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HARMONIC BLOCH FUNCTION SPACES AND THEIR COMPOSITION OPERATORS

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Abstract. In this paper we characterize some basic properties of composition operators on the spaces of harmonic Bloch functions. First we provide some equivalent conditions for boundedness and compactness of composition operators. In the sequel we investigate closed range composition operators. These results extends the similar results that were proven for composition operators on the Bloch spaces.

1. Introduction and Preliminaries

Let D be the open unit disk in the complex plane. For a continuously differentiable complex-valued function $f(z) = u(z) + iv(z)$, $z = x + iy$, we use the common notation for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y),$$
$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

A twice continuously differentiable complex-valued function $f = u + iv$ on D is called a harmonic function if and only if the real-valued function u and v satisfy Laplace's equations $\Delta u = \Delta v = 0$.

A direct calculation shows that the Laplacian of f is

$$\Delta f = 4f_{z\bar{z}}.$$

Thus for functions f with continuous second partial derivatives, it is clear that f is harmonic if and only if $\Delta f = 0$. We consider complex-valued harmonic function f defined in a simply connected domain $D \subset C$. The function f has a canonical

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decomposition $f = h+\bar{g}$, where h and g are analytic in D [6]. A planar complex-valued harmonic function f in D is called a harmonic Bloch function if and only if

$$\beta_f = \sup_{z,w\in D, z\neq w} \frac{|f(z)-f(w)|}{\varrho(z,w)} < \infty.$$

Here β_f is the Lipschitz number of f and

$$\varrho(z,w) = \arctan h \left| \frac{z-w}{1-\bar{z}w} \right|,$$

denotes the hyperbolic distance between z and w in D and also $\rho(z,w)$ is the pseudo-hyperbolic distance on D. In [3] Colonna proved that

$$\beta_f = \sup_{z\in D}(1-|z|^2)[|f_z(z)| + |f_{\bar{z}}(z)|].$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $HB(1)$ or HB, forms a complex Banach space with the norm $\|\cdot\|$ given by

$$\|f\|_{HB(1)} = |f(0)| + \sup_{z\in D}(1-|z|^2)[|f_z(z)| + |f_{\bar{z}}(z)|].$$

Definition 1.1. For $\alpha \in (0,\infty)$, the harmonic α-Bloch space $HB(\alpha)$ consists of complex-valued harmonic function f defined on D such that

$$|||f|||_{HB(\alpha)} = \sup_{z\in D}(1-|z|^2)^\alpha[|f_z(z)| + |f_{\bar{z}}(z)|] < \infty,$$

and the harmonic little α-Bloch space $HB_0(\alpha)$ consists of all function in $HB(\alpha)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha[|f_z(z)| + |f_{\bar{z}}(z)|] = 0.$$

Obviously, when $\alpha = 1$, we have $|||f|||_{HB(\alpha)} = \beta_f$. Each $HB(\alpha)$ is a Banach space with the norm given by

$$\|f\|_{HB(\alpha)} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha[|f_z(z)| + |f_{\bar{z}}(z)|]$$

and $HB_0(\alpha)$ is a closed subspace of $HB(\alpha)$. Now we define composition operators.

Definition 1.2. Let D be the open unit disk in the complex plane. Let φ be an analytic self-map of D, i. e., an analytic function φ in D such that $\varphi(D) \subset D$. The composition operator C_φ induced by such φ is the linear map on the spaces of all harmonic functions on the unit disk defined by

$$C_\varphi f = f \circ \varphi.$$

The composition operators on function spaces were studied by many authors. Some known results about composition operators can be found in [5] and [10]. In this paper we study composition operators on harmonic Bloch-type spaces $HB(\alpha)$. In section 2, by using of Theorem 2.1 in [8], we give a necessary and sufficient condition for boundedness of C_φ on $HB(\alpha)$ for $\alpha \in (0,\infty)$, which extends Theorem 3.1 in [8], by Lou. The compactness of C_φ on analytic Bloch-type spaces were characterized in [8,9].

In this paper, we deal the compactness of composition operators between the Banach spaces of harmonic functions $HB(\alpha)$ and $HB_0(\alpha)$.

Moreover, we investigate closed range composition operators. Closed range composition operators on the Bloch-type spaces have been studied in [2,4,7,11]). For $\alpha > 0$, and φ being an analytic self-map of D, let

$$\tau_{\varphi,\alpha}(z) = \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

We write τ_φ if $\alpha = 1$. We say that a subset $G \subset D$ is called sampling set for $HB(\alpha)$ if exists $S > 0$ such that for all $f \in HB(\alpha)$

$$\sup_{z\in G}(1-|z|^2)^\alpha[|f_z(z)|] + [|f_{\bar{z}}(z)|] \geq S|||f|||_{HB(\alpha)}.$$

To state the results obtained, we need the following definition. Let $\rho(z,w) = |\varphi_z(w)|$ denote the pseudohyperbolic distance (between z and w) on D, where φ_z is a disk automorphism of D that is

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

We say that subset $G \subset D$ is an r-net for D for some $r \in (0,1)$ if for each $z \in D$ exists $w \in G$ such that $\rho(z,w) < r$. For $c > 0$, let

$$\Omega_{c,\alpha} = \{z \in D : \tau_{\varphi,\alpha}(z) \geq c\}$$

and let $G_{c,\alpha} = \varphi(\Omega_{c,\alpha})$. If $\alpha = 1$, then we write Ω_c and G_c. Now we recall Montel's theorem for harmonic functions.

Theorem 1.1 ([1]). *If $\{u_n\}_{n=1}^\infty$ is a sequence of harmonic functions in the region Ω with $\sup_{n,x\in K}|u_n(x)| < \infty$ for every compact set $K \subset \Omega$, then there exists a subsequence, $\{u_{n_j}\}_{j=1}^\infty$ converging uniformly on every compact set $K \subset \Omega$.*

Also we recall a very useful theorem that we will use it a lot in this paper.

Theorem 1.2 ([8]). *Let $0 < \alpha < \infty$. Then there exist $f, g \in HB(\alpha)$ such that*

$$|f'(z)| + |g'(z)| \geq \frac{1}{(1-|z|)^\alpha},$$

for all $z \in D$.

2. Main Results

In this section we study bounded and compact composition operators on $HB(\alpha)$. And then we investigate closed range composition operators on $HB(\alpha)$. First we provide some equivalent conditions for boundedness of composition operator C_φ on $HB(\alpha)$.

Theorem 2.1. *If $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$, then the following statements are equivalent.*

a) *$C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded.*

b)

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. For the implication $a\to b$, by Theorem 2.1 of [8] we have that for $0<\alpha<\infty$ there exist $h,g\in B(\alpha)$ satisfying the inequality

$$|h'(z)|+|g'(z)|\geq\frac{1}{(1-|z|)^\alpha}.$$

If we set $f=h+\bar{g}\in HB(\alpha)$, then $f\circ\varphi(z)=h\circ\varphi(z)+\overline{g\circ\varphi(z)}$ and so by the same method of Theorem 3.1 of [8] we get the proof.

For the implication $b\to a$ we can do the same as Theorem 3.1 of [8]. □

In the next theorem we consider the composition operator from $HB_0(\alpha)$ into $HB(\alpha)$ and we find some conditions under which C_φ is bounded.

Theorem 2.2. *Let* $0<\alpha<\infty$, $\varphi\in H(D)$ *and* $\varphi(D)\subseteq D$. *Then the followings are equivalent.*

a) $C_\varphi:HB_0(\alpha)\to HB(\alpha)$ *is bounded.*

b)

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. The proof is similar to the proof of Theorem 3.3 of [8]. Hence we omit the proof. □

Now we consider the composition operator $C_\varphi:HB(\alpha)\to HB_0(\alpha)$ and we give an equivalent condition to boundedness of C_φ.

Theorem 2.3. *If* $0<\alpha<\infty$, $\varphi\in H(D)$ *and* $\varphi(D)\subseteq D$, *then the following are equivalent.*

a) $C_\varphi:HB(\alpha)\to HB_0(\alpha)$ *is bounded.*

b)

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|=0.$$

Proof. By a similar method of the proof of Theorem 3.4 of [8] we get the proof. □

Finally we provide some conditions for boundedness of the composition operator C_φ as an operator on $HB_0(\alpha)$.

Theorem 2.4. *If* $0<\alpha<\infty$, $\varphi\in H(D)$ *and* $\varphi(D)\subseteq D$, *then the followings are equivalent.*

a) $C_\varphi:HB_0(\alpha)\to HB_0(\alpha)$ *is bounded.*

b) $\varphi\in B_0(\alpha)$ *and*

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|<\infty.$$

Proof. By some simple calculations one can get the proof. □

A sequence $\{z_n\}$ in D is said to be R-separated if $\rho(z_n, z_m) = |\frac{z_m - z_n}{1 - \bar{z_m} z_n}| > R$ whenever $m \neq n$. Thus an R-separated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on D, or equivalently, the hyperbolic balls $D(z_n, r) = \{w : \rho(w, z_n) < r\}$ are pairwise disjoint for some $r > 0$. Evidently, any sequence $\{z_n\}$ in D which satisfies $|z_n| \to 1$ possesses an R-separated subsequence for any $R > 0$.

Another property of separated sequence is contained in the next proposition.

Proposition 2.1 ([9]). *There is an absolute constant $R > 0$ such that if $\{z_n\}$ is R-separated, then for every bounded sequence $\{\lambda_n\}$ there is an $f \in B$ such that $(1 - |z_n|^2) f'(z_n) = \lambda_n$ for all n.*

Since every sequence $\{z_n\}$ with $|z_n| \to 1$ contains an R-separated subsequence $\{z_{n_k}\}$, it follows that there is an $f \in B$ such that $(1 - |z_{n_k}|^2) f'(z_{n_k}) = 1$ for all k.

Now we begin investigating compactness of the composition operator C_φ in different cases. First we provide some equivalent conditions for compactness of C_φ as an operator on $HB(\alpha)$.

Theorem 2.5. *Let $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$. Then we have the followings conditions are equivalent.*

a) *$C_\varphi : HB(\alpha) \to HB(\alpha)$ is compact.*

b)

$$\lim_{|\varphi(z)| \to 1} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\alpha |\varphi'(z)| = 0$$

and

$$\sup_{z \in D} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\alpha |\varphi'(z)| < \infty.$$

Proof. By making use of the proof of Theorem 4.2 of [8] and the Proposition 1 of [9] we get the proof. □

Here we prove that the compactness of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ and $C_\varphi : HB(\alpha) \to HB_0(\alpha)$ are equivalent and we find an equivalent condition for compacness of C_φ in these cases.

Theorem 2.6. *Let $0 < \alpha < \infty$, $\varphi \in H(D)$ and $\varphi(D) \subseteq D$. Then the following statements are equivalent.*

a) *The operator $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is compact.*

b) *The operator $C_\varphi : HB(\alpha) \to HB_0(\alpha)$ is compact.*

c)

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$

Proof. First we prove the implication $a \to c$. If $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is compact, then the set $K = \overline{C_\varphi(S_{HB_0(\alpha)})} \subset HB_0(\alpha)$ is compact, in which $S_{HB_0(\alpha)} = \{f \in HB_0(\alpha) : \|f\|_{HB_0(\alpha)} \leq 1\}$. By the Theorem 2.5, we get that

$$\sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|z|^2)^\alpha [|f_z(z)| + |f_{\bar{z}}(z)|] = 1,$$

for all $z \in D$. Moreover we have

$$\begin{aligned} 0 &= \lim_{|z|\to 1} \sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|z|^2)^\alpha [|(f\circ\varphi)_z(z)| + |(f\circ\varphi)_{\bar{z}}(z)|] \\ &= \lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| \sup_{\|f\|_{HB(\alpha)} \leq 1} (1-|\varphi(z)|^2)^\alpha [|h'(\varphi(z)|) + |g'(\varphi(z))|]. \end{aligned}$$

So, we get the desired result.

Now we prove the implication $c \to b$. Let $\{f_n\}_{n\in\mathbb{N}} \subset HB(\alpha)$ and $\|f_n\|_{HB(\alpha)} \leq 1$, for all n. First we obtain that $\{C_\varphi f_n\}$ has a subsequence that converges in $HB_0(\alpha)$. By Montel's Theorem we have a subsequence $\{f_{n_k}\} \subset \{f_n\}$, that converges uniformly on subsets of D to a harmonic function f. Hence we have

$$\begin{aligned} (1-|z|^2)^\alpha [|f_z(z)| + |f_{\bar{z}}(z)|] &= \lim_{k\to\infty} (1-|z|^2)^\alpha [|(f_{n_k})_z(z)| + |(f_{n_k})_{\bar{z}}(z)|] \\ &\leq \lim_{k\to\infty} \|f_{n_k}\|_{HB(\alpha)} \\ &\leq 1. \end{aligned}$$

This means that $f \in HB(\alpha)$ with $\|f\|_{HB(\alpha)} \leq 1$. Also we have

$$\begin{aligned} (1-|z|^2)^\alpha [|(f\circ\varphi)_z(z)| + |(f\circ\varphi)_{\bar{z}}(z)|] &= \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|)^\alpha} |\varphi'(z)| \\ &\leq \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| \|f\|_{HB(\alpha)}. \end{aligned}$$

By these observations we conclude that $C_\varphi f \in HB_0(\alpha)$. Also we need to show that

$$\lim_{k\to\infty} \|C_\varphi f_{n_k} - C_\varphi f\|_{HB(\alpha)} = 0.$$

Since $\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0$, then for any $\varepsilon > 0$, there exists $r \in (0,1)$ such that for z with $r < |z| < 1$ we have

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \frac{\varepsilon}{4}.$$

And so for all z with $r < |z| < 1$ we have

$$\begin{aligned} (1-|z|^2)^\alpha |((f_{n_k} - f)\circ\varphi)'(z)| =& (1-|z|^2)^\alpha [|(f_{n_k})_z \varphi(z)| + |(f_{n_k})_{\bar{z}} \varphi(z)|] \\ &- (1-|z|^2)^\alpha [|f_z \varphi(z)| + |f_{\bar{z}} \varphi(z)|] \\ \leq& \frac{\varepsilon}{4} (\|f_{n_k}\|_{HB(\alpha)} + \|f\|_{HB(\alpha)}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

For z with $|z| \le r$, the set $\{\varphi(z) : |z| \le r\}$ is a compact subset of D. Since

$$(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar z}(z)|] = \lim_{k\to\infty}(1-|z|^2)^\alpha[|(f_{n_k})_z(z)|+|(f_{n_k})_{\bar z}(z)|]$$

and

$$\begin{aligned}(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| \le &(1-|z|^2)^\alpha\{[|(f_{n_k})_z\varphi(z)|+|(f_{n_k})_{\bar z}\varphi(z)|] \\ &-[|f_z\varphi(z)|+|f_{\bar z}\varphi(z)|]\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)|.\end{aligned}$$

Hence, we have $(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| \to 0$ uniformly on $\{z : |z| \le r\}$. Therefore, $(1-|z|^2)^\alpha|((f_{n_k}-f)\circ\varphi)'(z)| < \frac{\varepsilon}{2}$ for k sufficiently large and $\{z : |z| \le r\}$. This completes the proof.

The implication b) $\to$ a) is clear. □

Let (X,d) be a metric space and let $\varepsilon > 0$. We say that $A \subset X$ is an ε-net for (X,d), if for all $x \in X$ there exists an a in A such that $d(a,x) < \varepsilon$. We characterize the compact subsets of $HB_0(\alpha)$ in the next lemma.

Lemma 2.1. *A closed subset of $HB_0(\alpha)$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar z}(z)|] = 0.$$

Proof. Suppose that $K \subset HB_0(\alpha)$ is compact and $\varepsilon > 0$. Then we can choose an $\frac{\varepsilon}{2}$-net $f_1, f_2, \ldots, f_n \in K$. Hence, there exists δ, $0 < \delta < 1$, such that for all z with $|z| > \delta$ we have $(1-|z|^2)^\alpha[|(f_i)_z(z)|+|(f_i)_{\bar z}(z)|] < \frac{\varepsilon}{2}$ for all $1 \le i \le n$. If $f \in K$, then there exists some f_i such that $\|f-f_i\|_{HB(\alpha)} < \frac{\varepsilon}{2}$ and so for all z with $|z| > \delta$ we have

$$(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar z}(z)|] \le \|f-f_i\|_{HB(\alpha)} + (1-|z|^2)^\alpha[|(f_i)_z(z)|+|(f_i)_{\bar z}(z)|] < \varepsilon.$$

Therefore, we get that

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar z}(z)|] = 0.$$

Conversely, let K be a closed and bounded subset of $HB_0(\alpha)$ such that

$$\lim_{|z|\to 1}\sup_{f\in k}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar z}(z)|] = 0.$$

Since K is bounded, then it is relatively compact with respect to the topology of the uniform convergence on compact subsets of the unit disk. If (f_n) is a sequence in K, then by Montel's Theorem we have a subsequence $\{f_{n_k}\} \subset \{f_n\}$ which converges uniformly on compact subsets of D to a harmonic function f. Also $\{f'_{n_k}\}$ converges uniformly to f' on compact subsets of D. For every $\varepsilon > 0$ we can find $\delta > 0$ such that for all z with $|z| > \delta$ we have

$$(1-|z|^2)^\alpha[|(f_{n_k})_z(z)|+|(f_{n_k})_{\bar z}(z)|] < \frac{\varepsilon}{2},$$

for any integer $k > 0$. Therefore, $(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] < \frac{\varepsilon}{2}$, for all z with $|z| > \delta$. So,

$$\begin{aligned}\sup_{|z|>\delta}(1-|z|^2)^\alpha[|(f_{n_k}-f)_z(z)|+|(f_{n_k}-f)_{\bar{z}}(z)|] \le & \sup_{|z|>\delta}(1-|z|^2)^\alpha[|(f_{n_k})_z(z)| \\ & + |(f_{n_k})_{\bar{z}}(z)|] \\ & + \sup_{|z|>\delta}(1-|z|^2)^\alpha[|f_z(z)|+|f_{\bar{z}}(z)|] \\ < & \varepsilon.\end{aligned}$$

Moreover, since (f_{n_k}) converges uniformly on compact subsets of D to f and (f'_{n_k}) converges uniformly to f' on $\{z : |z| \le \delta\}$, we get that

$$\sup_{|z|\le\delta}(1-|z|^2)^\alpha[|(f_{n_k}-f)_z(z)|+|f_{n_k}-f)_{\bar{z}}(z)|] \le \varepsilon.$$

Consequently for k large enough, we have $\lim_{k\to\infty}\|f_{n_k}-f\|_{HB(\alpha)} \le \varepsilon$. This completes the proof. □

In the next theorem we prove that the norm convergence in $HB(\alpha)$ implies the uniform convergence.

Theorem 2.7. *The norm convergence in $HB(\alpha)$ implies the uniform convergence, that is if $\{f_n\} \subset HB(\alpha)$ such that $\|f_n - f\|_{HB(\alpha)} \to 0$, then $\{f_n\}$ converges uniformly to f.*

Proof. For $0 \ne z \in D$, we have

$$\begin{aligned}|f_n(z)-f(z)| &= \left|\int_0^1 \frac{d(f_n-f)}{dt}(zt)dt\right| \\ &= \left|z\int_0^1 \frac{d(f_n-f)}{d\varsigma(t)}(zt)dt + \bar{z}\int_0^1 \frac{d(f_n-f)}{d\bar{\varsigma}(t)}(zt)dt\right| \\ &\le |z|\int_0^1 [|(f_n-f)_{\varsigma(t)}(zt)|+|(f_n-f)_{\bar{\varsigma}(t)}(zt)|]dt,\end{aligned}$$

in which $\varsigma(t) = zt$. This gives us

$$\begin{aligned}|f_n(z)-f(z)| &\le \int_0^1 \frac{[|(f_n-f)_{\varsigma(t)}(zt)|+|(f_n-f)_{\bar{\varsigma}(t)}(zt)|]}{(1-|\varsigma(t)|^2)^\alpha}(1-|\varsigma(t)|^2)^\alpha dt \\ &\le (\|f_n-f\|_{HB(\alpha)})\int_0^1 \frac{1}{(1-|z|t)^\alpha}dt \to 0,\end{aligned}$$

when $n \to \infty$. So we get the proof. □

In the next theorem we provide some equivalent conditions for closedness of range of the composition operator on $HB(\alpha)$.

Theorem 2.8. *Let $\varphi : D \to D$, $\alpha > 0$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be a bounded operator. Then the range of $C_\varphi : HB(\alpha) \to HB(\alpha)$ is closed if and only if there exists $c > 0$ such that $G_{c,\alpha}$ is sampling for $HB(\alpha)$.*

Proof. Since $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded, then exists $K > 0$ such that $\sup_{z\in D} \tau_{\varphi,\alpha}(z) \le K$. Since every non-constant φ is an open map, then the composition operator C_φ is always one to one. By a basic operator theory result, a one-to-one operator has closed range if and only if it is bounded below. Hence, if C_φ has closed range, then C_φ is bounded below, that is exists $\varepsilon > 0$ such that for all $f \in HB(\alpha)$

$$\begin{aligned}\|C_\varphi f\|_{HB(\alpha)} &= \sup_{z\in D}(1-|z|^2)^\alpha[|(fo\varphi)_z(z)| + |(fo\varphi)_{\bar z}(z)|]\\ &= \sup_{z\in D}\tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\ge \varepsilon\|f\|_{HB(\alpha)}.\end{aligned}$$

Now we show that the set $G_{c,\alpha}$ is sampling for $HB(\alpha)$ with sampling constant $S = \frac{\varepsilon}{K}$. Since $\Omega_{c,\alpha} = \{z \in D : \tau_{\varphi,\alpha}(z) \ge c\}$, so for any $z \notin \Omega_{c,\alpha}$ and $c = \frac{\varepsilon}{2}$, we have

$$\sup_{z\notin\Omega_{c,\alpha}} \tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|] \le \frac{\varepsilon}{2}\|f\|_{HB(\alpha)}.$$

Therefore, we have

$$\begin{aligned}\varepsilon\|f\|_{HB(\alpha)} &\le \sup_{z\in D}\tau_{\varphi,\alpha}(z)|(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &= \sup_{z\in\Omega_{c,\alpha}}\tau_{\varphi,\alpha}(z)(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\le K\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|h'(w|) + |g'(w)|].\end{aligned}$$

Hence $\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|h'(w|) + |g'(w)|] \ge \frac{\varepsilon}{K}\|f\|_{HB(\alpha)}$. This means that $G_{c,\alpha}$ is a sampling set for $HB(\alpha)$ with sampling constant $S = \frac{\varepsilon}{K}$.

Conversely, suppose that $G_{c,\alpha}$ is a sampling set for $HB(\alpha)$, with sampling constant $S > 0$. So for all $f \in HB(\alpha)$ and $\varepsilon = cS$ we get the followings relations:

$$\begin{aligned}S\|f\|_{HB(\alpha)} &\le \sup_{z\in\Omega_{c,\alpha}}(1-|\varphi(z)|^2)^\alpha[|(f)_z(\varphi(z))| + |(f)_{\bar z}(\varphi(z))|]\\ &= \sup_{z\in\Omega_{c,\alpha}}(1-|\varphi(z)|^2)^\alpha[|h'(\varphi(z)|) + |g'(\varphi(z))|]\\ &\le \frac{1}{c}\sup_{z\in D}(1-|z|^2)^\alpha[|(h\circ\varphi)_z(z)| + |(g\circ\varphi)_{\bar z}(z)|]\\ &\le \frac{1}{c}\|f\circ\varphi\|_{HB(\alpha)}.\end{aligned}$$

Therefore,

$$\varepsilon\|f\|_{HB(\alpha)} \le \|f\circ\varphi\|_{HB(\alpha)} = \|C_\varphi f\|_{HB(\alpha)}.$$

Hence, C_φ is bounded below and so C_φ has closed range. □

Now we give some other necessary and sufficient conditions for closedness of range of $C_\varphi : HB(\alpha) \to HB(\alpha)$.

Theorem 2.9. *Let φ be a self-map of D, $\alpha > 0$, and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be a bounded operator. Then we have the following hold.*

a) *If the operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ has closed range, then there exist $c, r > 0$ with $r < 1$, such that $G_{c,\alpha}$ is an r-net for D.*

b) *If there exist $c, r > 0$ with $r < 1$, such that $G_{c,\alpha}$ contains an open annulus centered at the origin and with outer radius 1, then C_φ has closed range.*

Proof. a) For $a \in D$, let $\varphi_a(z)$ be a function such that $\varphi_a(0) = 0$ and $\varphi'_a(z) = (\psi'_a(z))^\alpha$, where ψ_a is the disc automorphism of D defined by $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$. Using the equalities

$$1 - \rho(z, w)^2 = 1 - |\psi_w(z)|^2 = (1 - |z|^2)|\psi'_w(z)|,$$

we get

$$\|\varphi_a + \bar{\varphi}_a\|_{HB(\alpha)} = \sup_{z\in D}(1 - |z|^2)^\alpha 2|\varphi'_a(z)| = 2\sup_{z\in D}(1 - |\psi_a(z)|^2)^\alpha = 2.$$

If we put $f = \varphi_a + \bar{\varphi}_a$, then we have

$$\begin{aligned}\|C_\varphi f\|_{HB(\alpha)} &= \|f \circ \varphi\|_{HB(\alpha)}\\ &= \sup_{z\in D}(1 - |z|^2)^\alpha[|(f \circ \varphi)_z(z)| + |(f \circ \varphi)_{\bar{z}}(z)|]\\ &= \sup_{z\in D}\tau_{\varphi,\alpha}(z)2(1 - |\psi_a(\varphi(z))|^2)^\alpha.\end{aligned}$$

Moreover, by assuming that C_φ is bounded and has closed range, then there exist K, $\varepsilon > 0$ such that $\sup_{z\in D}\tau_{\varphi,\alpha}(z) = K$ and

$$\|f \circ \varphi\|_{HB(\alpha)} = \sup_{z\in D}\tau_{\varphi,\alpha}(z)2(1 - |\psi_a(\varphi(z))|^2)^\alpha \geq \varepsilon\|\varphi_a + \bar{\varphi}_a\|_{HB(\alpha)}.$$

This implies that

$$\varepsilon \leq \sup_{z\in D}\tau_{\varphi,\alpha}(z)(1 - |\psi_a(\varphi(z))|^2)^\alpha \leq \sup_{z\in D}\tau_{\varphi,\alpha}(z) = K.$$

Since $1 - |\psi_a(\varphi(z))|^2 \leq 1$, then there exists $z_a \in D$ such that

$$\tau_{\varphi,\alpha}(z_a) \geq \frac{\varepsilon}{2}$$

and

$$(1 - |\psi_a(\varphi(z_a))|^2)^\alpha \geq \frac{\varepsilon}{2K}.$$

Thus, for $c = \frac{\varepsilon}{2}$ and $r = \sqrt{1 - (\frac{\varepsilon}{2K})^{\frac{1}{\alpha}}}$, we conclude that for all $a \in D$, there exists $z_a \in \Omega_{c,\alpha}$ such that $\rho(a, \varphi(z_a)) < r$ and so $G_{c,\alpha}$ is an r-net for D.

b) Let $G_{c,\alpha}$ contains the annulus $A = \{z : r_0 < |z| < 1\}$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ be bounded. Suppose that C_φ doesn't have closed range, then there exists a sequence $\{f_n\}$ with $\|f_n\|_{HB(\alpha)} = 1$ and $\|C_\varphi f_n\|_{HB(\alpha)} \to 0$. For each $\varepsilon > 0$, let $N_\varepsilon > 0$ such that for all $n > N_\varepsilon$ we have

$$\|C_\varphi f_n\|_{HB(\alpha)} < \varepsilon < c\varepsilon.$$

Since

$$\sup_{z\in D}(1-|z|^2)^\alpha[|(f_n)_z(z)|+|(f_n)_{\bar z}(z)|]=\sup_{z\in D}(1-|z|^2)^\alpha[|h_n'(z)|+|g_n'(z)|]=1,$$

then there exists a sequence $\{a_n\}$ in D such that for all n

$$(1-|a_n|^2)^\alpha[|h_n'(a_n)|+|g_n'(a_n)|]\geq\frac{1}{2}.$$

Moreover, we have

$$\begin{aligned}&\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|(f_n)_z(w)|+|(f_n)_{\bar z}(w)|]\\&=\sup_{z\in\Omega_{c,\alpha}}\tau_{\varphi,\alpha}^{-1}(z)\tau_{\varphi,\alpha}(z)(1-|\varphi(z)|^2)^\alpha[|(f_n)_z(\varphi(z))|+|(f_n)_{\bar z}(\varphi(z))|]\\&\leq\frac{1}{c}\sup_{z\in D}(1-|z|^2)^\alpha|\varphi'(z)|[|(f_n)_z(\varphi(z))|+|(f_n)_{\bar z}(\varphi(z))|]\\&<\frac{c\varepsilon}{c}=\varepsilon.\end{aligned}$$

If we take $\varepsilon<\frac{1}{2}$, then we get that each a_n with $n>N_\varepsilon$ belongs to $(G_{c,\alpha})^c$. Thus $|a_n|\leq r_0<1$ and $a_n\to a$ with $|a|\leq r_0$. On the other hand, by Montel's Theorem, there exists a subsequence $\{f_{n_k}\}$ such that converges uniformly on compact subsets of D to some function $f\in HB(\alpha)$. Hence $\{f_{n_k}'\}$ converges to f' uniformly on compact subsets of D, and since

$$\sup_{w\in G_{c,\alpha}}(1-|w|^2)^\alpha[|(f_n)_z(w)|+|(f_n)_{\bar z}(w)|]\to 0,$$

when $n\to\infty$ and $G_{c,\alpha}$ contains a compact subset of D, we conclude that $f'=0$. This contradicts the fact that

$$(1-|a|^2)^\alpha[|h'(a)|+|g'(a)|]\geq\frac{1}{2}.$$

Therefore, C_φ must be bounded below and consequently it has closed range. □

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