Kragujevac Journal of Mathematics
Volume 48(4) (2024), Pages 547–554.

# ON THE INITIAL VALUE PROBLEM FOR FUZZY NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

ALI EL MFADEL[1], SAID MELLIANI[1], AND M'HAMED ELOMARI[1]

Abstract. In this paper, we study the existence result of solutions for fuzzy nonlinear fractional differential equations involving Caputo differentiability of an arbitrary order $0 < q < 1$. As application, an example is included to show the applicability of our result.

## 1. Introduction

Fuzzy fractional differential equations were proposed to handle uncertainty due to incomplete information that appears in many mathematical or computer models of some deterministic real-world phenomena. In recent years, fractional differential equations have attracted a considerable interest both in mathematics and in applications as material theory, transport processes, fluid flow phenomena, earthquakes, solute transport, chemistry, wave propagation, signal theory, biology, electromagnetic theory, thermodynamics, mechanics, geology, astrophysics, economics and control theory (see [1–3]). For basic works related to the fuzzy fractional differential equations we refer the reader to [4, 16, 17].

Motivated by the above works, in this paper,we study the existence result of solution for the following fuzzy fractional initial value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in J = [t_0, t_0 + \delta], \\ x(t_0) = x_0. \end{cases} \tag{1.1}$$

Where ${}^cD^q$ is the Caputo derivative of $x(t)$ at order $q \in [0, 1]$ and $\delta > 0$.

*Key words and phrases.* Fuzzy numbers, fuzzy fractional integral, fuzzy fractional caputo derivative.

2010 *Mathematics Subject Classification.* Primary: 03E72. Secondary: 34K36.

DOI 10.46793/KgJMat2404.547M

*Received*: April 24, 2021.

*Accepted*: June 15, 2021.

To be more precise, we will show that problem (1.1) admits a solution on each locally compact subset of the space of a $E^1$ which is the space of all fuzzy numbers.

The paper is organized as follows. In Section 2, we give some basic properties of fuzzy sets, operations of fuzzy numbers and some detailed definitions of fuzzy fractional integral and fuzzy fractional derivative which will be used in the rest of this paper. In Section 3, we introduce the existence result of solution for the fuzzy fractional initial value problem by using Peano theorem. Illustrative example will be discussed in Section 4, followed by conclusion and futur works in Section 5.

## 2. Preliminaries

**Definition 2.1** ([18])**.** A fuzzy number is mapping $u : \mathbb{R}^n \to [0,1]$ such that

(a) $u$ is upper semi-continuous;
(b) $u$ is normal, that is, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;
(c) $u$ is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$;
(d) $\overline{\{x \in \mathbb{R}^n,\ u(x) > 0\}}$ is compact.

The $\alpha - cut$ of a fuzzy number $u$ is defined as follows:

$$[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}.$$

Moreover, we also can present the $\alpha - cut$ of fuzzy number $u$ by $[u]^\alpha = [u_l(\alpha), u_r(\alpha)]$. We denote by $E^n$ the collection of all fuzzy numbers.

*Example* 2.1. Let $u$ be a fuzzy number defined by the following function:

$$\mu_u(x) = \begin{cases} x-1, & x \in [1,2], \\ -x+3, & x \in [2,3], \\ 0, & \text{elswhere.} \end{cases}$$

Then we have $[u]^1 = \{2\}$.

**Definition 2.2** ([9])**.** Let $u \in E^1$ and $\alpha \in [0,1]$ we define the diameter of $\alpha - level$ set of the fuzzy set $u$ as follows

$$d([u]^\alpha) = l_r - l_l.$$

We denote by $\mathcal{C}(J, E^n)$ space of all fuzzy-valued functions which are continuous on $J$,and $\mathcal{P}_c(\mathbb{R}^n)$ the collection of all the compact subset of $\mathbb{R}^n$.

**Definition 2.3** ([9])**.** The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^n$ is defined as follows:

$$u \ominus_{gH} v = w \Leftrightarrow \text{i) } u = v + w \quad \text{or} \quad \text{ii) } v = u + (-1)w.$$

**Proposition 2.1.** *If $u \in E^1$ and $v \in E^1$, then the following properties hold.*

1) *If $u \ominus_{gH} v$ exists then it is unique.*
2) $u \ominus_{gH} u = 0_{E^1}$.
3) $(u+v) \ominus_{gH} v = u$.

4) $u \ominus_{gH} v = 0_{E^1} \Leftrightarrow u = v$.

**Definition 2.4** ([18])**.** According to the Zadeh's extension principle, the addition on $E^1$ is defined by:

$$(u \oplus v)(z) = \sup_{z=x+y} \min\{u(x), v(y)\}.$$

And scalar multiplication of a fuzzy number is given by:

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \widetilde{0}, & k = 0. \end{cases}$$

*Remark* 2.1 ([13]). Let $u, v \in E^1$ and $\alpha \in [0,1]$, then we have

$$\begin{aligned} [u+v]^\alpha =& [u]^\alpha + [v]^\alpha, \\ [u-v]^\alpha =& [u_1^\alpha - v_2^\alpha, u_2^\alpha - v_1^\alpha], \\ [ku]^\alpha =& ^k[u]^\alpha = \begin{cases} [\lambda u_1^\alpha, \lambda u_2^\alpha], & \text{if } \lambda \geq 0, \\ [\lambda u_2^\alpha, \lambda u_1^\alpha], & \text{if } \lambda < 0, \end{cases}, \\ [uv]^\alpha =& [\min u_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha, \max u_1^\alpha v_1^\alpha, u_1^\alpha v_2^\alpha, u_2^\alpha v_1^\alpha, u_2^\alpha v_2^\alpha]. \end{aligned}$$

**Definition 2.5** ([13])**.** Let $u, v \in E^n$ with $\alpha \in [0,1]$, then the Hausdorf distance between $u$ and $v$is given by:

$$D(u,v) = \sup_{\alpha \in [0,1]} d([u]^\alpha, [v]^\alpha),$$

where $d$ is the Hausdorff metric defined in $P_c(\mathbb{R}^n)$.

**Proposition 2.2** ([10])**.** *$D$ is a metric on $E^n$ and has the following properties:*

(a) *$(E^n; D)$ is a complete metric space;*
(b) *$D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$;*
(c) *$D(ku, kv) = |k| D(u,v)$ for all $u, v \in E^n$ and $k \in \mathbb{R}$;*
(d) *$D(u+w, v+z) \leq D(u,v) + D(w,z)$ for all $u, v, w, z \in E^n$.*

**Definition 2.6** ([7])**.** Let $f : [a,b] \to E^n$ and $t_0 \in [a,b]$. We say that $f$ is Hukuhara differentiable at $t_0$ if there exists $f'(t_0) \in E^n$ such that

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h} = \lim_{h \to 0^-} \frac{f(t_0) \ominus_{gH} f(t_0-h)}{h}.$$

*Remark* 2.2. Let $f : [a,b] \to E^n$ be a fuzzy function such that $[f(x)]^\alpha = \left[\underline{f}(x;\alpha), \overline{f}(x;\alpha)\right]$ for each $\alpha \in [0,1]$ then

$$[f'(x)]^\alpha = \left[\underline{f'}(x;\alpha), \overline{f'}(x;\alpha)\right].$$

**Definition 2.7.** $F : J \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$, the set-valued mapping $F_\alpha : J \to \mathcal{P}_c(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

A function $F : J \to E^n$ is called integrably bounded, if there exists an integrable function $h$ such that, $|x| < h(t)$ for all $x \in F_0(t)$.

**Definition 2.8.** Let $F : J \to E^n$. The integral of $F$ on $J$ denoted by $\int_I F(t)dt$, is given by

$$\left[\int_J F(t)dt\right]^\alpha = \int_J F_\alpha(t)dt = \left\{\int_J f(t)dt \mid f : J \to \mathbb{R}^n \text{ is a measurable selection for } F_\alpha\right\},$$

for all $\alpha \in [0,1]$.

**Proposition 2.3.** *If $u \in E^1$, then the following properties hold.*

(a) $[u]^\beta \subset [u]^\alpha$ *if* $0 \leq \alpha \leq \beta$.

(b) *If $\alpha_n \subset [0,1]$ is a nondecreasing sequence which converges to $\alpha$, then*

$$[u]^\alpha = \bigcap_{n\geq 1} [u]^{\alpha_n}.$$

*Conversely, if $A^\alpha = \{[u_1^\alpha, u_2^\alpha]; \alpha \in [0,1]\}$ is a family of closed real intervals verifying* (a) *and* (b), *then $A^\alpha$ defined a fuzzy number $u \in E^1$ such that $[u]^\alpha = A^\alpha$.*

2.1. **Fractional integral and fractional derivative of fuzzy function.** Let $q > 0$, the fractional integral of order $q$ of a real function $g : [t_0, t_0 + \delta] \to \mathbb{R}$ is given by

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} g(s)ds.$$

Let $f(t) \in L(J, E^1)$ such that $f(t) = [f_1^\alpha(t), f_2^\alpha(t)]$. Suppose that $f_1^\alpha, f_2^\alpha \in L(J, \mathbb{R})$ for all $\alpha \in [0,1]$ and let

$$A^\alpha = \left[\frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f_1^\alpha(s)ds, \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f_2^\alpha(s)ds\right], \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

We have the following lemma.

**Lemma 2.1** ([3])**.** *The family $\{A^\alpha \mid \alpha \in [0,1]\}$ given by* (2.1)*, defined a fuzzy number $u \in E^1$ such that $[u]^\alpha = A^\alpha$.*

**Definition 2.9** ([16])**.** Let $f(t) \in L(J, E^1)$. The fuzzy fractional integral of order $q \in [0,1]$ of $f$ denoted by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s)ds$$

is defined by

$$[I^q f(t)]^\alpha = [I^\alpha f_l(t;\alpha), I^q f_r(t;\alpha)].$$

**Proposition 2.4** ([16])**.** *Let $f, g \in L(J, E^1)$ and $b \in E^1$, then we have:*

(a) $I^q(bf)(t) = bI^q f(t)$;

(b) $I^q(f+g)(t) = I^q f(t) + I^q g(t)$;

(c) $I^{q_1} I^{q_2} f(t) = I^{q_1+q_2} f(t)$, *where* $(q_1, q_2) \in [0,1]^2$.

*Example* 2.2. Let $x : J \to E^1$ be a constant fuzzy function such that $x(t) = u \in E^1$. If $[u]^\alpha = [u_\alpha^1, u_\alpha^2]$, then

$$\begin{aligned}
[I^q x(t)]^\alpha &= \left[\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}u_\alpha^1(s)ds, \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}u_\alpha^2(s)ds\right],\\
[I^q x(t)]^\alpha &= \frac{t^q}{\Gamma(\alpha+1)}[u_\alpha^1, u_\alpha^2],\\
[I^q x(t)]^\alpha &= \frac{t^q}{\Gamma(\alpha+1)}[u]^\alpha.
\end{aligned}$$

**Definition 2.10** ([16])**.** Let $f \in C(J, E^1) \cap L(J, E^1)$.

The function $f$ is called fuzzy Caputo fractional differentiable of order $0 < q < 1$ at $t$ if there exists an element ${}^cD^q f(t) \in E^1$ such that

$${}^cD^q f(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f'(s)ds.$$

*Remark* 2.3 ([16]). Since $[f(t)]^\alpha = [f_l(t;\alpha), f_r(t;\alpha)]$ for each $\alpha \in [0,1]$, then

$$[{}^cD^q f(t)]^\alpha = [{}^cD^q f_l(t;\alpha), {}^c D^q f_r(t;\alpha)],$$

where

$$\begin{aligned}
{}^cD^q f_l(t;\alpha) &:= \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f_l'(s,\alpha)ds,\\
{}^cD^q f_r(t;\alpha) &:= \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f_r'(s,\alpha)ds.
\end{aligned}$$

*Example* 2.3. Let $x : [t_0;, t_0 + \delta] \to E^1$ be a constant fuzzy function such that $x(t) = u \in E^1$. If $[u]^\alpha = [u_\alpha^1, u_\alpha^2]$, then

$$\begin{aligned}
[{}^cD^q x(t)]^\alpha &= \left[\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}(u_\alpha^1)'ds, \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}(u_\alpha^2)'ds\right],\\
[{}^cD^q x(t)]^\alpha &= \{0\},\\
{}^cD^q x(t) &= 0_{E^1}.
\end{aligned}$$

**Theorem 2.1** ([5, 15])**.** *There exists a real Banach space $X$ such that $E^n$ can be embedded isometrically into a convex cone $C$ with vertex 0 in $X$. moreover we have:*

(a) *addition in $X$ induces addition in $E^n$;*
(b) *multiplication by real number in $X$ induces the corresponding operation in $E^n$;*
(c) *$C - C$ is dense in $X$;*
(d) *$C$ is closed.*

*Remark* 2.4. The structure of the normed space $X$ can be described as follows.

Define in $E^n \times E^n$ the following equivalence relation:

$$(u, v)R(u', v') \Leftrightarrow u + v' = v + u'.$$

We denote by $\langle u, v\rangle$ the equivalence class of $(u, v)$ and the space $X$ will be the set of equivalence classes. We define a vector space structure in $X$ by:

$$\langle u, v\rangle + \langle u, v\rangle \Leftrightarrow u + v' = v + u',$$
$$\lambda\langle u, v\rangle = \langle \lambda u, \lambda v\rangle, \quad \text{if } \lambda \geq 0,$$
$$\lambda\langle u, v\rangle = \langle (-\lambda)v, (-\lambda)u\rangle, \quad \text{if } \lambda < 0.$$

The isometry $j : E^n \to X$ is defined by

$$j(u) = \langle u, 0\rangle.$$

The norm in $X$ is defined by $\|\langle u, v\rangle\|_X = D(u, v)$.

**Theorem 2.2** ([10]). *Let $X$ be a Banach space and $j$ an embedding as in Theorem 2.1, $G : J \to E^n$ and assume that $j \circ G$ is Bochner integrable over $J$. Then we have*

1) *$I^q G(t) \in E^n$;*
2) *$j\left(I^q G(t)\right) = I^q j(G(t))$.*

## 3. The Fuzzy Fractional Initial Value Problem

Let $\tilde{C}$ be a closed subset of $(E^n, D)$, which is also closed under the addition and multiplication by a nonnegative real number and $f : J \times \tilde{C} \to \tilde{C}$ be a fuzzy continuous function.

In this section we show that the initial value problem (1) has a solution if and only if $\tilde{C}$ is locally compact.

**Definition 3.1** ([9]). A fuzzy function $x : J \to E^n$ is called $d$-increasing ($d$-decreasing) on $J$ if for every $\alpha \in [0, 1]$ the real function $t \mapsto d([x(t)]^\alpha)$ is nondecreasing (nonincreasing), respectively.

*Remark* 3.1. If $x : J \to E^n$ is $d$-increasing or $d$-decreasing on $J$, then we say that $x(t)$ is d-monotone on $J$.

**Lemma 3.1.** *A d-monotone fuzzy function $x(t)$ is a solution of initial value problem* (1.1) *if and only if*

1) *$x$ is continuous;*
2) *$x$ satisfies the integral equation $x(t) \ominus_{gH} x_0 = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(t, x(t)) ds$;*
3) *The function $t \mapsto I^q f(t, x(t))$is d-increasing on $J$.*

*Proof.* See the proof of Theorem 3 in [9]. □

We denote by $C(J, \tilde{C})$ the space of all continuous mappings from $J$ to $\tilde{C}$ and let $j$ be an embedding of $\tilde{C}$ into a Banach space $X$ allowed by Theorem 2.2.

**Theorem 3.1.** *The fuzzy fractional initial value problem* (1.1) *has a solution if and only if $\tilde{C}$ is locally compact.*

*Proof.* By Theorem 2.1 and Theorem 2.2 we can see that $x(t)$ is a solution of the problem (1.1) if and only if $j(x(t))$ is a continuous solution of the embedded equation

$$j(x(t)) \ominus_{gH} j(x_0) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} j(f(s, j^{-1}j(x(s)))). \tag{3.1}$$

Since $x(t) \in C(J, \tilde{C})$ then $j(f(s, x(s))$ is Bochner integrable.

It is known that the (3.1) has a solution if and only if $X$ is a finite dimensional space. Since a normed space is finite dimensional if and only if it is locally compact (see[12]) and we have $X = cl\{j(\tilde{C}) - j(\tilde{C})\}$ then the proof is completed. $\square$

## 4. Illustrative Example

*Example* 4.1. Let $m$ be a positive real number, then the following set,

$$E^1_m = \{u \in E^1 \mid d(\mathrm{supp}(u)) \leq m\}.$$

is a locally compact subset of $E^1$.

Indeed, for each $n = 1, 2, 3, \ldots$ let $\tilde{K}_n = K_n \cap E^1_m$, where $K_n = \{u \in E^1 \mid \mathrm{supp}(u) \subset [-n, n]\}$. Then since $E^1_m$ is closed in $E^1$, $\tilde{K}_n$ is compact for each $n$. Let $u \in E^1_m$, then $u$ belongs to the interior of $\tilde{K}_n$ for some $n$. Therefore, every element in $E^1_m$ has a compact neighborhood, it follows that $E^1_m$ is a locally compact.

## 5. Conclusion and Future Works

In this manuscript we established the existence results for fuzzy fractional differential equations by using Peano theorem. Our future work is to study the stability results for fuzzy fractional differential equations by using Mittag-Leffler stability notion.

**Acknowledgements.** The authors would like to express their sincere appreciation to the referees and editors for their very helpful suggestions and comments.

## References

[1] R. P. Agarwal, M. Belmekki and M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Difference Equ. (2009). https://doi.org/10.1155/2009/981728

[2] R. P. Agarwal, V. Lakshmikantham and J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Analysis: Theory, Methods and Applications **72**(6), (2010), 2859–2862.

[3] R. P. Agarwal, Y. Zhou, J. Wang and X. Luo, *Fractional functional differential equations with causal operators in Banach spaces*, Mathematical and Compututer Modelling **54**(5–6) (2011), 1440–1452.

[4] S. Arshad and V. Lupulescua, *On the fractional differential equations with uncertainty*, Nonlinear Analysis **74**(2) (2011), 3685–3693, https://doi.org/10.1016/j.na.2011.02.048

[5] P. Diamond and P. E. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems **35**(2) (1990), 241–250, https://doi.org/10.1016/0165-0114(90)90197-E

[6] J. Dieudonné, *Deux exemples singuliers d'équations différentielles*, Acta Scientiarum Mathematicarum **12**(6) (1950), 38–40.

[7] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980. https://doi.org/10.1057/jors.1982.38.
[8] M. Friedmana and A. Kandel, *Numerical solutions of fuzzy differential and integral equations*, Fuzzy Sets Systems **106**(1) (1999), 35–48. https://doi.org/10.1016/S0165-0114(98)00355-8
[9] N. V. Hoa, V. Lupulescu and D. O'Regan, *A note on initial value problems for fractional fuzzy differential equations*, Fuzzy Sets and Systems **347**(2) (2018), 54–69. https://doi.org/10.1016/j.fss.2017.10.002
[10] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems **24**(3) (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7
[11] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies **204**, Elsevier Science, Oxford, United Kingdom, 2006.
[12] S. Lang, *Analysis*, Addison-Wesley, Reading, MA, 1969.
[13] H. T. Nguyen, *A note on the extension principle for fuzzy sets*, J. Math. Anal. Appl. **64**(2) (1978), 369–380. https://doi.org/10.1016/0022-247X(78)90045-8
[14] M. L. Puri and D. A. Ralescu, *The concept of normality for fuzzy random variables*, Ann. Probab. **13**(4) (1985), 1373–1379.
[15] M. L. Puri and D. A. Ralescu, *Differential for fuzzy functions*, J. Math. Anal. Appl. **91**(3) (1983), 552–558.
[16] S. Salahshour, T. Allahviranloo, S. Abbasbandy and D. Baleanu, *Existence and uniqueness results for fractional differential equations with uncertainty*, Adv. Diff. Equ. **112**(1) (2012), 1687–1847. https://doi.org/10.1186/1687-1847-2012-112
[17] S. Salahshour, T. Allahviranloo and S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Communications in Nonlinear Science and Numerical Simulation **17**(3) (2012), 1372–1381. https://doi.org/10.1016/j.cnsns.2011.07.005
[18] L. Zadeh, *Fuzzy sets*, Information and Control **3**(1) (1965), 338–356.

[1] Laboratory of Applied Mathematics and Scientific Computing,
SULTAN Moulay Slimane University,
Address Beni Mellal, Morocco
*Email address*: elmfadelali@gmail.com
*Email address*: s.melliani@usms.ma
*Email address*: m.elomari@usms.ma