$$

Definition 2.6 ([9])**.** Let $f : [0,\infty) \to \mathbb{R}$ be a real valued function and Ψ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-Laplace transform of f is denoted by $L_\Psi\{f\}$ and is defined by

$$T(u) := L_\Psi\{f(t)\} := \int_0^\infty e^{-u\Psi(t)} \Psi^{'}(t) f(t) dt, \quad \text{for all } u.$$

3. Main Result

We consider functions in the set A is defined by

$$A = \{f(t) \mid \text{ exists } M, \tau_1, \tau_2 |f(t)| \leq M e^{|t|/T_j}, \text{ if } t \epsilon (-1^j) \times [0,\infty)\}.$$

Definition 3.1. Let $f : [0,\infty) \to \mathbb{R}$ be a real valued function and Ψ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-ST of f is denoted by $S_\Psi\{f\}$ and is defined by

$$T(u) := S_\Psi\{f(t)\} := \frac{1}{u} \int_0^\infty e^{-\frac{\Psi(t)}{u}} \Psi^{'}(t) f(t) dt, \quad \text{for all } u.$$

3.1. New definitions of Ψ-fractional derivatives.

Definition 3.2. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$ and $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-Prabhakar fractional integral and derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$\begin{aligned}\left(P^{\gamma,\Psi}_{\alpha,\mu,\omega} f\right)(x) &= \int_0^x (\Psi(x)-\Psi(t))^{\mu-1} E^{\gamma}_{\alpha,\mu}[\omega(\Psi(x)-\Psi(t))^{\alpha}]\Psi'(t)f(t)dt \\ &= (\varepsilon^{\gamma}_{\alpha,\mu,\omega} *_{\Psi} f)(x), \end{aligned} \tag{3.1}$$

where $*_{\Psi}$ denotes the convolution operation, $\alpha, \mu, \omega, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\mu) > 0$ and

$$\varepsilon^{\gamma}_{\alpha,\mu,\omega}\Psi(t) = \begin{cases} (\Psi(t))^{\mu-1}E^{\gamma}_{\alpha,\mu}(\omega(\Psi(t))^{\alpha}), & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{3.2}$$

For $\gamma = 0$, $\left(P^{0}_{\alpha,\mu,\omega} f\right)(x) = \left(I^{\mu,\Psi} f\right)(x)$ and for $\gamma = \mu = 0$, $\left(P^{0}_{\alpha,0,\omega} f\right)(x) = f(x)$,

$$D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = \left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m P^{-\gamma,\Psi}_{\rho,m-\mu,\omega} f(t). \tag{3.3}$$

Definition 3.3. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-Prabhakar fractional derivative of a function f of order μ is defined as

$$^{C}D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = P^{-\gamma,\Psi}_{\rho,m-\mu,\omega}\left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m f(t). \tag{3.4}$$

Definition 3.4. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$ and type $0 \leq \nu \leq 1$ and $\left(f * \varepsilon^{-\gamma(1-\nu)}_{\rho,(1-\nu)(m-\mu),\omega}\right)\Psi(t) \in AC^1[0,b]$. Then, the Ψ-Hilfer-Prabhakar fractional derivative of a function f of order μ defined as

$$\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+}\left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m \left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+} f\right)\right)(t). \tag{3.5}$$

Definition 3.5. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$^{C}\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+} P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+}\left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m f\right)(t). \tag{3.6}$$

Definition 3.6. Let μ be a real number and let $k \in \mathbb{R}^+$, such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, f be an integrable function defined on $[a,b]$ and $\Psi \in C^1([a,b])$ be increasing function such that $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-RL fractional integral and Ψ-k-RL fractional derivative of a function f of order μ are defined as

$$\begin{aligned} I_k^{\mu,\Psi} f(t) =& \frac{1}{k\Gamma_k(\mu)} \int_0^{\infty} (\Psi(t) - \Psi(s))^{\frac{\mu}{k}-1} \Psi^{'}(s) f(s) ds, \\ D_k^{\mu,\Psi} =& \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^m k^m I_k^{m-\mu,\Psi} f(t). \end{aligned}$$

Definition 3.7. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Caputo fractional derivative of a function f of order μ is defined as

$$ {}_k^C D_0^{\mu,\Psi} = k^m I_k^{m-\mu,\Psi} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m f(t).$$

Definition 3.8. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$D_k^{\mu,\nu,\Psi} = I_k^{\nu(m-\mu,\Psi)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^m I_k^{(1-\nu)(m-\mu),\Psi} f(t).$$

Definition 3.9. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Prabhakar fractional integral and derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$\begin{aligned} \left({}_k P_{\alpha,\mu,\omega}^{\gamma,\Psi} f\right)(x) &= \int_0^x \frac{(\Psi(x) - \Psi(t))^{\frac{\mu}{k}-1}}{k} E_{k,\alpha,\mu}^{\gamma} [\omega(\Psi(x) - \Psi(t))^{\frac{\alpha}{k}}] \Psi^{'}(t) f(t) dt \\ &= ({}_k\varepsilon_{\alpha,\mu,\omega}^{\gamma} *_{\Psi} f)(x), \end{aligned} \tag{3.7}$$

where $*_{\Psi}$ denotes the convolution operation, $\alpha, \mu, \omega, \gamma \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\mu) > 0$ and

$$ {}_k\varepsilon_{\alpha,\mu,\omega}^{\gamma} \Psi(t) = \begin{cases} \frac{(\Psi(t))^{\frac{\mu}{k}-1}}{k} E_{k,\alpha,\mu}^{\gamma} (\omega(\Psi(t))^{\frac{\alpha}{k}}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{3.8}$$

for $\gamma = 0$, $\left({}_k P_{\alpha,\mu,\omega}^{0} f\right)(x) = \left(I_k^{\mu,\Psi} f\right)(x)$ and for $\gamma = \mu = 0$, $\left({}_k P_{\alpha,0,\omega}^{0} f\right)(x) = f(x)$,

$$ {}_k D_{\rho,\mu,\omega}^{\gamma,\Psi} f(t) = \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m k^m {}_k P_{\rho,mk-\mu,\omega}^{-\gamma,\Psi} f(t). \tag{3.9}$$

Definition 3.10. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that

Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the regularized version of Ψ-k-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}_k{}^C D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = k^m {}_k P^{-\gamma,\Psi}_{\rho,mk-\mu,\omega} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f(t). \tag{3.10} $$

Definition 3.11. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$ and type $0 \leq \nu \leq 1$ and $\left(f * \varepsilon^{-\gamma(1-\nu)}_{\rho,(1-\nu)(mk-\mu),\omega} \right) \Psi(t) \in AC^1[0, b]$. Then, the Ψ-k-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = k^m \left({}_k P^{-\gamma\nu,\Psi}_{\alpha,\nu(mk-\mu),\omega,0^+} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m \left({}_k P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} f \right) \right)(t). \tag{3.11} $$

Definition 3.12. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}_k^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = k^m \left({}_k P^{-\gamma\nu,\Psi}_{\alpha,\nu(mk-\mu),\omega,0^+} {}_k P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f \right)(t). \tag{3.12} $$

4. Ψ-Laplace Transform and Ψ-Sumudu Transform of Ψ-Fractional Derivatives

Let $F(s)$ be the Ψ-Laplace transform of $f(t)$.

Lemma 4.1. *The* Ψ*-Laplace transform of* Ψ*-Prabhakar fractional integral equation* (3.1) *is*

$$ \mathcal{L}_\Psi \left(P^{\gamma}_{\alpha,\mu,\omega} f \right)(x) \ = s^{-\mu}(1 - \omega(s)^{-\alpha})^{-\gamma} F(s). \tag{4.1} $$

Lemma 4.2. *The* Ψ*-Laplace transform of* Ψ*-Prabhakar fractional derivative equation* (3.3) *is*

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = s^{\mu} \left(1 - \omega(s)^{-\alpha} \right)^{\gamma} F(s) - \sum_{k=0}^{m-1} s^{m-k-1} \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega} f(0^+) \right]. $$

For the case $[\mu] + 1 = m = 1$

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = s^{\mu} \left(1 - \omega(s)^{-\alpha} \right)^{\gamma} F(s) - \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t) \right]_{t=0^+}, \tag{4.2} $$

with $|\omega(s)^{-\alpha}| < 1$.

Proof. Taking Ψ-Laplace transforms of Ψ-Prabhakar fractional derivative in (3.3) and using (3.1), (3.2), (4.1) we get

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = \mathcal{L} \left(\left(\frac{1}{\Psi'} \cdot \frac{d}{dt} \right)^m P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t) \right)(s) $$

$$
\begin{aligned}
=&s^m \mathcal{L}_\Psi\bigg(\Big(\varepsilon^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} * f\Big)(t)\bigg)(s)\\
&-\sum_{k=0}^{m-1} s^{m-k-1}\left[\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+}\\
=&s^m \mathcal{L}_\Psi\bigg((\Psi(t))^{(m-\mu)-1} E^{-\gamma,\Psi}_{\alpha,(m-\mu)}\Big(\omega(\Psi(t))^\alpha\Big) F(s)\bigg)\\
&-\sum_{k=0}^{m-1} s^{m-k-1}\left[\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(0^+)\right]\\
=&s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{m-k-1}\left(P^{-\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega} f(0^+)\right).
\end{aligned}
$$

For the case $[\mu]+1=m=1$, we have

$$
\mathcal{L}_\Psi\left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\right)(s) = s^\mu\left(1-\omega(s)^{-\alpha}\right)^\gamma F(s) - \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+}. \qquad \square
$$

Lemma 4.3. *The Ψ-Laplace transform of regularized version of Ψ-Prabhakar fractional derivative equation* (3.4) *is*

(4.3)

$$
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^{(k)}(0^+),
$$

with $|\omega(s)^{-\alpha}|<1$.

Proof. Taking Ψ-Laplace transform of regularized version of Ψ-Prabhakar fractional derivative in (3.4) and using (3.1), (3.2), (4.1) we get

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s), &= \mathcal{L}_\Psi\left(\left(\varepsilon^{-\gamma}_{\alpha,(m-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'}\cdot\frac{d}{dt}f\right)^m(t)\right)(s)\right.\\
&= s^{-(m-\mu)}\Big(1-\omega(s)^\alpha\Big)^\gamma\left\{s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^k(0^+)\right\}\\
&= s^\mu\Big(1-\omega(s)^\alpha\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^\alpha\Big)^\gamma f^k(0^+). \qquad \square
\end{aligned}
$$

For absolutely continuous function $f\in AC^1[0,b]$,

$$
\left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+} = 0.
$$

Then in view of equation (4.2) and equation (4.3) $(m=1)$ we have

$$
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = \mathcal{L}\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) - (s)^{\mu-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f(0^+).
$$

Taking inverse Ψ-Laplace transform, we get

$$
{}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) = D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) - (\Psi(t))^{-\mu} E^{-\gamma}_{\alpha,m-\mu}(\omega(\Psi(t))^\alpha) f(0^+),
$$

for $f \in AC^1[0,b]$. This is the relation between Ψ-Prabhakar fractional derivative and its regularized version.

Lemma 4.4. *The Ψ-Laplace transform of Ψ-Hilfer-Prabhakar fractional derivative equation* (3.5) *is*

$$\mathcal{L}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{m(1-\nu)+\nu\mu-k-1}[1-\omega(s)^{-\alpha}]^{\gamma\nu} \times \Big(P^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(m-\mu)-k,\omega}, \Psi f(0^+)\Big).$$

Proof. Taking Ψ-Laplace transform of Ψ-Hilfer-Prabhakar fractional derivative in (3.5) and using (3.1), (3.2), (4.1) we have

$$\begin{aligned}\mathcal{L}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&\mathcal{L}_\Psi\left[\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^m \Big(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu),\omega,0^+} f\Big)\right)(t)\right](s)\\ =&s^m(s)^{-\nu(m-\mu)}\Big(1-\omega(s)^{-\alpha}\Big)^{\gamma\nu}\mathcal{L}_\Psi\left(\Big(\varepsilon^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(m-\mu),\omega} * f\Big)(t)\right)(s)\\ &-\sum_{k=0}^{m-1} s^m(s)^{-\nu(m-\mu)}\Big(1-\omega(s)^{-\alpha}\Big)^{\gamma\nu}\Big[P^{-\lambda(1-\nu),\Psi}_{\alpha,(1-\nu),(m-\mu)-k,\omega} f(0^+)\Big]\\ =&s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s)\\ &-\sum_{k=0}^{m-1} s^{m(1-\nu)+\nu\mu-k-1}[1-\omega(s)^{-\alpha}]^{\gamma\nu}\Big(P^{-\gamma(1-\nu)(m-\mu),\psi}_{\alpha,(1-\nu)(m-\mu)-k,\omega} f(0^+)\Big). \square\end{aligned}$$

Lemma 4.5. *The Ψ-Laplace transforms of the regularized version of Ψ-Hilfer-Prabhakar fractional derivative equation* (3.6) *of order* μ *is*

$$\mathcal{L}_\Psi\Big({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^k(0^+).$$

Proof. Taking Ψ-Laplace transforms of regularized version of Ψ-Hilfer-Prabhakar fractional derivative in (3.6) and using (3.1), (3.2), (4.1) we have

$$\begin{aligned}\mathcal{L}_\Psi\Big({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&\mathcal{L}_\Psi\left(\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega} *_\Psi \left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(k-\mu),\omega,0^+}\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^m f\right)\right)(t)\right)(s)\\ =& s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^k(0^+). \qquad \square\end{aligned}$$

Again for absolutely continuous function $f \in AC^1[0,b]$

$${}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) - (\Psi(t))^{-\mu} E^{-\gamma}_{\alpha,(m-\mu)}(\omega(\Psi(t))^\alpha) f(0^+).$$

This is the relation between Ψ-Hilfer-Prabhakar fractional derivative and its regularized version.

Let $F(u)$ be the Ψ-Sumudu transform of $f(t)$.

Lemma 4.6. *The* Ψ*-Sumudu transform of* Ψ*-Prabhakar integral equation* (3.1) *is*

$$S_\Psi\Big(P^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u)=u^{-1}(u)^{\mu}\Big(1-\omega(u)^{\alpha}\Big)^{-\gamma}F(u),$$

provided $|\omega(u)^{\alpha}|<1$.

Lemma 4.7. *The* Ψ*-Sumudu transform of* Ψ*-Prabhakar fractional derivative equation* (3.3) *is*

$$\mathcal{S}_\Psi\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u)=u^{-\mu}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}F(u)-\sum_{n=0}^{m-1}u^{-m+k}\Big[P^{\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega}f(0^{+})\Big].$$

For the case $[\mu]+1=m=1$

$$\mathcal{S}_\Psi\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u)=u^{-\mu}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}F(u)-\frac{1}{u}\left[P^{-\gamma,\Psi}_{\alpha,(1-\mu),\omega}f(t)\right]_{t=0^{+}},$$

with $|\omega(u)^{\alpha}|<1$.

Lemma 4.8. *The* Ψ*-Sumudu transform of regularized version of* Ψ*-Prabhakar fractional derivative equation* (3.4) *is*

$$\mathcal{S}_\Psi\Big({}^{C}D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u)=u^{-\mu}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}F(u)-\sum_{n=0}^{m-1}u^{-(\mu-k)}\left(1-\omega\left(\frac{1}{u}\right)^{\alpha}\right)^{\gamma}f^{(n)}(0),$$

with $|\omega(u)^{\alpha}|<1$.

Lemma 4.9. *The* Ψ*-Sumudu transform of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.5) *is*

$$\begin{aligned}\mathcal{S}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^{+}}f(t)\Big)(u)=&u^{-\mu}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}F(u)-\sum_{k=0}^{m-1}u^{m(\nu-1)-\nu\mu+k}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma\nu}\\&\times\Big[P^{-\gamma(1-\nu),\Psi}_{\alpha,\nu(1-\nu)(m-\mu)-k,\omega}f(0^{+})\Big].\end{aligned}$$

Lemma 4.10. *The* Ψ*-Sumudu transforms of the regularized version of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.6) *of order* μ *is*

$$\mathcal{S}_\Psi\Big({}^{C}\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^{+}}f(t)\Big)(u)=u^{-\mu}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}F(u)-\sum_{k=0}^{m-1}u^{-\mu+k}\Big(1-\omega(u)^{\alpha}\Big)^{\gamma}f^{k}(0^{+}).$$

4.1. Ψ-Laplace and Ψ-Sumudu transform of Ψ-k-fractional derivatives.

Lemma 4.11. *The* Ψ*-Laplace transform of* Ψ*-k-Prabhakar fractional integral equation* (3.7) *is*

$$\mathcal{L}_\Psi\Big({}_{k}P^{\gamma,\Psi}_{\alpha,\mu,\omega}f\Big)(x)\;=(ks)^{\frac{-\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{-\gamma}{k}}F(s). \tag{4.4}$$

Lemma 4.12. *The* Ψ*-Laplace transform of* Ψ*-k-Prabhakar fractional derivative equation* (3.9) *is*

$$\mathcal{L}_\Psi\Big({}_{k}D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(s)=(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}F(s)$$

$$-k^m \sum_{n=0}^{m-1} s^{m-n-1}\left[{}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega} f(0^+)\right].$$

For the case $[\frac{\mu}{k}]+1=m=1$

$$\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s)=(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}F(s)-k\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega} f(t)\Big]_{t=0^+}, \tag{4.5}$$

with $|\omega k(ks)^{\frac{-\alpha}{k}}|<1$.

Proof. Taking Ψ-Laplace transforms of Ψ-k-Prabhakar fractional derivative in (3.9) and using (3.7), (3.8), (4.4) we get

$$\begin{aligned}
\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) =&\mathcal{L}\left(\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^m k^m\, {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(t)\right)(s)\\
=&k^m s^m \mathcal{L}_\Psi\Big(\Big({}_k\varepsilon^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} * f\Big)(t)\Big)(s)\\
&-k^m\sum_{n=0}^{m-1} s^{m-n-1}\left[\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^n {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(t)\right]_{t=0^+}\\
=&(ks)^m\mathcal{L}_\Psi\left((\Psi(t))^{\frac{(mk-\mu)}{k}-1}E^{-\gamma}_{k,\alpha,(mk-\mu)}\Big(\omega(\Psi(t))^{\frac{\alpha}{k}}\Big)F(s)\right)\\
&-k^m\sum_{n=0}^{m-1} s^{m-n-1}\left[\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^k {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(0^+)\right]\\
=&(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}F(s)\\
&-k^m\sum_{n=0}^{m-1} s^{m-n-1}\Big({}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega} f(0^+)\Big).
\end{aligned}$$

□

For the case $[\frac{\mu}{k}]+1=m=1$, we have

$$\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s)=(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}F(s)-k\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega} f(t)\Big]_{t=0^+}.$$

Lemma 4.13. *The* Ψ*-Laplace transform of regularized version of* Ψ*-k-Prabhakar fractional derivative equation* (3.10) *is*

$$\begin{aligned}
\mathcal{L}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) =&(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}F(s)\\
&-k^m\sum_{n=0}^{m-1}(ks)^{\frac{\mu-n-1}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}f^{(n)}(0^+),
\end{aligned} \tag{4.6}$$

with $|\omega k(ks)^{\frac{-\alpha}{k}}|<1$.

Proof. Taking Ψ-Laplace transform of regularized version of Ψ-k-Prabhakar fractional derivative in (3.10) and using (3.7), (3.8), (3.9), (4.4) we get

$$\mathcal{L}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) =\mathcal{L}_\Psi\left({}_k\varepsilon^{-\gamma}_{\alpha,(mk-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}f\right)^m(t)\right)(s)$$

$$
\begin{aligned}
&=(ks)^m (ks)^{\frac{-(mk-\mu)}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} \\
&\quad \times \left\{ F(s) - k^m \sum_{k=0}^{m-1} (ks)^{\frac{\mu-n-1}{k}} f^n(0^+) \right\} \\
&=(ks)^{\frac{\mu}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} \\
&\quad - k^m \sum_{k=0}^{m-1} (ks)^{\frac{\mu-n-1}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^n(0^+). \qquad \square
\end{aligned}
$$

For absolutely continuous function $f \in AC^1[0,b]$

$$\left[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega} f(t)\right]_{t=0^+} = 0.$$

Then in view of (4.5) and (4.6) ($m=1$) we have

$$\mathcal{L}_\Psi\Big({}_k{}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = \mathcal{L}\Big({}_k D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) - k(ks)^{\frac{\mu-k}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f(0^+).$$

Taking inverse Ψ-Laplace transform, we get

$${}_k{}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) = {}_k D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) - (\Psi(t))^{\frac{-\mu}{k}} E^{-\gamma}_{k,\alpha,k-\mu}(\omega(\Psi(t))^{\frac{\alpha}{k}}) f(0^+),$$

for $f \in AC^1[0,b]$. This is the relation between Ψ-k–Prabhakar fractional derivative and its regularized version.

Lemma 4.14. *The Ψ-Laplace transform of Ψ-k-Hilfer-Prabhakar fractional derivative* (3.11) *is*

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&(ks)^{\frac{\mu}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
&- k^m \sum_{n=0}^{m-1} (ks)^{\frac{m(1-\nu)+\nu\mu-n-1}{k}} [1-\omega k (ks)^{\frac{-\alpha}{k}}]^{\frac{\gamma\nu}{k}} \\
&\times \Big({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu)-n,\omega} f(0^+)\Big).
\end{aligned} \tag{4.7}
$$

Proof. Taking Ψ-Laplace transform of Ψ-k-Hilfer-Prabhakar fractional derivative in (3.11) and using (3.7), (3.8), (3.9), (4.4) we have

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&\mathcal{L}_\Psi\Bigg[\Bigg({}_k\varepsilon^{-\gamma\nu}_{\alpha,\nu(mk-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m \\
&\times \Big({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} f\Big)\Bigg)(t)\Bigg](s) \\
=&(ks)^m (ks)^{\frac{-\nu(mk-\mu)}{k}} \Big(1-\omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}} \Bigg\{ s^m \Big((ks)^{-(1-\nu)(m-\mu)}\Big) \\
&\times \Bigg[\Big(\big(1-\omega k (ks)^{\frac{-\alpha}{k}}\big)^{\frac{\gamma(1-\nu)}{k}} F(s)\Big)(t)\Bigg)(s)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&- k^m \sum_{n=0}^{m-1} (ks)^{\frac{-\nu(mk-\mu)}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}} \\
&\times \Big[{}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu),(mk-\mu)-n,\omega} f(0^+)\Big]\Bigg] \\
=&(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
&- k^m \sum_{n=0}^{m-1} (ks)^{\frac{m(1-\nu)+\nu\mu-n-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}} \\
&\times \Big({}_kP^{-\gamma(1-\nu),\psi}_{\alpha,(1-\nu)(mk-\mu)-n,\omega} f(0^+)\Big). \qquad \square
\end{aligned}
$$

Lemma 4.15. *The* Ψ*-Laplace transforms of the regularized version of* Ψ*-*k*-Hilfer-Prabhakar fractional derivative equation* (3.12) *of order* μ *is*

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
&- k^m \sum_{n=0}^{m-1} (ks)^{\frac{\mu-n-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^n(0^+).
\end{aligned} \tag{4.8}
$$

Proof. Taking Ψ-Laplace transforms of regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative in (3.12) and using (3.8), (3.9), (4.4) we have

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =&k^m \mathcal{L}_\Psi\Bigg(\Bigg({}_k\varepsilon^{-\gamma\nu}_{\alpha,\nu(mk-\mu),\omega} *_\Psi \Bigg({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} \\
&\times \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m f\Bigg)\Bigg)(t)\Bigg)(s) \\
=&(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
&- k^m \sum_{n=0}^{m-1} (ks)^{\frac{\mu-k-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^n(0^+). \qquad \square
\end{aligned}
$$

Again for absolutely continuous function $f \in AC^1[0,b]$

$$
{}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) - (\Psi(t))^{\frac{-\mu}{k}} E^{-\gamma}_{k,\alpha,(mk-\mu)}(\omega(\Psi(t))^{\frac{\alpha}{k}}) f(0^+).
$$

This is the relation between Ψ-k-Hilfer-Prabhakar fractional derivative and its regularized version.

Let $F(s)$ be the Ψ-Sumudu transform of $f(t)$.

Lemma 4.16. *The* Ψ*-Sumudu transform of* Ψ*-*k*-Prabhakar integral equation* (3.7) *is*

$$
S_\Psi\Big({}_kP^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(u) = u^{-1} \left(\frac{u}{k}\right)^{\frac{\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{-\gamma}{k}} F(u),
$$

provided $|\omega k (\frac{u}{k})^{\frac{\alpha}{k}}| < 1$.

Lemma 4.17. *The Ψ-Sumudu transform of Ψ-k-Prabhakar fractional derivative equation* (3.9) *is*

$$\mathcal{S}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{-m+n}{k}}\Big[{}_kP^{\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega}f(0^+)\Big].$$

For the case $[\frac{\mu}{k}]+1=m=1$

$$\mathcal{S}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - \frac{k}{u}\left[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega}f(t)\right]_{t=0^+},$$

with $|\omega k(\frac{u}{k})^{\frac{\alpha}{k}}|<1$.

Lemma 4.18. *The Ψ-Sumudu transform of regularized version of Ψ-k-Prabhakar fractional derivative equation* (3.10) *is*

$$\mathcal{S}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{k}{u}\right)^{\frac{-(\mu-nk)}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}f^{(n)}(0),$$

with $|\omega k(\frac{u}{k})^{\frac{\alpha}{k}}|<1$.

Lemma 4.19. *The Ψ-Sumudu transform of Ψ-k-Hilfer-Prabhakar fractional derivative equation* (3.11) *is*

$$\mathcal{S}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{m(\nu-1)-\nu\mu+n}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \times\left[P^{-\gamma(1-\nu),\Psi}_{\alpha,\nu(1-\nu)(mk-\mu)-n,\omega}f(0^+)\right]. \tag{4.9}$$

Lemma 4.20. *The Ψ-Sumudu transforms of the regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative equation* (3.12) *of order μ is*

$$\mathcal{S}_\Psi\left({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{-\mu+nk}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}f^n(0^+).$$

5. Application

In this section we find the solutions of Cauchy problems involving Ψ-k-Hilfer-Prabhakar derivative and its regularised version.

Theorem 5.1. *The solution of Cauchy problem*

$$ {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}y(x) = \lambda_k P^{\delta,\Psi}_{\alpha,\mu,\omega,0^+}y(x) + f(x), \tag{5.1}$$

$$ \left[{}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(k-\mu),\omega,0^+}y(x)\right]_{t=0^+} = C, \quad C \geq 0, \tag{5.2}$$

where $x \in (0,\infty)$, $f(x) \in L^1[0,\infty)$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega,\lambda \in \mathbb{C}$, $\alpha > 0$, $\gamma,\delta \geq 0$ *is given by*

$$\begin{aligned} y(x) =& C\sum_{n=0}^{\infty}\lambda^n(\Psi(x))^{\frac{\nu(k-\mu)+\mu(1+2n)}{k}-1}E^{n(\delta+\gamma)-\gamma(\nu-1)}_{k,\alpha,\nu(k-\mu)+\mu(1+2n)}(\omega(\Psi(x))^{\frac{\alpha}{k}} \\ &+ \sum_{n=0}^{\infty}\lambda^n {}_kP^{\gamma+n(\delta+\gamma),\Psi}_{k,\alpha,\mu(1+2n),\omega,0^+}f(x), \end{aligned} \tag{5.3}$$

if the series on the right hand side of equation (5.3) *are convergent.*

Proof. Let $Y(u)$ and $F(u)$ denote the Ψ-Laplace transform of $y(x)$ and $f(x)$, respectively. Now taking Ψ-Laplace transform of (5.1) and using (3.7), (3.8), (4.4), (4.7), (5.2) we have

$$\begin{aligned} &(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}Y(u) - k(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}}C \\ =&\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}Y(u) + F(u). \end{aligned}$$

□

Thus, we have

$$\begin{aligned} Y(u) &= \left(\frac{Ck(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} + F(u)}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}} - \lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}\right) \\ &= \left[\frac{Ck(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} + F(u)}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right] \\ &\quad\times \frac{1}{\left[1-\frac{\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right]}. \end{aligned}$$

Hence, for $\left|\frac{\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right| < 1$, we get

$$\begin{aligned}Y(u) =&Ck\sum_{n=0}^{\infty}\lambda^n(ku)^{\frac{-\nu(k-\mu)-\mu(1+2n)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma(\nu-1)-n(\delta+\gamma)}{k}}\\ &+F(u)\sum_{n=0}^{\infty}\lambda^n(ku)^{\frac{-\mu(1+2n)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\gamma-n(\delta+\gamma)}{k}}.\end{aligned}$$

Now using inverse Laplace transform, we get the required result.

Theorem 5.2. *The solution of Cauchy problem*

$$ {}_k^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}u(x,t) = T\frac{\partial^2}{\partial x^2}u(x,t), \qquad t>0, x\in\mathbb{R}, \tag{5.4}$$

$$u(x,0^+) = g(x), \tag{5.5}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0. \tag{5.6}$$

with $\mu\in(0,1)$, $\omega\in\mathbb{R}$, $T,\alpha>0$, $\gamma\geq 0$ *is given by*

$$u(x,t) = \frac{1}{2k^2\pi}\int_{-\infty}^{\infty}dp\, e^{ipx}\widehat{g}(p)\sum_{n=0}^{\infty}(-T)^n p^{2n}(\Psi(t))^{\frac{n\mu}{k}}E^{n\gamma}_{k,\alpha,n\mu+k}(\omega(\Psi(t))^{\frac{\alpha}{k}}), \tag{5.7}$$

if the series on the right hand side of (5.7) *is convergent.*

Proof. Let $\overline{u}(x,q)$ and $\widehat{u}(p,t)$ denote the Ψ-Laplace transform and Fourier transform of $u(x,t)$, respectively. Taking Fourier transform of equation (5.4) and using (5.6) we get

$$ {}_k^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}\widehat{u}(p,t) = -Tp^2\widehat{u}(p,t). \tag{5.8}$$

Now taking Ψ-Laplace transform of (5.8) and using (4.8), (5.5) we get

$$\begin{aligned}&(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}\left(\overline{\widehat{u}}(p,s)-\frac{g(x)}{s}\right) = -Tp^2\overline{\widehat{u}}(p,s),\\ &\left((ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}+K\,p^2\right)s\overline{\widehat{u}}(p,s) = (ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}\widehat{g}(p).\end{aligned}$$

Thus, we have

$$\overline{\widehat{u}}(p,s) = \frac{\widehat{g}(p)}{s}\left(1+\frac{T\,p^2}{(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right)^{-1}.$$

Hence, for $\left|\frac{K\,p^2}{(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right| < 1$, we get

$$\overline{\widehat{u}}(p,s) = \frac{\widehat{g}(p)}{s}\sum_{n=0}^{\infty}(-T)^n p^{2n}(kq)^{\frac{-n\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{-n\gamma}{k}}$$

$$= \frac{\widehat{g}(p)}{k} \sum_{n=0}^{\infty} (-T)^n p^{2n} (ks)^{\frac{-n\mu-k}{k}} \left(1 - \omega k (ks)^{\frac{-\alpha}{k}}\right)^{\frac{-n\gamma}{k}}. \tag{5.9}$$

Taking inverse Laplace transform of (5.9) we get

$$\widehat{u}(x,s) = \frac{\widehat{g}(p)}{k^2} \sum_{n=0}^{\infty} (-T)^n p^{2n} (\Psi(t))^{\frac{n\mu}{k}} E^{n\gamma}_{k,\alpha,n\mu+k}(\omega(\Psi(t))^{\frac{\alpha}{k}}).$$

Using inverse Fourier transform, we get required result. □

The two above results can also be obtained using the Sumudu transform instead of Laplace transform and these are the generalizations of results discussed in [12].

Theorem 5.3 ([20]). *The solution of the differential equation*

$$-hM\Theta(x) = \rho V c_p \, {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}\Theta(x)\,, \tag{5.10}$$

$$\Theta(0) = \beta, \quad \text{for } \beta \geq 0, \tag{5.11}$$

where ρ-density, V-volume, c_p-specific heat of material, h-convection heat transfer coefficient, M-surface area of the body and $\Theta \in L^1[0,\infty)$, $0 < x < \infty$, $k, \alpha > 0$, $\gamma, \omega \in \mathbb{R}$, $\mu \in (0,1)$, $\nu \in [0,1]$ is given by

$$\Theta(x) = \beta \sum_{n=0}^{\infty} \left(\frac{-hM}{\rho V c_p}\right)^n (\Psi(x))^{\frac{\nu(k-\mu)+\mu(n+1)}{k}-1} E^{-\gamma(\nu-n-1)}_{k,\alpha,\nu(k-\mu)+\mu(n+1)}(\omega(\Psi(x))^{\frac{\alpha}{k}}), \tag{5.12}$$

if the series on the right hand side of (5.12) *is convergent.*

Proof. Let $\widehat{\Theta}(u)$ denote the Ψ-Sumudu transform of $\Theta(x)$. Now taking Ψ-Sumudu transform of (5.10) and using (4.9), (5.11) we have

$$\begin{aligned}
-hM\widehat{\Theta}(u) =& \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}} \widehat{\Theta}(u) \\
& - \rho V c_p \beta \left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \\
& \times \left[hM + \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right] \widehat{\Theta}(u), \\
=& \rho V c_p \beta \left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}}, \\
\widehat{\Theta}(u) =& \frac{\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \rho V c_p \beta}{\left[hM + \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right]}
\end{aligned}$$

$$
=\frac{\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}}\rho V c_p\beta}{\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}}
$$

$$
\times\left[1+\frac{hM}{\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right]^{-1}
$$

$$
=\beta\Big(\frac{u}{k}\Big)^{\frac{\nu(k-\mu)+\mu}{k}-1}\left(1-\omega k\Big(\frac{u}{k}\Big)^{\frac{\alpha}{k}}\right)^{\frac{\gamma(\nu-1)}{k}}
$$

$$
\times\sum_{n=0}^{\infty}\left(\frac{-hM}{\rho V c_p}\right)^n\left(\frac{u}{k}\right)^{\frac{n\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{-n\gamma}{k}},
$$

for

$$
\left|\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right|<1,
$$

$$
\widehat{\Theta}(u)=\beta\sum_{n=0}^{\infty}\left(\frac{-hM}{\rho V c_p}\right)^n\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)+\mu(n+1)}{k}-1}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma(\nu-n-1)}{k}}. \tag{5.13}
$$

Taking inverse Sumudu transform of (5.13), we get required solution of (5.12). □

6. Conclusion.

In the present paper, we investigate new fractional derivatives in the sense of Ψ-fractional calculus to find their generalized transforms called Ψ-Laplace and Ψ-Sumudu transforms. These derivatives are more generalization of fractional derivatives and effectively applicable for various applications like cauchy problems, heat transfer problem. In order to explain the obtained results, some examples were illustrated. It is noted that since generalized derivatives are global and contain a wide class of fractional derivatives.

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HARMONIC BLOCH FUNCTION SPACES AND THEIR COMPOSITION OPERATORS

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ABSTRACT. In this paper we characterize some basic properties of composition operators on the spaces of harmonic Bloch functions. First we provide some equivalent conditions for boundedness and compactness of composition operators. In the sequel we investigate closed range composition operators. These results extends the similar results that were proven for composition operators on the Bloch spaces.

1. INTRODUCTION AND PRELIMINARIES

Let D be the open unit disk in the complex plane. For a continuously differentiable complex-valued function $f(z) = u(z) + iv(z)$, $z = x + iy$, we use the common notation for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y),$$
$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

A twice continuously differentiable complex-valued function $f = u + iv$ on D is called a harmonic function if and only if the real-valued function u and v satisfy Laplace's equations $\Delta u = \Delta v = 0$.

A direct calculation shows that the Laplacian of f is

$$\Delta f = 4f_{z\bar{z}}.$$

Thus for functions f with continuous second partial derivatives, it is clear that f is harmonic if and only if $\Delta f = 0$. We consider complex-valued harmonic function f defined in a simply connected domain $D \subset C$. The function f has a canonical

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decomposition $f = h+\bar{g}$, where h and g are analytic in D [6]. A planar complex-valued harmonic function f in D is called a harmonic Bloch function if and only if

$$\beta_f = \sup_{z,w\in D, z\neq w} \frac{|f(z)-f(w)|}{\varrho(z,w)} < \infty.$$

Here β_f is the Lipschitz number of f and

$$\varrho(z,w) = \arctan h \left|\frac{z-w}{1-\bar{z}w}\right|,$$

denotes the hyperbolic distance between z and w in D and also $\rho(z,w)$ is the pseudo-hyperbolic distance on D. In [3] Colonna proved that

$$\beta_f = \sup_{z\in D}(1-|z|^2)[|f_z(z)|+|f_{\bar{z}}(z)|].$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $HB(1)$ or HB, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{HB(1)} = |f(0)| + \sup_{z\in D}(1-|z|^2)[|f_z(z)|+|f_{\bar{z}}(z)|].$$

Definition 1.1. For $\alpha \in (0,\infty)$, the harmonic α-Bloch space $HB(\alpha)$ consists of complex-valued harmonic function f defined on D such that

$$|||f|||_{HB(\alpha)} = \sup_{z\in D}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] < \infty,$$

and the harmonic little α-Bloch space $HB_0(\alpha)$ consists of all function in $HB(\alpha)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] = 0.$$

Obviously, when $\alpha = 1$, we have $|||f|||_{HB(\alpha)} = \beta_f$. Each $HB(\alpha)$ is a Banach space with the norm given by

$$\|f\|_{HB(\alpha)} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|]$$

and $HB_0(\alpha)$ is a closed subspace of $HB(\alpha)$. Now we define composition operators.

Definition 1.2. Let D be the open unit disk in the complex plane. Let φ be an analytic self-map of D, i. e., an analytic function φ in D such that $\varphi(D) \subset D$. The composition operator C_φ induced by such φ is the linear map on the spaces of all harmonic functions on the unit disk defined by

$$C_\varphi f = f \circ \varphi.$$

The composition operators on function spaces were studied by many authors. Some known results about composition operators can be found in [5] and [10]. In this paper we study composition operators on harmonic Bloch-type spaces $HB(\alpha)$. In section 2, by using of Theorem 2.1 in [8], we give a necessary and sufficient condition for boundedness of C_φ on $HB(\alpha)$ for $\alpha \in (0,\infty)$, which extends Theorem 3.1 in [8], by Lou. The compactness of C_φ on analytic Bloch-type spaces were characterized in [8,9].

In this paper, we deal the compactness of composition operators between the Banach spaces of harmonic functions $HB(\alpha)$ and $HB_0(\alpha)$.

Moreover, we investigate closed range composition operators. Closed range composition operators on the Bloch-type spaces have been studied in [2,4,7,11]). For $\alpha > 0$, and φ being an analytic self-map of D, let

$$\tau_{\varphi,\alpha}(z) = \frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

We write τ_φ if $\alpha = 1$. We say that a subset $G \subset D$ is called sampling set for $HB(\alpha)$ if exists $S > 0$ such that for all $f \in HB(\alpha)$

$$\sup_{z\in G}(1-|z|^2)^\alpha[|f_z(z)|] + [|f_{\bar{z}}(z)|] \geq S|||f|||_{HB(\alpha)}.$$

To state the results obtained, we need the following definition. Let $\rho(z,w) = |\varphi_z(w)|$ denote the pseudohyperbolic distance (between z and w) on D, where φ_z is a disk automorphism of D that is

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

We say that subset $G \subset D$ is an r-net for D for some $r \in (0,1)$ if for each $z \in D$ exists $w \in G$ such that $\rho(z,w) < r$. For $c > 0$, let

$$\Omega_{c,\alpha} = \{z \in D : \tau_{\varphi,\alpha}(z) \geq c\}$$

and let $G_{c,\alpha} = \varphi(\Omega_{c,\alpha})$. If $\alpha = 1$, then we write Ω_c and G_c. Now we recall Montel's theorem for harmonic functions.

Theorem 1.1 ([1]). *If $\{u_n\}_{n=1}^\infty$ is a sequence of harmonic functions in the region Ω with $\sup_{n,x\in K}|u_n(x)| < \infty$ for every compact set $K \subset \Omega$, then there exists a subsequence, $\{u_{n_j}\}_{j=1}^\infty$ converging uniformly on every compact set $K \subset \Omega$.*

Also we recall a very useful theorem that we will use it a lot in this paper.

Theorem 1.2 ([8]). *Let $0 < \alpha < \infty$. Then there exist $f, g \in HB(\alpha)$ such that*

$$|f'(z)| + |g'(z)| \geq \frac{1}{(1-|z|)^\alpha},$$

for all $z \in D$.

2. Main Results

In this section we study bounded and compact composition operators on $HB(\alpha)$. And then we investigate closed range composition operators on $HB(\alpha)$. First we provide some equivalent conditions for boundedness of composition operator C_φ on $HB(\alpha)$.

Theorem 2.1. *If $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$, then the following statements are equivalent.*

a) *$C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded.*

b)

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. For the implication $a\to b$, by Theorem 2.1 of [8] we have that for $0<\alpha<\infty$ there exist $h,g\in B(\alpha)$ satisfying the inequality

$$|h'(z)|+|g'(z)|\geq\frac{1}{(1-|z|)^\alpha}.$$

If we set $f=h+\bar{g}\in HB(\alpha)$, then $f\circ\varphi(z)=h\circ\varphi(z)+\overline{g\circ\varphi(z)}$ and so by the same method of Theorem 3.1 of [8] we get the proof.

For the implication $b\to a$ we can do the same as Theorem 3.1 of [8]. □

In the next theorem we consider the composition operator from $HB_0(\alpha)$ into $HB(\alpha)$ and we find some conditions under which C_φ is bounded.

Theorem 2.2. *Let $0<\alpha<\infty$, $\varphi\in H(D)$ and $\varphi(D)\subseteq D$. Then the followings are equivalent.*

a) $C_\varphi : HB_0(\alpha)\to HB(\alpha)$ *is bounded.*

b)

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. The proof is similar to the proof of Theorem 3.3 of [8]. Hence we omit the proof. □

Now we consider the composition operator $C_\varphi : HB(\alpha)\to HB_0(\alpha)$ and we give an equivalent condition to boundedness of C_φ.

Theorem 2.3. *If $0<\alpha<\infty$, $\varphi\in H(D)$ and $\varphi(D)\subseteq D$, then the following are equivalent.*

a) $C_\varphi : HB(\alpha)\to HB_0(\alpha)$ *is bounded.*

b)

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|=0.$$

Proof. By a similar method of the proof of Theorem 3.4 of [8] we get the proof. □

Finally we provide some conditions for boundedness of the composition operator C_φ as an operator on $HB_0(\alpha)$.

Theorem 2.4. *If $0<\alpha<\infty$, $\varphi\in H(D)$ and $\varphi(D)\subseteq D$, then the followings are equivalent.*

a) $C_\varphi : HB_0(\alpha)\to HB_0(\alpha)$ *is bounded.*

b) $\varphi\in B_0(\alpha)$ *and*

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. By some simple calculations one can get the proof. □

A sequence $\{z_n\}$ in D is said to be R-separated if $\rho(z_n, z_m) = |\frac{z_m - z_n}{1-\bar{z_m} z_n}| > R$ whenever $m \neq n$. Thus an R-separated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on D, or equivalently, the hyperbolic balls $D(z_n, r) = \{w : \rho(w, z_n) < r\}$ are pairwise disjoint for some $r > 0$. Evidently, any sequence $\{z_n\}$ in D which satisfies $|z_n| \to 1$ possesses an R-separated subsequence for any $R > 0$.

Another property of separated sequence is contained in the next proposition.

Proposition 2.1 ([9])**.** *There is an absolute constant $R > 0$ such that if $\{z_n\}$ is R-separated, then for every bounded sequence $\{\lambda_n\}$ there is an $f \in B$ such that $(1 - |z_n|^2)f'(z_n) = \lambda_n$ for all n.*

Since every sequence $\{z_n\}$ with $|z_n| \to 1$ contains an R-separated subsequence $\{z_{n_k}\}$, it follows that there is an $f \in B$ such that $(1 - |z_{n_k}|^2)f'(z_{n_k}) = 1$ for all k.

Now we begin investigating compactness of the composition operator C_φ in different cases. First we provide some equivalent conditions for compactness of C_φ as an operator on $HB(\alpha)$.

Theorem 2.5. *Let $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$. Then we have the followings conditions are equivalent.*

a) *$C_\varphi : HB(\alpha) \to HB(\alpha)$ is compact.*

b)

$$\lim_{|\varphi(z)| \to 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\alpha |\varphi'(z)| = 0$$

and

$$\sup_{z \in D} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\alpha |\varphi'(z)| < \infty.$$

Proof. By making use of the proof of Theorem 4.2 of [8] and the Proposition 1 of [9] we get the proof. □

Here we prove that the compactness of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ and $C_\varphi : HB(\alpha) \to HB_0(\alpha)$ are equivalent and we find an equivalent condition for compacness of C_φ in these cases.

Theorem 2.6. *Let $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$. Then the following statements are equivalent.*

a) *The operator $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is compact.*

b) *The operator $C_\varphi : HB(\alpha) \to HB_0(\alpha)$ is compact.*

c)

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$

Proof. First we prove the implication $a \to c$. If $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is compact, then the set $K = \overline{C_\varphi(S_{HB_0(\alpha)})} \subset HB_0(\alpha)$ is compact, in which $S_{HB_0(\alpha)} = \{f \in HB_0(\alpha) : \|f\|_{HB_0(\alpha)} \leq 1\}$. By the Theorem 2.5, we get that

$$\sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|z|^2)^\alpha [|f_z(z)| + |f_{\bar{z}}(z)|] = 1,$$

for all $z \in D$. Moreover we have

$$\begin{aligned} 0 &= \lim_{|z|\to 1} \sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|z|^2)^\alpha [|(f\circ\varphi)_z(z)| + |(f\circ\varphi)_{\bar{z}}(z)|] \\ &= \lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| \sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|\varphi(z)|^2)^\alpha [|h'(\varphi(z)|) + |g'(\varphi(z))|]. \end{aligned}$$

So, we get the desired result.

Now we prove the implication $c \to b$. Let $\{f_n\}_{n\in\mathbb{N}} \subset HB(\alpha)$ and $\|f_n\|_{HB(\alpha)} \leq 1$, for all n. First we obtain that $\{C_\varphi f_n\}$ has a subsequence that converges in $HB_0(\alpha)$. By Montel's Theorem we have a subsequence $\{f_{n_k}\} \subset \{f_n\}$, that converges uniformly on subsets of D to a harmonic function f. Hence we have

$$\begin{aligned} (1-|z|^2)^\alpha [|f_z(z)| + |f_{\bar{z}}(z)|] &= \lim_{k\to\infty} (1-|z|^2)^\alpha [|(f_{n_k})_z(z)| + |(f_{n_k})_{\bar{z}}(z)|] \\ &\leq \lim_{k\to\infty} \|f_{n_k}\|_{HB(\alpha)} \\ &\leq 1. \end{aligned}$$

This means that $f \in HB(\alpha)$ with $\|f\|_{HB(\alpha)} \leq 1$. Also we have

$$\begin{aligned} (1-|z|^2)^\alpha [|(f\circ\varphi)_z(z)| + |(f\circ\varphi)_{\bar{z}}(z)|] &= \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|)^\alpha} |\varphi'(z)| \\ &\leq \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| \|f\|_{HB(\alpha)}. \end{aligned}$$

By these observations we conclude that $C_\varphi f \in HB_0(\alpha)$. Also we need to show that

$$\lim_{k\to\infty} \|C_\varphi f_{n_k} - C_\varphi f\|_{HB(\alpha)} = 0.$$

Since $\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0$, then for any $\varepsilon > 0$, there exists $r \in (0,1)$ such that for z with $r < |z| < 1$ we have

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \frac{\varepsilon}{4}.$$

And so for all z with $r < |z| < 1$ we have

$$\begin{aligned} (1-|z|^2)^\alpha |((f_{n_k} - f)\circ\varphi)'(z)| =& (1-|z|^2)^\alpha [|(f_{n_k})_z \varphi(z)| + |(f_{n_k})_{\bar{z}} \varphi(z)|] \\ & - (1-|z|^2)^\alpha [|f_z \varphi(z)| + |f_{\bar{z}} \varphi(z)|] \\ \leq& \frac{\varepsilon}{4} (\|f_{n_k}\|_{HB(\alpha)} + \|f\|_{HB(\alpha)}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

For z with $|z| \leq r$, the set $\{\varphi(z) : |z| \leq r\}$ is a compact subset of D. Since

$$(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] = \lim_{k\to\infty}(1-|z|^2)^\alpha[|(f_{n_k})_z(z)|+|(f_{n_k})_{\bar{z}}(z)|]$$

and

$$\begin{aligned}(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| \leq &(1-|z|^2)^\alpha\{[|(f_{n_k})_z\varphi(z)|+|(f_{n_k})_{\bar{z}}\varphi(z)|] \\ &-[|f_z\varphi(z)|+|f_{\bar{z}}\varphi(z)|]\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|.\end{aligned}$$

Hence, we have $(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| \to 0$ uniformly on $\{z : |z| \leq r\}$. Therefore, $(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| < \frac{\varepsilon}{2}$ for k sufficiently large and $\{z : |z| \leq r\}$. This completes the proof.

The implication b) $\to$ a) is clear. □

Let (X,d) be a metric space and let $\varepsilon > 0$. We say that $A \subset X$ is an ε-net for (X,d), if for all $x \in X$ there exists an a in A such that $d(a,x) < \varepsilon$. We characterize the compact subsets of $HB_0(\alpha)$ in the next lemma.

Lemma 2.1. *A closed subset of $HB_0(\alpha)$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] = 0.$$

Proof. Suppose that $K \subset HB_0(\alpha)$ is compact and $\varepsilon > 0$. Then we can choose an $\frac{\varepsilon}{2}$-net $f_1, f_2, \ldots, f_n \in K$. Hence, there exists δ, $0 < \delta < 1$, such that for all z with $|z| > \delta$ we have $(1-|z|^2)^\alpha[|(f_i)_z(z)|+|(f_i)_{\bar{z}}(z)|] < \frac{\varepsilon}{2}$ for all $1 \leq i \leq n$. If $f \in K$, then there exists some f_i such that $\|f-f_i\|_{HB(\alpha)} < \frac{\varepsilon}{2}$ and so for all z with $|z| > \delta$ we have

$$(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] \leq \|f-f_i\|_{HB(\alpha)} + (1-|z|^2)^\alpha[|(f_i)_z(z)|+|(f_i)_{\bar{z}}(z)|] < \varepsilon.$$

Therefore, we get that

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] = 0.$$

Conversely, let K be a closed and bounded subset of $HB_0(\alpha)$ such that

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] = 0.$$

Since K is bounded, then it is relatively compact with respect to the topology of the uniform convergence on compact subsets of the unit disk. If (f_n) is a sequence in K, then by Montel's Theorem we have a subsequence $\{f_{n_k}\} \subset \{f_n\}$ which converges uniformly on compact subsets of D to a harmonic function f. Also $\{f'_{n_k}\}$ converges uniformly to f' on compact subsets of D. For every $\varepsilon > 0$ we can find $\delta > 0$ such that for all z with $|z| > \delta$ we have

$$(1-|z|^2)^\alpha[|(f_{n_k})_z(z)|+|(f_{n_k})_{\bar{z}}(z)|] < \frac{\varepsilon}{2},$$

for any integer $k > 0$. Therefore, $(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] < \frac{\varepsilon}{2}$, for all z with $|z| > \delta$. So,

$$\begin{aligned}\sup_{|z|>\delta}(1-|z|^2)^\alpha[|(f_{n_k}-f)_z(z)|+|(f_{n_k}-f)_{\bar{z}}(z)|] \leq & \sup_{|z|>\delta}(1-|z|^2)^\alpha[|(f_{n_k})_z(z)| \\ & +|(f_{n_k})_{\bar{z}}(z)|] \\ & +\sup_{|z|>\delta}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] \\ < & \varepsilon.\end{aligned}$$

Moreover, since (f_{n_k}) converges uniformly on compact subsets of D to f and (f'_{n_k}) converges uniformly to f' on $\{z : |z| \leq \delta\}$, we get that

$$\sup_{|z|\leq\delta}(1-|z|^2)^\alpha[|(f_{n_k}-f)_z(z)|+|f_{n_k}-f)_{\bar{z}}(z)|] \leq \varepsilon.$$

Consequently for k large enough, we have $\lim_{k\to\infty}\|f_{n_k}-f\|_{HB(\alpha)} \leq \varepsilon$. This completes the proof. □

In the next theorem we prove that the norm convergence in $HB(\alpha)$ implies the uniform convergence.

Theorem 2.7. *The norm convergence in $HB(\alpha)$ implies the uniform convergence, that is if $\{f_n\} \subset HB(\alpha)$ such that $\|f_n - f\|_{HB(\alpha)} \to 0$, then $\{f_n\}$ converges uniformly to f.*

Proof. For $0 \neq z \in D$, we have

$$\begin{aligned}|f_n(z)-f(z)| &= \left|\int_0^1 \frac{d(f_n-f)}{dt}(zt)dt\right| \\ &= \left|z\int_0^1 \frac{d(f_n-f)}{d\varsigma(t)}(zt)dt + \bar{z}\int_0^1 \frac{d(f_n-f)}{d\bar{\varsigma}(t)}(zt)dt\right| \\ &\leq |z|\int_0^1 [|(f_n-f)_{\varsigma(t)}(zt)|+|(f_n-f)_{\bar{\varsigma}(t)}(zt)|]dt,\end{aligned}$$

in which $\varsigma(t) = zt$. This gives us

$$\begin{aligned}|f_n(z)-f(z)| &\leq \int_0^1 \frac{[|(f_n-f)_{\varsigma(t)}(zt)|+|(f_n-f)_{\bar{\varsigma}(t)}(zt)|]}{(1-|\varsigma(t)|^2)^\alpha}(1-|\varsigma(t)|^2)^\alpha dt \\ &\leq (\|f_n-f\|_{HB(\alpha)})\int_0^1 \frac{1}{(1-|z|t)^\alpha}dt \to 0,\end{aligned}$$

when $n \to \infty$. So we get the proof. □

In the next theorem we provide some equivalent conditions for closedness of range of the composition operator on $HB(\alpha)$.

Theorem 2.8. *Let $\varphi : D \to D$, $\alpha > 0$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be a bounded operator. Then the range of $C_\varphi : HB(\alpha) \to HB(\alpha)$ is closed if and only if there exists $c > 0$ such that $G_{c,\alpha}$ is sampling for $HB(\alpha)$.*

Proof. Since $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded, then exists $K > 0$ such that $\sup_{z\in D} \tau_{\varphi,\alpha}(z) \leq K$. Since every non-constant φ is an open map, then the composition operator C_φ is always one to one. By a basic operator theory result, a one-to-one operator has closed range if and only if it is bounded below. Hence, if C_φ has closed range, then C_φ is bounded below, that is exists $\varepsilon > 0$ such that for all $f \in HB(\alpha)$

$$\begin{aligned}\|C_\varphi f\|_{HB(\alpha)} &= \sup_{z\in D}(1-|z|^2)^\alpha[|(fo\varphi)_z(z)| + |(fo\varphi)_{\bar z}(z)|]\\ &= \sup_{z\in D}\tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\geq \varepsilon\|f\|_{HB(\alpha)}.\end{aligned}$$

Now we show that the set $G_{c,\alpha}$ is sampling for $HB(\alpha)$ with sampling constant $S = \frac{\varepsilon}{K}$. Since $\Omega_{c,\alpha} = \{z \in D : \tau_{\varphi,\alpha}(z) \geq c\}$, so for any $z \notin \Omega_{c,\alpha}$ and $c = \frac{\varepsilon}{2}$, we have

$$\sup_{z\notin\Omega_{c,\alpha}} \tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|] \leq \frac{\varepsilon}{2}\|f\|_{HB(\alpha)}.$$

Therefore, we have

$$\begin{aligned}\varepsilon\|f\|_{HB(\alpha)} &\leq \sup_{z\in D}\tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &= \sup_{z\in\Omega_{c,\alpha}}\tau_{\varphi,\alpha}(z)(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\leq K\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|h'(w|) + |g'(w)|].\end{aligned}$$

Hence $\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|h'(w|) + |g'(w)|] \geq \frac{\varepsilon}{K}\|f\|_{HB(\alpha)}$. This means that $G_{c,\alpha}$ is a sampling set for $HB(\alpha)$ with sampling constant $S = \frac{\varepsilon}{K}$.

Conversely, suppose that $G_{c,\alpha}$ is a sampling set for $HB(\alpha)$, with sampling constant $S > 0$. So for all $f \in HB(\alpha)$ and $\varepsilon = cS$ we get the followings relations:

$$\begin{aligned}S\|f\|_{HB(\alpha)} &\leq \sup_{z\in\Omega_{c,\alpha}}(1-|\varphi(z)|^2)^\alpha[|(f)_z(\varphi(z))| + |(f)_{\bar z}(\varphi(z))|]\\ &= \sup_{z\in\Omega_{c,\alpha}}(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\leq \frac{1}{c}\sup_{z\in D}(1-|z|^2)^\alpha[|(h\circ\varphi)_z(z)| + |(g\circ\varphi)_{\bar z}(z)|]\\ &\leq \frac{1}{c}\|f\circ\varphi\|_{HB(\alpha)}.\end{aligned}$$

Therefore,

$$\varepsilon\|f\|_{HB(\alpha)} \leq \|f\circ\varphi\|_{HB(\alpha)} = \|C_\varphi f\|_{HB(\alpha)}.$$

Hence, C_φ is bounded below and so C_φ has closed range. □

Now we give some other necessary and sufficient conditions for closedness of range of $C_\varphi : HB(\alpha) \to HB(\alpha)$.

Theorem 2.9. *Let φ be a self-map of D, $\alpha > 0$, and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be a bounded operator. Then we have the following hold.*

a) *If the operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ has closed range, then there exist $c, r > 0$ with $r < 1$, such that $G_{c,\alpha}$ is an r-net for D.*

b) *If there exist $c, r > 0$ with $r < 1$, such that $G_{c,\alpha}$ contains an open annulus centered at the origin and with outer radius 1, then C_φ has closed range.*

Proof. a) For $a \in D$, let $\varphi_a(z)$ be a function such that $\varphi_a(0) = 0$ and $\varphi_a'(z) = (\psi_a'(z))^\alpha$, where ψ_a is the disc automorphism of D defined by $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$. Using the equalities

$$1 - \rho(z,w)^2 = 1 - |\psi_w(z)|^2 = (1-|z|^2)|\psi_w'(z)|,$$

we get

$$\|\varphi_a + \bar{\varphi_a}\|_{HB(\alpha)} = \sup_{z\in D}(1-|z|^2)^\alpha 2|\varphi_a'(z)| = 2\sup_{z\in D}(1-|\psi_a(z)|^2)^\alpha = 2.$$

If we put $f = \varphi_a + \bar{\varphi_a}$, then we have

$$\begin{aligned}\|C_\varphi f\|_{HB(\alpha)} &= \|f\circ\varphi\|_{HB(\alpha)}\\ &= \sup_{z\in D}(1-|z|^2)^\alpha[|(f\circ\varphi)_z(z)| + |(f\circ\varphi)_{\bar{z}}(z)|]\\ &= \sup_{z\in D}\tau_{\varphi,\alpha}(z)2(1-|\psi_a(\varphi(z))|^2)^\alpha.\end{aligned}$$

Moreover, by assuming that C_φ is bounded and has closed range, then there exist K, $\varepsilon > 0$ such that $\sup_{z\in D}\tau_{\varphi,\alpha}(z) = K$ and

$$\|f\circ\varphi\|_{HB(\alpha)} = \sup_{z\in D}\tau_{\varphi,\alpha}(z)2(1-|\psi_a(\varphi(z))|^2)^\alpha \geq \varepsilon\|\varphi_a + \bar{\varphi_a}\|_{HB(\alpha)}.$$

This implies that

$$\varepsilon \leq \sup_{z\in D}\tau_{\varphi,\alpha}(z)(1-|\psi_a(\varphi(z))|^2)^\alpha \leq \sup_{z\in D}\tau_{\varphi,\alpha}(z) = K.$$

Since $1-|\psi_a(\varphi(z))|^2 \leq 1$, then there exists $z_a \in D$ such that

$$\tau_{\varphi,\alpha}(z_a) \geq \frac{\varepsilon}{2}$$

and

$$(1-|\psi_a(\varphi(z_a))|^2)^\alpha \geq \frac{\varepsilon}{2K}.$$

Thus, for $c = \frac{\varepsilon}{2}$ and $r = \sqrt{1-(\frac{\varepsilon}{2K})^{\frac{1}{\alpha}}}$, we conclude that for all $a \in D$, there exists $z_a \in \Omega_{c,\alpha}$ such that $\rho(a, \varphi(z_a)) < r$ and so $G_{c,\alpha}$ is an r-net for D.

b) Let $G_{c,\alpha}$ contains the annulus $A = \{z : r_0 < |z| < 1\}$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be bounded. Suppose that C_φ doesn't have closed range, then there exists a sequence $\{f_n\}$ with $\|f_n\|_{HB(\alpha)} = 1$ and $\|C_\varphi f_n\|_{HB(\alpha)} \to 0$. For each $\varepsilon > 0$, let $N_\varepsilon > 0$ such that for all $n > N_\varepsilon$ we have

$$\|C_\varphi f_n\|_{HB(\alpha)} < \varepsilon < c\varepsilon.$$

Since

$$\sup_{z\in D}(1-|z|^2)^{\alpha}[|(f_n)_z(z)|+|(f_n)_{\bar z}(z)|]=\sup_{z\in D}(1-|z|^2)^{\alpha}[|h_n'(z)|+|g_n'(z)|]=1,$$

then there exists a sequence $\{a_n\}$ in D such that for all n

$$(1-|a_n|^2)^{\alpha}[|h_n'(a_n)|+|g_n'(a_n)|]\geq \frac{1}{2}.$$

Moreover, we have

$$\begin{aligned}&\sup_{w\in G_{c,\alpha}}(1-|w|^2)^{\alpha}[|(f_n)_z(w)|+|(f_n)_{\bar z}(w)|]\\ =&\sup_{z\in \Omega_{c,\alpha}}\tau_{\varphi,\alpha}^{-1}(z)\tau_{\varphi,\alpha}(z)(1-|\varphi(z)|^2)^{\alpha}[|(f_n)_z(\varphi(z))|+|(f_n)_{\bar z}(\varphi(z))|]\\ \leq&\frac{1}{c}\sup_{z\in D}(1-|z|^2)^{\alpha}|\varphi'(z)|[|(f_n)_z(\varphi(z))|+|(f_n)_{\bar z}(\varphi(z))|]\\ <&\frac{c\varepsilon}{c}=\varepsilon.\end{aligned}$$

If we take $\varepsilon<\frac{1}{2}$, then we get that each a_n with $n>N_\varepsilon$ belongs to $(G_{c,\alpha})^c$. Thus $|a_n|\leq r_0<1$ and $a_n\to a$ with $|a|\leq r_0$. On the other hand, by Montel's Theorem, there exists a subsequence $\{f_{n_k}\}$ such that converges uniformly on compact subsets of D to some function $f\in HB(\alpha)$. Hence $\{f_{n_k}'\}$ converges to f' uniformly on compact subsets of D, and since

$$\sup_{w\in G_{c,\alpha}}(1-|w|^2)^{\alpha}[|(f_n)_z(w)|+|(f_n)_{\bar z}(w)|]\to 0,$$

when $n\to\infty$ and $G_{c,\alpha}$ contains a compact subset of D, we conclude that $f'=0$. This contradicts the fact that

$$(1-|a|^2)^{\alpha}[|h'(a)|+|g'(a)|]\geq\frac{1}{2}.$$

Therefore, C_φ must be bounded below and consequently it has closed range. □

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ON THE INITIAL VALUE PROBLEM FOR FUZZY NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the existence result of solutions for fuzzy nonlinear fractional differential equations involving Caputo differentiability of an arbitrary order $0 < q < 1$. As application, an example is included to show the applicability of our result.

1. Introduction

Fuzzy fractional differential equations were proposed to handle uncertainty due to incomplete information that appears in many mathematical or computer models of some deterministic real-world phenomena. In recent years, fractional differential equations have attracted a considerable interest both in mathematics and in applications as material theory, transport processes, fluid flow phenomena, earthquakes, solute transport, chemistry, wave propagation, signal theory, biology, electromagnetic theory, thermodynamics, mechanics, geology, astrophysics, economics and control theory (see [1–3]). For basic works related to the fuzzy fractional differential equations we refer the reader to [4, 16, 17].

Motivated by the above works, in this paper,we study the existence result of solution for the following fuzzy fractional initial value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), & t \in J = [t_0, t_0 + \delta], \\ x(t_0) = x_0. \end{cases} \tag{1.1}$$

Where ${}^cD^q$ is the Caputo derivative of $x(t)$ at order $q \in [0, 1]$ and $\delta > 0$.

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To be more precise, we will show that problem (1.1) admits a solution on each locally compact subset of the space of a E^1 which is the space of all fuzzy numbers.

The paper is organized as follows. In Section 2, we give some basic properties of fuzzy sets, operations of fuzzy numbers and some detailed definitions of fuzzy fractional integral and fuzzy fractional derivative which will be used in the rest of this paper. In Section 3, we introduce the existence result of solution for the fuzzy fractional initial value problem by using Peano theorem. Illustrative example will be discussed in Section 4, followed by conclusion and futur works in Section 5.

2. Preliminaries

Definition 2.1 ([18])**.** A fuzzy number is mapping $u : \mathbb{R}^n \to [0, 1]$ such that

(a) u is upper semi-continuous;
(b) u is normal, that is, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
(c) u is fuzzy convex, that is, $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$;
(d) $\overline{\{x \in \mathbb{R}^n,\ u(x) > 0\}}$ is compact.

The $\alpha - cut$ of a fuzzy number u is defined as follows:

$$[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}.$$

Moreover, we also can present the $\alpha - cut$ of fuzzy number u by $[u]^\alpha = [u_l(\alpha), u_r(\alpha)]$. We denote by E^n the collection of all fuzzy numbers.

Example 2.1. Let u be a fuzzy number defined by the following function:

$$\mu_u(x) = \begin{cases} x - 1, & x \in [1, 2], \\ -x + 3, & x \in [2, 3], \\ 0, & \text{elswhere.} \end{cases}$$

Then we have $[u]^1 = \{2\}$.

Definition 2.2 ([9])**.** Let $u \in E^1$ and $\alpha \in [0, 1]$ we define the diameter of $\alpha - level$ set of the fuzzy set u as follows

$$d([u]^\alpha) = l_r - l_l.$$

We denote by $\mathcal{C}(J, E^n)$ space of all fuzzy-valued functions which are continuous on J,and $\mathcal{P}_c(\mathbb{R}^n)$ the collection of all the compact subset of $\mathbb{R}^n$.

Definition 2.3 ([9])**.** The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^n$ is defined as follows:

$$u \ominus_{gH} v = w \Leftrightarrow \text{i) } u = v + w \quad \text{or} \quad \text{ii) } v = u + (-1)w.$$

Proposition 2.1. *If $u \in E^1$ and $v \in E^1$, then the following properties hold.*

1) *If $u \ominus_{gH} v$ exists then it is unique.*
2) $u \ominus_{gH} u = 0_{E^1}$.
3) $(u + v) \ominus_{gH} v = u$.

4) $u \ominus_{gH} v = 0_{E^1} \Leftrightarrow u = v$.

Definition 2.4 ([18]). According to the Zadeh's extension principle, the addition on E^1 is defined by:

$$(u \oplus v)(z) = \sup_{z=x+y} \min\{u(x), v(y)\}.$$

And scalar multiplication of a fuzzy number is given by:

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \widetilde{0}, & k = 0. \end{cases}$$

Remark 2.1 ([13]). Let $u, v \in E^1$ and $\alpha \in [0,1]$, then we have

$$\begin{aligned} [u+v]^\alpha &= [u]^\alpha + [v]^\alpha, \\ [u-v]^\alpha &= [u_1^\alpha - v_2^\alpha, u_2^\alpha - v_1^\alpha], \\ [ku]^\alpha &= {}^k[u]^\alpha = \begin{cases} [\lambda u_1^\alpha, \lambda u_2^\alpha], & \text{if } \lambda \geq 0, \\ [\lambda u_2^\alpha, \lambda u_1^\alpha], & \text{if } \lambda < 0, \end{cases}, \\ [uv]^\alpha &= [\min u_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha, \max u_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha]. \end{aligned}$$

Definition 2.5 ([13]). Let $u, v \in E^n$ with $\alpha \in [0,1]$, then the Hausdorf distance between u and $v$is given by:

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_c(\mathbb{R}^n)$.

Proposition 2.2 ([10]). *D is a metric on E^n and has the following properties:*

(a) *$(E^n; D)$ is a complete metric space;*
(b) *$D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$;*
(c) *$D(ku, kv) = |k|D(u, v)$ for all $u, v \in E^n$ and $k \in \mathbb{R}$;*
(d) *$D(u + w, v + z) \leq D(u, v) + D(w, z)$ for all $u, v, w, z \in E^n$.*

Definition 2.6 ([7]). Let $f : [a, b] \to E^n$ and $t_0 \in [a, b]$. We say that f is Hukuhara differentiable at t_0 if there exists $f'(t_0) \in E^n$ such that

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus_{gH} f(t_0 - h)}{h}.$$

Remark 2.2. Let $f : [a, b] \to E^n$ be a fuzzy function such that $[f(x)]^\alpha = \left[\underline{f}(x; \alpha), \overline{f}(x; \alpha)\right]$ for each $\alpha \in [0, 1]$ then

$$[f'(x)]^\alpha = \left[\underline{f'}(x; \alpha), \overline{f'}(x; \alpha)\right].$$

Definition 2.7. $F : J \to E^n$ is strongly measurable if for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : J \to \mathcal{P}_c(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

A function $F : J \to E^n$ is called integrably bounded, if there exists an integrable function h such that, $|x| < h(t)$ for all $x \in F_0(t)$.

Definition 2.8. Let $F : J \to E^n$. The integral of F on J denoted by $\int_I F(t)dt$, is given by

$$\left[\int_J F(t)dt\right]^\alpha = \int_J F_\alpha(t)dt = \left\{\int_J f(t)dt \mid f : J \to \mathbb{R}^n \text{ is a measurable selection for } F_\alpha\right\},$$

for all $\alpha \in [0,1]$.

Proposition 2.3. *If $u \in E^1$, then the following properties hold.*

(a) $[u]^\beta \subset [u]^\alpha$ *if* $0 \leq \alpha \leq \beta$.

(b) *If* $\alpha_n \subset [0,1]$ *is a nondecreasing sequence which converges to* α, *then*

$$[u]^\alpha = \bigcap_{n\geq 1} [u]^{\alpha_n}.$$

Conversely, if $A^\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in [0,1]\}$ *is a family of closed real intervals verifying* (a) *and* (b), *then* A^α *defined a fuzzy number* $u \in E^1$ *such that* $[u]^\alpha = A^\alpha$.

2.1. **Fractional integral and fractional derivative of fuzzy function.** Let $q > 0$, the fractional integral of order q of a real function $g : [t_0, t_0 + \delta] \to \mathbb{R}$ is given by

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} g(s)ds.$$

Let $f(t) \in L(J, E^1)$ such that $f(t) = [f_1^\alpha(t), f_2^\alpha(t)]$. Suppose that $f_1^\alpha, f_2^\alpha \in L(J, \mathbb{R})$ for all $\alpha \in [0,1]$ and let

$$A^\alpha = \left[\frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f_1^\alpha(s)ds, \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f_2^\alpha(s)ds\right], \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

We have the following lemma.

Lemma 2.1 ([3])**.** *The family* $\{A^\alpha \mid \alpha \in [0,1]\}$ *given by* (2.1), *defined a fuzzy number* $u \in E^1$ *such that* $[u]^\alpha = A^\alpha$.

Definition 2.9 ([16])**.** Let $f(t) \in L(J, E^1)$. The fuzzy fractional integral of order $q \in [0,1]$ of f denoted by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s)ds$$

is defined by

$$[I^q f(t)]^\alpha = [I^\alpha f_l(t;\alpha), I^q f_r(t;\alpha)].$$

Proposition 2.4 ([16])**.** *Let* $f, g \in L(J, E^1)$ *and* $b \in E^1$, *then we have:*

(a) $I^q(bf)(t) = bI^q f(t)$;

(b) $I^q(f+g)(t) = I^q f(t) + I^q g(t)$;

(c) $I^{q_1} I^{q_2} f(t) = I^{q_1+q_2} f(t)$, *where* $(q_1, q_2) \in [0,1]^2$.

Example 2.2. Let $x : J \to E^1$ be a constant fuzzy function such that $x(t) = u \in E^1$. If $[u]^\alpha = [u^1_\alpha, u^2_\alpha]$, then

$$\begin{aligned}
[I^q x(t)]^\alpha &= \left[\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} u^1_\alpha(s) ds, \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} u^2_\alpha(s) ds\right],\\
[I^q x(t)]^\alpha &= \frac{t^q}{\Gamma(\alpha+1)}[u^1_\alpha, u^2_\alpha],\\
[I^q x(t)]^\alpha &= \frac{t^q}{\Gamma(\alpha+1)}[u]^\alpha.
\end{aligned}$$

Definition 2.10 ([16])**.** Let $f \in C(J, E^1) \cap L(J, E^1)$.

The function f is called fuzzy Caputo fractional differentiable of order $0 < q < 1$ at t if there exists an element ${}^cD^q f(t) \in E^1$ such that

$$ {}^cD^q f(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f'(s) ds.$$

Remark 2.3 ([16]). Since $[f(t)]^\alpha = [f_l(t;\alpha), f_r(t;\alpha)]$ for each $\alpha \in [0,1]$, then

$$[{}^cD^q f(t)]^\alpha = [{}^cD^q f_l(t;\alpha), {}^c D^q f_r(t;\alpha)],$$

where

$$\begin{aligned}
{}^cD^q f_l(t;\alpha) &:= \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f_l'(s,\alpha) ds,\\
{}^cD^q f_r(t;\alpha) &:= \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f_r'(s,\alpha) ds.
\end{aligned}$$

Example 2.3. Let $x : [t_0;, t_0 + \delta] \to E^1$ be a constant fuzzy function such that $x(t) = u \in E^1$. If $[u]^\alpha = [u^1_\alpha, u^2_\alpha]$, then

$$\begin{aligned}
[{}^cD^q x(t)]^\alpha &= \left[\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} (u^1_\alpha)' ds, \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} (u^2_\alpha)' ds\right],\\
[{}^cD^q x(t)]^\alpha &= \{0\},\\
{}^cD^q x(t) &= 0_{E^1}.
\end{aligned}$$

Theorem 2.1 ([5, 15])**.** *There exists a real Banach space X such that E^n can be embedded isometrically into a convex cone C with vertex 0 in X. moreover we have:*

(a) *addition in X induces addition in E^n;*
(b) *multiplication by real number in X induces the corresponding operation in E^n;*
(c) *$C - C$ is dense in X;*
(d) *C is closed.*

Remark 2.4. The structure of the normed space X can be described as follows.

Define in $E^n \times E^n$ the following equivalence relation:

$$(u, v)R(u', v') \Leftrightarrow u + v' = v + u'.$$

We denote by $\langle u, v\rangle$ the equivalence class of (u, v) and the space X will be the set of equivalence classes. We define a vector space structure in X by:

$$\langle u, v\rangle + \langle u, v\rangle \Leftrightarrow u + v' = v + u',$$
$$\lambda\langle u, v\rangle = \langle \lambda u, \lambda v\rangle, \quad \text{if } \lambda \geq 0,$$
$$\lambda\langle u, v\rangle = \langle (-\lambda)v, (-\lambda)u\rangle, \quad \text{if } \lambda < 0.$$

The isometry $j : E^n \to X$ is defined by

$$j(u) = \langle u, 0\rangle.$$

The norm in X is defined by $\|\langle u, v\rangle\|_X = D(u, v)$.

Theorem 2.2 ([10])**.** *Let X be a Banach space and j an embedding as in Theorem 2.1, $G : J \to E^n$ and assume that $j \circ G$ is Bochner integrable over J. Then we have*

1) *$I^q G(t) \in E^n$;*
2) *$j(I^q G(t)) = I^q j(G(t))$.*

3. The Fuzzy Fractional Initial Value Problem

Let $\tilde{C}$ be a closed subset of (E^n, D), which is also closed under the addition and multiplication by a nonnegative real number and $f : J \times \tilde{C} \to \tilde{C}$ be a fuzzy continuous function.

In this section we show that the initial value problem (1) has a solution if and only if $\tilde{C}$ is locally compact.

Definition 3.1 ([9])**.** A fuzzy function $x : J \to E^n$ is called d-increasing (d-decreasing) on J if for every $\alpha \in [0, 1]$ the real function $t \mapsto d([x(t)]^\alpha)$ is nondecreasing (nonincreasing), respectively.

Remark 3.1. If $x : J \to E^n$ is d-increasing or d-decreasing on J, then we say that $x(t)$ is d-monotone on J.

Lemma 3.1. *A d-monotone fuzzy function $x(t)$ is a solution of initial value problem* (1.1) *if and only if*

1) *x is continuous;*
2) *x satisfies the integral equation $x(t) \ominus_{gH} x_0 = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(t, x(t)) ds$;*
3) *The function $t \mapsto I^q f(t, x(t))$is d-increasing on J.*

Proof. See the proof of Theorem 3 in [9]. □

We denote by $C(J, \tilde{C})$ the space of all continuous mappings from J to $\tilde{C}$ and let j be an embedding of $\tilde{C}$ into a Banach space X allowed by Theorem 2.2.

Theorem 3.1. *The fuzzy fractional initial value problem* (1.1) *has a solution if and only if $\tilde{C}$ is locally compact.*

Proof. By Theorem 2.1 and Theorem 2.2 we can see that $x(t)$ is a solution of the problem (1.1) if and only if $j(x(t))$ is a continuous solution of the embedded equation

$$j(x(t)) \ominus_{gH} j(x_0) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} j(f(s, j^{-1}j(x(s)))). \tag{3.1}$$

Since $x(t) \in C(J, \tilde{C})$ then $j(f(s, x(s))$ is Bochner integrable.

It is known that the (3.1) has a solution if and only if X is a finite dimensional space. Since a normed space is finite dimensional if and only if it is locally compact (see[12]) and we have $X = cl\{j(\tilde{C}) - j(\tilde{C})\}$ then the proof is completed. □

4. Illustrative Example

Example 4.1. Let m be a positive real number, then the following set,

$$E^1_m = \{u \in E^1 \mid d(\text{supp}(u)) \leq m\}.$$

is a locally compact subset of E^1.

Indeed, for each $n = 1, 2, 3, \dots$ let $\tilde{K}_n = K_n \cap E^1_m$, where $K_n = \{u \in E^1 \mid \text{supp}(u) \subset [-n, n]\}$. Then since E^1_m is closed in E^1, $\tilde{K}_n$ is compact for each n. Let $u \in E^1_m$, then u belongs to the interior of $\tilde{K}_n$ for some n. Therefore, every element in E^1_m has a compact neighborhood, it follows that E^1_m is a locally compact.

5. Conclusion and Future Works

In this manuscript we established the existence results for fuzzy fractional differential equations by using Peano theorem. Our future work is to study the stability results for fuzzy fractional differential equations by using Mittag-Leffler stability notion.

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A COUPLED SYSTEM OF NONLINEAR LANGEVIN FRACTIONAL q-DIFFERENCE EQUATIONS ASSOCIATED WITH TWO DIFFERENT FRACTIONAL ORDERS IN BANACH SPACE

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Abstract. In this research article, we study the coupled system of nonlinear Langevin fractional q-difference equations associated with two different fractional orders in Banach Space. The existence, uniqueness, and stability in the sense of Ulam are established for the proposed system. Our approach is based on the technique of measure of noncompactness combined with Mönch fixed point theorem, the implementation Banach contraction principle fixed point theorem, and the employment of Urs's stability approach. Two examples illustrating the effectiveness of the theoretical results are presented.

1. Introduction

In understanding and developing a large class of systems, it is apparent that researchers and scientists have resorted to nature. Natural phenomena can be well understood both quantitatively and qualitatively. Mathematics plays a fundamental role in this respect because it is the science of patterns and relationships. Attempting to understand the quantitative and qualitative behavior of nature, mathematicians find out that evolution revolves from integer to fraction. Number theory, starting from integer and reaching to fractional as a result of division operation and eventually converging to real numbers, is well used to account for Quantitative behavior. Calculus which describes how things change offers a background for simulating structures undergoing change, and a means to infer the predictions of such structures. All these indicated that integer order calculus is a subcategory of fractional calculus

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which is defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer. New and many derivatives and fractional integrals theories have arisen since the end of the 17 century to the present day. The theory of derivation and fractional integration has long been regarded as a branch of mathematics without any real or practical explanation; it was considered as an abstract containing only little useful mathematical manipulations. During the past three decades, considerable interest was carried to fractional calculus by the application of these concepts in various fields of physics, engineering, biology, and mechanics, etc. in a much better form as compared to ordinary differential operators, which are local. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Hilfer [33], Kilbas et al. [36], Miller and Ross [39], Oldham [40], Pudlubny [41], Tarasov [45], Abbas et al. [1] and the references therein.

Fractional q-difference equations started toward the start of the nineteenth century [4, 30] and got big interested consideration lately and have attracted a large number of scientists and researchers [6, 14, 31]. Some fascinating insights concerning initial and boundary value problem of q-difference and Fractional q-difference equations can be found in [2, 7–11, 18, 24, 31] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [37]. Although the existing literature on solutions of fractional Langevin equations is quite wide (see, for example, [12, 13, 21, 46]). But, to the best of the author's knowledge, there is no literature to research the existence of weak solutions for fractional Langevin equations involving two fractional orders in Banach Spaces, so the research of this paper is new.

At the present day, there are numerous results on the existence and uniqueness of solutions for fractional differential equations. For greater details, the readers are cited the previous research [22, 23, 29, 36] and the references therein. However, due to the fact that in lots of conditions, which include nonlinear analysis and optimization, locating the exact solution of differential equations is almost tough or impossible, we don't forget approximate solutions. It is essential to observe that only stable approximate solutions are proper. various approaches of stability analysis are adopted for this reason. The HU-type stability concept has been taken into consideration in the severa literature. The said stability analysis is an clean and easy manner on this regard. This type idea of stability become formulated for the primary time by means of Ulam [47], and then the next year it become elaborated with the aid of Hyers [34, 48]. Impressive considerations have been provided to the investigation of the Ulam-Hyers (UH) stability of a wide range of FDEs, see [3, 16, 28, 43].

In this paper deals with the existence, uniqueness and Urs's stability of solutions for the following Langevin fractional q-difference system:

$$\begin{cases} \mathcal{D}_q^{\beta_i}\left(\mathcal{D}_q^{\alpha_i}+\lambda_i\right)\varpi(\varsigma)=f_i(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma)), \quad \varsigma\in J=[0,T], \\ \varpi_i(0)=0, \\ \varpi_i(T)+\lambda_i\mathcal{J}_q^{\alpha_i}\varpi_i(T)=0, \\ \mathcal{D}_q^{\alpha_i}\varpi_i(\xi_i)+\lambda_i\varpi_i(\xi_i)=0, \quad \xi_i\in\ ]\,0,T]\,, \end{cases} \tag{1.1}$$

where D_q^{ε} is the fractional q-derivative of the Reimann-Liouville type of order $\varepsilon\in\{\alpha_i,\beta_i\}$ such that $\alpha_i\in(0,1]$, $\beta_i\in(1,2]$ and $\mathcal{J}_q^{\alpha_i}$ is the fractional q-integral of the Reimann-Liouville type, $f_i: J\times\mathbb{E}^2\to\mathbb{E}$ are continuous functions, λ_i are real constants.

In this paper, we present existence results for the problem (1.1) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. That technique turns out to be a very useful tool in existence for several types of integral equations, details are found in Akhmerov et al. [15], Alvàrez [19], Banaȁ et al. [20], Benchohra et al. [22, 23], Boutiara et al. [25–27], Mönch [38], Szufla [44] and the references therein.

Here is a brief outline of the paper. The Section 2 provides the definitions and preliminary results that we will need to prove our main results and present an auxiliary lemma that provides solution representation for the solutions of system (1.1). In Section 3, we establish existence and uniqueness for stability in the sense of Ulam for system (1.1). In Section 4, we give some examples to illustrate the obtained results.

2. Preliminaries and Lemmas

We start this section by introducing some necessary definitions and basic results required for further developments.

In what follows, we recall some elementary definitions and properties related to fractional q-calculus. For $a\in\mathbb{R}$, we put

$$[a]_q=\frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is expressed by

$$(a-b)^{(0)}=1, \quad (a-b)^{(n)}=\prod_{k=0}^{n-1}\left(a-bq^k\right), \quad a,b\in\mathbb{R}, n\in\mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)}=a^\alpha\prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right), \quad a,b,\alpha\in\mathbb{R}.$$

Definition 2.1 ([35])**.** The q-gamma function is given by

$$\Gamma_q(\alpha)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha\in\mathbb{R}\setminus\{0,-1,-2,\dots\}.$$

The q-gamma function satisfies the classical recurrence relationship

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha).$$

Definition 2.2 ([35])**.** For any $\alpha, \beta > 0$, the q-beta function is defined by

$$B_q(\alpha, \beta) = \int_0^1 f^{(\alpha-1)}(1-qf)^{(\beta-1)} d_q f, \quad q \in (0,1),$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha, \beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Definition 2.3 ([35])**.** Let $f : J \to \mathbb{R}$ be a suitable function. We define the q-derivative of order $n \in \mathbb{N}$ of the function by $\mathcal{D}_q^0 f(\varsigma) = f(\varsigma)$,

$$\mathcal{D}_q f(\varsigma) := \mathcal{D}_q^1 f(\varsigma) = \frac{f(\varsigma) - f(q\varsigma)}{(1-q)\varsigma}, \quad \varsigma \neq 0, \quad \mathcal{D}_q f(0) = \lim_{\varsigma \to 0} \mathcal{D}_q f(\varsigma),$$

and

$$\mathcal{D}_q^n f(\varsigma) = \mathcal{D}_q \mathcal{D}_q^{n-1} f(\varsigma), \quad \varsigma \in \mathcal{J}, n \in \{1, 2, \ldots\}.$$

Set $\mathcal{J}_\varsigma := \{\varsigma q^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4 ([35])**.** For a given function $f : \mathcal{J}_\varsigma \to \mathbb{R}$, the expression defined by

$$\mathcal{J}_q f(\varsigma) = \int_0^\varsigma f(s)\, d_q s = \sum_{n=0}^{\infty} \varsigma(1-q) q^n f(tq^n),$$

is called q-integral, provided that the series converges.

We note that $\mathcal{D}_q \mathcal{J}_q f(\varsigma) = f(\varsigma)$, while if f is continuous at 0, then

$$\mathcal{J}_q \mathcal{D}_q f(\varsigma) = f(\varsigma) - f(0).$$

Definition 2.5 ([6])**.** The integral of a function $f : J \to \mathbb{R}$ defined by

$$\mathcal{J}_q^0 f(\varsigma) = f(\varsigma),$$

and

$$\mathcal{J}_q^\alpha f(\varsigma) = \int_0^\varsigma \frac{(\varsigma - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_q s, \quad \varsigma \in J,$$

is called Riemann-Liouville-fractional q-integral of order $\alpha \in \mathbb{R}_+$.

Lemma 2.1 ([42])**.** *Let $\alpha \in \mathbb{R}_+$ and $\beta \in (-1, \infty)$. One has*

$$\mathcal{J}_q^\alpha \varsigma^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} \varsigma^{\alpha+\beta}, \quad \beta \in (-1, \infty), \alpha \geq 0, \varsigma > 0.$$

In particular, if $f \equiv 1$, then

$$\mathcal{J}_q^\alpha 1(\varsigma) = \frac{1}{\Gamma_q(1+\alpha)} \varsigma^{(\alpha)}, \quad \textit{for all } \varsigma > 0.$$

Definition 2.6 ([14])**.** The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by $\mathcal{D}_q^0 f(\varsigma) = f(\varsigma)$ and

$$\mathcal{D}_q^\alpha f(\varsigma) = \mathcal{D}_q^{[\alpha]} \mathcal{I}_q^{[\alpha]-\alpha} f(\varsigma) = \frac{1}{\Gamma_q(n-\alpha)} \int_0^\varsigma \frac{f(s)}{(\varsigma - qs)^{\alpha-n+1}} d_q s,$$

where $[\alpha]$ is the integer part of α.

Lemma 2.2 ([32])**.** *Let $\alpha > 0$ and $n \in \mathbb{N}$ where $[\alpha]$ denotes the integer part of α. Then, the following fundamental identity holds*

$$\mathcal{I}_q^\alpha \mathcal{D}_q^n f(\varsigma) = \mathcal{D}_q^n \mathcal{I}_q^\alpha f(\varsigma) - \sum_{k=0}^{\alpha-1} \frac{\varsigma^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (\mathcal{D}_q^k h)(0).$$

Lemma 2.3 ([17])**.** *Let ϖ be a function defined on J and suppose that α, β are two real nonegative numbers. Then the following hold:*

$$\begin{aligned} \mathcal{I}_q^\alpha \mathcal{I}_q^\beta f(\varsigma) =& \mathcal{I}_q^{\alpha+\beta} f(\varsigma) = \mathcal{I}_q^\beta \mathcal{I}_q^\alpha f(\varsigma), \\ \mathcal{D}_q^\alpha \mathcal{I}_q^\alpha f(\varsigma) =& f(\varsigma). \end{aligned}$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.7 ([15, 20])**.** The mapping $\kappa : \mathfrak{M}_{\mathfrak{U}} \to [0, \infty)$ for Kuratowski measure of non-compactness is defined as:

$$\kappa(B) = \inf \Big\{ \varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \leq \varepsilon \Big\}.$$

Proposition 2.1. *The Kuratowski measure of noncompactness satisfies some properties* [15, 20]*:*

(a) $\mathcal{A} \subset \mathcal{B} \Rightarrow \kappa(\mathcal{A}) \leq \kappa(\mathcal{B})$;
(b) $\kappa(\mathcal{A}) = 0$ *if and only if* $\mathcal{A}$ *is relatively compact;*
(c) $\kappa(\mathcal{A}) = \kappa(\overline{\mathcal{A}}) = \kappa(\text{conv}(\mathcal{A}))$*, where* $\overline{\mathcal{A}}$ *and* $\text{conv}(\mathcal{A})$ *represent the closure and the convex hull of* $\mathcal{A}$*, respectively;*
(d) $\kappa(\mathcal{A} + \mathcal{B}) \leq \kappa(\mathcal{A}) + \kappa(\mathcal{B})$;
(e) $\kappa(\lambda \mathcal{A}) = |\lambda| \kappa(\mathcal{A}), \lambda \in \mathbb{R}$.

Definition 2.8. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $\varsigma \mapsto f(\varsigma, \varpi)$ is measurable for each $\varpi \in E$;
(ii) $\varpi \mapsto F(\varsigma, \varpi)$ is continuous for almost all $\varsigma \in J$.

Proposition 2.2. *For a given set V of functions $\omega : J \to E$, let us denote by*

$$V(\varsigma) = \{\omega(\varsigma) : \omega \in V\}, \quad \varsigma \in J,$$

and

$$V(J) = \{\omega(\varsigma) : \omega \in V, \varsigma \in J\}.$$

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.1 ([5,38,44]). *Let $\mathcal{D}$ be a bounded, closed and convex subset of a Banach space such that $0 \in \mathcal{D}$, and let N be a continuous mapping of $\mathcal{D}$ into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \kappa(V) = 0, \tag{2.1}$$

holds for every subset V of $\mathcal{D}$, then N has a fixed point.

Lemma 2.4 ([44]). *Let $\mathcal{D}$ be a bounded, closed and convex subset of the Banach space $\mathcal{U}$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $\varsigma \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \kappa(f(J_{\varsigma,h} \times B)) \leq p(\varsigma)\kappa(B),$$

where $J_{\varsigma,h} = [\varsigma - h, \varsigma] \cap J$.

If V is an equicontinuous subset of $\mathcal{D}$, then

$$\kappa\left(\left\{\int_J G(s,\varsigma) f(s,\varpi(s))ds : \varpi \in V\right\}\right) \leq \int_J \|G(\varsigma, s)\| p(s)\kappa(V(s))ds.$$

3. Main Results

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 3.1. *Let $\sigma_i \in \mathcal{C}$, $\alpha_i \in (0,1]$, $\beta_i \in (1,2]$, $i = 1,2$. Then the boundary value problem*

$$\begin{cases} \mathcal{D}_q^{\beta_i}\left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right)\varpi_i(\varsigma) = \sigma_i(\varsigma), \quad \varsigma \in (0,T), \\ \varpi_i(0) = 0, \\ \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i}\varpi_i(T) = 0, \\ \mathcal{D}_q^{\alpha_i}\varpi_i(\xi_i) + \lambda_i\varpi_i(\xi_i) = 0, \quad \xi_i \in \left]0, T\right], \end{cases} \tag{3.1}$$

has a unique solution defined by

$$\varpi_i(\varsigma) + \lambda_i \mathcal{I}_q^{\alpha_i}\varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(\varsigma) + \mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i) + \nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T), \quad i = 1,2, \tag{3.2}$$

where

$$\mu(\varsigma) = \frac{\Gamma_q(\beta-1)}{\Gamma_q(\beta+\alpha-1)}\left[\frac{(\beta-1)|\omega_4|\varsigma^{\alpha+\beta-1}}{(\beta+\alpha-1)|\Delta|} - \frac{|\omega_3|\varsigma^{\alpha+\beta-2}}{|\Delta|}\right] \tag{3.3}$$

and

$$\nu(\varsigma) = \frac{\Gamma_q(\beta-1)}{\Gamma_q(\beta+\alpha-1)}\left[\frac{|\omega_1|\varsigma^{\alpha+\beta-2}}{|\Delta|} - \frac{(\beta-1)|\omega_2|\varsigma^{\alpha+\beta-1}}{(\beta+\alpha-1)|\Delta|}\right], \tag{3.4}$$

with

$$\Delta = \omega_2\omega_3 - \omega_1\omega_4 \neq 0, \tag{3.5}$$

$$\omega_1 = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} T^{\beta+\alpha-1}, \quad \omega_3 = \xi^{\beta-1},$$
$$\omega_2 = \frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)} T^{\beta+\alpha-2}, \quad \omega_4 = \xi^{\beta-2}.$$

Proof. Applying the integrator operator $\mathcal{I}^\beta$ to (3.1) and using the Lemma 2.1 we get

$$(\mathcal{D}^\alpha + \lambda)\varpi(\varsigma) = c_1\varsigma^{\beta-1} + c_2\varsigma^{\beta-2} + \mathcal{I}^\beta\sigma(\varsigma), \quad \varsigma \in (0,T]. \tag{3.6}$$

We apply again the operator $\mathcal{I}^\alpha$ and use the results of Lemmas 2.1 to get the general solution representation of problem (3.1)

$$\varpi(\varsigma) = \mathcal{I}^{\alpha+\beta}\sigma(\varsigma) - \lambda\mathcal{I}^\alpha\varpi(\varsigma) + c_0\varsigma^{\alpha-1} + c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\varsigma^{\beta+\alpha-1} + c_2\frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)}\varsigma^{\beta+\alpha-2}, \tag{3.7}$$

where $c_0, c_1, c_2 \in \mathbb{R}$. By using the boundary conditions in problem (3.1) and the above equation, we observe that $c_0 = 0$ and

$$c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}T^{\beta+\alpha-1} + c_2\frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)}T^{\beta+\alpha-2} + \mathcal{I}^{\alpha+\beta}\sigma(T) = 0. \tag{3.8}$$

Moreover, we obtain

$$c_1\xi^{\beta-1} + c_2\xi^{\beta-2} + \mathcal{I}^\beta\sigma(\xi) = 0. \tag{3.9}$$

Also, by using (3.5), (3.8) and (3.9) can be written as

$$c_1\omega_1 + c_2\omega_2 = 0,$$
$$c_1\omega_3 + c_2\omega_4 = 0.$$

Solving the last two in c_1 and c_2, we end up with

$$c_1 = \frac{\omega_4}{\Delta}\mathcal{I}^{\alpha+\beta}\sigma(T) - \frac{\omega_4}{\Delta}\mathcal{I}^\beta\sigma(\xi),$$
$$c_2 = \frac{\omega_1}{\Delta}\mathcal{I}^\beta\sigma(\xi) - \frac{\omega_3}{\Delta}\mathcal{I}^{\alpha+\beta}\sigma(T).$$

Substituting c_1 and c_2 in (3.7), we get the desired solution representation (3.2). Besides and by the help of the results in Lemmas 2.1 one can easily figure out that (3.2) solves problem (3.1). This finishes the proof. □

We will need the following properties for the functions μ and ν defined in next lemma.

Lemma 3.2. *The functions μ and ν are continuous functions on J and satisfy the following properties:*

(1) $\mu_{\max,i} = \max_{0\le\varsigma\le T} |\mu_i(\varsigma)|$;
(2) $\nu_{\max,i} = \max_{0<\varsigma<T} |\omega(\varsigma)|$;
(3) $\overline{\mu}_{\max,i} = \max_{0\le\varsigma\le T} |\mu_i'(\varsigma)|$;
(4) $\overline{\nu}_{\max,i} = \max_{0<\varsigma<T} |\nu_i'(\varsigma)|$.

3.1. **Existance result.** In the following subsections, we establish the existence of solutions for the (1.1) by applying Mönch fixed point theorems.

Consider the space of real and continuous functions $\mathcal{U} = C(J, \mathbb{E})$ space with the norm

$$\|\varpi\|_\infty = \sup\{\|\varpi(\varsigma)\| : \varsigma \in J\}.$$

Then the product space $\mathcal{C} := \mathcal{U} \times \mathcal{V}$ defined by $\mathcal{C} = \{(\varpi, \omega) : \varpi \in \mathcal{U}, \omega \in \mathcal{V}\}$ is Banach space under the norm

$$\|(\varpi, \omega)\|_{\mathcal{C}} = \|\varpi\|_\infty + \|\omega\|_\infty,$$

and $\mathfrak{M}_{\mathcal{U}}$ represents the class of all bounded mappings in $\mathcal{U}$.

Let $L^1(J, \mathbb{E})$ be the Banach space of measurable functions $\varpi : J \to \mathbb{E}$ which are Bochner integrable, equipped with the norm

$$\|\varpi\|_{L^1} = \int_J |\varpi(\varsigma)| \, d\varsigma.$$

In what follows, we are concerned with the existence of solutions of (1.1).

Definition 3.1. By a solution of the coupled system (1.1) we mean a coupled measurable functions $(\varpi_1, \varpi_2) \in \mathcal{C}$ such that $\varpi_i(0) = 0, \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(T) = 0$ and $\mathcal{D}_q^{\alpha_i} \varpi_i(\xi_i) + \lambda_i \varpi_i(\xi_i) = 0$, $i = 1, 2$, and the equations $\mathcal{D}_q^{\beta_i} \left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right) \varpi_i(\varsigma) = f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma))$ are satisfied on J.

In what follows, we present the solution representation associated with System (1.1).

Lemma 3.3. *Let $\sigma_i \in \mathcal{U}$, $i = 1, 2$, be two given functions. Then, the following system of fractional differential equations*

$$\begin{cases} \mathcal{D}_q^{\beta_i} \left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right) \varpi_i(\varsigma) = \sigma_i(\varsigma), \quad \varsigma \in (0, T), \\ \varpi_i(1) = 0, \\ \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(T) = 0, \\ \mathcal{D}_q^{\alpha_i} \varpi_i(\xi_i) + \lambda_i \varpi_i(\xi_i) = 0, \quad \xi_i \in \,]0, T], \end{cases} \tag{3.10}$$

is equivalent to the integral equation

$$\varpi_i(\varsigma) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(\varsigma) + \mu_i(\varsigma) \mathcal{I}_q^{\beta_i} \sigma_i(\xi_i) + \nu_i(\varsigma) \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(T), \quad i = 1, 2. \tag{3.11}$$

Lemma 3.4. *Assume that $f_i : J \times \mathbb{E}^2 \to \mathbb{E}$ is continuous. A function $\varpi(\varsigma)$ solves the system* (1.1) *if and only if it is a fixed-point of the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ defined by*

$$\mathcal{G}_i \varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(\varsigma) - \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(\varsigma) + \mu_i(\varsigma) \mathcal{I}_q^{\beta_i} \sigma_i(\xi_i) + \nu_i(\varsigma) \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(T), \quad i = 1, 2. \tag{3.12}$$

3.1.1. *Existance result via Mönch fixed point theorem.* We further will use the following hypotheses.

(A1) For any $i = 1, 2$, $f_i : J \times \mathbb{E}^2 \to \mathbb{E}$ satisfies the Caratheodory conditions.

(A2) There exists $p_i, q_i \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$, such that

$$\|f(\varsigma, \varpi_1, \varpi_2)\| \leq p_i(\varsigma)\|\varpi_1\| + q_i(\varsigma)\|\varpi_2\|, \quad \text{for } \varsigma \in J \text{ and each } \varpi_i \in \mathbb{E}, i = 1, 2.$$

(A3) For any $\varsigma \in J$ and each bounded measurable sets $B_i \subset \mathbb{E}$, i=1,2, we have

$$\lim_{h \to 0^+} \kappa(f(J_{\varsigma,h} \times B_1, B_2), 0) \leq p_1(\varsigma)\kappa(B_1) + q_1(\varsigma)\kappa(B_2)$$

and

$$\lim_{h \to 0^+} \kappa(0, f(J_{\varsigma,h} \times B_1, B_2)) \leq p_2(\varsigma)\kappa(B_1) + q_2(\varsigma)\kappa(B_2),$$

where κ is the Kuratowski measure of compactness and $J_{\varsigma,h} = [\varsigma - h, \varsigma] \cap J$.

Set

$$p_i^* = \sup_{\varsigma \in J} p_i(\varsigma), \quad q_i^* = \sup_{\varsigma \in J} q_i(\varsigma), \quad i = 1, 2.$$

Theorem 3.1. *Assume that conditions* (A1)-(A3) *hold. If*

$$\Lambda < 1, \tag{3.13}$$

with

$$\Lambda := \sum_{i=1}^{2} \left(M_i(p_i^* + q_i^*) + N_i \right),$$

where

$$M_i = \left\{ \frac{(1 + \nu_{\max,i})\, T^{\alpha_i + \beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} + \frac{(\mu_{\max,i})\, \xi_i^{\beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} \right\}, \quad N_i = \frac{|\lambda_i|\, T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)}, \quad i = 1, 2,$$

then (1.1) *has at least one solution on* J.

Proof. We consider the operators $\mathcal{G}_i : \mathcal{C} \to \mathcal{C}$ defined by

$$\mathcal{G}\varpi = \mathcal{G}(\varpi_1, \varpi_2) = (\mathcal{G}_1\varpi_1, \mathcal{G}_2\varpi_2),$$

where the operators $\mathcal{G}_i$, $i = 1, 2$ are given by the formula (3.12). Clearly, the fixed points of the operators $\mathcal{G}_i$ are solutions of the system (1.1). Let we take

$$\mathcal{D}_r = \{\varpi_i \in \mathcal{C}, i = 1, 2 : \|(\varpi_1, \varpi_2)\| \leq r\}.$$

Clearly, the subset $\mathcal{D}_r$ is closed, bounded and convex. We shall show that $\mathcal{G}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step 1. First we show that $\mathcal{G}$ is sequentially continuous.

Let $\{\varpi_{1,n}, \varpi_{2,n}\}_n$ be a sequence such that $(\varpi_{1,n}, \varpi_{2,n}) \to (\varpi_1, \varpi_2)$ in $\mathcal{C}$. Then for any $\varsigma \in J$

$$\begin{aligned}\|\left(\mathcal{G}\varpi_{i,n} - \mathcal{G}\varpi_i\right)(\varsigma)\| \leq & \mathcal{I}_q^{\alpha_i + \beta_i}\|f_{i,n}(s, \varpi_{1,n}(s), \varpi_{2,n}(s)) - f_i(s, \varpi_1(s), \varpi_2(s))\|(\varsigma) \\ & - \lambda_i \mathcal{I}_q^{\alpha_i} \|\varpi_{i,n} - \varpi_i\| (\varsigma) \\ & + \mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\|f_{i,n}(s, \varpi_{1,n}(s), \varpi_{2,n}(s)) - f_i(s, \varpi_1(s), \varpi_2(s))\|(\xi_i)\end{aligned}$$

$$
\begin{aligned}
&+\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\|f_{i,n}(s,\varpi_{1,n}(s),\varpi_{2,n}(s))-f_i(s,\varpi_1(s),\varpi_2(s))\|(T)\\
\leq &\left\{\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma)+\mu_i(\varsigma)\mathcal{I}_q^{\beta_i}(1)(\xi_i)+\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\right\}\\
&\times\|f_{i,n}(s,\varpi_{1,n}(s),\varpi_{2,n}(s))-f_i(s,\varpi_1(s),\varpi_2(s))\|\\
&+\lambda_i\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\left\|\varpi_{i,n}-\varpi_i\right\|,\quad i=1,2.
\end{aligned}
$$

Since, for any $i=1,2$, the function f_i satisfies assumptions (A1), then we have $f_i(\varsigma,\varpi_{1,n}(\varsigma),\varpi_{2,n}(\varsigma))$ converges uniformly to $f_i(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma))$. Hence, the Lebesgue dominated convergence theorem implies that $(\mathcal{G}(\varpi_{1,n},\varpi_{2,n}))(\varsigma)$ converges uniformly to $(\mathcal{G}(\varpi_1,\varpi_{2,}))(\varsigma)$. Thus, $(\mathcal{G}(\varpi_{1,n},\varpi_{2,n}))\to(\mathcal{G}(\varpi_1,\varpi_{2,}))$. Hence, $\mathcal{G}:\mathcal{D}_r\to\mathcal{D}_r$ is sequentially continuous.

Step 2. Second we show that $\mathcal{G}$ maps $\mathcal{D}_r$ into itself.

Take $\varpi_i\in\mathcal{D}_r$, $i=1,2$, by (A2), we have, for each $\varsigma\in J$ and assume that $(\mathcal{G}(\varpi_i))(\varsigma)\neq 0$, $i=1,2$,

$$
\begin{aligned}
|\mathcal{G}_iu_i(\varsigma)|\leq&\left|\mathcal{I}_q^{\alpha_i+\beta_i}f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\varsigma)\right|+\left|\lambda_i\mathcal{I}_q^{\alpha_i}\varpi_i(s)(\varsigma)\right|\\
&+\left|\mu_i(\varsigma)\mathcal{I}_q^{\beta_i}f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\xi_i)\right|+\left|\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(T)\right|\\
\leq&(p_i^*+q_i^*)r\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma)+r\left|\lambda_i\right|\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\\
&+(p_i^*+q_i^*)r\mu_{\max,i}\mathcal{I}_q^{\beta_i}(1)(\xi_i)+(p_i^*+q_i^*)r\nu_{\max,i}\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\\
\leq&(p_i^*+q_i^*)r\left\{\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma)+\mu_{\max,i}\mathcal{I}_q^{\beta_i}(1)(\xi_i)+\nu_{\max,i}\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\right\}\\
&+r\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\left|\lambda_i\right|\\
\leq&(p_i^*+q_i^*)r\left\{\frac{(1+\nu_{\max,i})\,T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)}+\frac{(\mu_{\max,i})\,\xi_i^{\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)}\right\}+\frac{r\left|\lambda_i\right|T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\\
&=r(M_i(p_i^*+q_i^*)+N_i),\quad i=1,2.
\end{aligned}
$$

Hence we get

$$
\|(\mathcal{G}(\varpi_1,\varpi_2))\|_{\mathcal{C}}\leq\sum_{i=1}^{2}r\left(M_i(p_i^*+q_i^*)+N_i\right)\leq r.
$$

Step 3. We show that $\mathcal{G}(\mathcal{D}_r)$ is equicontinuous.

By Step 2, it is obvious that $\mathcal{G}(\mathcal{D}_r)\subset C(J,\mathbb{E})$ is bounded. For the equicontinuity of $\mathcal{G}(\mathcal{D}_r)$, let $\varsigma_1,\varsigma_2\in J$, $\varsigma_1<\varsigma_2$ and $\varpi\in\mathcal{D}_r$, so $\mathcal{G}\varpi(\varsigma_2)-\mathcal{G}\varpi(\varsigma_1)\neq 0$. Then

$$
\begin{aligned}
\|\mathcal{G}\varpi_i(\varsigma_2)-\mathcal{G}\varpi_i(\varsigma_1)\|\leq&\mathcal{I}_q^{\alpha_i+\beta_i}\left|f(s,\varpi_1(s),\varpi_2(s))(\varsigma_2)-f(s,\varpi_1(s),\varpi_2(s))(\varsigma_1)\right|\\
&+\left|\lambda_i\right|\mathcal{I}_q^{\alpha_i}\left|\varpi_i(s)(\varsigma_2)-\varpi_i(s)(\varsigma_1)\right|\\
&+\left|\mu_i(\varsigma_2)-\mu_i(\varsigma_1)\right|\mathcal{I}_q^{\beta_i}f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\xi_i)\\
&+\left|\nu_i(\varsigma_2)-\nu_i(\varsigma_1)\right|\mathcal{I}_q^{\alpha_i+\beta_i}f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(T),\\
\leq&(p_i^*+q_i^*)r\left|\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_2)-\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_1)\right|\\
&+r\left|\lambda_i\right|\left|\mathcal{I}_q^{\alpha_i}(1)(\varsigma_2)-\mathcal{I}_q^{\alpha_i}(1)(\varsigma_1)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ (p_i^* + q_i^*)r \left|\mu_i(\varsigma_2) - \mu_i(\varsigma_1)\right| \left|\mathcal{I}_q^{\beta_i}(1)(\varsigma_2) - \mathcal{I}_q^{\beta_i}(1)(\varsigma_1)\right| (\xi_i) \\
&+ (p_i^* + q_i^*)r \left|\nu_i(\varsigma_2) - \nu_i(\varsigma_1)\right| \left|\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_2) - \mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_1)\right| (T) \\
\leq & \frac{(p_i^* + q_i^*)r}{\Gamma_q(\alpha_i + \beta_i + 1)} \left\{(\varsigma_2^{\alpha_i+\beta_i} - \varsigma_1^{\alpha_i+\beta_i}) + 2(\varsigma_2 - \varsigma_1)^{\alpha_i+\beta_i}\right\} \\
&+ \frac{r\left|\lambda_i\right|}{\Gamma_q(\alpha_i + 1)} \left\{(\varsigma_2^{\alpha_i} - \varsigma_1^{\alpha_i}) + 2(\varsigma_2 - \varsigma_1)^{\alpha_i}\right\} + \frac{(p_i^* + q_i^*)r\xi_i^{\beta_i}}{\Gamma_q\left(\beta_i + 1\right)} \\
&\times \left|\mu_i(\varsigma_2) - \mu_i(\varsigma_1)\right| + \frac{(p_i^* + q_i^*)RT^{\alpha_i+\beta_i}}{\Gamma\left(\alpha_i + \beta_i + 1\right)} \left|\nu_i(\varsigma_2) - \nu_i(\varsigma_1)\right|.
\end{aligned}
$$

As $\varsigma_1 \to \varsigma_2$, the right hand side of the above inequality tends to zero. This means that $\mathcal{G}(\mathcal{D}_r) \subset \mathcal{D}_r$.

Finally we show that the implication (2.1) holds. Let $V \subset \mathcal{D}_r$ such that $V = \overline{conv}(\mathcal{G}(V) \cup \{(0,0)\})$. Since V is bounded and equicontinuous, and therefore the function $\omega \mapsto \omega(\varsigma) = \kappa(V(\varsigma))$ is continuous on J. By hypothesis (A2), and the properties of the measure κ, for any $\varsigma \in J$, we get

$$
\begin{aligned}
\omega(\varsigma) \leq & \kappa(\mathcal{G}(V)(\varsigma) \cup \{(0,0)\})) \leq \kappa((\mathcal{G}V)(\varsigma)) \\
\leq & \kappa\left(\left\{\left(\left(\mathcal{G}_1\omega_1\right)(\varsigma), \left(\mathcal{G}_2\omega_2\right)(\varsigma) : (\omega_1, \omega_2) \in V\right\}\right)\right. \\
\leq & \mathcal{I}_q^{\alpha_1+\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s, \omega_1(s), \omega_2(s)\right)(\varsigma)\right); 0\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
&+ \left|\lambda_1\right| \mathcal{I}_q^{\alpha_1}\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right) \\
&+ \left|\mu_1\right|(\varsigma)\mathcal{I}_q^{\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s, \omega_1(s), \omega_2(s)\right)(\varsigma)\right); 0\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
&+ \left|\nu_1\right|(\varsigma)\mathcal{I}_q^{\alpha_1+\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s, \omega_1(s), \omega_2(s)\right)(\varsigma)\right); 0\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
&+ \mathcal{I}_q^{\alpha_2+\beta_2}\kappa\left(\left\{\left(0, f_2\left(s, \omega_1(s), \omega_2(s)\right)\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
&+ \left|\lambda_2\right| \mathcal{I}_q^{\alpha_2}\kappa\left(\left\{(0, \omega_2(s)) : (0, \omega_2) \in V\right\}\right) \\
&+ \left|\mu_2\right|(\varsigma)\mathcal{I}_q^{\beta_2}\kappa\left(\left\{\left(0, f_2\left(s, \omega_1(s), \omega_2(s)\right)\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
&+ \left|\nu_2\right|(\varsigma)\mathcal{I}_q^{\alpha_2+\beta_2}\kappa\left(\left\{\left(0, f_2\left(s, \omega_1(s), \omega_2(s)\right)\right) : (\omega_1, \omega_2) \in V\right\}\right) \\
\leq & \mathcal{I}_q^{\alpha_1+\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right)\right. \\
&\left. + q_1(s)\kappa\left(\left\{(0, \omega_2(s)) : (0, \omega_2) \in V\right\}\right)\right] \\
&+ \left|\lambda_1\right| \mathcal{I}_q^{\alpha_1}\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right) \\
&+ \left|\mu_1\right|(\varsigma)\mathcal{I}_q^{\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right)\right. \\
&\left. + q_1(s)\kappa\left(\left\{(0, \omega_2(s)) : (0, \omega_2) \in V\right\}\right)\right] \\
&+ \left|\nu_1\right|(\varsigma)\mathcal{I}_q^{\alpha_1+\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right)\right. \\
&\left. + q_1(s)\kappa\left(\left\{(0, \omega_2(s)) : (0, \omega_2) \in V\right\}\right)\right] \\
&+ \mathcal{I}_q^{\alpha_2+\beta_2}\left[p_2(s)\kappa\left(\left\{(\omega_1(s), 0) : (\omega_1, 0) \in V\right\}\right)\right. \\
&\left. + q_2(s)\kappa\left(\left\{(0, \omega_2(s)) : (0, \omega_2) \in V\right\}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ |\lambda_2| \mathcal{I}_q^{\alpha_2} \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right) \\
&+ |\mu_2| (\varsigma) \mathcal{I}_q^{\beta_2} \left[p_2(s) \kappa \left(\{(\omega_1(s), 0) : (\omega_1, 0) \in V\}\right)\right. \\
&\left.+ q_2(s) \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right)\right] \\
&+ |\nu_2| (\varsigma) \\
&\times \mathcal{I}_q^{\alpha_2+\beta_2} \left[p_2(s) \kappa \left(\{(\omega_1(s), 0) : (\omega_1, 0) \in V\}\right)\right. \\
&\left.+ q_2(s) \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right)\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mu \left(V(\varsigma)\right) \leq & \mathcal{I}_q^{\alpha_1+\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + |\lambda_1| \mathcal{I}_q^{\alpha_1} \left((1)(s)\right) \times \kappa \left(V(s)\right) \\
& + |\mu_1| (\varsigma) \mathcal{I}_q^{\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + |\nu_1| (\varsigma) \mathcal{I}_q^{\alpha_1+\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + \mathcal{I}_q^{\alpha_2+\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right) \\
& + |\lambda_2| \mathcal{I}_q^{\alpha_2} \left((1)(s)\right) \times \kappa \left(V(s)\right) \\
& + |\mu_2| (\varsigma) \mathcal{I}_q^{\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right) \\
& + |\nu_2| (\varsigma) \mathcal{I}_q^{\alpha_2+\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mu \left(V(\varsigma)\right) \leq \sum_{i=1}^{n=2} & \left(\left\{ \frac{(p_i^* + q_i^*) T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} \left(1 + \nu_{\max,i}\right) + \frac{(p_i^* + q_i^*) T^{\alpha_i+1}}{\Gamma_q(\alpha_i + 1)} |\mu_{\max,i}| \right\} \right. \\
& \left. + \left\{ \frac{|\lambda_i| \xi_i^{\beta_i}}{\Gamma_q(\beta_i + 1)} \right\} \right) \sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right).
\end{aligned}
$$

This means that

$$
\sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right) \leq \Lambda \sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right).
$$

By (3.13) it follows that $\sup_{\varsigma \in J} \kappa((V(\varsigma)) = 0$, that is $\kappa(V(\varsigma)) = 0$ for each $\varsigma \in J$, and then $V(\varsigma)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in $\mathcal{D}_r$. Applying now Theorem 2.4, we conclude that $\mathcal{G}$ has a fixed point, which is a solution of (1.1). □

3.2. **Uniqueness Result.** Let $X = \{\varpi : \varpi \in C'(J)\}$ be the Banach space of functions whose first derivatives are continuous on J, endowed with the $\|\varpi\|_X = \|\varpi\| + \|\varpi'\| = \max_{\varsigma \in J} |\varpi(\varsigma)| + \max_{\varsigma \in J} |\varpi'(\varsigma)|$. Obviously, the product space $(X \times X, \| \cdot \|_X)$ is also a Banach space with the norm $\|(\varpi_1, \varpi_2)\|_{X \times X} = \|\varpi_1\|_X + \|\varpi_2\|_X$. A closed ball with radius R centered on the zero function in $X \times X$ is defined by $B_R(0,0) = B_R = \left\{(\varpi_1, \varpi_2) \in X \times X : \|(\varpi_1, \varpi_2)\|_{X \times X} \leq R\right\}$ Define the operator $\mathcal{G} : X \times X \to X \times X$

by

$$\mathcal{G}(\varpi_1,\varpi_2)(\varsigma)=\begin{pmatrix}\mathcal{G}_1(\varpi_1,\varpi_2)(\varsigma)\\ \mathcal{G}_2(\varpi_1,\varpi_2)(\varsigma)\end{pmatrix},\quad \varsigma\in J,$$

where

$$\mathcal{G}_i\varpi_i(\varsigma)=\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(\varsigma)-\lambda_i\mathcal{I}_q^{\alpha_i}\varpi_i(\varsigma)+\mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i)+\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T).$$

Clearly, (ϖ_1,ϖ_2) is a fixed point of $\mathcal{G}$ if and only if (ϖ_1,ϖ_2) is a solution of system (1.1). Furthermore, we have

$$\mathcal{G}_i'\varpi_i(\varsigma)=\mathcal{I}_q^{\alpha_i+\beta_i-1}\sigma_i(\varsigma)-\lambda_i\mathcal{I}_q^{\alpha_i-1}\varpi_i(\varsigma)+\mu_i'(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i)+\nu_i'(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T).$$

Throughout the remaining part of the paper, we make use of the following assumptions and notations:

(H1) $f_1,f_2:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous;

(H2) there exist constants L_i and K_i such that

$$|f_i(\varsigma,\varpi_1,\varpi_2)-f_i(\varsigma,\omega_1,\omega_2)|\le L_i|\varpi_1-\omega_1|+K_i|\varpi_2-\omega_2|,$$

for all $(\varsigma,\varpi_1,\varpi_2),(\varsigma,\omega_1,\omega_2)\in[0,T]\times\mathbb{R}^2$;

(H3) $A_i=\max_{0\le\varsigma\le T}|f_i(\varsigma,0,0)|$.

Further, we use the following notations:

$$\begin{aligned}
\Theta_i&=\left[\frac{T^{\alpha_i+\beta_i}(1+\nu_{\max,i})}{\Gamma_q(\alpha_i+\beta_i+1)}+\frac{\xi_i^{\beta_i}\mu_{\max,i}}{\Gamma_q(\beta_i+1)}\right],\\
\Omega_i&=\frac{|\lambda_i|T_i^{\alpha_i}}{\Gamma_q(\alpha_i+1)},\\
\overline{\Theta}_i&=\left[\frac{T^{\alpha_i+\beta_i-1}}{\Gamma_q(\alpha_i+\beta_i)}+\frac{\overline{\nu}_{\max,i}T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)}+\frac{\overline{\mu}_{\max,i}\xi_i^{\beta_i}}{\Gamma_q(\beta_i+1)}\right],\\
\overline{\Omega}_i&=\frac{|\lambda_i|T_i^{\alpha_i-1}}{\Gamma_q(\alpha_i)},\\
L&=\sum_{i=1}^{2}\left[(L_i+K_i)\left(\Theta_i+\overline{\Theta}_i\right)+\left(\Omega_i+\overline{\Omega}_i\right)\right],\\
A&=\sum_{i=1}^{2}A_i\left(\Theta_i+\overline{\Theta}_i\right).
\end{aligned}$$

To this end, we also use this assumption:

(H4) $(L+A)\le 1$.

3.2.1. *Uniqueness via Banach fixed point theorem.*

Theorem 3.2. *Assume* (H1)-(H4) *holds. Then,* (1.1) *has a unique solution* $(\varpi_1,\varpi_2)\in B_R$.

Proof. Clearly, $\mathcal{G} : B_R \to X \times X$. First, we show that $\mathcal{G}$ is a contraction mapping. To see this, let $(\varpi_1, \varpi_2), (\omega_1, \omega_2) \in B_R$ $\varsigma \in J$, and consider

$$\begin{aligned}
&|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)| \\
\leq &\mathcal{J}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (\varsigma) + |\lambda_i| \mathcal{J}_q^{\alpha_i} |\varpi_i - \omega_i| (\varsigma) \\
&+ \mu_{\max,i} \mathcal{J}_q^{\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (\xi_i) \\
&+ \nu_{\max,i} \mathcal{J}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (T) \\
\leq &\left[L_i \mathcal{J}_q^{\alpha_i+\beta_i} \|\varpi_1 - \omega_1\| (\varsigma) + K_i \mathcal{J}_q^{\alpha_i+\beta_i} \|\varpi_2 - \omega_2\| (\varsigma) \right] + |\lambda_i| \mathcal{J}_q^{\alpha_i} |\varpi_i - \omega_i| (\varsigma) \\
&+ \mu_{\max,i} \left[L_i \mathcal{J}_q^{\beta_i} \|\varpi_1 - \omega_1\| (\xi_i) + K_i \mathcal{J}_q^{\beta_i} \|\varpi_2 - \omega_2\| (\xi_i) \right] \\
&+ \nu_{\max,i} \left[L_i \mathcal{J}_q^{\beta_i} \|\varpi_1 - \omega_1\| (T) + K_i \mathcal{J}_q^{\beta_i} \|\varpi_2 - \omega_2\| (T) \right] \\
\leq &\left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} \right] + |\lambda_i| \frac{\|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + 1)} T_i^{\alpha_i} \\
&+ \mu_{\max,i} \left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\beta_i + 1)} \xi_i^{\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\beta_i + 1)} \xi_i^{\beta_i} \right] \\
&+ \nu_{\max,i} \left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} \right] \\
= &(L_i + K_i) \left[\frac{T^{\alpha_i+\beta_i} (1 + \nu_{\max,i})}{\Gamma_q(\alpha_i + \beta_i + 1)} + \frac{\xi_i^{\beta_i} \mu_{\max,i}}{\Gamma_q(\beta_i + 1)} \right] [|\varpi_1 - \omega_1| + |\varpi_2 - \omega_2|] \\
&+ \frac{|\lambda_i| T_i^{\alpha_i}}{\Gamma_q(\alpha_i + 1)} \|\varpi_1 - \omega_1\| \\
= &[(L_i + K_i) \Theta_i + \Omega_i] |\varpi_1 - \omega_1| + \Theta_i |\varpi_2 - \omega_2|,
\end{aligned}$$

implying that

$$\|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)\| \leq [(L_i + K_i) \Theta_i + \Omega_i] \|\varpi_1 - \omega_1\|_X + \Theta_i \|\varpi_2 - \omega_2\|_X. \tag{3.14}$$

Likewise, and by using the precedent technique, we have

$$\left\|\mathcal{G}_i' u_i(\varsigma) - \mathcal{G}_i' v_i(\varsigma)\right\| \leq \left[(L_i + K_i) \overline{\Theta}_i + \overline{\Omega}_i\right] \|\varpi_1 - \omega_1\|_X + \overline{\Theta}_i \|\varpi_2 - \omega_2\|_X. \tag{3.15}$$

Then, from (3.14) and (3.15), we have

$$\begin{aligned}
\|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)\| \leq &\left[(L_i + K_i) \left(\Theta_i + \overline{\Theta}_i\right) + \left(\Omega_i + \overline{\Omega}_i\right)\right] \|\varpi_1 - \omega_1\|_X \\
&+ \left(\Theta_i + \overline{\Theta}_i\right) \|\varpi_2 - \omega_2\|_X.
\end{aligned} \tag{3.16}$$

Consequently,

$$\|\mathcal{G}(\varpi_1, \varpi_2) - \mathcal{G}(\omega_1, \omega_2)\|_{X \times X} \leq L \|(\varpi_1, \varpi_2) - (\omega_1, \omega_2)\|_{X \times X}.$$

Because $L < 1$, $\mathcal{G}$ is a contraction mapping with contraction constant L.

Next, we show that

$$\mathcal{G}(\partial B_R) \subseteq B_R. \tag{3.17}$$

To see this, let $(\varpi_1, \varpi_2) \in \partial B_R, \varsigma \in J$, and consider

$$
\begin{aligned}
|\mathcal{G}_i u_i(\varsigma)| \leq & \mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) \right| (\varsigma) + |\lambda_i| \, \mathcal{I}_q^{\alpha_i} \left| \varpi_i \right| (\varsigma) \\
& + \mu_{\max,i} \mathcal{I}_q^{\beta_i} \left| f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) \right| (\xi_i) + \nu_{\max,i} \mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) \right| (T) \\
\leq & \left[\mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i\left(s, \varpi_1(s), \varpi_2(s)\right) - f_i(s,0,0) \right| (\varsigma) + \mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i(s,0,0) \right| (\varsigma) \right] \\
& + |\lambda_i| \left[\mathcal{I}_q^{\alpha_i} \left| \varpi_i \right| (\varsigma) \right] \\
& + \mu_{\max,i} \left[\mathcal{I}_q^{\beta_i} \left| f_i\left(s, \varpi_1(s), \varpi_2(s)\right) - f_i(s,0,0) \right| (\xi_i) + \mathcal{I}_q^{\beta_i} \left| f_i(s,0,0) \right| (\xi_i) \right] \\
& + \nu_{\max,i} \Big[\mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i\left(s, \varpi_1(s), \varpi_2(s)\right) - f_i(s,0,0) \right| (T) \\
& + \mathcal{I}_q^{\alpha_i+\beta_i} \left| f_i(s,0,0) \right| (T) \Big] \\
\leq & \left[L_i \mathcal{I}_q^{\alpha_i+\beta_i} \left| \varpi_1 \right| (\varsigma) + K_i \mathcal{I}_q^{\alpha_i+\beta_i} \left| \varpi_2 \right| (\varsigma) + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)} \right] + \left[\frac{|\lambda_i| \, T^{\alpha_i} R}{\Gamma_q(\alpha_i+1)} \right] \\
& + \mu_{\max,i} \left[L_i \mathcal{I}_q^{\beta_i} \left| \varpi_1 \right| (\xi_i) + K_i \mathcal{I}_q^{\beta_i} \left| \varpi_2 \right| (\xi_i) + \frac{\xi_i^{\beta_i} A_i}{\Gamma_q(\beta_i+1)} \right] \\
& + \nu_{\max,i} \left[L_i \mathcal{I}_q^{\alpha_i+\beta_i} \left| \varpi_1 \right| (\varsigma) + K_i \mathcal{I}_q^{\alpha_i+\beta_i} \left| \varpi_2 \right| (\varsigma) + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)} \right] \\
\leq & \left[\frac{T^{\alpha_i+\beta_i} L_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} K_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)} \right] \\
& + |\lambda_i| \left[\frac{T^{\alpha_i} R}{\Gamma_q(\alpha_i+1)} \right] + \mu_{\max,i} \left[\frac{\xi_i^{\beta_i} L_i R}{\Gamma_q(\beta_i+1)} + \frac{\xi_i^{\beta_i} K_i R}{\Gamma_q(\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\beta_i+1)} \right] \\
& + \nu_{\max,i} \left[\frac{T^{\alpha_i+\beta_i} L_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} K_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)} \right] \\
= & \left[R\left(L_i + K_i + A_i\right) \right] \left[\frac{\left[\nu_{\max,i}+1\right] T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{\mu_{\max,i} \xi_i^{\beta_i}}{\Gamma_q(\beta_i+1)} \right] + \left[\frac{R \left|\lambda_i\right| T^{\alpha_i}}{\Gamma_q(\alpha_i+1)} \right] \\
= & \left[\left(L_i + K_i + A_i\right) \Theta_i + \Omega_i \right] R,
\end{aligned}
$$

implying that

$$\|\mathcal{G}_i u_i(\varsigma)\| \leq \left[\left(L_i + K_i + A_i\right) \Theta_i + \Omega_i \right] R. \tag{3.18}$$

Likewise, and by using the precedent technique, we have

$$\|\mathcal{G}_i' \varpi_i(\varsigma)\| \leq \left[\left(L_i + K_i + A_i\right) \overline{\Theta}_i + \overline{\Omega}_i \right] R. \tag{3.19}$$

Then, from (3.18) and (3.19), we have

$$\|\mathcal{G}_i \varpi_i(\varsigma)\| \leq \left[\left(L_i + K_i + A_i\right) \left(\Theta_i + \overline{\Theta}_i \right) + \left(\Omega_i + \overline{\Omega}_i \right) \right] R. \tag{3.20}$$

Consequently,

$$\|\mathcal{G}\left(\varpi_1, \varpi_2\right) - \mathcal{G}\left(\omega_1, \omega_2\right)\|_{X \times X} \leq (L + A) \, R \leq R,$$

implying that (3.17) holds. Therefore, by the Banach fixed-point theorem, $\mathcal{G}$ has a unique fixed-point $(\varpi_1, \varpi_2) \in B_R$. The proof is complete. □

3.3. **Stability of the solutions of (1.1).** We use Urs's [48] approach to establish the Ulam-Hyers stability of the solutions of (1.1).

Theorem 3.3 ([48]). *Let X be a Banach space and $T_1, T_2 : X \times X \to X$ be two operators. Then, the operational equations system*

$$\begin{cases} \varpi_1 = T_1(\varpi_1, \varpi_2), \\ \varpi_2 = T_2(\varpi_1, \varpi_2), \end{cases}$$

is said to be Ulam-Hyers stable if there exist $C_1, C_2, C_3, C_4 > 0$ such that for each $\epsilon_1, \varepsilon_2 > 0$ and each solution-pair $(\varpi_1^*, \varpi_2^*) \in X \times X$ of the in-equations:

$$\begin{cases} \|\varpi_1 - T_1(\varpi_1, \varpi_2)\|_X \leq \epsilon_1, \\ \|\varpi_2 - T_2(\varpi_1, \varpi_2)\|_X \leq \varepsilon_2, \end{cases}$$

there exists a solution $(\omega_1^*, \omega_2^*) \in X \times X$ of (1.1) such that

$$\begin{cases} \|\varpi_1^* - \omega_1^*\|_X \leq C_1\varepsilon_1 + C_2\epsilon_2, \\ \|\varpi_2^* - \omega_2^*\|_X \leq C_3\varepsilon_1 + C_4\epsilon_2. \end{cases}$$

Theorem 3.4 ([48]). *Let X be a Banach space, $T_1, T_2 : X \times X \to X$ be two operators such that*

$$\begin{cases} \|T_1(\varpi_1, \varpi_2) - T_1(\omega_1, \omega_2)\|_X \leq k_1 \|\varpi_1 - \omega_1\|_x + k_2 \|\varpi_2 - \omega_2\|_X, \\ \|T_2(\varpi_1, \varpi_2) - T_2(\omega_1, \omega_2)\|_X \leq k_3 \|\varpi_1 - \omega_1\|_x + k_4 \|\varpi_2 - \omega_2\|_X, \end{cases}$$

for all $(\varpi_1, \varpi_2), (\omega_1, \omega_2) \in X \times X$. Suppose

$$H = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

converges to zer0. Then, the operational equations (1.1) is Ulam-Hyers stable.

Set

$$\begin{aligned} C_1 &= \left[(L_1 + K_1)\left(\Theta_1 + \overline{\Theta}_1\right) + \left(\Omega_1 + \overline{\Omega}_1\right)\right], \\ C_2 &= (L_1 + K_1)\left(\Theta_1 + \overline{\Theta}_1\right), \\ C_3 &= \left[(L_2 + K_2)\left(\Theta_2 + \overline{\Theta}_2\right) + \left(\Omega_2 + \overline{\Omega}_2\right)\right], \\ C_4 &= (L_2 + K_2)\left(\Theta_2 + \overline{\Theta}_2\right). \end{aligned}$$

Theorem 3.5. *Assume (H1)-(H4) hold. Further, assume the spectral radius of H is less than one. Then, the solution of (1.1) is Ulam-Hyers stable.*

Proof. In view of Theorem 3.2 we have

$$\begin{cases} \|A_1(\varpi_1, \varpi_2) - A_1(\omega_1, \omega_2)\|_X \leq C_1 \|\varpi_1 - \omega_1\|_X + C_2 \|\varpi_2 - \omega_2\|_X, \\ \|A_2(\varpi_1, \varpi_2) - A_2(\omega_1, \omega_2)\|_X \leq C_3 \|\varpi_2 - \omega_1\|_X + C_4 \|\varpi_2 - \omega_2\|_X, \end{cases}$$

which implies that

$$\|A(\varpi_1,\varpi_2)-A(\omega_1,\omega_2)\|_{X\times X}\le A\begin{pmatrix}\|\varpi_1-\omega_1\|_X\\ \|\varpi_2-\omega_2\|_X\end{pmatrix}. \tag{3.21}$$

Because the spectral radius of H is less than one, the solution of (1.1) is Ulam-Hyers stable. □

4. Example

This section is devoted to the illustration of the results derived in the last section.

Example 4.1. In this section, we present some examples to illustrate our results.

Let $\mathbb{E}=l^1=\{\varpi=(\varpi_1,\varpi_2,\dots,\varpi_n,\dots):\sum_{n=1}^{\infty}|\varpi_n|<\infty\}$ with the norm

$$\|\varpi\|_{\mathbb{E}}=\sum_{n=1}^{\infty}|\varpi_n|.$$

Consider the following nonlinear Langevin $\frac{1}{4}$-fractional equation:

(4.1)

$$\begin{cases}\mathcal{D}_{1/4}^{1/4}\left(\mathcal{D}_{1/4}^{4/3}-\dfrac{1}{10}\right)\varpi(\varsigma)=\dfrac{\sqrt{3}|\varpi|\cos^2(2\pi\varsigma)}{3(27-\varsigma)}+\dfrac{\sqrt{2}\pi|y|}{(7\pi-\varsigma)^2}\left(\dfrac{|y|}{|y|+3}+1\right), & \varsigma\in J,\\ \mathcal{D}_{1/4}^{1/2}\left(\mathcal{D}_{1/4}^{5/3}-\dfrac{2}{5}\right)y(\varsigma)=\dfrac{\sqrt{2}\pi|\varpi|}{4(4\pi-\varsigma)^2}\left(\dfrac{|\varpi|}{|\varpi|+3}+1\right)+\dfrac{|y|\sin^2(2\pi\varsigma)}{(10-\varsigma)^2}, & \varsigma\in J,\\ \varpi(0)=0,\quad \varpi(1)+\dfrac{1}{10}\mathcal{J}_q^{1/4}\varpi(1)=0,\quad \mathcal{D}_q^{1/4}\varpi(1/2)+\dfrac{1}{10}\varpi(1/2)=0, \\ y(0)=0,\quad \varpi_1(1)+\dfrac{2}{5}\mathcal{J}_q^{1/2}y(1)=0,\quad \mathcal{D}_q^{1/2}y(3/4)+\dfrac{2}{5}y(3/4)=0.\end{cases}$$

Here $J=[0,1]$, $\alpha_1=1/4$, $\alpha_2=1/2$, $\beta_1=4/3$, $\beta_2=5/3$, $\xi_1=3/4$, $\xi_2=1/2$, $\lambda_1=1/10$, $\lambda_2=2/5$, with

$$f(\varsigma,\varpi)=(((\sin\varsigma+1)e^{-\varsigma})/24)(\varpi^2/(1+|\varpi|)).$$

Clearly, the function f is continuous. For each $\varpi\in\mathbb{E}$ and $\varsigma\in[0,1]$, we have

$$|f(\varsigma,\varpi_1,\varpi_2)|\le\frac{\sqrt{3}}{81}|\varpi_1|+\frac{\sqrt{2}}{49\pi}|\varpi_2|$$

and

$$|g(\varsigma,\varpi_1,\varpi_2)|\le\frac{\sqrt{2}}{64\pi}|\varpi_1|+\frac{1}{100}|\varpi_2|.$$

Hence, the hypothesis (H2) is satisfied with $p_1^*=\frac{\sqrt{3}}{81}$, $q_1^*=\frac{\sqrt{2}}{49\pi}$, $p_2^*=\frac{\sqrt{2}}{64\pi}$ and $q_2^*=\frac{1}{100}$. We shall show that condition (3.13) holds with $J=[0,1]$. Indeed,

$$\Lambda_1=0.1687\quad\Lambda_2=0.1985,\quad\Lambda\simeq0.3672<1.$$

Simple computations show that all conditions of Theorem 3.1 are satisfied. It follows that the coupled (4.1) has at least one solution defined on J.

Example 4.2. Consider the following coupled system:

$$\begin{cases} \mathcal{D}^{1/4}\left(\mathcal{D}^{4/3}+\dfrac{1}{237}\right)\varpi_1(\varsigma)=\dfrac{1}{100}+\dfrac{\varsigma}{10e^{\varsigma}}\dfrac{|\varpi_1|}{10+|\varpi_1|}+\dfrac{\varsigma\,|\varpi_2|}{(9+e^{\varsigma})^2\left(|\varpi_2|+1\right)}, \quad \varsigma\in[0,1],\\ \mathcal{D}^{1/2}\left(\mathcal{D}^{5/3}+\dfrac{1}{100}\right)\varpi_2(\varsigma)=\dfrac{\varsigma}{100e^{\varsigma}}+\dfrac{\sin|\varpi_1|\,\varsigma+\sin|\varpi_2|}{e^{\varsigma}+99}, \quad \varsigma\in[0,1],\\ \varpi_1(0)=0, \quad \varpi_1(1)+\dfrac{1}{10}\mathcal{I}_q^{1/4}\varpi_1(1)=0, \quad \mathcal{D}_q^{1/4}\varpi_1(1/2)+\dfrac{1}{10}\varpi_1(1/2)=0,\\ \varpi_2(0)=0, \quad \varpi_2(1)+\dfrac{2}{5}\mathcal{I}_q^{1/2}\varpi_2(1)=0, \quad \mathcal{D}_q^{1/2}\varpi_2(3/4)+\dfrac{2}{5}\varpi_1(3/4)=0. \end{cases}$$

Using the given data, we find that

$$\begin{aligned} &|\eta_1(\varsigma,\varpi_1,\varpi_2)-f_1(\varsigma,\omega_1,\omega_2)|\le\frac{1}{100}|\varpi_1-\varpi_2|+\frac{1}{100}|\omega_1-\omega_2|,\\ &|f_2(\varsigma,\varpi_1,\varpi_2)-f_2(\varsigma,\omega_1,\omega_2)|\le\frac{1}{100}|\varpi_1-\varpi_2|+\frac{1}{100}|\omega_1-\omega_2|,\\ &|\eta_1(\varsigma,0,0)|\le\frac{1}{10}, \quad |\eta_1(\varsigma,\varpi_1,\varpi_2)|\le\frac{1}{10}+\frac{\varsigma}{5e^{\varsigma}}+\frac{\varsigma}{(1+e^{\varsigma})^2},\\ &|\eta_2(\varsigma,0,0)|\le\frac{\varsigma}{10e^{\varsigma}}, \quad |\eta_2(\varsigma,\varpi_1,\varpi_2)|\le\frac{\varsigma}{10e^{\varsigma}}+\frac{\varsigma+1}{e^{\varsigma}+10}, \end{aligned}$$

for any $\varsigma\in[0,1]$. Then η_i, $i=1,2$ satisfying (H1)-(H4), with $L_i=\frac{1}{100}$, $K_i=\frac{1}{100}$, $i=1,2$, $A_i=\frac{1}{100}$, $i=1,2$, We find that

$$\begin{aligned} &\Theta_1=1.3850, \quad \Theta_2=1.1300, \quad \Omega_1=0.0207, \quad \Omega_2=0.0354,\\ &\overline{\Theta}_1=6.0050, \quad \overline{\Theta}_2=2.3900, \quad \overline{\Omega}_1=0.2048, \quad \overline{\Omega}_2=0.1992. \end{aligned}$$

Hence, $L\simeq0.6783$, and $A\simeq0.1091$. Therefore, $L+A<1$, and then all conditions of Theorem (3.2) are satisfied, which implies the existence of a unique solution for system (3.21) in $[0,1]$. On the other hand, we find that

$$C_1=0.3733, \quad C_2=0.3050, \quad C_3=0.1478, \quad C_4=0.0704.$$

The spectral radius of the matrix

$$H=\begin{pmatrix}0.3733 & 0.3050\\ 0.1478 & 0.0704\end{pmatrix}$$

is 0.48. Hence, by Theorem 3.5, the solution of (3.21) is Ulam-Hyers stable.

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ANALYTIC STUDIES OF A CLASS OF LANGEVIN DIFFERENTIAL EQUATIONS DOMINATED BY A CLASS OF JULIA FRACTAL FUNCTIONS

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ABSTRACT. In this investigation, we study a class of analytic functions of type Carathéodory style in the open unit disk connected with some fractal domains. This class of analytic functions is formulated based on a kind of Langevin differential equations (LDEs). We aim to study the analytic solvability of LDEs in the advantage of geometric function theory consuming the geometric properties of the Julia fractal (JF) and other fractal connected with the logarithmic function. The analytic solutions of the LDEs are obtainable by employing the subordination theory.

1. INTRODUCTION

Recently, analysis on fractals has been established by numerous investigators studying various problems in engineering (fractal antennas), physics (material processing), chemistry (chemical processing), biology (DNA) and computer science (image processing) [1–8]. Harmonic analysis is employed to describe derivatives and integrals on fractal sets. Probability theory is utilized to formulate Laplacians on fractals [9]. Fractional spaces are plotted to continuous real space in order to explain differential equations on fractals [10–15]. Fractional calculus is smeared in fractal spaces to clarify anomalous diffusion [16–20]. Extended fractional Langevin equations to complex domain are indicated by special types of fractal [21]. The fractal Langevin equation is studied presenting the dynamics of Brownian elements in the long time boundary [22]. Other studies such as an approximate fractal Langevin differential equation are

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consumed and an approximate solution is indicated [23–27]. In the present study, we aim to investigate the analytic solution of Langevin differential equation by using a Julia fractal functions and other fractal [28–30].

1.1. **Differential equation formula.** The second order LDE of a complex variable z is structured by [31]

$$f''(z) + \lambda f'(z) = \Lambda(f(z)), \tag{1.1}$$

where $\lambda > 0$ indicates the damping connection parameter and Λ is the noise term. To investigate the geometric properties of (1.1), we consume the analytic function $f(z)$ in $\cup$ achieving the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. This class of analytic functions is known as the normalized class denoting by $\wedge$. Extend (1.1) with complex coefficient, then we have equivalent equation

$$F(z) := \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right), \quad z \in \cup, \tag{1.2}$$

where $\lambda(z)$ is analytic function in the open unit disk $\cup$. Evidently $F(0) = 1$, for all $\lambda(z) \in \cup$ (see the following example).

Example 1.1. • Assume the function $f(z) = \frac{z}{1-z}$, $\lambda(z) = z$. Then we get the series $F(z) = 1 + z + 3z^2 + 5z^3 + 7z^4 + 9z^5 + O(z^6)$.

• Let $\lambda(z) = 1$ and $f(z) = \frac{z}{1-z}$. This implies the series $F(z) = 1 + 3z + 5z^2 + 7z^3 + 9z^4 + 11z^5 + O(z^6)$.

We demand the following preliminaries.

Definition 1.1. • Two analytic functions f and g in $\cup$ are called subordinate denoting by $f \prec g$, if for a function h is selected such that $|h(z)| \le |z|$ indicating the equation $f = g(h)$ [32].

• The Ma-Minda construction inequalities signified by $S^*(p)$ and $K(p)$ of starlike and convex functions are structured by $\left(\frac{zf'(z)}{f(z)}\right) \prec p(z)$ and $\left(1 + \frac{z f''(z)}{f'(z)}\right) \prec p(z)$, respectively, where p achieves the existing in the class $\mathcal{P}$ where $\operatorname{Re}(p(z)) > 0$, $p(0) = 1$, $|p'(0)| > 1$.

By utilizing the definition of LDEs, we formulate a new class of analytic functions as follows.

Definition 1.2. A function of the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \cup,$$

is in the class $\mathbf{M}_\lambda(p)$ if and only if

$$F(z) = \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right) \prec p(z), \quad z \in \cup,\ p(0) = 1,\ p'(0) > 1,\ \lambda(z) \in \cup. \tag{1.3}$$

We study the analytic solvability of (1.3) by using different types of the parametric Julia fractal formulas taking the construction (see Figure 1)

$$J_\kappa(z) = 1 + z - \kappa\, z^3, \quad z \in \cup,$$
$$\Upsilon_\kappa(z) = \frac{1+z^2}{1-\kappa\, z^2} = 1 + (\kappa+1)z^2 + (\kappa^2+\kappa)z^4 + O(z^6), \quad z \in \cup,$$

and

$$L_\kappa(z) = z^2 + \frac{1}{1-\kappa z^2} = 1 + (\kappa+1)z^2 + \kappa^2 z^4 + O(z^6), \quad z \in \cup, |z| < 1/\sqrt{(|\kappa|)}.$$

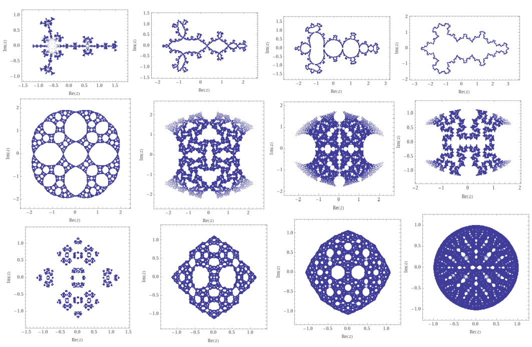

FIGURE 1. The plot of J_κ, $\kappa = 1, 1/2, 1/3, 1/4$, $\Upsilon_\kappa, \kappa = 1, 2, 3, 4$ and $L_\kappa, \kappa = 3, 5, 10, 100$, respectively

The technique is to find the optimal value of κ which satisfies the inequality subordination

$$1 + \kappa\left(\frac{z\, p'(z)}{[p(z)]^k}\right) \prec (1+z)^\kappa, \quad z \in \cup,$$

to satisfy one of the following inequalities

$$p(z) \prec J_\kappa, \quad p(z) \prec \Upsilon_\kappa, \quad p(z) \prec L_\kappa.$$

As an application, we consider the LDEs to investigate the solvability by using the Julia fractal functions

$$F(z) \prec J_\kappa, \quad F(z) \prec \Upsilon_\kappa, \quad F(z) \prec L_\kappa.$$

Special cases are investigated for some well known classes of analytic functions.

2. Computational Results

This section deals with consequences regarding $p(z)$ and $F(z)$.

Theorem 2.1. *Let the function $p \in \mathcal{P}$ admitting the inequalities*

$$1+\kappa\left(\frac{z\,p'(z)}{(\,p(z)\,)^k}\right) \prec \Sigma_\kappa(z), \quad z \in \cup,$$

where $k = 0, 1, 2$ and $\Sigma_\kappa(z) = (1+z)^\kappa$, $z \in \cup$. Then

(A) $p(z) \prec J_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = 1.3247$;
(B) $p(z) \prec \Upsilon_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = \frac{1}{2}$;
(C) $p(z) \prec L_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = 0.550667$.

Proof. Firstly, we aim to prove the inequality $p(z) \prec J_\kappa(z)$, therefore we have the following cases.

Case I. $k = 0 \Rightarrow 1+\kappa\,(z\,p'(z)) \prec (1+z)^\kappa$. Let $T_\kappa : \cup \to \mathbb{C}$ admitting the structure

$$T_\kappa(z) = \frac{(\kappa^2+\kappa+1) - (z+1)^{\kappa+1}\,{}_2F_1(1,\kappa+1,\kappa+2,z)}{\kappa^2+\kappa}, \quad z \in \cup,$$

where ${}_2F_1$ indicates the hypergeometric function for all $z \in \cup$ with the power series

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!}.$$

Clearly, $T_\kappa(z)$ is analytic in $\cup$ satisfying $T_\kappa(0) = 1$ and it is an approximate solution by a hypergeometric function of the differential equation

$$1+\kappa\,(z\,T_\kappa'(z)) \;=\; (z+1)^\kappa, \quad z \in \cup. \tag{2.1}$$

Let

$$\mathfrak{W}(z) := -\frac{\kappa}{3}\,(z\,T_\kappa'(z)) = \frac{((z+1)^\kappa((z-1)2F1(1,\kappa+1,\kappa+2,z)+z+1))}{z-1}.$$

Then by [32, Lemma 4.5e], where

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^\alpha, \quad \beta \leq \gamma, \tag{2.2}$$

we have for $\kappa > 0$

$$\begin{aligned}
\mathfrak{W}(z) &= -\frac{\kappa}{3}\,(z\,T_\kappa'(z)) \\
&= -\frac{1}{3}\cdot\frac{((z+1)^\kappa((z-1)_2F_1(1,\kappa+1,\kappa+2,z)+z+1))}{z-1} \\
&= -\frac{1}{3}\cdot\frac{((z+1)^\kappa((z-1)(1-z)+z+1))}{z-1} \\
&= \left(\frac{z(1+z)^\kappa}{1-z}\right)\left(1-\frac{z}{3}\right) \\
&= z + \left(\kappa+\frac{2}{3}\right)z^2 + \frac{1}{6}(3\kappa^2+k+4)z^3 + \frac{1}{6}(\kappa^3-\kappa^2+4\kappa+4)z^4 \\
&\quad + \frac{1}{72}(3\kappa^4-10\kappa^3+33\kappa^2+22\kappa+48)z^5 + O(z^6).
\end{aligned}$$

By the assumption of the theorem, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{z\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right) =&\operatorname{Re}\left(1+\left(\kappa+\frac{2}{3}\right)z+\left(\frac{8}{9}-\kappa\right)z^2+\left(\kappa+\frac{26}{27}\right)z^3\right.\\ &\left.+\left(\frac{80}{81}-\kappa\right)z^4+\left(\kappa+\frac{242}{243}\right)z^5+O(z^6)\right)\\ >&0\end{aligned}$$

provided $0<\kappa\leq 4.27772$. That is, $\mathfrak{W}(z)$ is starlike function. Thus, by using $\mathfrak{G}(z):=\mathfrak{W}(z)+1/(-3)$, one can obtain

$$\operatorname{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right)=\operatorname{Re}\left(\frac{z\,\mathfrak{G}'(z)}{\mathfrak{W}(z)}\right)>0.$$

Thus, Miller-Mocanu Lemma (see [32, page 132]) admits

$$1+\kappa\,(z\,p'(z))\prec 1+\kappa\,(zT'_\kappa(z))\Rightarrow p(z)\prec\,T_\kappa(z).$$

Our aim is to prove that $p(z)\prec J_\kappa(z)$, which indicates if $T_\kappa(z)\prec J_\kappa(z)$. To complete this conclusion, we have to prove that $T_\kappa(z)\prec(1+z)^\kappa$. By using (2.2), we have

$$\frac{\kappa^2+\kappa+1}{\kappa^2+\kappa}=T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}.$$

Since

$$0=\Sigma_\kappa(-1)\leq\Sigma_\kappa(1)=2^\kappa,\quad \kappa>0,$$

thus, we obtain

$$T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}\leq 2^\kappa,$$

whenever $\kappa>0.78124$. As a conclusion, we have $T_\kappa(z)\prec J_\kappa(z)$ when

$$\kappa=J_\kappa(-1)\leq T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}\leq J_\kappa(1)=2-\kappa,$$

which is provided

$$0.7812<\kappa<\frac{1}{3}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)^{1/3}+\frac{\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/3}}{3^{2/3}}\approx 1.3247.$$

This implies the relation

$$T_\kappa(z)\prec J_\kappa(z)\Rightarrow p(z)\prec\,J_\kappa(z),\quad z\in\cup.$$

Case II. $k=1\Rightarrow 1+\kappa\left(\frac{z\,p'(z)}{p(z)}\right)\prec(1+z)^\kappa$. Define a function $E_\kappa:\cup\to\mathbb{C}$ formulating the structure

$$E_\kappa(z)=\exp\left(\frac{1-(z+1)^{\kappa+1}\;{}_2F_1(1,\kappa+1,\kappa+2,z)}{\kappa^2+\kappa}\right).$$

Obviously, $E_\kappa(z)$ is analytic in $\cup$ satisfying $E_\kappa(0) = 1$ and it is an approximated solution by a hypergeometric function satisfying the differential equation

$$1+\kappa\left(\frac{z\,E'_\kappa(z)}{E_\kappa(z)}\right) \;=\; (1+z)^\kappa, \quad z\in\cup.$$

By considering $\mathfrak{W}(z) = \kappa\left(\frac{z\,E'_\kappa(z)}{E_\kappa(z)}\right) = (1+z)^\kappa - 1$, which is starlike function with $\kappa \neq 0$ and $\mathfrak{T}(z) = \mathfrak{W}(z) + 1$, we attain

$$\mathrm{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right) = \mathrm{Re}\left(\frac{z\,\mathfrak{T}'(z)}{\mathfrak{W}(z)}\right) > 0, \quad z\in\cup.$$

Thus, Miller-Mocanu Lemma, yields

$$1+\kappa\left(\frac{z\,p'(z)}{p(z)}\right) \;\prec 1+\kappa\left(\frac{zE'_\kappa(z)}{E_\kappa(z)}\right) \Rightarrow p(z) \prec\; E_\kappa(z).$$

Consequently, one can recognize the next equality

$$\exp\left(\frac{1}{\kappa^2+\kappa}\right) = E_\kappa(-1) = E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right).$$

Moreover, this implies $E_\kappa(z) \prec (1+z)^\kappa$ such that for $\kappa \neq 0$ the inequality

$$0 = \Sigma_\kappa(-1) \le E_\kappa(-1) = E_\kappa(1) \le \Sigma_\kappa(1) = 2^\kappa, \quad \kappa > 0.876764,$$

holds. Thus, we get $E_\kappa(z) \prec J_\kappa(z)$ when

$$\kappa = J_\kappa(-1) \le E_\kappa(-1) \le E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right) \le J_\kappa(+1) = 2-\kappa.$$

This leads to the following subordination for $\kappa \approx 1$

$$E_\kappa(z) \prec\; J_\kappa(z) \Rightarrow p(z) \prec J_\kappa(z), \quad z\in\cup.$$

Case III: $k = 2 \Rightarrow 1+\kappa\left(\frac{z\,p'(z)}{p^2(z)}\right) \prec (1+z)^\kappa$. Consume that $H_\kappa : \cup \to \mathbb{C}$ satisfies the formula

$$H_\kappa(z) = \frac{\kappa(\kappa+1)}{\kappa^2+\kappa+(z+1)^{\kappa+1}\ {}_2F_1(1,\kappa+1,\kappa+2,z)}.$$

Clearly, $H_\kappa(z)$ is analytic in $\cup$ admitting $H_\kappa(0) = 1$ and it is the approximated outcome in terms of the hypegoemetric function

$$1+\kappa\left(\frac{z\,H'_\kappa(z)}{H^2_\kappa(z)}\right) \;=\; (1+z)^\kappa, \quad z\in\cup.$$

Similarly, we use the starlike function $\mathfrak{W}(z) = \Sigma_\kappa(z) - 1$ and $\mathfrak{Y}(z) = \mathfrak{W}(z) + 1$, we get

$$\mathrm{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right) = \mathrm{Re}\left(\frac{z\,\mathfrak{Y}'(z)}{\mathfrak{U}(z)}\right) > 0, \quad z\in\cup.$$

Hence, the Miller-Mocanu Lemma yields

$$1+\kappa\left(\frac{z\,p'(z)}{p^2(z)}\right) \;\prec 1+\kappa\left(\frac{zH'_\kappa(z)}{H^2_\kappa(z)}\right) \Rightarrow p(z) \prec\; H_\kappa(z).$$

Accordingly, for $\kappa \geq 1$, we obtain

$$1 = H_\kappa(-1) = H_\kappa(1) = 1.$$

Moreover, for $\kappa = 1$, we have

$$\kappa = J_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq J_\kappa(+1) = 2 - \kappa.$$

Thus, one can realize that

$$H_\kappa(z) \prec J_\kappa(z) \Rightarrow p(z) \prec J_\kappa(z), \quad z \in \cup.$$

For the second and third part, we proceed in the same manner of above construction of the functions $T_\kappa(z), E_\kappa(z)$ and $H_\kappa(z)$. We conclude that for the second part,

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq T_\kappa(-1) = T_\kappa(1) = \frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa} \leq \Upsilon_\kappa(1) = \frac{2}{1-\kappa},$$

whenever

$$\kappa = \frac{1}{3}\left((-2 - 2(\frac{2}{(47+3\sqrt{249}))^{1/3}} + \left(\frac{1}{2}(47+3\sqrt{249})\right)^{1/3}\right) \approx 0.3532099\ldots$$

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq E_\kappa(-1) \leq E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right) \leq \Upsilon_\kappa(+1) = \frac{2}{1-\kappa},$$

whenever $\kappa \approx 0.490561$ and

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq \Upsilon_\kappa(+1) = \frac{2}{1-\kappa}, \quad \kappa \approx \frac{1}{2}.$$

Then we get $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $z \in \cup$. For the last part, we obtain

$$\frac{2-\kappa}{1-\kappa} = L_\kappa(-1) \leq T_\kappa(-1) = T_\kappa(1) = \frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa} \leq L_\kappa(1) = \frac{2-\kappa}{1-\kappa},$$

whenever, $\kappa = \sqrt{(2)} - 1 \approx 0.414213\ldots$,

$$\frac{2-\kappa}{1-\kappa} = L_\kappa(-1) \leq E_\kappa(-1) \leq E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right) \leq L_\kappa(+1) = \frac{2-\kappa}{1-\kappa},$$

whenever $\kappa \approx 0.550667$ and

$$\frac{2-\kappa}{1+\kappa} < L_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq L_\kappa(+1) = \frac{2-\kappa}{1-\kappa}, \quad \kappa \approx \frac{1}{2}.$$

Then, we conclude that $p(z) \prec L_\kappa(z)$, $\kappa > 0.550667$, $z \in \cup$ □

As an application of Theorem 2.1, we let $p(z) = \frac{zf'(z)}{f(z)}$, $f\wedge$. Thus, one can recognize the following consequence.

Corollary 2.1. *Let $f\wedge$. If one of the inequalities is indicted*

(a) $1 + \kappa\left(\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2\right) \prec (1+z)^\kappa$;

(b) $1 + \kappa\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec (1+z)^\kappa$;

(c) $1 + \kappa\left(\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}\right)^{-1} + \left(\frac{zf'(z)}{f(z)}\right)^{-1} - 1\right) \prec (1+z)^\kappa$;

then $f \in S^*(J_\kappa)$, $\kappa > 1.3247$, $f \in S^*(\Upsilon_\kappa)$, $\kappa > \frac{1}{2}$ *and* $f \in S^*(L_\kappa)$, $\kappa > 0.550667$.

Corollary 2.2. *Let* $p(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$. *If one of the inequalities is indicted*

(a)

$$1 + \frac{\kappa z(A-B)}{(Bz+1)^2} \prec (1+z)^\kappa, \quad \frac{A+1}{B+1} < 0.676, \quad \kappa = \frac{(-A+2B+1)}{(B+1)} > 1.3247;$$

(b)

$$1 + \frac{\kappa z(A-B)}{((Az+1)(Bz+1))} \prec (1+z)^\kappa,$$

where

$$A+1 \neq 0, \; \kappa = \frac{(A-2B-1)}{(A+1)} > \frac{1}{2}, \; B+1 \neq 0, \; A > 4B+3;$$

(c)

$$1 + \frac{\kappa z(A-B)}{(Az+1)^2} \prec (1+z)^\kappa,$$

where

$$A \neq B, \kappa = \frac{(A-2B-1)}{(A-B)}, \; B+1 \neq 0, \; \frac{A}{(A-B)} - \frac{(2B)}{(A-B)} - \frac{1}{(A-B)} > 0.55;$$

then

$$\frac{1+Az}{1+Bz} \prec J_\kappa(z), \quad \kappa > 1.324,$$
$$\frac{1+Az}{1+Bz} \prec \Upsilon_\kappa(z), \quad \kappa > 1/2,$$

and

$$\frac{1+Az}{1+Bz} \prec L_\kappa(z), \quad \kappa > 0.55066.$$

Corollary 2.3. *Let* $p(z) = 1 + \sin(z)$. *If one of the inequalities is indicted*

(a) $1 + \kappa z \cos(z) \prec (1+z)^\kappa$, $\kappa > 1.324$;
(b) $1 + \frac{\kappa z \cos(z)}{\sin(z)+1} \prec (1+z)^\kappa$, $\kappa > 0.5$;
(c) $1 + \frac{(\kappa z \cos(z))}{(\sin(z)+1)^2} \prec (1+z)^\kappa$, $\kappa > 0.55066$;

then $p(z) \prec J_\kappa(z)$, $\kappa > 1.324$, $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $p(z) \prec L_\kappa(z)$, $\kappa > 0.55066$.

Corollary 2.4. *Let* $p(z) = e^z$. *If one of the inequalities is indicted*

(a) $1 + \kappa z e^z \prec (1+z)^\kappa$, $\kappa > 1.324$;
(b) $1 + \kappa z \prec (1+z)^\kappa$, $\kappa > 0.5$;
(c) $1 + \kappa z\, e^{-z} \prec (1+z)^\kappa$, $\kappa > 0.55066$;

then $p(z) \prec J_\kappa(z)$, $\kappa > 1.324$, $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $p(z) \prec L_\kappa(z)$, $\kappa > 0.55066$.

Next result admits some properties of LDE.

Theorem 2.2. *Consider two functions* $\Sigma_\kappa(z) = (1+z)^\kappa$, $\kappa \in \mathbb{R}$, *and*

$$F(z) = \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right).$$

If one of the inequalities

$$1 + \kappa\left(\frac{z\,F'(z)}{[F(z)]^k}\right) \prec (1+z)^\kappa$$

is occurred, where $k = 0, 1, 2$, *then*

- $F(z) \prec J_\kappa(z)$, $\kappa > 1.324$;
- $F(z) \prec \Upsilon_\kappa(z), \kappa > 0.5$;
- $F(z) \prec J_\kappa(z), \kappa > 0.55066$.

Furthermore, if $\operatorname{Re}(F(z)) > 0$ *and* $\lambda(z)$ *satisfies*

$$\operatorname{Re}(\lambda(z)) > 0, \quad [\Im(1-\lambda(z))]^2 \le 3[\operatorname{Re}(\lambda(z))]^2,$$

then f *is starlike in* $\cup$.

Proof. Since for all $\lambda(z), z \in \cup$, we have $F(0) = 1$, then by using the same technique in Theorem 2.1, we have the first part regarding the subordinated inequalities. For the second part, we assume that

$$p(z) = \frac{zf'(z)}{f(z)}, \quad f \in \wedge,\ z \in \cup.$$

Then a computation implies that

$$F(z) = \lambda(z)zp'(z) + \lambda(z)p^2 + [1-\lambda(z)]p(z).$$

Then by the assumptions and in view of [32, Example 2.4], we have $\operatorname{Re}(p(z)) > 0$ which implies that $f(z)$ is starlike. □

3. Examples

In this section, we deal with special cases of the LDEs depending on the formula of $\lambda(z)$.

Case I. Let $\lambda(z) = 1$. The construction of LDE becomes

$$\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right) = F(z), \quad z \in \cup,\ f \in \wedge. \tag{3.1}$$

Then by using $J_\kappa(z) = 1 + z - \kappa z^3$, Figure 2 shows the solution for different values of $\kappa > 1.324$, Figure 3 indicates the solution by using $\Upsilon_\kappa(z) = \frac{1+z^2}{1-\kappa z^2}$ for $\kappa > 0.5$. It can be seen that the solution satisfies when

- $\kappa = 0.6$, we have

$$\begin{aligned} f(z) =& c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1-0.6i, 1+0.6i \\ -0.5, 0.5 \end{matrix}\,\middle|\, 0.6z^2\right) \\ &+ 0.774\, c_1\, z\, \left({}_2F_1(0.5-0.6i, 0.5+0.6i; 2; 0.6z^2)\right), \end{aligned}$$

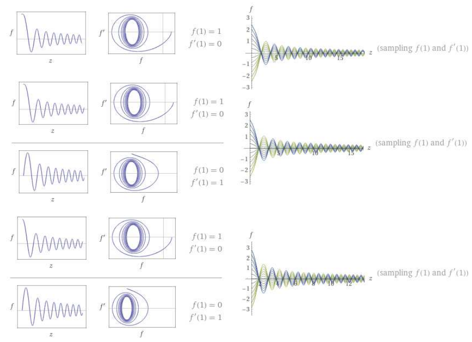

FIGURE 2. The plot of (3.1) by using $J_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 1.5, 2, 4$, respectively

where $f(0) = 0$ provided $c_2 = 0$;

- $\kappa = 2$, we obtain

$$f(z) = c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1 - \frac{i}{2\sqrt{2}}, 1 + \frac{i}{2\sqrt{2}} \\ -0.5, 0.5 \end{matrix} \middle| \, 2z^2 \right) + i\sqrt{(2)} c_1 z \left({}_2F_1(1/4(2 - i\sqrt{(2)}), 1/4(2 + i\sqrt{(2)}); 2; 2z^2) \right);$$

- $\kappa = 4$, we get

$$f(z) = c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1 - \frac{i}{4}, 1 + \frac{i}{4} \\ -0.5, 0.5 \end{matrix} \middle| \, 4z^2 \right) + 2ic_1 z \left({}_2F_1\left(\frac{1}{2} - \frac{i}{4}, \frac{1}{2} - \frac{i}{4}; 2; 2z^2\right)\right).$$

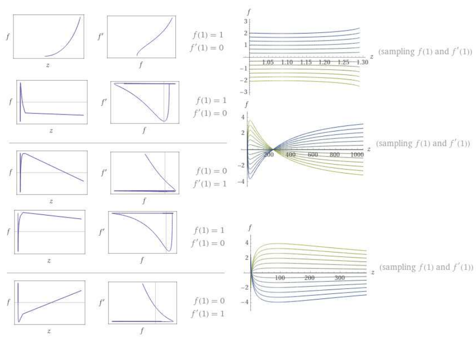

FIGURE 3. The plot of (3.1) using $\Upsilon_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

Figure 4 imposes the behavior of (3.1) by using $L_\kappa(z)$.

Case II. Let $\lambda(z) = z$. The construction of LDE becomes

$$z\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{z f'(z)}{f(z)}\right) = F(z), \quad z \in \cup,\ f \in \wedge. \tag{3.2}$$

A computation implies the following constructions of $F(z)$.

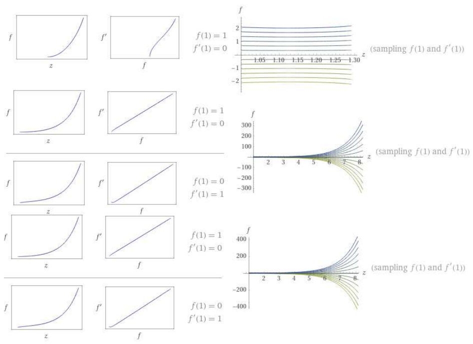

FIGURE 4. The plot of (3.1) using $L_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

- For

$$\begin{aligned} J_{1.5} \Rightarrow & F(z) = -1.5(z - 1.12271)(z^2 + 1.12271z + 0.593803), \\ J_2(z) \Rightarrow & F(z) = 1 + z \end{aligned}$$

and $J_4(z) \Rightarrow F(z) = 1 + z$;
- for

$$\begin{aligned} \Upsilon_{0.6} \Rightarrow & F(z) = \frac{1.66667z^2 + 1.66667}{1.66667 - z^2}, \\ \Upsilon_2(z) \Rightarrow & F(z) = \frac{z^2}{1 - 2z^2} + \frac{1}{1 - 2z^2} \end{aligned}$$

and

$$\Upsilon_4(z) \Rightarrow F(z) = \frac{z^2}{1 - 4z^2} + \frac{1}{1 - 4z^2};$$

- for

$$\begin{aligned} L_{0.6} \Rightarrow & F(z) = \frac{(z - 1.53946)(z + 1.53946)(z^2 + 0.703257)}{(z - 1.29099)(z + 1.29099)}, \\ L_2 \Rightarrow & F(z) = \frac{(z - 1)(z + 1)(2z^2 + 1)}{2z^2 - 1} \end{aligned}$$

and

$$L_4 \Rightarrow F(z) = \frac{4z^4 - z^2 - 1}{(2z - 1)(2z + 1)}.$$

Figures 5–7 show the behavior of (3.2) for $J_\kappa(z), \Upsilon_\kappa(z)$ and $L_\kappa(z)$, respectively.

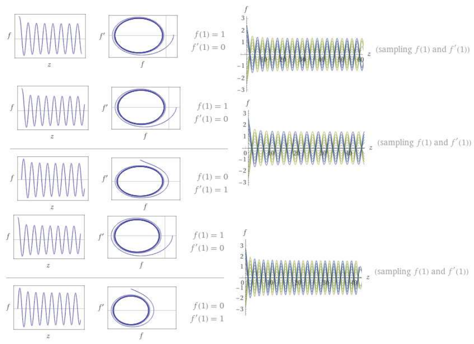

FIGURE 5. The plot of (3.2) by using $J_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 1.5, 2, 4$, respectively

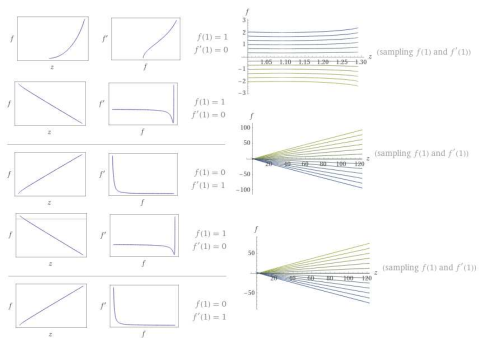

FIGURE 6. The plot of (3.2) using $\Upsilon_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

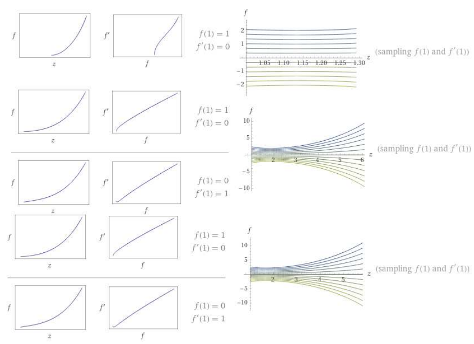

FIGURE 7. The plot of (3.2) using $L_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

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WELL-POSEDNESS AND ASYMPTOTIC STABILITY OF A NON-LINEAR POROUS SYSTEM WITH A DELAY TERM

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Abstract. Our interest in this work is to treat a one-dimensional Porous system with a non-linear damping and a delay in the non-linear internal feedback. We prove the global existence and uniqueness of its solution in suitable function spaces by means of the Faedo-Galerkin procedure combined with the energy method under a suitable relation between the weight of the delayed feedback and the weight of the non-delayed feedback. Also, we give an explicit and general decay rate estimate by applying the well-known multiplier method integrated with some properties of convex functions and for two opposites cases with respect to the speeds of wave propagation.

1. Introduction

In the present paper, we study the well-posedness and asymptotic behavior of solutions of the following Porous system

(1.1)

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x,t-\tau(t))) = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0 \quad \text{in }]0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in }]0,1[, \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \text{in }]0,1[, \\ \phi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), \quad \text{in }]0,1[\times]0,\tau[, \end{cases}$$

Key words and phrases. Non-linear Porous system, global existence, delay term, general decay, Faedo-Glaerkin method, multiplier method.

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where x denotes the space variable, t is the time variable, $\tau(\cdot) > 0$ is a time varying delay, μ_1 is a positive constant and μ_2 is a real number. The functions $u = u(x,t)$ and $\phi = \phi(x,t)$ represent, respectively, the displacement of the solid elastic material and the volume fraction and the initial data $(u_0, u_1, \phi_0, \phi_1, f_0)$ belongs to a suitable Sobolev space. The original Porous system is governed by the following evolution equations

$$\begin{aligned} \rho_1 u_{tt} =& T_x, \\ \rho_2 \phi_{tt} =& H_x + G, \end{aligned}$$

where T, H and G denote, respectively, the stress, the equilibrated stress and the equilibrated body force. The constitutive equations are as follows

$$T = \kappa u_x + b\phi, \quad H = \delta \phi_x, \quad G = -b u_x - \xi \phi,$$

where ρ_1, ρ_2, κ, b, δ and ξ are positive constants satisfying in the one-dimensional case, the following inequality

$$\kappa \xi > b^2.$$

If we consider $\kappa = b = \xi$, we find the well-known Timoshenko system which is introduced by S.Timoshenko [17] and it has been widely considered in the literature. For the better comprehension of our motivation, we appeal to keep in mind that the system

$$\begin{cases} \rho_1 u_{tt} - \kappa(u_{xx} - \phi_x) = 0, & \text{in }]0,L[\times]0,\infty[, \\ \rho_2 \phi_{tt} - \delta \phi_{xx} + k(u_x + \phi) = 0, & \text{in }]0,L[\times]0,\infty[, \end{cases} \tag{1.2}$$

is conservative. Namely, by taking any suitable boundary conditions into consideration, the energy of (1.2) given by

$$E(t) = \frac{1}{2} \int_0^L \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa (u_x + \phi)^2 + \delta \phi_x^2 \Big] dx,$$

satisfies the energy's conservation property, that is, for all $t > 0$, $E(t) = E(0)$. In this vein, various damping such as viscoelastic damping, frictional damping and thermal dissipation are employed to stabilize the vibrations. It has been shown that the stability depends on the position and nature of the controls and some relations between the constants ρ_1, ρ_2, κ and δ. Let us recall some known results on the stability of the Timoshenko system with frictional dampings. Soufyane and Wehbe [16] used the unique damping $a(x)\phi_t$ in the shear angle displacement and showed that the solution is uniformly stable. This one has been obtained in the case of the equal-speeds, i.e.,

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}. \tag{1.3}$$

Raposo et al. [15] examined (1.2) by setting two linear frictional dampings u_t and ϕ_t where they realized an exponential decay result without imposing any condition on the coefficients. In [1], Alabau Boussouira extended [16] to a problem with a non-linear damping acting in the second equation. Under the condition (1.3), she established a general and semi-explicit formula for the decay rate of the solutions. This result was later improved by Mustafa and Messaoudi [11] where they obtained a general and explicit decay estimate. In the other hand, for the Porous system, Quintanilla [13] proved that the damping $a\phi_t$ is not strong enough to obtain the exponential stability result. However, Apalara [3] got the exponential decay of the solutions for the same problem provided (1.3) holds true. Furthermore, in the nonequal-speeds case, he [3] established a general decay result when he employed a weak non-linear damping $\mu(t)g(\phi_t)$.

In the recent years, the Timoshenko system with time delay has been discussed by several researchers. In particular, we consider the following model with a delay term

(1.4)
$$\begin{cases}\rho_1 u_{tt} - \kappa(u_{xx} - \phi_x) + a_1 f_1(u_t) + a_2 f_2(u_t(x, t-\tau(t))) = 0, & \text{in }]0, L[\times]0, \infty[,\\ \rho_2\phi_{tt} - \delta\phi_{xx} + \kappa(u_x + \phi) + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x, t-\tau(t))) = 0, & \text{in }]0, L[\times]0, \infty[.\end{cases}$$

Here, f_i and g_i are real functions, a_i and μ_i are positive numbers for $i = 1, 2$. If $a_i = 0$, $g_i(x) = x$ and $\mu_2 < \mu_1$, then the exponential stability has been proved by Kiran et al. [6] in the case of equal-speeds. In the case of a constant delay, Apalara [2] considered (1.4) when $\mu_i = 0$, $f_i(x) = x$ and $a_2 < a_1$ and established an exponential stability result provided $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$. In the opposite case, only a polynomial decay is obtained. As far as we know, the first work investigated the Timoshenko beam with a nonlinear delay term is the one of Benaissa and Bahlil [5]. The problem treated is (1.4) with $a_i = 0$. They considered only the equal-speeds case where they obtained an explicit decay estimate under a suitable relation between μ_1 and μ_2 and some additional assumptions. For the Porous system with delay term, the subject of this article, we cite the works [10, 14] and [7]. The authors of [7] examined a non-linear Porous system of the form

$$\begin{cases}\rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, & \text{in }]0, 1[\times]0, \infty[,\\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t + \mu_2\phi_t(x, t-\tau) + \alpha(t)g(\phi_t) = 0, & \text{in }]0, 1[\times]0, \infty[,\end{cases}$$

and established, under the assumption $|\mu_2| < \mu_1$, a general decay of solution when $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$.

As a consequence of the works cited above, if only one equation of a Timoshenko system is damped then the uniform stability may be achieved for weak solutions if and only if $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$. However, in the situation when $\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}$, a weaker decay rate result is achieved for strong solutions. According to this results, three questions naturally arise.

1. Is it possible to consider the Porous system with a non-linear damping term and a time varying delay in the internal feedback acting only in the second equation and get the same result as in the Timoshenko system?

2. In the equal-speeds case, is it possible to get the stability result with same hypotheses on μ_1, μ_2, g_1 and g_2 as in the Timoshenko system?

3. As we have mentioned above, the nonequal-speeds case is not considered for the non-linear Timoshenko system with delay (see [5]). So, is it possible to obtain the stability result under the same conditions imposed for the equal-speeds case?

The main aim of this manuscript is to give positive answers to theses three questions by investigating (1.1).

The rest of our paper is as follows. In the next section, we provide some assumptions and materials needed in our work. In Section 3, we state and prove the existence and the uniqueness results. The last section is devoted to the study of the asymptotic behavior of the solutions. We use c throughout this paper to denote a generic fixed positive constant, which may be different in different estimates.

2. Preliminaries

In this section, we present some assumptions, materials and notations that will be used later. Firstly, following the same arguments of Nicaise and Pignotti [12], we introduce the new variable

$$z(x,\rho,t) = \phi_t(x, t-\rho\tau(t)), \quad x \in [0,1], \rho \in [0,1], t > 0.$$

It is clear that

$$\tau(t) z_t(x,\rho,t) + (1-\rho\tau'(t)) z_\rho(x,\rho,t) = 0, \quad \text{in } ([0,1])^2 \times [0,\infty].$$

Hence, our problem (1.1) becomes

(2.1)
$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(z(x,1)) = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ \tau(t) z_t(x,\rho,t) + (1-\rho\tau'(t)) z_\rho(x,\rho,t) = 0, \quad \text{in } (]0,1[)^2\times]0,\infty[, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad \text{in }]0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in }]0,1[, \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \text{in }]0,1[, \\ z(x,\rho,0) = f_0(x, -\rho\tau(0)), \quad \text{in } (]0,1[)^2. \end{cases}$$

In order to deal with the new variable z, we define the Hilbert space

$$L_z^2(0,1) = L^2\Big(0,1; L^2(0,1)\Big) = \left\{ z :]0,1[\to L^2(0,1), \ \int_0^1 \int_0^1 z^2(x,\rho) d\rho dx < \infty \right\},$$

which endowed with the inner product

$$(z, \widetilde{z}) = \int_0^1 \int_0^1 z(x,\rho,t) \widetilde{z}(x,\rho,t) d\rho dx.$$

We consider now the following assumptions.

(A_1) $g_1 : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function of class C^1 and $g_2 : \mathbb{R} \to \mathbb{R}$ is an increasing function of class C^1 such that it exist $\epsilon < 1$, c_1, c_2 and a convex and non-decreasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(2.2)
$$\begin{cases} H(0) = 0 \text{ and } H \text{ is linear on } [0,\epsilon] \textbf{ or } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\epsilon] \text{ such that} \\ c_1|s_1| \le |g_1(s_1)| + |g_2(s_2)| \le c_2\Big(|s_1| + |s_2|\Big), \quad \text{if } |s_1| + |s_2| \ge \epsilon, \\ s_1^2 + g_1^2(s_1) + g_2^2(s_2) \le H^{-1}\Big(s_1 g_1(s_1) + s_2 g_2(s_2)\Big), \quad \text{if } |s_1| + |s_2| \le \epsilon. \end{cases}$$

Also, for any $s \in \mathbb{R}$, we assume that it exist some positive constants $\widetilde{c}_2$, α_1 and α_2 satisfying

$$|g_2'(s)| \le \widetilde{c}_2 \tag{2.3}$$

and

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{2.4}$$

where G is a primitive of g_2.

(A_2) τ is a function in $W^{2,\infty}([0,T])$, $T > 0$, such that

$$\begin{cases} 0 < \tau_0 \le \tau(t) \le \tau_1, \quad \text{for all } t > 0, \\ \tau'(t) \le \theta < 1, \quad \text{for all } t > 0, \end{cases}$$

where τ_0 and τ_1 are a positive numbers.

(A_3) With respect to the weights of feedbacks μ_i, $i = 1, 2$, we assume that

$$|\mu_2| < \frac{\alpha_1(1-\theta)}{\alpha_2(1-\alpha_1\theta)}\mu_1.$$

We define the energy associated with the solution of (2.1) as

(2.5)
$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2b u_x\phi + 2\tau(t)\gamma \int_0^1 G(z(x,\rho))d\rho\right] dx,$$

where γ is a positive number such that

$$\frac{(1-\alpha_1)|\mu_2|}{\alpha_1(1-\theta)} < \gamma < \frac{\mu_1 - \alpha_2|\mu_2|}{\alpha_2}.$$

Remark 2.1. The energy functional $E(t)$ defined in (2.5) is positive. In fact, we can easily show that

$$\kappa u_x^2 + 2b u_x\phi + \xi\phi^2 = \frac{1}{2}\left[\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2 + 2\kappa_1 u_x^2 + 2\xi_1\phi^2\right],$$

where $2\kappa_1 = \kappa - \frac{b^2}{\xi}$ and $2\xi_1 = \xi - \frac{b^2}{\kappa}$ are positives from $\kappa\xi > b^2$. Thus,

$$\kappa u_x^2 + 2b u_x\phi + \xi\phi^2 > \frac{1}{2}\left[\kappa\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2\right] > 0,$$

which implies the positivity of $E(t)$ and

$$E(t) > \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2\phi_t^2 + \kappa_1 u_x^2 + \xi_1\phi^2 + 2\gamma\tau(t)\int_0^1 G(z(x,\rho))d\rho\right] dx. \tag{2.6}$$

Remark 2.2. • The strict non-decreasing property of g_1 implies the existence of a positive constant $\widetilde{c}_1$ satisfying

$$\widetilde{c}_1 < g_1'(s). \tag{2.7}$$

• Assumption (2.2) implies that $s_1 g_1(s_1) + s_2 g_2(s_2) > 0$ for all $s_1, s_2 \in \mathbb{R}$.

• By the mean value theorem for integrals and the monotonicity of g_2, we deduce that

$$G(s) = \int_0^s g_2(\sigma)d\sigma \leq s g_2(s),$$

then $\alpha_1 < \alpha_2 \leq 1$.

Remark 2.3. Let Ψ^* be the conjugate function of the differential convex function Ψ, i.e.,

$$\Psi^*(s) = \sup(st - \Psi(t)),$$

then Ψ^* is the Legendre transform of Ψ, which is given by (see Arnold [4])

$$\Psi^*(s) = s(\Psi')^{-1}(s) - \Psi[(\Psi')^{-1}(s)], \quad \text{if } s \in [0, \Psi'(r)],$$

satisfies the generalized Young inequality

$$AB \leq \Psi^*(A) + \Psi(B), \quad \text{if } A \in [0, \Psi'(r)], B \in [0, r]. \tag{2.8}$$

A starting point will be to give a derivative's upper bounded of the functional E_1 defined as

$$E_1(t) = E(t) + \varepsilon \int_0^1 \int_0^1 z^2(x, \rho)d\rho dx, \quad \text{for } \varepsilon \geq 0. \tag{2.9}$$

Lemma 2.1. *For any $\varepsilon \geq 0$, the functional E_1 satisfies along the solution of* (2.1) *the following estimate*

(2.10)

$$E_1'(t) \leq -\beta_1 \int_0^1 \phi_t g_1(\phi_t)dx - \beta_2 \int_0^1 z(x, 1)g_2(z(x, 1))dx + \varepsilon \int_0^1 \phi_t^2 dx - \varepsilon \int_0^1 z^2(x, 1)dx,$$

where $\beta_1 = \mu_1 - \gamma\alpha_2 - \alpha_2|\mu_2|$ and $\beta_2 = \gamma(1 - \theta)\alpha_1 - (1 - \alpha_1)|\mu_2|$.

Proof. Multiplying $(2.1)_1$ and $(2.1)_2$ by u_t and ϕ_t, respectively, and using integration by parts over $[0, 1]$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\Big]dx \\ &+ \mu_1 \int_0^1 \phi_t g_1(\phi_t)dx + \mu_2 \int_0^1 \phi_t g_2(z(x, 1))dx = 0. \end{aligned} \tag{2.11}$$

Multiplying $(2.1)_3$ by $\gamma g_2(z(x, \rho))$ and integrating the product over $([0, 1])^2$, we get

$$\gamma\tau(t) \int_0^1 \int_0^1 z_t(x, \rho)g_2(z(x, \rho))d\rho dx + \gamma(1 - \rho\tau'(t)) \int_0^1 \int_0^1 z_\rho(x, \rho)g_2(z(x, \rho))d\rho dx = 0.$$

This means that

$$\gamma\frac{d}{dt}\int_0^1 \int_0^1 \tau(t)G(z(x, \rho))d\rho dx + \gamma\int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\Big((1 - \rho\tau'(t))G(z(x, \rho))\Big)d\rho dx = 0.$$

Consequently, using the fact that $z_t(x,0,t)=\phi_t$, we get

$$\gamma\frac{d}{dt}\int_0^1\int_0^1\tau(t)G(z(x,\rho)d\rho dx=-\gamma\int_0^1\Big[(1-\tau'(t))G(z(x,1))-G(\phi_t)\Big]dx. \tag{2.12}$$

Also, we have

$$\varepsilon\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho)d\rho dx=-\varepsilon\int_0^1\Big[z^2(x,1)-\phi_t^2\Big]dx. \tag{2.13}$$

The last equality has been obtained by applying the same previous arguments and after multiplying $(2.1)_3$ by $2\varepsilon z(x,\rho)$. Combining the estimates (2.11)–(2.13) and using (2.4), we get

$$\begin{aligned} E_1'(t)\leq &-(\mu_1-\gamma\alpha_2)\int_0^1\phi_t g_1(\phi_t)dx-\gamma(1-\theta)\alpha_1\int_0^1 z(x,1)g_2(z(x,1))dx\\ &-\varepsilon\int_0^1 z^2(x,1)dx+\varepsilon\int_0^1\phi_t^2dx-\mu_2\int_0^1\phi_t g_2(z(x,1))dx. \end{aligned} \tag{2.14}$$

From Remark 2.3, we have

$$G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s)),\quad \text{for all } s\geq 0.$$

Hence,

$$G^*(g_2(z(x,1)))=z(x,1)g_2(z(x,1)-G(z(x,1)).$$

Taking (2.8) with $A=g_2(z(x,1))$ and $B=\phi_t$, and using (2.4) again, we obtain

$$\mu_2\phi_t g_2(z(x,1))\leq\alpha_2|\mu_2|\phi_t g_1(\phi_t)+(1-\alpha_1)|\mu_2|z(x,1)g_2(z(x,1)). \tag{2.15}$$

By inserting (2.15) into (2.14), we arrive at the desired inequality. This finishes the proof. □

3. The Well-posedness of the Probem

In the current section, we prove the existence and the uniqueness results to system (2.1). Firstly, we prove the existence of a unique strong solution, next, using a density argument, we extend the obtained result for weak solutions. For this, let $U=U(t)=\Big(u,u_t,\phi,\phi_t,z)^T$ and $U_0=U(0)=(u_0,u_1,\phi_0,\phi_1,f_0(\cdot,-\cdot\tau(0))\Big)^T$. We then consider the following spaces

$$\mathcal{H}=H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L_z^2(0,1)$$

and

$$\mathcal{H}_0=\Big(H^2\cap H_0^1(0,1)\Big)\times H_0^1(0,1)\times\Big(H^2\cap H_0^1(0,1)\Big)\times H_0^1(0,1)\times L^2\Big(0,1;H^1(0,1)\Big).$$

Our first main result is given by the following theorem.

Theorem 3.1. *Assuming that the assumptions (A_1)-(A_3) hold and that $\kappa\xi>b^2$. Then for any $U\in\mathcal{H}$ satisfying the compatibility condition*

$$f_0(\cdot,0)=\phi_1,$$

problem (2.1) *admits only one global weak solution*

$$U \in C\Big([0,+\infty);\mathcal{H}\Big).$$

Moreover, if $U_0 \in \mathcal{H}_0$*, the solution of* (2.1) *is strong solution, and satisfies*

$$U \in C\Big([0,+\infty);\mathcal{H}_0\Big) \cap C^1([0,+\infty);\mathcal{H}\Big).$$

Proof. The proof will be established by implementing the Faedo-Galerkin method. For, let $U \in \mathcal{H}_0$, $T > 0$ be fixed and for $m = 1, 2, \dots,$ let $\{\Phi^m\}_{m\in\mathbb{N}}$ be a Hilbertian basis of $H_0^1(0,1)$ and F^m the vector space generated by $\Phi^1, \Phi^2, \dots, \Phi^m$. Defining, for $1 \leq i \leq m$, the sequence $\Psi^i(x,\rho)$ as

$$\Psi^i(x,0) = \Phi^i(x).$$

Then, we may extend $\Psi^i(x,0)$ by $\Psi^i(x,\rho)$ over $L_z^2(0,1)$ and denote Z^m the space generated by $\Psi^1, \Psi^2, \dots, \Psi^m$. We will construct an approximate solution (u^m, ϕ^m, z^m), $i = 1, 2, \dots,$ in the form

$$(u^m(x,t), \phi^m(x,t)) = \left(\sum_{i=1}^m c^{im}(t), \sum_{i=1}^m d^{im}(t)\right)\Phi^i(x),$$

$$z^m(x,\rho) = \sum_{i=1}^m e^{im}(t)\Psi^i(x,\rho),$$

where c^{im}, d^{im} and e^{im}, $i = 1, 2, \dots, m$, are determined by the following finite dimensional problem

$$\begin{cases} \Big(\kappa u_x^m + b\phi^m, \Phi_x^i\Big) + \Big(\rho_1 u_{tt}^m, \Phi^i\Big) = 0, \\ \Big(\delta\phi_x^m, \Phi_x^i\Big) + \Big(\rho_2\phi_{tt}^m + bu_x^m + \xi\phi^m + \mu_1 g_1(\phi_t^m) + \mu_2 g_2(z^m(\cdot,1)), \Phi^i\Big) = 0, \\ \Big(\tau(t) z_t^m(\cdot,\rho) + (1-\rho\tau'(t)) z_\rho^m(\cdot,\rho), \Psi^i(\cdot,\rho)\Big) = 0, \end{cases} \tag{3.1}$$

with

$$\begin{aligned} u^m(\cdot,0) =& u_0^m = \sum_{i=1}^m (u_0, \Phi^i)\Phi^i \to u_0, \quad \text{in } H^2 \cap H_0^1(0,1), \\ u_t^m(\cdot,0) =& u_1^m = \sum_{i=1}^m (u_1, \Phi^i)\Phi^i \to u_1, \quad \text{in } H_0^1(0,1), \\ \phi^m(0) =& \phi_0^m = \sum_{i=1}^m (\phi_0, \Phi^i)\Phi^i \to \phi_0, \quad \text{in } H^2 \cap H_0^1(0,1), \\ \phi_t^m(\cdot,0) =& \phi_1^m = \sum_{i=1}^m (\phi_1, \Phi^i)\Phi^i \to \phi_1, \quad \text{in } H_0^1(0,1), \\ z^m(\cdot,\cdot,\cdot,0) =& z_0^m = \sum_{i=1}^m (f_0, \Psi^i)\Psi^i \to f_0, \quad \text{in } L^2\Big(0,1; H^1(0,1)\Big), \end{aligned} \tag{3.2}$$

as $m \to +\infty$.

The standard methods of ODEs give the existence of a unique solution of (3.1) on the inertval $[0, T_m]$, $0 < T_m < T$. In the next step, we will prove that T_m is independent of m. In other words, the approximate solution becomes global and defined for all $t > 0$.

1.The first priori estimate. As for Lemma 2.1, the functional

$$\begin{aligned} E_1^m(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1|u_t^m|^2 + \rho_2|\phi_t^m|^2 + \kappa|u_x^m|^2 + \delta|\phi_x^m|^2 + \xi|\phi^m|^2 + 2bu_x^m\phi^m \\ & + 2\gamma\tau(t)\int_0^1 G(z^m(x,\rho))d\rho + 2\varepsilon\int_0^1 |z^m(x,\rho)|^2 d\rho\Big]dx \end{aligned}$$

satisfies, for any $\varepsilon \geq 0$,

$$\begin{aligned} &(E_1^m(t))' + \beta_1\int_0^1 \phi_t^m g_1(\phi_t^m)dx + \beta_2\int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ &+ \varepsilon\int_0^1 |z^m(x,1)|^2dx \leq \varepsilon\int_0^1 |\phi_t^m|^2 dx. \end{aligned}$$

Choosing $\varepsilon > 0$, then integrating over $[0, t]$ and taking the convergences (3.2) into account, we get

$$\begin{aligned} &E_1^m(t) + \beta_1\int_0^t\int_0^1 \phi_t^m g_1(\phi_t^m)dxdt \\ &+ \beta_2\int_0^t\int_0^1 z^m(x,1)g_2(z^m(x,1))dxdt + \varepsilon\int_0^t\int_0^1 |z^m(x,1)|^2dxdt \\ \leq& c + \varepsilon\int_0^t\int_0^1 |\phi_t^m|^2dxdt. \end{aligned}$$

The Gronwall's Lemma yields the following first priori estimate

$$\begin{aligned} E_1^m(t) &+ \int_0^t\int_0^1 \phi_t^m g_1(\phi_t^m)dxdt + \int_0^t\int_0^1 z^m(x,1)g_2(z^m(x,1))dxdt \\ &+ \int_0^t\int_0^1 |z^m(x,1)|^2dxdt \leq c. \end{aligned} \tag{3.3}$$

This estimate gives us the global existence of (u^m, ϕ^m, z^m) in $[0, +\infty)$ and

z^m is uniformly bounded in $L^\infty_{\text{loc}}\Big(0,\infty; L^2_z(0,1)\Big)$,

u^m, ϕ^m are uniformly bounded in $L^\infty_{\text{loc}}\Big(0,\infty; H^1_0(0,1)\Big)$,

u^m_t, ϕ^m_t are uniformly bounded in $L^\infty_{\text{loc}}\Big(0,\infty, L^2(0,1)\Big)$,

$\phi_t^m g_1(\phi_t^m)$ is uniformly bounded in $L^1\Big((0,T)\times(0,1)\Big)$,

$z^m(x,1)g_2(z^m(x,1))$ is uniformly bounded in $L^1\Big((0,T)\times(0,1)\Big)$.

2. The second priori estimate. Firstly, we are going to estimate $u_{tt}^m(0)$ and $\phi_{tt}^m(0)$ in the L^2-norm. Also, we need to estimate $z_t^m(x,\rho,0)$ in the L^2_z-norm. For that, we replace Φ^i in $(3.1)_1$ by u_{tt}^m, Φ^i in $(3.1)_2$ by ϕ_{tt}^m and using Young's inequality to get

$$\begin{aligned}\int_0^1 \Big[|u_{tt}^m(0)|^2 + |\phi_{tt}^m(0)|^2\Big]dx \leq & c\int_0^1 \Big[|u_{xx}^m(0)|^2 + |u_x^m(0)|^2 + |\phi_{xx}^m(0)|^2 + |\phi_x^m(0)|^2 \\ & + |\phi^m(0)|^2 + g_1^2(\phi_t^m(0)) + g_2^2(z^m(x,1,0))\Big]dx.\end{aligned} \tag{3.4}$$

Replacing Ψ^i in $(2.1)_3$ by $z_t^m(x,\rho,t)$ and using Cauchy-Schwarz and Young's inequalities, we get

$$\int_0^1\int_0^1 |z_t^m(x,\rho,0)|^2 d\rho dx \leq c\int_0^1\int_0^1 |z_\rho^m(x,\rho,0)|^2 d\rho dx. \tag{3.5}$$

The sum of (3.4)–(3.5) with (3.2) yields

$$\int_0^1 \Big[|u_{tt}^m(0)|^2 + |\phi_{tt}^m(0)|^2 + \int_0^1 |z_t^m(x,\rho,0)|^2 d\rho\Big]dx \leq c. \tag{3.6}$$

Now, we derivate $(3.1)_1$ and $(3.1)_2$ with respect to t. Then, we set $\Phi^i = 2u_{tt}^m$ and $\Phi^i = 2\phi_{tt}^m$, respectively, in the first and the second resulting equations and using the non-decreasing property of g_1, we find

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m\Big]dx \\ \leq & -\mu_2\int_0^1 z_t^m(x,1)g_2'(z^m(x,1))\phi_{tt}^m dx.\end{aligned}$$

The boundedness of g_2' and the Young's inequality imply that

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m\Big]dx \\ \leq & \epsilon_1\int_0^1 |z_t^m(x,1)|^2 dx + c\int_0^1 |\phi_{tt}^m|^2 dx.\end{aligned} \tag{3.7}$$

In the other hand, taking the derivative of $(3.1)_3$ with respect to t and then setting $\Psi^i = 2z_t^m(x,\rho,t)$ in the resulting equation, it follows that

$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))}|z_t^m(x,\rho,t)|^2 d\rho dx + \int_0^1\int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'|z_t^m(x,\rho,t)|^2 d\rho dx \\ &+ \int_0^1\int_0^1 \frac{d}{d\rho}|z_t^m(x,\rho,t)|^2 d\rho dx = 0.\end{aligned}$$

As $z_t^m(x,0,t) = \phi_{tt}^m(x,t)$, it comes

$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))}|z_t^m(x,\rho,t)|^2 d\rho dx + \int_0^1\int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))}\right)'|z_t^m(x,\rho,t)|^2 d\rho dx \\ &+ \int_0^1 |z_t^m(x,1,t)|^2 d\rho dx = \int_0^1 |\phi_{tt}^m|^2 dx.\end{aligned} \tag{3.8}$$

Let I^m be defined by

$$I^m(t)=\int_0^1\Big[\rho_1|u_{tt}^m|^2+\rho_2|\phi_{tt}^m|^2+\kappa|u_{xt}^m|^2+\delta|\phi_{xt}^m|^2 \\ +\xi|\phi_t^m|^2+2bu_{xt}^m\phi_t^m+\frac{\tau(t)}{(1-\rho\tau'(t))}\int_0^1|z_t^m(x,\rho)|^2d\rho\Big]dx,$$

hence from the estimates (3.7)–(3.8), we find

$$(I^m(t))'+(1-\epsilon_1)\int_0^1|z_t^m(x,1))|^2dx\le c\int_0^1|\phi_{tt}^m|^2dx.$$

Choosing $\epsilon_1<1$, then integrating over $[0,t]$, we get

$$I^m(t)+\int_0^t\int_0^1|z_t^m(x,1)|^2dxdt\le cI^m(0)+c\int_0^t\int_0^1|\phi_{tt}^m|^2dxdt.$$

Employing Gronwall's lemma with (3.2) and (3.6), we obtain the second estimate below

$$I^m(t)+\int_0^t\int_0^1|z_t^m(x,1))|^2dxdt\le c. \tag{3.9}$$

We, therefore, deduce that

$$\begin{aligned}&z_t^m \quad \text{is uniformly bounded in} \quad L^2\Big(0,T;L_z^2(0,1)\Big),\\ &u_t^m,\phi_t^m \quad \text{are uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty;H_0^1(0,1)\Big),\\ &u_{tt}^m,\phi_{tt}^m \quad \text{are uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty;L^2(0,1)\Big),\end{aligned}$$

Hence it follows from the estimates (3.3) and (3.9) that it exist subsequences $\{u^n\}_{n=1}^\infty\subset\{u^m\}_{m=1}^\infty$, $\{\phi^n\}_{n=1}^\infty\subset\{\phi^m\}_{m=1}^\infty$ and $\{z^n\}_{n=1}^\infty\subset\{z^m\}_{m=1}^\infty$ verify for all $T\ge 0$ the following convergences

$$\begin{cases} g_1(\phi_t^n)\to f \quad \text{and} \quad g_2(z^n)\to h \quad \text{weakly-star in} \quad L^2\Big(0,T;L^2\Big),\\ u^n\to u \quad \text{and} \quad \phi^n\to\phi \quad \text{weakly-star in} \quad L^2\Big(0,T;H_0^1\Big),\\ u_t^n\to u_t \quad \text{and} \quad \phi_t^n\to\phi_t \quad \text{weakly-star in} \quad L^\infty\Big(0,T;H_0^1\Big),\\ u_{tt}^n\to u_{tt} \quad \text{and} \quad \phi_{tt}^n\to\phi_{tt} \quad \text{weakly-star in} \quad L^\infty\Big(0,T;L^2\Big),\\ z^n\to z \quad \text{and} \quad z_t^n\to z_t \quad \text{weakly-star in} \quad L^\infty\Big(0,T;L_z^2\Big),\end{cases} \tag{3.10}$$

We will show that (u,ϕ,z) is a strong solution of system (2.1). Firstly, we prove that $f=g_1(\phi_t)$ and $h=g_2(z(x,1))$ which will be given in the following lemma.

Lemma 3.1. *For each $T>0$, $g_1(\phi_t^n)\to g_1(\phi_t)$ weakly-star in $L^2\Big((0,1)\times(0,T)\Big)$ and $g_2(z^n(x,1))\to g_2(z(x,1))$ weakly-star in $L^2\Big((0,1)\times(0,T)\Big)$.*

Proof. From (3.9), we have ϕ_t^n is bounded in $L^\infty(0,T;H_0^1)$ and ϕ_{tt}^n is bounded in $L^\infty(0,T;L^2)$. Then, the injection by continuity in L^p gives us the boundedness of ϕ_t^n in $L^2(0,T;H_0^1)$ and ϕ_{tt}^n in $L^2(0,T;L^2)$. Hence, ϕ_t^n is bounded in $H^1(Q)$, where

$Q = (0,1) \times (0,T)$. It is known that the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact. This permit us to extract a subsequence ϕ^n, still represented by the same notation, such that

$$\phi_t^n \to \phi_t \quad \text{strongly in} \quad L^2\Big(0,T;L^2(0,1)\Big),$$

which gives

$$\phi_t^n \to \phi_t, \quad \text{a.e. on } Q.$$

Then, by the continuity of g_1,

$$g_1(\phi_t^n) \to g_1(\phi_t), \qquad \text{a.e. on } Q. \tag{3.11}$$

Similarly,

$$g_2(z^n(x,1)) \to g_2(z(x,1)), \quad \text{a.e. on } Q. \tag{3.12}$$

On the other hand, with $\mathcal{R}^m(x,t)$ defined as

$$\mathcal{R}^m(x,t) = \phi_t^m g_1(\phi_t^m) + z^m(x,1) g_2(z^m(x,1)),$$

we assert by using Jensen's inequality and the concavity of H^{-1} that

$$\begin{aligned} \int_0^1 H^{-1}\Big(\mathcal{R}^m(x,t)\Big)dx \leq & cH^{-1}\left(\int_0^1 \mathcal{R}^m(x,t)dx\right) \\ \leq & cH^*(1) + c\int_0^1 \mathcal{R}^m(x,t)dx. \end{aligned} \tag{3.13}$$

For $r^m = |\phi_t^m| + |z^m(x,1)|$, we write

$$\begin{aligned} \int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \leq & \int_{r^m \leq \epsilon} \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \\ & + \int_{r^m \geq \epsilon} \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx. \end{aligned}$$

Then, by using (2.2) and (3.13), we get

$$\int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \leq cH^*(1) + c\int_0^1 \mathcal{R}^m(x,t)dx.$$

Thus, by (3.3), it results

$$\int_0^t \int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dxdt \leq c,$$

which implies that $g_1(\phi_t^n), g_2(z^n(x,1)) \in L^2(Q)$. Combining these with (3.11)–(3.12) and using Lemma 1.3 in [9] page 12, we derive to

$$g_1(\phi_t^n) \to g_1(\phi_t) \quad \text{weakly-star in} \quad L^2((0,1)\times(0,T)),$$

$$g_2(z^n(x,1)) \to g_2(z(x,1)) \quad \text{weakly-star in} \quad L^2((0,1)\times(0,T)).$$

This shows that $f = g_1(\phi_t)$ and $h = g_2(z(x,1))$. $\square$

Passage to the limit. To prove that (u, ϕ, z) is a strong solution of problem (2.1) we discuss as in [9]. For this, we consider functions $v, \omega \in C\Big(0, T; H_0^1(0,1)\Big)$ and $y \in C\Big(0, T; L_z^2(0,1)\Big)$ having the forms

$$(v(x,t), \omega(x,t)) = \Big(\sum_{i=1}^{N} \widetilde{c}^{in}(t), \sum_{i=1}^{N} \widetilde{d}^{in}(t)\Big)\Phi^i(x), \tag{3.14}$$

$$y(x,\rho,t) = \sum_{i=1}^{N} \widetilde{e}^{in}(t)\Psi^i(x,\rho), \tag{3.15}$$

where $N \geq n$ is a fixed integer.

Then we multiply $(3.1)_1$, $(3.1)_2$ and $(3.1)_3$ by $\widetilde{c}^{in}(t)$, $\widetilde{d}^{in}$ and $\widetilde{c}^{in}$, respectively, and summing the resultants over i from 1 to N, we find that

$$\begin{cases} \displaystyle\int_0^T \int_0^1 \Big[\Big(\kappa u_x^n + b\phi^n\Big)v_x + \rho_1 u_{tt}^n v\Big] dxdt = 0, \\ \displaystyle\int_0^T \int_0^1 \Big[\delta\phi_x^n \omega_x + \Big(\rho_2 \phi_{tt}^n + b u_x^n + \xi \phi_x^n + \mu_1 g_1(\phi_t^n) + \mu_2 g_2(z^n(x,1))\Big)\omega\Big] dxdt = 0, \\ \displaystyle\int_0^T \int_0^1 \int_0^1 \Big[\tau(t) z_t^n(x,\rho) + (1-\rho\tau'(t)) z_\rho^n(x,\rho)\Big] y(x,\rho) d\rho dxdt = 0. \end{cases} \tag{3.16}$$

After passing to the limit in (3.16) as $n \to +\infty$ and using (3.10), we arrive at

$$\begin{cases} \displaystyle\int_0^T \int_0^1 \Big[\Big(\kappa u_x + b\phi\Big)v_x + \rho_1 u_{tt} v\Big] dxdt = 0, \\ \displaystyle\int_0^T \int_0^1 \Big[\delta\phi_x \omega_x + \Big(\rho_2 \phi_{tt} + b u_x + \xi \phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(z(x,1))\Big)\omega\Big] dxdt = 0, \\ \displaystyle\int_0^T \int_0^1 \int_0^1 \Big[\tau(t) z_t(x,\rho) + (1-\rho\tau'(t)) z_\rho(x,\rho)\Big] y(x,\rho) d\rho dxdt = 0. \end{cases} \tag{3.17}$$

The above equations hold for all $(v, \omega, y) \in \Big(L^2\Big(0, T; H_0^1\Big)\Big)^2 \times L^2\Big(0, T; L_z^2\Big)$ since the functions of the forms (3.14) and (3.15) are dense, respectively, in $L^2\Big(0, T; H_0^1\Big)$ and $L^2\Big(0, T; L_z^2\Big)$. Next, we must verify that the limit functions u, ϕ, z satisfy the initial conditions, i.e.,

$$u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad \phi(\cdot,0) = \phi_0, \quad \phi_t(\cdot,0) = \phi_1, \quad z(\cdot,0) = f_0. \tag{3.18}$$

For, we let $v, \omega \in C^2\Big(0, T; H_0^1\Big)$ and $y \in C^1(0, T, L_z^2)$ with

$$u(x,T) = u_t(x,T) = \phi(x,T) = \phi_t(x,T) = y(x,\rho,T) = 0.$$

Then we integrate with respect to t in (3.17), we get

(3.19)

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u v_{tt} - \Big(\kappa u_x + b\phi\Big) v_x\Big] dxdt + \rho_1 \int_0^1 \Big[u(0)v_t(0) - u_t(0)v(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi \omega_{tt} + \delta \phi_x \omega_x + \Big(b u_x + \xi \phi_x + \mu_1 g_1(\phi_t) + \mu_2 g_2(z(x,1))\Big)\omega\Big] dxdt \\ +\rho_2 \int_0^1 \Big[\phi(0)\omega_t(0) - \phi_t(0)\omega(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[- z(x,\rho,t) y_t(x,\rho,t) + \dfrac{1-\rho\tau'(t)}{\tau(t)} z_\rho(x,\rho,t) y(x,\rho,t)\Big] d\rho dx dt \\ - \int_0^1 \int_0^1 z(x,\rho,0) y(x,\rho,0) d\rho dx = 0. \end{cases}$$

Similarly from (3.16), we have

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u^n v_{tt} + \Big(\kappa u^n_x + b\phi^n\Big) v_x\Big] dxdt + \rho_1 \int_0^1 \Big[u^n(0)v_t(0) - u^n_t(0)v(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi^n \omega_{tt} + \delta \phi^n_x \omega_x + \Big(b u^n_x + \xi \phi^n_x + \mu_1 g_1(\phi^n_t) + \mu_2 g_2(z^n(x,1))\Big)\omega\Big] dxdt \\ +\rho_2 \int_0^1 \Big[\phi^n(0)\omega_t(0) - \phi^n_t(0)\omega(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[z^n(x,\rho,t) y_t(x,\rho,t) + \dfrac{1-\rho\tau'(t)}{\tau(t)} z^n_\rho(x,\rho,t) y(x,\rho,t)\Big] d\rho dx dt \\ - \int_0^1 \int_0^1 z^n(x,\rho,0) y(x,\rho,0) d\rho dx = 0. \end{cases}$$

Recalling (3.10) and (3.2), we obtain

$$\begin{cases} \int_0^T \int_0^1 \Big[\rho_1 u v_{tt} + \Big(\kappa u_x + b\phi\Big) v_x\Big] dxdt + \rho_1 \int_0^1 \Big[u_0 v_t(0) - u_1 v(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \Big[\rho_2 \phi \omega_{tt} + \delta \phi_x \omega_x + \Big(b u_x + \xi \phi_x + \mu_1 g_1(\phi_t) + \mu_2 g_2(z(x,1))\Big)\omega\Big] dxdt \\ +\rho_2 \int_0^1 \Big[\phi_0 \omega_t(0) - \phi_1 \omega(0)\Big] dx = 0, \\ \int_0^T \int_0^1 \int_0^1 \Big[- z(x,\rho,t) y_t(x,\rho,t) + \dfrac{1-\rho\tau'(t)}{\tau(t)} z_\rho(x,\rho,t) y(x,\rho,t)\Big] d\rho dx dt \\ - \int_0^1 \int_0^1 f_0 y(x,\rho,0) d\rho dx = 0. \end{cases} \tag{3.20}$$

As $v(x,0)$, $v_t(x,0)$, $\omega(x,0)$, $\omega_t(x,0)$, $y(x,\rho,0)$ are arbitrary, comparing identities (3.19) and (3.20), we deduce (3.18). Consequently, (2.1) admits at least one global strong solution (u,ϕ,z).

For the uniqueness, we assume that $(\widetilde{u}, \widetilde{\phi}, \widetilde{z})$ and $(\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ are two solutions of system (2.1). Then $(u, \phi, z) = (\widetilde{u}, \widetilde{\phi}, \widetilde{z}) - (\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ verifies the following system

(3.21)
$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \\ \rho_2 \phi_{tt} - \delta \phi_{xx} + b u_x + \xi \phi + \mu_1 \Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t) \Big) + \mu_2 \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1)) \Big) = 0, \\ \tau(t) z_t(x,\rho,t) + (1 - \rho\tau'(t)) z_\rho(x,\rho,t) = 0, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(0,t) = 0, \\ u(x,0) = u_t(x,0) = \phi(x,0) = \phi_t(x,0) = z(x,\rho,0) = 0. \end{cases}$$

To get the uniqueness of solution of (2.1), we must verify that $(u, \phi, z) = (0,0,0)$ is the solution of (3.21). For that, a multiplication of $(3.21)_1$ by $2u_t$ and $(3.21)_2$ by $2\phi_t$, yields

(3.22)
$$\frac{d}{dt} \int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2b u_x \phi \Big] dx + 2\mu_1 \int_0^1 \phi_t \Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t) \Big) dx$$
$$+ 2\mu_2 \int_0^1 \phi_t \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1)) \Big) dx = 0.$$

Then, we multiply $(3.21)_3$ by $2z$, we get

$$\frac{d}{dt} \int_0^1 \int_0^1 \tau(t) z^2(x,\rho) d\rho dx + \int_0^1 (1 - \tau'(t)) z^2(x,1) dx - \int_0^1 \phi_t^2 dx = 0. \tag{3.23}$$

By setting

$$\Lambda(t) = \int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2b u_x \phi + \tau(t) \int_0^1 z^2(x,\rho) d\rho \Big] dx$$

and summing the estimates (3.22)–(3.23), we obtain

$$\begin{aligned} \Lambda'(t) =& 2\mu_1 \int_0^1 \omega_t \Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t) \Big) dx + \int_0^1 \phi_t^2 dx - \int_0^1 (1 - \tau'(t)) z^2(x,1) dx \\ & - 2\mu_2 \int_0^1 \phi_t \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1)) \Big) dx. \end{aligned} \tag{3.24}$$

As g_1 is an increasing function, we can easily see that

$$\Big(s_0 - s \Big) \Big(g_1(s_0) - g_1(s) \Big) > 0, \quad \text{for all } s_0, s \in \mathbb{R}.$$

Thus, (3.24) becomes

$$\Lambda'(t) \leq \int_0^1 \phi_t^2 dx - (1 - \theta) \int_0^1 z^2(x,1) dx - 2\mu_2 \int_0^1 \phi_t \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1)) \Big) dx.$$

Using Young's inequality, we get

$$\Lambda'(t) \leq c \int_0^1 \phi_t^2 dx - \int_0^1 z^2(x,1) dx + \epsilon_2 \int_0^1 \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1)) \Big)^2 dx.$$

Since g_2 is C^1 then g_2 is Lipschitzien function, this leads us to

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx - (1-c\epsilon_2)\int_0^1 z^2(x,1)dx.$$

Hence, for a suitable ϵ_2, we have

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx.$$

As $\Lambda(t)$ is positive (for the same raison given in Remark 2.1) and $\Lambda(0)=0$, Gronwall's Lemma forces that $\Lambda(t)=0$ $(0\leq t\leq T)$, which means that $u=\phi=z=0$.

Consequently, (2.1) has only one global strong solution.

If $U_0 \in \mathcal{H}$, then it results from the density of $\mathcal{H}_0$ in $\mathcal{H}$ that the system (2.1) has a unique global weak solution. □

4. Asymptotic Behavior

This section will be concerned with the study of the solution's asymptotic behavior of system (2.1). In fact, using the Lyapunov method, we will prove that, under equal wave speeds and non-equal wave speeds cases, the solution of (2.1) converges to zero as t tends to infinity.

We start with this important notation. By setting $\varepsilon = 0$ in (2.9) and under the assumption (A$_1$), we have

$$E'(t) \leq -\beta_1\int_0^1 \phi_t g_1(\phi_t)dx - \beta_2\int_0^1 z(x,1)g_2(z(x,1))dx \leq 0, \quad \text{for all } t\geq 0. \tag{4.1}$$

Then (2.1) is dissipative with respect to E.

4.1. **Technical lemmas.**

In this subsection, we state and prove various lemmas given for (u,ϕ,z) a solution of (2.1). It would help us to estimate the derivative of the Lyapunov functional.

Lemma 4.1. *The functional*

$$F_1(t) = -\rho_1\int_0^1 u_t u dx$$

satisfies

$$F_1'(t) \leq -\rho_1\int_0^1 u_t^2 dx + \frac{3\kappa}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{4.2}$$

Proof. A simple differentiation with respect to t, using $(2.1)_1$, yields

$$F_1'(t) = -\rho_1\int_0^1 u_t^2 dx + \kappa\int_0^1 u_x^2 dx + b\int_0^1 u_x\phi dx.$$

The Young's and Poincaré's inequalities lead to (4.2). □

Lemma 4.2. *The functional defined by*

$$F_2(t) = \rho_2 \int_0^1 \phi_t u_x dx + \frac{\delta \rho_1}{\kappa} \int_0^1 u_t \phi_x dx$$

satisfies for any $\eta > 0$

$$\begin{aligned} F_2'(t) \leq & -\frac{b}{2} \int_0^1 u_x^2 dx + \eta \Big(u_x^2(1,t) + u_x^2(0,t) \Big) + \frac{\delta^2}{4\eta} \Big(\phi_x^2(1,t) + \phi_x^2(0,t) \Big) \\ & + c \int_0^1 \phi_x^2 dx + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx + \left(\frac{\delta \rho_1}{\kappa} - \rho_2 \right) \int_0^1 \phi_{xt} u_t dx. \end{aligned} \tag{4.3}$$

Proof. Direct computations, using $(2.1)_1$–$(2.1)_2$, lead to

$$\begin{aligned} F_2'(t) = & \int_0^1 u_x \Big[\delta \phi_{xx} - b u_x - \xi \phi - \mu_1 g_1(\phi_t) - \mu_2 g_2(z(x,1)) \Big] dx \\ & + \frac{\delta}{\kappa} \int_0^1 \phi_x \Big[\kappa u_{xx} + b \phi_x \Big] dx + \left(\frac{\delta \rho_1}{\kappa} - \rho_2 \right) \int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} F_2'(t) = & \Big[\delta u_x \phi_x \Big]_{x=0}^{x=1} - b \int_0^1 u_x^2 dx + \frac{b\delta}{\kappa} \int_0^1 \phi_x^2 dx - \xi \int_0^1 u_x \phi dx - \mu_1 \int_0^1 g_1(\phi_t) u_x dx \\ & - \mu_2 \int_0^1 g_2(z(x,1)) u_x dx + \left(\frac{\delta \rho_1}{\kappa} - \rho_2 \right) \int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

Using Young's and Poincaré's inequalities, (4.3) is established. □

Lemma 4.3. *Let* χ *be a solution of*

$$\begin{cases} \chi_{xx} = -\phi_x, \\ \chi(0) = \chi(1) = 0. \end{cases}$$

Then the functional

$$F_3(t) = \int_0^1 \left(\rho_2 \phi_t \phi + \frac{b \rho_1}{\kappa} u_t \chi \right) dx$$

satisfies the following estimate

$$\begin{aligned} F_3'(t) \leq & -\delta \int_0^1 \phi_x^2 dx - \frac{1}{2} \left(\xi - \frac{b^2}{\kappa} \right) \int_0^1 \phi^2 dx + \eta_0 \int_0^1 u_t^2 dx + c \int_0^1 \phi_t^2 dx \\ & + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx, \quad \textit{for all } \eta_0 > 0. \end{aligned} \tag{4.4}$$

Proof. Differentiating F_3 and using $(2.1)_1$–$(2.1)_2$, we get

$$\begin{aligned} F_3'(t) = & -\xi \int_0^1 \phi^2 dx + \frac{b^2}{\kappa} \int_0^1 \chi_x^2 dx - \delta \int_0^1 \phi_x^2 dx + \rho_2 \int_0^1 \phi_t^2 dx + \frac{b \rho_1}{\kappa} \int_0^1 u_t \chi_t dx \\ & - \mu_1 \int_0^1 \phi g_1(\phi_t) dx - \mu_2 \int_0^1 \phi g_2(z(x,1)) dx. \end{aligned} \tag{4.5}$$

By exploiting Young's inequality, we have

$$\frac{b\rho_1}{\kappa}\int_0^1 u_t\chi_t dx \leq \eta_0\int_0^1 u_t^2 dx + c\int_0^1 \chi_t^2 dx, \tag{4.6}$$

$$\mu_1\int_0^1 \phi g_1(\phi_t)dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2 dx + c\int_0^1 g_1^2(\phi_t)dx, \tag{4.7}$$

$$\mu_2\int_0^1 \phi g_2(z(x,1))dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2 dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{4.8}$$

Inserting (4.6)–(4.8) into (4.5) and using the fact that

$$\int_0^1 \chi_x^2 dx \leq \int_0^1 \phi^2 dx,$$
$$\int_0^1 \chi_t^2 dx \leq \int_0^1 \chi_{tx}^2 dx \leq \int_0^1 \phi_t^2 dx,$$

we obtain (4.4). □

Next, in order to eliminate the boundary terms, appearing in (4.3), we introduce the following function

$$m(x) = -4x + 2, \quad x \in [0,1]. \tag{4.9}$$

Then, we have the following result.

Lemma 4.4. *For any $\eta > 0$, the functional F_4 defined by*

$$F_4(t) = \frac{\eta}{\kappa}\int_0^1 \rho_1 m(x)u_t u_x dx + \frac{\delta}{4\eta}\int_0^1 \rho_2 m(x)\phi_t\phi_x dx$$

satisfies

$$\begin{aligned} F_4'(t) \leq & -\eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) - \frac{\delta^2}{4\eta}\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big) \\ & + c\eta\rho_1\int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx + \left(\left(\frac{1}{4} + \frac{\eta}{4}\right)b + 2\eta\right)\int_0^1 u_x^2 dx \\ & + c\int_0^1 \phi_x^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \end{aligned} \tag{4.10}$$

Proof. By using $(2.1)_1$, $(2.1)_2$ and (4.9), it holds that

$$\begin{aligned} F_4'(t) = & \frac{\eta}{\kappa}\bigg[-\kappa\Big(u_x^2(1,t) + u_x^2(0,t)\Big) + 2\rho_1\int_0^1 u_t^2 dx + b\int_0^1 m(x)u_x\phi_x dx \\ & + 2\kappa\int_0^1 u_x^2 dx\bigg] + \frac{\delta}{4\eta}\bigg[-\delta\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big) + 2\rho_2\int_0^1 \phi_t^2 dx \\ & + 2\delta\int_0^1 \phi_x^2 dx - b\int_0^1 m(x)\phi_x u_x dx - \mu_1\int_0^1 m(x)\phi_x g_1(\phi_t)dx \\ & - \mu_2\int_0^1 m(x)\phi_x g_2(z(x,1))dx - 2\xi\int_0^1 \phi^2 dx\bigg]. \end{aligned}$$

The estimate (4.10) follows by exploiting Young's and Poincaré's inequalities. □

Lemma 4.5. *The functional*

$$F_5(t) = \tau(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx$$

satisfies

(4.11)

$$\begin{aligned}F_5'(t) \le &-\tau(t)e^{-\tau_1}\int_0^1\int_0^1 G(z(x,\rho,t))d\rho dx - \alpha_1(1-\theta)e^{-\tau_1}\int_0^1 z(x,1)g_2(z(x,1))dx\\ &+ c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx.\end{aligned}$$

Proof. Taking the derivative of F_5 and using $(2.1)_3$, we have

$$\begin{aligned}F_5'(t) =&\tau'(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx\\ &+\int_0^1\int_0^1(1-\rho\tau'(t))e^{-\tau(t)\rho}z_\rho(x,\rho,t)g_2(z(x,\rho,t))d\rho dx.\end{aligned}$$

Then

$$\begin{aligned}F_5'(t) =&-\int_0^1\int_0^1 \frac{d}{d\rho}\Big[(1-\rho\tau'(t))e^{-\tau(t)\rho}G(z(x,\rho,t))\Big]d\rho dx\\ &-\int_0^1\int_0^1 \tau(t)e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx\\ =&-\int_0^1\Big[(1-\tau'(t))e^{-\tau(t)}G(z(x,1,t)) - G(z(x,0,t))\Big]dx\\ &-\tau(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx.\end{aligned}$$

Using (2.4) with the fact that $z(x,0,t)=\phi_t$, $e^{-\tau(t)} \le e^{-\tau_1\rho} \le 1$ for all $\rho\in[0,1]$ and $\tau\in[\tau_0,\tau_1]$, we obtain

$$\begin{aligned}F_5'(t) \le &-\tau(t)\int_0^1\int_0^1 e^{-\tau_1}G(z(x,\rho,t))d\rho dx. - e^{-\tau_1}(1-\theta)\alpha_1\int_0^1 z(x,1)g_2(z(x,1))dx\\ &+\alpha_2\int_0^1 \phi_t g_1(\phi_t)dx.\end{aligned}$$

The estimate (4.11) follows by exploiting Young's inequality. □

Lemma 4.6. *For a suitable choice of N and N_i, $i=1,2,\dots,5$, the functional defined by*

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{5} N_iF_i(t). \qquad (4.12)$$

satisfies, for a fixed positive constant m_0, the estimate

$$\mathcal{L}'(t) \leq -m_0 E(t) + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{4.13}$$

Proof. From (4.1), (4.2), (4.3), (4.4), (4.10) and (4.11), it follows that for any $t \geq 0$

$$\begin{aligned}
\mathcal{L}'(t) \leq & -(N_4 - N_2)\Big[\eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) + \frac{\delta^2}{4\eta}\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big)\Big] \\
& - \Big[\rho_1 N_1 - \eta_0 N_3 - \eta c \rho_1 N_4\Big] \int_0^1 u_t^2 dx + \Big[N_3 + N_4 + N_5\Big] c \int_0^1 \phi_t^2 dx \\
& - \Big[\frac{b}{2} N_2 - \frac{3\kappa}{2} N_1 - \Big(\Big(\frac{1}{4} + \frac{\eta}{4}\Big) b + 2\eta\Big) N_4\Big] \int_0^1 u_x^2 dx \\
& - \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big) N_3 \int_0^1 \phi^2 dx - \Big[\delta N_3 - \Big(N_1 + N_2 + N_4\Big) c\Big] \int_0^1 \phi_x^2 dx \\
& - \tau e^{-\tau} N_5 \int_0^1 \int_0^1 G(z(x,\rho)) d\rho dx + \Big[N_2 + N_3 + N_4 + N_5\Big] c \int_0^1 g_1^2(\phi_t) dx \\
& + \Big[N_2 + N_3 + N_4\Big] c \int_0^1 g_2^2(z(x,1)) dx + N_2 \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx.
\end{aligned}$$

Furthermore, we take

$$N_1 = 3\eta c, \quad N_2 = N_4 = N_5 = 1, \quad \eta_0 = \frac{\eta c \rho_1}{N_3},$$

to get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\eta c \rho_1 \int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx - \frac{1}{4}\Big(b - \eta\Big(18\kappa c + b + 8\Big)\Big) \int_0^1 u_x^2 dx \\
& - \Big(\delta N_3 - c\Big) \int_0^1 \phi_x^2 dx - \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big) N_3 \int_0^1 \phi^2 dx + c\int_0^1 g_2^2(z(x,1)) dx \\
& - \tau e^{-\tau} \int_0^1 \int_0^1 G(z(x,\rho)) d\rho dx + c\int_0^1 g_1^2(\phi_t) dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx.
\end{aligned} \tag{4.14}$$

Now, we select $\eta < \frac{b}{18\kappa c + b + 8}$ and then we choose N_3 large enough such that

$$\delta N_3 - c > 0.$$

Hence, the estimate (4.14) with the fact that $\kappa\xi > b^2$ and (2.6) gives us (4.13). □

4.2. **General decay rates for equal of wave speeds.** In this subsection, we show that the solution have a general decay rate in the case of equal speeds of wave propagation.

Theorem 4.1. *Let $U \in \mathcal{H}$. Assuming that (A_1), (A_2) and (A_3) are fulfilled, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}.$$

Then, there exist positive constants a, a_1 and a_2 such that the solution of (2.1) *satisfies*

$$E(t) \leq a H_1^{-1}\Big(a_1 t + a_2\Big), \quad \text{for all } t > 0, \tag{4.15}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds \quad \text{and} \quad H_2(t) = tH'(\epsilon_0 t).$$

Proof. Since $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$, then we can easily show for N sufficiently large, that the functional $\mathcal{L}$ given by (4.12) is equivalent to E, i.e.,

$$\mathcal{L}(t) \sim E(t).$$

We consider, as is [8], the following two partitions of $[0, 1]$

$$\mathcal{D}_1 = \Big\{x \in [0,1] : |\phi_t| + |z(x,1)| \leq \epsilon\Big\}, \quad \mathcal{D}_2 = \Big\{x \in [0,1] : |\phi_t| + |z(x,1)| > \epsilon\Big\}$$

and we define $\mathcal{R}(x, t)$ by

$$\mathcal{R}(x,t) = \phi_t g_1(\phi_t) + z(x,1,t) g_2(z(x,1,t)).$$

Then by recalling (2.2) and (4.1), we obtain

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) + \int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big)dx. \tag{4.16}$$

Now, we discuss two cases.

1. H **is linear on** $[0, \epsilon]$. In this case, we obtain, for some positive constant c',

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) - c'E'(t).$$

Hence, $\mathcal{L}_0 = \mathcal{L} + (c + c')E \sim E$ satisfies

$$\mathcal{L}_0(t) \leq -\mathcal{L}_0(0)e^{-ct},$$

which leads to

$$E(t) \leq -cE(0)e^{-ct}.$$

2. H **is non linear on** $[0, \epsilon]$. By using Jensen's inequality and the concavity of H^{-1}, we find that

$$\int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big)dx \leq cH^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right).$$

Thus, (4.16) rewrites as

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) + cH^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right). \tag{4.17}$$

For $\epsilon_0 < \epsilon$ and $m_1 > 0$, the functional given by

$$\mathcal{L}_1(t) = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + m_1 E(t)$$

satisfies, for some fixed positive constants ζ_0 and ζ_1,

$$\zeta_0 \mathcal{L}_1(t) \leq E(t) \leq \zeta_1 \mathcal{L}_1(t) \tag{4.18}$$

and

$$\mathcal{L}_1'(t) = \epsilon_0 \frac{E'(t)}{E(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}(t) + H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{L}'(t) + m_1 E'(t).$$

Next, by recaling the fact that $E' \leq 0$, $H' > 0$ and $H'' > 0$ on $[0, \epsilon]$ and using (4.17), we get

$$\mathcal{L}_1'(t) \leq -m_0 E(t) H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + cH'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right) + m_1 E'(t). \tag{4.19}$$

Let H^* be the convex conjugate of H, then by testing (2.8) with

$$A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = H^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right),$$

we get

$$H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right) \leq H^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + \int_{\mathcal{D}_1} \mathcal{R}(x,t)dx.$$

Using (4.1) with the fact $H^* \leq s(H')^{-1}(s)$, we have that

$$H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right) \leq \epsilon_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t). \tag{4.20}$$

The substitution of (4.20) into (4.19) provides

$$\mathcal{L}_1'(t) \leq -\Big(m_0 E(0) - c\epsilon_0\Big) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + (m_1 - c)E'(t).$$

Fixing ϵ_0 sufficiently small, so that $m_0 E(0) - c\epsilon_0 > 0$, then for $m_1 > c$, we can find a positive constant a_0 such that

$$\mathcal{L}_1'(t) \leq -a_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -a_0 H_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right), \tag{4.21}$$

where $H_2(t) = tH'(\epsilon_0 t)$ is a positive non-decreasing function on $[0, 1]$. Next, by setting $\mathcal{L}_2 = \frac{\zeta_0 \mathcal{L}_1}{E(0)}$, we can easily show, by (4.18), that $\mathcal{L}_2 \sim E$. And, from (4.21), we discover that

$$\mathcal{L}_2'(t) \leq -a_1 H_2(L(t)). \tag{4.22}$$

From the definition of H_1, we have

$$H_1'(t) = -\frac{1}{H_2(t)},$$

whereupon the inequality (4.22) becomes

$$\mathcal{L}_2'(t) \leq a_1 \frac{1}{H_1'(\mathcal{L}_2(t))},$$

which implies

$$\Big[H_1(\mathcal{L}_2(t))\Big]' \le a_1.$$

An integration over $[0, t]$ yields that

$$H_1(\mathcal{L}_2(t)) \le a_1 t + H_1(\mathcal{L}_2(0)).$$

Then, using the non-decreasing property of H^{-1}, we infer that

$$\mathcal{L}_2(t) \le H^{-1}\Big(a_1 t + a_2\Big).$$

The use of $\mathcal{L}_2 \sim E$ leads us to (4.15). Hence, the proof is completed. □

4.3. **General decay rates for non-equal of wave speeds.** In this subsection, we investigate the situation when $\frac{\rho_1}{\kappa} \ne \frac{\rho_2}{\delta}$, which is more realistic in the view of physics.

Theorem 4.2. *Let $U_0 \in \mathcal{H}_0$. Assume that (A_1) and (A_2) hold, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} \ne \frac{\rho_2}{\delta}.$$

Then, for

$$|\mu_2| < \min\left\{\frac{\alpha_1}{\alpha_2}, \frac{2\widetilde{c}_1}{\widetilde{c}_2(2-\theta)}\right\}(1-\theta)\mu_1, \tag{4.23}$$

there exist some positive numbers w and w_1 such that for any $t > 0$

$$E(t) \le w H_2^{-1}\Big(\frac{w_1}{t}\Big). \tag{4.24}$$

Proof. Differentiating (2.1) with respect to x, we obtain

$$\begin{cases} \rho_1 u_{xtt} - \kappa u_{xxx} - b\phi_{xx} = 0, \\ \rho_2 \phi_{xtt} - \delta\phi_{xxx} + b u_{xx} + \xi\phi_x + \mu_1 \phi_{xt} g_1'(\phi_t) + \mu_2 z_x(x,1) g_2'(z(x,1)) = 0, \\ \tau(t) z_{xt}(x,\rho,t) + (1-\rho\tau'(t)) z_{x\rho}(x,\rho,t) = 0, \\ u_x(0,t) = u_x(1,t) = \phi_x(0,t) = \phi_x(1,t) = 0, \\ u_x(x,0) = u_x^0(x), \quad u_t(x,0) = u_x^1(x), \\ \phi_x(x,0) = \phi_x^1(x), \quad \phi_{xt}(x,0) = \phi_x^1(x), \\ z_x(x,\rho,0) = f_x^0(x, -\rho\tau(0)). \end{cases} \tag{4.25}$$

Then, for a fixed positive constant $\widetilde{\gamma}$ satisfying

$$\frac{\widetilde{c}_2|\mu_2|}{(1-\theta)} < \widetilde{\gamma} < \Big(2\widetilde{c}_1\mu_1 - \widetilde{c}_2|\mu_2|\Big), \tag{4.26}$$

where $\widetilde{c}_1$ and $\widetilde{c}_2$ are introduced in (2.7) and (2.3), we define the modified energy functional to system (4.25) as

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 \\ &+ 2bu_{xx}\phi_x + 2\widetilde{\gamma}\tau(t)\int_0^1 z_{xt}^2(x,\rho,t)d\rho\Big]dx.\end{aligned}$$

Our point of departure will be to show that the modified energy functional $\mathcal{E}$ is non-increasing. So, we have the following result.

Lemma 4.7. *Under the assumptions in Theorem* 4.2*, the modified energy functional* $\mathcal{E}$ *is non-increasing and satisfies for any* $t \geq 0$

$$\mathcal{E}'(t) \leq -c\int_0^1 \phi_{xt}^2 dx - c\int_0^1 z_x^2(x,1)dx. \tag{4.27}$$

Proof. Multiplying $(4.25)_1$ and $(4.25)_2$ by u_{xt} and ϕ_{xt}, respectively, and integrating by parts over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{2}\cdot\frac{d}{dt}\int_0^1 \Big[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 + 2bu_{xx}\phi_x\Big]dx \\ &+\mu_1\int_0^1 \phi_{xt}^2 g_1'(\phi_t)dx + \mu_2\int_0^1 \phi_{xt}z_x(x,1)g_2'(z(x,1))dx = 0.\end{aligned} \tag{4.28}$$

Similarly, we multiply $(4.25)_3$ by $\widetilde{\gamma} z_x(x,\rho,t)$, we get

$$\frac{\widetilde{\gamma}}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau(t)z_{xt}^2(x,\rho,t)d\rho dx = -\frac{\widetilde{\gamma}}{2}(1-\tau'(t))\int_0^1 z_x^2(x,1)dx + \frac{\widetilde{\gamma}}{2}\int_0^1 \phi_{xt}^2 dx. \tag{4.29}$$

Combining the estimates (4.28)–(4.29) and using the fact that $\widetilde{c}_1 < g_1'(s)$ and (A_2), we yield that

$$\begin{aligned}\mathcal{E}'(t) \leq& -\left(\widetilde{c}_1\mu_1 - \frac{\widetilde{\gamma}}{2}\right)\int_0^1 \phi_{xt}^2 dx - \frac{\widetilde{\gamma}}{2}(1-\tau'(t))\int_0^1 z_x^2(x,1)dx \\ &-\mu_2\int_0^1 \phi_{xt}z_x(x,1)g_2'(z(x,1))dx.\end{aligned}$$

By using Young's inequality with the fact that $|g_2'(s)| < \widetilde{c}_2$, we arrive at

$$\mathcal{E}'(t) \leq -\left(\widetilde{c}_1\mu_1 - \frac{\widetilde{\gamma}}{2} - \frac{\widetilde{c}_2|\mu_2|}{2}\right)\int_0^1 \phi_{xt}^2 dx - \left(\frac{\widetilde{\gamma}}{2}(1-\theta) - \frac{\widetilde{c}_2|\mu_2|}{2}\right)\int_0^1 z_x^2(x,1)dx.$$

Estimate (4.27) follows by using (4.23) and (4.26). □

Now, going back to the proof of Theorem 4.2. Defining, as in (4.12), a Lyapunov functional L by

$$L(t) = M\mathcal{E}(t) + \mathcal{L}(t).$$

It should be mentioned that L is not equivalent to E. Then, using (4.13) and (4.27), we get

$$\begin{aligned} L'(t) \leq & - m_0 E(t) - cM \int_0^1 \phi_{xt}^2 dx + \left(\frac{\delta \rho_1}{\kappa} - \rho_2 \right) \int_0^1 \phi_{xt} u_t dx \\ & + c \int_0^1 \phi_t^2 dx + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx. \end{aligned}$$

Utilizing Young's inequality and the definition of $E(t)$, we get

$$\begin{aligned} L'(t) \leq & - (m_0 - \eta_1) E(t) - (cM - c_{\eta_1}) \int_0^1 \phi_{xt}^2 dx + c \int_0^1 \phi_t^2 dx \\ & + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx. \end{aligned}$$

Fixing $\eta_1 < m_0$ and then taking M sufficiently large, so that $cM - c_{\eta_1} \leq 0$, we obtain for $d_0 > 0$

$$L'(t) \leq -d_0 E(t) + c \int_0^1 \phi_t^2 dx + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx.$$

Consequently by exploiting (2.2) and (4.1), it holds that

$$L'(t) \leq -d_0 E(t) - cE'(t) + \int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big) dx. \tag{4.30}$$

As in the proof of Theorem 4.1, we distinguish the following two cases.

1. ***H*** **is linear on** $[0, \epsilon]$**.** From (4.30) and by using (4.1), we have, for some positive constant c',

$$L'(t) \leq -d_0 E(t) - (c + c') E'(t).$$

Then, the functional $L_0 = L + (c + c')E$, satisfies

$$L_0'(t) \leq -d_0 E(t).$$

Integrating over $[0, t]$ and using the non-increasing property of E, we yield that

$$tE(t) \leq \int_0^t E(s) ds \leq \frac{1}{d_0} L_0(0).$$

Hence, for $d > 0$ we have

$$E(t) \leq \frac{d}{t}, \quad \text{for all } t > 0.$$

2. ***H*** **is non-linear on** $[0, \epsilon]$**.** By repeating the same arguments as in the second part of the proof of Theorem 4.1, we find that the functional

$$L_1(t) = H' \left(\epsilon_0 \frac{E(t)}{E(0)} \right) L(t) + d_1 E(t)$$

satisfies, for a fixed positive constant w_0, the following property

$$L_1'(t) \leq -w_0 H_2 \left(\epsilon_0 \frac{E(t)}{E(0)} \right).$$

An integration over $[0,t]$ gives

$$\int_0^t H_2\left(\epsilon_0 \frac{E(s)}{E(0)}\right) ds \leq \frac{1}{w_0} L_1(0). \tag{4.31}$$

It follows from the fact that $E' \leq 0$ and $H_2' > 0$ that the map

$$t \mapsto H_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$$

is non-increasing. Thus, from (4.31), we obtain

$$tH_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \leq \int_0^t H_2\left(\epsilon_0 \frac{E(s)}{E(0)}\right) ds \leq \frac{1}{w_0} L_1(0).$$

Consequently, for $w, w_1 > 0$ we have

$$E(t) \leq wH_2^{-1}\left(\frac{w_1}{t}\right), \quad \text{for all } t > 0,$$

which finishes the proof. □

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ON SIMULTANEOUS APPROXIMATION AND COMBINATIONS OF LUPAS TYPE OPERATORS

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ABSTRACT. The purpose of the present paper is to study a sequence of linear and positive operators which was introduced by A. Lupas. First, we obtain estimate of moments of the operators and then prove a basic convergence theorem for simultaneous approximation. Further, we find error in approximation in terms of modulus of continuity of function. Finally, we establish a Voronovskaja asymptotic formula for linear combination of the above operators.

1. INTRODUCTION

At the International Dortmund Meeting held in Written (Germany, March, 1995), A. Lupas [11] introduced the following Linear positive operators for $f : [0, \infty) \to \mathbb{R}$ as

$$L_n(f, x) = (1 - a)^{nx} \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{\nu!} a^\nu f\left(\frac{\nu}{n}\right), \quad x \geq 0, \tag{1.1}$$

$$(1 - a)^{-\alpha} = \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} a^\nu, \tag{1.2}$$

where

$$|a| < 1, \quad (\alpha)_0 = 1, \quad (\alpha)_\nu = \alpha(\alpha + 1) \cdots (\alpha + \nu - 1), \quad \nu \geq 1.$$

If we impose that $L_n(t, x) = x$, we find that $a = 1/2$. Therefore, operator (1.1) becomes

$$L_n(f, x) = 2^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{2^\nu \nu!} f\left(\frac{\nu}{n}\right), \quad x \geq 0.$$

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It was seen that these opeartors are positive and linear and preseve linear functions. Bernstein polynomials [10] exhibit the property of simultaneous approximation. Simultaneous approximation for Baskakov operators, modified by Durrmeyer, was studied by Heilmann and Müller [8]. Another modification of Baskakov operators for simultaneous approximation was investigated by Sinha et al. [18]. Yet another modification of Baskakov operators viz., integral modification of Baskakov operators shows simultaneous approximation property in Thamer et al. [20]. This was studied for Durrmeyer modification of Bernstein polynomials by Gonska and Zhou [4]. In the summation-integral type operators Gupta et al. [7] explored the simultanoeus approximation. So far research work was done for linear positive operators ([3],[6],[9],[12]-[15], [19]). Singh and Agrawal [17] proved simultaneous approximation by a linear combination of Bernstein-Durrmeyer type polynomials. Gupta [5] studied the differences of operators of Lupas type. So, the Lupas operators play very important role to approximate functions for $f \in C[0, \infty)$.

It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. Therefore, in order to improve the order of approximation by the operators (1.1), we apply the technique of linear combination introduced by Butzer [2] and Rathore [16].

The approximation process for linear combination is defined as follows.

Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then, the linear combination $L_n(f, k, x)$ of $L_{d_j n}(f, x), j = 0, 1, 2, \ldots, k$, is given by

$$L_n(f,k,x) = \frac{1}{\Delta} \begin{vmatrix} L_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ L_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix},$$

where Δ is the Vandermonde determinant defined as

$$\Delta = \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \cdots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \cdots & d_1^{-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_k^{-1} & d_k^{-2} & \cdots & d_k^{-k} \end{vmatrix}.$$

On simplification, we have

$$L_n(f,k,x) = \sum_{j=0}^{k} C(j,k) L_{d_j n}(f,x), \tag{1.3}$$

where

$$C(j,k) = \prod_{i=0, i \neq j}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0,0) = 1.$$

2. Moment Estimates

Lemma 2.1 ([5]). *The following relations hold:*

$$L_n(1,x)=1,\quad L_n(t-x,x)=\frac{2a-1}{1-a}x, L_n((t-x)^2,x)=\frac{n^2x^2(2a-1)^2+nax}{n^2(1-a)^2}.$$

Now, we define mth order moment

$$\mu_m(x)=L_n((t-x)^m,x)=(1-a)^{nx}\left\{\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu\left(\frac{\nu}{n}-x\right)^m\right\}.$$

Lemma 2.2. *$\mu_m(x)$ is a polynomial in x of degree $[m/2]$. Moreover*

$$\mu_m(x)=O\left(\frac{1}{n^{[\frac{m+1}{2}]}}\right),\quad n\to\infty.$$

Proof. By definition of moments of mth order, we have

$$\begin{aligned}\mu_m(x)=&(1-a)^{nx}\sum_{\nu=0}^{\infty}(-1)^m a^\nu\frac{(nx)_\nu}{\nu!}\left\{\sum_{r=0}^{m}\binom{m}{r}(-1)^r x^{m-r}\left(\frac{\nu}{n}\right)^r\right\}\\ =&(-1)^m(1-a)^{nx}\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}a^\nu\left\{\sum_{r=0}^{m}\binom{m}{r}\left(-\frac{1}{n}\right)^r x^{m-r}\right.\\ &\left.\times\left(\nu^{(r)}+p_2\nu^{(r-1)}+p_4\nu^{(r-2)}+\cdots\right)\right\},\end{aligned}\tag{2.1}$$

where $\nu^{(r)}=\nu(\nu-1)(\nu-2)\cdots(\nu-r+1)$, p_2 is a polynomial in r of second degree, p_4 is a polynomial in r of fourth degree and so on.

It follows from (1.2) upon s times differentiation in a that

$$\sum_{\nu=0}^{\infty}\frac{(nx)_\nu}{\nu!}\nu(\nu-1)\cdots(\nu-s+1)a^{\nu-s}=(nx)_s(1-a)^{-nx-s}.\tag{2.2}$$

Making an use of (2.2) in (2.1)

$$\begin{aligned}\mu_m(x)=&(-1)^m\sum_{r=0}^{m}\binom{m}{r}\left(-\frac{1}{n}\right)^r x^{m-r}\left\{\frac{a^r}{(1-a)^r}(nx)_r+p_2\frac{a^{r-1}}{(1-a)^{r-1}}(nx)_{r-1}\right.\\ &\left.+p_4\frac{a^{r-2}}{(1-a)^{r-2}}(nx)_{r-2}+\cdots\right\}.\end{aligned}$$

Again,

$$\frac{(nx)_r}{n^r x^r}=1+\frac{q_2}{nx}+\frac{q_4}{(nx)^2}+\frac{q_6}{(nx)^3}+\cdots,$$

where q_j as before is a polynomial in r of degree j.

Therefore, taking $a = 1/2$ and using fact that $\sum_{r=0}^{m} \binom{m}{r}(-1)^r r^s = 0$, $s < m$, we find that

$$\mu_m(x) = (-1)^m x^m \left\{ \frac{C}{(nx)^{[\frac{m+1}{2}]}} + \cdots \text{higher order terms} \right\}.$$

Therefore, $\mu_m(x)$ is a polynomial in x of degree $[m/2]$. This completes the proof of lemma. □

3. Simultaneous Approximation

Theorem 3.1. *Let $f' \in C_B[0,\infty)$. Then, sequence $\left\{\frac{d}{dx}(L_n(f,x))\right\}_{n=1}^{\infty}$ converges to $f'(x)$ pointwise on $[0,\infty)$. Moreover, if S is a compact subset of $[0,\infty)$ then sequence $\left\{\frac{d}{dx}(L_n(f,x))\right\}_{n=1}^{\infty}$ converges to $f'(x)$ uniformly on S.*

Proof. We expand

$$f(w) = f(x) + (w-x)f'(x) + \int_x^w (f'(t) - f'(x))dt.$$

Operating $L_n(\cdot, y)$ on both sides of above equation and in view of Lemma 2.1, we obtain

$$L_n(f,y) = f(x) + \left(\frac{ay}{1-a} - x\right) f'(x) + (1-a)^{ny} \left\{ \sum_{\nu=0}^{\infty} \frac{(ny)_\nu}{\nu!} a^\nu R_\nu \right\},$$

where $R_\nu = \int\limits_x^{\nu/n} (f'(t) - f'(x))dt$. Thus,

$$\begin{aligned} \frac{d}{dx} L_n(f,x) =& \frac{a}{1-a} f'(x) + n(1-a)^{nx} \\ &\times \left\{ \ln(1-a) \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{\nu!} a^\nu R_\nu + \sum_{\nu=1}^{\infty} \frac{d(nx)_\nu}{d(nx)} \frac{a^\nu}{\nu!} R_\nu \right\}. \end{aligned} \tag{3.1}$$

We put $nx = \alpha$ and differentiate (1.2) w.r.t. α. Further, we equate coefficient of a^ν on both sides, we get

$$\frac{1}{\nu!} \cdot \frac{d(\alpha)_\nu}{d\alpha} = \frac{\alpha_{\nu-1}}{(\nu-1)!} + \frac{1}{2} \cdot \frac{\alpha_{\nu-2}}{(\nu-2)!} + \frac{1}{3} \cdot \frac{\alpha_{\nu-3}}{(\nu-3)!} + \cdots + \frac{1}{\nu} \cdot \frac{\alpha_0}{0!}. \tag{3.2}$$

Using (3.2) in (3.1), we get

$$\begin{aligned}
&\frac{d}{dx}L_n(f,x)-\frac{a}{1-a}f'(x)\\
=&n(1-a)^\alpha\Bigg[a(R_1-R_0)+a^2\left\{\frac{(\alpha)_1}{1!}(R_2-R_1)+\frac{(\alpha)_0}{2}(R_2-R_0)\right\}\\
&+a^3\left\{\frac{(\alpha)_2}{2!}(R_3-R_2)+\frac{1}{2}\cdot\frac{(\alpha)_1}{1!}(R_3-R_1)+\frac{1}{3}(\alpha)_0(R_3-R_0)\right\}+\cdots\\
&+a^\nu\Bigg\{\frac{(\alpha)_{\nu-1}}{(\nu-1)!}(R_\nu-R_{\nu-1})+\frac{(\alpha)_{\nu-2}}{(\nu-2)!}\cdot\frac{1}{2}(R_\nu-R_{\nu-2})\\
&+\frac{(\alpha)_{\nu-3}}{(\nu-3)!}\cdot\frac{1}{3}(R_\nu-R_{\nu-3})+\cdots+\frac{(\alpha)_1}{1!}\cdot\frac{1}{\nu-1}(R_\nu-R_1)\\
&+\frac{1}{\nu}\cdot\frac{(\alpha)_0}{1}(R_\nu-R_0)\Bigg\}+\cdots\Bigg]\\
=&n(1-a)^\alpha\Bigg[a\Bigg\{(\alpha)_0(R_1-R_0)+\frac{a(\alpha)_1}{1!}(R_2-R_1)+\frac{a^2(\alpha)_2}{2!}(R_3-R_2)\\
&+\frac{a^3(\alpha)_3}{3!}(R_4-R_3)+\cdots\Bigg\}+a^2\Bigg\{\frac{(\alpha)_0}{2}(R_2-R_0)+\frac{a(\alpha)_1}{1!}\cdot\frac{1}{2}(R_3-R_1)\\
&+\frac{a^2(\alpha)_2}{2!}\cdot\frac{1}{2}(R_4-R_2)+\frac{a^3(\alpha)_3}{3!}\cdot\frac{1}{2}(R_5-R_3)+\cdots\Bigg\}\\
&+a^3\Bigg\{(\alpha)_0\frac{1}{3}(R_3-R_0)+\frac{a(\alpha)_1}{1!}\cdot\frac{1}{3}(R_4-R_1)+\frac{a^2(\alpha)_2}{2!}\cdot\frac{1}{3}(R_5-R_2)+\cdots\Bigg\}\\
&+\cdots\Bigg]\\
=&n(1-a)^\alpha\left[\Sigma_1+\Sigma_2+\Sigma_3+\cdots\right],\quad\text{say.}
\end{aligned}\tag{3.3}$$

The continuity of $f'(\cdot)$ at point x implies that for a given $\epsilon>0$ there exists a $\delta=\delta(x)$, (depending on x) such that $|f'(t)-f'(x)|<\epsilon$ if $|t-x|<\delta$. We break R_p-R_q in two parts depending upon $|t-x|<\delta$ and $|t-x|\geq\delta$. In the second part, there may be two terms, where $|f'(t)-f'(x)|\leq 2\|f'\|_{C_B[0,\infty)}\cdot\frac{1}{\delta^2}(t-x)^2$.

Using Lemma 2.1, we get

$$\begin{aligned}
|\Sigma_1|&\leq a\frac{\epsilon}{n}\left(\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\right)+\frac{2\cdot 2\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a}{n}\left\{\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\left(\frac{k}{n}-x\right)^2\right\}\\
&=a\frac{\epsilon}{n}(1-a)^{-\alpha}+\frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a}{n}\cdot\left\{\frac{nx^2(2a-1)^2+ax}{n(1-a)^2}\right\}(1-a)^{-\alpha}.
\end{aligned}\tag{3.4}$$

Now,

$$|\Sigma_2| \leq a^2\frac{\epsilon}{n}\left\{\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\right\} + \frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a^2}{n}\left\{\sum_{k=0}^{\infty}\frac{a^k}{k!}(\alpha)_k\left(\frac{k}{n}-x\right)^2\right\}$$

$$= a^2\frac{\epsilon}{n}(1-a)^{-\alpha} + \frac{4\|f'\|_{C_B[0,\infty)}}{\delta^2}\cdot\frac{a^2}{n}\left\{\frac{nx^2(2a-1)^2+ax}{n(1-a)^2}\right\}(1-a)^{-\alpha}. \tag{3.5}$$

The similar estimates for $\Sigma_3, \Sigma_4, \dots$ are combined in (3.3) and we take $a = \frac{1}{2}$ due to Agratini [1]. Finally,

$$\left|\frac{d}{dx}L_n(f,x) - f'(x)\right| \leq Cn\left(\frac{\epsilon}{n}+\frac{1}{n^2}\right).$$

This completes the proof of the first part.

Proof of second part of Theorem 3.1. Let S be a compact subset of $[0,\infty)$. The pointwise continuity of function $f'(\cdot)$ at points of S, imply, by virtue of compactness of S, that $f'(\cdot)$ is now uniformly continuous on S. Thus, δ is now independent of x. Moreover S, being compact, is a bounded subset of $[0,\infty)$. Thus $x \in S$ implies $|x| < C_1$, a constant. This implies by (3.4) and (3.5) that convergence is uniform. □

Theorem 3.2. *Let $f' \in C_B[0,\infty)$. Then for $\delta > 0$ and $[a,b] \subset (a_1,b_1)$ we have*

$$\sup_{x\in[a,b]}|L_n^{'}(f,x) - f'(x)| \leq \omega(f',\delta,[a_1,b_1]) + \frac{C}{n}\|f'\|_{C_B[0,\infty)}.$$

Proof. We proceed in similar way as in the proof of Theorem 3.1. In the steps following (3.3) if $|t-x| < \delta$, then $|f'(t) - f'(x)| \leq \omega(f',\delta,[a_1,b_1])$. When $|t-x| \geq \delta$, using boundedness of f' the total contribution is of order $\|f'\|_{C_B[0,\infty)}O\left(\frac{1}{n}\right)$ as $n \to \infty$, by Lemma 2.1. Hence, the proof follows. □

4. Linear Combinations

Theorem 4.1. *Let $f^{(2k+2)} \in C_B[0,\infty)$. Then there holds for each $x \in [0,\infty)$, pointwise:*

$$\frac{d}{dx}L_n(f,k,x) - f'(x) = \frac{1}{n^{k+1}}\left\{\sum_{j=k+2}^{2k+2} q_j(x)f^{(j)}(x)\right\} + o\left(\frac{1}{n^{k+1}}\right), \quad n\to\infty. \tag{4.1}$$

Moreover, if S is a compact subset of $[0,\infty)$, then convergence (4.1) is uniform on S.

Proof. Using Taylor's series expansion, we write

$$\begin{aligned} f(w) =& f(x) + (w-x)f'(x) + \frac{(w-x)^2}{2!} f^{(2)}(x) + \cdots \\ &+ \frac{(w-x)^{2k+2}}{(2k+2)!} f^{(2k+2)}(x) \\ &+ \int_x^w \int_x^{t_1} \int_x^{t_2} \cdots \int_x^{t_{2k+1}} (f^{(2k+2)}(u) - f^{(2k+2)}(x)) dt_{2k+1}\, dt_{2k} \cdots dt_1 du. \end{aligned}$$

Operating $L_n(\cdot, y)$ on both sides of above equation and in view of Lemma 2.1, we obtain

$$\begin{aligned} L_n(f,y) =& f(x) + \left(\frac{ay}{1-a} - x\right) f'(x) + \frac{f^{(2)}(x)}{2!} p_2(1/n, y) \\ &+ \frac{f^{(3)}(x)}{3!} p_3(1/n, y) + \cdots + \frac{f^{(2k+2)}(x)}{(2k+2)} p_{2k+2}(1/n, y) \\ &+ (1-a)^{ny} \left(\sum_{\nu=0}^{\infty} \frac{(ny)_\nu}{\nu!} a^\nu R_\nu \right), \end{aligned}$$

where

$$R_\nu = \int_x^{\nu/n} \int_x^{t_1} \int_x^{t_2} \cdots \int_x^{t_{2k+1}} (f^{(2k+2)}(u) - f^{(2k+2)}(x)) dt_{2k+1} dt_{2k} \cdots dt_1 du$$

and $p_j\left(\frac{1}{n}, y\right)$ is a polynomial in y of degree j and in $\frac{1}{n}$ of degree $(j-1)$.

This implies that

$$\begin{aligned} \frac{d}{dx} L_n(f,x) =& \frac{a}{1-a} f'(x) + \frac{f^{(2)}(x)}{2!} p_2'(1/n, x) \\ &+ \frac{f^{(3)}(x)}{3!} p_3'(1/n, x) + \cdots + \frac{f^{(2k+2)}(x)}{(2k+2)} p_{2k+2}'(1/n, x) \\ &+ n(1-a)^{nx} \left\{ \log(1-a) \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{\nu!} a^\nu R_\nu + \sum_{\nu=1}^{\infty} \frac{d(nx)_\nu}{d(nx)} \cdot \frac{a^\nu}{\nu!} R_\nu \right\}. \end{aligned} \tag{4.2}$$

Let $\phi(n,x) = n(1-a)^{nx} \left\{ \log(1-a) \sum_{\nu=0}^{\infty} \frac{(nx)_\nu}{\nu!} a^\nu R_\nu + \sum_{\nu=1}^{\infty} \frac{d(nx)_\nu}{d(nx)} \frac{a^\nu}{\nu!} R_\nu \right\}$. Now, taking linear combinations on (4.2) and using their properties (1.3), we have

$$\frac{d}{dx} L_n(f,k,x) - \left(\frac{a}{1-a}\right) f'(x) = \left\{ \sum_{j=k+2}^{2k+2} q_j(x) f^{(j)}(x) \right\} \frac{1}{n^{k+1}} + \sum_{j=0}^{k} C(j,k) \phi(d_j n, x).$$

We analyze last term as in (3.1) and obtain the required result.

The proof of the second part of theorem follows from the proof of the second part of Theorem 3.1. ☐

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APPROXIMATION BY A COMPOSITION OF APOSTOL-GENOCCHI AND PĂLTĂNEA-DURRMEYER OPERATORS

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Abstract. The present paper deals with the Durrmeyer construction of operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. For the proposed operators, we first establish a global approximation result followed by its convergence estimate in terms of usual, r-th and weighted modulus of continuity. We further study the asymptotic type results such as the Voronovskaya theorem and quantitative Voronovskaya theorem. Moreover, we estimate the rate of pointwise convergence of the proposed operators for functions of bounded variation defined on the interval $(0, \infty)$. Finally, the results are validated through graphical representations and an absolute error table.

1. Introduction

Recently, Prakash et al. [20] proposed the following positive linear sequence of operators:

$$G_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.1}$$

where $s_{n,k}^{\alpha,\lambda}(x) = e^{-nx}\left(\frac{1+\lambda e}{2}\right)^{\alpha} \frac{g_k^{\alpha}(nx;\lambda)}{k!}$ and $g_k^{\alpha}(x;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, which belong to the class of orthogonal polynomials. These polynomials were defined for a complex variable z, $|z| < \pi$ in [16]. However, in this study we limit ourselves to a real variable $t \in [0,\infty)$. The generalized Apostol

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Genocchi polynomial of order α, i.e., $g_k^{\alpha}(x;\lambda)$ can be estimated with the help of following generating function:

$$\left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} g_k^{\alpha}(x;\lambda)\frac{t^k}{k!}. \tag{1.2}$$

The more explicit form of $g_k^{\alpha}(x;\lambda)$ was proposed by Luo and Srivastava in [17]. They presented some elementary properties of these polynomials and derived explicit series representation of $g_k^{\alpha}(x;\lambda)$ in terms of hypergeometric function defined by Gauss. The series is given as follows:

$$\begin{aligned} g_k^{\alpha}(x;\lambda) =& 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{n=0}^{k-\alpha}\binom{k-\alpha}{n}\binom{\alpha+n-1}{n}\frac{\lambda^n}{(1+\lambda)^{\alpha+n}} \\ &\times\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n(x+j)^{k-n-\alpha}{}_2F_1\left(\alpha+n-k,n;n+1;\frac{j}{x+j}\right), \end{aligned}$$

where $\{k,\alpha\}\in\mathbb{N}\cup\{0\}$, $\lambda\in\mathbb{R}\setminus\{-1\}$, $x\in\mathbb{R}$ and ${}_2F_1(a,b;c;t)$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = {}_2F_1(b,a;c;t) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\cdot\frac{t^k}{k!}.$$

In particular, for $\alpha = 1$ and $\lambda = 1$, these operators reduce to classical Genocchi polynomials which are obtained by the following generating function:

$$\frac{2te^{xt}}{e^t+1} = \sum_{k=0}^{\infty} g_k(x)\frac{t^k}{k!}, \quad |t|<2\pi,\ x\in\mathbb{R},$$

where $g_k(x) = g_k(x;1)$. It can be clearly seen that $g_k(x)$ are the k^{th}-degree polynomials, few terms of which are given as follows:

$$\begin{gathered} g_1(x) = 1, \quad g_2(x) = 2x-1, \quad g_3(x) = 3x(x-1), \\ g_4(x) = 4x^3-6x^2+1, \quad g_5(x) = 5x^4-10x^3+5x, \ldots \end{gathered}$$

For the case $x = 0$, one can obtain the so-called Genocchi numbers g_k using the relation:

$$g_k(x) = \sum_{i=0}^{k}\binom{k}{i}g_i x^{k-i}.$$

Genocchi numbers can be defined in many ways depending on the field where they are intended to be applied. They find a wide range of application in numerical analysis, combinatorics, number theory, graph theory etc. Luo [15,16] defined Apostol-Genocchi polynomials of higher order and also introduced q-Apostol-Genocchi polynomials. He studied the relationship of these polynomials with Zeta function. In the last two decades, a surprising number of papers appeared studying Genocchi numbers, their combinatorial relations, Genocchi polynomials and their generalisations along with their various expansions and integral representations. To the readers, we suggest following articles [4, 18, 21] and references therein.

In the recent past, much work has been dedicated towards the Durrmeyer type modification of linear positive operators. For instance, Dhamija and Deo [8] introduced the Durrmeyer form of Jain operators based on inverse Pólya-Eggenberger distribution. They studied its moments with the aid of Vandermonde convolution formula and analysed other approximation properties. Heilmann and Raşa [12] studied a link between Baskakov-Durrmeyer type operators and their corresponding classical Kantorovich variants. Acu and Radu [3] introduced and studied a class of operators which link α-Bernstein operators and genuine α-Bernstein Durrmeyer operators. To see more work relevant to this area, one may refer [2, 5, 7, 10, 11, 13].

Inspired by above stated researches, we now consider a Durrmeyer type modification of Apostol-Genocchi operators based on Păltănea basis on positive real line. For $f \in C[0,\infty)$ and $\rho > 0$, the operators are defined as follows:

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) \int_0^{\infty} l_{n,k}^{\rho}(t) f(t)\, dt, \quad x \in [0,\infty), \tag{1.3}$$

where $l_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $s_{n,k}^{\alpha,\lambda}(x)$ is defined in (1.1).

The outline of the study is as follows. We consider a Durrmeyer type construction of Apostol-Genocchi operators based on the basis function due to Păltănea [19] with real parameters α, λ and ρ. We establish approximation estimates such as a global approximation theorem and rate of approximation in terms of usual, r-th and weighted modulus of continuity. We further study asymptotic formulae such as Voronovskaya theorem and quantitative Voronovskaya theorem. The last theorem is an application of the proposed operators for the functions whose derivatives are of bounded variation. Moreover, the approximation and the absolute error therein has been shown graphically by varying the values of various parameters using Mathematica software.

2. Preliminaries

Before proceeding to our main results, we state some general lemmas which are useful throughout this paper. In addition, we have used Mathematica software wherever necessary for complex and tedious calculations such as for moments and central moments etc.

Lemma 2.1. *For $e_s(t) = t^s$, $s \in \mathbb{N} \cup \{0\}$ and $\rho > 0$, we have*

$$\int_0^{\infty} l_{n,k}^{\rho}(t) t^s = \frac{(k\rho + s - 1)!}{(n\rho)^s (k\rho - 1)!} = \frac{(k\rho)_s}{(n\rho)^s},$$

where the symbol $(\beta)_n = \beta(\beta+1)(\beta+2)\cdots(\beta+n-1)$, $(\beta)_0 = 1$ denotes the rising factorial.

Lemma 2.2. *For operators* (1.3)*, the moments are obtained as follows:*

$\mathcal{M}_n^{\alpha,\lambda}(e_0;x) = 1,$

$$\mathcal{M}_n^{\alpha,\lambda}(e_1;x) = x + \frac{\alpha}{n(1+\lambda e)},$$

$$\mathcal{M}_n^{\alpha,\lambda}(e_2;x) = x^2 + \frac{x}{n}\left[\frac{1+2\alpha+\lambda e}{(1+\lambda e)} + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2 - 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right],$$

$$\begin{aligned}
&\mathcal{M}_n^{\alpha,\lambda}(e_3;x)\\
=&x^3 + \frac{x^2}{n}\left[\frac{3(\alpha+\lambda e+1)}{(1+\lambda e)} + \frac{3}{\rho}\right]\\
&+ \frac{x}{n^2}\left[\frac{3\alpha^2 - 3\alpha\lambda^2 e^2 - 3\alpha\lambda e + 3\alpha + \lambda^2 e^2 + 2\lambda e + 1}{(1+\lambda e)^2} + \frac{3(2\alpha+\lambda e+1)}{\rho(1+\lambda e)} + \frac{2}{\rho^2}\right]\\
&+ \frac{1}{n^3}\left[\frac{\alpha^3 - 3\alpha^2\lambda^2 e^2 - 6\alpha^2\lambda e - \alpha\lambda^3 e^3 - 4\alpha\lambda^2 e^2 - 5\alpha\lambda e}{(1+\lambda e)^3} + \frac{2\alpha}{\rho^2(1+\lambda e)}\right.\\
&\left.+\frac{3(\alpha^2 - \alpha\lambda^2 e^2 - 2\alpha\lambda e)}{\rho(1+\lambda e)^2}\right],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}_n^{\alpha,\lambda}(e_4;x)\\
=&x^4 + \frac{x^3}{n}\left[\frac{2\alpha+3\lambda e+3}{(1+\lambda e)} + \frac{6}{\rho}\right]\\
&+ \frac{x^2}{n^2}\left[\frac{25 + 12\alpha + 6\alpha^2 + 50\mathrm{e}\lambda + 25\mathrm{e}^2\lambda^2 - 6\alpha\mathrm{e}^2\lambda^2}{(1+\lambda e)^2} + \frac{18(1+\alpha+\mathrm{e}\lambda)}{(1+\lambda e)\rho} + \frac{11}{\rho^2}\right]\\
&+ \frac{x}{n^3}\left[\frac{7 + 20\alpha + 63\alpha^2 + 2\alpha^3 - 6\alpha^2 e^2\lambda^2 - 9\alpha^2\lambda e}{(1+\lambda e)^3}\right.\\
&+ \frac{-5\alpha e^3\lambda^3 + 4\alpha e^2\lambda^2 + 24\alpha\lambda e + 7e^3\lambda^3 + 21\lambda e}{(1+\lambda e)^3}\\
&\left.+\frac{6(3\alpha^2 - 3\alpha e^2\lambda^2 - 3\alpha\lambda e + 3\alpha + e^2\lambda^2 + 2\lambda e + 1)}{(1+\lambda e)^2\rho} + \frac{11(1+2\alpha+\lambda e)}{(1+\lambda e)\rho^2} + \frac{6}{\rho^3}\right]\\
&\times \frac{1}{n^4}\left[\frac{\alpha^4 - 6e^2\alpha^3\lambda - 12e\alpha^3\lambda - 16e^4\alpha\lambda^4 + 8e^3\alpha\lambda^3 - 82e^3\alpha\lambda^2 - 118e^2\alpha\lambda^2 - 66\alpha + \lambda e}{(1+\lambda e)^4}\right.\\
&+ \frac{6(\alpha^3 - 3e^2\alpha^2\lambda^2 - 6e\alpha^2\lambda - e^3\alpha\lambda^3 - 4e^2\alpha\lambda^2 - 5e\alpha\lambda)}{(1+\lambda e)^3\rho}\\
&\left.+ \frac{11(\alpha^2 - \alpha e^2\lambda^2 - 2\alpha\lambda e)}{(1+\lambda e)^2\rho^2} + \frac{6\alpha}{(1+\lambda e)\rho^3}\right].
\end{aligned}$$

Proof. In the proposed operators (1.3), for $s = 0, 1, 2$ respectively we have 1.

2. For $s = 1$, again using Lemma 2.1 we have

$$\mathcal{M}_n^{\alpha,\lambda}(e_1;x) = \frac{e^{-nx}}{n}\left(\frac{1+\lambda e}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{g_k^{\alpha}(nx;\lambda)}{k!}k. \tag{2.1}$$

Differentiating both sides of (1.2) with respect to t and taking limits $t \to 1$ and $x \to nx$, we have

$$\sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k = 2^{\alpha} e^{nx} \left(\frac{1}{1+\lambda e}\right)^{1+\alpha} (\alpha + nx(1+\lambda e)).$$

Making use of this value in equation (2.1), we obtain the first moment.

3. Similarly for $s=2$, we have

$$\mathcal{M}_n^{\alpha,\lambda}(e_2;x) = \frac{e^{-nx}}{n^2}\left(\frac{1+\lambda e}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k^2. \tag{2.2}$$

On differentiating both sides of (1.2) with respect to t and taking limits $t \to 1$ and $x \to nx$, we have

$$\sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k(k-1) = 2^{\alpha} e^{nx} \left(\frac{1}{1+\lambda e}\right)^{1+\alpha} \Big\{2ns\alpha(1+\lambda e) + n^2x^2(1+\lambda e)^2 - (\lambda e(3+\lambda e) - \alpha + 1)\Big\}.$$

Combining this with the first order moment and equation (2.2) we obtain the third moment.

We can obtain the higher order moments in a similar way. □

Lemma 2.3. *Let us define* $\delta_n^s(x) = \mathcal{M}_n^{\alpha,\lambda}(\Phi_s;x)$, *where* $\Phi_s(t) = (e_1 - x)^s$ *and* $s = 1, 2$. *Then, from Lemma* 2.2 *we have*

$$\delta_n^{(1)}(x) = \frac{\alpha}{n(1+\lambda e)},$$

$$\delta_n^{(2)}(x) = \frac{x}{n}\left[1 + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2 - 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right].$$

Furthermore,

$$\lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_1;x) = \frac{\alpha}{(1+\lambda e)},$$

$$\lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_2;x) = \left(1+\frac{1}{\rho}\right)x,$$

$$\lim_{n\to\infty} n^2\mathcal{M}_n^{\alpha,\lambda}(\Phi_4;x) = \frac{3(1+(6+8\alpha)\rho - (3+8\alpha)\rho^2 + \lambda e(1+6\rho-3\rho^2))x^2}{(1+\lambda e)\rho^2}$$

and

$$\lim_{n\to\infty} n^3\mathcal{M}_n^{\alpha,\lambda}(\Phi_6;x) = \frac{(1+\rho)(2+\rho+3\alpha\rho+\lambda e(2+\rho))x^5}{(1+\lambda e)^3\rho^4}.$$

Remark 2.1. Since fourth and sixth central moments are too lengthy and unnecessarily space consuming, we are omitting their values here. Instead we choose to write their limiting values, which is useful in the proofs of our main theorems.

3. Main Theorems

Theorem 3.1. *For any* $f \in C_B[0,\infty)$, *where* $C_B[0,\infty)$ *is the class of all continuous and bounded functions, we have*

$$\mathcal{M}_n^{\alpha,\lambda}(f(t);x) = f(x)$$

uniformly on any compact subset of $[0,\infty)$.

Proof. Taking into account Lemma 2.2, we can easily see that $\mathcal{M}_n^{\alpha,\lambda}(e_r;x) \to x^r$ for each $r = 0,1,2$ and hence using the well known Korovkin's theorem due to [14], operators $\mathcal{M}_n^{\alpha,\lambda}$ converge uniformly on each compact subset of $[0,\infty)$. ☐

3.1. **Global approximation.** Let us denote $B_f[0,\infty)$ the space of all functions f on positive real axis that satisfy the condition $|f(x)| \leq H_f(1+x^2)$ where H_f is a constant depending only on f but independent of x.

Let $C_f[0,\infty)$ be the subspace of $B_f[0,\infty)$ containing all continuous f on $[0,\infty)$. The norm in $C_f[0,\infty)$ is defined by

$$||f||_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Also, let $C_f^l[0,\infty) := \left\{ f \in C_f[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} \text{ is finite} \right\}$.

Theorem 3.2. *For each* $f \in C_f^l[0,\infty)$, *we have*

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(f;x) - f \right\|_2 = 0.$$

Proof. The proof of this theorem can be given by application of Korovkin theorem [9] on the interval $[0,\infty)$. Therefore, it would suffice if we prove that

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(e_l;x) - e_l \right\|_2 = 0, \quad l = 0,1,2. \tag{3.1}$$

For $l = 0$, condition (3.1) holds as operators $\mathcal{M}_n^{\alpha,\lambda}$ preserve constant functions. Next, we can write

$$\left\| \mathcal{M}_n^{\alpha,\lambda}(e_1;x) - x \right\|_2 \leq \sup_{x\in[0,\infty)} \frac{\alpha}{n(1+\lambda e)(1+x^2)} \to 0,$$

for adequately large n. Therefore, the condition (3.1) is satisfied for $l = 1$.

Finally, we write

$$\left\| \mathcal{M}_n^{\alpha,\lambda}(e_2;x) - x^2 \right\|_2 \leq \sup_{x\in[0,\infty)} \frac{1}{(1+x^2)} \left\{ \frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2 + 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)} \right) \right\},$$

which suggests

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(e_2;x) - e_2 \right\|_2 = 0.$$

Hence, the theorem follows. ☐

Let $C_B[0,\infty)$ be the class of all continuous and bounded real valued functions. We define the r-th order modulus of continuity by $\omega_r(f,\delta)$ and define it as

$$\omega_r(f,\delta) = \sup_{x\in[0,\infty)} \sup_{0\le h\le\delta} \left|\Delta_h^r f(x)\right|,$$

where Δ denotes the forward difference. In particular, the usual modulus of continuity is defined for $r=1$ and is denoted by $\omega(f,\delta)$. Moreover, we define the norm as $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$.

Also, the Peetre's K-functional for the function $g \in C_B^2[0,\infty)$ is defined as:

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\right\},$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

The next theorem establishes the degree of approximation of the operators $\mathcal{M}_n^{\alpha,\lambda}$ in terms of the usual and second order modulus of continuity for the functions in the space $C_B[0,\infty)$.

Theorem 3.3. *For $\hbar \in C_B^2[0,\infty)$, define the auxiliary operators $\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}$ as*

$$\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar;x) = \mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar\left(x + \frac{\alpha}{n(1+\lambda e)}\right) + \hbar(x). \tag{3.2}$$

Then there exists a constant $C>0$ such that

$$|\mathcal{M}_n^{\alpha,\lambda}(\hbar,x) - \hbar(x)| \le C\omega_2(\hbar,\sqrt{\delta}) + \omega(\hbar,\delta_n^{(1)}(x)),$$

where

$$\delta = \delta_n^{(2)}(x) + \left(\frac{\alpha}{n(1+\lambda e)}\right)^2.$$

Proof. Using Lemma 2.2, one can easily observe that $\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}((t-x);x) = 0$.

Let $f \in C_B^2[0,\infty)$, then by Taylor's expansion we have

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du.$$

Moreover, we can write

$$\begin{aligned}\left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(f;x) - f(x)\right| &= \left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right| \\ &\le \left|\mathcal{M}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right| \\ &\quad + \left|\int_x^{\left(x+\frac{\alpha}{n(1+\lambda e)}\right)} \left(\left(x + \frac{\alpha}{n(1+\lambda e)}\right) - u\right) f''(u)du\right|\end{aligned}$$

$$\leq \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n\,(1+\lambda e)}\right)^2\right)\|f''\|. \tag{3.3}$$

Since we know that

$$\left|\mathcal{M}_n^{\alpha,\lambda}\,(\hbar;x)\right| \leq \|\hbar\|,$$

therefore

$$\left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar;x)\right| \leq \left|\mathcal{M}_n^{\alpha,\lambda}\,(\hbar;x)\right| + \left|\hbar\left(x+\frac{\alpha}{n\,(1+\lambda e)}\right)\right| + |\hbar(x)| \leq 3\,\|\hbar\|. \tag{3.4}$$

Finally, combining equations (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
\left|\mathcal{M}_n^{\alpha,\lambda}\,(\hbar;x)-\hbar(x)\right| \leq& \left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar-f);x)-(\hbar-f)(x)\right| + \left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(f;x)-f(x)\right| \\
&+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right| \\
\leq& 4\,\|\hbar-f\| + \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n\,(1+\lambda e)}\right)^2\right)\|f''\| \\
&+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right| \\
\leq& C\left\{\|\hbar-f\| + \left(\delta_n^{(2)}\,(x)+\left(\frac{\alpha}{n\,(1+\lambda e)}\right)^2\right)\|f''\|\right\} \\
&+\omega\left(\hbar,\frac{\alpha}{n\,(1+\lambda e)}\right).
\end{aligned}$$

Taking infimum over all $f \in C_B^2[0,\infty)$ and using the result $K_2\,(f,\delta) \leq \omega_2\left(f,\sqrt{\delta}\right)$ due to [6], we get the desired outcome. ☐

Theorem 3.4. *Let $\hbar \in C_B(0,\infty)$, then for any $r>0$, $x\in[0,r]$ and adequately large n, we have*

$$\left|\mathcal{M}_n^{\alpha,\lambda}\,(\hbar;x)-\hbar\,(x)\right| \leqslant 4H_\hbar\left(1+x^2\right)\frac{D}{n}+2\omega_{r+1}\left(\hbar,\sqrt{\frac{D}{n}}\right),$$

where D is a positive constant.

Proof. If $x\in[0,r]$ and $t>r+1$, then $t-x>1$. Therefore, we have the following inequality:

$$|\hbar\,(t)-\hbar\,(x)| \leqslant 4H_\hbar\left(1+x^2\right)(t-x)^2.$$

Again for $x\in[0,r]$ and $t\in[0,r+1]$ and using the well known inequality $\omega\,(f,\beta\delta) \leqslant (\beta+1)\,\omega\,(f,\delta)\,,\beta\in(0,\infty)$, one can obtain

$$|\hbar\,(t)-\hbar\,(x)| \leqslant \left(1+\frac{|t-x|}{\delta}\right)\omega_{r+1}\,(\hbar,\delta)\,.$$

From (3.5) and (3.5), we can write

$$\left|\hbar\left(t\right)-\hbar\left(x\right)\right|\leqslant 4H_{\hbar}\left(1+x^{2}\right)\left(t-x\right)^{2}+\left(1+\frac{\left|t-x\right|}{\delta}\right)\omega_{r+1}\left(\hbar,\delta\right).$$

Applying operator $\mathcal{M}_{n}^{\alpha,\lambda}$ in the above relation and making use of Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{M}_{n}^{\alpha,\lambda}\left(\hbar;x\right)-\hbar\left(x\right)\right| \leqslant & 4H_{\hbar}\left(1+x^{2}\right)\mathcal{M}_{n}^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\\ &+\left(1+\frac{1}{\delta}\mathcal{M}_{n}^{\alpha,\lambda}\left(\left|t-x\right|;x\right)\right)\omega_{r+1}\left(\hbar,\delta\right)\\ \leqslant & 4H_{\hbar}\left(1+x^{2}\right)\mathcal{M}_{n}^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\\ &+2\omega_{r+1}\left(\hbar,\sqrt{\mathcal{M}_{n}^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)}\right).\end{aligned}$$

Since $\mathcal{M}_{n}^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\leqslant\frac{D}{n}$, where D is a positive constant, it follows that for adequately large n, we have

$$\left|\mathcal{M}_{n}^{\alpha,\lambda}\left(\hbar;x\right)-\hbar\left(x\right)\right|\leqslant 4H_{\hbar}\left(1+x^{2}\right)\frac{D}{n}+2\omega_{r+1}\left(\hbar,\sqrt{\frac{D}{n}}\right),$$

which is the required result. □

3.2. **Quantitative Voronovskaya theorem and Voronovoskaya theorem.** Let $C_{B}\left[0,\infty\right)$ be the subspace of $B_{f}\left[0,\infty\right)$ containing all continuous and bounded functions f for which $\lim\limits_{x\to\infty}\left|f\left(x\right)\right|\left(1+x^{2}\right)^{-1}$ is finite.

The weighted modulus of continuity $\Omega\left(f,\delta\right)$ due to [1] for each $f\in C_{B}\left[0,\infty\right)$ is defined as

$$\Omega\left(f,\delta\right)=\sup_{x\in[0,\infty),|h|<\delta}\frac{\left|f\left(x+h\right)-f\left(x\right)\right|}{\left(1+h^{2}+x^{2}+h^{2}x^{2}\right)}.$$

In the next theorem, we discuss the quantitative Voronovskaya theorem for the proposed operators (1.3) and derive a Voronovoskaya asymptotic result as a resulting corollary, making use of the following properties of weighted modulus of continuity. For every $f\in C_{f}^{l}[0,\infty)$,

(a) $\Omega\left(f,\delta\right)\to 0$ for $\delta\to 0$;
(b) $\left|f\left(t\right)-f\left(x\right)\right|\leq\left(1+\left(t-x\right)^{2}\right)\left(1+x^{2}\right)\Omega\left(f,\left|t-x\right|\right).$

Theorem 3.5. *Let $\hbar''\in C_{2}^{l}\left[0,\infty\right)$ and $x\in\left[0,\infty\right)$. Then we have*

$$\left|\mathcal{M}_{n}^{\alpha,\lambda}(\hbar;x)-\hbar(x)-\frac{\alpha}{n\left(1+\lambda e\right)}\hbar'\left(x\right)\right.$$
$$\left.-\frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right)+\frac{1}{n^{2}}\left(\frac{\alpha^{2}+2\alpha\lambda e-\alpha e^{2}\lambda^{2}}{\left(1+\lambda e\right)^{2}}+\frac{\alpha}{\rho\left(1+\lambda e\right)}\right)\right)\hbar''\left(x\right)\right|$$

$$\leq \frac{8\left(1+x^{2}\right)}{n} \Omega\left(\hbar'', \frac{1}{\sqrt{n}}\right).$$

Proof. By Taylor's expansion, we may write

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)-\hbar(x) &= \mathcal{M}_n^{\alpha,\lambda}\left(\left(\hbar(t)-\hbar(x)\right);x\right)\\ &= \mathcal{M}_n^{\alpha,\lambda}\left((t-x)\,\hbar'(x)+\frac{(t-x)^2}{2}\hbar''(x)+\lambdabar(t,x)\,(t-x)^2;x\right),\end{aligned}$$

where $\lambdabar(t,x)=(\hbar''(\varsigma)-\hbar''(x))/2$ is a continuous function which tends to zero at 0 and ς lies between x and t. Using Lemma 2.3, we get

$$\begin{aligned}&\Bigg|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)-\hbar(x)-\frac{\alpha}{n(1+\lambda e)}\hbar'(x)\\ &-\frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right)+\frac{1}{n^2}\left(\frac{\alpha^2+2\alpha\lambda e-\alpha e^2\lambda^2}{(1+\lambda e)^2}+\frac{\alpha}{\rho(1+\lambda e)}\right)\right)\hbar''(x)\Bigg|\\ &\leq \mathcal{M}_n^{\alpha,\lambda}\left(\left|\lambdabar(t,x)\right|(t-x)^2;x\right).\end{aligned}$$

With simple manipulations in property (b) of weighted modulus of continuity and using $|\varsigma-x|\leq|t-x|$, we can write

$$\left|\lambdabar(t,x)\right|\leq 8\left(1+x^2\right)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega\left(\hbar'',\delta\right),$$

which implies that

$$\left|\lambdabar(t,x)\right|(t-x)^2\leq 8\left(1+x^2\right)\left((t-x)^2+\frac{(t-x)^6}{\delta^4}\right)\Omega\left(\hbar'',\delta\right).$$

Therefore, in view of Lemma 2.3, we can write

$$\mathcal{M}_n^{\alpha,\lambda}\left(\left|\lambdabar(t,x)\right|(t-x)^2;x\right)\leq 8\left(1+x^2\right)\Omega\left(\hbar'',\delta\right)\left\{\delta_n^{(2)}(x)+\frac{1}{\delta^4}\delta_n^{(6)}(x)\right\},$$

as $n\to\infty$. Choosing $\delta=\frac{1}{\sqrt{n}}$, we get the desired outcome. □

Corollary 3.1. *Let f be a bounded and integrable function on the interval* $[0,\infty)$ *such that the second derivative of f exists at a fixed point $x\in[0,\infty)$. Then*

$$\lim_{n\to\infty} n(\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x))=\frac{\alpha}{(1+\lambda e)}f'(x)+x\left(1+\frac{1}{\rho}\right)f''(x).$$

3.3. **Functions of derivatives of bounded variation.** Next we estimate the rate of convergence of the operators (1.3) for functions with derivatives of bounded variation defined on $[0,\infty)$.

Let $f\in \mathrm{DBV}_\tau[0,\infty)$ be the class of functions whose derivatives are of bounded variation on any finite subinterval of $[0,\infty)$ and satisfy the growth condition $|f(t)|\leq$

Kt^{τ}, $\tau > 0$ for all $t > 0$ and constant $K > 0$. For such functions, let us represent our proposed operators (1.3) in the following form

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \int_0^{\infty} \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)f(t)dt, \tag{3.5}$$

where $\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) l_{n,k}^{\rho}(t)$.

Lemma 3.1. *For $x \in [0,\infty)$ and adequately large n, we have*

(i) *if $0 \leq y < x$, then*

$$\vartheta_n(x,y) = \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \leq \frac{K\delta_n^2(x)}{n(x-y)^2};$$

(ii) *if $x < z \leq \infty$, then*

$$1 - \vartheta_n(x,z) = \int_z^{\infty} \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \leq \frac{K\delta_n^2(x)}{n(z-x)^2}.$$

Proof. (i) Taking into account Lemma 2.2 and proposed operators (1.3), we have

$$\begin{aligned}\vartheta_n(x,z) &= \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \\ &\leq \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\left(\frac{x-t}{x-y}\right)^2 dt = \frac{1}{(x-y)^2}\int_0^y (t-x)^2 \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \\ &\leq \frac{1}{(x-y)^2}\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right) \leq \frac{K\delta_n^2(x)}{n(x-y)^2}.\end{aligned}$$

Proof of (ii) is similar to (i). □

Theorem 3.6. *Consider a function f of bounded variation on every sub-interval of $[0,\infty)$ that satisfies the growth condition $|f(t)| \leq Kt^{\tau}$ for some absolute constant K and $\tau > 0$. If there exists an integer γ, $(2\gamma \geq \tau)$ such that $f(t) \leq O(t^{\gamma})$ for every $t > 0$, then for $\gamma > 0$, $x \in [0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)\right| \leq &\frac{1}{2}\left(f'(x+) + f'(x-)\right)\delta_n^1(x) \\ &+ \sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+) - f'(x-)\right| + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(f'_x) \\ &+ \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{x}{\sqrt{k}}}(f'_x) + \frac{K\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x)\right| \\ &+ \wp(\gamma,\tau,x) + \frac{K\delta_n^2(x)}{nx^2}|f(x)| + \sqrt{\frac{K\delta_n^2(x)}{n}}f'(x+),\end{aligned}$$

where $\bigvee_b^a (f)$ *denotes the total variation of* f *on any finite subinterval* $[a,b]$ *of* $[0,\infty)$ *and* $\wp(\gamma,\tau,x) := 2^\gamma Ç\left(\int\limits_0^\infty (t-x)^{2\tau}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right)^{\frac{\gamma}{2\tau}}$.

Proof. For $x\in[0,\infty)$, we can write for our proposed operators (3.5) that

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x) &= \int\limits_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,(f(t)-f(x))\,dt\\ &= \int\limits_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\left(\int\limits_x^t f'(x)\,du\right)dt.\end{aligned}\tag{3.6}$$

Also for any $f\in DBV_\gamma[0,\infty)$, equality (3.7) holds true, i.e.,

$$\begin{aligned}f'(u) =& \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\\ &+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right),\end{aligned}\tag{3.7}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x,\\ 0, & u\neq x.\end{cases}$$

It can be easily verified that:

$$\int\limits_0^\infty\left(\int\limits_x^t\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)\delta_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt=0.$$

Now in view of our proposed operators (3.5), we may write

$$\begin{aligned}&\int\limits_0^\infty\left(\int\limits_x^t\left(\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ =&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right).\end{aligned}$$

Moreover

$$\begin{aligned}&\int\limits_0^\infty\left(\int\limits_x^t\left(\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\right)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ \le&\frac{1}{2}\left|f'(x+)-f'(x-)\right|\int\limits_0^\infty|t-x|\,\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ \le&\frac{1}{2}\left|f'(x+)-f'(x-)\right|\left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)\right)^{1/2}.\end{aligned}\tag{3.8}$$

Making use of equations (3.7)–(3.8) and Lemma 2.2 in equation (3.6), we get

$$\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)\le\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left|f'(x+)-f'(x-)\right|\left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)\right)^{1/2}\\
&+\int_0^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\
\leq&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)\\
&+\sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+)-f'(x-)\right|\\
&+\int_0^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt.
\end{aligned}
$$

Taking absolute values on both sides and rewriting equation we have:

$$
\begin{aligned}
\left|\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)\right|\leq&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)\\
&+\sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+)-f'(x-)\right|+P_{n_1}(x)+P_{n_2}(x),
\end{aligned}
\tag{3.9}
$$

where

$$
P_{n_1}(x)=\left|\int_0^x\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|
$$

and

$$
P_{n_2}(x)=\left|\int_x^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|.
$$

Integrating by parts after applying Lemma 3.1, and taking $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$
P_{n_1}(x)\leq\int_0^{x-\frac{x}{\sqrt{n}}}\vartheta_n(x;t)\left|f'_x(t)\right|dt+\int_{x-\frac{x}{\sqrt{n}}}^{x}\vartheta_n(x;t)\left|f'_x(t)\right|dt.
$$

Since $f'_x(x)=0$ and $\vartheta_n(x;t)\leq 1$, it implies

$$
\begin{aligned}
\int_{x-\frac{x}{\sqrt{n}}}^{x}\vartheta_n(x;t)\left|f'_x(t)\right|dt=&\int_{x-\frac{x}{\sqrt{n}}}^{x}\vartheta_n(x;t)\left|f'_x(t)-f'_x(x)\right|dt\\
\leq&\int_{x-\frac{x}{\sqrt{n}}}^{x}\bigvee_t^x\left(f'_x\right)dt\leq\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\left(f'_x\right).
\end{aligned}
$$

Again using Lemma 3.1 and substituting $y=x-\frac{x}{u}$ we obtain

$$
\int_0^{x-\frac{x}{\sqrt{n}}}\vartheta_n(x;t)\left|f'_x(t)\right|dt\leq\frac{K\delta_n^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\left|f'_x(t)\right|}{(x-t)^2}dt\leq\frac{K\delta_n^2(x)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^{x}\left(f'_x\right)du
$$

$$\leq \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f'_x).$$

Thus, we can write $P_{n_1}(x)$ as

$$P_{n_1}(x) \leq \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f'_x) + \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f'_x). \tag{3.10}$$

Next, to estimate $P_{n_2}(x)$, we have

$$\begin{aligned}P_{n_2}(x) &\leq \left|\int_x^{2x}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right| + \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|\\ &\leq A_n(x) + B_n(x),\end{aligned}$$

where

$$A_n(x) = \left|\int_x^{2x}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|$$

and

$$B_n(x) = \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|.$$

Since $1-\vartheta_n(x,t)\leq 1$, by putting $t = x+\frac{x}{u}$ successively, we have

$$\begin{aligned}A_n(x) =& \left|\int_x^{2x} f'_x(u)\,(1-\vartheta_n(x,2x))\,du - \int_x^{2x} f'_x(t)\,(1-\vartheta_n(x,t)\,dt\right|\\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \int_x^{x+\frac{x}{\sqrt{n}}}\left|f'_x(t)\right|\left|1-\vartheta_n(x,t)\right|dt\\ &+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\left|f'_x(t)\right|\left|1-\vartheta_n(x,t)\right|dt\\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \frac{K\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{V_x^t(f'_x)}{(t-x)^2}dt + \int_x^{x+\frac{x}{\sqrt{n}}}\bigvee_x^t(f'_x)dt\\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \frac{K\delta_n^2(x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{\sqrt{k}}}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(f'_x).\end{aligned}$$

Further we estimate the value of $B_n(x)$ as follows:

$$B_n(x) = \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)dt\right| \tag{3.11}$$

$$\leq Ç \int_{2x}^{\infty} t^{\gamma} \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt + |f(x)| \int_{2x}^{\infty} \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt + \sqrt{\frac{K\delta_n^2(x)}{n}} f'(x+). \tag{3.12}$$

It is obvious that $t \leq 2(t-x)$ and $x \leq t-x$, when $t \geq 2x$. Now applying Hölder's inequality in the first term of equation (3.11), we get

$$B_n(x) = 2^{\gamma} Ç \left(\int_0^{\infty} (t-x)^{2\tau} \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt \right)^{\frac{\gamma}{2\tau}} + \frac{K\delta_n^2(x)}{nx^2}|f(x)| + \sqrt{\frac{K\delta_n^2(x)}{n}} f'(x+)$$

$$= \wp(\gamma,\tau,x) + \frac{K\delta_n^2(x)}{nx^2}|f(x)| + \sqrt{\frac{K\delta_n^2(x)}{n}} f'(x+). \tag{3.13}$$

Finally, combining equations (3.10)–(3.13) and putting values of $P_{n_1}(x)$ and $P_{n_2}(x)$ in (3.9), we get the required result and the theorem is proved. □

Example 3.1. Let $f(x) = x^4 - 3x^3 + 2x^2 + 1$. We choose parameters $\alpha = \lambda = 2$ and $\rho = 3$. For $n = 10, 50, 100, 200$, we have the following representations.

(a) Figure 1 shows the rate of approximation of the operators $\mathcal{M}_n^{\alpha,\lambda}$ towards the function f. Clearly the proposed operators (1.3) converge to the function f for sufficiently large n.

(b) In Figure 2, the associated absolute error $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ is represented graphically for arbitrary values of x in interval $[0,\infty)$. It can be observed that error is monotonically decreasing for increasing n.

(c) An error estimation table is provided in Table 1 which depicts that for higher value of n, the error approaches to zero.

Therefore, it can be concluded that proposed operators (1.3) provide good approximation for n adequately large.

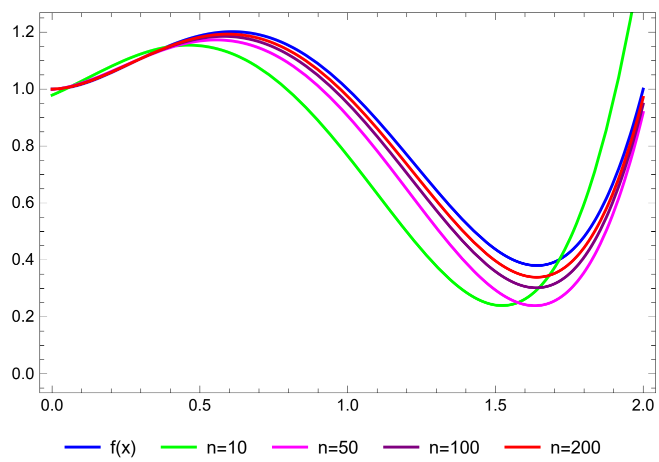

FIGURE 1. Convergence of $\mathcal{M}_n^{\alpha,\lambda}(f;x)$ for the polynomial function $f(x) = x^4 - 3x^3 + 2x^2 + 1$ with parameters $\alpha = \lambda = 2$, $\rho = 3$.

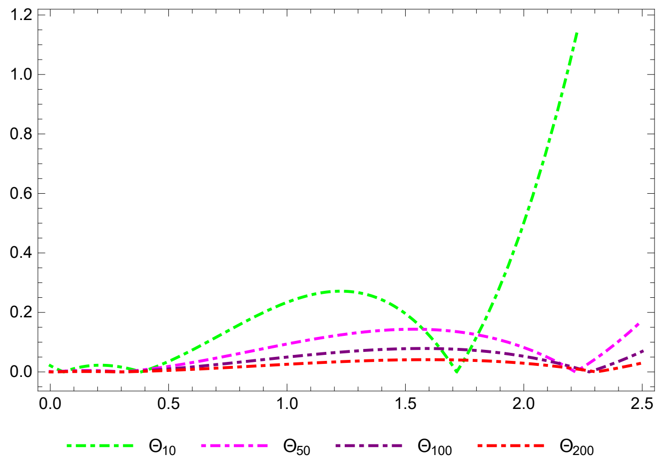

FIGURE 2. Absolute error $\Theta_n = \mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators for $f(x) = x^4 - 3x^3 + 2x^2 + 1$ with parameters $\alpha = \lambda = 2$, $\rho = 3$.

TABLE 1. Table for Absolute error $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators $\mathcal{M}_n^{\alpha,\lambda}$.

x	Θ_{10}	Θ_{50}	Θ_{100}	Θ_{200}
0.4	0.0100759	0.00856603	0.0045899	0.00236582
0.8	0.166213	0.0652568	0.0345534	0.0177532
1.2	0.232451	0.120425	0.0647845	0.0335316
1.6	0.0454756	0.123219	0.0698567	0.0369876
2.0	0.921829	0.022785	0.0243437	0.0154083
2.4	2.65087	0.231729	0.0971809	0.0439198
2.8	5.48687	0.691177	0.320143	0.15371
3.2	9.68409	1.40641	0.66997	0.326674
3.6	15.4968	2.42828	1.17209	0.575527
4.0	23.1792	3.80765	1.85192	0.912982

Example 3.2. Figure 3 illustrates the effect of increase in values of parameter ρ on the rate of convergence of proposed operators $\mathcal{M}_n^{\alpha,\lambda}$ for the function $f(x) = 4x(x - 1.1)(x - 1.9)$ while keeping the value of α, λ and n fixed. Here we chose $n = 10$ and $\alpha = \lambda = 2$ to show the impact of the parameter ρ clearly. One can easily deduce from the figure that as we increase the value of ρ the rate of convergence gets relatively faster.

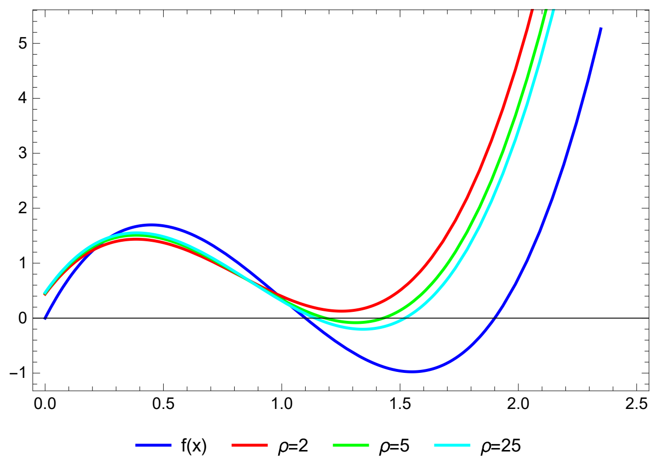

FIGURE 3. Effect of increase in parametric value of ρ for given $n = 10$, $\alpha = \lambda = 2$ on the convergence rate of proposed operators.

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