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A COUPLED SYSTEM OF NONLINEAR LANGEVIN FRACTIONAL q-DIFFERENCE EQUATIONS ASSOCIATED WITH TWO DIFFERENT FRACTIONAL ORDERS IN BANACH SPACE

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Abstract. In this research article, we study the coupled system of nonlinear Langevin fractional q-difference equations associated with two different fractional orders in Banach Space. The existence, uniqueness, and stability in the sense of Ulam are established for the proposed system. Our approach is based on the technique of measure of noncompactness combined with Mönch fixed point theorem, the implementation Banach contraction principle fixed point theorem, and the employment of Urs's stability approach. Two examples illustrating the effectiveness of the theoretical results are presented.

1. Introduction

In understanding and developing a large class of systems, it is apparent that researchers and scientists have resorted to nature. Natural phenomena can be well understood both quantitatively and qualitatively. Mathematics plays a fundamental role in this respect because it is the science of patterns and relationships. Attempting to understand the quantitative and qualitative behavior of nature, mathematicians find out that evolution revolves from integer to fraction. Number theory, starting from integer and reaching to fractional as a result of division operation and eventually converging to real numbers, is well used to account for Quantitative behavior. Calculus which describes how things change offers a background for simulating structures undergoing change, and a means to infer the predictions of such structures. All these indicated that integer order calculus is a subcategory of fractional calculus

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which is defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer. New and many derivatives and fractional integrals theories have arisen since the end of the 17 century to the present day. The theory of derivation and fractional integration has long been regarded as a branch of mathematics without any real or practical explanation; it was considered as an abstract containing only little useful mathematical manipulations. During the past three decades, considerable interest was carried to fractional calculus by the application of these concepts in various fields of physics, engineering, biology, and mechanics, etc. in a much better form as compared to ordinary differential operators, which are local. To get a couple of developments about the theory of fractional differential equations, one can allude to the monographs of Hilfer [33], Kilbas et al. [36], Miller and Ross [39], Oldham [40], Pudlubny [41], Tarasov [45], Abbas et al. [1] and the references therein.

Fractional q-difference equations started toward the start of the nineteenth century [4, 30] and got big interested consideration lately and have attracted a large number of scientists and researchers [6, 14, 31]. Some fascinating insights concerning initial and boundary value problem of q-difference and Fractional q-difference equations can be found in [2, 7–11, 18, 24, 31] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [37]. Although the existing literature on solutions of fractional Langevin equations is quite wide (see, for example, [12, 13, 21, 46]). But, to the best of the author's knowledge, there is no literature to research the existence of weak solutions for fractional Langevin equations involving two fractional orders in Banach Spaces, so the research of this paper is new.

At the present day, there are numerous results on the existence and uniqueness of solutions for fractional differential equations. For greater details, the readers are cited the previous research [22, 23, 29, 36] and the references therein. However, due to the fact that in lots of conditions, which include nonlinear analysis and optimization, locating the exact solution of differential equations is almost tough or impossible, we don't forget approximate solutions. It is essential to observe that only stable approximate solutions are proper. various approaches of stability analysis are adopted for this reason. The HU-type stability concept has been taken into consideration in the severa literature. The said stability analysis is an clean and easy manner on this regard. This type idea of stability become formulated for the primary time by means of Ulam [47], and then the next year it become elaborated with the aid of Hyers [34, 48]. Impressive considerations have been provided to the investigation of the Ulam-Hyers (UH) stability of a wide range of FDEs, see [3, 16, 28, 43].

In this paper deals with the existence, uniqueness and Urs's stability of solutions for the following Langevin fractional q-difference system:

$$
\begin{cases}
\mathcal{D}_q^{\beta_i}\left(\mathcal{D}_q^{\alpha_i}+\lambda_i\right)\varpi(\varsigma)=f_i(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma)), & \varsigma\in J=[0,T],\\
\varpi_i(0)=0,\\
\varpi_i(T)+\lambda_i\mathcal{J}_q^{\alpha_i}\varpi_i(T)=0,\\
\mathcal{D}_q^{\alpha_i}\varpi_i(\xi_i)+\lambda_i\varpi_i(\xi_i)=0, & \xi_i\in\ ]0,T],
\end{cases}
\tag{1.1}
$$

where D_q^{ε} is the fractional q-derivative of the Reimann-Liouville type of order $\varepsilon\in\{\alpha_i,\beta_i\}$ such that $\alpha_i\in(0,1]$, $\beta_i\in(1,2]$ and $\mathcal{J}_q^{\alpha_i}$ is the fractional q-integral of the Reimann-Liouville type, $f_i: J\times\mathbb{E}^2\to\mathbb{E}$ are continuous functions, λ_i are real constants.

In this paper, we present existence results for the problem (1.1) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. That technique turns out to be a very useful tool in existence for several types of integral equations, details are found in Akhmerov et al. [15], Alvàrez [19], Banaś et al. [20], Benchohra et al. [22, 23], Boutiara et al. [25–27], Mönch [38], Szufla [44] and the references therein.

Here is a brief outline of the paper. The Section 2 provides the definitions and preliminary results that we will need to prove our main results and present an auxiliary lemma that provides solution representation for the solutions of system (1.1). In Section 3, we establish existence and uniqueness for stability in the sense of Ulam for system (1.1). In Section 4, we give some examples to illustrate the obtained results.

2. Preliminaries and Lemmas

We start this section by introducing some necessary definitions and basic results required for further developments.

In what follows, we recall some elementary definitions and properties related to fractional q-calculus. For $a\in\mathbb{R}$, we put

$$[a]_q=\frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is expressed by

$$(a-b)^{(0)}=1,\quad (a-b)^{(n)}=\prod_{k=0}^{n-1}\left(a-bq^k\right),\quad a,b\in\mathbb{R}, n\in\mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)}=a^{\alpha}\prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right),\quad a,b,\alpha\in\mathbb{R}.$$

Definition 2.1 ([35])**.** The q-gamma function is given by

$$\Gamma_q(\alpha)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\quad \alpha\in\mathbb{R}\setminus\{0,-1,-2,\dots\}.$$

The q-gamma function satisfies the classical recurrence relationship

$$\Gamma_q(1+\alpha) = [\alpha]_q \Gamma_q(\alpha).$$

Definition 2.2 ([35])**.** For any $\alpha, \beta > 0$, the q-beta function is defined by

$$B_q(\alpha,\beta) = \int_0^1 f^{(\alpha-1)}(1-qf)^{(\beta-1)} d_q f, \quad q \in (0,1),$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Definition 2.3 ([35])**.** Let $f : J \to \mathbb{R}$ be a suitable function. We define the q-derivative of order $n \in \mathbb{N}$ of the function by $\mathcal{D}_q^0 f(\varsigma) = f(\varsigma)$,

$$\mathcal{D}_q f(\varsigma) := \mathcal{D}_q^1 f(\varsigma) = \frac{f(\varsigma) - f(q\varsigma)}{(1-q)\varsigma}, \quad \varsigma \neq 0, \quad \mathcal{D}_q f(0) = \lim_{\varsigma \to 0} \mathcal{D}_q f(\varsigma),$$

and

$$\mathcal{D}_q^n f(\varsigma) = \mathcal{D}_q \mathcal{D}_q^{n-1} f(\varsigma), \quad \varsigma \in \mathcal{J}, n \in \{1,2,\dots\}.$$

Set $\mathcal{J}_\varsigma := \{\varsigma q^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4 ([35])**.** For a given function $f : \mathcal{J}_\varsigma \to \mathbb{R}$, the expression defined by

$$\mathcal{J}_q f(\varsigma) = \int_0^\varsigma f(s)\, d_q s = \sum_{n=0}^{\infty} \varsigma(1-q)q^n f(tq^n),$$

is called q-integral, provided that the series converges.

We note that $\mathcal{D}_q \mathcal{J}_q f(\varsigma) = f(\varsigma)$, while if f is continuous at 0, then

$$\mathcal{J}_q \mathcal{D}_q f(\varsigma) = f(\varsigma) - f(0).$$

Definition 2.5 ([6])**.** The integral of a function $f : J \to \mathbb{R}$ defined by

$$\mathcal{J}_q^0 f(\varsigma) = f(\varsigma),$$

and

$$\mathcal{J}_q^\alpha f(\varsigma) = \int_0^\varsigma \frac{(\varsigma - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_q s, \quad \varsigma \in J,$$

is called Riemann-Liouville-fractional q-integral of order $\alpha \in \mathbb{R}_+$.

Lemma 2.1 ([42])**.** *Let $\alpha \in \mathbb{R}_+$ and $\beta \in (-1,\infty)$. One has*

$$\mathcal{J}_q^\alpha \varsigma^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} \varsigma^{\alpha+\beta}, \quad \beta \in (-1,\infty), \alpha \geq 0, \varsigma > 0.$$

In particular, if $f \equiv 1$, then

$$\mathcal{J}_q^\alpha 1(\varsigma) = \frac{1}{\Gamma_q(1+\alpha)} \varsigma^{(\alpha)}, \quad \text{for all } \varsigma > 0.$$

Definition 2.6 ([14])**.** The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : J \to \mathbb{R}$ is defined by $\mathcal{D}_q^0 f(\varsigma) = f(\varsigma)$ and

$$\mathcal{D}_q^\alpha f(\varsigma) = \mathcal{D}_q^{[\alpha]} \mathcal{I}_q^{[\alpha]-\alpha} f(\varsigma) = \frac{1}{\Gamma_q(n-\alpha)} \int_0^\varsigma \frac{f(s)}{(\varsigma - qs)^{\alpha-n+1}} d_q s,$$

where $[\alpha]$ is the integer part of α.

Lemma 2.2 ([32])**.** *Let $\alpha > 0$ and $n \in \mathbb{N}$ where $[\alpha]$ denotes the integer part of α. Then, the following fundamental identity holds*

$$\mathcal{I}_q^\alpha \mathcal{D}_q^n f(\varsigma) = \mathcal{D}_q^n \mathcal{I}_q^\alpha f(\varsigma) - \sum_{k=0}^{\alpha-1} \frac{\varsigma^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (\mathcal{D}_q^k h)(0).$$

Lemma 2.3 ([17])**.** *Let ϖ be a function defined on J and suppose that α, β are two real nonegative numbers. Then the following hold:*

$$\begin{aligned} \mathcal{I}_q^\alpha \mathcal{I}_q^\beta f(\varsigma) =& \mathcal{I}_q^{\alpha+\beta} f(\varsigma) = \mathcal{I}_q^\beta \mathcal{I}_q^\alpha f(\varsigma), \\ \mathcal{D}_q^\alpha \mathcal{I}_q^\alpha f(\varsigma) =& f(\varsigma). \end{aligned}$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.7 ([15, 20])**.** The mapping $\kappa : \mathfrak{M}_\mathcal{U} \to [0, \infty)$ for Kuratowski measure of non-compactness is defined as:

$$\kappa(B) = \inf \Big\{ \varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \leq \varepsilon \Big\}.$$

Proposition 2.1. *The Kuratowski measure of noncompactness satisfies some properties* [15, 20]*:*

(a) $\mathcal{A} \subset \mathcal{B} \Rightarrow \kappa(\mathcal{A}) \leq \kappa(\mathcal{B})$;
(b) $\kappa(\mathcal{A}) = 0$ *if and only if* $\mathcal{A}$ *is relatively compact;*
(c) $\kappa(\mathcal{A}) = \kappa(\overline{\mathcal{A}}) = \kappa(\operatorname{conv}(\mathcal{A}))$, *where* $\overline{\mathcal{A}}$ *and* $\operatorname{conv}(\mathcal{A})$ *represent the closure and the convex hull of* $\mathcal{A}$*, respectively;*
(d) $\kappa(\mathcal{A} + \mathcal{B}) \leq \kappa(\mathcal{A}) + \kappa(\mathcal{B})$;
(e) $\kappa(\lambda\mathcal{A}) = |\lambda|\kappa(\mathcal{A}), \lambda \in \mathbb{R}$.

Definition 2.8. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $\varsigma \mapsto f(\varsigma, \varpi)$ is measurable for each $\varpi \in E$;
(ii) $\varpi \mapsto F(\varsigma, \varpi)$ is continuous for almost all $\varsigma \in J$.

Proposition 2.2. *For a given set V of functions $\omega : J \to E$, let us denote by*

$$V(\varsigma) = \{\omega(\varsigma) : \omega \in V\}, \quad \varsigma \in J,$$

and

$$V(J) = \{\omega(\varsigma) : \omega \in V, \varsigma \in J\}.$$

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.1 ([5,38,44]). *Let $\mathcal{D}$ be a bounded, closed and convex subset of a Banach space such that $0 \in \mathcal{D}$, and let N be a continuous mapping of $\mathcal{D}$ into itself. If the implication*

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \kappa(V) = 0, \tag{2.1}$$

holds for every subset V of $\mathcal{D}$, then N has a fixed point.

Lemma 2.4 ([44]). *Let $\mathcal{D}$ be a bounded, closed and convex subset of the Banach space $\mathcal{U}$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $\varsigma \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \kappa(f(J_{\varsigma,h} \times B)) \leq p(\varsigma)\kappa(B),$$

where $J_{\varsigma,h} = [\varsigma - h, \varsigma] \cap J$.

If V is an equicontinuous subset of $\mathcal{D}$, then

$$\kappa\left(\left\{\int_J G(s,\varsigma) f(s, \varpi(s))ds : \varpi \in V\right\}\right) \leq \int_J \|G(\varsigma, s)\| p(s) \kappa(V(s))ds.$$

3. Main Results

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 3.1. *Let $\sigma_i \in \mathcal{C}$, $\alpha_i \in (0,1]$, $\beta_i \in (1,2]$, $i = 1,2$. Then the boundary value problem*

$$\begin{cases} \mathcal{D}_q^{\beta_i}\left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right)\varpi_i(\varsigma) = \sigma_i(\varsigma), & \varsigma \in (0,T), \\ \varpi_i(0) = 0, \\ \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i}\varpi_i(T) = 0, \\ \mathcal{D}_q^{\alpha_i}\varpi_i(\xi_i) + \lambda_i \varpi_i(\xi_i) = 0, & \xi_i \in \left]0, T\right], \end{cases} \tag{3.1}$$

has a unique solution defined by

$$\varpi_i(\varsigma) + \lambda_i \mathcal{I}_q^{\alpha_i}\varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(\varsigma) + \mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i) + \nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T), \quad i = 1,2, \tag{3.2}$$

where

$$\mu(\varsigma) = \frac{\Gamma_q(\beta - 1)}{\Gamma_q(\beta + \alpha - 1)}\left[\frac{(\beta - 1)|\omega_4|\varsigma^{\alpha+\beta-1}}{(\beta + \alpha - 1)|\Delta|} - \frac{|\omega_3|\varsigma^{\alpha+\beta-2}}{|\Delta|}\right] \tag{3.3}$$

and

$$\nu(\varsigma) = \frac{\Gamma_q(\beta - 1)}{\Gamma_q(\beta + \alpha - 1)}\left[\frac{|\omega_1|\varsigma^{\alpha+\beta-2}}{|\Delta|} - \frac{(\beta - 1)|\omega_2|\varsigma^{\alpha+\beta-1}}{(\beta + \alpha - 1)|\Delta|}\right], \tag{3.4}$$

with

$$\Delta = \omega_2\omega_3 - \omega_1\omega_4 \neq 0, \tag{3.5}$$

$$\omega_1 = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} T^{\beta+\alpha-1}, \quad \omega_3 = \xi^{\beta-1},$$
$$\omega_2 = \frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)} T^{\beta+\alpha-2}, \quad \omega_4 = \xi^{\beta-2}.$$

Proof. Applying the integrator operator $\mathcal{I}^\beta$ to (3.1) and using the Lemma 2.1 we get

$$(\mathcal{D}^\alpha + \lambda)\,\varpi(\varsigma) = c_1\varsigma^{\beta-1} + c_2\varsigma^{\beta-2} + \mathcal{I}^\beta\sigma(\varsigma), \quad \varsigma \in (0,T]. \tag{3.6}$$

We apply again the operator $\mathcal{I}^\alpha$ and use the results of Lemmas 2.1 to get the general solution representation of problem (3.1)

$$\varpi(\varsigma) = \mathcal{I}^{\alpha+\beta}\sigma(\varsigma) - \lambda\mathcal{I}^\alpha\varpi(\varsigma) + c_0\varsigma^{\alpha-1} + c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\varsigma^{\beta+\alpha-1} + c_2\frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)}\varsigma^{\beta+\alpha-2}, \tag{3.7}$$

where $c_0, c_1, c_2 \in \mathbb{R}$. By using the boundary conditions in problem (3.1) and the above equation, we observe that $c_0 = 0$ and

$$c_1\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}T^{\beta+\alpha-1} + c_2\frac{\Gamma(\beta-1)}{\Gamma(\beta+\alpha-1)}T^{\beta+\alpha-2} + \mathcal{I}^{\alpha+\beta}\sigma(T) = 0. \tag{3.8}$$

Moreover, we obtain

$$c_1\xi^{\beta-1} + c_2\xi^{\beta-2} + \mathcal{I}^\beta\sigma(\xi) = 0. \tag{3.9}$$

Also, by using (3.5), (3.8) and (3.9) can be written as

$$c_1\omega_1 + c_2\omega_2 = 0,$$
$$c_1\omega_3 + c_2\omega_4 = 0.$$

Solving the last two in c_1 and c_2, we end up with

$$c_1 = \frac{\omega_4}{\Delta}\mathcal{I}^{\alpha+\beta}\sigma(T) - \frac{\omega_4}{\Delta}\mathcal{I}^\beta\sigma(\xi),$$
$$c_2 = \frac{\omega_1}{\Delta}\mathcal{I}^\beta\sigma(\xi) - \frac{\omega_3}{\Delta}\mathcal{I}^{\alpha+\beta}\sigma(T).$$

Substituting c_1 and c_2 in (3.7), we get the desired solution representation (3.2). Besides and by the help of the results in Lemmas 2.1 one can easily figure out that (3.2) solves problem (3.1). This finishes the proof. □

We will need the following properties for the functions μ and ν defined in next lemma.

Lemma 3.2. *The functions μ and ν are continuous functions on J and satisfy the following properties:*

(1) $\mu_{\max,i} = \max_{0\le\varsigma\le T}|\mu_i(\varsigma)|$;
(2) $\nu_{\max,i} = \max_{0<\varsigma<T}|\omega(\varsigma)|$;
(3) $\overline{\mu}_{\max,i} = \max_{0\le\varsigma\le T}|\mu_i'(\varsigma)|$;
(4) $\overline{\nu}_{\max,i} = \max_{0<\varsigma<T}|\nu_i'(\varsigma)|$.

3.1. **Existance result.** In the following subsections, we establish the existence of solutions for the (1.1) by applying Mönch fixed point theorems.

Consider the space of real and continuous functions $\mathcal{U} = C(J, \mathbb{E})$ space with the norm

$$\|\varpi\|_\infty = \sup\{\|\varpi(\varsigma)\| : \varsigma \in J\}.$$

Then the product space $\mathcal{C} := \mathcal{U} \times \mathcal{V}$ defined by $\mathcal{C} = \{(\varpi, \omega) : \varpi \in \mathcal{U}, \omega \in \mathcal{V}\}$ is Banach space under the norm

$$\|(\varpi, \omega)\|_{\mathcal{C}} = \|\varpi\|_\infty + \|\omega\|_\infty,$$

and $\mathfrak{M}_{\mathcal{U}}$ represents the class of all bounded mappings in $\mathcal{U}$.

Let $L^1(J, \mathbb{E})$ be the Banach space of measurable functions $\varpi : J \to \mathbb{E}$ which are Bochner integrable, equipped with the norm

$$\|\varpi\|_{L^1} = \int_J |\varpi(\varsigma)|\, d\varsigma.$$

In what follows, we are concerned with the existence of solutions of (1.1).

Definition 3.1. By a solution of the coupled system (1.1) we mean a coupled measurable functions $(\varpi_1, \varpi_2) \in \mathcal{C}$ such that $\varpi_i(0) = 0, \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(T) = 0$ and $\mathcal{D}_q^{\alpha_i} \varpi_i(\xi_i) + \lambda_i \varpi_i(\xi_i) = 0$, $i = 1, 2$, and the equations $\mathcal{D}_q^{\beta_i} \left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right) \varpi_i(\varsigma) = f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma))$ are satisfied on J.

In what follows, we present the solution representation associated with System (1.1).

Lemma 3.3. *Let $\sigma_i \in \mathcal{U}$, $i = 1, 2$, be two given functions. Then, the following system of fractional differential equations*

$$\begin{cases} \mathcal{D}_q^{\beta_i} \left(\mathcal{D}_q^{\alpha_i} + \lambda_i\right) \varpi_i(\varsigma) = \sigma_i(\varsigma), & \varsigma \in (0, T), \\ \varpi_i(1) = 0, \\ \varpi_i(T) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(T) = 0, \\ \mathcal{D}_q^{\alpha_i} \varpi_i(\xi_i) + \lambda_i \varpi_i(\xi_i) = 0, & \xi_i \in \left]0, T\right], \end{cases} \tag{3.10}$$

is equivalent to the integral equation

$$\varpi_i(\varsigma) + \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(\varsigma) + \mu_i(\varsigma) \mathcal{I}_q^{\beta_i} \sigma_i(\xi_i) + \nu_i(\varsigma) \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(T), \quad i = 1, 2. \tag{3.11}$$

Lemma 3.4. *Assume that $f_i : J \times \mathbb{E}^2 \to \mathbb{E}$ is continuous. A function $\varpi(\varsigma)$ solves the system* (1.1) *if and only if it is a fixed-point of the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ defined by*

$$\mathcal{G}_i \varpi_i(\varsigma) = \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(\varsigma) - \lambda_i \mathcal{I}_q^{\alpha_i} \varpi_i(\varsigma) + \mu_i(\varsigma) \mathcal{I}_q^{\beta_i} \sigma_i(\xi_i) + \nu_i(\varsigma) \mathcal{I}_q^{\alpha_i+\beta_i} \sigma_i(T), \quad i = 1, 2. \tag{3.12}$$

3.1.1. *Existance result via Mönch fixed point theorem.* We further will use the following hypotheses.

(A1) For any $i = 1, 2$, $f_i : J \times \mathbb{E}^2 \to \mathbb{E}$ satisfies the Caratheodory conditions.

(A2) There exists $p_i, q_i \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$, such that

$$\|f(\varsigma, \varpi_1, \varpi_2)\| \leq p_i(\varsigma)\|\varpi_1\| + q_i(\varsigma)\|\varpi_2\|, \quad \text{for } \varsigma \in J \text{ and each } \varpi_i \in \mathbb{E}, i = 1, 2.$$

(A3) For any $\varsigma \in J$ and each bounded measurable sets $B_i \subset \mathbb{E}$, i=1,2, we have

$$\lim_{h \to 0^+} \kappa(f(J_{\varsigma,h} \times B_1, B_2), 0) \leq p_1(\varsigma)\kappa(B_1) + q_1(\varsigma)\kappa(B_2)$$

and

$$\lim_{h \to 0^+} \kappa(0, f(J_{\varsigma,h} \times B_1, B_2)) \leq p_2(\varsigma)\kappa(B_1) + q_2(\varsigma)\kappa(B_2),$$

where κ is the Kuratowski measure of compactness and $J_{\varsigma,h} = [\varsigma - h, \varsigma] \cap J$.

Set

$$p_i^* = \sup_{\varsigma \in J} p_i(\varsigma), \quad q_i^* = \sup_{\varsigma \in J} q_i(\varsigma), \quad i = 1, 2.$$

Theorem 3.1. *Assume that conditions* (A1)-(A3) *hold. If*

$$\Lambda < 1, \tag{3.13}$$

with

$$\Lambda := \sum_{i=1}^{2} \left(M_i(p_i^* + q_i^*) + N_i\right),$$

where

$$M_i = \left\{ \frac{(1 + \nu_{\max,i})\, T^{\alpha_i + \beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} + \frac{(\mu_{\max,i})\, \xi_i^{\beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} \right\}, \quad N_i = \frac{|\lambda_i|\, T^{\alpha_i}}{\Gamma_q(\alpha_i + 1)}, \quad i = 1, 2,$$

then (1.1) *has at least one solution on* J.

Proof. We consider the operators $\mathcal{G}_i : \mathcal{C} \to \mathcal{C}$ defined by

$$\mathcal{G}\varpi = \mathcal{G}(\varpi_1, \varpi_2) = (\mathcal{G}_1\varpi_1, \mathcal{G}_2\varpi_2),$$

where the operators $\mathcal{G}_i$, $i = 1, 2$ are given by the formula (3.12). Clearly, the fixed points of the operators $\mathcal{G}_i$ are solutions of the system (1.1). Let we take

$$\mathcal{D}_r = \{\varpi_i \in \mathcal{C}, i = 1, 2 : \|(\varpi_1, \varpi_2)\| \leq r\}.$$

Clearly, the subset $\mathcal{D}_r$ is closed, bounded and convex. We shall show that $\mathcal{G}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step 1. First we show that $\mathcal{G}$ is sequentially continuous.

Let $\{\varpi_{1,n}, \varpi_{2,n}\}_n$ be a sequence such that $(\varpi_{1,n}, \varpi_{2,n}) \to (\varpi_1, \varpi_2)$ in $\mathcal{C}$. Then for any $\varsigma \in J$

$$\begin{aligned}\|\left(\mathcal{G}\varpi_{i,n} - \mathcal{G}\varpi_i\right)(\varsigma)\| \leq & \mathcal{I}_q^{\alpha_i + \beta_i}\|f_{i,n}(s, \varpi_{1,n}(s), \varpi_{2,n}(s)) - f_i(s, \varpi_1(s), \varpi_2(s))\|(\varsigma) \\ & - \lambda_i \mathcal{I}_q^{\alpha_i}\|\varpi_{i,n} - \varpi_i\|(\varsigma) \\ & + \mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\|f_{i,n}(s, \varpi_{1,n}(s), \varpi_{2,n}(s)) - f_i(s, \varpi_1(s), \varpi_2(s))\|(\xi_i)\end{aligned}$$

$$
\begin{aligned}
&+ \nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\|f_{i,n}(s,\varpi_{1,n}(s),\varpi_{2,n}(s)) - f_i(s,\varpi_1(s),\varpi_2(s))\|(T)\\
\leq &\left\{\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma) + \mu_i(\varsigma)\mathcal{I}_q^{\beta_i}(1)(\xi_i) + \nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\right\}\\
&\times \|f_{i,n}(s,\varpi_{1,n}(s),\varpi_{2,n}(s)) - f_i(s,\varpi_1(s),\varpi_2(s))\|\\
&+ \lambda_i\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\left\|\varpi_{i,n} - \varpi_i\right\|, \quad i = 1,2.
\end{aligned}
$$

Since, for any $i = 1,2$, the function f_i satisfies assumptions (A1), then we have $f_i(\varsigma,\varpi_{1,n}(\varsigma),\varpi_{2,n}(\varsigma))$ converges uniformly to $f_i(\varsigma,\varpi_1(\varsigma),\varpi_2(\varsigma))$. Hence, the Lebesgue dominated convergence theorem implies that $(\mathcal{G}(\varpi_{1,n},\varpi_{2,n}))(\varsigma)$ converges uniformly to $(\mathcal{G}(\varpi_1,\varpi_{2,}))(\varsigma)$. Thus, $(\mathcal{G}(\varpi_{1,n},\varpi_{2,n})) \to (\mathcal{G}(\varpi_1,\varpi_{2,}))$. Hence, $\mathcal{G} : \mathcal{D}_r \to \mathcal{D}_r$ is sequentially continuous.

Step 2. Second we show that $\mathcal{G}$ maps $\mathcal{D}_r$ into itself.

Take $\varpi_i \in \mathcal{D}_r$, $i = 1,2$, by (A2), we have, for each $\varsigma \in J$ and assume that $(\mathcal{G}(\varpi_i))(\varsigma) \neq 0$, $i = 1,2$,

$$
\begin{aligned}
|\mathcal{G}_i u_i(\varsigma)| \leq &\left|\mathcal{I}_q^{\alpha_i+\beta_i} f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\varsigma)\right| + \left|\lambda_i\mathcal{I}_q^{\alpha_i}\varpi_i(s)(\varsigma)\right|\\
&+ \left|\mu_i(\varsigma)\mathcal{I}_q^{\beta_i} f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\xi_i)\right| + \left|\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i} f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(T)\right|\\
\leq &(p_i^* + q_i^*)r\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma) + r\left|\lambda_i\right|\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\\
&+ (p_i^* + q_i^*)r\mu_{\max,i}\mathcal{I}_q^{\beta_i}(1)(\xi_i) + (p_i^* + q_i^*)r\nu_{\max,i}\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\\
\leq &(p_i^* + q_i^*)r\left\{\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma) + \mu_{\max,i}\mathcal{I}_q^{\beta_i}(1)(\xi_i) + \nu_{\max,i}\mathcal{I}_q^{\alpha_i+\beta_i}(1)(T)\right\}\\
&+ r\mathcal{I}_q^{\alpha_i}(1)(\varsigma)\left|\lambda_i\right|\\
\leq &(p_i^* + q_i^*)r\left\{\frac{(1+\nu_{\max,i})\,T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{(\mu_{\max,i})\,\xi_i^{\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)}\right\} + \frac{r\left|\lambda_i\right|T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\\
&= r(M_i(p_i^* + q_i^*) + N_i), \quad i = 1,2.
\end{aligned}
$$

Hence we get

$$
\|(\mathcal{G}(\varpi_1,\varpi_2))\|_{\mathcal{C}} \leq \sum_{i=1}^{2} r\left(M_i(p_i^* + q_i^*) + N_i\right) \leq r.
$$

Step 3. We show that $\mathcal{G}(\mathcal{D}_r)$ is equicontinuous.

By Step 2, it is obvious that $\mathcal{G}(\mathcal{D}_r) \subset C(J,\mathbb{E})$ is bounded. For the equicontinuity of $\mathcal{G}(\mathcal{D}_r)$, let $\varsigma_1,\varsigma_2 \in J$, $\varsigma_1 < \varsigma_2$ and $\varpi \in \mathcal{D}_r$, so $\mathcal{G}\varpi(\varsigma_2) - \mathcal{G}\varpi(\varsigma_1) \neq 0$. Then

$$
\begin{aligned}
\|\mathcal{G}\varpi_i(\varsigma_2) - \mathcal{G}\varpi_i(\varsigma_1)\| \leq &\mathcal{I}_q^{\alpha_i+\beta_i}\left|f(s,\varpi_1(s),\varpi_2(s))(\varsigma_2) - f(s,\varpi_1(s),\varpi_2(s))(\varsigma_1)\right|\\
&+ \left|\lambda_i\right|\mathcal{I}_q^{\alpha_i}\left|\varpi_i(s)(\varsigma_2) - \varpi_i(s)(\varsigma_1)\right|\\
&+ \left|\mu_i(\varsigma_2) - \mu_i(\varsigma_1)\right|\mathcal{I}_q^{\beta_i} f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(\xi_i)\\
&+ \left|\nu_i(\varsigma_2) - \nu_i(\varsigma_1)\right|\mathcal{I}_q^{\alpha_i+\beta_i} f_i\left(s,\varpi_1(s),\varpi_2(s)\right)(T),\\
\leq &(p_i^* + q_i^*)r\left|\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_2) - \mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_1)\right|\\
&+ r\left|\lambda_i\right|\left|\mathcal{I}_q^{\alpha_i}(1)(\varsigma_2) - \mathcal{I}_q^{\alpha_i}(1)(\varsigma_1)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+(p_i^*+q_i^*)r\left|\mu_i(\varsigma_2)-\mu_i(\varsigma_1)\right|\left|\mathcal{I}_q^{\beta_i}(1)(\varsigma_2)-\mathcal{I}_q^{\beta_i}(1)(\varsigma_1)\right|(\xi_i)\\
&+(p_i^*+q_i^*)r\left|\nu_i(\varsigma_2)-\nu_i(\varsigma_1)\right|\left|\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_2)-\mathcal{I}_q^{\alpha_i+\beta_i}(1)(\varsigma_1)\right|(T)\\
\leq&\frac{(p_i^*+q_i^*)r}{\Gamma_q(\alpha_i+\beta_i+1)}\left\{(\varsigma_2^{\alpha_i+\beta_i}-\varsigma_1^{\alpha_i+\beta_i})+2(\varsigma_2-\varsigma_1)^{\alpha_i+\beta_i}\right\}\\
&+\frac{r\left|\lambda_i\right|}{\Gamma_q(\alpha_i+1)}\left\{(\varsigma_2^{\alpha_i}-\varsigma_1^{\alpha_i})+2(\varsigma_2-\varsigma_1)^{\alpha_i}\right\}+\frac{(p_i^*+q_i^*)r\xi_i^{\beta_i}}{\Gamma_q\left(\beta_i+1\right)}\\
&\times\left|\mu_i(\varsigma_2)-\mu_i(\varsigma_1)\right|+\frac{(p_i^*+q_i^*)RT^{\alpha_i+\beta_i}}{\Gamma\left(\alpha_i+\beta_i+1\right)}\left|\nu_i(\varsigma_2)-\nu_i(\varsigma_1)\right|.
\end{aligned}
$$

As $\varsigma_1\to\varsigma_2$, the right hand side of the above inequality tends to zero. This means that $\mathcal{G}(\mathcal{D}_r)\subset\mathcal{D}_r$.

Finally we show that the implication (2.1) holds. Let $V\subset\mathcal{D}_r$ such that $V=\overline{conv}(\mathcal{G}(V)\cup\{(0,0)\})$. Since V is bounded and equicontinuous, and therefore the function $\omega\mapsto\omega(\varsigma)=\kappa(V(\varsigma))$ is continuous on J. By hypothesis (A2), and the properties of the measure κ, for any $\varsigma\in J$, we get

$$
\begin{aligned}
\omega(\varsigma)\leq&\kappa(\mathcal{G}(V)(\varsigma)\cup\{(0,0)\}))\leq\kappa((\mathcal{G}V)(\varsigma))\\
\leq&\kappa\left(\left\{\left(\left(\mathcal{G}_1\omega_1\right)(\varsigma),\left(\mathcal{G}_2\omega_2\right)(\varsigma):(\omega_1,\omega_2)\in V\right\}\right)\\
\leq&\mathcal{I}_q^{\alpha_1+\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s,\omega_1(s),\omega_2(s)\right)(\varsigma)\right);0\right):(\omega_1,\omega_2)\in V\right\}\right)\\
&+\left|\lambda_1\right|\mathcal{I}_q^{\alpha_1}\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\\
&+\left|\mu_1\right|(\varsigma)\mathcal{I}_q^{\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s,\omega_1(s),\omega_2(s)\right)(\varsigma)\right);0\right):(\omega_1,\omega_2)\in V\right\}\right)\\
&+\left|\nu_1\right|(\varsigma)\mathcal{I}_q^{\alpha_1+\beta_1}\kappa\left(\left\{\left(\left(f_1\left(s,\omega_1(s),\omega_2(s)\right)(\varsigma)\right);0\right):(\omega_1,\omega_2)\in V\right\}\right)\\
&+\mathcal{I}_q^{\alpha_2+\beta_2}\kappa\left(\left\{\left(0,f_2\left(s,\omega_1(s),\omega_2(s)\right)\right):(\omega_1,\omega_2)\in V\right\}\right)\\
&+\left|\lambda_2\right|\mathcal{I}_q^{\alpha_2}\kappa\left(\left\{(0,\omega_2(s)):(0,\omega_2)\in V\right\}\right)\\
&+\left|\mu_2\right|(\varsigma)\mathcal{I}_q^{\beta_2}\kappa\left(\left\{\left(0,f_2\left(s,\omega_1(s),\omega_2(s)\right)\right):(\omega_1,\omega_2)\in V\right\}\right)\\
&+\left|\nu_2\right|(\varsigma)\mathcal{I}_q^{\alpha_2+\beta_2}\kappa\left(\left\{\left(0,f_2\left(s,\omega_1(s),\omega_2(s)\right)\right):(\omega_1,\omega_2)\in V\right\}\right)\\
\leq&\mathcal{I}_q^{\alpha_1+\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\right.\\
&\left.+q_1(s)\kappa\left(\left\{(0,\omega_2(s)):(0,\omega_2)\in V\right\}\right)\right]\\
&+\left|\lambda_1\right|\mathcal{I}_q^{\alpha_1}\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\\
&+\left|\mu_1\right|(\varsigma)\mathcal{I}_q^{\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\right.\\
&\left.+q_1(s)\kappa\left(\left\{(0,\omega_2(s)):(0,\omega_2)\in V\right\}\right)\right]\\
&+\left|\nu_1\right|(\varsigma)\mathcal{I}_q^{\alpha_1+\beta_1}\left[p_1(s)\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\right.\\
&\left.+q_1(s)\kappa\left(\left\{(0,\omega_2(s)):(0,\omega_2)\in V\right\}\right)\right]\\
&+\mathcal{I}_q^{\alpha_2+\beta_2}\left[p_2(s)\kappa\left(\left\{(\omega_1(s),0):(\omega_1,0)\in V\right\}\right)\right.\\
&\left.+q_2(s)\kappa\left(\left\{(0,\omega_2(s)):(0,\omega_2)\in V\right\}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ |\lambda_2| \mathcal{I}_q^{\alpha_2} \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right) \\
&+ |\mu_2| (\varsigma) \mathcal{I}_q^{\beta_2} \left[p_2(s) \kappa \left(\{(\omega_1(s), 0) : (\omega_1, 0) \in V\}\right)\right. \\
&\left.+ q_2(s) \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right)\right] \\
&+ |\nu_2| (\varsigma) \\
&\times \mathcal{I}_q^{\alpha_2+\beta_2} \left[p_2(s) \kappa \left(\{(\omega_1(s), 0) : (\omega_1, 0) \in V\}\right)\right. \\
&\left.+ q_2(s) \kappa \left(\{(0, \omega_2(s)) : (0, \omega_2) \in V\}\right)\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mu \left(V(\varsigma)\right) \leq & \mathcal{I}_q^{\alpha_1+\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + |\lambda_1| \mathcal{I}_q^{\alpha_1} \left((1)(s)\right) \times \kappa \left(V(s)\right) \\
& + |\mu_1| (\varsigma) \mathcal{I}_q^{\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + |\nu_1| (\varsigma) \mathcal{I}_q^{\alpha_1+\beta_1} \left(p_1(s) + q_1(s)\right) \times \kappa \left(V(s)\right) \\
& + \mathcal{I}_q^{\alpha_2+\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right) \\
& + |\lambda_2| \mathcal{I}_q^{\alpha_2} \left((1)(s)\right) \times \kappa \left(V(s)\right) \\
& + |\mu_2| (\varsigma) \mathcal{I}_q^{\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right) \\
& + |\nu_2| (\varsigma) \mathcal{I}_q^{\alpha_2+\beta_2} \left(p_2(s) + q_2(s)\right) \times \kappa \left(V(s)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mu \left(V(\varsigma)\right) \leq \sum_{i=1}^{n=2} & \left(\left\{ \frac{(p_i^* + q_i^*) T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i + \beta_i + 1)} \left(1 + \nu_{\max,i}\right) + \frac{(p_i^* + q_i^*) T^{\alpha_i+1}}{\Gamma_q(\alpha_i + 1)} \left|\mu_{\max,i}\right| \right\} \right. \\
& \left. + \left\{ \frac{|\lambda_i| \xi_i^{\beta_i}}{\Gamma_q(\beta_i + 1)} \right\} \right) \sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right).
\end{aligned}
$$

This means that

$$
\sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right) \leq \Lambda \sup_{\varsigma \in \mathcal{J}} \kappa \left(V(\varsigma)\right).
$$

By (3.13) it follows that $\sup_{\varsigma \in J} \kappa((V(\varsigma)) = 0$, that is $\kappa(V(\varsigma)) = 0$ for each $\varsigma \in J$, and then $V(\varsigma)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in $\mathcal{D}_r$. Applying now Theorem 2.4, we conclude that $\mathcal{G}$ has a fixed point, which is a solution of (1.1). □

3.2. **Uniqueness Result.** Let $X = \{\varpi : \varpi \in C'(J)\}$ be the Banach space of functions whose first derivatives are continuous on J, endowed with the $\|\varpi\|_X = \|\varpi\| + \|\varpi'\| = \max_{\varsigma \in J} |\varpi(\varsigma)| + \max_{\varsigma \in J} |\varpi'(\varsigma)|$. Obviously, the product space $(X \times X, \|\cdot\|_X)$ is also a Banach space with the norm $\|(\varpi_1, \varpi_2)\|_{X \times X} = \|\varpi_1\|_X + \|\varpi_2\|_X$. A closed ball with radius R centered on the zero function in $X \times X$ is defined by $B_R(0,0) = B_R = \left\{(\varpi_1, \varpi_2) \in X \times X : \|(\varpi_1, \varpi_2)\|_{X \times X} \leq R\right\}$ Define the operator $\mathcal{G} : X \times X \to X \times X$

by

$$\mathcal{G}(\varpi_1,\varpi_2)(\varsigma)=\begin{pmatrix}\mathcal{G}_1(\varpi_1,\varpi_2)(\varsigma)\\ \mathcal{G}_2(\varpi_1,\varpi_2)(\varsigma)\end{pmatrix},\quad \varsigma\in J,$$

where

$$\mathcal{G}_i\varpi_i(\varsigma)=\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(\varsigma)-\lambda_i\mathcal{I}_q^{\alpha_i}\varpi_i(\varsigma)+\mu_i(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i)+\nu_i(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T).$$

Clearly, (ϖ_1,ϖ_2) is a fixed point of $\mathcal{G}$ if and only if (ϖ_1,ϖ_2) is a solution of system (1.1). Furthermore, we have

$$\mathcal{G}_i'\varpi_i(\varsigma)=\mathcal{I}_q^{\alpha_i+\beta_i-1}\sigma_i(\varsigma)-\lambda_i\mathcal{I}_q^{\alpha_i-1}\varpi_i(\varsigma)+\mu_i'(\varsigma)\mathcal{I}_q^{\beta_i}\sigma_i(\xi_i)+\nu_i'(\varsigma)\mathcal{I}_q^{\alpha_i+\beta_i}\sigma_i(T).$$

Throughout the remaining part of the paper, we make use of the following assumptions and notations:

(H1) $f_1,f_2:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous;

(H2) there exist constants L_i and K_i such that

$$|f_i(\varsigma,\varpi_1,\varpi_2)-f_i(\varsigma,\omega_1,\omega_2)|\le L_i|\varpi_1-\omega_1|+K_i|\varpi_2-\omega_2|,$$

for all $(\varsigma,\varpi_1,\varpi_2),(\varsigma,\omega_1,\omega_2)\in[0,T]\times\mathbb{R}^2$;

(H3) $A_i=\max_{0\le\varsigma\le T}|f_i(\varsigma,0,0)|$.

Further, we use the following notations:

$$\begin{aligned}
\Theta_i &=\left[\frac{T^{\alpha_i+\beta_i}\left(1+\nu_{\max,i}\right)}{\Gamma_q(\alpha_i+\beta_i+1)}+\frac{\xi_i^{\beta_i}\mu_{\max,i}}{\Gamma_q(\beta_i+1)}\right],\\
\Omega_i &=\frac{|\lambda_i|\,T_i^{\alpha_i}}{\Gamma_q(\alpha_i+1)},\\
\overline{\Theta}_i &=\left[\frac{T^{\alpha_i+\beta_i-1}}{\Gamma_q(\alpha_i+\beta_i)}+\frac{\overline{\nu}_{\max,i}T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)}+\frac{\overline{\mu}_{\max,i}\xi_i^{\beta_i}}{\Gamma_q(\beta_i+1)}\right],\\
\overline{\Omega}_i &=\frac{|\lambda_i|\,T_i^{\alpha_i-1}}{\Gamma_q(\alpha_i)},\\
L &=\sum_{i=1}^{2}\left[(L_i+K_i)\left(\Theta_i+\overline{\Theta}_i\right)+\left(\Omega_i+\overline{\Omega}_i\right)\right],\\
A &=\sum_{i=1}^{2}A_i\left(\Theta_i+\overline{\Theta}_i\right).
\end{aligned}$$

To this end, we also use this assumption:

(H4) $(L+A)\le 1$.

3.2.1. *Uniqueness via Banach fixed point theorem.*

Theorem 3.2. *Assume* (H1)-(H4) *holds. Then,* (1.1) *has a unique solution* $(\varpi_1,\varpi_2)\in B_R$.

Proof. Clearly, $\mathcal{G} : B_R \to X \times X$. First, we show that $\mathcal{G}$ is a contraction mapping. To see this, let $(\varpi_1, \varpi_2), (\omega_1, \omega_2) \in B_R$ $\varsigma \in J$, and consider

$$\begin{aligned}
&|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)| \\
\leq& \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (\varsigma) + |\lambda_i| \mathcal{I}_q^{\alpha_i} |\varpi_i - \omega_i| (\varsigma) \\
&+ \mu_{\max,i} \mathcal{I}_q^{\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (\xi_i) \\
&+ \nu_{\max,i} \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma)) - f_i(\varsigma, \omega_1(\varsigma), \omega_2(\varsigma))| (T) \\
\leq& \left[L_i \mathcal{I}_q^{\alpha_i+\beta_i} \|\varpi_1 - \omega_1\| (\varsigma) + K_i \mathcal{I}_q^{\alpha_i+\beta_i} \|\varpi_2 - \omega_2\| (\varsigma)\right] + |\lambda_i| \mathcal{I}_q^{\alpha_i} |\varpi_i - \omega_i| (\varsigma) \\
&+ \mu_{\max,i} \left[L_i \mathcal{I}_q^{\beta_i} \|\varpi_1 - \omega_1\| (\xi_i) + K_i \mathcal{I}_q^{\beta_i} \|\varpi_2 - \omega_2\| (\xi_i)\right] \\
&+ \nu_{\max,i} \left[L_i \mathcal{I}_q^{\beta_i} \|\varpi_1 - \omega_1\| (T) + K_i \mathcal{I}_q^{\beta_i} \|\varpi_2 - \omega_2\| (T)\right] \\
\leq& \left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i}\right] + |\lambda_i| \frac{\|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + 1)} T_i^{\alpha_i} \\
&+ \mu_{\max,i} \left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\beta_i + 1)} \xi_i^{\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\beta_i + 1)} \xi_i^{\beta_i}\right] \\
&+ \nu_{\max,i} \left[\frac{L_i \|\varpi_1 - \omega_1\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i} + \frac{K_i \|\varpi_2 - \omega_2\|}{\Gamma_q(\alpha_i + \beta_i + 1)} T^{\alpha_i+\beta_i}\right] \\
=& (L_i + K_i) \left[\frac{T^{\alpha_i+\beta_i} (1 + \nu_{\max,i})}{\Gamma_q(\alpha_i + \beta_i + 1)} + \frac{\xi_i^{\beta_i} \mu_{\max,i}}{\Gamma_q(\beta_i + 1)}\right] [|\varpi_1 - \omega_1| + |\varpi_2 - \omega_2|] \\
&+ \frac{|\lambda_i| T_i^{\alpha_i}}{\Gamma_q(\alpha_i + 1)} \|\varpi_1 - \omega_1\| \\
=& [(L_i + K_i) \Theta_i + \Omega_i] |\varpi_1 - \omega_1| + \Theta_i |\varpi_2 - \omega_2|,
\end{aligned}$$

implying that

$$\|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)\| \leq [(L_i + K_i) \Theta_i + \Omega_i] \|\varpi_1 - \omega_1\|_X + \Theta_i \|\varpi_2 - \omega_2\|_X. \tag{3.14}$$

Likewise, and by using the precedent technique, we have

$$\left\|\mathcal{G}_i' u_i(\varsigma) - \mathcal{G}_i' v_i(\varsigma)\right\| \leq \left[(L_i + K_i) \overline{\Theta}_i + \overline{\Omega}_i\right] \|\varpi_1 - \omega_1\|_X + \overline{\Theta}_i \|\varpi_2 - \omega_2\|_X. \tag{3.15}$$

Then, from (3.14) and (3.15), we have

$$\begin{aligned}
\|\mathcal{G}_i u_i(\varsigma) - \mathcal{G}_i v_i(\varsigma)\| \leq& \left[(L_i + K_i) \left(\Theta_i + \overline{\Theta}_i\right) + \left(\Omega_i + \overline{\Omega}_i\right)\right] \|\varpi_1 - \omega_1\|_X \\
&+ \left(\Theta_i + \overline{\Theta}_i\right) \|\varpi_2 - \omega_2\|_X.
\end{aligned} \tag{3.16}$$

Consequently,

$$\|\mathcal{G}(\varpi_1, \varpi_2) - \mathcal{G}(\omega_1, \omega_2)\|_{X \times X} \leq L \|(\varpi_1, \varpi_2) - (\omega_1, \omega_2)\|_{X \times X}.$$

Because $L < 1$, $\mathcal{G}$ is a contraction mapping with contraction constant L.

Next, we show that

$$\mathcal{G}(\partial B_R) \subseteq B_R. \tag{3.17}$$

To see this, let $(\varpi_1, \varpi_2) \in \partial B_R, \varsigma \in J$, and consider

$$\begin{aligned}
|\mathcal{G}_i u_i(\varsigma)| \leq & \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma))| (\varsigma) + |\lambda_i| \mathcal{I}_q^{\alpha_i} |\varpi_i| (\varsigma) \\
& + \mu_{\max,i} \mathcal{I}_q^{\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma))| (\xi_i) + \nu_{\max,i} \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(\varsigma, \varpi_1(\varsigma), \varpi_2(\varsigma))| (T) \\
\leq & \left[\mathcal{I}_q^{\alpha_i+\beta_i} |f_i(s, \varpi_1(s), \varpi_2(s)) - f_i(s,0,0)| (\varsigma) + \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(s,0,0)| (\varsigma)\right] \\
& + |\lambda_i| \left[\mathcal{I}_q^{\alpha_i} |\varpi_i| (\varsigma)\right] \\
& + \mu_{\max,i} \left[\mathcal{I}_q^{\beta_i} |f_i(s, \varpi_1(s), \varpi_2(s)) - f_i(s,0,0)| (\xi_i) + \mathcal{I}_q^{\beta_i} |f_i(s,0,0)| (\xi_i)\right] \\
& + \nu_{\max,i} \left[\mathcal{I}_q^{\alpha_i+\beta_i} |f_i(s, \varpi_1(s), \varpi_2(s)) - f_i(s,0,0)| (T) \right.\\
& \left. + \mathcal{I}_q^{\alpha_i+\beta_i} |f_i(s,0,0)| (T)\right] \\
\leq & \left[L_i \mathcal{I}_q^{\alpha_i+\beta_i} |\varpi_1| (\varsigma) + K_i \mathcal{I}_q^{\alpha_i+\beta_i} |\varpi_2| (\varsigma) + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)}\right] + \left[\frac{|\lambda_i| T^{\alpha_i} R}{\Gamma_q(\alpha_i+1)}\right] \\
& + \mu_{\max,i} \left[L_i \mathcal{I}_q^{\beta_i} |\varpi_1| (\xi_i) + K_i \mathcal{I}_q^{\beta_i} |\varpi_2| (\xi_i) + \frac{\xi_i^{\beta_i} A_i}{\Gamma_q(\beta_i+1)}\right] \\
& + \nu_{\max,i} \left[L_i \mathcal{I}_q^{\alpha_i+\beta_i} |\varpi_1| (\varsigma) + K_i \mathcal{I}_q^{\alpha_i+\beta_i} |\varpi_2| (\varsigma) + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)}\right] \\
\leq & \left[\frac{T^{\alpha_i+\beta_i} L_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} K_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)}\right] \\
& + |\lambda_i| \left[\frac{T^{\alpha_i} R}{\Gamma_q(\alpha_i+1)}\right] + \mu_{\max,i} \left[\frac{\xi_i^{\beta_i} L_i R}{\Gamma_q(\beta_i+1)} + \frac{\xi_i^{\beta_i} K_i R}{\Gamma_q(\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\beta_i+1)}\right] \\
& + \nu_{\max,i} \left[\frac{T^{\alpha_i+\beta_i} L_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} K_i R}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{T^{\alpha_i+\beta_i} A_i}{\Gamma_q(\alpha_i+\beta_i+1)}\right] \\
= & \left[R(L_i + K_i + A_i)\right] \left[\frac{[\nu_{\max,i}+1] T^{\alpha_i+\beta_i}}{\Gamma_q(\alpha_i+\beta_i+1)} + \frac{\mu_{\max,i} \xi_i^{\beta_i}}{\Gamma_q(\beta_i+1)}\right] + \left[\frac{R|\lambda_i| T^{\alpha_i}}{\Gamma_q(\alpha_i+1)}\right] \\
= & \left[(L_i + K_i + A_i)\Theta_i + \Omega_i\right] R,
\end{aligned}$$

implying that

$$\|\mathcal{G}_i u_i(\varsigma)\| \leq \left[(L_i + K_i + A_i)\Theta_i + \Omega_i\right] R. \tag{3.18}$$

Likewise, and by using the precedent technique, we have

$$\|\mathcal{G}_i' \varpi_i(\varsigma)\| \leq \left[(L_i + K_i + A_i)\overline{\Theta}_i + \overline{\Omega}_i\right] R. \tag{3.19}$$

Then, from (3.18) and (3.19), we have

$$\|\mathcal{G}_i \varpi_i(\varsigma)\| \leq \left[(L_i + K_i + A_i)\left(\Theta_i + \overline{\Theta}_i\right) + \left(\Omega_i + \overline{\Omega}_i\right)\right] R. \tag{3.20}$$

Consequently,

$$\|\mathcal{G}(\varpi_1, \varpi_2) - \mathcal{G}(\omega_1, \omega_2)\|_{X \times X} \leq (L + A) R \leq R,$$

implying that (3.17) holds. Therefore, by the Banach fixed-point theorem, $\mathcal{G}$ has a unique fixed-point $(\varpi_1, \varpi_2) \in B_R$. The proof is complete. ☐

3.3. **Stability of the solutions of (1.1).** We use Urs's [48] approach to establish the Ulam-Hyers stability of the solutions of (1.1).

Theorem 3.3 ([48]). *Let X be a Banach space and $T_1, T_2 : X \times X \to X$ be two operators. Then, the operational equations system*

$$\begin{cases} \varpi_1 = T_1(\varpi_1, \varpi_2), \\ \varpi_2 = T_2(\varpi_1, \varpi_2), \end{cases}$$

is said to be Ulam-Hyers stable if there exist $C_1, C_2, C_3, C_4 > 0$ such that for each $\epsilon_1, \varepsilon_2 > 0$ and each solution-pair $(\varpi_1^*, \varpi_2^*) \in X \times X$ of the in-equations:

$$\begin{cases} \|\varpi_1 - T_1(\varpi_1, \varpi_2)\|_X \le \epsilon_1, \\ \|\varpi_2 - T_2(\varpi_1, \varpi_2)\|_X \le \varepsilon_2, \end{cases}$$

there exists a solution $(\omega_1^*, \omega_2^*) \in X \times X$ of (1.1) such that

$$\begin{cases} \|\varpi_1^* - \omega_1^*\|_X \le C_1\varepsilon_1 + C_2\epsilon_2, \\ \|\varpi_2^* - \omega_2^*\|_X \le C_3\varepsilon_1 + C_4\epsilon_2. \end{cases}$$

Theorem 3.4 ([48]). *Let X be a Banach space, $T_1, T_2 : X \times X \to X$ be two operators such that*

$$\begin{cases} \|T_1(\varpi_1, \varpi_2) - T_1(\omega_1, \omega_2)\|_X \le k_1 \|\varpi_1 - \omega_1\|_x + k_2 \|\varpi_2 - \omega_2\|_X, \\ \|T_2(\varpi_1, \varpi_2) - T_2(\omega_1, \omega_2)\|_X \le k_3 \|\varpi_1 - \omega_1\|_x + k_4 \|\varpi_2 - \omega_2\|_X, \end{cases}$$

for all $(\varpi_1, \varpi_2), (\omega_1, \omega_2) \in X \times X$. Suppose

$$H = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix},$$

converges to zer0. Then, the operational equations (1.1) is Ulam-Hyers stable.

Set

$$\begin{aligned} C_1 &= \left[(L_1 + K_1)\left(\Theta_1 + \overline{\Theta}_1\right) + \left(\Omega_1 + \overline{\Omega}_1\right)\right], \\ C_2 &= (L_1 + K_1)\left(\Theta_1 + \overline{\Theta}_1\right), \\ C_3 &= \left[(L_2 + K_2)\left(\Theta_2 + \overline{\Theta}_2\right) + \left(\Omega_2 + \overline{\Omega}_2\right)\right], \\ C_4 &= (L_2 + K_2)\left(\Theta_2 + \overline{\Theta}_2\right). \end{aligned}$$

Theorem 3.5. *Assume (H1)-(H4) hold. Further, assume the spectral radius of H is less than one. Then, the solution of (1.1) is Ulam-Hyers stable.*

Proof. In view of Theorem 3.2 we have

$$\begin{cases} \|A_1(\varpi_1, \varpi_2) - A_1(\omega_1, \omega_2)\|_X \le C_1 \|\varpi_1 - \omega_1\|_X + C_2 \|\varpi_2 - \omega_2\|_X, \\ \|A_2(\varpi_1, \varpi_2) - A_2(\omega_1, \omega_2)\|_X \le C_3 \|\varpi_2 - \omega_1\|_X + C_4 \|\varpi_2 - \omega_2\|_X, \end{cases}$$

which implies that

$$\|A(\varpi_1,\varpi_2)-A(\omega_1,\omega_2)\|_{X\times X}\leq A\begin{pmatrix}\|\varpi_1-\omega_1\|_X\\ \|\varpi_2-\omega_2\|_X\end{pmatrix}. \tag{3.21}$$

Because the spectral radius of H is less than one, the solution of (1.1) is Ulam-Hyers stable. □

4. Example

This section is devoted to the illustration of the results derived in the last section.

Example 4.1. In this section, we present some examples to illustrate our results.

Let $\mathbb{E}=l^1=\{\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n,\ldots):\sum_{n=1}^{\infty}|\varpi_n|<\infty\}$ with the norm

$$\|\varpi\|_{\mathbb{E}}=\sum_{n=1}^{\infty}|\varpi_n|.$$

Consider the following nonlinear Langevin $\frac{1}{4}$-fractional equation:

(4.1)

$$\begin{cases}\mathcal{D}^{1/4}_{1/4}\left(\mathcal{D}^{4/3}_{1/4}-\dfrac{1}{10}\right)\varpi(\varsigma)=\dfrac{\sqrt{3}|\varpi|\cos^2(2\pi\varsigma)}{3(27-\varsigma)}+\dfrac{\sqrt{2}\pi|y|}{(7\pi-\varsigma)^2}\left(\dfrac{|y|}{|y|+3}+1\right), & \varsigma\in J,\\ \mathcal{D}^{1/2}_{1/4}\left(\mathcal{D}^{5/3}_{1/4}-\dfrac{2}{5}\right)y(\varsigma)=\dfrac{\sqrt{2}\pi|\varpi|}{4(4\pi-\varsigma)^2}\left(\dfrac{|\varpi|}{|\varpi|+3}+1\right)+\dfrac{|y|\sin^2(2\pi\varsigma)}{(10-\varsigma)^2}, & \varsigma\in J,\\ \varpi(0)=0,\quad \varpi(1)+\dfrac{1}{10}\mathcal{J}^{1/4}_q\varpi(1)=0,\quad \mathcal{D}^{1/4}_q\varpi(1/2)+\dfrac{1}{10}\varpi(1/2)=0,\\ y(0)=0,\quad \varpi_1(1)+\dfrac{2}{5}\mathcal{J}^{1/2}_q y(1)=0,\quad \mathcal{D}^{1/2}_q y(3/4)+\dfrac{2}{5}y(3/4)=0.\end{cases}$$

Here $J=[0,1]$, $\alpha_1=1/4$, $\alpha_2=1/2$, $\beta_1=4/3$, $\beta_2=5/3$, $\xi_1=3/4$, $\xi_2=1/2$, $\lambda_1=1/10$, $\lambda_2=2/5$, with

$$f(\varsigma,\varpi)=(((\sin\varsigma+1)e^{-\varsigma})/24)(\varpi^2/(1+|\varpi|)).$$

Clearly, the function f is continuous. For each $\varpi\in\mathbb{E}$ and $\varsigma\in[0,1]$, we have

$$|f(\varsigma,\varpi_1,\varpi_2)|\leq\frac{\sqrt{3}}{81}|\varpi_1|+\frac{\sqrt{2}}{49\pi}|\varpi_2|$$

and

$$|g(\varsigma,\varpi_1,\varpi_2)|\leq\frac{\sqrt{2}}{64\pi}|\varpi_1|+\frac{1}{100}|\varpi_2|.$$

Hence, the hypothesis (H2) is satisfied with $p_1^*=\frac{\sqrt{3}}{81}$, $q_1^*=\frac{\sqrt{2}}{49\pi}$, $p_2^*=\frac{\sqrt{2}}{64\pi}$ and $q_2^*=\frac{1}{100}$. We shall show that condition (3.13) holds with $J=[0,1]$. Indeed,

$$\Lambda_1=0.1687\quad\Lambda_2=0.1985,\quad\Lambda\simeq0.3672<1.$$

Simple computations show that all conditions of Theorem 3.1 are satisfied. It follows that the coupled (4.1) has at least one solution defined on J.

Example 4.2. Consider the following coupled system:

$$\begin{cases} \mathcal{D}^{1/4}\left(\mathcal{D}^{4/3}+\dfrac{1}{237}\right)\varpi_1(\varsigma)=\dfrac{1}{100}+\dfrac{\varsigma}{10e^{\varsigma}}\dfrac{|\varpi_1|}{10+|\varpi_1|}+\dfrac{\varsigma\,|\varpi_2|}{(9+e^{\varsigma})^2\,(|\varpi_2|+1)}, \quad \varsigma\in[0,1],\\ \mathcal{D}^{1/2}\left(\mathcal{D}^{5/3}+\dfrac{1}{100}\right)\varpi_2(\varsigma)=\dfrac{\varsigma}{100e^{\varsigma}}+\dfrac{\sin|\varpi_1|\,\varsigma+\sin|\varpi_2|}{e^{\varsigma}+99}, \quad \varsigma\in[0,1],\\ \varpi_1(0)=0, \quad \varpi_1(1)+\dfrac{1}{10}\mathcal{I}_q^{1/4}\varpi_1(1)=0, \quad \mathcal{D}_q^{1/4}\varpi_1(1/2)+\dfrac{1}{10}\varpi_1(1/2)=0,\\ \varpi_2(0)=0, \quad \varpi_2(1)+\dfrac{2}{5}\mathcal{I}_q^{1/2}\varpi_2(1)=0, \quad \mathcal{D}_q^{1/2}\varpi_2(3/4)+\dfrac{2}{5}\varpi_1(3/4)=0. \end{cases}$$

Using the given data, we find that

$$\begin{aligned} &|\eta_1(\varsigma,\varpi_1,\varpi_2)-f_1(\varsigma,\omega_1,\omega_2)|\le\frac{1}{100}|\varpi_1-\varpi_2|+\frac{1}{100}|\omega_1-\omega_2|,\\ &|f_2(\varsigma,\varpi_1,\varpi_2)-f_2(\varsigma,\omega_1,\omega_2)|\le\frac{1}{100}|\varpi_1-\varpi_2|+\frac{1}{100}|\omega_1-\omega_2|,\\ &|\eta_1(\varsigma,0,0)|\le\frac{1}{10}, \quad |\eta_1(\varsigma,\varpi_1,\varpi_2)|\le\frac{1}{10}+\frac{\varsigma}{5e^{\varsigma}}+\frac{\varsigma}{(1+e^{\varsigma})^2},\\ &|\eta_2(\varsigma,0,0)|\le\frac{\varsigma}{10e^{\varsigma}}, \quad |\eta_2(\varsigma,\varpi_1,\varpi_2)|\le\frac{\varsigma}{10e^{\varsigma}}+\frac{\varsigma+1}{e^{\varsigma}+10}, \end{aligned}$$

for any $\varsigma\in[0,1]$. Then η_i, $i=1,2$ satisfying (H1)-(H4), with $L_i=\frac{1}{100}$, $K_i=\frac{1}{100}$, $i=1,2$, $A_i=\frac{1}{100}$, $i=1,2$, We find that

$$\begin{aligned} &\Theta_1=1.3850, \quad \Theta_2=1.1300, \quad \Omega_1=0.0207, \quad \Omega_2=0.0354,\\ &\overline{\Theta}_1=6.0050, \quad \overline{\Theta}_2=2.3900, \quad \overline{\Omega}_1=0.2048, \quad \overline{\Omega}_2=0.1992. \end{aligned}$$

Hence, $L\simeq 0.6783$, and $A\simeq 0.1091$. Therefore, $L+A<1$, and then all conditions of Theorem (3.2) are satisfied, which implies the existence of a unique solution for system (3.21) in $[0,1]$. On the other hand, we find that

$$C_1=0.3733, \quad C_2=0.3050, \quad C_3=0.1478, \quad C_4=0.0704.$$

The spectral radius of the matrix

$$H=\begin{pmatrix}0.3733 & 0.3050\\ 0.1478 & 0.0704\end{pmatrix}$$

is 0.48. Hence, by Theorem 3.5, the solution of (3.21) is Ulam-Hyers stable.

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