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ANALYTIC STUDIES OF A CLASS OF LANGEVIN DIFFERENTIAL EQUATIONS DOMINATED BY A CLASS OF JULIA FRACTAL FUNCTIONS

RABHA W. IBRAHIM[1] AND DUMITRU BALEANU[2,3,4]

Abstract. In this investigation, we study a class of analytic functions of type Carathéodory style in the open unit disk connected with some fractal domains. This class of analytic functions is formulated based on a kind of Langevin differential equations (LDEs). We aim to study the analytic solvability of LDEs in the advantage of geometric function theory consuming the geometric properties of the Julia fractal (JF) and other fractal connected with the logarithmic function. The analytic solutions of the LDEs are obtainable by employing the subordination theory.

1. Introduction

Recently, analysis on fractals has been established by numerous investigators studying various problems in engineering (fractal antennas), physics (material processing), chemistry (chemical processing), biology (DNA) and computer science (image processing) [1–8]. Harmonic analysis is employed to describe derivatives and integrals on fractal sets. Probability theory is utilized to formulate Laplacians on fractals [9]. Fractional spaces are plotted to continuous real space in order to explain differential equations on fractals [10–15]. Fractional calculus is smeared in fractal spaces to clarify anomalous diffusion [16–20]. Extended fractional Langevin equations to complex domain are indicated by special types of fractal [21]. The fractal Langevin equation is studied presenting the dynamics of Brownian elements in the long time boundary [22]. Other studies such as an approximate fractal Langevin differential equation are

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consumed and an approximate solution is indicated [23–27]. In the present study, we aim to investigate the analytic solution of Langevin differential equation by using a Julia fractal functions and other fractal [28–30].

1.1. **Differential equation formula.** The second order LDE of a complex variable z is structured by [31]

$$f''(z) + \lambda f'(z) = \Lambda(f(z)), \tag{1.1}$$

where $\lambda > 0$ indicates the damping connection parameter and Λ is the noise term. To investigate the geometric properties of (1.1), we consume the analytic function $f(z)$ in $\cup$ achieving the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. This class of analytic functions is known as the normalized class denoting by $\wedge$. Extend (1.1) with complex coefficient, then we have equivalent equation

$$F(z) := \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right), \quad z \in \cup, \tag{1.2}$$

where $\lambda(z)$ is analytic function in the open unit disk $\cup$. Evidently $F(0) = 1$, for all $\lambda(z) \in \cup$ (see the following example).

Example 1.1. • Assume the function $f(z) = \frac{z}{1-z}$, $\lambda(z) = z$. Then we get the series $F(z) = 1 + z + 3z^2 + 5z^3 + 7z^4 + 9z^5 + O(z^6)$.

• Let $\lambda(z) = 1$ and $f(z) = \frac{z}{1-z}$. This implies the series $F(z) = 1 + 3z + 5z^2 + 7z^3 + 9z^4 + 11z^5 + O(z^6)$.

We demand the following preliminaries.

Definition 1.1. • Two analytic functions f and g in $\cup$ are called subordinate denoting by $f \prec g$, if for a function h is selected such that $|h(z)| \leq |z|$ indicating the equation $f = g(h)$ [32].

• The Ma-Minda construction inequalities signified by $S^*(p)$ and $K(p)$ of starlike and convex functions are structured by $\left(\frac{zf'(z)}{f(z)}\right) \prec p(z)$ and $\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p(z)$, respectively, where p achieves the existing in the class $\mathcal{P}$ where $\operatorname{Re}(p(z)) > 0$, $p(0) = 1$, $|p'(0)| > 1$.

By utilizing the definition of LDEs, we formulate a new class of analytic functions as follows.

Definition 1.2. A function of the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \cup,$$

is in the class $\mathbf{M}_\lambda(p)$ if and only if

$$F(z) = \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right) \prec p(z), \quad z \in \cup,\ p(0) = 1,\ p'(0) > 1,\ \lambda(z) \in \cup. \tag{1.3}$$

We study the analytic solvability of (1.3) by using different types of the parametric Julia fractal formulas taking the construction (see Figure 1)

$$\begin{aligned} J_\kappa(z) =& 1 + z - \kappa\, z^3, \quad z \in \cup, \\ \Upsilon_\kappa(z) =& \frac{1+z^2}{1-\kappa\, z^2} = 1 + (\kappa+1)z^2 + (\kappa^2+\kappa)z^4 + O(z^6), \quad z \in \cup, \end{aligned}$$

and

$$\begin{aligned} L_\kappa(z) &= z^2 + \frac{1}{1-\kappa z^2} \\ &= 1 + (\kappa+1)z^2 + \kappa^2 z^4 + O(z^6), \quad z \in \cup, |z| < 1/\sqrt{(|\kappa|)}. \end{aligned}$$

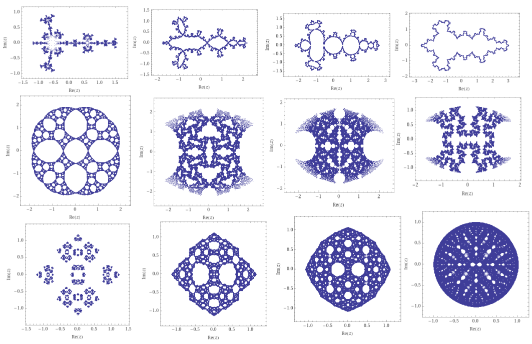

FIGURE 1. The plot of J_κ, $\kappa = 1, 1/2, 1/3, 1/4$, $\Upsilon_\kappa, \kappa = 1, 2, 3, 4$ and $L_\kappa, \kappa = 3, 5, 10, 100$, respectively

The technique is to find the optimal value of κ which satisfies the inequality subordination

$$1 + \kappa \left(\frac{z\, p'(z)}{[p(z)]^k} \right) \prec (1+z)^\kappa, \quad z \in \cup,$$

to satisfy one of the following inequalities

$$p(z) \prec J_\kappa, \quad p(z) \prec \Upsilon_\kappa, \quad p(z) \prec L_\kappa.$$

As an application, we consider the LDEs to investigate the solvability by using the Julia fractal functions

$$F(z) \prec J_\kappa, \quad F(z) \prec \Upsilon_\kappa, \quad F(z) \prec L_\kappa.$$

Special cases are investigated for some well known classes of analytic functions.

2. Computational Results

This section deals with consequences regarding $p(z)$ and $F(z)$.

Theorem 2.1. *Let the function* $p \in \mathcal{P}$ *admitting the inequalities*

$$1+\kappa\left(\frac{z\,p'(z)}{(\,p(z)\,)^{k}}\right) \prec \Sigma_\kappa(z), \quad z \in \cup,$$

where $k = 0, 1, 2$ *and* $\Sigma_\kappa(z) = (1+z)^\kappa$, $z \in \cup$. *Then*

(A) $p(z) \prec J_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = 1.3247$*;*
(B) $p(z) \prec \Upsilon_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = \frac{1}{2}$*;*
(C) $p(z) \prec L_\kappa(z)$, $z \in \cup$, *for* $\kappa \geq \max \kappa_k = 0.550667$.

Proof. Firstly, we aim to prove the inequality $p(z) \prec J_\kappa(z)$, therefore we have the following cases.

Case I. $k = 0 \Rightarrow 1 + \kappa\,(z\,p'(z)) \prec (1+z)^\kappa$. Let $T_\kappa : \cup \to \mathbb{C}$ admitting the structure

$$T_\kappa(z) = \frac{(\kappa^2+\kappa+1) - (z+1)^{\kappa+1}\,{}_2F_1(1, \kappa+1, \kappa+2, z)}{\kappa^2+\kappa}, \quad z \in \cup,$$

where ${}_2F_1$ indicates the hypergeometric function for all $z \in \cup$ with the power series

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!}.$$

Clearly, $T_\kappa(z)$ is analytic in $\cup$ satisfying $T_\kappa(0) = 1$ and it is an approximate solution by a hypergeometric function of the differential equation

$$1 + \kappa\,(z\,T'_\kappa(z)) = (z+1)^\kappa, \quad z \in \cup. \tag{2.1}$$

Let

$$\mathfrak{W}(z) := -\frac{\kappa}{3}\,(z\,T'_\kappa(z)) = \frac{((z+1)^\kappa((z-1)2F1(1, \kappa+1, \kappa+2, z) + z + 1))}{z-1}.$$

Then by [32, Lemma 4.5e], where

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^\alpha, \quad \beta \leq \gamma, \tag{2.2}$$

we have for $\kappa > 0$

$$\begin{aligned}
\mathfrak{W}(z) &= -\frac{\kappa}{3}\,(z\,T'_\kappa(z)) \\
&= -\frac{1}{3} \cdot \frac{((z+1)^\kappa((z-1)_2F_1(1, \kappa+1, \kappa+2, z) + z + 1))}{z-1} \\
&= -\frac{1}{3} \cdot \frac{((z+1)^\kappa((z-1)(1-z) + z + 1))}{z-1} \\
&= \left(\frac{z(1+z)^\kappa}{1-z}\right)\left(1 - \frac{z}{3}\right) \\
&= z + \left(\kappa + \frac{2}{3}\right) z^2 + \frac{1}{6}(3\kappa^2 + k + 4)z^3 + \frac{1}{6}(\kappa^3 - \kappa^2 + 4\kappa + 4)z^4 \\
&+ \frac{1}{72}(3\kappa^4 - 10\kappa^3 + 33\kappa^2 + 22\kappa + 48)z^5 + O(z^6).
\end{aligned}$$

By the assumption of the theorem, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{z\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right) =&\mathrm{Re}\left(1+\left(\kappa+\frac{2}{3}\right)z+\left(\frac{8}{9}-\kappa\right)z^2+\left(\kappa+\frac{26}{27}\right)z^3\right.\\ &\left.+\left(\frac{80}{81}-\kappa\right)z^4+\left(\kappa+\frac{242}{243}\right)z^5+O(z^6)\right)\\ >&0\end{aligned}$$

provided $0<\kappa\leq 4.27772$. That is, $\mathfrak{W}(z)$ is starlike function. Thus, by using $\mathfrak{G}(z):=\mathfrak{W}(z)+1/(-3)$, one can obtain

$$\mathrm{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right)=\mathrm{Re}\left(\frac{z\,\mathfrak{G}'(z)}{\mathfrak{W}(z)}\right)>0.$$

Thus, Miller-Mocanu Lemma (see [32, page 132]) admits

$$1+\kappa\,(z\,p'(z))\prec 1+\kappa\,(zT_\kappa'(z))\Rightarrow p(z)\prec\ T_\kappa(z).$$

Our aim is to prove that $p(z)\prec J_\kappa(z)$, which indicates if $T_\kappa(z)\prec J_\kappa(z)$. To complete this conclusion, we have to prove that $T_\kappa(z)\prec(1+z)^\kappa$. By using (2.2), we have

$$\frac{\kappa^2+\kappa+1}{\kappa^2+\kappa}=T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}.$$

Since

$$0=\Sigma_\kappa(-1)\leq\Sigma_\kappa(1)=2^\kappa,\quad \kappa>0,$$

thus, we obtain

$$T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}\leq 2^\kappa,$$

whenever $\kappa>0.78124$. As a conclusion, we have $T_\kappa(z)\prec J_\kappa(z)$ when

$$\kappa=J_\kappa(-1)\leq T_\kappa(-1)=T_\kappa(1)=\frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa}\leq J_\kappa(1)=2-\kappa,$$

which is provided

$$0.7812<\kappa<\frac{1}{3}\left(\frac{27}{2}-\frac{3\sqrt{69}}{2}\right)^{1/3}+\frac{\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/3}}{3^{2/3}}\approx 1.3247.$$

This implies the relation

$$T_\kappa(z)\prec J_\kappa(z)\Rightarrow p(z)\prec\ J_\kappa(z),\quad z\in\cup.$$

Case II. $k=1\Rightarrow 1+\kappa\left(\frac{z\,p'(z)}{p(z)}\right)\prec(1+z)^\kappa$. Define a function $E_\kappa:\cup\to\mathbb{C}$ formulating the structure

$$E_\kappa(z)=\exp\left(\frac{1-(z+1)^{\kappa+1}\ {}_2F_1(1,\kappa+1,\kappa+2,z)}{\kappa^2+\kappa}\right).$$

Obviously, $E_\kappa(z)$ is analytic in $\cup$ satisfying $E_\kappa(0) = 1$ and it is an approximated solution by a hypergeometric function satisfying the differential equation

$$1+\kappa\left(\frac{z\,E'_\kappa(z)}{E_\kappa(z)}\right) = (1+z)^\kappa, \quad z\in\cup.$$

By considering $\mathfrak{W}(z)=\kappa\left(\frac{z\,E'_\kappa(z)}{E_\kappa(z)}\right)=(1+z)^\kappa-1$, which is starlike function with $\kappa\neq 0$ and $\mathfrak{T}(z)=\mathfrak{W}(z)+1$, we attain

$$\mathrm{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right)=\mathrm{Re}\left(\frac{z\,\mathfrak{T}'(z)}{\mathfrak{W}(z)}\right)>0, \quad z\in\cup.$$

Thus, Miller-Mocanu Lemma, yields

$$1+\kappa\left(\frac{z\,p'(z)}{p(z)}\right) \prec 1+\kappa\left(\frac{zE'_\kappa(z)}{E_\kappa(z)}\right)\Rightarrow p(z)\prec E_\kappa(z).$$

Consequently, one can recognize the next equality

$$\exp\left(\frac{1}{\kappa^2+\kappa}\right)=E_\kappa(-1)=E_\kappa(1)=\exp\left(\frac{1}{\kappa^2+\kappa}\right).$$

Moreover, this implies $E_\kappa(z)\prec(1+z)^\kappa$ such that for $\kappa\neq 0$ the inequality

$$0=\Sigma_\kappa(-1)\le E_\kappa(-1)=E_\kappa(1)\le\Sigma_\kappa(1)=2^\kappa, \quad \kappa>0.876764,$$

holds. Thus, we get $E_\kappa(z)\prec J_\kappa(z)$ when

$$\kappa=J_\kappa(-1)\le E_\kappa(-1)\le E_\kappa(1)=\exp\left(\frac{1}{\kappa^2+\kappa}\right)\le J_\kappa(+1)=2-\kappa.$$

This leads to the following subordination for $\kappa\approx 1$

$$E_\kappa(z)\prec J_\kappa(z)\Rightarrow p(z)\prec J_\kappa(z), \quad z\in\cup.$$

Case III: $k=2\Rightarrow 1+\kappa\left(\frac{z\,p'(z)}{p^2(z)}\right)\prec(1+z)^\kappa$. Consume that $H_\kappa:\cup\to\mathbb{C}$ satisfies the formula

$$H_\kappa(z)=\frac{\kappa(\kappa+1)}{\kappa^2+\kappa+(z+1)^{\kappa+1}\ {}_2F_1(1,\kappa+1,\kappa+2,z)}.$$

Clearly, $H_\kappa(z)$ is analytic in $\cup$ admitting $H_\kappa(0)=1$ and it is the approximated outcome in terms of the hypegoemetric function

$$1+\kappa\left(\frac{z\,H'_\kappa(z)}{H^2_\kappa(z)}\right) = (1+z)^\kappa, \quad z\in\cup.$$

Similarly, we use the starlike function $\mathfrak{W}(z)=\Sigma_\kappa(z)-1$ and $\mathfrak{Y}(z)=\mathfrak{W}(z)+1$, we get

$$\mathrm{Re}\left(\frac{z\,\mathfrak{W}'(z)}{\mathfrak{W}(z)}\right)=\mathrm{Re}\left(\frac{z\,\mathfrak{Y}'(z)}{\mathfrak{U}(z)}\right)>0, \quad z\in\cup.$$

Hence, the Miller-Mocanu Lemma yields

$$1+\kappa\left(\frac{z\,p'(z)}{p^2(z)}\right) \prec 1+\kappa\left(\frac{zH'_\kappa(z)}{H^2_\kappa(z)}\right)\Rightarrow p(z)\prec H_\kappa(z).$$

Accordingly, for $\kappa \geq 1$, we obtain

$$1 = H_\kappa(-1) = H_\kappa(1) = 1.$$

Moreover, for $\kappa = 1$, we have

$$\kappa = J_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq J_\kappa(+1) = 2 - \kappa.$$

Thus, one can realize that

$$H_\kappa(z) \prec J_\kappa(z) \Rightarrow p(z) \prec J_\kappa(z), \quad z \in \cup.$$

For the second and third part, we proceed in the same manner of above construction of the functions $T_\kappa(z), E_\kappa(z)$ and $H_\kappa(z)$. We conclude that for the second part,

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq T_\kappa(-1) = T_\kappa(1) = \frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa} \leq \Upsilon_\kappa(1) = \frac{2}{1-\kappa},$$

whenever

$$\kappa = \frac{1}{3}\left((-2 - 2(\frac{2}{(47+3\sqrt{249}))^{1/3}} + \left(\frac{1}{2}(47+3\sqrt{249})\right)^{1/3}\right) \approx 0.3532099\ldots$$

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq E_\kappa(-1) \leq E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right) \leq \Upsilon_\kappa(+1) = \frac{2}{1-\kappa},$$

whenever $\kappa \approx 0.490561$ and

$$\frac{2}{1-\kappa} = \Upsilon_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq \Upsilon_\kappa(+1) = \frac{2}{1-\kappa}, \quad \kappa \approx \frac{1}{2}.$$

Then we get $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $z \in \cup$. For the last part, we obtain

$$\frac{2-\kappa}{1-\kappa} = L_\kappa(-1) \leq T_\kappa(-1) = T_\kappa(1) = \frac{(\kappa^2+\kappa+1)}{\kappa^2+\kappa} \leq L_\kappa(1) = \frac{2-\kappa}{1-\kappa},$$

whenever, $\kappa = \sqrt{(2)} - 1 \approx 0.414213\ldots$,

$$\frac{2-\kappa}{1-\kappa} = L_\kappa(-1) \leq E_\kappa(-1) \leq E_\kappa(1) = \exp\left(\frac{1}{\kappa^2+\kappa}\right) \leq L_\kappa(+1) = \frac{2-\kappa}{1-\kappa},$$

whenever $\kappa \approx 0.550667$ and

$$\frac{2-\kappa}{1+\kappa} < L_\kappa(-1) \leq H_\kappa(-1) \leq H_\kappa(1) \leq L_\kappa(+1) = \frac{2-\kappa}{1-\kappa}, \quad \kappa \approx \frac{1}{2}.$$

Then, we conclude that $p(z) \prec L_\kappa(z)$, $\kappa > 0.550667$, $z \in \cup$ □

As an application of Theorem 2.1, we let $p(z) = \frac{zf'(z)}{f(z)}$, $f\wedge$. Thus, one can recognize the following consequence.

Corollary 2.1. *Let $f\wedge$. If one of the inequalities is indicted*

(a) $1 + \kappa\left(\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2\right) \prec (1+z)^\kappa$;

(b) $1 + \kappa\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec (1+z)^\kappa$;

(c) $1 + \kappa\left(\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}\right)^{-1} + \left(\frac{zf'(z)}{f(z)}\right)^{-1} - 1\right) \prec (1+z)^\kappa$;

then $f \in S^*(J_\kappa)$, $\kappa > 1.3247$, $f \in S^*(\Upsilon_\kappa)$, $\kappa > \frac{1}{2}$ *and* $f \in S^*(L_\kappa)$, $\kappa > 0.550667$.

Corollary 2.2. *Let* $p(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$. *If one of the inequalities is indicted*

(a)

$$1 + \frac{\kappa z(A-B)}{(Bz+1)^2} \prec (1+z)^\kappa, \quad \frac{A+1}{B+1} < 0.676, \quad \kappa = \frac{(-A+2B+1)}{(B+1)} > 1.3247;$$

(b)

$$1 + \frac{\kappa z(A-B)}{((Az+1)(Bz+1))} \prec (1+z)^\kappa,$$

where

$$A+1 \neq 0,\ \kappa = \frac{(A-2B-1)}{(A+1)} > \frac{1}{2},\ B+1 \neq 0,\ A > 4B+3;$$

(c)

$$1 + \frac{\kappa z(A-B)}{(Az+1)^2} \prec (1+z)^\kappa,$$

where

$$A \neq B, \kappa = \frac{(A-2B-1)}{(A-B)},\ B+1 \neq 0,\ \frac{A}{(A-B)} - \frac{(2B)}{(A-B)} - \frac{1}{(A-B)} > 0.55;$$

then

$$\frac{1+Az}{1+Bz} \prec J_\kappa(z), \quad \kappa > 1.324,$$

$$\frac{1+Az}{1+Bz} \prec \Upsilon_\kappa(z), \quad \kappa > 1/2,$$

and

$$\frac{1+Az}{1+Bz} \prec L_\kappa(z), \quad \kappa > 0.55066.$$

Corollary 2.3. *Let* $p(z) = 1 + \sin(z)$. *If one of the inequalities is indicted*

(a) $1 + \kappa z \cos(z) \prec (1+z)^\kappa$, $\kappa > 1.324$;
(b) $1 + \frac{\kappa z \cos(z)}{\sin(z)+1} \prec (1+z)^\kappa$, $\kappa > 0.5$;
(c) $1 + \frac{(\kappa z \cos(z))}{(\sin(z)+1)^2} \prec (1+z)^\kappa$, $\kappa > 0.55066$;

then $p(z) \prec J_\kappa(z)$, $\kappa > 1.324$, $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $p(z) \prec L_\kappa(z)$, $\kappa > 0.55066$.

Corollary 2.4. *Let* $p(z) = e^z$. *If one of the inequalities is indicted*

(a) $1 + \kappa z e^z \prec (1+z)^\kappa$, $\kappa > 1.324$;
(b) $1 + \kappa z \prec (1+z)^\kappa$, $\kappa > 0.5$;
(c) $1 + \kappa z\, e^{-z} \prec (1+z)^\kappa$, $\kappa > 0.55066$;

then $p(z) \prec J_\kappa(z)$, $\kappa > 1.324$, $p(z) \prec \Upsilon_\kappa(z)$, $\kappa > 0.5$, $p(z) \prec L_\kappa(z)$, $\kappa > 0.55066$.

Next result admits some properties of LDE.

Theorem 2.2. *Consider two functions* $\Sigma_\kappa(z) = (1+z)^\kappa$, $\kappa \in \mathbb{R}$, *and*

$$F(z) = \lambda(z)\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right).$$

If one of the inequalities

$$1 + \kappa\left(\frac{z\,F'(z)}{[F(z)]^k}\right) \prec (1+z)^\kappa$$

is occurred, where $k = 0, 1, 2$, *then*

- $F(z) \prec J_\kappa(z)$, $\kappa > 1.324$;
- $F(z) \prec \Upsilon_\kappa(z), \kappa > 0.5$;
- $F(z) \prec J_\kappa(z), \kappa > 0.55066$.

Furthermore, if $\mathrm{Re}\,(F(z)) > 0$ *and* $\lambda(z)$ *satisfies*

$$\mathrm{Re}\,(\lambda(z)) > 0, \quad [\Im(1-\lambda(z))]^2 \leq 3[\mathrm{Re}\,(\lambda(z))]^2,$$

then f *is starlike in* $\cup$.

Proof. Since for all $\lambda(z), z \in \cup$, we have $F(0) = 1$, then by using the same technique in Theorem 2.1, we have the first part regarding the subordinated inequalities. For the second part, we assume that

$$p(z) = \frac{zf'(z)}{f(z)}, \quad f \in \wedge,\ z \in \cup.$$

Then a computation implies that

$$F(z) = \lambda(z)zp'(z) + \lambda(z)p^2 + [1-\lambda(z)]p(z).$$

Then by the assumptions and in view of [32, Example 2.4], we have $\mathrm{Re}\,(p(z)) > 0$ which implies that $f(z)$ is starlike. □

3. Examples

In this section, we deal with special cases of the LDEs depending on the formula of $\lambda(z)$.

Case I. Let $\lambda(z) = 1$. The construction of LDE becomes

$$\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right) = F(z), \quad z \in \cup,\ f \in \wedge. \tag{3.1}$$

Then by using $J_\kappa(z) = 1 + z - \kappa z^3$, Figure 2 shows the solution for different values of $\kappa > 1.324$, Figure 3 indicates the solution by using $\Upsilon_\kappa(z) = \frac{1+z^2}{1-\kappa z^2}$ for $\kappa > 0.5$. It can be seen that the solution satisfies when

- $\kappa = 0.6$, we have

$$\begin{aligned} f(z) =& c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1-0.6i, 1+0.6i \\ -0.5, 0.5 \end{matrix}\;\middle|\; 0.6z^2\right) \\ &+ 0.774\, c_1\, z\left({}_2F_1(0.5-0.6i, 0.5+0.6i; 2; 0.6z^2)\right), \end{aligned}$$

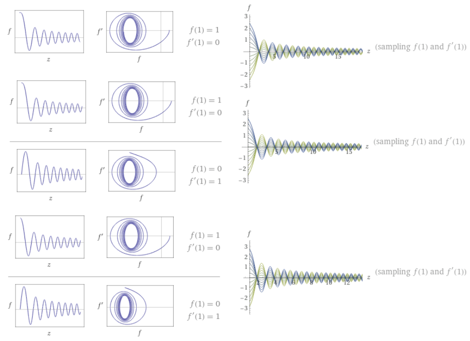

FIGURE 2. The plot of (3.1) by using $J_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 1.5, 2, 4$, respectively

where $f(0) = 0$ provided $c_2 = 0$;

- $\kappa = 2$, we obtain

$$\begin{aligned} f(z) =& c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1 - \dfrac{i}{2\sqrt{2}}, 1 + \dfrac{i}{2\sqrt{2}} \\ -0.5, 0.5 \end{matrix} \middle| \, 2z^2 \right) \\ &+ i\sqrt{(2)}c_1 z \left({}_2F_1(1/4(2 - i\sqrt{(2)}), 1/4(2 + i\sqrt{(2)}); 2; 2z^2) \right); \end{aligned}$$

- $\kappa = 4$, we get

$$f(z) = c_2 G^{(2,0)}_{(2,2)}\left(\begin{matrix} 1 - \dfrac{i}{4}, 1 + \dfrac{i}{4} \\ -0.5, 0.5 \end{matrix} \middle| \, 4z^2 \right) + 2ic_1 z \left({}_2F_1\left(\frac{1}{2} - \frac{i}{4}, \frac{1}{2} - \frac{i}{4}; 2; 2z^2 \right) \right).$$

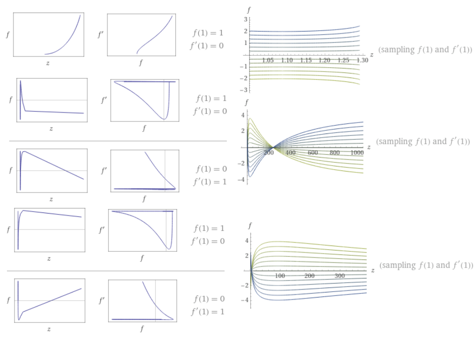

FIGURE 3. The plot of (3.1) using $\Upsilon_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

Figure 4 imposes the behavior of (3.1) by using $L_\kappa(z)$.

Case II. Let $\lambda(z) = z$. The construction of LDE becomes

$$z\left(\frac{z^2 f''(z)}{f(z)}\right) + \left(\frac{z f'(z)}{f(z)}\right) = F(z), \quad z \in \cup, \ f \in \wedge. \tag{3.2}$$

A computation implies the following constructions of $F(z)$.

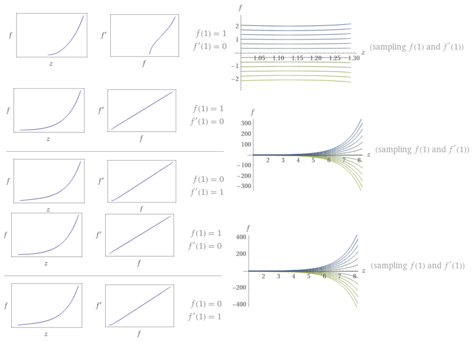

FIGURE 4. The plot of (3.1) using $L_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

- For

$$\begin{aligned} J_{1.5} \Rightarrow & F(z) = -1.5(z - 1.12271)(z^2 + 1.12271z + 0.593803), \\ J_2(z) \Rightarrow & F(z) = 1 + z \end{aligned}$$

and $J_4(z) \Rightarrow F(z) = 1 + z$;
- for

$$\begin{aligned} \Upsilon_{0.6} \Rightarrow & F(z) = \frac{1.66667z^2 + 1.66667}{1.66667 - z^2}, \\ \Upsilon_2(z) \Rightarrow & F(z) = \frac{z^2}{1 - 2z^2} + \frac{1}{1 - 2z^2} \end{aligned}$$

and

$$\Upsilon_4(z) \Rightarrow F(z) = \frac{z^2}{1 - 4z^2} + \frac{1}{1 - 4z^2};$$

- for

$$\begin{aligned} L_{0.6} \Rightarrow & F(z) = \frac{(z - 1.53946)(z + 1.53946)(z^2 + 0.703257)}{(z - 1.29099)(z + 1.29099)}, \\ L_2 \Rightarrow & F(z) = \frac{(z-1)(z+1)(2z^2+1)}{2z^2 - 1} \end{aligned}$$

and

$$L_4 \Rightarrow F(z) = \frac{4z^4 - z^2 - 1}{(2z-1)(2z+1)}.$$

Figures 5–7 show the behavior of (3.2) for $J_\kappa(z), \Upsilon_\kappa(z)$ and $L_\kappa(z)$, respectively.

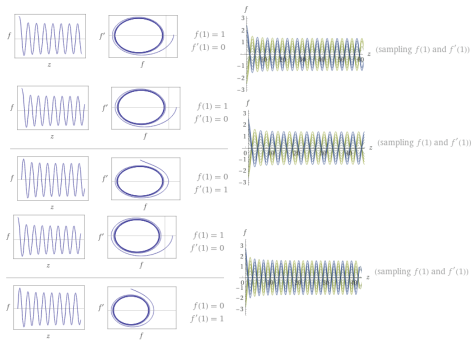

Figure 5. The plot of (3.2) by using $J_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 1.5, 2, 4$, respectively

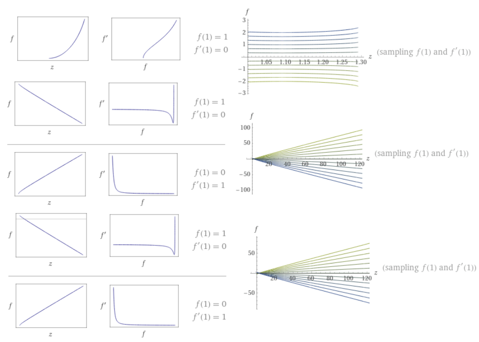

Figure 6. The plot of (3.2) using $\Upsilon_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

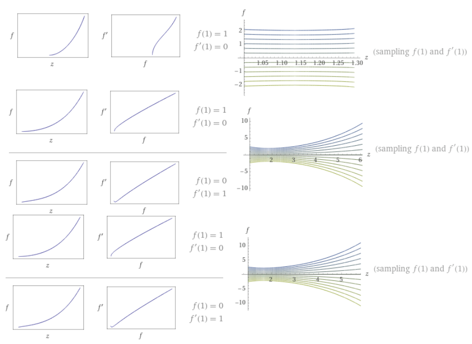

Figure 7. The plot of (3.2) using $L_\kappa(z)$ for $\lambda(z) = 1$, $\kappa = 0.6, 2, 4$, respectively

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[1]IEEE: 94086547, Kuala Lumpur, 59200, Malaysia
Email address: rabhaibrahim@yahoo.com

[2]Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey

[3]Institute of Space Sciences, R76900 Magurele-Bucharest, Romania

[4]Department of Medical Research, China Medical University, Taichung 40402, Taiwan
Email address: dumitru@cankaya.edu.tr