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PRABHAKAR AND HILFER-PRABHAKAR FRACTIONAL DERIVATIVES IN THE SETTING OF Ψ-FRACTIONAL CALCULUS AND ITS APPLICATIONS

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Abstract. The aim of this paper is to study to fractional calculus for class of Ψ function. The present study is designed to study generalized fractional derivatives and find their generalized transforms called Ψ-Laplace transform and Ψ-Sumudu transform. Moreover, find the analytical solutions of some applications in physics the form of generalized fractional derivatives by transform technique.

1. Introduction

In recent years, many researchers investigated generalization of integration and differentiation operators in the field of fractional calculus. In literature several different definitions of fractional integrals and derivatives are available, like Riemann-Liouville integral and derivative Caputo fractional derivative etc. (see [3,18,19]). In [13] defined new fractional derivative called Hilfer fractional derivative which is generalization of Riemann and Caputo fractional derivative. The first investigated generalized Mittage-Leffler function by Prabhakar [17]. The so-called Prabhakar integral is defined in a similar way Riemann-Lioville integral [12,14,17]. Roberto Garra et al. [12] introduced fractional derivative by definition of Hilfer derivative replacing Riemann-Liouville integral operator by Prabhakar integral operator called Hilfer-Prabhakar derivative also defined Prabhakar and Hilfer-Prabhakar derivatives regularized version. Dorrego defined generalization of Prabhakar integral and derivative called k-Prabhakar integral

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and k-Prabhakar derivative [7]. In [5,6] Dorrego and Cerutti defined the kernel of k-Mittage-Leffler function and generalized derivative called k-fractional Hilfer derivative. Recently Dole et al. [15,16] defined generalized fractional derivative like k-Hilfer-Prabhakar derivative as well as defined regularized version of k-Prabhakar derivative. Moreover, find Laplace and Sumudu transform to regularized version of k-Prabhakar derivative also k-Hilfer-Prabhakar derivative and its regularized version. In [11] Sausa and Oliviera introduced new fractional derivative in the setting of Ψ-fractional operator called Ψ-Hilfer fractional derivative defined as

$$D^{\mu,\nu,\Psi} = I^{\nu(m-\mu),\Psi}\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^{m} I^{(1-\nu)(m-\mu),\Psi} f(t).$$

The new generalized integral transform called Ψ-Laplace transform published on an arXiv, by Fahad et al. and obtain for Ψ-Hilfer fractional derivative as follows

$$L_{\Psi}\Big\{D^{\mu,\nu,\Psi} f(t)\Big\} = s^{\mu} L_{\Psi}\Big\{f(t)\Big\} - \sum_{k=0}^{m-1} s^{m(1-\nu)+\mu\nu-k-1}\big(I^{(1-\nu)(m-\mu)-k,\Psi} f(t)\big).$$

In this paper, we define new integral transform called Ψ-Sumudu transform. Define some generalized definitions of fractional derivatives in the setting of Ψ-fractional calculus like Ψ-Prabhakar, Ψ-Hilfer-Prabhakar, Ψ-k-RL-fractional, Ψ-k-Hilfer, Ψ-k-Prabhakar, Ψ-k-Hilfer-Prabhakar fractional integrals and derivatives as well as define all these new fractional derivatives regularized versions. These results are used to obtain the relation between Ψ-Prabhakar fractional derivative and its regularized version and also the relation between Ψ-Hilfer-Prabhakar fractional derivative and its regularized version involving Mittag-Leffler function. Moreover, we obtain Ψ-Laplace transform [9] and Ψ-Sumudu transform to find solutions of fractional differential equations.

2. Preliminaries

Definition 2.1 ([17])**.** Let $n \in \mathbb{N}$, $\alpha, \mu, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\mu) > 0$. The Mittag-Leffler function is defined as

$$E^{\gamma}_{\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\mu)}\cdot\frac{z^n}{n!},$$

where $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+(n-1))$ is the Pochhammer symbol.

Definition 2.2 ([5])**.** Let $n \in \mathbb{N}$, $\alpha, \mu, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\mu) > 0$. The k-Mittag-Leffler function is defined as

$$E^{\gamma}_{k,\alpha,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\mu)}\cdot\frac{z^n}{n!},$$

where $(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\cdots(\gamma+(n-1)k)$ is the Pochhammer symbol.

Definition 2.3 ([8,10])**.** Let μ be a real number such that $\mu > 0$, $-\infty \le a < b \le \infty$, $m = \mu+1$, f be an integrable function defined on $[a,b]$ and $\Psi \in C^1([a,b])$ be increasing function such that $\Psi^{'}(t) \neq 0$ for all t $\in [a,b]$. Then, the Ψ-RL-fractional integral and Ψ-RL fractional derivative of a function f of order μ are defined as

$$I_0^{\mu,\Psi} f(t) = \frac{1}{\Gamma(\mu)} \int_0^{\infty} (\Psi(t) - \Psi(s))^{\mu-1} \Psi^{'}(s) f(s) ds, \tag{2.1}$$

$$D_0^{\mu,\Psi} = \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m I_0^{m-\mu,\Psi} f(t). \tag{2.2}$$

It is to be noted that for $\Psi(t) \to t$, $I_0^{\mu,\Psi} f(t) \to I_0^{\mu} f(t)$ which is the standard Riemann- Liouville integral. Moreover for $\Psi(t) \to \ln(t)$ the integral defined in equation (2.1) towards Hadamard fractional integral.

Inspired by Caputo's concept [2] of fractional derivative, Almeidea [1] presents the following Caputo version of equation (2.2) and studies some important properties of fractional calculus.

Definition 2.4 ([1]). Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-C-fractional derivative of a function f of order μ is defined as

$${}^{C}D_0^{\mu,\Psi} = I_0^{m-\mu,\Psi} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f(t).$$

Definition 2.5 ([11]). Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, and $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$D_0^{\mu,\nu,\Psi} = I_0^{\nu(m-\mu,\Psi)} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m I_0^{(1-\nu)(m-\mu),\Psi} f(t).$$

Definition 2.6 ([9]). Let $f : [0,\infty) \to \mathbb{R}$ be a real valued function and Ψ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-Laplace transform of f is denoted by $L_{\Psi}\{f\}$ and is defined by

$$T(u) := L_{\Psi}\{f(t)\} := \int_0^{\infty} e^{-u\Psi(t)} \Psi^{'}(t) f(t) dt, \quad \text{for all } u.$$

3. Main Result

We consider functions in the set A is defined by

$$A = \{f(t) \mid \text{ exists } M, \tau_1, \tau_2 |f(t)| \leq M e^{|t|/T_j}, \text{ if } t\epsilon(-1^j) \times [0,\infty)\}.$$

Definition 3.1. Let $f : [0,\infty) \to \mathbb{R}$ be a real valued function and Ψ be a non-negative increasing function such that $\Psi(0) = 0$. Then the Ψ-ST of f is denoted by $S_{\Psi}\{f\}$ and is defined by

$$T(u) := S_{\Psi}\{f(t)\} := \frac{1}{u} \int_0^{\infty} e^{-\frac{\Psi(t)}{u}} \Psi^{'}(t) f(t) dt, \quad \text{for all } u.$$

3.1. New definitions of Ψ-fractional derivatives.

Definition 3.2. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$ and $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the Ψ-Prabhakar fractional integral and derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$\begin{aligned}\left(P^{\gamma,\Psi}_{\alpha,\mu,\omega} f\right)(x) &= \int_0^x (\Psi(x) - \Psi(t))^{\mu-1} E^{\gamma}_{\alpha,\mu}[\omega(\Psi(x) - \Psi(t))^{\alpha}]\Psi'(t) f(t) dt \\ &= (\varepsilon^{\gamma}_{\alpha,\mu,\omega} *_{\Psi} f)(x), \end{aligned} \tag{3.1}$$

where $*_{\Psi}$ denotes the convolution operation, $\alpha, \mu, \omega, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\mu) > 0$ and

$$\varepsilon^{\gamma}_{\alpha,\mu,\omega} \Psi(t) = \begin{cases} (\Psi(t))^{\mu-1} E^{\gamma}_{\alpha,\mu}(\omega(\Psi(t))^{\alpha}), & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{3.2}$$

For $\gamma = 0$, $\left(P^{0}_{\alpha,\mu,\omega} f\right)(x) = \left(I^{\mu,\Psi} f\right)(x)$ and for $\gamma = \mu = 0$, $\left(P^{0}_{\alpha,0,\omega} f\right)(x) = f(x)$,

$$D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = \left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m P^{-\gamma,\Psi}_{\rho,m-\mu,\omega} f(t). \tag{3.3}$$

Definition 3.3. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-Prabhakar fractional derivative of a function f of order μ is defined as

$$^{C}D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = P^{-\gamma,\Psi}_{\rho,m-\mu,\omega} \left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m f(t). \tag{3.4}$$

Definition 3.4. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$ and type $0 \leq \nu \leq 1$ and $\left(f * \varepsilon^{-\gamma(1-\nu)}_{\rho,(1-\nu)(m-\mu),\omega}\right) \Psi(t) \in AC^1[0,b]$. Then, the Ψ-Hilfer-Prabhakar fractional derivative of a function f of order μ defined as

$$\mathfrak{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+} \left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m \left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+} f\right)\right)(t). \tag{3.5}$$

Definition 3.5. Let μ be a real number such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = \mu + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi'(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$^{C}\mathfrak{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = \left(P^{-\gamma\nu,\Psi}_{\alpha,\nu(m-\mu),\omega,0^+} P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu,\omega,0^+} \left(\frac{1}{\Psi'} \cdot \frac{d}{dt}\right)^m f\right)(t). \tag{3.6}$$

Definition 3.6. Let μ be a real number and let $k \in \mathbb{R}^+$, such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, f be an integrable function defined on $[a,b]$ and $\Psi \in C^1([a,b])$ be increasing function such that $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-RL fractional integral and Ψ-k-RL fractional derivative of a function f of order μ are defined as

$$I_k^{\mu,\Psi} f(t) = \frac{1}{k\Gamma_k(\mu)} \int_0^{\infty} (\Psi(t) - \Psi(s))^{\frac{\mu}{k}-1} \Psi^{'}(s) f(s) ds,$$

$$D_k^{\mu,\Psi} = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m k^m I_k^{m-\mu,\Psi} f(t).$$

Definition 3.7. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Caputo fractional derivative of a function f of order μ is defined as

$${}_k^C D_0^{\mu,\Psi} = k^m I_k^{m-\mu,\Psi} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f(t).$$

Definition 3.8. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Hilfer fractional derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$D_k^{\mu,\nu,\Psi} = I_k^{\nu(m-\mu,\Psi)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^m I_k^{(1-\nu)(m-\mu),\Psi} f(t).$$

Definition 3.9. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $\mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a, b]$. Then, the Ψ-k-Prabhakar fractional integral and derivative of a function f of order μ and type $0 \leq \nu \leq 1$ is given by

$$\begin{aligned} \left({}_k P_{\alpha,\mu,\omega}^{\gamma,\Psi} f \right)(x) &= \int_0^x \frac{(\Psi(x) - \Psi(t))^{\frac{\mu}{k}-1}}{k} E_{k,\alpha,\mu}^{\gamma} [\omega(\Psi(x) - \Psi(t))^{\frac{\alpha}{k}}] \Psi^{'}(t) f(t) dt \\ &= ({}_k\varepsilon_{\alpha,\mu,\omega}^{\gamma} *_{\Psi} f)(x), \end{aligned} \tag{3.7}$$

where $*_{\Psi}$ denotes the convolution operation, $\alpha, \mu, \omega, \gamma \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\mu) > 0$ and

$${}_k\varepsilon_{\alpha,\mu,\omega}^{\gamma} \Psi(t) = \begin{cases} \frac{(\Psi(t))^{\frac{\mu}{k}-1}}{k} E_{k,\alpha,\mu}^{\gamma} (\omega(\Psi(t))^{\frac{\alpha}{k}}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{3.8}$$

for $\gamma = 0$, $\left({}_k P_{\alpha,\mu,\omega}^{0} f \right)(x) = \left(I_k^{\mu,\Psi} f \right)(x)$ and for $\gamma = \mu = 0$, $\left({}_k P_{\alpha,0,\omega}^{0} f \right)(x) = f(x)$,

$${}_k D_{\rho,\mu,\omega}^{\gamma,\Psi} f(t) = \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m k^m {}_k P_{\rho,mk-\mu,\omega}^{-\gamma,\Psi} f(t). \tag{3.9}$$

Definition 3.10. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a, b])$ be the functions such that

Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-k-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}_k{}^C D^{\gamma,\Psi}_{\rho,\mu,\omega} f(t) = k^m {}_k P^{-\gamma,\Psi}_{\rho,mk-\mu,\omega} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f(t). \tag{3.10}$$

Definition 3.11. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$ and type $0 \leq \nu \leq 1$ and $\left(f * \varepsilon^{-\gamma(1-\nu)}_{\rho,(1-\nu)(mk-\mu),\omega} \right) \Psi(t) \in AC^1[0,b]$. Then, the Ψ-k-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = k^m \left({}_k P^{-\gamma\nu,\Psi}_{\alpha,\nu(mk-\mu),\omega,0^+} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m \left({}_k P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} f \right) \right)(t). \tag{3.11}$$

Definition 3.12. Let μ be a real number and let $k \in \mathbb{R}^+$ such that $k, \mu > 0$, $-\infty \leq a < b \leq \infty$, $m = [\frac{\mu}{k}] + 1$, $f, \Psi \in C^m([a,b])$ be the functions such that Ψ is increasing and $\Psi^{'}(t) \neq 0$ for all $t \in [a,b]$. Then, the regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative of a function f of order μ is defined as

$$ {}^C_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = k^m \left({}_k P^{-\gamma\nu,\Psi}_{\alpha,\nu(mk-\mu),\omega,0^+} {}_k P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+} \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} \right)^m f \right)(t). \tag{3.12}$$

4. Ψ-Laplace Transform and Ψ-Sumudu Transform of Ψ-Fractional Derivatives

Let $F(s)$ be the Ψ-Laplace transform of $f(t)$.

Lemma 4.1. *The Ψ-Laplace transform of Ψ-Prabhakar fractional integral equation* (3.1) *is*

$$ \mathcal{L}_\Psi \left(P^\gamma_{\alpha,\mu,\omega} f \right)(x) = s^{-\mu}(1 - \omega(s)^{-\alpha})^{-\gamma} F(s). \tag{4.1}$$

Lemma 4.2. *The Ψ-Laplace transform of Ψ-Prabhakar fractional derivative equation* (3.3) *is*

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = s^\mu \left(1 - \omega(s)^{-\alpha} \right)^\gamma F(s) - \sum_{k=0}^{m-1} s^{m-k-1} \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega} f(0^+) \right].$$

For the case $[\mu] + 1 = m = 1$

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = s^\mu \left(1 - \omega(s)^{-\alpha} \right)^\gamma F(s) - \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t) \right]_{t=0^+}, \tag{4.2}$$

with $|\omega(s)^{-\alpha}| < 1$.

Proof. Taking Ψ-Laplace transforms of Ψ-Prabhakar fractional derivative in (3.3) and using (3.1), (3.2), (4.1) we get

$$ \mathcal{L}_\Psi \left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) \right)(s) = \mathcal{L} \left(\left(\frac{1}{\Psi'} \cdot \frac{d}{dt} \right)^m P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t) \right)(s)$$

$$
\begin{aligned}
&= s^m \mathcal{L}_\Psi\left(\left(\varepsilon^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} * f\right)(t)\right)(s) \\
&\quad - \sum_{k=0}^{m-1} s^{m-k-1}\left[\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+} \\
&= s^m \mathcal{L}_\Psi\left((\Psi(t))^{(m-\mu)-1} E^{-\gamma,\Psi}_{\alpha,(m-\mu)}\Big(\omega(\Psi(t))^\alpha\Big) F(s)\right) \\
&\quad - \sum_{k=0}^{m-1} s^{m-k-1}\left[\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^k P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(0^+)\right] \\
&= s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{m-k-1}\left(P^{-\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega} f(0^+)\right).
\end{aligned}
$$

For the case $[\mu]+1=m=1$, we have

$$
\mathcal{L}_\Psi\left(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\right)(s) = s^\mu\left(1-\omega(s)^{-\alpha}\right)^\gamma F(s) - \left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+}. \qquad \square
$$

Lemma 4.3. *The* Ψ*-Laplace transform of regularized version of* Ψ*-Prabhakar fractional derivative equation* (3.4) *is*

(4.3)

$$
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^{(k)}(0^+),
$$

with $|\omega(s)^{-\alpha}| < 1$.

Proof. Taking Ψ-Laplace transform of regularized version of Ψ-Prabhakar fractional derivative in (3.4) and using (3.1), (3.2), (4.1) we get

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s), &= \mathcal{L}_\Psi\left(\left(\varepsilon^{-\gamma}_{\alpha,(m-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'}\cdot\frac{d}{dt}f\right)^m(t)\right)(s)\right. \\
&= s^{-(m-\mu)}\Big(1-\omega(s)^\alpha\Big)^\gamma\left\{s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^k(0^+)\right\} \\
&= s^\mu\Big(1-\omega(s)^\alpha\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^\alpha\Big)^\gamma f^k(0^+). \qquad \square
\end{aligned}
$$

For absolutely continuous function $f \in AC^1[0,b]$,

$$
\left[P^{-\gamma,\Psi}_{\alpha,(m-\mu),\omega} f(t)\right]_{t=0^+} = 0.
$$

Then in view of equation (4.2) and equation (4.3) ($m=1$) we have

$$
\mathcal{L}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = \mathcal{L}\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) - (s)^{\mu-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f(0^+).
$$

Taking inverse Ψ-Laplace transform, we get

$$
{}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) = D^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t) - (\Psi(t))^{-\mu} E^{-\gamma}_{\alpha,m-\mu}(\omega(\Psi(t))^\alpha) f(0^+),
$$

for $f \in AC^1[0,b]$. This is the relation between Ψ-Prabhakar fractional derivative and its regularized version.

Lemma 4.4. *The* Ψ*-Laplace transform of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.5) *is*

$$\mathcal{L}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{m(1-\nu)+\nu\mu-k-1}[1-\omega(s)^{-\alpha}]^{\gamma\nu} \times \Big(P^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(m-\mu)-k,\omega}, \Psi f(0^+)\Big).$$

Proof. Taking Ψ-Laplace transform of Ψ-Hilfer-Prabhakar fractional derivative in (3.5) and using (3.1), (3.2), (4.1) we have

$$\begin{aligned}\mathcal{L}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s) =&\mathcal{L}_\Psi\left[\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^m \Big(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(m-\mu),\omega,0^+}f\Big)\right)(t)\right](s)\\
=&s^m(s)^{-\nu(m-\mu)}\Big(1-\omega(s)^{-\alpha}\Big)^{\gamma\nu}\mathcal{L}_\Psi\left(\Big(\varepsilon^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(m-\mu),\omega} * f\Big)(t)\right)(s)\\
&-\sum_{k=0}^{m-1} s^m(s)^{-\nu(m-\mu)}\Big(1-\omega(s)^{-\alpha}\Big)^{\gamma\nu}\Big[P^{-\lambda(1-\nu),\Psi}_{\alpha,(1-\nu),(m-\mu)-k,\omega}f(0^+)\Big]\\
=&s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s)\\
&-\sum_{k=0}^{m-1} s^{m(1-\nu)+\nu\mu-k-1}[1-\omega(s)^{-\alpha}]^{\gamma\nu}\Big(P^{-\gamma(1-\nu)(m-\mu),\psi}_{\alpha,(1-\nu)(m-\mu)-k,\omega}f(0^+)\Big).\end{aligned}$$ □

Lemma 4.5. *The* Ψ*-Laplace transforms of the regularized version of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.6) *of order* μ *is*

$$\mathcal{L}_\Psi\Big({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s) = s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^k(0^+).$$

Proof. Taking Ψ-Laplace transforms of regularized version of Ψ-Hilfer-Prabhakar fractional derivative in (3.6) and using (3.1), (3.2), (4.1) we have

$$\begin{aligned}\mathcal{L}_\Psi\Big({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s) =&\mathcal{L}_\Psi\left(\left(\varepsilon^{-\gamma\nu}_{\alpha,\nu(m-\mu),\omega} *_\Psi \left(P^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(k-\mu),\omega,0^+}\left(\frac{1}{\Psi'}\cdot\frac{d}{dt}\right)^m f\right)\right)(t)\right)(s)\\
=& s^\mu\Big(1-\omega(s)^{-\alpha}\Big)^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}\Big(1-\omega(s)^{-\alpha}\Big)^\gamma f^k(0^+).\end{aligned}$$ □

Again for absolutely continuous function $f \in AC^1[0,b]$

$${}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t) = \mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t) - (\Psi(t))^{-\mu}E^{-\gamma}_{\alpha,(m-\mu)}(\omega(\Psi(t))^\alpha)f(0^+).$$

This is the relation between Ψ-Hilfer-Prabhakar fractional derivative and its regularized version.

Let $F(u)$ be the Ψ-Sumudu transform of $f(t)$.

Lemma 4.6. *The* Ψ*-Sumudu transform of* Ψ*-Prabhakar integral equation* (3.1) *is*

$$S_\Psi\Big(P^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = u^{-1}(u)^\mu\Big(1-\omega(u)^\alpha\Big)^{-\gamma}F(u),$$

provided $|\omega(u)^\alpha| < 1$.

Lemma 4.7. *The* Ψ*-Sumudu transform of* Ψ*-Prabhakar fractional derivative equation* (3.3) *is*

$$\mathcal{S}_\Psi\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = u^{-\mu}\Big(1-\omega(u)^\alpha\Big)^{\gamma}F(u) - \sum_{n=0}^{m-1} u^{-m+k}\Big[P^{\gamma,\Psi}_{\alpha,(m-\mu)-k,\omega}f(0^+)\Big].$$

For the case $[\mu]+1=m=1$

$$\mathcal{S}_\Psi\Big(D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = u^{-\mu}\Big(1-\omega(u)^\alpha\Big)^{\gamma}F(u) - \frac{1}{u}\left[P^{-\gamma,\Psi}_{\alpha,(1-\mu),\omega}f(t)\right]_{t=0^+},$$

with $|\omega(u)^\alpha| < 1$.

Lemma 4.8. *The* Ψ*-Sumudu transform of regularized version of* Ψ*-Prabhakar fractional derivative equation* (3.4) *is*

$$\mathcal{S}_\Psi\Big({}^C D^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = u^{-\mu}\Big(1-\omega(u)^\alpha\Big)^{\gamma}F(u) - \sum_{n=0}^{m-1} u^{-(\mu-k)}\left(1-\omega\left(\frac{1}{u}\right)^{\alpha}\right)^{\gamma} f^{(n)}(0),$$

with $|\omega(u)^\alpha| < 1$.

Lemma 4.9. *The* Ψ*-Sumudu transform of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.5) *is*

$$\begin{aligned}\mathcal{S}_\Psi\Big(\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(u) =& u^{-\mu}\Big(1-\omega(u)^\alpha\Big)^{\gamma}F(u) - \sum_{k=0}^{m-1} u^{m(\nu-1)-\nu\mu+k}\Big(1-\omega(u)^\alpha\Big)^{\gamma\nu}\\ &\times\Big[P^{-\gamma(1-\nu),\Psi}_{\alpha,\nu(1-\nu)(m-\mu)-k,\omega}f(0^+)\Big].\end{aligned}$$

Lemma 4.10. *The* Ψ*-Sumudu transforms of the regularized version of* Ψ*-Hilfer-Prabhakar fractional derivative equation* (3.6) *of order* μ *is*

$$\mathcal{S}_\Psi\Big({}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(u) = u^{-\mu}\Big(1-\omega(u)^\alpha\Big)^{\gamma}F(u) - \sum_{k=0}^{m-1} u^{-\mu+k}\Big(1-\omega(u)^\alpha\Big)^{\gamma} f^k(0^+).$$

4.1. Ψ-Laplace and Ψ-Sumudu transform of Ψ-k-fractional derivatives.

Lemma 4.11. *The* Ψ*-Laplace transform of* Ψ*-*k*-Prabhakar fractional integral equation* (3.7) *is*

$$\mathcal{L}_\Psi\Big({}_kP^{\gamma,\Psi}_{\alpha,\mu,\omega}f\Big)(x) = (ks)^{\frac{-\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{-\gamma}{k}}F(s). \tag{4.4}$$

Lemma 4.12. *The* Ψ*-Laplace transform of* Ψ*-*k*-Prabhakar fractional derivative equation* (3.9) *is*

$$\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(s) =(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}F(s)$$

$$- k^m \sum_{n=0}^{m-1} s^{m-n-1} \left[{}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega} f(0^+)\right].$$

For the case $[\frac{\mu}{k}] + 1 = m = 1$

$$\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = (ks)^{\frac{\mu}{k}}\Big(1 - \omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) - k\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega} f(t)\Big]_{t=0^+}, \tag{4.5}$$

with $|\omega k(ks)^{\frac{-\alpha}{k}}| < 1$.

Proof. Taking Ψ-Laplace transforms of Ψ-k-Prabhakar fractional derivative in (3.9) and using (3.7), (3.8), (4.4) we get

$$\begin{aligned}
\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) =& \mathcal{L}\left(\left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m k^m {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(t)\right)(s) \\
=& k^m s^m \mathcal{L}_\Psi\Big(\Big({}_k\varepsilon^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} * f\Big)(t)\Big)(s) \\
& - k^m \sum_{n=0}^{m-1} s^{m-n-1} \left[\left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^n {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(t)\right]_{t=0^+} \\
=& (ks)^m \mathcal{L}_\Psi\left((\Psi(t))^{\frac{(mk-\mu)}{k}-1} E^{-\gamma}_{k,\alpha,(mk-\mu)}\Big(\omega(\Psi(t))^{\frac{\alpha}{k}}\Big) F(s)\right) \\
& - k^m \sum_{n=0}^{m-1} s^{m-n-1} \left[\left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^k {}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu),\omega} f(0^+)\right] \\
=& (ks)^{\frac{\mu}{k}}\Big(1 - \omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
& - k^m \sum_{n=0}^{m-1} s^{m-n-1}\Big({}_kP^{-\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega} f(0^+)\Big).
\end{aligned}$$

□

For the case $[\frac{\mu}{k}] + 1 = m = 1$, we have

$$\mathcal{L}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = (ks)^{\frac{\mu}{k}}\Big(1 - \omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) - k\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega} f(t)\Big]_{t=0^+}.$$

Lemma 4.13. *The* Ψ*-Laplace transform of regularized version of* Ψ*-*k*-Prabhakar fractional derivative equation* (3.10) *is*

$$\begin{aligned}
\mathcal{L}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) =& (ks)^{\frac{\mu}{k}}\Big(1 - \omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) \\
& - k^m \sum_{n=0}^{m-1} (ks)^{\frac{\mu-n-1}{k}}\Big(1 - \omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^{(n)}(0^+),
\end{aligned} \tag{4.6}$$

with $|\omega k(ks)^{\frac{-\alpha}{k}}| < 1$.

Proof. Taking Ψ-Laplace transform of regularized version of Ψ-k-Prabhakar fractional derivative in (3.10) and using (3.7), (3.8), (3.9), (4.4) we get

$$\mathcal{L}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(s) = \mathcal{L}_\Psi\left({}_k\varepsilon^{-\gamma}_{\alpha,(mk-\mu),\omega} *_\Psi \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt} f\right)^m (t)\right)(s)$$

$$
\begin{aligned}
&=(ks)^m (ks)^{\frac{-(mk-\mu)}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}\\
&\quad\times\left\{F(s)-k^m\sum_{k=0}^{m-1}(ks)^{\frac{\mu-n-1}{k}}f^n(0^+)\right\}\\
&=(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}\\
&\quad-k^m\sum_{k=0}^{m-1}(ks)^{\frac{\mu-n-1}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}f^n(0^+). \qquad \square
\end{aligned}
$$

For absolutely continuous function $f\in AC^1[0,b]$

$$\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega}f(t)\Big]_{t=0^+}=0.$$

Then in view of (4.5) and (4.6) ($m=1$) we have

$$\mathcal{L}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(s)=\mathcal{L}\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(s)-k(ks)^{\frac{\mu-k}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}f(0^+).$$

Taking inverse Ψ-Laplace transform, we get

$${}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)={}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)-(\Psi(t))^{\frac{-\mu}{k}}E^{-\gamma}_{k,\alpha,k-\mu}(\omega(\Psi(t))^{\frac{\alpha}{k}})f(0^+),$$

for $f\in AC^1[0,b]$. This is the relation between Ψ-k–Prabhakar fractional derivative and its regularized version.

Lemma 4.14. *The Ψ-Laplace transform of Ψ-k-Hilfer-Prabhakar fractional derivative* (3.11) *is*

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s)=&(ks)^{\frac{\mu}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}}F(s)\\
&-k^m\sum_{n=0}^{m-1}(ks)^{\frac{m(1-\nu)+\nu\mu-n-1}{k}}[1-\omega k(ks)^{\frac{-\alpha}{k}}]^{\frac{\gamma\nu}{k}}\\
&\times\Big({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu)-n,\omega}f(0^+)\Big).
\end{aligned}
\tag{4.7}
$$

Proof. Taking Ψ-Laplace transform of Ψ-k-Hilfer-Prabhakar fractional derivative in (3.11) and using (3.7), (3.8), (3.9), (4.4) we have

$$
\begin{aligned}
\mathcal{L}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(s)=&\mathcal{L}_\Psi\Bigg[\Bigg({}_k\varepsilon^{-\gamma\nu}_{\alpha,\nu(mk-\mu),\omega}*_\Psi\left(\frac{1}{\Psi'(t)}\cdot\frac{d}{dt}\right)^m\\
&\times\Big({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+}f\Big)\Bigg)(t)\Bigg](s)\\
=&(ks)^m(ks)^{\frac{-\nu(mk-\mu)}{k}}\Big(1-\omega k(ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}}\Bigg\{s^m\Big((ks)^{-(1-\nu)(m-\mu)}\Big)\\
&\times\Bigg[\Big((1-\omega k(ks)^{\frac{-\alpha}{k}})^{\frac{\gamma(1-\nu)}{k}}F(s)\Big)(t)\Bigg)(s)\Bigg\}
\end{aligned}
$$

$$- k^m \sum_{n=0}^{m-1} (ks)^{\frac{-\nu(mk-\mu)}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}}$$
$$\times \Big[{}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu),(mk-\mu)-n,\omega} f(0^+)\Big]\Bigg]$$
$$=(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s)$$
$$- k^m \sum_{n=0}^{m-1} (ks)^{\frac{m(1-\nu)+\nu\mu-n-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma\nu}{k}}$$
$$\times \Big({}_kP^{-\gamma(1-\nu),\psi}_{\alpha,(1-\nu)(mk-\mu)-n,\omega} f(0^+)\Big). \qquad \square$$

Lemma 4.15. *The* Ψ*-Laplace transforms of the regularized version of* Ψ*-k-Hilfer-Prabhakar fractional derivative equation* (3.12) *of order* μ *is*

$$\mathcal{L}_\Psi\Big({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s) - k^m \sum_{n=0}^{m-1} (ks)^{\frac{\mu-n-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^n(0^+). \tag{4.8}$$

Proof. Taking Ψ-Laplace transforms of regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative in (3.12) and using (3.8), (3.9), (4.4) we have

$$\mathcal{L}_\Psi\Big({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t)\Big)(s) =k^m \mathcal{L}_\Psi\Bigg(\Bigg({}_k\varepsilon^{-\gamma\nu}_{\alpha,\nu(mk-\mu),\omega} *_\Psi \Bigg({}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(mk-\mu),\omega,0^+}$$
$$\times \left(\frac{1}{\Psi'(t)} \cdot \frac{d}{dt}\right)^m f\Bigg)\Bigg)(t)\Bigg)(s)$$
$$=(ks)^{\frac{\mu}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} F(s)$$
$$- k^m \sum_{n=0}^{m-1} (ks)^{\frac{\mu-k-1}{k}} \Big(1 - \omega k (ks)^{\frac{-\alpha}{k}}\Big)^{\frac{\gamma}{k}} f^n(0^+). \qquad \square$$

Again for absolutely continuous function $f \in AC^1[0, b]$

$${}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) = {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} f(t) - (\Psi(t))^{\frac{-\mu}{k}} E^{-\gamma}_{k,\alpha,(mk-\mu)}(\omega(\Psi(t))^{\frac{\alpha}{k}}) f(0^+).$$

This is the relation between Ψ-k-Hilfer-Prabhakar fractional derivative and its regularized version.

Let $F(s)$ be the Ψ-Sumudu transform of $f(t)$.

Lemma 4.16. *The* Ψ*-Sumudu transform of* Ψ*-k-Prabhakar integral equation* (3.7) *is*

$$S_\Psi\Big({}_kP^{\gamma,\Psi}_{\alpha,\mu,\omega} f(t)\Big)(u) = u^{-1}\left(\frac{u}{k}\right)^{\frac{\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{-\gamma}{k}} F(u),$$

provided $|\omega k (\frac{u}{k})^{\frac{\alpha}{k}}| < 1$.

Lemma 4.17. *The Ψ-Sumudu transform of Ψ-k-Prabhakar fractional derivative equation* (3.9) *is*

$$\mathcal{S}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{-m+n}{k}}\Big[{}_kP^{\gamma,\Psi}_{\alpha,(mk-\mu)-n,\omega}f(0^+)\Big].$$

For the case $[\frac{\mu}{k}]+1=m=1$

$$\mathcal{S}_\Psi\Big({}_kD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - \frac{k}{u}\Big[{}_kP^{-\gamma,\Psi}_{\alpha,(k-\mu),\omega}f(t)\Big]_{t=0^+},$$

with $|\omega k(\frac{u}{k})^{\frac{\alpha}{k}}|<1$.

Lemma 4.18. *The Ψ-Sumudu transform of regularized version of Ψ-k-Prabhakar fractional derivative equation* (3.10) *is*

$$\mathcal{S}_\Psi\Big({}_k{}^CD^{\gamma,\Psi}_{\alpha,\mu,\omega}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{k}{u}\right)^{\frac{-(\mu-nk)}{k}}\left(1-\omega k\left(\frac{k}{u}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}f^{(n)}(0),$$

with $|\omega k(\frac{u}{k})^{\frac{\alpha}{k}}|<1$.

Lemma 4.19. *The Ψ-Sumudu transform of Ψ-k-Hilfer-Prabhakar fractional derivative equation* (3.11) *is*

$$\mathcal{S}_\Psi\Big({}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\Big)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{m(\nu-1)-\nu\mu+n}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \times\Big[P^{-\gamma(1-\nu),\Psi}_{\alpha,\nu(1-\nu)(mk-\mu)-n,\omega}f(0^+)\Big]. \tag{4.9}$$

Lemma 4.20. *The Ψ-Sumudu transforms of the regularized version of Ψ-k-Hilfer-Prabhakar fractional derivative equation* (3.12) *of order μ is*

$$\mathcal{S}_\Psi\left({}_k{}^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}f(t)\right)(u) = \left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}F(u) - k^m\sum_{n=0}^{m-1}\left(\frac{u}{k}\right)^{\frac{-\mu+nk}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}f^n(0^+).$$

5. Application

In this section we find the solutions of Cauchy problems involving Ψ-k-Hilfer-Prabhakar derivative and its regularised version.

Theorem 5.1. *The solution of Cauchy problem*

$$
{}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}y(x) = \lambda {}_kP^{\delta,\Psi}_{\alpha,\mu,\omega,0^+}y(x) + f(x), \tag{5.1}
$$

$$
\left[{}_kP^{-\gamma(1-\nu),\Psi}_{\alpha,(1-\nu)(k-\mu),\omega,0^+}y(x)\right]_{t=0^+} = C, \quad C \geq 0, \tag{5.2}
$$

where $x \in (0,\infty)$, $f(x) \in L^1[0,\infty)$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega, \lambda \in \mathbb{C}$, $\alpha > 0$, $\gamma, \delta \geq 0$ *is given by*

$$
\begin{aligned}
y(x) =& C\sum_{n=0}^{\infty} \lambda^n (\Psi(x))^{\frac{\nu(k-\mu)+\mu(1+2n)}{k}-1} E^{n(\delta+\gamma)-\gamma(\nu-1)}_{k,\alpha,\nu(k-\mu)+\mu(1+2n)}(\omega(\Psi(x))^{\frac{\alpha}{k}} \\
&+ \sum_{n=0}^{\infty} \lambda^n {}_kP^{\gamma+n(\delta+\gamma),\Psi}_{k,\alpha,\mu(1+2n),\omega,0^+} f(x),
\end{aligned} \tag{5.3}
$$

if the series on the right hand side of equation (5.3) *are convergent.*

Proof. Let $Y(u)$ and $F(u)$ denote the Ψ-Laplace transform of $y(x)$ and $f(x)$, respectively. Now taking Ψ-Laplace transform of (5.1) and using (3.7), (3.8), (4.4), (4.7), (5.2) we have

$$
\begin{aligned}
&(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}} Y(u) - k(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} C \\
=&\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}} Y(u) + F(u).
\end{aligned}
$$

□

Thus, we have

$$
\begin{aligned}
Y(u) &= \left(\frac{Ck(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} + F(u)}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}} - \lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}\right) \\
&= \left[\frac{Ck(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} + F(u)}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right] \\
&\quad \times \frac{1}{\left[1-\frac{\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right]}.
\end{aligned}
$$

Hence, for $\left|\frac{\lambda(ku)^{\frac{-\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\delta}{k}}}{(ku)^{\frac{\mu}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right| < 1$, we get

$$\begin{aligned} Y(u) =& Ck\sum_{n=0}^{\infty}\lambda^n(ku)^{\frac{-\nu(k-\mu)-\mu(1+2n)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma(\nu-1)-n(\delta+\gamma)}{k}} \\ &+ F(u)\sum_{n=0}^{\infty}\lambda^n(ku)^{\frac{-\mu(1+2n)}{k}}\left(1-\omega k(ku)^{\frac{-\alpha}{k}}\right)^{\frac{-\gamma-n(\delta+\gamma)}{k}}. \end{aligned}$$

Now using inverse Laplace transform, we get the required result.

Theorem 5.2. *The solution of Cauchy problem*

$$ {}_k^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}u(x,t) = T\frac{\partial^2}{\partial x^2}u(x,t), \qquad t>0, x\in\mathbb{R}, \tag{5.4}$$

$$u(x,0^+) = g(x), \tag{5.5}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0. \tag{5.6}$$

with $\mu\in(0,1)$, $\omega\in\mathbb{R}$, $T,\alpha>0$, $\gamma\geq 0$ *is given by*

$$u(x,t) = \frac{1}{2k^2\pi}\int_{-\infty}^{\infty}dp\, e^{ipx}\widehat{g}(p)\sum_{n=0}^{\infty}(-T)^n p^{2n}(\Psi(t))^{\frac{n\mu}{k}}E^{n\gamma}_{k,\alpha,n\mu+k}(\omega(\Psi(t))^{\frac{\alpha}{k}}), \tag{5.7}$$

if the series on the right hand side of (5.7) *is convergent.*

Proof. Let $\overline{u}(x,q)$ and $\widehat{u}(p,t)$ denote the Ψ-Laplace transform and Fourier transform of $u(x,t)$, respectively. Taking Fourier transform of equation (5.4) and using (5.6) we get

$$ {}_k^C\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+}\widehat{u}(p,t) = -Tp^2\widehat{u}(p,t). \tag{5.8}$$

Now taking Ψ-Laplace transform of (5.8) and using (4.8), (5.5) we get

$$\begin{aligned} &(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}\left(\overline{\widehat{u}}(p,s)-\frac{g(x)}{s}\right) = -Tp^2\overline{\widehat{u}}(p,s), \\ &\left((ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}+K\,p^2\right)s\overline{\widehat{u}}(p,s) = (ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}\widehat{g}(p). \end{aligned}$$

Thus, we have

$$\overline{\widehat{u}}(p,s) = \frac{\widehat{g}(p)}{s}\left(1+\frac{T\,p^2}{(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right)^{-1}.$$

Hence, for $\left|\frac{K\,p^2}{(ks)^{\frac{\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right| < 1$, we get

$$\overline{\widehat{u}}(p,s) = \frac{\widehat{g}(p)}{s}\sum_{n=0}^{\infty}(-T)^n p^{2n}(kq)^{\frac{-n\mu}{k}}\left(1-\omega k(ks)^{\frac{-\alpha}{k}}\right)^{\frac{-n\gamma}{k}}$$

$$= \frac{\widehat{g}(p)}{k} \sum_{n=0}^{\infty} (-T)^n p^{2n} (ks)^{\frac{-n\mu-k}{k}} \left(1 - \omega k (ks)^{\frac{-\alpha}{k}}\right)^{\frac{-n\gamma}{k}}. \tag{5.9}$$

Taking inverse Laplace transform of (5.9) we get

$$\widehat{u}(x,s) = \frac{\widehat{g}(p)}{k^2} \sum_{n=0}^{\infty} (-T)^n p^{2n} (\Psi(t))^{\frac{n\mu}{k}} E^{n\gamma}_{k,\alpha,n\mu+k}(\omega(\Psi(t))^{\frac{\alpha}{k}}).$$

Using inverse Fourier transform, we get required result. □

The two above results can also be obtained using the Sumudu transform instead of Laplace transform and these are the generalizations of results discussed in [12].

Theorem 5.3 ([20]). *The solution of the differential equation*

$$-hM\Theta(x) = \rho V c_p \, {}_k\mathcal{D}^{\gamma,\mu,\nu,\Psi}_{\alpha,\omega,0^+} \Theta(x)\,, \tag{5.10}$$

$$\Theta(0) = \beta, \quad \text{for } \beta \geq 0, \tag{5.11}$$

where ρ-density, V-volume, c_p-specific heat of material, h-convection heat transfer coefficient, M-surface area of the body and $\Theta \in L^1[0,\infty)$, $0 < x < \infty$, $k, \alpha > 0$, $\gamma, \omega \in \mathbb{R}$, $\mu \in (0,1)$, $\nu \in [0,1]$ is given by

$$\Theta(x) = \beta \sum_{n=0}^{\infty} \left(\frac{-hM}{\rho V c_p}\right)^n (\Psi(x))^{\frac{\nu(k-\mu)+\mu(n+1)}{k}-1} E^{-\gamma(\nu-n-1)}_{k,\alpha,\nu(k-\mu)+\mu(n+1)}(\omega(\Psi(x))^{\frac{\alpha}{k}}), \tag{5.12}$$

if the series on the right hand side of (5.12) *is convergent.*

Proof. Let $\widehat{\Theta}(u)$ denote the Ψ-Sumudu transform of $\Theta(x)$. Now taking Ψ-Sumudu transform of (5.10) and using (4.9), (5.11) we have

$$\begin{aligned}
-hM\widehat{\Theta}(u) =& \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}} \widehat{\Theta}(u) \\
& - \rho V c_p \beta \left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \\
& \times \left[hM + \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right] \widehat{\Theta}(u), \\
=& \rho V c_p \beta \left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}}, \\
\widehat{\Theta}(u) =& \frac{\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}} \rho V c_p \beta}{\left[hM + \rho V c_p \left(\frac{u}{k}\right)^{\frac{-\mu}{k}} \left(1 - \omega k \left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right]}
\end{aligned}$$

$$=\frac{\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)}{k}-1}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma\nu}{k}}\rho V c_p\beta}{\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}}$$

$$\times\left[1+\frac{hM}{\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}}\right]^{-1}$$

$$=\beta\Big(\frac{u}{k}\Big)^{\frac{\nu(k-\mu)+\mu}{k}-1}\left(1-\omega k\Big(\frac{u}{k}\Big)^{\frac{\alpha}{k}}\right)^{\frac{\gamma(\nu-1)}{k}}$$

$$\times\sum_{n=0}^{\infty}\left(\frac{-hM}{\rho V c_p}\right)^n\left(\frac{u}{k}\right)^{\frac{n\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{-n\gamma}{k}},$$

for

$$\left|\rho V c_p\left(\frac{u}{k}\right)^{\frac{-\mu}{k}}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}\right|<1,$$

$$\widehat{\Theta}(u)=\beta\sum_{n=0}^{\infty}\left(\frac{-hM}{\rho V c_p}\right)^n\left(\frac{u}{k}\right)^{\frac{\nu(k-\mu)+\mu(n+1)}{k}-1}\left(1-\omega k\left(\frac{u}{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{\gamma(\nu-n-1)}{k}}. \tag{5.13}$$

Taking inverse Sumudu transform of (5.13), we get required solution of (5.12). □

6. Conclusion.

In the present paper, we investigate new fractional derivatives in the sense of Ψ-fractional calculus to find their generalized transforms called Ψ-Laplace and Ψ-Sumudu transforms. These derivatives are more generalization of fractional derivatives and effectively applicable for various applications like cauchy problems, heat transfer problem. In order to explain the obtained results, some examples were illustrated. It is noted that since generalized derivatives are global and contain a wide class of fractional derivatives.

References

[1] R. A. Almeida, *Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci Numer. Simul. **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006

[2] M. Caputo, *Linear model of model of dissipation whose q is almost frequency independent-II*, Geophysical Journal International **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x

[3] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, 2nd Ed., Chapman and Hall/CRC, 2007. https://doi.org/10.1201/9781420010916

[4] R. Diaz and E. Pariguan, *On hypergeometric functions and k-Pochammer symbol*, Divulgaciones Matematica **15**(2) (2007), 179–192.
[5] G. A. Dorrego and R. A. Cerutti, *The k-Mittag-Leffler Function*, International Journal of Contemporary Mathematical Sciences **7**(15) (2012), 705–716.
[6] G. A. Dorrego and R. A. Cerutti, *The k-Fractional Hilfer Derivative*, IInternational Journal of Mathematical Analysis **7**(11) (2013), 543–550. http://dx.doi.org/10.12988/ijma.2013.13051
[7] G. A. Dorrego, *Generalized Riemann-Liouville fractional operators associated with a generalization of the Prabhakar integral operator*, Progress in Fractional Differentiation and Applications **2**(2) (2016), 131–140. http://dx.doi.org/10.18576/pfda/020206
[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier (North-Holland) Science Publishers, Amsterdam, London, New York. https://doi.org/10.1016/S0304-0208(06)80003-4
[9] H. M. Fahad, M. Rehaman and A. Fernandez, *On Laplace transforms with respect to functions and their applications to fractional differential equations*, arXiv:1907.04541v2 [math.CA] 10 Aug 2020.
[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applicatios*, Gordon and Breach, New York, 1993.
[11] J. D. V. Sousa and E. C. de Oliveira, *On the Ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005
[12] R. Garra, R. Gorenflo, F. Polito and Z. Tomovski, *Hilfer-Prabhakar derivative and some applications*, Appl. Math. Comput. **242** (2014), 576–589. http://dx.doi.org/10.1016/j.amc.2014.05.129
[13] R. Hilfer, *Applications of Fractional Calculus in Physics*, Word Scientific, Singapore, 2000. https://doi.org/10.1142/3779
[14] A. A. Kilbas, M. Saigo and R. K. Saxena, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transforms Spec. Funct. **15**(1) (2004), 31–49. https://doi.org/10.1080/10652460310001600717
[15] S. K. Panchal, A. D. Khandagale and P. V. Dole, *Sumudu transform of Hilfer-Prabhakar fractional derivatives with applications*, Proceeding of National Conference on Recent Trends in Mathematics **1** (2016), 60–66.
[16] S. K. Panchal, P. V. Dole and A. D. Khandagale, *k-Hilfer-Prabhakar fractional derivatives and its applications*, Indian J. Math. **59**(3) (2017), 367–383.
[17] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J. **19** (1971), 7–15.
[18] I. Podlubny, *Fractional Differential Equations*, Academy Press, San-Diego, 1999. https://doi.org/10.1016/S0076-5392(99)80021-6
[19] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill International Editions, New York, 1972.
[20] X. J. Yang, *A new integral transform with an application in heat-transfer*, Thermal Science **20**(3) (2016), 677–681. https://doi.org/10.2298/TSCI16S3677Y

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