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APPROXIMATION BY A COMPOSITION OF APOSTOL-GENOCCHI AND PĂLTĂNEA-DURRMEYER OPERATORS

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Abstract. The present paper deals with the Durrmeyer construction of operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. For the proposed operators, we first establish a global approximation result followed by its convergence estimate in terms of usual, r-th and weighted modulus of continuity. We further study the asymptotic type results such as the Voronovskaya theorem and quantitative Voronovskaya theorem. Moreover, we estimate the rate of pointwise convergence of the proposed operators for functions of bounded variation defined on the interval $(0,\infty)$. Finally, the results are validated through graphical representations and an absolute error table.

1. Introduction

Recently, Prakash et al. [20] proposed the following positive linear sequence of operators:

$$G_n^{\alpha,\lambda}(f;x)=\sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) f\left(\frac{k}{n}\right), \quad x\in[0,\infty), \tag{1.1}$$

where $s_{n,k}^{\alpha,\lambda}(x)=e^{-nx}\left(\frac{1+\lambda e}{2}\right)^{\alpha}\frac{g_k^{\alpha}(nx;\lambda)}{k!}$ and $g_k^{\alpha}(x;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, which belong to the class of orthogonal polynomials. These polynomials were defined for a complex variable z, $|z|<\pi$ in [16]. However, in this study we limit ourselves to a real variable $t\in[0,\infty)$. The generalized Apostol

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Genocchi polynomial of order α, i.e., $g_k^{\alpha}(x;\lambda)$ can be estimated with the help of following generating function:

$$\left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} g_k^{\alpha}(x;\lambda)\frac{t^k}{k!}. \tag{1.2}$$

The more explicit form of $g_k^{\alpha}(x;\lambda)$ was proposed by Luo and Srivastava in [17]. They presented some elementary properties of these polynomials and derived explicit series representation of $g_k^{\alpha}(x;\lambda)$ in terms of hypergeometric function defined by Gauss. The series is given as follows:

$$\begin{aligned} g_k^{\alpha}(x;\lambda) =& 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{n=0}^{k-\alpha}\binom{k-\alpha}{n}\binom{\alpha+n-1}{n}\frac{\lambda^n}{(1+\lambda)^{\alpha+n}} \\ &\times\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n(x+j)^{k-n-\alpha}{}_2F_1\left(\alpha+n-k,n;n+1;\frac{j}{x+j}\right), \end{aligned}$$

where $\{k,\alpha\}\in\mathbb{N}\cup\{0\}$, $\lambda\in\mathbb{R}\setminus\{-1\}$, $x\in\mathbb{R}$ and ${}_2F_1(a,b;c;t)$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = {}_2F_1(b,a;c;t) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\cdot\frac{t^k}{k!}.$$

In particular, for $\alpha = 1$ and $\lambda = 1$, these operators reduce to classical Genocchi polynomials which are obtained by the following generating function:

$$\frac{2te^{xt}}{e^t+1} = \sum_{k=0}^{\infty} g_k(x)\frac{t^k}{k!}, \quad |t|<2\pi,\ x\in\mathbb{R},$$

where $g_k(x) = g_k(x;1)$. It can be clearly seen that $g_k(x)$ are the k^{th}-degree polynomials, few terms of which are given as follows:

$$\begin{gathered} g_1(x) = 1, \quad g_2(x) = 2x-1, \quad g_3(x) = 3x(x-1), \\ g_4(x) = 4x^3-6x^2+1, \quad g_5(x) = 5x^4-10x^3+5x, \ldots \end{gathered}$$

For the case $x = 0$, one can obtain the so-called Genocchi numbers g_k using the relation:

$$g_k(x) = \sum_{i=0}^{k}\binom{k}{i}g_i x^{k-i}.$$

Genocchi numbers can be defined in many ways depending on the field where they are intended to be applied. They find a wide range of application in numerical analysis, combinatorics, number theory, graph theory etc. Luo [15,16] defined Apostol-Genocchi polynomials of higher order and also introduced q-Apostol-Genocchi polynomials. He studied the relationship of these polynomials with Zeta function. In the last two decades, a surprising number of papers appeared studying Genocchi numbers, their combinatorial relations, Genocchi polynomials and their generalisations along with their various expansions and integral representations. To the readers, we suggest following articles [4, 18, 21] and references therein.

In the recent past, much work has been dedicated towards the Durrmeyer type modification of linear positive operators. For instance, Dhamija and Deo [8] introduced the Durrmeyer form of Jain operators based on inverse Pólya-Eggenberger distribution. They studied its moments with the aid of Vandermonde convolution formula and analysed other approximation properties. Heilmann and Raşa [12] studied a link between Baskakov-Durrmeyer type operators and their corresponding classical Kantorovich variants. Acu and Radu [3] introduced and studied a class of operators which link α-Bernstein operators and genuine α-Bernstein Durrmeyer operators. To see more work relevant to this area, one may refer [2, 5, 7, 10, 11, 13].

Inspired by above stated researches, we now consider a Durrmeyer type modification of Apostol-Genocchi operators based on Păltănea basis on positive real line. For $f \in C[0,\infty)$ and $\rho > 0$, the operators are defined as follows:

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) \int_0^{\infty} l_{n,k}^{\rho}(t) f(t)\, dt, \quad x \in [0,\infty), \tag{1.3}$$

where $l_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $s_{n,k}^{\alpha,\lambda}(x)$ is defined in (1.1).

The outline of the study is as follows. We consider a Durrmeyer type construction of Apostol-Genocchi operators based on the basis function due to Păltănea [19] with real parameters α, λ and ρ. We establish approximation estimates such as a global approximation theorem and rate of approximation in terms of usual, r-th and weighted modulus of continuity. We further study asymptotic formulae such as Voronovskaya theorem and quantitative Voronovskaya theorem. The last theorem is an application of the proposed operators for the functions whose derivatives are of bounded variation. Moreover, the approximation and the absolute error therein has been shown graphically by varying the values of various parameters using Mathematica software.

2. Preliminaries

Before proceeding to our main results, we state some general lemmas which are useful throughout this paper. In addition, we have used Mathematica software wherever necessary for complex and tedious calculations such as for moments and central moments etc.

Lemma 2.1. *For $e_s(t) = t^s$, $s \in \mathbb{N} \cup \{0\}$ and $\rho > 0$, we have*

$$\int_0^{\infty} l_{n,k}^{\rho}(t) t^s = \frac{(k\rho + s - 1)!}{(n\rho)^s (k\rho - 1)!} = \frac{(k\rho)_s}{(n\rho)^s},$$

where the symbol $(\beta)_n = \beta(\beta+1)(\beta+2)\cdots(\beta+n-1)$, $(\beta)_0 = 1$ denotes the rising factorial.

Lemma 2.2. *For operators (1.3), the moments are obtained as follows:*

$\mathcal{M}_n^{\alpha,\lambda}(e_0;x) = 1,$

$$\mathcal{M}_n^{\alpha,\lambda}(e_1;x) = x + \frac{\alpha}{n(1+\lambda e)},$$

$$\mathcal{M}_n^{\alpha,\lambda}(e_2;x) = x^2 + \frac{x}{n}\left[\frac{1+2\alpha+\lambda e}{(1+\lambda e)} + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2 - 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right],$$

$$\begin{aligned}
&\mathcal{M}_n^{\alpha,\lambda}(e_3;x)\\
=&x^3 + \frac{x^2}{n}\left[\frac{3(\alpha+\lambda e+1)}{(1+\lambda e)} + \frac{3}{\rho}\right]\\
&+ \frac{x}{n^2}\left[\frac{3\alpha^2 - 3\alpha\lambda^2e^2 - 3\alpha\lambda e + 3\alpha + \lambda^2e^2 + 2\lambda e + 1}{(1+\lambda e)^2} + \frac{3(2\alpha+\lambda e+1)}{\rho(1+\lambda e)} + \frac{2}{\rho^2}\right]\\
&+ \frac{1}{n^3}\left[\frac{\alpha^3 - 3\alpha^2\lambda^2e^2 - 6\alpha^2\lambda e - \alpha\lambda^3e^3 - 4\alpha\lambda^2e^2 - 5\alpha\lambda e}{(1+\lambda e)^3} + \frac{2\alpha}{\rho^2(1+\lambda e)}\right.\\
&\left.+\frac{3(\alpha^2 - \alpha\lambda^2e^2 - 2\alpha\lambda e)}{\rho(1+\lambda e)^2}\right],\\
&\mathcal{M}_n^{\alpha,\lambda}(e_4;x)\\
=&x^4 + \frac{x^3}{n}\left[\frac{2\alpha+3\lambda e+3}{(1+\lambda e)} + \frac{6}{\rho}\right]\\
&+ \frac{x^2}{n^2}\left[\frac{25+12\alpha+6\alpha^2+50\mathrm{e}\lambda+25\mathrm{e}^2\lambda^2-6\alpha\mathrm{e}^2\lambda^2}{(1+\lambda e)^2} + \frac{18(1+\alpha+\mathrm{e}\lambda)}{(1+\lambda e)\rho} + \frac{11}{\rho^2}\right]\\
&+ \frac{x}{n^3}\left[\frac{7+20\alpha+63\alpha^2+2\alpha^3-6\alpha^2e^2\lambda^2-9\alpha^2\lambda e}{(1+\lambda e)^3}\right.\\
&+ \frac{-5\alpha e^3\lambda^3+4\alpha e^2\lambda^2+24\alpha\lambda e+7e^3\lambda^3+21\lambda e}{(1+\lambda e)^3}\\
&\left.+\frac{6(3\alpha^2-3\alpha e^2\lambda^2-3\alpha\lambda e+3\alpha+e^2\lambda^2+2\lambda e+1)}{(1+\lambda e)^2\rho} + \frac{11(1+2\alpha+\lambda e)}{(1+\lambda e)\rho^2} + \frac{6}{\rho^3}\right]\\
&\times \frac{1}{n^4}\left[\frac{\alpha^4 - 6e^2\alpha^3\lambda - 12e\alpha^3\lambda - 16e^4\alpha\lambda^4 + 8e^3\alpha\lambda^3 - 82e^3\alpha\lambda^2 - 118e^2\alpha\lambda^2 - 66\alpha + \lambda e}{(1+\lambda e)^4}\right.\\
&+ \frac{6(\alpha^3-3e^2\alpha^2\lambda^2-6e\alpha^2\lambda-e^3\alpha\lambda^3-4e^2\alpha\lambda^2-5e\alpha\lambda)}{(1+\lambda e)^3\rho}\\
&\left.+ \frac{11(\alpha^2-\alpha e^2\lambda^2-2\alpha\lambda e)}{(1+\lambda e)^2\rho^2} + \frac{6\alpha}{(1+\lambda e)\rho^3}\right].
\end{aligned}$$

Proof. In the proposed operators (1.3), for $s = 0, 1, 2$ respectively we have 1.

2. For $s = 1$, again using Lemma 2.1 we have

$$\mathcal{M}_n^{\alpha,\lambda}(e_1;x) = \frac{e^{-nx}}{n}\left(\frac{1+\lambda e}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{g_k^{\alpha}(nx;\lambda)}{k!}k. \tag{2.1}$$

Differentiating both sides of (1.2) with respect to t and taking limits $t \to 1$ and $x \to nx$, we have

$$\sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k = 2^{\alpha} e^{nx} \left(\frac{1}{1+\lambda e}\right)^{1+\alpha} (\alpha + nx(1+\lambda e)).$$

Making use of this value in equation (2.1), we obtain the first moment.

3. Similarly for $s = 2$, we have

$$\mathcal{M}_n^{\alpha,\lambda}(e_2;x) = \frac{e^{-nx}}{n^2}\left(\frac{1+\lambda e}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k^2. \tag{2.2}$$

On differentiating both sides of (1.2) with respect to t and taking limits $t \to 1$ and $x \to nx$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k(k-1) =& 2^{\alpha} e^{nx} \left(\frac{1}{1+\lambda e}\right)^{1+\alpha} \Big\{2ns\alpha(1+\lambda e) \\ &+ n^2x^2(1+\lambda e)^2 - (\lambda e(3+\lambda e) - \alpha + 1)\Big\}.\end{aligned}$$

Combining this with the first order moment and equation (2.2) we obtain the third moment.

We can obtain the higher order moments in a similar way. □

Lemma 2.3. *Let us define* $\delta_n^s(x) = \mathcal{M}_n^{\alpha,\lambda}(\Phi_s;x)$, *where* $\Phi_s(t) = (e_1 - x)^s$ *and* $s = 1, 2$. *Then, from Lemma* 2.2 *we have*

$$\begin{aligned}\delta_n^{(1)}(x) &= \frac{\alpha}{n(1+\lambda e)},\\ \delta_n^{(2)}(x) &= \frac{x}{n}\left[1 + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2 - 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}\lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_1;x) =& \frac{\alpha}{(1+\lambda e)},\\ \lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_2;x) =& \left(1 + \frac{1}{\rho}\right)x,\\ \lim_{n\to\infty} n^2\mathcal{M}_n^{\alpha,\lambda}(\Phi_4;x) =& \frac{3\left(1 + (6+8\alpha)\rho - (3+8\alpha)\rho^2 + \lambda e\left(1 + 6\rho - 3\rho^2\right)\right)x^2}{(1+\lambda e)\rho^2}\end{aligned}$$

and

$$\lim_{n\to\infty} n^3\mathcal{M}_n^{\alpha,\lambda}(\Phi_6;x) = \frac{(1+\rho)(2+\rho+3\alpha\rho+\lambda e(2+\rho))x^5}{(1+\lambda e)^3\rho^4}.$$

Remark 2.1. Since fourth and sixth central moments are too lengthy and unnecessarily space consuming, we are omitting their values here. Instead we choose to write their limiting values, which is useful in the proofs of our main theorems.

3. Main Theorems

Theorem 3.1. *For any $f \in C_B[0,\infty)$, where $C_B[0,\infty)$ is the class of all continuous and bounded functions, we have*

$$\mathcal{M}_n^{\alpha,\lambda}(f(t);x) = f(x)$$

uniformly on any compact subset of $[0,\infty)$.

Proof. Taking into account Lemma 2.2, we can easily see that $\mathcal{M}_n^{\alpha,\lambda}(e_r;x) \to x^r$ for each $r = 0,1,2$ and hence using the well known Korovkin's theorem due to [14], operators $\mathcal{M}_n^{\alpha,\lambda}$ converge uniformly on each compact subset of $[0,\infty)$. □

3.1. **Global approximation.** Let us denote $B_f[0,\infty)$ the space of all functions f on positive real axis that satisfy the condition $|f(x)| \leq H_f(1+x^2)$ where H_f is a constant depending only on f but independent of x.

Let $C_f[0,\infty)$ be the subspace of $B_f[0,\infty)$ containing all continuous f on $[0,\infty)$. The norm in $C_f[0,\infty)$ is defined by

$$||f||_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Also, let $C_f^l[0,\infty) := \left\{ f \in C_f[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} \text{ is finite} \right\}$.

Theorem 3.2. *For each $f \in C_f^l[0,\infty)$, we have*

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(f;x) - f \right\|_2 = 0.$$

Proof. The proof of this theorem can be given by application of Korovkin theorem [9] on the interval $[0,\infty)$. Therefore, it would suffice if we prove that

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(e_l;x) - e_l \right\|_2 = 0, \quad l = 0,1,2. \tag{3.1}$$

For $l = 0$, condition (3.1) holds as operators $\mathcal{M}_n^{\alpha,\lambda}$ preserve constant functions. Next, we can write

$$\left\| \mathcal{M}_n^{\alpha,\lambda}(e_1;x) - x \right\|_2 \leq \sup_{x\in[0,\infty)} \frac{\alpha}{n(1+\lambda e)(1+x^2)} \to 0,$$

for adequately large n. Therefore, the condition (3.1) is satisfied for $l = 1$.

Finally, we write

$$\left\| \mathcal{M}_n^{\alpha,\lambda}(e_2;x) - x^2 \right\|_2 \leq \sup_{x\in[0,\infty)} \frac{1}{(1+x^2)} \left\{ \frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2 + 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)} \right) \right\},$$

which suggests

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(e_2;x) - e_2 \right\|_2 = 0.$$

Hence, the theorem follows. □

Let $C_B[0,\infty)$ be the class of all continuous and bounded real valued functions. We define the r-th order modulus of continuity by $\omega_r(f,\delta)$ and define it as

$$\omega_r(f,\delta) = \sup_{x\in[0,\infty)} \sup_{0\le h\le\delta} \left|\Delta_h^r f(x)\right|,$$

where Δ denotes the forward difference. In particular, the usual modulus of continuity is defined for $r=1$ and is denoted by $\omega(f,\delta)$. Moreover, we define the norm as $\|f\| = \sup\limits_{x\in[0,\infty)} |f(x)|$.

Also, the Peetre's K-functional for the function $g\in C_B^2[0,\infty)$ is defined as:

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta\|g''\| : g\in C_B^2[0,\infty)\right\},$$

where

$$C_B^2[0,\infty) = \{g\in C_B[0,\infty) : g', g''\in C_B[0,\infty)\}.$$

The next theorem establishes the degree of approximation of the operators $\mathcal{M}_n^{\alpha,\lambda}$ in terms of the usual and second order modulus of continuity for the functions in the space $C_B[0,\infty)$.

Theorem 3.3. *For $\hbar\in C_B^2[0,\infty)$, define the auxiliary operators $\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}$ as*

$$\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar;x) = \mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right) + \hbar(x). \tag{3.2}$$

Then there exists a constant $C>0$ such that

$$|\mathcal{M}_n^{\alpha,\lambda}(\hbar,x) - \hbar(x)| \le C\omega_2(\hbar,\sqrt{\delta}) + \omega(\hbar,\delta_n^{(1)}(x)),$$

where

$$\delta = \delta_n^{(2)}(x) + \left(\frac{\alpha}{n(1+\lambda e)}\right)^2.$$

Proof. Using Lemma 2.2, one can easily observe that $\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}((t-x);x) = 0$.

Let $f\in C_B^2[0,\infty)$, then by Taylor's expansion we have

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du.$$

Moreover, we can write

$$\begin{aligned}\left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(f;x) - f(x)\right| &= \left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right| \\ &\le \left|\mathcal{M}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right| \\ &\quad + \left|\int_x^{\left(x+\frac{\alpha}{n(1+\lambda e)}\right)} \left(\left(x+\frac{\alpha}{n(1+\lambda e)}\right) - u\right) f''(u)du\right|\end{aligned}$$

$$\leq \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n\left(1+\lambda e\right)}\right)^2\right)\left\|f''\right\|. \tag{3.3}$$

Since we know that

$$\left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)\right| \leq \left\|\hbar\right\|,$$

therefore

$$\left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar;x)\right| \leq \left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)\right|+\left|\hbar\left(x+\frac{\alpha}{n\left(1+\lambda e\right)}\right)\right|+\left|\hbar(x)\right| \leq 3\left\|\hbar\right\|. \tag{3.4}$$

Finally, combining equations (3.2), (3.3) and (3.4), we get

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)-\hbar(x)\right| \leq& \left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(\hbar-f);x)-(\hbar-f)(x)\right|+\left|\widetilde{\mathcal{Q}}_n^{\alpha,\lambda}(f;x)-f(x)\right|\\ &+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right|\\ \leq& 4\left\|\hbar-f\right\|+\left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n\left(1+\lambda e\right)}\right)^2\right)\left\|f''\right\|\\ &+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right|\\ \leq& C\left\{\left\|\hbar-f\right\|+\left(\delta_n^{(2)}\left(x\right)+\left(\frac{\alpha}{n\left(1+\lambda e\right)}\right)^2\right)\left\|f''\right\|\right\}\\ &+\omega\left(\hbar,\frac{\alpha}{n\left(1+\lambda e\right)}\right).\end{aligned}$$

Taking infimum over all $f \in C_B^2[0,\infty)$ and using the result $K_2\left(f,\delta\right) \leq \omega_2\left(f,\sqrt{\delta}\right)$ due to [6], we get the desired outcome. □

Theorem 3.4. *Let $\hbar \in C_B(0,\infty)$, then for any $r>0$, $x \in [0,r]$ and adequately large n, we have*

$$\left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)-\hbar\left(x\right)\right| \leqslant 4H_\hbar\left(1+x^2\right)\frac{D}{n}+2\omega_{r+1}\left(\hbar,\sqrt{\frac{D}{n}}\right),$$

where D is a positive constant.

Proof. If $x \in [0,r]$ and $t > r+1$, then $t-x>1$. Therefore, we have the following inequality:

$$\left|\hbar\left(t\right)-\hbar\left(x\right)\right| \leqslant 4H_\hbar\left(1+x^2\right)\left(t-x\right)^2.$$

Again for $x \in [0,r]$ and $t \in [0,r+1]$ and using the well known inequality $\omega\left(f,\beta\delta\right) \leqslant \left(\beta+1\right)\omega\left(f,\delta\right), \beta \in (0,\infty)$, one can obtain

$$\left|\hbar\left(t\right)-\hbar\left(x\right)\right| \leqslant \left(1+\frac{|t-x|}{\delta}\right)\omega_{r+1}\left(\hbar,\delta\right).$$

From (3.5) and (3.5), we can write

$$\left|\hbar\left(t\right)-\hbar\left(x\right)\right|\leqslant 4H_{\hbar}\left(1+x^{2}\right)\left(t-x\right)^{2}+\left(1+\frac{\left|t-x\right|}{\delta}\right)\omega_{r+1}\left(\hbar,\delta\right).$$

Applying operator $\mathcal{M}_n^{\alpha,\lambda}$ in the above relation and making use of Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)-\hbar\left(x\right)\right| \leqslant & 4H_{\hbar}\left(1+x^{2}\right)\mathcal{M}_n^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\\ & +\left(1+\frac{1}{\delta}\mathcal{M}_n^{\alpha,\lambda}\left(\left|t-x\right|;x\right)\right)\omega_{r+1}\left(\hbar,\delta\right)\\ \leqslant & 4H_{\hbar}\left(1+x^{2}\right)\mathcal{M}_n^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\\ & +2\omega_{r+1}\left(\hbar,\sqrt{\mathcal{M}_n^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)}\right).\end{aligned}$$

Since $\mathcal{M}_n^{\alpha,\lambda}\left(\left(t-x\right)^{2};x\right)\leqslant\frac{D}{n}$, where D is a positive constant, it follows that for adequately large n, we have

$$\left|\mathcal{M}_n^{\alpha,\lambda}\left(\hbar;x\right)-\hbar\left(x\right)\right|\leqslant 4H_{\hbar}\left(1+x^{2}\right)\frac{D}{n}+2\omega_{r+1}\left(\hbar,\sqrt{\frac{D}{n}}\right),$$

which is the required result. □

3.2. **Quantitative Voronovskaya theorem and Voronovoskaya theorem.** Let $C_B\left[0,\infty\right)$ be the subspace of $B_f\left[0,\infty\right)$ containing all continuous and bounded functions f for which $\lim\limits_{x\to\infty}\left|f\left(x\right)\right|\left(1+x^{2}\right)^{-1}$ is finite.

The weighted modulus of continuity $\Omega\left(f,\delta\right)$ due to [1] for each $f\in C_B\left[0,\infty\right)$ is defined as

$$\Omega\left(f,\delta\right)=\sup_{x\in[0,\infty),|h|<\delta}\frac{\left|f\left(x+h\right)-f\left(x\right)\right|}{\left(1+h^{2}+x^{2}+h^{2}x^{2}\right)}.$$

In the next theorem, we discuss the quantitative Voronovskaya theorem for the proposed operators (1.3) and derive a Voronovoskaya asymptotic result as a resulting corollary, making use of the following properties of weighted modulus of continuity. For every $f\in C_f^l[0,\infty)$,

(a) $\Omega\left(f,\delta\right)\to 0$ for $\delta\to 0$;
(b) $\left|f\left(t\right)-f\left(x\right)\right|\leq\left(1+\left(t-x\right)^{2}\right)\left(1+x^{2}\right)\Omega\left(f,\left|t-x\right|\right).$

Theorem 3.5. *Let $\hbar''\in C_2^l\left[0,\infty\right)$ and $x\in\left[0,\infty\right)$. Then we have*

$$\begin{aligned}&\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)-\hbar(x)-\frac{\alpha}{n\left(1+\lambda e\right)}\hbar'\left(x\right)\right.\\ &\left.-\frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right)+\frac{1}{n^{2}}\left(\frac{\alpha^{2}+2\alpha\lambda e-\alpha e^{2}\lambda^{2}}{\left(1+\lambda e\right)^{2}}+\frac{\alpha}{\rho\left(1+\lambda e\right)}\right)\right)\hbar''\left(x\right)\right|\end{aligned}$$

$$\leq \frac{8\left(1+x^2\right)}{n}\Omega\left(\hbar'', \frac{1}{\sqrt{n}}\right).$$

Proof. By Taylor's expansion, we may write

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(\hbar; x) - \hbar(x) &= \mathcal{M}_n^{\alpha,\lambda}\left((\hbar(t) - \hbar(x)); x\right)\\ &= \mathcal{M}_n^{\alpha,\lambda}\left((t-x)\,\hbar'(x) + \frac{(t-x)^2}{2}\hbar''(x) + \lambda\!\!\!{}^-(t,x)\,(t-x)^2; x\right),\end{aligned}$$

where $\bar{\lambda}(t,x) = (\hbar''(\varsigma) - \hbar''(x))/2$ is a continuous function which tends to zero at 0 and ς lies between x and t. Using Lemma 2.3, we get

$$\begin{aligned}&\Bigg|\mathcal{M}_n^{\alpha,\lambda}(\hbar; x) - \hbar(x) - \frac{\alpha}{n(1+\lambda e)}\hbar'(x)\\ &-\frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2 + 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right)\right)\hbar''(x)\Bigg|\\ &\leq \mathcal{M}_n^{\alpha,\lambda}\left(|\bar{\lambda}(t,x)|\,(t-x)^2; x\right).\end{aligned}$$

With simple manipulations in property (b) of weighted modulus of continuity and using $|\varsigma - x| \leq |t-x|$, we can write

$$|\bar{\lambda}(t,x)| \leq 8\left(1+x^2\right)\left(1 + \frac{(t-x)^4}{\delta^4}\right)\Omega\left(\hbar'', \delta\right),$$

which implies that

$$|\bar{\lambda}(t,x)|\,(t-x)^2 \leq 8\left(1+x^2\right)\left((t-x)^2 + \frac{(t-x)^6}{\delta^4}\right)\Omega\left(\hbar'', \delta\right).$$

Therefore, in view of Lemma 2.3, we can write

$$\mathcal{M}_n^{\alpha,\lambda}\left(|\bar{\lambda}(t,x)|\,(t-x)^2; x\right) \leq 8\left(1+x^2\right)\Omega\left(\hbar'', \delta\right)\left\{\delta_n^{(2)}(x) + \frac{1}{\delta^4}\delta_n^{(6)}(x)\right\},$$

as $n \to \infty$. Choosing $\delta = \frac{1}{\sqrt{n}}$, we get the desired outcome. □

Corollary 3.1. *Let f be a bounded and integrable function on the interval* $[0,\infty)$ *such that the second derivative of f exists at a fixed point $x \in [0,\infty)$. Then*

$$\lim_{n\to\infty} n(\mathcal{M}_n^{\alpha,\lambda}(f; x) - f(x)) = \frac{\alpha}{(1+\lambda e)}f'(x) + x\left(1+\frac{1}{\rho}\right)f''(x).$$

3.3. **Functions of derivatives of bounded variation.** Next we estimate the rate of convergence of the operators (1.3) for functions with derivatives of bounded variation defined on $[0,\infty)$.

Let $f \in \mathrm{DBV}_\tau[0,\infty)$ be the class of functions whose derivatives are of bounded variation on any finite subinterval of $[0,\infty)$ and satisfy the growth condition $|f(t)| \leq$

Kt^{τ}, $\tau > 0$ for all $t > 0$ and constant $K > 0$. For such functions, let us represent our proposed operators (1.3) in the following form

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \int_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)f(t)dt, \tag{3.5}$$

where $\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t) = \sum\limits_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) l_{n,k}^{\rho}(t)$.

Lemma 3.1. *For $x \in [0,\infty)$ and adequately large n, we have*

(i) *if $0 \le y < x$, then*

$$\vartheta_n(x,y) = \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \le \frac{K\delta_n^2(x)}{n(x-y)^2};$$

(ii) *if $x < z \le \infty$, then*

$$1 - \vartheta_n(x,z) = \int_z^\infty \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \le \frac{K\delta_n^2(x)}{n(z-x)^2}.$$

Proof. (i) Taking into account Lemma 2.2 and proposed operators (1.3), we have

$$\begin{aligned}\vartheta_n(x,z) &= \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \\ &\le \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\left(\frac{x-t}{x-y}\right)^2 dt = \frac{1}{(x-y)^2}\int_0^y (t-x)^2 \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \\ &\le \frac{1}{(x-y)^2}\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right) \le \frac{K\delta_n^2(x)}{n(x-y)^2}.\end{aligned}$$

Proof of (ii) is similar to (i). □

Theorem 3.6. *Consider a function f of bounded variation on every sub-interval of $[0,\infty)$ that satisfies the growth condition $|f(t)| \le Kt^{\tau}$ for some absolute constant K and $\tau > 0$. If there exists an integer γ, $(2\gamma \ge \tau)$ such that $f(t) \le O(t^{\gamma})$ for every $t > 0$, then for $\gamma > 0$, $x \in [0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)\right| \le& \frac{1}{2}\left(f'(x+) + f'(x-)\right)\delta_n^1(x) \\ &+ \sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+) - f'(x-)\right| + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(f'_x) \\ &+ \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{x}{\sqrt{k}}}(f'_x) + \frac{K\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x)\right| \\ &+ \wp(\gamma,\tau,x) + \frac{K\delta_n^2(x)}{nx^2}|f(x)| + \sqrt{\frac{K\delta_n^2(x)}{n}}f'(x+),\end{aligned}$$

where $\bigvee\limits_{b}^{a}(f)$ *denotes the total variation of* f *on any finite subinterval* $[a,b]$ *of* $[0,\infty)$ *and* $\wp(\gamma,\tau,x) := 2^{\gamma}Ç\left(\int\limits_0^{\infty}(t-x)^{2\tau}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right)^{\frac{\gamma}{2\tau}}$.

Proof. For $x\in[0,\infty)$, we can write for our proposed operators (3.5) that

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x) &= \int\limits_0^{\infty}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\left(f(t)-f(x)\right)dt\\ &= \int\limits_0^{\infty}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\left(\int\limits_x^{t}f'(x)\,du\right)dt. \qquad (3.6)\end{aligned}$$

Also for any $f\in DBV_{\gamma}[0,\infty)$, equality (3.7) holds true, i.e.,

$$\begin{aligned}f'(u) =&\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\\ &+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right), \qquad (3.7)\end{aligned}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x,\\ 0, & u\neq x.\end{cases}$$

It can be easily verified that:

$$\int\limits_0^{\infty}\left(\int\limits_x^{t}\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)\delta_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt=0.$$

Now in view of our proposed operators (3.5), we may write

$$\begin{aligned}&\int\limits_0^{\infty}\left(\int\limits_x^{t}\left(\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ =&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right).\end{aligned}$$

Moreover

$$\begin{aligned}&\int\limits_0^{\infty}\left(\int\limits_x^{t}\left(\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\right)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ \leq&\frac{1}{2}\left|f'(x+)-f'(x-)\right|\int\limits_0^{\infty}|t-x|\,\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\ \leq&\frac{1}{2}\left|f'(x+)-f'(x-)\right|\left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)\right)^{1/2}. \qquad (3.8)\end{aligned}$$

Making use of equations (3.7)–(3.8) and Lemma 2.2 in equation (3.6), we get

$$\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)\leq\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left|f'(x+)-f'(x-)\right|\left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)\right)^{1/2}\\
&+\int_0^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\
\leq&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)\\
&+\sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+)-f'(x-)\right|\\
&+\int_0^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt.
\end{aligned}
$$

Taking absolute values on both sides and rewriting equation we have:

$$
\begin{aligned}
\left|\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)\right|\leq&\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}\left((t-x);x\right)\\
&+\sqrt{\frac{K\delta_n^2(x)}{4n}}\left|f'(x+)-f'(x-)\right|+P_{n_1}(x)+P_{n_2}(x),
\end{aligned}
\tag{3.9}
$$

where

$$P_{n_1}(x)=\left|\int_0^x\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|$$

and

$$P_{n_2}(x)=\left|\int_x^\infty\left(\int_x^t f'_x(x)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|.$$

Integrating by parts after applying Lemma 3.1, and taking $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$P_{n_1}(x)\leq\int_0^{x-\frac{x}{\sqrt{n}}}\vartheta_n(x;t)\left|f'_x(t)\right|dt+\int_{x-\frac{x}{\sqrt{n}}}^x\vartheta_n(x;t)\left|f'_x(t)\right|dt.$$

Since $f'_x(x)=0$ and $\vartheta_n(x;t)\leq 1$, it implies

$$
\begin{aligned}
\int_{x-\frac{x}{\sqrt{n}}}^x\vartheta_n(x;t)\left|f'_x(t)\right|dt=&\int_{x-\frac{x}{\sqrt{n}}}^x\vartheta_n(x;t)\left|f'_x(t)-f'_x(x)\right|dt\\
\leq&\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_t^x(f'_x)\,dt\leq\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(f'_x).
\end{aligned}
$$

Again using Lemma 3.1 and substituting $y=x-\frac{x}{u}$ we obtain

$$\int_0^{x-\frac{x}{\sqrt{n}}}\vartheta_n(x;t)\left|f'_x(t)\right|dt\leq\frac{K\delta_n^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\left|f'_x(t)\right|}{(x-t)^2}dt\leq\frac{K\delta_n^2(x)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x(f'_x)\,du$$

$$\leq \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f'_x).$$

Thus, we can write $P_{n_1}(x)$ as

$$P_{n_1}(x) \leq \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f'_x) + \frac{K\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(f'_x). \tag{3.10}$$

Next, to estimate $P_{n_2}(x)$, we have

$$\begin{aligned} P_{n_2}(x) &\leq \left|\int_x^{2x}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right| + \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right| \\ &\leq A_n(x) + B_n(x), \end{aligned}$$

where

$$A_n(x) = \left|\int_x^{2x}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|$$

and

$$B_n(x) = \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)\,du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right|.$$

Since $1-\vartheta_n(x,t) \leq 1$, by putting $t = x + \frac{x}{u}$ successively, we have

$$\begin{aligned} A_n(x) =& \left|\int_x^{2x} f'_x(u)\,(1-\vartheta_n(x,2x))\,du - \int_x^{2x} f'_x(t)\,(1-\vartheta_n(x,t)\,dt\right| \\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \int_x^{x+\frac{x}{\sqrt{n}}}\left|f'_x(t)\right|\left|1-\vartheta_n(x,t)\right|dt \\ &+ \int_{x+\frac{x}{\sqrt{n}}}^{2x}\left|f'_x(t)\right|\left|1-\vartheta_n(x,t)\right|dt \\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \frac{K\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{V_x^t(f'_x)}{(t-x)^2}dt + \int_x^{x+\frac{x}{\sqrt{n}}}\bigvee_x^t(f'_x)dt \\ \leq& \frac{K\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right| + \frac{K\delta_n^2(x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{\sqrt{k}}}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(f'_x). \end{aligned}$$

Further we estimate the value of $B_n(x)$ as follows:

$$B_n(x) = \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)dt\right| \tag{3.11}$$

$$\leq Ç\int_{2x}^{\infty} t^{\gamma}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt+|f(x)|\int_{2x}^{\infty}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt+\sqrt{\frac{K\delta_n^2(x)}{n}}f'(x+). \tag{3.12}$$

It is obvious that $t \leq 2(t-x)$ and $x \leq t-x$, when $t \geq 2x$. Now applying Hölder's inequality in the first term of equation (3.11), we get

$$B_n(x) = 2^{\gamma}Ç\left(\int_0^{\infty}(t-x)^{2\tau}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right)^{\frac{\gamma}{2\tau}}+\frac{K\delta_n^2(x)}{nx^2}|f(x)|+\sqrt{\frac{K\delta_n^2(x)}{n}}f'(x+)$$

$$= \wp(\gamma,\tau,x)+\frac{K\delta_n^2(x)}{nx^2}|f(x)|+\sqrt{\frac{K\delta_n^2(x)}{n}}f'(x+). \tag{3.13}$$

Finally, combining equations (3.10)–(3.13) and putting values of $P_{n_1}(x)$ and $P_{n_2}(x)$ in (3.9), we get the required result and the theorem is proved. $\square$

Example 3.1. Let $f(x) = x^4 - 3x^3 + 2x^2 + 1$. We choose parameters $\alpha = \lambda = 2$ and $\rho = 3$. For $n = 10, 50, 100, 200$, we have the following representations.

(a) Figure 1 shows the rate of approximation of the operators $\mathcal{M}_n^{\alpha,\lambda}$ towards the function f. Clearly the proposed operators (1.3) converge to the function f for sufficiently large n.

(b) In Figure 2, the associated absolute error $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ is represented graphically for arbitrary values of x in interval $[0,\infty)$. It can be observed that error is monotonically decreasing for increasing n.

(c) An error estimation table is provided in Table 1 which depicts that for higher value of n, the error approaches to zero.

Therefore, it can be concluded that proposed operators (1.3) provide good approximation for n adequately large.

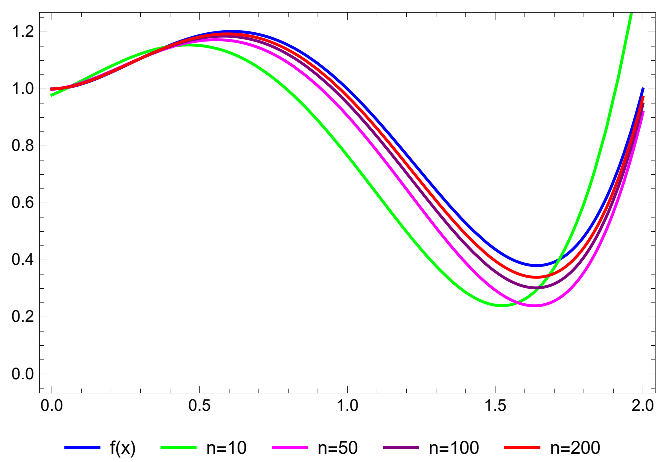

FIGURE 1. Convergence of $\mathcal{M}_n^{\alpha,\lambda}(f;x)$ for the polynomial function $f(x) = x^4 - 3x^3 + 2x^2 + 1$ with parameters $\alpha = \lambda = 2$, $\rho = 3$.

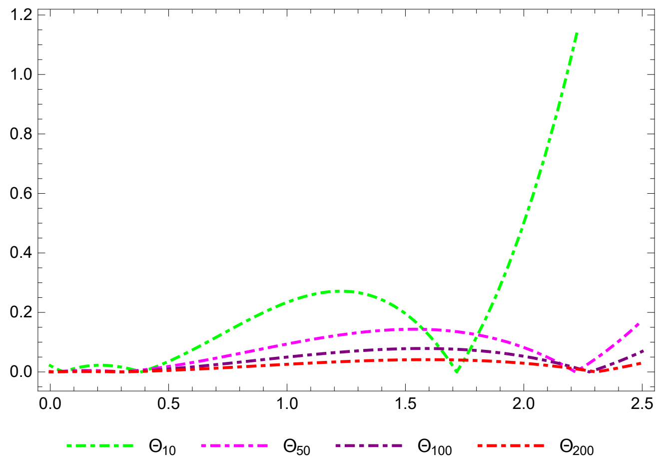

FIGURE 2. Absolute error $\Theta_n = \mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators for $f(x) = x^4 - 3x^3 + 2x^2 + 1$ with parameters $\alpha = \lambda = 2$, $\rho = 3$.

TABLE 1. Table for Absolute error $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators $\mathcal{M}_n^{\alpha,\lambda}$.

x	Θ_{10}	Θ_{50}	Θ_{100}	Θ_{200}
0.4	0.0100759	0.00856603	0.0045899	0.00236582
0.8	0.166213	0.0652568	0.0345534	0.0177532
1.2	0.232451	0.120425	0.0647845	0.0335316
1.6	0.0454756	0.123219	0.0698567	0.0369876
2.0	0.921829	0.022785	0.0243437	0.0154083
2.4	2.65087	0.231729	0.0971809	0.0439198
2.8	5.48687	0.691177	0.320143	0.15371
3.2	9.68409	1.40641	0.66997	0.326674
3.6	15.4968	2.42828	1.17209	0.575527
4.0	23.1792	3.80765	1.85192	0.912982

Example 3.2. Figure 3 illustrates the effect of increase in values of parameter ρ on the rate of convergence of proposed operators $\mathcal{M}_n^{\alpha,\lambda}$ for the function $f(x) = 4x(x - 1.1)(x - 1.9)$ while keeping the value of α, λ and n fixed. Here we chose $n = 10$ and $\alpha = \lambda = 2$ to show the impact of the parameter ρ clearly. One can easily deduce from the figure that as we increase the value of ρ the rate of convergence gets relatively faster.

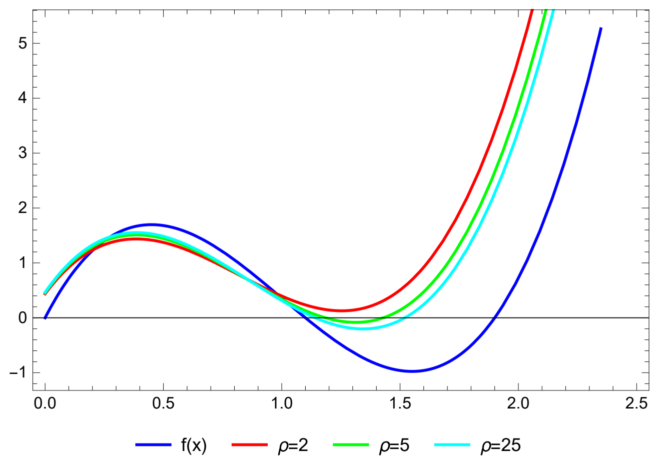

Figure 3. Effect of increase in parametric value of ρ for given $n = 10$, $\alpha = \lambda = 2$ on the convergence rate of proposed operators.

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References

[1] T. Acar, A. Aral and I. Raşa, *The new forms of Voronovskayas theorem in weighted spaces*, Positivity **20**(1) (2016), 25–40. https://doi.org/10.1007/s11117-015-0338-4

[2] A. M. Acu, T. Acar and V. A. Radu, *Approximation by modified U_n^ρ operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.). Serie A. Matemáticas **113**(3) (2019), 2715–2729. https://doi.org/10.1007/s13398-019-00655-y

[3] A. M. Acu and V. A. Radu, *Approximation by certain operators linking the α-Bernstein and the Genuine α-Bernstein-Durrmeyer operators*, in: N.Deo et al. (Eds.), *Mathematical Analysis I: Approximation Theory*, Springer, Proceedings in Mathematics & Statistics **306**, 2020, 77–88.

[4] I. N. Cangul, H. Ozden and Y. Simsek, *A new approach to q-Genocchi numbers and their interpolation functions*, Nonlinear Analysis: Theory, Methods & Applications **71**(12) (2009), e793–e799. https://doi.org/10.1016/j.na.2008.11.040

[5] N. Deo and S. Kumar, *Durrmeyer variant of Apostol-Genocchi-Baskakov operators*, Quaest. Math. **44**(4) (2020), 1–18. https://doi.org/10.2989/16073606.2020.1834000

[6] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[7] M. Dhamija, *Durrmeyer modification of Lupas type Baskakov operators based on IPED*, in: N. Deo et al. (Eds.), *Mathematical Analysis I: Approximation Theory*, Springer, Proceedings in Mathematics & Statistics **306**, 2020, 111–120.

[8] M. Dhamija and N. Deo, *Jain-Durrmeyer operators associated with the inverse Pólya–Eggenberger distribution*, Appl. Math. Comput. **286** (2016), 15–22. https://doi.org/10.1016/j.amc.2016.03.015

[9] A. D. Gadjiev, *Theorems of the type of PP Korovkins theorems*, Mat. Zametki. **20**(5) (1976), 781–786.

[10] T. Garg, A. M. Acu and P. N. Agrawal, *Further results concerning some general Durrmeyer type operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(3) (2019), 2373–2390. https://doi.org/10.1007/s13398-019-00628-1

[11] V. Gupta, *A note on the general family of operators preserving linear functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(4) (2019), 3717–3725. https://doi.org/10.1007/s13398-019-00727-z

[12] M. Heilmann and I. Raşa, *A nice representation for a link between Baskakov and Szász-Mirakjan-Durrmeyer operators and their Kantorovich variants*, Results Math. **74**(1) (2019), 1–12. https://doi.org/10.1007/s00025-018-0932-4

[13] A. Kajla, N. Ispir, P. N. Agrawal and M. Goyal, *q-Bernstein-Schurer-Durrmeyer type operators for functions of one and two variables*, Appl. Math. Comput. **275** (2016), 372–385. https://doi.org/10.1016/j.amc.2015.11.048

[14] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publication Co., Delhi 1960.

[15] Q. M. Luo, *q-Extensions for the Apostol-Genocchi polynomials*, General Mathematics **17**(2) (2009), 113–125.

[16] Q. M. Luo, *Extensions of the Genocchi polynomials and their Fourier expansions and integral representations*, Osaka J. Math. **48** (2011), 291–309.

[17] Q. M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind*, Appl. Math. Comput. **217**(12) (2011), 5702–5728. https://doi.org/10.1016/j.amc.2010.12.048

[18] M. A. Özarslan, *Unified Apostol-Bernoulli, Euler and Genocchi polynomials*, Comput. Math. Appl. **62**(6) (2011), 2452–2462. https://doi.org/10.1016/j.camwa.2011.07.031

[19] R. Păltănea, *Modified Szász-Mirakjan operators of integral form*, Carpathian J. Math. **24**(3) (2008), 378–385.

[20] C. Prakash, D. K. Verma and N. Deo, *Approximation by a new sequence of operators involving Apostol-Genocchi polynomials*, Math. Slovaca (to appear).

[21] H. M. Srivastava, B. Kurt and V. Kurt, *Identities and relations involving the modified degenerate hermite-based Apostol-Bernoulli and Apostol-Euler polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.). Serie A. Matemáticas **113**(2) (2019), 1299–1313. https://doi.org/10.1007/s13398-018-0549-1

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