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WELL-POSEDNESS AND ASYMPTOTIC STABILITY OF A NON-LINEAR POROUS SYSTEM WITH A DELAY TERM

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Abstract. Our interest in this work is to treat a one-dimensional Porous system with a non-linear damping and a delay in the non-linear internal feedback. We prove the global existence and uniqueness of its solution in suitable function spaces by means of the Faedo-Galerkin procedure combined with the energy method under a suitable relation between the weight of the delayed feedback and the weight of the non-delayed feedback. Also, we give an explicit and general decay rate estimate by applying the well-known multiplier method integrated with some properties of convex functions and for two opposites cases with respect to the speeds of wave propagation.

1. Introduction

In the present paper, we study the well-posedness and asymptotic behavior of solutions of the following Porous system

(1.1)

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ \rho_2\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x,t-\tau(t))) = 0, \quad \text{in }]0,1[\times]0,\infty[, \\ u(0,t) = u(1,t) = \phi(0,t) = \phi(1,t) = 0 \quad \text{in }]0,\infty[, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in }]0,1[, \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \text{in }]0,1[, \\ \phi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), \quad \text{in }]0,1[\times]0,\tau[, \end{cases}$$

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where x denotes the space variable, t is the time variable, $\tau(\cdot) > 0$ is a time varying delay, μ_1 is a positive constant and μ_2 is a real number. The functions $u = u(x,t)$ and $\phi = \phi(x,t)$ represent, respectively, the displacement of the solid elastic material and the volume fraction and the initial data $(u_0, u_1, \phi_0, \phi_1, f_0)$ belongs to a suitable Sobolev space. The original Porous system is governed by the following evolution equations

$$\begin{aligned} \rho_1 u_{tt} =& T_x, \\ \rho_2 \phi_{tt} =& H_x + G, \end{aligned}$$

where T, H and G denote, respectively, the stress, the equilibrated stress and the equilibrated body force. The constitutive equations are as follows

$$T = \kappa u_x + b\phi, \quad H = \delta \phi_x, \quad G = -bu_x - \xi\phi,$$

where ρ_1, ρ_2, κ, b, δ and ξ are positive constants satisfying in the one-dimensional case, the following inequality

$$\kappa\xi > b^2.$$

If we consider $\kappa = b = \xi$, we find the well-known Timoshenko system which is introduced by S.Timoshenko [17] and it has been widely considered in the literature. For the better comprehension of our motivation, we appeal to keep in mind that the system

$$\begin{cases} \rho_1 u_{tt} - \kappa(u_{xx} - \phi_x) = 0, & \text{in }]0, L[\times]0, \infty[, \\ \rho_2 \phi_{tt} - \delta\phi_{xx} + k(u_x + \phi) = 0, & \text{in }]0, L[\times]0, \infty[, \end{cases} \tag{1.2}$$

is conservative. Namely, by taking any suitable boundary conditions into consideration, the energy of (1.2) given by

$$E(t) = \frac{1}{2}\int_0^L \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa(u_x + \phi)^2 + \delta\phi_x^2\Big]dx,$$

satisfies the energy's conservation property, that is, for all $t > 0$, $E(t) = E(0)$. In this vein, various damping such as viscoelastic damping, frictional damping and thermal dissipation are employed to stabilize the vibrations. It has been shown that the stability depends on the position and nature of the controls and some relations between the constants ρ_1, ρ_2, κ and δ. Let us recall some known results on the stability of the Timoshenko system with frictional dampings. Soufyane and Wehbe [16] used the unique damping $a(x)\phi_t$ in the shear angle displacement and showed that the solution is uniformly stable. This one has been obtained in the case of the equal-speeds, i.e.,

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}. \tag{1.3}$$

Raposo et al. [15] examined (1.2) by setting two linear frictional dampings u_t and ϕ_t where they realized an exponential decay result without imposing any condition on the coefficients. In [1], Alabau Boussouira extended [16] to a problem with a non-linear damping acting in the second equation. Under the condition (1.3), she established a general and semi-explicit formula for the decay rate of the solutions. This result was later improved by Mustafa and Messaoudi [11] where they obtained a general and explicit decay estimate. In the other hand, for the Porous system, Quintanilla [13] proved that the damping $a\phi_t$ is not strong enough to obtain the exponential stability result. However, Apalara [3] got the exponential decay of the solutions for the same problem provided (1.3) holds true. Furthermore, in the nonequal-speeds case, he [3] established a general decay result when he employed a weak non-linear damping $\mu(t)g(\phi_t)$.

In the recent years, the Timoshenko system with time delay has been discussed by several researchers. In particular, we consider the following model with a delay term

$$\begin{cases} \rho_1 u_{tt} - \kappa(u_{xx} - \phi_x) + a_1 f_1(u_t) + a_2 f_2(u_t(x, t - \tau(t))) = 0, & \text{in }]0, L[\times]0, \infty[, \\ \rho_2 \phi_{tt} - \delta\phi_{xx} + \kappa(u_x + \phi) + \mu_1 g_1(\phi_t) + \mu_2 g_2(\phi_t(x, t - \tau(t))) = 0, & \text{in }]0, L[\times]0, \infty[. \end{cases} \tag{1.4}$$

Here, f_i and g_i are real functions, a_i and μ_i are positive numbers for $i = 1, 2$. If $a_i = 0$, $g_i(x) = x$ and $\mu_2 < \mu_1$, then the exponential stability has been proved by Kiran et al. [6] in the case of equal-speeds. In the case of a constant delay, Apalara [2] considered (1.4) when $\mu_i = 0$, $f_i(x) = x$ and $a_2 < a_1$ and established an exponential stability result provided $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$. In the opposite case, only a polynomial decay is obtained. As far as we know, the first work investigated the Timoshenko beam with a nonlinear delay term is the one of Benaissa and Bahlil [5]. The problem treated is (1.4) with $a_i = 0$. They considered only the equal-speeds case where they obtained an explicit decay estimate under a suitable relation between μ_1 and μ_2 and some additional assumptions. For the Porous system with delay term, the subject of this article, we cite the works [10, 14] and [7]. The authors of [7] examined a non-linear Porous system of the form

$$\begin{cases} \rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, & \text{in }]0, 1[\times]0, \infty[, \\ \rho_2 \phi_{tt} - \delta\phi_{xx} + b u_x + \xi\phi + \mu_1 \phi_t + \mu_2 \phi_t(x, t - \tau) + \alpha(t) g(\phi_t) = 0, & \text{in }]0, 1[\times]0, \infty[, \end{cases}$$

and established, under the assumption $|\mu_2| < \mu_1$, a general decay of solution when $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$.

As a consequence of the works cited above, if only one equation of a Timoshenko system is damped then the uniform stability may be achieved for weak solutions if and only if $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$. However, in the situation when $\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}$, a weaker decay rate result is achieved for strong solutions. According to this results, three questions naturally arise.

1. Is it possible to consider the Porous system with a non-linear damping term and a time varying delay in the internal feedback acting only in the second equation and get the same result as in the Timoshenko system?

2. In the equal-speeds case, is it possible to get the stability result with same hypotheses on μ_1, μ_2, g_1 and g_2 as in the Timoshenko system?

3. As we have mentioned above, the nonequal-speeds case is not considered for the non-linear Timoshenko system with delay (see [5]). So, is it possible to obtain the stability result under the same conditions imposed for the equal-speeds case?

The main aim of this manuscript is to give positive answers to theses three questions by investigating (1.1).

The rest of our paper is as follows. In the next section, we provide some assumptions and materials needed in our work. In Section 3, we state and prove the existence and the uniqueness results. The last section is devoted to the study of the asymptotic behavior of the solutions. We use c throughout this paper to denote a generic fixed positive constant, which may be different in different estimates.

2. Preliminaries

In this section, we present some assumptions, materials and notations that will be used later. Firstly, following the same arguments of Nicaise and Pignotti [12], we introduce the new variable

$$z(x,\rho,t)=\phi_t(x,t-\rho\tau(t)),\quad x\in[0,1],\rho\in[0,1],t>0.$$

It is clear that

$$\tau(t)z_t(x,\rho,t)+(1-\rho\tau'(t))z_\rho(x,\rho,t)=0,\quad \text{in } ([0,1])^2\times[0,\infty].$$

Hence, our problem (1.1) becomes

(2.1)
$$\begin{cases}\rho_1u_{tt}-\kappa u_{xx}-b\phi_x=0, & \text{in }]0,1[\times]0,\infty[,\\ \rho_2\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\mu_1g_1(\phi_t)+\mu_2g_2(z(x,1))=0, & \text{in }]0,1[\times]0,\infty[,\\ \tau(t)z_t(x,\rho,t)+(1-\rho\tau'(t))z_\rho(x,\rho,t)=0, & \text{in } (]0,1[)^2\times]0,\infty[,\\ u(0,t)=u(1,t)=\phi(0,t)=\phi(1,t)=0, & \text{in }]0,\infty[,\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & \text{in }]0,1[,\\ \phi(x,0)=\phi_0(x),\quad \phi_t(x,0)=\phi_1(x), & \text{in }]0,1[,\\ z(x,\rho,0)=f_0(x,-\rho\tau(0)), & \text{in } (]0,1[)^2.\end{cases}$$

In order to deal with the new variable z, we define the Hilbert space

$$L^2_z(0,1)=L^2\Big(0,1;L^2(0,1)\Big)=\Big\{z:]0,1[\to L^2(0,1),\ \int_0^1\int_0^1 z^2(x,\rho)d\rho dx<\infty\Big\},$$

which endowed with the inner product

$$(z,\widetilde{z})=\int_0^1\int_0^1 z(x,\rho,t)\widetilde{z}(x,\rho,t)d\rho dx.$$

We consider now the following assumptions.

(A_1) $g_1:\mathbb{R}\to\mathbb{R}$ is a strictly increasing function of class C^1 and $g_2:\mathbb{R}\to\mathbb{R}$ is an increasing function of class C^1 such that it exist $\epsilon<1$, c_1, c_2 and a convex and non-decreasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

(2.2)
$$\begin{cases} H(0) = 0 \text{ and } H \text{ is linear on } [0,\epsilon] \textbf{ or } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\epsilon] \text{ such that} \\ c_1|s_1| \le |g_1(s_1)| + |g_2(s_2)| \le c_2\Big(|s_1| + |s_2|\Big), \quad \text{if } |s_1| + |s_2| \ge \epsilon, \\ s_1^2 + g_1^2(s_1) + g_2^2(s_2) \le H^{-1}\Big(s_1 g_1(s_1) + s_2 g_2(s_2)\Big), \quad \text{if } |s_1| + |s_2| \le \epsilon. \end{cases}$$

Also, for any $s \in \mathbb{R}$, we assume that it exist some positive constants $\widetilde{c}_2$, α_1 and α_2 satisfying

$$|g_2'(s)| \le \widetilde{c}_2 \tag{2.3}$$

and

$$\alpha_1 s g_2(s) \le G(s) \le \alpha_2 s g_1(s), \tag{2.4}$$

where G is a primitive of g_2.

(A$_2$) τ is a function in $W^{2,\infty}([0,T])$, $T > 0$, such that

$$\begin{cases} 0 < \tau_0 \le \tau(t) \le \tau_1, \quad \text{for all } t > 0, \\ \tau'(t) \le \theta < 1, \quad \text{for all } t > 0, \end{cases}$$

where τ_0 and τ_1 are a positive numbers.

(A$_3$) With respect to the weights of feedbacks μ_i, $i = 1, 2$, we assume that

$$|\mu_2| < \frac{\alpha_1(1-\theta)}{\alpha_2(1-\alpha_1\theta)}\mu_1.$$

We define the energy associated with the solution of (2.1) as

(2.5)
$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi + 2\tau(t)\gamma\int_0^1 G(z(x,\rho))d\rho\right]dx,$$

where γ is a positive number such that

$$\frac{(1-\alpha_1)|\mu_2|}{\alpha_1(1-\theta)} < \gamma < \frac{\mu_1 - \alpha_2|\mu_2|}{\alpha_2}.$$

Remark 2.1. The energy functional $E(t)$ defined in (2.5) is positive. In fact, we can easily show that

$$\kappa u_x^2 + 2bu_x\phi + \xi\phi^2 = \frac{1}{2}\left[\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2 + 2\kappa_1 u_x^2 + 2\xi_1\phi^2\right],$$

where $2\kappa_1 = \kappa - \frac{b^2}{\xi}$ and $2\xi_1 = \xi - \frac{b^2}{\kappa}$ are positives from $\kappa\xi > b^2$. Thus,

$$\kappa u_x^2 + 2bu_x\phi + \xi\phi^2 > \frac{1}{2}\left[\kappa\Big(u_x + \frac{b}{\kappa}\phi\Big)^2 + \xi\Big(\phi + \frac{b}{\xi}u_x\Big)^2\right] > 0,$$

which implies the positivity of $E(t)$ and

$$E(t) > \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2\phi_t^2 + \kappa_1 u_x^2 + \xi_1\phi^2 + 2\gamma\tau(t)\int_0^1 G(z(x,\rho))d\rho\right]dx. \tag{2.6}$$

Remark 2.2. • The strict non-decreasing property of g_1 implies the existence of a positive constant $\widetilde{c}_1$ satisfying

$$\widetilde{c}_1 < g_1'(s). \tag{2.7}$$

• Assumption (2.2) implies that $s_1 g_1(s_1) + s_2 g_2(s_2) > 0$ for all $s_1, s_2 \in \mathbb{R}$.

• By the mean value theorem for integrals and the monotonicity of g_2, we deduce that

$$G(s) = \int_0^s g_2(\sigma) d\sigma \leq s g_2(s),$$

then $\alpha_1 < \alpha_2 \leq 1$.

Remark 2.3. Let Ψ^* be the conjugate function of the differential convex function Ψ, i.e.,

$$\Psi^*(s) = \sup(st - \Psi(t)),$$

then Ψ^* is the Legendre transform of Ψ, which is given by (see Arnold [4])

$$\Psi^*(s) = s(\Psi')^{-1}(s) - \Psi[(\Psi')^{-1}(s)], \quad \text{if } s \in [0, \Psi'(r)],$$

satisfies the generalized Young inequality

$$AB \leq \Psi^*(A) + \Psi(B), \quad \text{if } A \in [0, \Psi'(r)], B \in [0, r]. \tag{2.8}$$

A starting point will be to give a derivative's upper bounded of the functional E_1 defined as

$$E_1(t) = E(t) + \varepsilon \int_0^1 \int_0^1 z^2(x, \rho) d\rho dx, \quad \text{for } \varepsilon \geq 0. \tag{2.9}$$

Lemma 2.1. *For any $\varepsilon \geq 0$, the functional E_1 satisfies along the solution of* (2.1) *the following estimate*

$$E_1'(t) \leq -\beta_1 \int_0^1 \phi_t g_1(\phi_t) dx - \beta_2 \int_0^1 z(x,1) g_2(z(x,1)) dx + \varepsilon \int_0^1 \phi_t^2 dx - \varepsilon \int_0^1 z^2(x,1) dx, \tag{2.10}$$

where $\beta_1 = \mu_1 - \gamma\alpha_2 - \alpha_2|\mu_2|$ and $\beta_2 = \gamma(1-\theta)\alpha_1 - (1-\alpha_1)|\mu_2|$.

Proof. Multiplying $(2.1)_1$ and $(2.1)_2$ by u_t and ϕ_t, respectively, and using integration by parts over $[0,1]$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1 u_t^2 + \rho_2 \phi_t^2 + \kappa u_x^2 + \delta \phi_x^2 + \xi \phi^2 + 2b u_x \phi\Big] dx \\ &+ \mu_1 \int_0^1 \phi_t g_1(\phi_t) dx + \mu_2 \int_0^1 \phi_t g_2(z(x,1)) dx = 0. \end{aligned} \tag{2.11}$$

Multiplying $(2.1)_3$ by $\gamma g_2(z(x,\rho))$ and integrating the product over $([0,1])^2$, we get

$$\gamma\tau(t) \int_0^1 \int_0^1 z_t(x,\rho) g_2(z(x,\rho)) d\rho dx + \gamma(1-\rho\tau'(t)) \int_0^1 \int_0^1 z_\rho(x,\rho) g_2(z(x,\rho)) d\rho dx = 0.$$

This means that

$$\gamma \frac{d}{dt} \int_0^1 \int_0^1 \tau(t) G(z(x,\rho)) d\rho dx + \gamma \int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\Big((1-\rho\tau'(t)) G(z(x,\rho))\Big) d\rho dx = 0.$$

Consequently, using the fact that $z_t(x,0,t)=\phi_t$, we get

$$\gamma\frac{d}{dt}\int_0^1\int_0^1\tau(t)G(z(x,\rho)d\rho dx=-\gamma\int_0^1\Big[(1-\tau'(t))G(z(x,1))-G(\phi_t)\Big]dx. \tag{2.12}$$

Also, we have

$$\varepsilon\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho)d\rho dx=-\varepsilon\int_0^1\Big[z^2(x,1)-\phi_t^2\Big]dx. \tag{2.13}$$

The last equality has been obtained by applying the same previous arguments and after multiplying $(2.1)_3$ by $2\varepsilon z(x,\rho)$. Combining the estimates (2.11)–(2.13) and using (2.4), we get

$$\begin{aligned}E_1'(t)\leq&-(\mu_1-\gamma\alpha_2)\int_0^1\phi_t g_1(\phi_t)dx-\gamma(1-\theta)\alpha_1\int_0^1 z(x,1)g_2(z(x,1))dx\\&-\varepsilon\int_0^1 z^2(x,1)dx+\varepsilon\int_0^1\phi_t^2dx-\mu_2\int_0^1\phi_t g_2(z(x,1))dx.\end{aligned} \tag{2.14}$$

From Remark 2.3, we have

$$G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s)),\quad \text{for all } s\geq 0.$$

Hence,

$$G^*(g_2(z(x,1)))=z(x,1)g_2(z(x,1)-G(z(x,1)).$$

Taking (2.8) with $A=g_2(z(x,1))$ and $B=\phi_t$, and using (2.4) again, we obtain

$$\mu_2\phi_t g_2(z(x,1))\leq\alpha_2|\mu_2|\phi_t g_1(\phi_t)+(1-\alpha_1)|\mu_2|z(x,1)g_2(z(x,1)). \tag{2.15}$$

By inserting (2.15) into (2.14), we arrive at the desired inequality. This finishes the proof. □

3. The Well-posedness of the Probem

In the current section, we prove the existence and the uniqueness results to system (2.1). Firstly, we prove the existence of a unique strong solution, next, using a density argument, we extend the obtained result for weak solutions. For this, let $U=U(t)=\Big(u,u_t,\phi,\phi_t,z)^T$ and $U_0=U(0)=(u_0,u_1,\phi_0,\phi_1,f_0(\cdot,-\cdot\,\tau(0))\Big)^T$. We then consider the following spaces

$$\mathcal{H}=H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L_z^2(0,1)$$

and

$$\mathcal{H}_0=\Big(H^2\cap H_0^1(0,1)\Big)\times H_0^1(0,1)\times\Big(H^2\cap H_0^1(0,1)\Big)\times H_0^1(0,1)\times L^2\Big(0,1;H^1(0,1)\Big).$$

Our first main result is given by the following theorem.

Theorem 3.1. *Assuming that the assumptions (A_1)-(A_3) hold and that $\kappa\xi>b^2$. Then for any $U\in\mathcal{H}$ satisfying the compatibility condition*

$$f_0(\cdot,0)=\phi_1,$$

problem (2.1) *admits only one global weak solution*

$$U \in C\Big([0,+\infty);\mathcal{H}\Big).$$

Moreover, if $U_0 \in \mathcal{H}_0$*, the solution of* (2.1) *is strong solution, and satisfies*

$$U \in C\Big([0,+\infty);\mathcal{H}_0\Big) \cap C^1([0,+\infty);\mathcal{H}\Big).$$

Proof. The proof will be established by implementing the Faedo-Galerkin method. For, let $U \in \mathcal{H}_0$, $T > 0$ be fixed and for $m = 1, 2, \dots$, let $\{\Phi^m\}_{m\in\mathbb{N}}$ be a Hilbertian basis of $H_0^1(0,1)$ and F^m the vector space generated by $\Phi^1, \Phi^2, \dots, \Phi^m$. Defining, for $1 \leq i \leq m$, the sequence $\Psi^i(x,\rho)$ as

$$\Psi^i(x,0) = \Phi^i(x).$$

Then, we may extend $\Psi^i(x,0)$ by $\Psi^i(x,\rho)$ over $L_z^2(0,1)$ and denote Z^m the space generated by $\Psi^1, \Psi^2, \dots, \Psi^m$. We will construct an approximate solution (u^m, ϕ^m, z^m), $i = 1, 2, \dots$, in the form

$$(u^m(x,t), \phi^m(x,t)) = \left(\sum_{i=1}^m c^{im}(t), \sum_{i=1}^m d^{im}(t)\right) \Phi^i(x),$$

$$z^m(x,\rho) = \sum_{i=1}^m e^{im}(t)\Psi^i(x,\rho),$$

where c^{im}, d^{im} and e^{im}, $i = 1, 2, \dots, m$, are determined by the following finite dimensional problem

$$\begin{cases} \Big(\kappa u_x^m + b\phi^m, \Phi_x^i\Big) + \Big(\rho_1 u_{tt}^m, \Phi^i\Big) = 0, \\ \Big(\delta\phi_x^m, \Phi_x^i\Big) + \Big(\rho_2\phi_{tt}^m + bu_x^m + \xi\phi^m + \mu_1 g_1(\phi_t^m) + \mu_2 g_2(z^m(\cdot,1)), \Phi^i\Big) = 0, \\ \Big(\tau(t) z_t^m(\cdot,\rho) + (1-\rho\tau'(t)) z_\rho^m(\cdot,\rho), \Psi^i(\cdot,\rho)\Big) = 0, \end{cases} \tag{3.1}$$

with

$$\begin{aligned} u^m(\cdot,0) =& u_0^m = \sum_{i=1}^m (u_0, \Phi^i)\Phi^i \to u_0, \quad \text{in } H^2 \cap H_0^1(0,1), \\ u_t^m(\cdot,0) =& u_1^m = \sum_{i=1}^m (u_1, \Phi^i)\Phi^i \to u_1, \quad \text{in } H_0^1(0,1), \\ \phi^m(0) =& \phi_0^m = \sum_{i=1}^m (\phi_0, \Phi^i)\Phi^i \to \phi_0, \quad \text{in } H^2 \cap H_0^1(0,1), \\ \phi_t^m(\cdot,0) =& \phi_1^m = \sum_{i=1}^m (\phi_1, \Phi^i)\Phi^i \to \phi_1, \quad \text{in } H_0^1(0,1), \\ z^m(\cdot,\cdot,\cdot,0) =& z_0^m = \sum_{i=1}^m (f_0, \Psi^i)\Psi^i \to f_0, \quad \text{in } L^2\Big(0,1; H^1(0,1)\Big), \end{aligned} \tag{3.2}$$

as $m \to +\infty$.

The standard methods of ODEs give the existence of a unique solution of (3.1) on the inertval $[0, T_m]$, $0 < T_m < T$. In the next step, we will prove that T_m is independent of m. In other words, the approximate solution becomes global and defined for all $t > 0$.

1.The first priori estimate. As for Lemma 2.1, the functional

$$\begin{aligned} E_1^m(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1|u_t^m|^2 + \rho_2|\phi_t^m|^2 + \kappa|u_x^m|^2 + \delta|\phi_x^m|^2 + \xi|\phi^m|^2 + 2bu_x^m\phi^m \\ &+ 2\gamma\tau(t)\int_0^1 G(z^m(x,\rho))d\rho + 2\varepsilon\int_0^1 |z^m(x,\rho)|^2 d\rho\Big]dx \end{aligned}$$

satisfies, for any $\varepsilon \geq 0$,

$$\begin{aligned} &(E_1^m(t))' + \beta_1\int_0^1 \phi_t^m g_1(\phi_t^m)dx + \beta_2\int_0^1 z^m(x,1)g_2(z^m(x,1))dx \\ &+ \varepsilon\int_0^1 |z^m(x,1)|^2 dx \leq \varepsilon\int_0^1 |\phi_t^m|^2 dx. \end{aligned}$$

Choosing $\varepsilon > 0$, then integrating over $[0, t]$ and taking the convergences (3.2) into account, we get

$$\begin{aligned} &E_1^m(t) + \beta_1\int_0^t\int_0^1 \phi_t^m g_1(\phi_t^m)dxdt \\ &+ \beta_2\int_0^t\int_0^1 z^m(x,1)g_2(z^m(x,1))dxdt + \varepsilon\int_0^t\int_0^1 |z^m(x,1)|^2 dxdt \\ \leq& c + \varepsilon\int_0^t\int_0^1 |\phi_t^m|^2 dxdt. \end{aligned}$$

The Gronwall's Lemma yields the following first priori estimate

$$\begin{aligned} E_1^m(t) &+ \int_0^t\int_0^1 \phi_t^m g_1(\phi_t^m)dxdt + \int_0^t\int_0^1 z^m(x,1)g_2(z^m(x,1))dxdt \\ &+ \int_0^t\int_0^1 |z^m(x,1)|^2 dxdt \leq c. \end{aligned} \tag{3.3}$$

This estimate gives us the global existence of (u^m, ϕ^m, z^m) in $[0, +\infty)$ and

$$\begin{aligned} &z^m \quad \text{is uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty; L^2_z(0,1)\Big), \\ &u^m, \phi^m \quad \text{are uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty; H^1_0(0,1)\Big), \\ &u_t^m, \phi_t^m \quad \text{are uniformly bounded in} \quad L^\infty_{\text{loc}}\Big(0,\infty, L^2(0,1)\Big), \\ &\phi_t^m g_1(\phi_t^m) \quad \text{is uniformly bounded in} \quad L^1\Big((0,T)\times(0,1)\Big), \\ &z^m(x,1)g_2(z^m(x,1)) \quad \text{is uniformly bounded in} \quad L^1\Big((0,T)\times(0,1)\Big). \end{aligned}$$

2. The second priori estimate. Firstly, we are going to estimate $u_{tt}^m(0)$ and $\phi_{tt}^m(0)$ in the L^2-norm. Also, we need to estimate $z_t^m(x,\rho,0)$ in the L^2_z-norm. For that, we replace Φ^i in $(3.1)_1$ by u_{tt}^m, Φ^i in $(3.1)_2$ by ϕ_{tt}^m and using Young's inequality to get

$$\begin{aligned}(3.4)\quad \int_0^1 \Big[|u_{tt}^m(0)|^2+|\phi_{tt}^m(0)|^2\Big]dx \leq & c\int_0^1 \Big[|u_{xx}^m(0)|^2+|u_x^m(0)|^2+|\phi_{xx}^m(0)|^2+|\phi_x^m(0)|^2\\ & +|\phi^m(0)|^2+g_1^2(\phi_t^m(0))+g_2^2(z^m(x,1,0))\Big]dx.\end{aligned}$$

Replacing Ψ^i in $(2.1)_3$ by $z_t^m(x,\rho,t)$ and using Cauchy-Schwarz and Young's inequalities, we get

$$(3.5)\qquad \int_0^1\int_0^1 |z_t^m(x,\rho,0)|^2 d\rho dx \leq c\int_0^1\int_0^1 |z_\rho^m(x,\rho,0)|^2 d\rho dx.$$

The sum of (3.4)–(3.5) with (3.2) yields

$$(3.6)\qquad \int_0^1 \Big[|u_{tt}^m(0)|^2+|\phi_{tt}^m(0)|^2+\int_0^1 |z_t^m(x,\rho,0)|^2 d\rho\Big]dx \leq c.$$

Now, we derivate $(3.1)_1$ and $(3.1)_2$ with respect to t. Then, we set $\Phi^i = 2u_{tt}^m$ and $\Phi^i = 2\phi_{tt}^m$, respectively, in the first and the second resulting equations and using the non-decreasing property of g_1, we find

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2+\rho_2|\phi_{tt}^m|^2+\kappa|u_{xt}^m|^2+\delta|\phi_{xt}^m|^2+\xi|\phi_t^m|^2+2bu_{xt}^m\phi_t^m\Big]dx\\ \leq & -\mu_2\int_0^1 z_t^m(x,1)g_2'(z^m(x,1))\phi_{tt}^m dx.\end{aligned}$$

The boundedness of g_2' and the Young's inequality imply that

$$\begin{aligned}(3.7)\qquad &\frac{d}{dt}\int_0^1 \Big[\rho_1|u_{tt}^m|^2+\rho_2|\phi_{tt}^m|^2+\kappa|u_{xt}^m|^2+\delta|\phi_{xt}^m|^2+\xi|\phi_t^m|^2+2bu_{xt}^m\phi_t^m\Big]dx\\ \leq & \epsilon_1\int_0^1 |z_t^m(x,1)|^2 dx + c\int_0^1 |\phi_{tt}^m|^2 dx.\end{aligned}$$

In the other hand, taking the derivative of $(3.1)_3$ with respect to t and then setting $\Psi^i = 2z_t^m(x,\rho,t)$ in the resulting equation, it follows that

$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))}|z_t^m(x,\rho,t)|^2 d\rho dx+\int_0^1\int_0^1 \Big(\frac{\tau(t)}{(1-\rho\tau'(t))}\Big)'|z_t^m(x,\rho,t)|^2 d\rho dx\\ &+\int_0^1\int_0^1 \frac{d}{d\rho}|z_t^m(x,\rho,t)|^2 d\rho dx = 0.\end{aligned}$$

As $z_t^m(x,0,t) = \phi_{tt}^m(x,t)$, it comes

(3.8)
$$\begin{aligned}&\frac{d}{dt}\int_0^1\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))}|z_t^m(x,\rho,t)|^2 d\rho dx+\int_0^1\int_0^1 \Big(\frac{\tau(t)}{(1-\rho\tau'(t))}\Big)'|z_t^m(x,\rho,t)|^2 d\rho dx\\ &+\int_0^1 |z_t^m(x,1,t)|^2 d\rho dx = \int_0^1 |\phi_{tt}^m|^2 dx.\end{aligned}$$

Let I^m be defined by

$$I^m(t) = \int_0^1 \Big[\rho_1|u_{tt}^m|^2 + \rho_2|\phi_{tt}^m|^2 + \kappa|u_{xt}^m|^2 + \delta|\phi_{xt}^m|^2 + \xi|\phi_t^m|^2 + 2bu_{xt}^m\phi_t^m + \frac{\tau(t)}{(1-\rho\tau'(t))}\int_0^1 |z_t^m(x,\rho)|^2 d\rho\Big]dx,$$

hence from the estimates (3.7)–(3.8), we find

$$(I^m(t))' + (1-\epsilon_1)\int_0^1 |z_t^m(x,1))|^2 dx \leq c\int_0^1 |\phi_{tt}^m|^2 dx.$$

Choosing $\epsilon_1 < 1$, then integrating over $[0,t]$, we get

$$I^m(t) + \int_0^t\int_0^1 |z_t^m(x,1)|^2 dxdt \leq cI^m(0) + c\int_0^t\int_0^1 |\phi_{tt}^m|^2 dxdt.$$

Employing Gronwall's lemma with (3.2) and (3.6), we obtain the second estimate below

$$I^m(t) + \int_0^t\int_0^1 |z_t^m(x,1))|^2 dxdt \leq c. \tag{3.9}$$

We, therefore, deduce that

$$\begin{aligned}
&z_t^m \quad \text{is uniformly bounded in} \quad L^2\Big(0,T;L_z^2(0,1)\Big),\\
&u_t^m, \phi_t^m \quad \text{are uniformly bounded in} \quad L^\infty_{\mathrm{loc}}\Big(0,\infty;H_0^1(0,1)\Big),\\
&u_{tt}^m, \phi_{tt}^m \quad \text{are uniformly bounded in} \quad L^\infty_{\mathrm{loc}}\Big(0,\infty;L^2(0,1)\Big),
\end{aligned}$$

Hence it follows from the estimates (3.3) and (3.9) that it exist subsequences $\{u^n\}_{n=1}^\infty \subset \{u^m\}_{m=1}^\infty$, $\{\phi^n\}_{n=1}^\infty \subset \{\phi^m\}_{m=1}^\infty$ and $\{z^n\}_{n=1}^\infty \subset \{z^m\}_{m=1}^\infty$ verify for all $T \geq 0$ the following convergences

$$\begin{cases}
g_1(\phi_t^n) \to f \quad \text{and} \quad g_2(z^n) \to h \quad \text{weakly-star in} \quad L^2\Big(0,T;L^2\Big),\\
u^n \to u \quad \text{and} \quad \phi^n \to \phi \quad \text{weakly-star in} \quad L^2\Big(0,T;H_0^1\Big),\\
u_t^n \to u_t \quad \text{and} \quad \phi_t^n \to \phi_t \quad \text{weakly-star in} \quad L^\infty\Big(0,T;H_0^1\Big),\\
u_{tt}^n \to u_{tt} \quad \text{and} \quad \phi_{tt}^n \to \phi_{tt} \quad \text{weakly-star in} \quad L^\infty\Big(0,T;L^2\Big),\\
z^n \to z \quad \text{and} \quad z_t^n \to z_t \quad \text{weakly-star in} \quad L^\infty\Big(0,T;L_z^2\Big),
\end{cases} \tag{3.10}$$

We will show that (u,ϕ,z) is a strong solution of system (2.1). Firstly, we prove that $f = g_1(\phi_t)$ and $h = g_2(z(x,1))$ which will be given in the following lemma.

Lemma 3.1. *For each $T > 0$, $g_1(\phi_t^n) \to g_1(\phi_t)$ weakly-star in $L^2\Big((0,1)\times(0,T)\Big)$ and $g_2(z^n(x,1)) \to g_2(z(x,1))$ weakly-star in $L^2\Big((0,1)\times(0,T)\Big)$.*

Proof. From (3.9), we have ϕ_t^n is bounded in $L^\infty(0,T;H_0^1)$ and ϕ_{tt}^n is bounded in $L^\infty(0,T;L^2)$. Then, the injection by continuity in L^p gives us the boundedness of ϕ_t^n in $L^2(0,T;H_0^1)$ and ϕ_{tt}^n in $L^2(0,T;L^2)$. Hence, ϕ_t^n is bounded in $H^1(Q)$, where

$Q = (0,1) \times (0,T)$. It is known that the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact. This permit us to extract a subsequence ϕ^n, still represented by the same notation, such that

$$\phi_t^n \to \phi_t \quad \text{strongly in} \quad L^2\Big(0,T;L^2(0,1)\Big),$$

which gives

$$\phi_t^n \to \phi_t, \quad \text{a.e. on } Q.$$

Then, by the continuity of g_1,

$$g_1(\phi_t^n) \to g_1(\phi_t), \qquad \text{a.e. on } Q. \tag{3.11}$$

Similarly,

$$g_2(z^n(x,1)) \to g_2(z(x,1)), \quad \text{a.e. on } Q. \tag{3.12}$$

On the other hand, with $\mathcal{R}^m(x,t)$ defined as

$$\mathcal{R}^m(x,t) = \phi_t^m g_1(\phi_t^m) + z^m(x,1) g_2(z^m(x,1)),$$

we assert by using Jensen's inequality and the concavity of H^{-1} that

$$\begin{aligned} \int_0^1 H^{-1}\Big(\mathcal{R}^m(x,t)\Big)dx \leq & cH^{-1}\left(\int_0^1 \mathcal{R}^m(x,t)dx\right) \\ \leq & cH^*(1) + c\int_0^1 \mathcal{R}^m(x,t)dx. \end{aligned} \tag{3.13}$$

For $r^m = |\phi_t^m| + |z^m(x,1)|$, we write

$$\begin{aligned} \int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \leq & \int_{r^m \leq \epsilon} \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \\ & + \int_{r^m \geq \epsilon} \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx. \end{aligned}$$

Then, by using (2.2) and (3.13), we get

$$\int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dx \leq cH^*(1) + c\int_0^1 \mathcal{R}^m(x,t)dx.$$

Thus, by (3.3), it results

$$\int_0^t \int_0^1 \Big[g_1^2(\phi_t^m) + g_2^2(z^m(x,1))\Big]dxdt \leq c,$$

which implies that $g_1(\phi_t^n), g_2(z^n(x,1)) \in L^2(Q)$. Combining these with (3.11)–(3.12) and using Lemma 1.3 in [9] page 12, we derive to

$$g_1(\phi_t^n) \to g_1(\phi_t) \quad \text{weakly-star in} \quad L^2((0,1)\times(0,T)),$$

$$g_2(z^n(x,1)) \to g_2(z(x,1)) \quad \text{weakly-star in} \quad L^2((0,1)\times(0,T)).$$

This shows that $f = g_1(\phi_t)$ and $h = g_2(z(x,1))$. $\square$

Passage to the limit. To prove that (u,ϕ,z) is a strong solution of problem (2.1) we discuss as in [9]. For this, we consider functions $v,\omega\in C\Big(0,T;H_0^1(0,1)\Big)$ and $y\in C\Big(0,T;L_z^2(0,1)\Big)$ having the forms

$$(v(x,t),\omega(x,t))=\Big(\sum_{i=1}^{N}\widetilde{c}^{in}(t),\sum_{i=1}^{N}\widetilde{d}^{in}(t)\Big)\Phi^i(x), \tag{3.14}$$

$$y(x,\rho,t)=\sum_{i=1}^{N}\widetilde{e}^{in}(t)\Psi^i(x,\rho), \tag{3.15}$$

where $N\geq n$ is a fixed integer.

Then we multiply $(3.1)_1$, $(3.1)_2$ and $(3.1)_3$ by $\widetilde{c}^{in}(t)$, $\widetilde{d}^{in}$ and $\widetilde{c}^{in}$, respectively, and summing the resultants over i from 1 to N, we find that

$$\begin{cases}\displaystyle\int_0^T\int_0^1\Big[\Big(\kappa u_x^n+b\phi^n\Big)v_x+\rho_1u_{tt}^nv\Big]dxdt=0,\\ \displaystyle\int_0^T\int_0^1\Big[\delta\phi_x^n\omega_x+\Big(\rho_2\phi_{tt}^n+bu_x^n+\xi\phi_x^n+\mu_1g_1(\phi_t^n)+\mu_2g_2(z^n(x,1))\Big)\omega\Big]dxdt=0,\\ \displaystyle\int_0^T\int_0^1\int_0^1\Big[\tau(t)z_t^n(x,\rho)+(1-\rho\tau'(t))z_\rho^n(x,\rho)\Big]y(x,\rho)d\rho dxdt=0.\end{cases} \tag{3.16}$$

After passing to the limit in (3.16) as $n\to+\infty$ and using (3.10), we arrive at

$$\begin{cases}\displaystyle\int_0^T\int_0^1\Big[\Big(\kappa u_x+b\phi\Big)v_x+\rho_1u_{tt}v\Big]dxdt=0,\\ \displaystyle\int_0^T\int_0^1\Big[\delta\phi_x\omega_x+\Big(\rho_2\phi_{tt}+bu_x+\xi\phi+\mu_1g_1(\phi_t)+\mu_2g_2(z(x,1))\Big)\omega\Big]dxdt=0,\\ \displaystyle\int_0^T\int_0^1\int_0^1\Big[\tau(t)z_t(x,\rho)+(1-\rho\tau'(t))z_\rho(x,\rho)\Big]y(x,\rho)d\rho dxdt=0.\end{cases} \tag{3.17}$$

The above equations hold for all $(v,\omega,y)\in\Big(L^2\Big(0,T;H_0^1\Big)\Big)^2\times L^2\Big(0,T;L_z^2\Big)$ since the functions of the forms (3.14) and (3.15) are dense, respectively, in $L^2\Big(0,T;H_0^1\Big)$ and $L^2\Big(0,T;L_z^2\Big)$. Next, we must verify that the limit functions u, ϕ, z satisfy the initial conditions, i.e.,

$$u(\cdot,0)=u_0,\quad u_t(\cdot,0)=u_1,\quad \phi(\cdot,0)=\phi_0,\quad \phi_t(\cdot,0)=\phi_1,\quad z(\cdot,0)=f_0. \tag{3.18}$$

For, we let $v,\omega\in C^2\Big(0,T;H_0^1\Big)$ and $y\in C^1(0,T,L_z^2)$ with

$$u(x,T)=u_t(x,T)=\phi(x,T)=\phi_t(x,T)=y(x,\rho,T)=0.$$

Then we integrate with respect to t in (3.17), we get

(3.19)
$$\begin{cases}\int_0^T\int_0^1\Big[\rho_1 uv_{tt}-\Big(\kappa u_x+b\phi\Big)v_x\Big]dxdt+\rho_1\int_0^1\Big[u(0)v_t(0)-u_t(0)v(0)\Big]dx=0,\\ \int_0^T\int_0^1\Big[\rho_2\phi\omega_{tt}+\delta\phi_x\omega_x+\Big(bu_x+\xi\phi_x+\mu_1g_1(\phi_t)+\mu_2g_2(z(x,1))\Big)\omega\Big]dxdt\\ +\rho_2\int_0^1\Big[\phi(0)\omega_t(0)-\phi_t(0)\omega(0)\Big]dx=0,\\ \int_0^T\int_0^1\int_0^1\Big[-z(x,\rho,t)y_t(x,\rho,t)+\dfrac{1-\rho\tau'(t)}{\tau(t)}z_\rho(x,\rho,t)y(x,\rho,t)\Big]d\rho dxdt\\ -\int_0^1\int_0^1 z(x,\rho,0)y(x,\rho,0)d\rho dx=0.\end{cases}$$

Similarly from (3.16), we have

$$\begin{cases}\int_0^T\int_0^1\Big[\rho_1 u^n v_{tt}+\Big(\kappa u^n_x+b\phi^n\Big)v_x\Big]dxdt+\rho_1\int_0^1\Big[u^n(0)v_t(0)-u^n_t(0)v(0)\Big]dx=0,\\ \int_0^T\int_0^1\Big[\rho_2\phi^n\omega_{tt}+\delta\phi^n_x\omega_x+\Big(bu^n_x+\xi\phi^n_x+\mu_1g_1(\phi^n_t)+\mu_2g_2(z^n(x,1))\Big)\omega\Big]dxdt\\ +\rho_2\int_0^1\Big[\phi^n(0)\omega_t(0)-\phi^n_t(0)\omega(0)\Big]dx=0,\\ \int_0^T\int_0^1\int_0^1\Big[z^n(x,\rho,t)y_t(x,\rho,t)+\dfrac{1-\rho\tau'(t)}{\tau(t)}z^n_\rho(x,\rho,t)y(x,\rho,t)\Big]d\rho dxdt\\ -\int_0^1\int_0^1 z^n(x,\rho,0)y(x,\rho,0)d\rho dx=0.\end{cases}$$

Recalling (3.10) and (3.2), we obtain

$$\begin{cases}\int_0^T\int_0^1\Big[\rho_1 uv_{tt}+\Big(\kappa u_x+b\phi\Big)v_x\Big]dxdt+\rho_1\int_0^1\Big[u_0v_t(0)-u_1v(0)\Big]dx=0,\\ \int_0^T\int_0^1\Big[\rho_2\phi\omega_{tt}+\delta\phi_x\omega_x+\Big(bu_x+\xi\phi_x+\mu_1g_1(\phi_t)+\mu_2g_2(z(x,1))\Big)\omega\Big]dxdt\\ +\rho_2\int_0^1\Big[\phi_0\omega_t(0)-\phi_1\omega(0)\Big]dx=0,\\ \int_0^T\int_0^1\int_0^1\Big[-z(x,\rho,t)y_t(x,\rho,t)+\dfrac{1-\rho\tau'(t)}{\tau(t)}z_\rho(x,\rho,t)y(x,\rho,t)\Big]d\rho dxdt\\ -\int_0^1\int_0^1 f_0y(x,\rho,0)d\rho dx=0.\end{cases}\tag{3.20}$$

As $v(x,0)$, $v_t(x,0)$, $\omega(x,0)$, $\omega_t(x,0)$, $y(x,\rho,0)$ are arbitrary, comparing identities (3.19) and (3.20), we deduce (3.18). Consequently, (2.1) admits at least one global strong solution (u,ϕ,z).

For the uniqueness, we assume that $(\widetilde{u}, \widetilde{\phi}, \widetilde{z})$ and $(\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ are two solutions of system (2.1). Then $(u, \phi, z) = (\widetilde{u}, \widetilde{\phi}, \widetilde{z}) - (\widetilde{\widetilde{u}}, \widetilde{\widetilde{\phi}}, \widetilde{\widetilde{z}})$ verifies the following system

$$
\begin{cases}
\rho_1 u_{tt} - \kappa u_{xx} - b\phi_x = 0, \\
\rho_2 \phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\Big) + \mu_2\Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\Big) = 0, \\
\tau(t) z_t(x,\rho,t) + (1 - \rho\tau'(t)) z_\rho(x,\rho,t) = 0, \\
u(0,t) = u(1,t) = \phi(0,t) = \phi(0,t) = 0, \\
u(x,0) = u_t(x,0) = \phi(x,0) = \phi_t(x,0) = z(x,\rho,0) = 0.
\end{cases}
\tag{3.21}
$$

To get the uniqueness of solution of (2.1), we must verify that $(u, \phi, z) = (0, 0, 0)$ is the solution of (3.21). For that, a multiplication of $(3.21)_1$ by $2u_t$ and $(3.21)_2$ by $2\phi_t$, yields

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \Big[\rho_1 u_t^2 + \rho_2\phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\Big]dx + 2\mu_1\int_0^1 \phi_t\Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\Big)dx \\
&+ 2\mu_2\int_0^1 \phi_t\Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\Big)dx = 0.
\end{aligned}
\tag{3.22}
$$

Then, we multiply $(3.21)_3$ by $2z$, we get

$$
\frac{d}{dt}\int_0^1\int_0^1 \tau(t) z^2(x,\rho)d\rho dx + \int_0^1 (1 - \tau'(t)) z^2(x,1)dx - \int_0^1 \phi_t^2 dx = 0.
\tag{3.23}
$$

By setting

$$
\Lambda(t) = \int_0^1 \Big[\rho_1 u_t^2 + \rho_2\phi_t^2 + \kappa u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi + \tau(t)\int_0^1 z^2(x,\rho)d\rho\Big]dx
$$

and summing the estimates (3.22)–(3.23), we obtain

$$
\begin{aligned}
\Lambda'(t) = & 2\mu_1\int_0^1 \omega_t\Big(g_1(\widetilde{\phi}_t) - g_1(\widetilde{\widetilde{\phi}}_t)\Big)dx + \int_0^1 \phi_t^2 dx - \int_0^1 (1 - \tau'(t)) z^2(x,1)dx \\
& - 2\mu_2\int_0^1 \phi_t\Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\Big)dx.
\end{aligned}
\tag{3.24}
$$

As g_1 is an increasing function, we can easily see that

$$
\Big(s_0 - s\Big)\Big(g_1(s_0) - g_1(s)\Big) > 0, \quad \text{for all } s_0, s \in \mathbb{R}.
$$

Thus, (3.24) becomes

$$
\Lambda'(t) \le \int_0^1 \phi_t^2 dx - (1-\theta)\int_0^1 z^2(x,1)dx - 2\mu_2\int_0^1 \phi_t\Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\Big)dx.
$$

Using Young's inequality, we get

$$
\Lambda'(t) \le c\int_0^1 \phi_t^2 dx - \int_0^1 z^2(x,1)dx + \epsilon_2\int_0^1 \Big(g_2(\widetilde{z}(x,1)) - g_2(\widetilde{\widetilde{z}}(x,1))\Big)^2 dx.
$$

Since g_2 is C^1 then g_2 is Lipschitzien function, this leads us to

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx - (1 - c\epsilon_2)\int_0^1 z^2(x,1)dx.$$

Hence, for a suitable ϵ_2, we have

$$\Lambda'(t) \leq c\int_0^1 \phi_t^2 dx.$$

As $\Lambda(t)$ is positive (for the same raison given in Remark 2.1) and $\Lambda(0) = 0$, Gronwall's Lemma forces that $\Lambda(t) = 0$ $(0 \leq t \leq T)$, which means that $u = \phi = z = 0$.

Consequently, (2.1) has only one global strong solution.

If $U_0 \in \mathcal{H}$, then it results from the density of $\mathcal{H}_0$ in $\mathcal{H}$ that the system (2.1) has a unique global weak solution. □

4. Asymptotic Behavior

This section will be concerned with the study of the solution's asymptotic behavior of system (2.1). In fact, using the Lyapunov method, we will prove that, under equal wave speeds and non-equal wave speeds cases, the solution of (2.1) converges to zero as t tends to infinity.

We start with this important notation. By setting $\varepsilon = 0$ in (2.9) and under the assumption (A_1), we have

$$E'(t) \leq -\beta_1\int_0^1 \phi_t g_1(\phi_t)dx - \beta_2\int_0^1 z(x,1)g_2(z(x,1))dx \leq 0, \quad \text{for all } t \geq 0. \tag{4.1}$$

Then (2.1) is dissipative with respect to E.

4.1. Technical lemmas.

In this subsection, we state and prove various lemmas given for (u, ϕ, z) a solution of (2.1). It would help us to estimate the derivative of the Lyapunov functional.

Lemma 4.1. *The functional*

$$F_1(t) = -\rho_1\int_0^1 u_t u dx$$

satisfies

$$F_1'(t) \leq -\rho_1\int_0^1 u_t^2 dx + \frac{3\kappa}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{4.2}$$

Proof. A simple differentiation with respect to t, using $(2.1)_1$, yields

$$F_1'(t) = -\rho_1\int_0^1 u_t^2 dx + \kappa\int_0^1 u_x^2 dx + b\int_0^1 u_x \phi dx.$$

The Young's and Poincaré's inequalities lead to (4.2). □

Lemma 4.2. *The functional defined by*

$$F_2(t) = \rho_2 \int_0^1 \phi_t u_x dx + \frac{\delta \rho_1}{\kappa} \int_0^1 u_t \phi_x dx$$

satisfies for any $\eta > 0$

(4.3)

$$\begin{aligned} F_2'(t) \leq & -\frac{b}{2} \int_0^1 u_x^2 dx + \eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) + \frac{\delta^2}{4\eta}\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big) \\ & + c\int_0^1 \phi_x^2 dx + c\int_0^1 g_1^2(\phi_t) dx + c\int_0^1 g_2^2(z(x,1)) dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

Proof. Direct computations, using $(2.1)_1$–$(2.1)_2$, lead to

$$\begin{aligned} F_2'(t) =& \int_0^1 u_x \Big[\delta\phi_{xx} - bu_x - \xi\phi - \mu_1 g_1(\phi_t) - \mu_2 g_2(z(x,1))\Big] dx \\ & + \frac{\delta}{\kappa} \int_0^1 \phi_x \Big[\kappa u_{xx} + b\phi_x\Big] dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} F_2'(t) =& \Big[\delta u_x \phi_x\Big]_{x=0}^{x=1} - b\int_0^1 u_x^2 dx + \frac{b\delta}{\kappa} \int_0^1 \phi_x^2 dx - \xi \int_0^1 u_x \phi dx - \mu_1 \int_0^1 g_1(\phi_t) u_x dx \\ & - \mu_2 \int_0^1 g_2(z(x,1)) u_x dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right) \int_0^1 \phi_{xt} u_t dx. \end{aligned}$$

Using Young's and Poincaré's inequalities, (4.3) is established. □

Lemma 4.3. *Let* χ *be a solution of*

$$\begin{cases} \chi_{xx} = -\phi_x, \\ \chi(0) = \chi(1) = 0. \end{cases}$$

Then the functional

$$F_3(t) = \int_0^1 \left(\rho_2 \phi_t \phi + \frac{b\rho_1}{\kappa} u_t \chi\right) dx$$

satisfies the following estimate

$$\begin{aligned} F_3'(t) \leq & -\delta \int_0^1 \phi_x^2 dx - \frac{1}{2}\left(\xi - \frac{b^2}{\kappa}\right) \int_0^1 \phi^2 dx + \eta_0 \int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx \\ & + c\int_0^1 g_1^2(\phi_t) dx + c\int_0^1 g_2^2(z(x,1)) dx, \quad \textit{for all } \eta_0 > 0. \end{aligned} \tag{4.4}$$

Proof. Differentiating F_3 and using $(2.1)_1$–$(2.1)_2$, we get

$$\begin{aligned} F_3'(t) =& -\xi \int_0^1 \phi^2 dx + \frac{b^2}{\kappa} \int_0^1 \chi_x^2 dx - \delta \int_0^1 \phi_x^2 dx + \rho_2 \int_0^1 \phi_t^2 dx + \frac{b\rho_1}{\kappa} \int_0^1 u_t \chi_t dx \\ & - \mu_1 \int_0^1 \phi g_1(\phi_t) dx - \mu_2 \int_0^1 \phi g_2(z(x,1)) dx. \end{aligned} \tag{4.5}$$

By exploiting Young's inequality, we have

$$\frac{b\rho_1}{\kappa}\int_0^1 u_t\chi_t dx \leq \eta_0\int_0^1 u_t^2 dx + c\int_0^1 \chi_t^2 dx, \tag{4.6}$$

$$\mu_1\int_0^1 \phi g_1(\phi_t)dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2 dx + c\int_0^1 g_1^2(\phi_t)dx, \tag{4.7}$$

$$\mu_2\int_0^1 \phi g_2(z(x,1))dx \leq \frac{1}{4}\left(\xi - \frac{b^2}{\kappa}\right)\int_0^1 \phi^2 dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{4.8}$$

Inserting (4.6)–(4.8) into (4.5) and using the fact that

$$\int_0^1 \chi_x^2 dx \leq \int_0^1 \phi^2 dx,$$
$$\int_0^1 \chi_t^2 dx \leq \int_0^1 \chi_{tx}^2 dx \leq \int_0^1 \phi_t^2 dx,$$

we obtain (4.4). □

Next, in order to eliminate the boundary terms, appearing in (4.3), we introduce the following function

$$m(x) = -4x + 2, \quad x \in [0,1]. \tag{4.9}$$

Then, we have the following result.

Lemma 4.4. *For any $\eta > 0$, the functional F_4 defined by*

$$F_4(t) = \frac{\eta}{\kappa}\int_0^1 \rho_1 m(x)u_t u_x dx + \frac{\delta}{4\eta}\int_0^1 \rho_2 m(x)\phi_t\phi_x dx$$

satisfies

$$\begin{aligned} F_4'(t) \leq & -\eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) - \frac{\delta^2}{4\eta}\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big) \\ & + c\eta\rho_1\int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx + \left(\Big(\frac{1}{4} + \frac{\eta}{4}\Big)b + 2\eta\right)\int_0^1 u_x^2 dx \\ & + c\int_0^1 \phi_x^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \end{aligned} \tag{4.10}$$

Proof. By using $(2.1)_1$, $(2.1)_2$ and (4.9), it holds that

$$\begin{aligned} F_4'(t) = & \frac{\eta}{\kappa}\left[-\kappa\Big(u_x^2(1,t) + u_x^2(0,t)\Big) + 2\rho_1\int_0^1 u_t^2 dx + b\int_0^1 m(x)u_x\phi_x dx \right. \\ & \left. + 2\kappa\int_0^1 u_x^2 dx\right] + \frac{\delta}{4\eta}\left[-\delta\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big) + 2\rho_2\int_0^1 \phi_t^2 dx \right. \\ & + 2\delta\int_0^1 \phi_x^2 dx - b\int_0^1 m(x)\phi_x u_x dx - \mu_1\int_0^1 m(x)\phi_x g_1(\phi_t)dx \\ & \left. - \mu_2\int_0^1 m(x)\phi_x g_2(z(x,1))dx - 2\xi\int_0^1 \phi^2 dx\right]. \end{aligned}$$

The estimate (4.10) follows by exploiting Young's and Poincaré's inequalities. □

Lemma 4.5. *The functional*

$$F_5(t) = \tau(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx$$

satisfies

(4.11)
$$\begin{aligned}F_5'(t) \le & -\tau(t)e^{-\tau_1}\int_0^1\int_0^1 G(z(x,\rho,t))d\rho dx - \alpha_1(1-\theta)e^{-\tau_1}\int_0^1 z(x,1)g_2(z(x,1))dx\\ &+ c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx.\end{aligned}$$

Proof. Taking the derivative of F_5 and using $(2.1)_3$, we have

$$\begin{aligned}F_5'(t) =&\tau'(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx\\ &+\int_0^1\int_0^1 (1-\rho\tau'(t))e^{-\tau(t)\rho}z_\rho(x,\rho,t)g_2(z(x,\rho,t))d\rho dx.\end{aligned}$$

Then

$$\begin{aligned}F_5'(t) =& -\int_0^1\int_0^1 \frac{d}{d\rho}\Big[(1-\rho\tau'(t))e^{-\tau(t)\rho}G(z(x,\rho,t))\Big]d\rho dx\\ &-\int_0^1\int_0^1 \tau(t)e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx\\ =& -\int_0^1 \Big[(1-\tau'(t))e^{-\tau(t)}G(z(x,1,t)) - G(z(x,0,t))\Big]dx\\ &-\tau(t)\int_0^1\int_0^1 e^{-\tau(t)\rho}G(z(x,\rho,t))d\rho dx.\end{aligned}$$

Using (2.4) with the fact that $z(x,0,t)=\phi_t$, $e^{-\tau(t)} \le e^{-\tau_1\rho} \le 1$ for all $\rho\in[0,1]$ and $\tau\in[\tau_0,\tau_1]$, we obtain

$$\begin{aligned}F_5'(t) \le & -\tau(t)\int_0^1\int_0^1 e^{-\tau_1}G(z(x,\rho,t))d\rho dx. - e^{-\tau_1}(1-\theta)\alpha_1\int_0^1 z(x,1)g_2(z(x,1))dx\\ &+\alpha_2\int_0^1 \phi_t g_1(\phi_t)dx.\end{aligned}$$

The estimate (4.11) follows by exploiting Young's inequality. □

Lemma 4.6. *For a suitable choice of N and N_i, $i=1,2,\dots,5$, the functional defined by*

(4.12)
$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{5} N_iF_i(t).$$

satisfies, for a fixed positive constant m_0, the estimate

$$\mathcal{L}'(t) \leq -m_0 E(t) + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt} u_t dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 g_1^2(\phi_t)dx + c\int_0^1 g_2^2(z(x,1))dx. \tag{4.13}$$

Proof. From (4.1), (4.2), (4.3), (4.4), (4.10) and (4.11), it follows that for any $t \geq 0$

$$\begin{aligned}
\mathcal{L}'(t) \leq &- (N_4 - N_2)\Big[\eta\Big(u_x^2(1,t) + u_x^2(0,t)\Big) + \frac{\delta^2}{4\eta}\Big(\phi_x^2(1,t) + \phi_x^2(0,t)\Big)\Big] \\
&- \Big[\rho_1 N_1 - \eta_0 N_3 - \eta c\rho_1 N_4\Big]\int_0^1 u_t^2 dx + \Big[N_3 + N_4 + N_5\Big]c\int_0^1 \phi_t^2 dx \\
&- \Big[\frac{b}{2}N_2 - \frac{3\kappa}{2}N_1 - \Big(\Big(\frac{1}{4} + \frac{\eta}{4}\Big)b + 2\eta\Big)N_4\Big]\int_0^1 u_x^2 dx \\
&- \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big)N_3\int_0^1 \phi^2 dx - \Big[\delta N_3 - \Big(N_1 + N_2 + N_4\Big)c\Big]\int_0^1 \phi_x^2 dx \\
&- \tau e^{-\tau} N_5 \int_0^1\int_0^1 G(z(x,\rho))d\rho dx + \Big[N_2 + N_3 + N_4 + N_5\Big]c\int_0^1 g_1^2(\phi_t)dx \\
&+ \Big[N_2 + N_3 + N_4\Big]c\int_0^1 g_2^2(z(x,1))dx + N_2\left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt} u_t dx.
\end{aligned}$$

Furthermore, we take

$$N_1 = 3\eta c, \quad N_2 = N_4 = N_5 = 1, \quad \eta_0 = \frac{\eta c\rho_1}{N_3},$$

to get

$$\begin{aligned}
\mathcal{L}'(t) \leq &- \eta c\rho_1 \int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx - \frac{1}{4}\Big(b - \eta\Big(18\kappa c + b + 8\Big)\Big)\int_0^1 u_x^2 dx \\
&- \Big(\delta N_3 - c\Big)\int_0^1 \phi_x^2 dx - \frac{1}{2}\Big(\xi - \frac{b^2}{\kappa}\Big)N_3\int_0^1 \phi^2 dx + c\int_0^1 g_2^2(z(x,1))dx \\
&- \tau e^{-\tau}\int_0^1\int_0^1 G(z(x,\rho))d\rho dx + c\int_0^1 g_1^2(\phi_t)dx + \left(\frac{\delta\rho_1}{\kappa} - \rho_2\right)\int_0^1 \phi_{xt} u_t dx.
\end{aligned} \tag{4.14}$$

Now, we select $\eta < \frac{b}{18\kappa c + b + 8}$ and then we choose N_3 large enough such that

$$\delta N_3 - c > 0.$$

Hence, the estimate (4.14) with the fact that $\kappa\xi > b^2$ and (2.6) gives us (4.13). □

4.2. **General decay rates for equal of wave speeds.** In this subsection, we show that the solution have a general decay rate in the case of equal speeds of wave propagation.

Theorem 4.1. *Let $U \in \mathcal{H}$. Assuming that (A_1), (A_2) and (A_3) are fulfilled, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}.$$

Then, there exist positive constants a, a_1 and a_2 such that the solution of (2.1) *satisfies*

$$E(t) \leq aH_1^{-1}\Big(a_1 t + a_2\Big), \quad \text{for all } t > 0, \tag{4.15}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds \quad \text{and} \quad H_2(t) = tH'(\epsilon_0 t).$$

Proof. Since $\frac{\rho_1}{\kappa} = \frac{\rho_2}{\delta}$, then we can easily show for N sufficiently large, that the functional $\mathcal{L}$ given by (4.12) is equivalent to E, i.e.,

$$\mathcal{L}(t) \sim E(t).$$

We consider, as is [8], the following two partitions of $[0,1]$

$$\mathcal{D}_1 = \Big\{x \in [0,1] : |\phi_t| + |z(x,1)| \leq \epsilon\Big\}, \quad \mathcal{D}_2 = \Big\{x \in [0,1] : |\phi_t| + |z(x,1)| > \epsilon\Big\}$$

and we define $\mathcal{R}(x,t)$ by

$$\mathcal{R}(x,t) = \phi_t g_1(\phi_t) + z(x,1,t) g_2(z(x,1,t)).$$

Then by recalling (2.2) and (4.1), we obtain

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) + \int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big)dx. \tag{4.16}$$

Now, we discuss two cases.

1. H **is linear on** $[0,\epsilon]$. In this case, we obtain, for some positive constant c',

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) - c'E'(t).$$

Hence, $\mathcal{L}_0 = \mathcal{L} + (c + c')E \sim E$ satisfies

$$\mathcal{L}_0(t) \leq -\mathcal{L}_0(0)e^{-ct},$$

which leads to

$$E(t) \leq -cE(0)e^{-ct}.$$

2. H **is non linear on** $[0,\epsilon]$. By using Jensen's inequality and the concavity of H^{-1}, we find that

$$\int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big)dx \leq cH^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right).$$

Thus, (4.16) rewrites as

$$\mathcal{L}'(t) \leq -m_0 E(t) - cE'(t) + cH^{-1}\left(\int_{\mathcal{D}_1} \mathcal{R}(x,t)dx\right). \tag{4.17}$$

For $\epsilon_0 < \epsilon$ and $m_1 > 0$, the functional given by

$$\mathcal{L}_1(t) = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + m_1 E(t)$$

satisfies, for some fixed positive constants ζ_0 and ζ_1,

$$\zeta_0\mathcal{L}_1(t) \leq E(t) \leq \zeta_1\mathcal{L}_1(t) \tag{4.18}$$

and

$$\mathcal{L}_1'(t) = \epsilon_0\frac{E'(t)}{E(0)}H''\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}(t) + H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathcal{L}'(t) + m_1E'(t).$$

Next, by recaling the fact that $E' \leq 0$, $H' > 0$ and $H'' > 0$ on $[0,\epsilon]$ and using (4.17), we get

$$\mathcal{L}_1'(t) \leq -m_0E(t)H'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + cH'\left(\epsilon_0\frac{E(t)}{E(0)}\right)H^{-1}\left(\int_{\mathcal{D}_1}\mathcal{R}(x,t)dx\right) + m_1E'(t). \tag{4.19}$$

Let H^* be the convex conjugate of H, then by testing (2.8) with

$$A = H'\left(\epsilon_0\frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = H^{-1}\left(\int_{\mathcal{D}_1}\mathcal{R}(x,t)dx\right),$$

we get

$$H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)H^{-1}\left(\int_{\mathcal{D}_1}\mathcal{R}(x,t)dx\right) \leq H^*\left(H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right) + \int_{\mathcal{D}_1}\mathcal{R}(x,t)dx.$$

Using (4.1) with the fact $H^* \leq s(H')^{-1}(s)$, we have that

$$H'\left(\epsilon_0\frac{E(t)}{E(0)}\right)H^{-1}\left(\int_{\mathcal{D}_1}\mathcal{R}(x,t)dx\right) \leq \epsilon_0\frac{E(t)}{E(0)}H'\left(\epsilon_0\frac{E(t)}{E(0)}\right) - cE'(t). \tag{4.20}$$

The substitution of (4.20) into (4.19) provides

$$\mathcal{L}_1'(t) \leq -\Big(m_0E(0) - c\epsilon_0\Big)\frac{E(t)}{E(0)}H'\left(\epsilon_0\frac{E(t)}{E(0)}\right) + (m_1 - c)E'(t).$$

Fixing ϵ_0 sufficiently small, so that $m_0E(0) - c\epsilon_0 > 0$, then for $m_1 > c$, we can find a positive constant a_0 such that

$$\mathcal{L}_1'(t) \leq -a_0\frac{E(t)}{E(0)}H'\left(\epsilon_0\frac{E(t)}{E(0)}\right) = -a_0H_2\left(\epsilon_0\frac{E(t)}{E(0)}\right), \tag{4.21}$$

where $H_2(t) = tH'(\epsilon_0 t)$ is a positive non-decreasing function on $[0,1]$. Next, by setting $\mathcal{L}_2 = \frac{\zeta_0\mathcal{L}_1}{E(0)}$, we can easily show, by (4.18), that $\mathcal{L}_2 \sim E$. And, from (4.21), we discover that

$$\mathcal{L}_2'(t) \leq -a_1H_2(L(t)). \tag{4.22}$$

From the definition of H_1, we have

$$H_1'(t) = -\frac{1}{H_2(t)},$$

whereupon the inequality (4.22) becomes

$$\mathcal{L}_2'(t) \leq a_1\frac{1}{H_1'(\mathcal{L}_2(t))},$$

which implies

$$\Big[H_1(\mathcal{L}_2(t))\Big]' \leq a_1.$$

An integration over $[0,t]$ yields that

$$H_1(\mathcal{L}_2(t)) \leq a_1 t + H_1(\mathcal{L}_2(0)).$$

Then, using the non-decreasing property of H^{-1}, we infer that

$$\mathcal{L}_2(t) \leq H^{-1}\Big(a_1 t + a_2\Big).$$

The use of $\mathcal{L}_2 \sim E$ leads us to (4.15). Hence, the proof is completed. □

4.3. **General decay rates for non-equal of wave speeds.** In this subsection, we investigate the situation when $\dfrac{\rho_1}{\kappa} \neq \dfrac{\rho_2}{\delta}$, which is more realistic in the view of physics.

Theorem 4.2. *Let $U_0 \in \mathcal{H}_0$. Assume that (A_1) and (A_2) hold, $\kappa\xi > b^2$ and that*

$$\frac{\rho_1}{\kappa} \neq \frac{\rho_2}{\delta}.$$

Then, for

$$|\mu_2| < \min\left\{\frac{\alpha_1}{\alpha_2}, \frac{2\widetilde{c}_1}{\widetilde{c}_2(2-\theta)}\right\}(1-\theta)\mu_1, \tag{4.23}$$

there exist some positive numbers w and w_1 such that for any $t > 0$

$$E(t) \leq w H_2^{-1}\Big(\frac{w_1}{t}\Big). \tag{4.24}$$

Proof. Differentiating (2.1) with respect to x, we obtain

$$\begin{cases}
\rho_1 u_{xtt} - \kappa u_{xxx} - b\phi_{xx} = 0,\\
\rho_2 \phi_{xtt} - \delta\phi_{xxx} + b u_{xx} + \xi\phi_x + \mu_1\phi_{xt} g_1'(\phi_t) + \mu_2 z_x(x,1) g_2'(z(x,1)) = 0,\\
\tau(t) z_{xt}(x,\rho,t) + (1-\rho\tau'(t)) z_{x\rho}(x,\rho,t) = 0,\\
u_x(0,t) = u_x(1,t) = \phi_x(0,t) = \phi_x(1,t) = 0,\\
u_x(x,0) = u_x^0(x), \quad u_t(x,0) = u_x^1(x),\\
\phi_x(x,0) = \phi_x^1(x), \quad \phi_{xt}(x,0) = \phi_x^1(x),\\
z_x(x,\rho,0) = f_x^0(x,-\rho\tau(0)).
\end{cases} \tag{4.25}$$

Then, for a fixed positive constant $\widetilde{\gamma}$ satisfying

$$\frac{\widetilde{c}_2|\mu_2|}{(1-\theta)} < \widetilde{\gamma} < \Big(2\widetilde{c}_1\mu_1 - \widetilde{c}_2|\mu_2|\Big), \tag{4.26}$$

where $\widetilde{c}_1$ and $\widetilde{c}_2$ are introduced in (2.7) and (2.3), we define the modified energy functional to system (4.25) as

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\int_0^1 \Big[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 \\ &+ 2bu_{xx}\phi_x + 2\widetilde{\gamma}\tau(t)\int_0^1 z_{xt}^2(x,\rho,t)d\rho\Big]dx.\end{aligned}$$

Our point of departure will be to show that the modified energy functional $\mathcal{E}$ is non-increasing. So, we have the following result.

Lemma 4.7. *Under the assumptions in Theorem* 4.2*, the modified energy functional $\mathcal{E}$ is non-increasing and satisfies for any $t \geq 0$*

$$\mathcal{E}'(t) \leq -c\int_0^1 \phi_{xt}^2 dx - c\int_0^1 z_x^2(x,1)dx. \tag{4.27}$$

Proof. Multiplying $(4.25)_1$ and $(4.25)_2$ by u_{xt} and ϕ_{xt}, respectively, and integrating by parts over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{2}\cdot\frac{d}{dt}\int_0^1 \Big[\rho_1 u_{xt}^2 + \rho_2\phi_{xt}^2 + \kappa u_{xx}^2 + \delta\phi_{xx}^2 + \xi\phi_x^2 + 2bu_{xx}\phi_x\Big]dx \\ &+\mu_1\int_0^1 \phi_{xt}^2 g_1'(\phi_t)dx + \mu_2\int_0^1 \phi_{xt} z_x(x,1)g_2'(z(x,1))dx = 0.\end{aligned} \tag{4.28}$$

Similarly, we multiply $(4.25)_3$ by $\widetilde{\gamma} z_x(x,\rho,t)$, we get

$$\frac{\widetilde{\gamma}}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau(t)z_{xt}^2(x,\rho,t)d\rho dx = -\frac{\widetilde{\gamma}}{2}(1-\tau'(t))\int_0^1 z_x^2(x,1)dx + \frac{\widetilde{\gamma}}{2}\int_0^1 \phi_{xt}^2 dx. \tag{4.29}$$

Combining the estimates (4.28)–(4.29) and using the fact that $\widetilde{c}_1 < g_1'(s)$ and (A_2), we yield that

$$\begin{aligned}\mathcal{E}'(t) \leq& -\left(\widetilde{c}_1\mu_1 - \frac{\widetilde{\gamma}}{2}\right)\int_0^1 \phi_{xt}^2 dx - \frac{\widetilde{\gamma}}{2}(1-\tau'(t))\int_0^1 z_x^2(x,1)dx \\ &- \mu_2\int_0^1 \phi_{xt} z_x(x,1)g_2'(z(x,1))dx.\end{aligned}$$

By using Young's inequality with the fact that $|g_2'(s)| < \widetilde{c}_2$, we arrive at

$$\mathcal{E}'(t) \leq -\left(\widetilde{c}_1\mu_1 - \frac{\widetilde{\gamma}}{2} - \frac{\widetilde{c}_2|\mu_2|}{2}\right)\int_0^1 \phi_{xt}^2 dx - \left(\frac{\widetilde{\gamma}}{2}(1-\theta) - \frac{\widetilde{c}_2|\mu_2|}{2}\right)\int_0^1 z_x^2(x,1)dx.$$

Estimate (4.27) follows by using (4.23) and (4.26). □

Now, going back to the proof of Theorem 4.2. Defining, as in (4.12), a Lyapunov functional L by

$$L(t) = M\mathcal{E}(t) + \mathcal{L}(t).$$

It should be mentioned that L is not equivalent to E. Then, using (4.13) and (4.27), we get

$$\begin{aligned} L'(t) \leq & - m_0 E(t) - cM \int_0^1 \phi_{xt}^2 dx + \left(\frac{\delta \rho_1}{\kappa} - \rho_2 \right) \int_0^1 \phi_{xt} u_t dx \\ & + c \int_0^1 \phi_t^2 dx + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx. \end{aligned}$$

Utilizing Young's inequality and the definition of $E(t)$, we get

$$\begin{aligned} L'(t) \leq & - (m_0 - \eta_1) E(t) - (cM - c_{\eta_1}) \int_0^1 \phi_{xt}^2 dx + c \int_0^1 \phi_t^2 dx \\ & + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx. \end{aligned}$$

Fixing $\eta_1 < m_0$ and then taking M sufficiently large, so that $cM - c_{\eta_1} \leq 0$, we obtain for $d_0 > 0$

$$L'(t) \leq -d_0 E(t) + c \int_0^1 \phi_t^2 dx + c \int_0^1 g_1^2(\phi_t) dx + c \int_0^1 g_2^2(z(x,1)) dx.$$

Consequently by exploiting (2.2) and (4.1), it holds that

$$L'(t) \leq -d_0 E(t) - cE'(t) + \int_{\mathcal{D}_1} H^{-1}\Big(\mathcal{R}(x,t)\Big) dx. \tag{4.30}$$

As in the proof of Theorem 4.1, we distinguish the following two cases.

1. H **is linear on** $[0, \epsilon]$**.** From (4.30) and by using (4.1), we have, for some positive constant c',

$$L'(t) \leq -d_0 E(t) - (c + c') E'(t).$$

Then, the functional $L_0 = L + (c + c')E$, satisfies

$$L_0'(t) \leq -d_0 E(t).$$

Integrating over $[0, t]$ and using the non-increasing property of E, we yield that

$$tE(t) \leq \int_0^t E(s) ds \leq \frac{1}{d_0} L_0(0).$$

Hence, for $d > 0$ we have

$$E(t) \leq \frac{d}{t}, \quad \text{for all } t > 0.$$

2. H **is non-linear on** $[0, \epsilon]$**.** By repeating the same arguments as in the second part of the proof of Theorem 4.1, we find that the functional

$$L_1(t) = H'\left(\epsilon_0 \frac{E(t)}{E(0)} \right) L(t) + d_1 E(t)$$

satisfies, for a fixed positive constant w_0, the following property

$$L_1'(t) \leq -w_0 H_2 \left(\epsilon_0 \frac{E(t)}{E(0)} \right).$$

An integration over $[0,t]$ gives

$$\int_0^t H_2\left(\epsilon_0 \frac{E(s)}{E(0)}\right) ds \leq \frac{1}{w_0} L_1(0). \tag{4.31}$$

It follows from the fact that $E' \leq 0$ and $H_2' > 0$ that the map

$$t \mapsto H_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$$

is non-increasing. Thus, from (4.31), we obtain

$$tH_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \leq \int_0^t H_2\left(\epsilon_0 \frac{E(s)}{E(0)}\right) ds \leq \frac{1}{w_0} L_1(0).$$

Consequently, for $w, w_1 > 0$ we have

$$E(t) \leq wH_2^{-1}\left(\frac{w_1}{t}\right), \quad \text{for all } t > 0,$$

which finishes the proof. □

References

[1] F. Alabau-Boussouira, *Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), 643–699. https://doi.org/10.1007/s00030-007-5033-0

[2] T. A. Apalara, *Asymptotic behavior of weakly dissipative Timoshenko system with internal constant delay feedbacks*, Appl. Anal. **10** (2015), 187–202. https://doi.org/10.1080/00036811.2014.1000314

[3] T. A. Apalara, *A general decay for a weakly nonlinearly damped porous system*, J. Dyn. Control Syst. **25** (2019), 311–322. https://doi.org/10.1007/s10883-018-9407-x

[4] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.

[5] A. Benaissa and M. Bahlil, *Global existence and energy decay of solutions to a nonlinear Timoshenko beam system with a delay term*, Taiwanese J. Math. **18**(5) (2014), 1411–1437. https://doi.org/10.11650/tjm.18.2014.3586

[6] M. Kirane, B. Said-Houari, and M. N. Anwar, *Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks*, Commun. Pure Appl. Anal. **10**(2) (2011), 667–686. https://doi.org/10.3934/cpaa.2011.10.667

[7] H. Khochemane, S. Zitouni and L. Bouzettouta, *Stability result for a nonlinear damping Porous-elastic system with delay term*, Nonlinear Stud. **27**(2) (2020), 487–503. https://doi.org/10.46793/KgJMat2003.443S

[8] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson Wiley, Paris, 1994.

[9] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, 1969.

[10] W. J. Liu and M. M. Chen, *Well-posedness and exponential decay for a porous thermoelastic system with second sound and a time varying delay term in the internal feedback*, Contin. Mech. Thermodyn. **29** (2017), 731–746. https://doi.org/10.1007/s00161-017-0556-z

[11] M. I. Mustafa and S. A. Messaoudi, *General energy decay rates for a weakly damped Timoshenko system*, J. Dyn. Control Sys. **16**(2) (2010), 211–226. https://doi.org/10.1007/s10883-010-9090-z

[12] S. Nicaise and C. Pignotti, *Interior feedback stabilization of wave equations with time dependent delay*, Electron. J. Differential Equations **41** (2011), 1–20.
[13] R. Quintanilla, *Slow decay for one-dimensional porous dissipation elasticity*, Appl. Math. Lett. **16**(4) (2003), 487–491.
[14] C. A. Raposo T. A. Apalara and J. O. Ribeiro, *Analyticity to transmission problem with delay in Porous-elasticity*, J. Math. Anal. App. **466** (2018), 819–834. `https://doi.org/10.1016/J.JMAA.2018.06.017`
[15] C. A. Raposo J. Ferreira, M. L. Santos and N. N. O. Castro, *Exponential stability for the Timoshenko system with two weak dampings*, Appl. Math. Lett. **18**(5) (2005), 535–541. `https://doi.org/10.1016/j.aml.2004.03.017`
[16] A. Soufyane and A. Wehbe, *Uniform stabilization for the Timoshenko beam by a locally distributed damping*, Electron. J. Differential Equations **29** (2003), 1–14.
[17] S. P. Timoshenko, *On the correction for shear of the differential equation for transverse vibrations of prismaticbars*, Philisophical Magazin **41**(245) (1921), 744–746. `https://doi.org/10.1080/14786442108636264`

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