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# LOCAL $K$-CONVOLUTED $C$-GROUPS AND ABSTRACT CAUCHY PROBLEMS

CHUNG-CHENG KUO[1]

Abstract. We first present a new form of a local $K$-convoluted $C$-group on a Banach space $X$, and then deduce some basic properties of a nondegenerate local $K$-convoluted $C$-group on $X$ and some generation theorems of local $K$-convoluted $C$-groups, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local $K$-convoluted $C$-group on $X$ with subgenerator $A$ and the unique existence of solutions of the abstract Cauchy problem ACP$(A, f, x)$.

## 1. Introduction

Let $X$ be a Banach space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|$, and let $L(X)$ denote the family of all bounded linear operators from $X$ into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\mathrm{ACP}(A, f, x)\ \begin{cases} u'(t) = Au(t) + f(t), & \text{for } t \in (-T_0, T_0), \\ u(0) = x, \end{cases}$$

where $x \in X$, $A$ is a closed linear operator in $X$, and $f \in L^1_{loc}((-T_0, T_0), X)$ (the family of all locally integrable functions from $(-T_0, T_0)$ into $X$). A function $u$ is called a solution of ACP$(A, f, x)$ if $u \in \mathrm{C}((-T_0, T_0), X)$ satisfies ACP$(A, f, x)$ (that is, $u(0) = x$ and for a.e. $t \in (-T_0, T_0)$, $u(t)$ is differentiable and $u(t) \in D(A)$, and $u'(t)=Au(t)+f(t)$ for a.e. $t \in (-T_0, T_0)$). For each $C \in L(X)$ and $K \in L^1_{loc}([0, T_0), \mathbb{F})$, a family $S(\cdot)(= \{S(t) \,|\, |t| < T_0\})$ in $L(X)$ is called a local $K$-convoluted $C$-group on

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$X$ if $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x$$
$$= \left(\operatorname{sgn} t \operatorname{sgn} s \operatorname{sgn}(t+s)\int_0^{t+s} - \operatorname{sgn} s\int_0^t - \operatorname{sgn} t\int_0^s\right) K(|t+s-r|)S(r)Cx dr,$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$. In particular, $S(\cdot)$ is called a local (0-times integrated) $C$-group on $X$ if $K = j_{-1}$ (the Dirac measure at 0) or equivalently, $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x = S(t+s)Cx, \quad \text{for all } x \in X \text{ and } |t|, |s|, |t+s| < T_0,$$

(see [2]). Moreover, we say that $S(\cdot)$ is nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $|t| < T_0$. The nondegeneracy of a local $K$-convoluted $C$-group $S(\cdot)$ on $X$ implies that

$$S(0) = C \text{ if } K = j_{-1} \text{ and } S(0) = 0 \text{ (the zero operator on X) otherwise,}$$

and the (integral) generator $A : \mathrm{D}(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in $X$ defined by

$$\mathrm{D}(A) = \{x \in X \mid S(\cdot)x - K_0(|\cdot|)Cx = \widetilde{S}(\cdot)y_x \text{ on } (-T_0, T_0) \text{ for some } y_x \in X\}$$

and $Ax = y_x$ for all $x \in \mathrm{D}(A)$. Here $\widetilde{S}(t)z = \int_0^t S(s)z ds$. In general, a local $K$-convoluted $C$-group on $X$ is called a $K$-convoluted $C$-group on $X$ if $T_0 = \infty$; a (local) $K$-convoluted $C$-group on $X$ is called a (local) $K$-convoluted group on $X$ if $C = I$ (the identity operator on $X$) or a (local) $\alpha$-times integrated $C$-group on $X$ if $K$ is equal to the function $j_{\alpha-1}$ for some $\alpha \geq 0$, defined by $j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ (see [4, 7, 21]). Here $\Gamma(\cdot)$ denotes the Gamma function, a (local) $\alpha$-times integrated $C$-group on $X$ is called a (local) $\alpha$-times integrated group on $X$ if $C = I$; and a (local) $C$-group on $X$ is called a $c_0$-group on $X$ if $C = I$ (see [1, 5]). Some basic properites of a nondegenerate (local) $\alpha$-times integrated $C$-semigroup on $X$ have been established by many authors (in [2, 3, 26–28] for $\alpha = 0$, and in [6, 10, 17–20, 22, 23, 25, 29, 30] for $\alpha > 0$), which can be extended to the case of local $K$-convoluted $C$-semigroup just as results in [7–10, 13–16]. Some equivalence relations between the generation of a nondegenerate (local) $K$-convoluted $C$-semigroup on $X$ with subgenerator $A$ and the unique existence of solutions of the abstract Cauchy problem $\mathrm{ACP}(A, f, x)$ are also discussed in [2, 26, 27] for the case $K = j_{\alpha-1}$ with $\alpha = 0$ and in [11–13, 30, 31] with $\alpha > 0$, and in [8, 13, 16] for the general case. The purpose of this paper is to investigate the following basic properties of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ on $X$ just as results in [13] concerning local $K$-convoluted $C$-semigroups on $X$ when $C$ is injective and some additional conditions are taken into consideration

(1.1) $C^{-1}AC = A$,

(1.2) $\widetilde{S}(t)x \in D(A)$ and $A\widetilde{S}(t)x = S(t)x - K_0(|t|)Cx, \quad$ for all $x \in X$ and $|t| < T_0$,

(1.3) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax, \quad$ for all $x \in D(A)$ and $|t| < T_0$;

and

$$(1.4) \qquad S(t)S(s) = S(s)S(t), \quad \text{on } X, \text{ for all } |t|, |s| < T_0,$$

(see Theorems 2.5, 2.6 and 2.7 below), which have been established partially in [8] by another method, and then deduce some equivalence relations between the generation of a nondegenerate local $K$-convoluted $C$-group on $X$ with subgenerator $A$ and the unique existence of solutions of ACP$(A, f, x)$, which are similar to some results in [13] concerning equivalence relations between the generation of a nondegenerate local $K$-convoluted $C$-semigroup on $X$ with subgenerator $A$ and the unique existence of solutions of ACP$(A, f, x)$. To do these, we will first prove an important lemma which shows that a strongly continuous family $S(\cdot)$ in $\mathrm{L}(X)$ is a local $K$-convoluted $C$-group on $X$ is equivalent to $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$ is a local $K_0$-convoluted $C$-group on $X$ (see Lemma 2.1 below), and then show that a strongly continuous family $S(\cdot)$ in $L(X)$ which commutes with $C$ on $X$ is a local $K$-convoluted $C$-group on $X$ is equivalent to $\widetilde{S}(t)[S(s)-K_0(|s|)C]=[S(t)-K_0(|t|)C]\widetilde{S}(s)$ for all $|t|, |s|, |t+s| < T_0$ (see Theorem 2.1 below). In order to show that $\mathrm{sgn}(\cdot)b*S(\cdot)$ is a local $a*K$-convoluted $C$-group on $X$ if $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ and $b(\cdot) = a(|\cdot|)$ for some $a \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$. In particular, $\mathrm{sgn}(\cdot)J_\beta * S(\cdot)$ is a local $K_\beta$-convoluted $C$-group on $X$ if $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ and $\beta > -1$, which can be applied to show that its only if part is also true when $\beta$ is a nonnegative integer (see Proposition 2.1 below). Here $K_\beta(t) = K * j_\beta(t)$ for $\beta > -1$, $J_\beta(\cdot) = j_\beta(|\cdot|)$, $f * S(t)x = \int_0^t f(t-s)S(s)x ds$ for all $x \in X$ and $f \in L^1_{loc}((-T_0, T_0), \mathbb{F})$. We also show that a strongly continuous family $S(\cdot)$ in $L(X)$ which commutes with $C$ on $X$ is a local $K$-convoluted $C$-group on $X$ when it has a subgenerator (see Theorem 2.4 below). Moreover, $S(\cdot)$ is nondegenerate if $C$ is injective and the generator of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)$ on $X$ is the unique subgenerator of $S(\cdot)$ which contains all its subgenerators, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$ when $S(\cdot)$ has a subgenerator (see Theorems 2.5, 2.6 and 2.7 below). This can be applied to show that $CA \subset AC$ and $S(\cdot)$ is a nondegenerate local $K$-convoluted $C$-group on $X$ with generator $C^{-1}AC$ when $C$ is injective, $K_0$ a kernel on $[0, T_0)$ (that is, $f = 0$ on $[0, T_0)$ whenever $f \in C([0, T_0), \mathbb{F})$ with $\int_0^t K_0(t-s)f(s)ds = 0$ for all $0 \le t < T_0$) and $S(\cdot)$ a strongly continuous family in $\mathrm{L}(X)$ with closed subgenerator $A$. In this case, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator $A_0$ of $S(\cdot)$ (see Theorem 2.8 below). Some illustrative examples concerning these theorems are also presented in the final part of this paper.

## 2. Basic Properties of Local $K$-Convoluted $C$-Groups

In the following we will note some facts concerning local $K$-convoluted $C$-groups which can be expansively appled in this paper.

*Remark* 2.1. Let $S(\cdot)(= \{S(t) \,|\, |t| < T_0\})$ be a strongly continuous family in $L(X)$. Then the following are equivalent.

(i) $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$.

(ii) (see [8]) $S_+(\cdot)$ and $S_-(\cdot)$ are local $K$-convoluted $C$-semigroups on $X$, $S(t)S(s)x = S(s)S(t)x$ on $X$ for all $-T_0 < t \leq 0 \leq s < T_0$,

$$S(t)S(s)x = \int_{t+s}^{s} K(r-t-s)S(r)Cxdr + \int_{t}^{0} K(t+s-r)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$, and

$$S(t)S(s)x = \int_{t}^{t+s} K(t+s-r)S(r)Cxdr + \int_{0}^{s} K(r-t-s)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$.

(iii)

$$T(t)T(s)x = (\int_0^{t+s} - \int_0^t - \int_0^s)K(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$.

Here $T(\cdot) = \mathrm{sgn}(\cdot)S(\cdot)$ on $(-T_0, T_0)$, $S_+(\cdot) = S(\cdot)$ and $S_-(\cdot) = S(-\cdot)$ on $[0, T_0)$.

Next we will deduce an important lemma which can be used to obtain a new equivalence relation between the generation of a local $K$-convoluted $C$-group $S(\cdot)$ on $X$ and the equality of

$$\widetilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\widetilde{S}(s),$$

on $X$ for all $|t|, |s|, |t+s| < T_0$ when $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ is a strongly continuous family in $L(X)$ commuting with $C$ on $X$ just as a result in [13] for the case of local $K$-convoluted $C$-semigroup and in [19] for the case of local $\alpha$-times integrated $C$-semigroup.

**Lemma 2.1.** *Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $L(X)$. Then $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ if and only if* $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$ *is a local $K_0$-convoluted $C$-group on $X$. In this case,*

(i) *$S(\cdot)$ is nondegenerate if and only if $\widetilde{S}(\cdot)$ is;*

(ii) *$A$ is the generator of $S(\cdot)$ if and only if it is the generator of* $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$.

*Proof.* Let $x \in$ X be given. We set $T(\cdot) = \mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$. Then

$$\begin{aligned} &\frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \\ &= -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(s)\widetilde{S}(t)Cx \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\frac{d}{ds}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \\ &= -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(|t|)\widetilde{S}(s)Cx, \end{aligned} \tag{2.2}$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$. Using integration by parts to the right-hand sides of (2.1) and (2.2), we obtain

$$\begin{aligned}(2.3)\quad & -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(s)\widetilde{S}(t)Cx \\ &= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx\end{aligned}$$

and

$$\begin{aligned}(2.4)\quad & -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(|t|)\widetilde{S}(s)Cx \\ &= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,\end{aligned}$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$. Combining (2.1)–(2.4), we have

$$\begin{aligned}(2.5)\quad & \frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \\ &= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx\end{aligned}$$

and

$$\begin{aligned}(2.6)\quad & \frac{d}{ds}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \\ &= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,\end{aligned}$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$. Similarly, we can show that

$$\begin{aligned}(2.7)\quad & \frac{d}{dt}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right] \\ &= -\int_{t}^{t+s} K_0(t+s-r)S(r)Cxdr + \int_{0}^{s} K_0(r-t-s)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad & \frac{d}{ds}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right] \\ &= -\int_{t}^{t+s} K_0(t+s-r)S(r)Cxdr + \int_{0}^{s} K_0(r-t-s)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,\end{aligned}$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. By (2.6) and (2.8), we have

$$\begin{aligned}(2.9)\quad & \frac{d}{ds}\frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \\ &= -\int_{t+s}^{s} K(r-t-s)S(r)Cxdr + \int_{t}^{0} K(t+s-r)S(r)Cxdr,\end{aligned}$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$ and

$$(2.10)\quad \frac{d}{dt}\frac{d}{ds}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right]$$

$$= -\int_t^{t+s} K(t+s-r)S(r)Cxdr + \int_0^s K(r-t-s)S(r)Cxdr,$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Suppose that $T(\cdot)$ is a local $K_0$-convoluted $C$-group on $X$. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local $K_0$-convoluted $C$-semigroups on $X$, $T_+(\cdot) = \widetilde{S_+}(\cdot)$ and $T_-(\cdot) = \widetilde{S_-}(\cdot)$ on $[0, T_0)$, $T(t)T(s) = T(s)T(t)$ on $X$ for all $-T_0 < t \le 0 \le s < T_0$,

$$T(t)T(s)x = \int_{t+s}^s K_0(r-t-s)T(r)Cxdr + \int_t^0 K_0(t+s-r)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$ and

$$T(t)T(s)x = \int_t^{t+s} K_0(t+s-r)T(r)Cxdr + \int_0^s K_0(r-t-s)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$ or equivalently, $S_+(\cdot)$ and $S_-(\cdot)$ both are local $K$-convoluted $C$-semigroups on $X$, $S(t)S(s) = S(s)S(t)$ on $X$ for all $-T_0 < t \le 0 \le s < T_0$

$$(2.11) \qquad -\widetilde{S}(t)\widetilde{S}(s)x = \int_{t+s}^s K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\widetilde{S}(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$, and

$$(2.12) \qquad -\widetilde{S}(t)\widetilde{S}(s)x = -\int_t^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_0^s K_0(r-t-s)\widetilde{S}(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Combining (2.7)–(2.10), we have

$$(2.13) \qquad S(t)S(s)x = \int_{t+s}^s K(r-t-s)S(r)Cxdr + \int_t^0 K(t+s-r)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$ and

$$(2.14) \qquad S(t)S(s)x = \int_t^{t+s} K(t+s-r)S(r)Cxdr + \int_0^s K(r-t-s)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Consequently, $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$. Conversely, suppose that $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local $K_0$-convoluted $C$-semigroups on $X$, $T(t)T(s) = T(s)T(t)$ on $X$ for all $-T_0 < t \le 0 \le s < T_0$, and (2.13)–(2.14) both hold. By (2.9) and (2.10), we have (2.11) and (2.12) both hold. Consequently, $T(\cdot)$ is a local $K_0$-convoluted $C$-group on $X$. □

**Theorem 2.1.** *Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $L(X)$ which commutes with $C$ on $X$. Then $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ if and only if*

$$(2.15) \qquad \widetilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\widetilde{S}(s), \quad \text{on } X,$$

*for all $|t|, |s|, |t+s| < T_0$.*

*Proof.* We set $T(\cdot) = \mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$. Suppose that $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$. Then $S_+(\cdot)$ and $S_-(\cdot)$ both are local $K$-convoluted $C$-semigroups on $X$. To show that (2.15) holds for all $|t|, |s|, |t+s| < T_0$, we observe from [13, Theorem 2.2] that we need only to show that $\widetilde{S}(t)[S(s) - K_0(|s|)C]x{=}[S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $|t|, |s| < T_0$ with $ts \leq 0$. Let $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ be given with $t + s \geq 0$. By Lemma 2.1, (2.1) and (2.2), we have

$$\begin{aligned}
-S(t)\widetilde{S}(s)x - K_0(|s|)\widetilde{S}(t)Cx =& \frac{d}{dt}T(t)T(s)x - K_0(|s|)\widetilde{S}(t)Cx \\
=& \frac{d}{ds}T(t)T(s)x - K_0(|t|)\widetilde{S}(s)Cx \\
=& -\widetilde{S}(t)S(s)x - K_0(|t|)\widetilde{S}(s)Cx,
\end{aligned}$$

or equivalently, $\widetilde{S}(t)[S(s) - K_0(|s|)C]x{=}[S(t) - K_0(|t|)C]\widetilde{S}(s)x$. Similarly, we can show that $\widetilde{S}(t)[S(s) - K_0(|s|)C]x{=}[S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$. Since $S(t)S(s) = S(s)S(t)$ on $X$ for all $|t|, |s|, |t+s| < T_0$, we also have $\widetilde{S}(t)[S(s) - K_0(|s|)C]x{=}[S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $-T_0 < s \leq 0 \leq t < T_0$. Consequently, (2.15) holds for all $|t|, |s|, |t+s| < T_0$. Conversely, suppose that (2.15) holds for all $|t|, |s|, |t+s| < T_0$. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local $K_0$-convoluted $C$-semigroups on $X$ and $\widetilde{S}(t)S(s)x - S(t)\widetilde{S}(s)x = K_0(|s|)\widetilde{S}(t)Cx - K_0(|t|)\widetilde{S}(s)Cx$ for all $x \in X$ and $|t|, |s|, |t+s| < T_0$ with $t+s \geq 0$. Fix $x \in X$ and $-T_0 < t < 0 \leq s < T_0$ with $t+s \geq 0$, we have

$$\begin{aligned}
&\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x \\
=&K_0(|r|)\widetilde{S}(t+s-r)Cx - K_0(|t+s-r|)\widetilde{S}(r)Cx,
\end{aligned} \tag{2.16}$$

for all $t \leq r \leq 0$. Using integration by parts to the left-hand side of (2.16) over $[t, 0]$ and change of variables to the right-hand side of (2.16) over $[t, 0]$, we obtain

$$\begin{aligned}
T(t)T(s)x =& -\widetilde{S}(t)\widetilde{S}(s)x \\
=& \int_t^0 [\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x]dr \\
=& \int_t^0 [K_0(|r|)\widetilde{S}(t+s-r)Cx - K_0(|t+s-r|)\widetilde{S}(r)Cx]dr \\
=& \int_s^{t+s} K_0(|t+s-r|)\widetilde{S}(r)Cxdr - \int_t^0 K_0(|t+s-r|)\widetilde{S}(r)Cxdr \\
=& \int_{t+s}^{s} K_0(|t+s-r|)T(r)Cxdr + \int_t^0 K_0(|t+s-r|)T(r)Cxdr.
\end{aligned} \tag{2.17}$$

Using change of variables to the left-hand side of (2.16) over $[t, 0]$, we also have

$$T(s)T(t)x = -\widetilde{S}(s)\widetilde{S}(t)x = \int_t^0 [\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x]dr. \tag{2.18}$$

Combining (2.17) with (2.18), we have $T(t)T(s) = T(s)T(t)$ on $X$ for all $|t|, |s|, |t+s| < T_0$ with $ts \leq 0$ and

$$T(t)T(s)x = \int_{t+s}^{s} K_0(|t+s-r|)T(r)Cxdr + \int_{t}^{0} K_0(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$. Similarly, we can show that

$$T(t)T(s)x = \int_{t}^{t+s} K_0(|t+s-r|)T(r)Cxdr + \int_{0}^{s} K_0(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$ when the interval $[t, 0]$ of the integration of (2.16) is replaced by $[t, t+s]$. Consequently, $T(\cdot)$ is a local $K_0$-convoluted $C$-group on X. Combining this with Lemma 2.1, we get that $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$. □

**Proposition 2.1.** *Let $S(\cdot)$ be a local $K$-convoluted $C$-group on $X$, $a \in L^1_{loc}([0,T_0),\mathbb{F})$, and $b(\cdot) = a(|\cdot|)$. Then $\operatorname{sgn}(\cdot)b * S(\cdot)$ is a local $a * K$-convoluted $C$-group on $X$. In particular, for each $\beta > -1$, $\operatorname{sgn}(\cdot)J_\beta * S(\cdot)$ is a local $K_\beta$-convoluted $C$-group on $X$. Here $J_\beta(\cdot) = j_\beta(|\cdot|)$. Moreover, $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ if it is a strongly continuous family in $L(X)$ such that $\operatorname{sgn}^k(\cdot)j_{k-1} * S(\cdot) = \operatorname{sgn}(\cdot)J_{k-1} * S(\cdot)$ is a local $K_{k-1}$-convoluted $C$-group on $X$ for some nonnegative integer $k$.*

*Proof.* Clearly, $\operatorname{sgn}(\cdot)b * S(\cdot)$ is strongly continuous family in $L(X)$ which commutes with $C$ on $X$. To show that $\operatorname{sgn}(\cdot)b * S(\cdot)$ is a local $a * K$-convoluted $C$-group on $X$, we remain only to show that

$$\begin{aligned}&[(\operatorname{sgn} t)b * S(t) - \widetilde{a * K}(|t|)C]j_0 * [\operatorname{sgn}(\cdot)b * S(\cdot)](s)\\ =&j_0 * [\operatorname{sgn}(\cdot)b * S(\cdot)](t)[(\operatorname{sgn} s)b * S(s) - \widetilde{a * K}(|s|)C],\end{aligned}$$

on $X$ for all $|t|, |s|, |t+s| < T_0$. Here $\widetilde{a * K} = j_0 * (a * K)$. Clearly,

$$b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)j_0 * (b * K)(|t|),$$

on $X$ for all $0 \leq t < T_0$. Next we will show that $b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)j_0 * b * K(|t|)$ on $X$ for all $-T_0 < t \leq 0$. Let $-T_0 < t \leq 0$ be given, then

$$\begin{aligned}b * K_0(|\cdot|)(t) &= \int_0^t b(s)K_0(|t-s|)ds = \int_0^t b(s)K_0(s-t)ds\\ &= -\int_0^t a(-s)\int_s^t K(s-r)drds = -\int_t^0\int_s^t a(-s)K(s-r)drds\\ &= -\int_0^t\int_r^t a(-r)K(r-s)dsdr = \int_t^0\int_r^t a(-r)K(r-s)dsdr\end{aligned}$$

and

$$\begin{aligned}\int_t^0\int_r^t a(-r)K(r-s)dsdr &= -\int_t^0\int_s^0 a(-r)K(r-s)drds\\ &= -\int_0^t\int_0^s a(|r|)K(r-s)drds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_0^{-s} a(|r|)K(-r-s)drds \\
&= \int_0^t b * K(-s)ds = -\int_0^{-t} b * K(s)ds \\
&=(\operatorname{sgn} t) j_0 * (b * K)(|t|).
\end{aligned}
$$

Since $b * K(|t|) = a * K(|t|)$ for all $|t| < T_0$, we have $b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)\widetilde{a * K}(|t|)$ for all $|t| < T_0$. Clearly, $b * \widetilde{S}(t) = j_0 * (b * S)(t)$ on $X$ for all $|t| < T_0$. Since $j_0 * [\operatorname{sgn}(\cdot) b * S(\cdot)](t) = (\operatorname{sgn} t) j_0 * (b * S)(t) = (\operatorname{sgn} t) b * \widetilde{S}(t)$ on $X$ for all $|t| < T_0$, we also have

$$
\begin{aligned}
&[(\operatorname{sgn} t)(b * S)(t) - \widetilde{a * K}(|t|)C](\operatorname{sgn} s)\widetilde{b * S}(s)x \\
=&[(\operatorname{sgn} t)(b * S)(t) - (\operatorname{sgn} t) b * K_0(|\cdot|)(t)C](\operatorname{sgn} s) b * \widetilde{S}(s)x \\
=&(\operatorname{sgn} t)[(b * S)(t) - b * K_0(|\cdot|)(t)C](\operatorname{sgn} s) b * \widetilde{S}(s)x \\
=&(\operatorname{sgn} t)\int_0^t b(t-s)[S(r) - K_0(|r|)C](\operatorname{sgn} s) b * \widetilde{S}(s)xdr \\
=&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S}\right](s)(\operatorname{sgn} s)xdr
\end{aligned}
$$

and

$$
\begin{aligned}
&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S}\right](s)(\operatorname{sgn} s)xdr \\
=&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)\widetilde{S}(r)(S(\cdot) - K_0(|\cdot|)C)\right](s)(\operatorname{sgn} s)xdr \\
=&(\operatorname{sgn} t) b * \widetilde{S}(t) b * [S(\cdot) - K_0(|\cdot|)C](s)(\operatorname{sgn} s)x \\
=&(\operatorname{sgn} t) b * \widetilde{S}(t)[b * S(s) - b * K_0(|\cdot|)(s)C](\operatorname{sgn} s)x \\
=&(\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s) b * S(s) - (\operatorname{sgn} s) b * K_0(|\cdot|)(s)C]x \\
=&(\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s) b * S(s) - \widetilde{a * K}(|s|)C]x,
\end{aligned}
$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$. □

**Definition 2.1.** Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$. A linear operator $A$ in $X$ is called a subgenerator of $S(\cdot)$ if

$$
S(t)x - K_0(|t|)Cx = \int_0^t S(r)Axdr,
$$

for all $x \in D(A)$ and $|t| < T_0$, and

$$
\int_0^t S(r)xdr \in D(A) \quad \text{and} \quad A\int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx, \tag{2.19}
$$

for all $x \in X$ and $|t| < T_0$. A subgenerator $A$ of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$.

*Remark* 2.2. Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$, and $A$ a linear operator in $X$. Then $A$ is a subgenerator of $S(\cdot)$ if and only if $A$ is a subgenerator of $S_+(\cdot)$ and $-A$ a subgenerator of $S_-(\cdot)$.

*Remark* 2.3. Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$, and $A$ a (closed) linear operator in $X$. Then $A$ is the maximal subgenerator of $S(\cdot)$ if $A$ is the maximal subgenerator of $S_+(\cdot)$ and $-A$ the maximal subgenerator of $S_-(\cdot)$.

**Theorem 2.2.** *Let $S(\cdot)$ be a local $K$-convoluted $C$-group on $X$ and $K_0$ not the zero function on $[0,T_0)$, or a $K$-convoluted $C$-group on $X$. Assume that $C$ is injective. Then $S(\cdot)$ is nondegenerate if and only if $S_+(\cdot)$ and $S_-(\cdot)$ both are nondegenerate if and only if $S_+(\cdot)$ or $S_-(\cdot)$ is nondegenerate.*

*Proof.* Clearly, $S(\cdot)$ is nondegenerate if either $S_+(\cdot)$ or $S_-(\cdot)$ is nondegenerate. Conversely, suppose that $S(\cdot)$ is nondegenerate and $S_+(\cdot)x=0$ on $[0,T_0)$ for some $x\in X$. By Theorem 2.1, we have $\widetilde{S}(t)[S(s)-K_0(|s|)C]x=[S(t)-K_0(|t|)C]\widetilde{S}(s)x=0$ for all $-T_0<t\le 0\le s<T_0$, and so $\widetilde{S}(t)K_0(|s|)Cx=0$. Hence, $\widetilde{S}(t)x=0$. Since $-T_0<t\le 0$ is arbitrary, we have $S(\cdot)x=0$ on $(-T_0,0]$, which together with the nondegeneracy of $S(\cdot)$ implies that $x=0$. Consequently, $S_+(\cdot)$ is nondegenerate. Similarly, we can show that $S_-(\cdot)$ is nondegenerate when $S(\cdot)$ is nondegenerate. □

**Theorem 2.3.** *Let $S(\cdot)$ be a nondegenerate local $K$-convoluted $C$-group on $X$ and $K_0$ not the zero function on $[0,T_0)$, or a $K$-convoluted $C$-group on $X$. Assume that $C$ is injective. Then $A$ is the generator of $S(\cdot)$ if and only if $A$ is the generator of $S_+(\cdot)$ and $-A$ the generator of $S_-(\cdot)$ if and only if $A$ is the generator of $S_+(\cdot)$ or $-A$ the generator of $S_-(\cdot)$.*

*Proof.* Suppose that $A$ is the generator of $S_+(\cdot)$ and $-A$ is the generator of $S_-(\cdot)$. We set $B$ to denote the generator of $S(\cdot)$. Then $S(\cdot)x-K_0(|\cdot|)Cx=\widetilde{S}(\cdot)Ax$ on $(-T_0,T_0)$ for all $x\in D(A)$ or equivalently, $A\subset B$. Since $S(\cdot)x-K_0(|\cdot|)Cx=\widetilde{S}(\cdot)Bx$ on $(-T_0,T_0)$ for all $x\in D(B)$, we have $B\subset A$. Consequently, $A=B$ is the generator of $S(\cdot)$. Suppose that $A$ is the generator of $S(\cdot)$. We set $B_+$ and $B_-$ to denote the generators of $S_+(\cdot)$ and $S_-(\cdot)$, respectively. To show that $B_+=A$ and $B_-=-A$, we observe from the preceding argument, we need only to show that $B_+=-B_-$. Let $x\in D(B_-)$ be given, then

$$\begin{aligned}\widetilde{S}(t)[S(s)-K_0(|s|)C]x&=[S(t)-K_0(|t|)C]\widetilde{S}(s)x=\widetilde{S}(s)[S(t)-K_0(|t|)C]x\\&=\widetilde{S}(s)[-\widetilde{S}(t)B_-x]=\widetilde{S}(s)[\widetilde{S}(t)(-B_-)x]\\&=\widetilde{S}(t)[\widetilde{S}(s)(-B_-)x],\end{aligned}$$

for all $-T_0<t\le 0\le s<T_0$. By the nondegeneracy of $S_-(\cdot)$, we have $[S(s)-K_0(|s|)C]=\widetilde{S}(s)[-B_-x]$ for all $0\le s<T_0$, and so $x\in D(B_+)$ and $B_+x=-B_-x$. Hence, $-B_-\subset B_+$. By symmetry, we also have $B_+\subset -B_-$. Consequently, $B_+=-B_-$. □

**Theorem 2.4.** *Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $L(X)$ which commutes with $C$ on $X$. Assume that $S(\cdot)$ has a subgenerator. Then $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$. Moreover, $S(\cdot)$ is nondegenerate if the injectivity of $C$ is added and $K_0$ is a nonzero function on $[0,T_0)$.*

Combining Remark 2.2 with [13, Lemma 2.8], the next lemma is also obtained.

**Lemma 2.2.** *Let $A$ be a closed subgenerator of a strongly continuous family $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ in $L(X)$, and $K_0$ a kernel on $[0,t_0)$ (or equivalently, $K$ is a kernel on $[0,t_0)$) for some $0<t_0\leq T_0$. Assume that $C$ is injective, and $u\in C((-t_0,t_0),X)$ satisfies $u(\cdot)=Aj_0*u(\cdot)$ on $(-t_0,t_0)$. Then $u=0$ on $(-t_0,t_0)$.*

By slightly modifying the proof of [13, Theorem 2.7], we can apply Lemma 2.2 to deduce the next theorem concerning nondegenerate $K$-convoluted $C$-groups, and so its proof is omitted.

**Theorem 2.5.** *Let $S(\cdot)$ be a nondegenerate local $K$-convoluted $C$-group on $X$ with generator $A$. Assume that $S(\cdot)$ has a subgenerator. Then $A$ is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover, if $C$ is injective. Then (1.1)–(1.3) hold, and (1.4) also holds when $K_0$ is a kernel on $[0,T_0)$ or $T_0=\infty$.*

**Lemma 2.3.** *Let $S(\cdot)$ be a local $K$-convoluted $C$-group on $X$ and $0\in\operatorname{supp}K_0$ (the support of $K_0$), or a $K$-convoluted $C$-group on $X$ and $K_0$ not the zero function on $[0,\infty)$. Assume that $S(\cdot)x=0$ on $[0,t_0)$ or on $(-t_0,0]$ for some $x\in X$ and $0<t_0\leq T_0$. Then $CS(\cdot)x=0$ on $(-T_0,T_0)$. In particular, $S(t)x=0$ for all $|t|<T_0$ if the injectivity of $C$ is added.*

*Proof.* Let $S(\cdot)x=0$ on $[0,t_0)$ and $|t|<T_0$ be given, then $|t|+s<T_0$ and $K_0(s)\neq 0$ for some $0<s<t_0$, so that $\widetilde{S}(s)S(t)x{=}S(t)\widetilde{S}(s)x=0$, $S(s)\widetilde{S}(t)x{=}\widetilde{S}(t)S(s)x=0$, and $\widetilde{S}(s)K_0(|t|)Cx=K_0(|t|)C\widetilde{S}(s)x=0$. By Theorem 2.3, we have $\widetilde{S}(s)[S(t)-K_0(|t|)C]x$ $=[S(s)-K_0(s)C]\widetilde{S}(t)x$. Hence, $K_0(s)\widetilde{S}(t)Cx=K_0(s)C\widetilde{S}(t)x=0$, which implies that $\widetilde{S}(t)Cx=0$. Since $|t|<T_0$ is arbitrary, we have $CS(t)x=S(t)Cx=0$ for all $|t|<T_0$. In particular, $S(t)x=0$ for all $|t|<T_0$ if the injectivity of $C$ is added. □

**Lemma 2.4.** *Let $S(\cdot)$ be a nondegenerate local $K$-convoluted $C$-group on $X$ with generator $A$ and $0\in\operatorname{supp}K_0$. Assume that $C$ is injective. Then $A$ is a subgenerator of $S(\cdot)$.*

*Proof.* By Theorems 2.2 and 2.3, $A$ is the generator of $S_+(\cdot)$ and $-A$ is the generator of $S_-(\cdot)$. It follows from [13, Theorem 2.9] that $A$ is a subgenerator of $S_+(\cdot)$ and $-A$ is a subgenerator of $S_-(\cdot)$, which together with Remark 2.2 implies that $A$ is a subgenerator of $S(\cdot)$. □

By slightly modifying the proof of Lemma 2.4, the next lemma concerning nondegenerate $K$-convoluted $C$-groups is also attained.

**Lemma 2.5.** *Let $S(\cdot)$ be a nondegenerate $K$-convoluted $C$-group on $X$ with generator $A$. Then $C$ is injective, and $A$ is a subgenerator of $S(\cdot)$.*

Combining Theorem 2.5 with Lemma 2.5, the next theorem concerning nondegenerate $K$-convoluted $C$-groups is also obtained.

**Theorem 2.6.** *Let $S(\cdot)$ be a nondegenerate $K$-convoluted $C$-group on $X$ with generator $A$. Then $A$ is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover,* (1.1)–(1.4) *hold.*

Since $0 \in \mathrm{supp} K_0$ implies that $K_0$ is a kernel on $[0, T_0)$, we can apply Theorem 2.5 and Lemma 2.4 to obtain the next theorem.

**Theorem 2.7.** *Let $S(\cdot)$ be a nondegenerate local $K$-convoluted $C$-group on $X$ with generator $A$ and $0 \in \mathrm{supp}\, K_0$. Assume that $C$ is injective. Then $A$ is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover,* (1.1)–(1.4) *hold.*

**Theorem 2.8.** *Let $S(\cdot)(= \{S(t) \,|\, |t| < T_0\})$ be a strongly continuous family in $L(X)$ which has a subgenerator and $K_0$ a kernel on $[0, T_0)$. Assume that $C$ is injective. Then $S(\cdot)$ is a nondegenerate local $K$-convoluted $C$-group on $X$, $CA \subset AC$ and $C^{-1}AC$ is the generator of $S(\cdot)$ for each closed subgenerator $A$ of $S(\cdot)$. In particular, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator $A_0$ of $S(\cdot)$.*

*Proof.* Suppose that $A$ is a closed subgenerator of $S(\cdot)$. By Remark 2.2, $A$ is a closed subgenerator of $S_+(\cdot)$. By [13], Theorem 2.13, we have $CA \subset AC$ and $C^{-1}AC$ is the generator of $S_+(\cdot)$. By Theorem 2.3, $C^{-1}AC$ is the generator of $S(\cdot)$. Similarly, we can show that $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator $A_0$ of $S(\cdot)$. □

**Corollary 2.1.** *Let $S(\cdot)$ be a nondegenerate local $K$-convoluted $C$-group on $X$ which has a subgenerator and $K_0$ a kernel on $[0, T_0)$. Assume that $C$ is injective and $R(C)$ is dense in $X$. Then $A$ is a closed subgenerator of $S_+(\cdot)$ if and only if $-A$ is a closed subgenerator of $S_-(\cdot)$.*

*Proof.* By Remark 2.2, we need only to show that $A$ is a closed subgenerator of $S(\cdot)$ when $A$ is a closed subgenerator of $S_+(\cdot)$. Since $\int_0^t S(r)Axdr = \int_0^t S(r)C^{-1}ACxdr = S(t)x - K_0(|t|)Cx$ for all $x \in D(A)$ and $|t| < T_0$, we remain only to show that (2.19) holds for all $x \in X$ and $|t| < T_0$. Suppose that $x \in X$ and $|t| < T_0$ are given. By [13], Theorem 2.13, $C^{-1}AC$ is the generator of $S_+(\cdot)$. By Theorem 2.3, $C^{-1}AC$ is the generator of $S(\cdot)$. By Theorems 2.5 and 2.8, $C^{-1}AC$ is the maximal subgenerator of $S(\cdot)$, and so $C^{-1}AC \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$. Hence, $AC \int_0^t S(r)xdr = A \int_0^t S(r)Cxdr = S(t)Cx - K_0(|t|)CCx$, which together with the denseness of $R(C)$ implies that $A \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$ for all $x \in X$ and $|t| < T_0$.

□

*Remark* 2.4. Let $S(\cdot)(= \{S(t) \,|\, |t| < T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$. Then $S(\cdot)$ is a local $K$-convoluted $C$-group on $X$ with closed subgenerator $A$ if and only if $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$ is a local $K_0$-convoluted $C$-group on $X$ with closed subgenerator $A$.

## 3. Abstract Cauchy Problems

In the following, we always assume that $C \in L(X)$ is injective, $K_0$ a kernel on $[0, T_0)$, and $A$ a closed linear operator in $X$ such that $CA \subset AC$. We first note some basic properties concerning the solutions of ACP$(A, f, x)$ just as results in [13] for the case of $A$ is the generator of a nondegenerate local $K_0$-convoluted $C$-semigroup on $X$.

**Proposition 3.1.** *Let $A$ be a subgenerator of a nondegenerate local $K_0$-convoluted $C$-group $S(\cdot)$ on $X$. Then for each $x \in D(A)$, $\operatorname{sgn}(\cdot)S(\cdot)x$ is the unique solution of ACP$(A, K_0(|\cdot|)Cx, 0)$ in $C((-T_0, T_0), [D(A)])$. Here $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$.*

**Proposition 3.2.** *Let $A$ be a subgenerator of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)$ on $X$ and $C^1 = \{x \in X \mid S(\cdot)x$ is continuously differentiable on $(-T_0, T_0)\}$. Then*

(i) *for each $x \in C^1$, $S(t)x \in D(A)$ for a.e. $t \in (-T_0, T_0)$;*
(ii) *for each $x \in C^1$, $S(\cdot)x$ is the unique solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)Cx, 0)$;*
(iii) *for each $x \in D(A)$, $S(\cdot)x$ is the unique solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)Cx, 0)$ in $C((-T_0, T_0), [D(A)])$.*

**Proposition 3.3.** *Let $A$ be the generator of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)$ on $X$ and $x \in X$. Assume that $S(t)x \in R(C)$ for all $|t| < T_0$ and $C^{-1}S(\cdot)x \in C((-T_0, T_0), X)$ is differentiable a.e. on $(-T_0, T_0)$. Then $C^{-1}S(t)x \in D(A)$ for a.e. $t \in (-T_0, T_0)$ and $C^{-1}S(\cdot)x$ is the unique solution of*

$$\text{ACP}(A, \operatorname{sgn}(\cdot)K(|\cdot|)x, 0).$$

*Proof.* Clearly, $S(\cdot)x = CC^{-1}S(\cdot)x$ is differentiable a.e. on $(-T_0, T_0)$. By (1.1)–(1.4), we have

$$\begin{aligned} C\frac{d}{dt}C^{-1}S(t)x =& \frac{d}{dt}S(t)x \\ =& AS(t)x + (\operatorname{sgn} t)K(|t|)Cx = ACC^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)Cx, \end{aligned}$$

for a.e. $t \in (-T_0, T_0)$. Hence, for a.e. $t \in (-T_0, T_0)$, $C^{-1}S(t)x \in \mathrm{D}(C^{-1}AC) = \mathrm{D}(A)$ and

$$\frac{d}{dt}\mathrm{C}^{-1}S(t)x = (C^{-1}AC)C^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)x = AC^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)x,$$

which implies that $C^{-1}S(\cdot)x$ is a solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)x, 0)$. □

Applying Theorem 2.8, we can investigate an important result concerning the relation between the generation of a nondegenerate local $K$-convoluted $C$-group on $X$ with subgenerator $A$ and the unique existence of solutions of ACP$(A, f, x)$, which extends some results in [13] for the case of local $K$-convoluted $C$-semigroup

**Theorem 3.1.** *The following statements are equivalent.*

(i) *$A$ is a subgenerator of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)$ on $X$.*

(ii) *For each* $x \in X$ *and* $g \in L^1_{loc}((-T_0, T_0), X)$*, ACP*$(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ *has a unique solution in* $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$.

(iii) *For each* $x \in X$ *the problem ACP*$(A, K_0(|\cdot|)Cx, 0)$ *has a unique solution in* $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$.

(iv) *For each* $x \in X$ *the integral equation* $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$ *has a unique solution* $v(\cdot; x)$ *in* $C((-T_0, T_0), X)$.

*In this case,* $\widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$ *is the unique solution of ACP*$(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ *and* $v(\cdot; x) = S(\cdot)x$.

*Proof.* We will first prove that (i)⇒(ii) holds. Let $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ be given. We set $u(\cdot) = \widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$, then $u \in \mathrm{C}^1((-T_0, T_0), X) \cap \mathrm{C}((-T_0, T_0), [\mathrm{D}(A)])$, $u(0) = 0$, and

$$\begin{aligned} Au(t) =& A\widetilde{S}(t)x + A\int_0^t \widetilde{S}(t-s)g(s)ds \\ =& S(t)x - K_0(|t|)Cx + \int_0^t [S(t-s) - K_0(|t-s|)C]g(s)ds \\ =& S(t)x + \int_0^t S(t-s)g(s)ds - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)] \\ =& u'(t) - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)], \end{aligned}$$

for all $0 \le t < T_0$. Hence, $u$ is a solution of $\mathrm{ACP}(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ in $\mathrm{C}^1((-T_0, T_0), X) \cap \mathrm{C}((-T_0, T_0), [\mathrm{D}(A)])$. The uniqueness of solutions for $\mathrm{ACP}(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ follows directly from the uniqueness of solutions for $\mathrm{ACP}(A, 0, 0)$.

Clearly, (ii)⇒(iii) holds, and (iii) and (iv) both are equivalent. We remain only to show that (iv)⇒(i) holds. The assumption of *(iv)* implies that for each $x \in X$, $v_+(\cdot) = v(\cdot; x)$ on $[0, T_0)$ is a unique solution of the integral equation $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$ on $[0, T_0)$, which together with [13, Theorem 3.4] implies that $A$ is a subgenerator of a nondegenerate local $K$-convoluted $C$-semigroup on $X$. Similarly, we can show that $-A$ is a subgenerator of a nondegenerate local $K$-convoluted $C$-semigroup on $X$. It follows from Remark 2.2 and Theorem 2.2 that $A$ is a subgenerator of a nondegenerate local $K$-convoluted $C$-group on $X$. □

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.5] to obtain the next result, and so its proof is omitted.

**Theorem 3.2.** *Assume that* $R(C) \subset R(\lambda - A)$ *for some* $\lambda \in \mathbb{F}$ *and*

$$ACP(A, \mathrm{sgn}(\cdot)K(|\cdot|)x, 0)$$

*has a unique solution in* $C((-T_0, T_0), [D(A)])$ *for each* $x \in D(A)$ *with* $(\lambda - A)x \in R(C)$. *Then* $A$ *is a subgenerator of a nondegenerate local* $K$*-convoluted* $C$*-group on* $X$.

Since $C^{-1}AC = A$ and $R((\lambda - A)^{-1}C) = C(D(A))$ if $\rho(A) \neq \emptyset$, we can apply Theorem 3.2 to obtain the next corollary.

**Corollary 3.1.** *Assume that the resolvent set of $A : D(A) \to X$ is nonempty. Then $A$ is the generator of a nondegenerate local $K$-convoluted $C$-group on $X$ if and only if for each $x \in D(A)$ ACP$(A, \mathrm{sgn}(\cdot)K(|\cdot|)Cx, 0)$ has a unique solution in $C((-T_0, T_0), [D(A)])$.*

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.7] to obtain the next result, and so its proof is omitted.

**Theorem 3.3.** *Assume that $A$ is densely defined. Then the following are equivalent.*

(i) *$A$ is a subgenerator of a nondegenerate local $K$-convoluted $C$-group $S(\cdot)$ on $X$.*
(ii) *For each $x \in D(A)$ ACP$(A, \mathrm{sgn}(\cdot)K(|\cdot|)Cx, 0)$ has a unique solution $u(\cdot; Cx)$ in $C((-T_0, T_0), [D(A)])$ which depends continuously on $x$. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $(-T_0, T_0)$.*

We end this paper with several illustrative examples.

*Example* 3.1. Let $X = C_b(\mathbb{R})$, and $S(t)$ for $t \in \mathbb{R}$ be bounded linear operators on $X$ defined by $S(t)f(x) = f(x + t)$ for all $x \in \mathbb{R}$. Then for each $K \in L^1_{loc}([0, T_0), \mathbb{F})$ and $\beta > -1$, $\mathrm{sgn}(\cdot)K_\beta(|\cdot|) * S(\cdot) = \{\mathrm{sgn}(t)K_\beta(||) * S(t) \mid |t| < T_0\}$ is local a $K_\beta$-convoluted group on $X$ which is also nondegenerate with a closed subgenerator $\frac{d}{dx}$ when $K_0$ is not the zero function on $[0, T_0)$ (or equivalently, $K$ is not the zero in $\mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$), but $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$ may not be a local $K$-convoluted group on $X$ except for $K \in L^1_{loc}([0, T_0), \mathbb{F})$ so that $K * S(\cdot)$ is a strongly continuous family in $L(X)$ for which $\frac{d}{dx}$ is a closed subgenerator of $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$ when $K_0$ is not the zero function on $[0, T_0)$. Moreover, (1.1)–(1.4) hold and $\frac{d}{dx}$ is its generator and maximal subgenerator when $K_0$ is a kernel on $[0, T_0)$. In this case, $\frac{d}{dx} = \overline{A_0}$ for each subgenerator $A_0$ of $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$.

*Example* 3.2. Let $X = \mathrm{C}_b(\mathbb{R})$(or $L^\infty(\mathbb{R})$), and $A$ be the maximal differential operator in $X$ defined by $Au = \sum\limits_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $\mathrm{UC}_b(\mathbb{R})$ (or $\mathrm{C}_0(\mathbb{R})$) $= \overline{D(A)}$. Here $a_0, a_1, \ldots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2, 19] that $\{S(t) \mid |t| < T_0\}$ defined by

$$(S(t)f)(x) = \frac{1}{\sqrt{2\pi}} \mathrm{sgn}(t) \int_0^t \int_{-\infty}^{\infty} K(|t - s|)\widetilde{\phi_s}(x - y) f(y) dy ds,$$

for all $f \in X$ and $|t| < T_0$, is a norm continuous local $K_0$-convoluted group on X with closed subgenerator $A$ if the real-valued polynomial $p(x) = \sum_{j=0}^{k} a_j (ix)^j$ satisfies $\sup_{x \in \mathbb{R}} p(x) < \infty$, and $K \in L^1_{loc}([0, T_0), \mathbb{F})$ is not the zero function on $[0, T_0)$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of $\phi_t$ with $\phi_t(x) = \int_0^t e^{p(x)s} ds$ for all $t \geq 0$. Now if $K_0$ is a kernel on $[0, T_0)$, then $A$ is its generator and maximal subgenerator. Applying Theorem 3.1, we get that for each $f \in X$ and continuous function $g$ on $(-T_0, T_0) \times \mathbb{R}$ with $\int_{-t}^{t} \sup_{x \in \mathbb{R}} |g(s, x)| ds < \infty$ for all $0 \leq t < T_0$, the function $u$ on

$(-T_0, T_0) \times \mathbb{R}$ defined by

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} K_0(|t-s|)\widetilde{\phi_s}(x-y)f(y)dyds + \frac{1}{\sqrt{2\pi}} \int_0^t \int_0^{t-r} \int_{-\infty}^{\infty} K_0(|t-r-s|)\widetilde{\phi_s}(x-y)g(r,y)dydsdr,$$

for all $|t| < T_0$ and $x \in \mathbb{R}$, is the unique solution of

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=0}^{k} a_j \left(\frac{\partial}{\partial x}\right)^j u(t,x) + K_1(|t|)f(x) + \int_0^t K_1(|t-s|)g(s,x)ds,$$
$$u(0,x) = 0, \quad \text{for } t \in (-T_0, T_0) \text{ and a.e. } x \in \mathbb{R},$$

in $\mathrm{C}^1((-T_0,T_0),\mathrm{X}) \cap \mathrm{C}((-T_0,T_0),[D(\mathrm{A})])$.

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# ON NORMALIZED SIGNLESS LAPLACIAN RESOLVENT ENERGY

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Abstract. Let $G$ be a simple connected graph with $n$ vertices. Denote by $\mathcal{L}^{+}(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2}$ the normalized signless Laplacian matrix of graph $G$, where $Q(G)$ and $D(G)$ are the signless Laplacian and diagonal degree matrices of $G$, respectively. The eigenvalues of matrix $\mathcal{L}^{+}(G)$, $2 = \gamma_1^{+} \geq \gamma_2^{+} \geq \cdots \geq \gamma_n^{+} \geq 0$, are normalized signless Laplacian eigenvalues of $G$. In this paper, we introduce the normalized signless Laplacian resolvent energy of $G$ as $ERNS(G) = \sum_{i=1}^{n} \frac{1}{3-\gamma_i^{+}}$. We also obtain some lower and upper bounds for $ERNS(G)$ as well as its relationships with other energies and signless Kemeny's constant.

## 1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with $n$ vertices and $m$ edges, where $|V| = n$ and $|E| = m$. Denote by $d_i$ the degree of the vertex $v_i$ of $G$, $i = 1, 2, \ldots, n$. If $v_i$ and $v_j$ are two adjacent vertices of $G$, then we denote this by $i \sim j$.

Let $A(G)$ be the adjacency matrix of $G$. Eigenvalues of $A(G)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are said to be the (ordinary) eigenvalues of $G$ [11]. Then the energy of the graph $G$ is defined as [15]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

Various properties and bounds on $E(G)$ may be found in the monographs [19, 22] and references cited therein.

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In line with concept of graph energy, the resolvent energy of $G$ is put forward in [18] as

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}.$$

For the basic properties and bounds of $ER(G)$, the reader may refer to [1,13,34,35].

Let $D(G) = \mathrm{diag}(d_1, d_2, \dots, d_n)$ denote the diagonal degree matrix of $G$. The Laplacian and signless Laplacian matrices of $G$ are, respectively, defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ the eigenvalues of $L(G)$ and $Q(G)$, respectively [26]. Recently, Laplacian resolvent and signless Laplacian resolvent energies of $G$ are, respectively, introduced as [7]

$$RL(G) = \sum_{i=1}^{n} \frac{1}{n+1-\mu_i}$$

and

$$RQ(G) = \sum_{i=1}^{n} \frac{1}{2n-1-q_i}.$$

Since graph $G$ is connected, the matrix $D(G)^{-1/2}$ is well defined. Then, the normalized Laplacian matrix of $G$ is defined by [10]

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G),$$

where $I_n$ is the $n \times n$ unity matrix and $R(G)$ is the Randić matrix [2]. The following properties for the normalized Laplacian eigenvalues, $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- > \gamma_n^- = 0$, are valid [36]

$$\sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G), \tag{1.1}$$

where

$$R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j},$$

is a kind of topological index of $G$ called as general Randić index [8,31].

The matrix $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G)$ is defined to be the normalized signless Laplacian matrix of $G$ [10]. Some well known identities concerning the normalized signless Laplacian eigenvalues, $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, are [9]

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G). \tag{1.2}$$

For $i = 1, 2, \dots, n$, the following relations (see [14,24]) exist

$$\gamma_i^- = 1 - \rho_{n-i+1} \quad \text{and} \quad \gamma_i^+ = 1 + \rho_i. \tag{1.3}$$

Here, $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the Randić eigenvalues of $G$ [2,24].

Motivated by the definitions of graph resolvent energies $ER$, $RL$ and $RQ$ and considering the fact that $\gamma_i^- \leq 2$, $1 \leq i \leq n$, Sun and Das [33] defined the normalized Laplacian resolvent energy of $G$ as

$$ERN\,(G) = \sum_{i=1}^{n} \frac{1}{3-\gamma_i^-}.$$

Since the property $\gamma_i^+ \leq 2$, $1 \leq i \leq n$, is also satisfied by the normalized signless Laplacian eigenvalues, we now introduce the normalized signless Laplacian resolvent energy of $G$ as follows

$$ERNS\,(G) = \sum_{i=1}^{n} \frac{1}{3-\gamma_i^+}.$$

Notice that in the case of bipartite graph the normalized Laplacian and normalized signless Laplacian eigenvalues coincide [3]. From hence, for bipartite graphs, $ERN\,(G)$ is equal to $ERNS\,(G)$.

Before we proceed, let us recall another graph invariant closely related to normalized Laplacian eigenvalues and so called Kemeny's constant. It is defined as [6]

$$K\,(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^-}.$$

For more information on $K\,(G)$, see [21,27].

Since for connected non-bipartite graphs $\gamma_i^+ > 0$ for $i = 1, 2, \ldots, n$, [4], very recently, in an analogous manner with Kemeny's constant, signless Kemeny's constant of connected non-bipartite graphs is considered as [28]

$$K^+\,(G) = \sum_{i=1}^{n} \frac{1}{\gamma_i^+}.$$

In [28], it is also emphasized that $K\,(G)$ coincides with $K^+\,(G)$ in the case of bipartite graphs.

In this paper, we obtain some lower and upper bounds for $ERNS\,(G)$ as well as its relationships with other energies and $K^+\,(G)$.

## 2. Lemmas

We now recall some known results on graph spectra and analytical inequalities that will be used in our main results.

**Lemma 2.1** ([14]). *For any connected graph $G$, the largest normalized signless Laplacian eigenvalue is $\gamma_1^+ = 2$.*

**Lemma 2.2** ([14]). *Let $G$ be a graph of order $n \geq 2$ with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$$

*if and only if $G \cong K_n$.*

**Lemma 2.3** ([4]). *If $G$ is a connected non–bipartite graph of order $n$, then $\gamma_i^+ > 0$ for $i = 1, 2, \dots, n$.*

**Lemma 2.4** ([3]). *If $G$ is a bipartite graph, then the eigenvalues of $\mathcal{L}$ and $\mathcal{L}^+$ coincide.*

**Lemma 2.5** ([23]). *Let $G$ be a connected graph of order $n$. Then $\gamma_2^- \geq 1$, the equality holds if and only if $G$ is a complete bipartite graph.*

**Lemma 2.6** ([12]). *Let $G$ be a connected graph with $n > 2$ vertices. Then $\gamma_2^- = \gamma_3^- = \cdots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

**Lemma 2.7** ([10]). *Let $G$ be a bipartite graph with $n$ vertices Then for $i = 1, 2, \dots, n$, $\gamma_i^- + \gamma_{n-i+1}^- = 2$.*

**Lemma 2.8** ([24]). *For any connected graph $G$, the largest Randić eigenvalue is $\rho_1 = 1$.*

**Lemma 2.9** ([2]). *Let $G$ be a graph with $n$ vertices and Randić matrix $R(G)$. Then*

$$tr\left(R(G)^2\right) = 2R_{-1}$$

*and*

$$tr\left(R(G)^3\right) = 2\sum_{i\sim j} \frac{1}{d_i d_j} \left( \sum_{k\sim i,\, k\sim j} \frac{1}{d_k} \right).$$

**Lemma 2.10** ([30]). *Let $x = (x_i)$ and $a = (a_i)$ be two sequences of positive real numbers, $i = 1, 2, \dots, n$. Then for any $r \geq 0$*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum\limits_{i=1}^{n} a_i\right)^{r}}. \tag{2.1}$$

*Equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.*

**Lemma 2.11** ([20]). *Let $a = (a_i)$ and $p = (p_i)$ be two sequences of positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$ and $0 < r \leq a_i \leq R < +\infty$, $i = 1, 2, \dots, n$, $r, R \in \mathbb{R}$. Then*

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i} \leq \frac{1}{4} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right)^2. \tag{2.2}$$

*Equality holds if and only if $R = a_1 = a_2 = \cdots = a_n = r$.*

## 3. Lower and Upper Bounds on $ERNS(G)$

In this section, we establish some lower and upper bounds for $ERNS(G)$.

**Theorem 3.1.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then, for any real $\alpha$, such that $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$,*

$$ERNS(G) \geq 1 + \frac{1}{3-\alpha} + \frac{(n-2)^2}{2n-4+\alpha}. \tag{3.1}$$

*If $\alpha = \frac{n-2}{n-1}$, equality holds if and only if $G \cong K_n$.*

*Proof.* By arithmetic-harmonic mean inequality [29], we have

$$\sum_{i=3}^{n} a_i \sum_{i=3}^{n} \frac{1}{a_i} \geq (n-2)^2,$$

where $a_i > 0$, $i = 3, 4, \ldots, n$, are arbitrary real numbers. For $a_i = 3 - \gamma_i^+$, $i = 3, 4, \ldots, n$, the above inequality transforms into

$$\sum_{i=3}^{n} \left(3 - \gamma_i^+\right) \sum_{i=3}^{n} \frac{1}{3-\gamma_i^+} \geq (n-2)^2,$$

that is

$$\sum_{i=1}^{n} \frac{1}{3-\gamma_i^+} \geq \frac{1}{3-\gamma_1^+} + \frac{1}{3-\gamma_2^+} + \frac{(n-2)^2}{\sum_{i=3}^{n}\left(3-\gamma_i^+\right)}.$$

Then, it follows from the above, (1.2) and Lemma 2.1 that

$$ERNS(G) \geq 1 + \frac{1}{3-\gamma_2^+} + \frac{(n-2)^2}{2n-4+\gamma_2^+}. \tag{3.2}$$

Now, consider the function defined as follows

$$f(x) = \frac{1}{3-x} + \frac{(n-2)^2}{2n-4+x}.$$

It can be easily seen that $f$ is increasing for $x \geq \frac{n-2}{n-1}$. Then for any real $\alpha$, $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$,

$$f\left(\gamma_2^+\right) \geq f(\alpha) = \frac{1}{3-\alpha} + \frac{(n-2)^2}{2n-4+\alpha}.$$

Based on this inequailty and (3.2), we obtain the lower bound (3.1). Equality in (3.1) holds if and only if

$$\gamma_2^+ = \alpha \quad \text{and} \quad \gamma_3^+ = \cdots = \gamma_n^+.$$

If $\alpha = \frac{n-2}{n-1}$, then from the above and Lemma 2.2, one can easily conclude that the equality in (3.1) holds if and only if $G \cong K_n$. □

**Corollary 3.1.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices.Then*

$$ERNS(G) \geq 1 + \frac{(n-1)^2}{2n-1}.$$

*Equality holds if and only if $G \cong K_n$.*

Considering the techniques in Theorem 3.1 with Lemmas 2.1, 2.4 and 2.6, we obtain the following result for bipartite graphs.

**Theorem 3.2.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real $\alpha$, such that $\gamma_2^+ = \gamma_2^- \geq \alpha \geq 1$*

$$ERNS(G) = ERN(G) \geq \frac{4}{3} + \frac{1}{3-\alpha} + \frac{(n-3)^2}{2n-7+\alpha}.$$

*If $\alpha = 1$, equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

In [5], it was obtained that

$$\gamma_2^+ = \gamma_2^- \geq 1 + \sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}. \tag{3.3}$$

From Theorem 3.2 and (3.3), we directly have the following.

**Corollary 3.2.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{4}{3} + \frac{1}{2-\sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}} + \frac{(n-3)^2}{2n-6+\sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}}. \tag{3.4}$$

From Theorem 3.2 and Lemma 2.5, we have the following result. It was proven in Theorem 3.8 of [33].

**Corollary 3.3** ([33])**.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{n}{2} + \frac{1}{3}. \tag{3.5}$$

*Equality holds if and only if $G \cong K_{p,q}$.*

*Remark* 3.1. Note that the lower bound (3.4) is stronger than the lower bound (3.5).

**Theorem 3.3.** *Let $G$ be a connected graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3}\left(n+2+\frac{(n-2)^2}{2(n-1-R_{-1}(G))}\right). \tag{3.6}$$

*Equality holds if and only if $G \cong K_n$ or $G \cong K_{p,q}$, $p+q=n$.*

*Proof.* Suppose $G$ is a connected non–bipartite graph with $n \geq 3$ vertices. Then, by Lemmas 2.1 and 2.3, $\gamma_1^+ = 2$ and $\gamma_i^+ > 0$, $i = 2,3,\ldots,n$. For $r=1$ the inequality (2.1) transforms into

$$\sum_{i=2}^{n} \frac{x_i^2}{a_i} \geq \frac{\left(\sum\limits_{i=2}^{n} x_i\right)^2}{\sum\limits_{i=2}^{n} a_i}. \tag{3.7}$$

Setting $x_i = \gamma_i^+$, $a_i = \gamma_i^+(3-\gamma_i^+)$, $i = 2,3,\dots,n$, in (3.7) and using (1.2) and Lemma 2.1, we have

$$\sum_{i=2}^{n} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} \geq \frac{\left(\sum\limits_{i=2}^{n} \gamma_i^+\right)^2}{\sum\limits_{i=2}^{n} \gamma_i^+(3-\gamma_i^+)} = \frac{(n-2)^2}{2(n-1-R_{-1}(G))}. \tag{3.8}$$

On the other hand, from the above and Lemma 2.1, we also have

$$\begin{aligned}\sum_{i=2}^{n} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} &= \sum_{i=2}^{n} \frac{\gamma_i^+}{3-\gamma_i^+} = \sum_{i=2}^{n} \frac{\gamma_i^+ - 3 + 3}{3-\gamma_i^+} \\ &= -(n-1) + 3\left(ERNS(G) - 1\right) \\ &= 3ERNS(G) - n - 2. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), the inequality (3.6) is obtained.

Equality in (3.8) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_n^+},$$

that is $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. By Lemma 2.2, when $G$ is non–bipartite graph, equality in (3.6) holds if and only if $G \cong K_n$.

Now, suppose $G$ is a connected bipartite graph with $n \geq 3$ vertices. Then, by Lemmas 2.1 and 2.4, $\gamma_1^+ = 2$ and $\gamma_2^+ \geq \gamma_3^+ \geq \cdots \geq \gamma_{n-1}^+ > \gamma_n^+ = 0$. The inequality (2.1) can be considered as

$$\sum_{i=2}^{n-1} \frac{x_i^2}{a_i} \geq \frac{\left(\sum\limits_{i=2}^{n-1} x_i\right)^2}{\sum\limits_{i=2}^{n-1} a_i}.$$

Taking $x_i = \gamma_i^+$, $a_i = \gamma_i^+(3-\gamma_i^+)$, $i = 2,3,\dots,n-1$, in the above inequality and considering (1.2), we get

$$\sum_{i=2}^{n-1} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} \geq \frac{\left(\sum\limits_{i=2}^{n-1} \gamma_i^+\right)^2}{\sum\limits_{i=2}^{n-1} \gamma_i^+(3-\gamma_i^+)} = \frac{(n-2)^2}{2(n-1-R_{-1}(G))}. \tag{3.10}$$

Observe that

$$\begin{aligned}\sum_{i=2}^{n-1} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} &= \sum_{i=2}^{n-1} \frac{\gamma_i^+}{3-\gamma_i^+} = \sum_{i=2}^{n-1} \frac{\gamma_i^+ - 3 + 3}{3-\gamma_i^+} = \\ &= -(n-2) + 3\left(ERNS(G) - 1 - \frac{1}{3}\right) = \\ &= 3ERNS(G) - n - 2\,. \end{aligned}$$

From the above and inequality (3.10) we arrive at (3.6).

Equality in (3.10) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_{n-1}^+},$$

that is when $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+$. Since $G$ is a bipartite graph, by Lemmas 2.4 and 2.6, equality in (3.6) holds if and only if $G \cong K_{p,q}$, $p+q=n$. ☐

**Corollary 3.4.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3}\left(n+2+\frac{(n-2)^2}{2(n-1-R_{-1}(G))}\right).$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

**Theorem 3.4.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \leq 1 + \frac{4n-5-(n-1)(\gamma_2^+ + \gamma_n^+)}{(3-\gamma_2^+)(3-\gamma_n^+)}. \tag{3.11}$$

*Equality holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for $i = 2,3,\dots,n$.*

*Proof.* For every $i = 2,3,\dots,n$, the following inequalities are valid

$$(3-\gamma_2^+ - 3 + \gamma_i^+)(3-\gamma_n^+ - 3 + \gamma_i^+) \leq 0,$$

$$(3-\gamma_i^+)^2 + (3-\gamma_2^+)(3-\gamma_n^+) \leq (6-\gamma_2^+ - \gamma_n^+)(3-\gamma_i^+),$$

$$(3-\gamma_i^+) + \frac{(3-\gamma_2^+)(3-\gamma_n^+)}{3-\gamma_i^+} \leq 6-\gamma_2^+ - \gamma_n^+. \tag{3.12}$$

After summation of (3.12) over $i$, $i = 2,3,\dots,n$, we obtain

$$\sum_{i=2}^{n}(3-\gamma_i^+) + (3-\gamma_2^+)(3-\gamma_n^+)\sum_{i=2}^{n}\frac{1}{3-\gamma_i^+} \leq (6-\gamma_2^+ - \gamma_n^+)\sum_{i=2}^{n} 1,$$

that is

$$2n-1+(3-\gamma_2^+)(3-\gamma_n^+)(ERNS(G)-1) \leq (6-\gamma_2^+ - \gamma_n^+)(n-1), \tag{3.13}$$

from which (3.11) is obtained.

Equality in (3.12) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for every $i = 2,3,\dots,n$, which implies that equality in (3.11) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for every $i = 2,3,\dots,n$. ☐

**Corollary 3.5.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \leq 1 + \frac{\left((n-1)(6-\gamma_2^+ - \gamma_n^+)\right)^2}{4(2n-1)(3-\gamma_2^+)(3-\gamma_n^+)}. \tag{3.14}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* After applying the arithmetic-geometric mean inequality, AM-GM, on (3.13) we obtain

$$2\sqrt{(2n-1)(3-\gamma_2^+)(3-\gamma_n^+)(ERNS(G)-1)} \leq (6-\gamma_2^+-\gamma_n^+)(n-1)\,,$$

from which (3.14) is obtained. □

The proof of the next theorem is fully analogous to that of Theorem 3.4, thus omitted.

**Theorem 3.5.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \leq \frac{4}{3} + \frac{2(n-2)}{(3-\gamma_2^+)(3-\gamma_{n-1}^+)}\,.$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

**Theorem 3.6.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) \leq \frac{1}{3}\left(n+2+\frac{\left((n-2)(6-\gamma_2^+-\gamma_n^+)\right)^2}{8\,(n-1-R_{-1}\,(G))\left(3-\gamma_2^+\right)(3-\gamma_n^+)}\right). \tag{3.15}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* Setting $p_i = \frac{\gamma_i^+}{n-2}$, $a_i = \frac{1}{3-\gamma_i^+}$, $i = 2,3,\ldots,n$, $R = \frac{1}{3-\gamma_2^+}$, $r = \frac{1}{3-\gamma_n^+}$ in (2.2), we have that

$$\sum_{i=2}^{n}\left(\frac{\gamma_i^+}{n-2}\right)\left(\frac{1}{3-\gamma_i^+}\right)\sum_{i=2}^{n}\left(\frac{\gamma_i^+}{n-2}\right)\left(3-\gamma_i^+\right) \leq \frac{1}{4}\left(\sqrt{\frac{3-\gamma_n^+}{3-\gamma_2^+}}+\sqrt{\frac{3-\gamma_2^+}{3-\gamma_n^+}}\right)^2.$$

Considering this with (1.2) and (3.9) and Lemma 2.1, we obtain that

$$\frac{2\,(n-1-R_{-1}\,(G))}{(n-2)^2}\,(3ERNS\,(G)-n-2) \leq \frac{1}{4}\left(\frac{\left(6-\gamma_2^+-\gamma_n^+\right)^2}{\left(3-\gamma_2^+\right)(3-\gamma_n^+)}\right).$$

From the above result, we arrive at the upper bound (3.15). The equality in (3.15) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_n^+},$$

that is

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+.$$

Thus, in view of Lemma 2.2, we conclude that the equality in (3.15) holds if and only if $G \cong K_n$. □

Using the techniques in Theorem 3.6 with Lemmas 2.1, 2.4, 2.6, 2.7 and 2.11, we have the following.

**Theorem 3.7.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\left(G\right) = ERN(G) \leq \frac{1}{3}\left(n + 2 + \frac{2\left(n-2\right)^2}{\left(n-1-R_{-1}\left(G\right)\right)\left(3-\gamma_2^+\right)\left(3-\gamma_{n-1}^+\right)}\right).$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

## 4. Relations Between $ERNS(G)$ and other Energies

One of the chemically/mathematically most important graph spectrum–based invariants in graph theory is the concept of graph energy introduced in [15]. Due to the evident success of graph energy, a number of graph energies and energy-like graph invariants have been put forward in the literature. We first recall some of them.

For a graph $G$, in full analogy with the graph energy [15], Randić (normalized Laplacian or normalized signless Laplacian) energy is defined as [2,8,17]

$$RE\left(G\right) = \sum_{i=1}^{n} \left|\rho_i\right|,$$

where $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the Randić eigenvalues of $G$ [2,24].

In analogous manner with Laplacian energy-like invariant [25], Laplacian incidence energy is introduced as [32]

$$LIE\left(G\right) = \sum_{i=1}^{n-1} \sqrt{\gamma_i^-}$$

and by analogy with incidence energy [16], the Randić (normalized) incidence energy is put forward in [9,14] as

$$I_R E\left(G\right) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}.$$

Here, $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- > \gamma_n^- = 0$ and $2 = \gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ are, respectively, the normalized Laplacian and normalized signless Laplacian eigenvalues of $G$ [10,14]. Note that $LIE$ is equal to $I_RE$, for bipartite graphs [3].

Now, we are ready to give some relationships between $ERNS\left(G\right)$ and other energies emphasized in the above.

**Theorem 4.1.** *Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\left(G\right) \geq 1 + \frac{\left(RE\left(G\right)-1\right)^2}{4R_{-1} - 2\sum\limits_{i\sim j}\frac{1}{d_i d_j}\left(\sum\limits_{k\sim i,\ k\sim j}\frac{1}{d_k}\right) - 1}. \tag{4.1}$$

*Equality is achieved for $G \cong K_n$.*

*Proof.* For $x_i = \left|\gamma_i^+ - 1\right|$ and $a_i = \frac{1}{3-\gamma_i^+}$, $i = 2, 3, \dots, n$, the inequality (3.7) becomes

$$\sum_{i=2}^{n} \left(\gamma_i^+ - 1\right)^2 \left(3 - \gamma_i^+\right) \geq \frac{\left(\sum\limits_{i=2}^{n} \left|\gamma_i^+ - 1\right|\right)^2}{\sum\limits_{i=2}^{n} \frac{1}{3-\gamma_i^+}}. \tag{4.2}$$

From (1.3) and Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
\sum_{i=2}^{n} \left(\gamma_i^+ - 1\right)^2 \left(3 - \gamma_i^+\right) =& \sum_{i=2}^{n} \rho_i^2 \left(2 - \rho_i\right) \\
=& 2 \sum_{i=2}^{n} \rho_i^2 - \sum_{i=2}^{n} \rho_i^3 \\
=& 2 \left(2R_{-1} - 1\right) - \left(2 \sum_{i\sim j} \frac{1}{d_i d_j} \left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1\right) \\
=& 4R_{-1} - 2 \sum_{i\sim j} \frac{1}{d_i d_j} \left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1. \qquad (4.3)
\end{aligned}$$

Then by (4.2) and (4.3) and Lemma 2.1, we get that

$$4R_{-1} - 2 \sum_{i\sim j} \frac{1}{d_i d_j} \left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1 \geq \frac{\left(RE\left(G\right) - 1\right)^2}{ERNS\left(G\right) - 1}.$$

From the above, the inequality (4.1) follows. One can easily check that the equality in (4.1) is achieved for $G \cong K_n$. ☐

**Theorem 4.2.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3} \left(n + 2 + \frac{(I_R E(G) - \sqrt{2})^2}{2n - 1}\right). \tag{4.4}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* Taking $x_i = \sqrt{\gamma_i^+}$, $a_i = 3 - \gamma_i^+$, $i = 2, 3, \dots, n$, in (3.7)

$$\sum_{i=2}^{n} \frac{\gamma_i^+}{3 - \gamma_i^+} \geq \frac{\left(\sum_{i=2}^{n} \sqrt{\gamma_i^+}\right)^2}{\sum_{i=2}^{n} (3 - \gamma_i^+)}.$$

Considering this with (1.2) and (3.9) and Lemma 2.1

$$3ERNS(G) - n - 2 \geq \frac{\left(I_R E(G) - \sqrt{2}\right)^2}{2n - 1}.$$

From the above we obtain (4.4). Equality in (4.4) holds if and only if

$$\frac{\sqrt{\gamma_2^+}}{3 - \gamma_2^+} = \frac{\sqrt{\gamma_3^+}}{3 - \gamma_3^+} = \dots = \frac{\sqrt{\gamma_n^+}}{3 - \gamma_n^+},$$

that is if and only if

$$\left(\sqrt{\gamma_i^+}-\sqrt{\gamma_j^+}\right)\left(3+\sqrt{\gamma_i^+\gamma_j^+}\right)=0,\quad i\neq j,$$

which implies that equality in (4.4) holds if and only if $G\cong K_n$. □

**Theorem 4.3.** *Let $G$ be a connected bipartite graph with $n\geq 3$ vertices. Then*

$$ERNS(G)=ERN(G)\geq\frac{1}{3}\left(n+2+\frac{\left(LIE(G)-\sqrt{2}\right)^2}{2(n-2)}\right).$$

*Equality holds if $G\cong K_{p,q}$, $p+q=n$.*

## 5. Relationships Between $ERNS(G)$ and $K^+(G)$

In this section, we present some relationships between $ERNS(G)$ and $K^+(G)$.

**Theorem 5.1.** *Let $G$ be a connected non-bipartite graph wit $n\geq 3$ vertices. Then*

$$ERNS(G)\geq\frac{3}{2}-K^+(G)+\frac{3(n-1)^2}{2(n-1-R_{-1}(G))}. \tag{5.1}$$

*Equality holds if and only if $G\cong K_n$.*

*Proof.* The arithmetic-harmonic mean inequality can be considered as [29]

$$\sum_{i=2}^n a_i\sum_{i=2}^n\frac{1}{a_i}\geq(n-1)^2. \tag{5.2}$$

For $a_i=\gamma_i^+(3-\gamma_i^+)$, $i=2,3,\dots,n$, the above inequality transforms into

$$\sum_{i=2}^n\gamma_i^+(3-\gamma_i^+)\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}\geq(n-1)^2. \tag{5.3}$$

From the above, (1.2) and Lemma 2.1,

$$\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}\geq\frac{(n-1)^2}{2(n-1-R_{-1}(G))}. \tag{5.4}$$

On the other hand, by Lemma 2.1, we have that

$$\begin{aligned}\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}&=\frac{1}{3}\left(\sum_{i=2}^n\frac{1}{\gamma_i^+}+\sum_{i=2}^n\frac{1}{3-\gamma_i^+}\right)\\&=\frac{1}{3}\left(K^+(G)-\frac{1}{2}+ERNS(G)-1\right)\\&=\frac{1}{3}\left(K^+(G)-\frac{3}{2}+ERNS(G)\right)\end{aligned}$$

Combining this with (5.4) we arrive at (5.1). Equality in (5.3) holds if and only if $\gamma_2^+(3-\gamma_2^+)=\gamma_3^+(3-\gamma_3^+)=\cdots=\gamma_n^+(3-\gamma_n^+)$. Suppose $i\neq j$. Then, from the identity $\gamma_i^+(3-\gamma_i^+)=\gamma_j^+(3-\gamma_j^+)$, follows that $(\gamma_i^+-\gamma_j^+)(3-\gamma_i^+-\gamma_j^+)=0$. Thus, we conclude that equality in (5.3) holds if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. Having this in mind and Lemma 2.2, we conclude that equality in (5.1) holds if and only if $G \cong K_n$. $\square$

Using the similar idea in Theorem 5.1 with Lemmas 2.1, 2.4 and 2.6, we get the following.

**Theorem 5.2.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{11}{6} - K(G) + \frac{3(n-2)^2}{2(n-1-R_{-1}(G))}\,.$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

**Theorem 5.3.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3}\left(n+2+\frac{2(n-1)^2}{6K^+(G)-2n-1}\right)\,. \tag{5.5}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* For $a_i = \frac{\gamma_i^+}{3-\gamma_i^+}$, $i = 2,3,\ldots,n$, the inequality (5.2) transforms into

$$\sum_{i=2}^{n} \frac{\gamma_i^+}{3-\gamma_i^+} \sum_{i=2}^{n} \frac{3-\gamma_i^+}{\gamma_i^+} \geq (n-1)^2\,. \tag{5.6}$$

On the other hand, by Lemma 2.1, we have that

$$\sum_{i=2}^{n} \frac{3-\gamma_i^+}{\gamma_i^+} = 3\left(K^+(G)-\frac{1}{2}\right) - (n-1) = 3K^+(G) - n - \frac{1}{2}\,.$$

From the above, (3.9) and (5.6), we obtain (5.5). Equality in (5.6) holds if and only if

$$\frac{\gamma_2^+}{3-\gamma_2^+} = \frac{\gamma_3^+}{3-\gamma_3^+} = \cdots = \frac{\gamma_n^+}{3-\gamma_n^+}.$$

Suppose $i \neq j$. Then equality in (5.6) holds if and only if $\frac{\gamma_i^+}{3-\gamma_i^+} = \frac{\gamma_j^+}{3-\gamma_j^+}$, that is if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. By Lemma 2.2, equality in (5.5) holds if and only if $G \cong K_n$. $\square$

Conisdering the same idea in Theorem 5.3 together with Lemma 2.1, 2.4 and 2.6, we have the following.

**Theorem 5.4.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3}\left(n+2+\frac{2(n-2)^2}{6K(G)-2n+1}\right)\,.$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

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# THE NEW INEQUALITIES FOR $tgs$-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some Hadamard-Hadamard type inequalities for $tgs$-convex functions. Our results are the generalizations of some known results. The new generalized estimate of the midpoints product of two $tgs$-convex functions is also considered.

## 1. Introduction

**Definition 1.1.** A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that $f$ is concave if $(-f)$ is convex.

For convex functions, we have the following inequality which is known in the literature as Hermite-Hadamard inequality.

**Theorem 1.1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

*If $f$ is a positive concave function, then the inequality is reversed.*

In 1906, Fejér [1] showed the following weighted generalization of inequality (1.2).

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**Theorem 1.2.** *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b q(t)dt \leq \frac{1}{b-a}\int_a^b f(t)q(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b q(t)dt, \tag{1.3}$$

*where $q : [a,b] \to \mathbb{R}$ is positive, integrable, and symmetric with respect to $\frac{a+b}{2}$.*

Some refinements, variations, generalizations and improvements of inequalities (1.2) and (1.3) can be seen [2, 3] and [4].

**Definition 1.2** ([5])**.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. $f$ is called a $tgs$-convex function on $I$ if the inequality

$$f(tx + (1-t)y) \leq t(1-t)(f(x) + f(y)) \tag{1.4}$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that $f$ is $tgs$-concave if $(-f)$ is $tgs$-convex.

For $tgs$-convex functions, the following results hold [5].

**Theorem 1.3.** *Assume that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a $tgs$-convex function and $a, b \in I$ with $a < b$, then we have*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{6}. \tag{1.5}$$

**Theorem 1.4.** *Assume that $f$ and $g$ are real valued, nonnegative $tgs$-convex functions on $[a,b]$, then we have*

$$8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)g(t)dt + \frac{1}{30}\left[M(a,b) + N(a,b)\right], \tag{1.6}$$

*where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.*

The recent results on $tgs$-convex functions can be seen in [5, 6] and [7].

In this paper, we give the improvements of (1.5) and (1.6). The weighted generalization of inequality (1.5) are also established.

## 2. Main Results

The following result is an improvement of (1.5).

**Theorem 2.1.** *Assume that $f : [a,b] \to \mathbb{R}$ is a $tgs$-convex function, then we have*

$$\begin{aligned} 4f\left(\frac{a+b}{2}\right) \leq & f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \\ \leq & \frac{1}{b-a}\int_a^b f(t)dt \\ \leq & \frac{f(a)+f(b)}{12} + \frac{f(\frac{a+b}{2})}{6} \end{aligned} \tag{2.1}$$

$$\leq \frac{f(a)+f(b)}{8}.$$

*Proof.* Using (1.5) in $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$, we have

$$2f\left(\frac{3a+b}{4}\right) \leq \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt \leq \frac{f(a)+f(\frac{a+b}{2})}{6},$$
$$2f\left(\frac{a+3b}{4}\right) \leq \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(t)dt \leq \frac{f(b)+f(\frac{a+b}{2})}{6}.$$

Form the above inequalities, we have

$$f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)+2f(\frac{a+b}{2})}{12}.$$

A combination of the above inequality and the following results

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{(3a+b)/4+(a+3b)/4}{2}\right) \leq \frac{1}{4}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right),$$
$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{4},$$

deduces the desired inequality (2.1). □

The following Hadamard-Hadamard-Fejér type inequality for $tgs$-convex function holds.

**Theorem 2.2.** *Assume that $f : [a, b] \to \mathbb{R}$ is a $tgs$-convex function, then we have*

$$2f\left(\frac{a+b}{2}\right)\int_a^b q(x)dx \leq \int_a^b f(x)q(x)dx \leq (f(a)+f(b))\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}q(x)dx, \tag{2.2}$$

*where $q : [a, b] \to \mathbb{R}$ is positive, integrable, and symmetric with respect to $\frac{a+b}{2}$.*

*Proof.* Since $q(x) = q(a+b-x)$, we have

$$2f\left(\frac{a+b}{2}\right)\int_a^b q(x)dx \leq 2\int_a^b f\left(\frac{x}{2}+\frac{a+b-x}{2}\right)q(x)dx,$$
$$\leq \frac{1}{2}\int_a^b f(x)q(x)dx + \frac{1}{2}\int_a^b f(a+b-x)q(a+b-x)dx$$
$$= \int_a^b f(x)q(x)dx.$$

On the other hand,

$$\int_a^b f(x)q(x)dx = (b-a)\int_0^1 f(tb+(1-t)a)q(tb+(1-t)a)dt$$

$$\le (b-a)(f(a)+f(b))\int_0^1 t(1-t)q(tb+(1-t)a)dt$$
$$=(f(a)+f(b))\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}q(x)dx. \qquad \square$$

*Remark* 2.1. We get (1.5) by putting $q(x)=1$ in (2.2).

The following inequalities are improvements of (1.6).

**Theorem 2.3.** *Assume that $f$ and $g$ are real valued, nonnegative tgs-convex functions on $[a,b]$, then we have*

$$\begin{aligned}8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le& \frac{1}{b-a}\int_a^b f(t)q(t)dt\\ &+\frac{1}{60}(N((a+b)/2,(a+b)/2)+N(a,b)\\ &+N(a,(a+b)/2)+N((a+b)/2,b)]\\ \le& \frac{1}{b-a}\int_a^b f(t)dt\\ &+\frac{1}{480}[5M(a,b)+13N(a,b)],\end{aligned}$$

*where $M(a,b)$ and $N(a,b)$ are defined in Theorem* 1.4.

*Proof.* For $\lambda\in[0,1]$, we have

$$\begin{aligned}&8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\ =&8f\left(\frac{(1-\lambda)b+\lambda(a+b)/2}{2}+\frac{(1-\lambda)a+\lambda(a+b)/2}{2}\right)\\ &\times g\left(\frac{(1-\lambda)b+\lambda(a+b)/2}{2}+\frac{(1-\lambda)a+\lambda(a+b)/2}{2}\right)\\ \le&\frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)\\ &+\frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\\ &+\frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\\ &+\frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)\\ \le&\frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)\\ &+\frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-\lambda)^2\lambda^2[f(a)+f((a+b)/2))(g((a+b)/2)+g(b))\\
&+(f((a+b)/2)+f(b))(g((a+b)/2)+g(a))]\\
=&\frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)\\
&+\frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\\
&+\frac{1}{2}(1-\lambda)^2\lambda^2[N((a+b)/2,(a+b)/2))\\
&+N(a,b)+N(a,(a+b)/2)+N((a+b)/2,b)].
\end{aligned}
$$

Integrating both sides of the above inequality with respect to $\lambda$ over $[0,1]$, we have

$$
\begin{aligned}
&8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
\leq&\frac{1}{2}\int_0^1 f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)d\lambda\\
&+\frac{1}{2}\int_0^1 f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)d\lambda\\
&+\frac{1}{2}\int_0^1(1-\lambda)^2\lambda^2[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2))+N((a+b)/2,b)]d\lambda\\
=&\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b f(x)g(x)dx+\int_a^{\frac{a+b}{2}} f(x)g(x)dx\right]\\
&+\frac{1}{60}[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2)+N((a+b)/2,b)]\\
=&\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{60}[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2)+N((a+b)/2,b)].
\end{aligned}
$$

On the other hand, since

$$N((a+b)/2,(a+b)/2)\leq\frac{1}{8}[M(a,b)+N(a,b)] \tag{2.3}$$

and

$$N(a,(a+b)/2)+N((a+b)/2,b)\leq\frac{1}{2}[M(a,b)+N(a,b)], \tag{2.4}$$

we have

$$
\begin{aligned}
8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq&\frac{1}{b-a}\int_a^b f(x)g(x)dx\\
&+\frac{1}{60}(N((a+b)/2,(a+b)/2)+N(a,b)
\end{aligned}
$$

$$+ N(a,(a+b)/2) + N((a+b)/2,b)]$$
$$\leq \frac{1}{b-a}\int_a^b f(t)dt$$
$$+ \frac{1}{480}[5M(a,b) + 13N(a,b)]. \qquad \square$$

## 3. Applications to Probability Density Function

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x) = Pr(X \leq x) = \int_a^x f(t)dt$.

**Theorem 3.1.** *With the assumptions of Theorem* 2.1*, we have the inequality*

$$4F\left(\frac{a+b}{2}\right) \leq F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \tag{3.1}$$
$$\leq \frac{b-E(X)}{b-a}$$
$$\leq \frac{F(a)+F(b)}{12} + \frac{F((a+b)/2)}{6}$$
$$\leq \frac{F(a)+F(b)}{8}.$$

*Proof.* In the proof of Theorem 2.1, letting $f = F$, and taking into account that

$$E(x) = \int_a^b tdF(t) = b - \int_a^b F(t)dt,$$

we obtain (3.1). $\square$

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# INEQUALITIES FOR HYPERBOLIC TYPE HARMONIC PREINVEX FUNCTION

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Abstract. In the present paper, we have introduced a new class of preinvexity namely hyperbolic type harmonic preinvex functions and to support this new definition, some of its algebraic properties are elaborated. By using this new class of preinvexity, we have established a few Hermite-Hadamard type integral inequalities. Some novel refinements of Hemite-Hadamard type inequalities for hyperbolic type harmonic preinvex functions are presented as well. Finally, the Riemann-Liouville fractional version of the Hermite-Hadamard Inequality is established.

## 1. Preliminaries

Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $p < q$ and $p, q \in \mathbb{K}$. Then the Hermite-Hadamard inequality is expressed as follows (see [1]):

$$\varphi\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q \varphi(x)dx \leq \frac{\varphi(p)+\varphi(q)}{2}. \tag{1.1}$$

The Hermite-Hadamard inequality which was proved separately by Hermite in 1883 and Hadamard in 1896 is extensively studied in the convex theory. The double inequality is known as Hermite-Hadamard integral inequality for convex function in the literature. It deals with a necessary and sufficient condition for a function to be convex. For some recent results associated with the inequality (1.1) we recommend interested readers to go through [2–5] and the references therein.

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Recently, the concept of convexity has experienced very interesting developments. Many researchers generalised the classical concepts of convex sets and functions in different directions. A significant extension of convex function is invex function, introduced by Hanson [6]. Consequently, preinvex function is introduced by Ben Israel et al [7] and Weir et al. [8]. Rita Pini [9], introduced the concept of prequasi-invex as an extension of invex function.

**Definition 1.1** ([10])**.** A function $\varphi : \mathbb{K} = [p, p + \eta(q, p)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic preinvex function if, the inequality

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq (1-k)\varphi(p) + k\varphi(q), \quad \text{for all } p, q \in K,\ k \in [0,1], \tag{1.2}$$

holds, where $\eta(\cdot,\cdot) : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is a bifunction.

For $\eta(q, p) = q - p$, (1.2) reduces to the inequality for harmonic convex function.

If the inequality is reversed in (1.2), then $f$ is said to be harmonically preconcave function.

**Condition C** ([11]). Let $\mathbb{K} \subseteq \mathbb{R}$ be an invex set with respect to bi-function $\eta(\cdot,\cdot)$. Then for any $p, q \in \mathbb{K}$ and $k \in [0, 1]$

$$\eta\left(p, p + k\eta(q,p)\right) = -k\eta(q,p),$$
$$\eta\left(q, p + k\eta(q,p)\right) = (1-k)\eta(q,p),$$

for every $p, q \in \mathbb{K}$, $k_1, k_2 \in [0, 1]$ and using Condition C, we get

$$\eta(p + k_2\eta(q,p), p + k_1\eta(q,p)) = (k_2 - k_1)\eta(q,p).$$

In [11], Mohan and Neogy proved that a differentiable function which is invex on $\mathbb{K}$, w.r.t $\eta$, is also preinvex under Condition C.

İşcan proved the Hermite-Hadamard type inequality for the harmonically convex function.

**Theorem 1.1** ([12, Theorem 2.4])**.** *Let $\mathbb{K} \subseteq (0, \infty)$ be an interval and $\varphi : \mathbb{K} \to \mathbb{R}$ be a harmonically convex function with $p < q$ and $p, q \in \mathbb{K}$. Then the Hermite-Hadamard type inequality*

$$\varphi\left(\frac{2pq}{p+q}\right) \leq \frac{pq}{q-p}\int_p^q \frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(p)+\varphi(q)}{2} \tag{1.3}$$

*holds.*

Noor [10], has proved that a function $\varphi$ is harmonic preinvex if and only if $\varphi$ satisfies the following inequality

$$\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(p)+\varphi(q)}{2}.$$

Toplu [13], introduced the concept of Hyperbolic type convexity as follows.

**Definition 1.2.** A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type convex function if for every $p, q \in \mathbb{K}$ and $k \in [0, 1]$, the inequality

$$\varphi(kp + (1-k)q) \leq \frac{\sinh k}{\sinh 1}\varphi(p) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(q)$$

holds.

**Theorem 1.2** ([13, Theorem 3.1])**.** *Let $\varphi : [p, q] \to \mathbb{R}$ be a hyperbolic type convex function. If $p < q$ and $\varphi \in \mathcal{L}[p, q]$, then the following Hermite-Hadamard type inequality holds.*

$$\varphi\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q \varphi(x)dx \leq \frac{\cosh 1 - 1}{\sinh 1}\varphi(p) + \frac{e-1}{e\sinh 1}\varphi(q).$$

## 2. Main Result

In this section, we introduce new classes of hyperbolic type harmonic preinvex function. The main purpose of this paper is to introduce the concept of preinvexity for hyperbolic type harmonic convex functions and establish some results associated with the right hand side of the inequalities similar to (1.3) for the classes of hyperbolic type harmonic preinvex functions. For some recent results connected with preinvexity see [14–20] and the references therein.

**Definition 2.1.** A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type preinvex function if and only if for every $p, q \in K$ and $k \in [0, 1]$

$$\varphi(p + k\eta(q,p)) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)$$

holds.

**Definition 2.2.** A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type harmonic preinvex function if and only if for every $p, q \in \mathbb{K}$ and $k \in [0, 1]$

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) \tag{2.1}$$

holds.

**Definition 2.3.** Let $h : (0, 1) \subseteq \mathbb{K} \to \mathbb{R}$ be a non negative function, then a real valued function $\varphi : \mathbb{K} \subseteq [0, \infty] \to \mathbb{R}$ is called hyperbolic type $h$-harmonic preinvex function if and only if for every $p, q \in \mathbb{K}$ and $k \in [0, 1]$

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq h\left(\frac{\sinh k}{\sinh 1}\right)\varphi(q) + h\left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p) \tag{2.2}$$

holds.

*Remark* 2.1. If $h(k) = k$, then (2.2) reduces to (2.1).

**Theorem 2.1.** *Consider $\varphi$ and $\psi$ be two real valued hyperbolic type harmonic preinvex functions, then*

(i) *$\varphi+\psi$ is hyperbolic type harmonic preinvex function;*

(ii) *for $c \in \mathbb{R},\ c \geq 0$, the function $c\varphi$ is hyperbolic type harmonic preinvex function.*

*Proof.* (i) Let $\varphi$ and $\psi$ be two hyperbolic type harmonic preinvex functions, then

$$\begin{aligned}
&(\varphi+\psi)\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\\
=&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)+\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\\
\leq&\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)+\frac{\sinh k}{\sinh 1}\psi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(p)\\
=&\frac{\sinh k}{\sinh 1}[\varphi(q)+\psi(q)]+\frac{\sinh 1-\sinh k}{\sinh 1}[\varphi(p)+\psi(p)]\\
=&\frac{\sinh k}{\sinh 1}(\varphi+\psi)(q)+\frac{\sinh 1-\sinh k}{\sinh 1}(\varphi+\psi)(p).
\end{aligned}$$

(ii) Let $\varphi$ be hyperbolic type harmonic preinvex functions and $c \in \mathbb{R},\ c \geq 0$, then

$$\begin{aligned}
(c\varphi)\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &\leq c\left(\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right)\\
&=\frac{\sinh k}{\sinh 1}c\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}c\varphi(p)\\
&=\frac{\sinh k}{\sinh 1}(c\varphi)(q)+\frac{\sinh 1-\sinh k}{\sinh 1}(c\varphi)(p).
\end{aligned}$$

□

**Theorem 2.2.** *If $\varphi : \mathbb{K} \to \mathbb{K}$ is a hyperbolic type harmonic convex and $\psi : \mathbb{K} \to \mathbb{R}$ is a nondecreasing convex function, then $\psi \circ \varphi : \mathbb{K} \to \mathbb{R}$ is a hyperbolic type harmonic preinvex function.*

*Proof.* For $\alpha, \beta \in \mathbb{K}$ and $k \in [0,1]$

$$\begin{aligned}
\psi\circ\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &= \psi\left(\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right)\\
&\leq \frac{\sinh k}{\sinh 1}\psi(\varphi(q))+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(\varphi(p))\\
&\leq \frac{\sinh k}{\sinh 1}\psi\circ\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi\circ\varphi(p).
\end{aligned}$$

□

**Theorem 2.3.** *Let $\varphi : [p,q] \to \mathbb{R}$ be an arbitrary family of hyperbolic type harmonic preinvex functions and let $\varphi(x) = \sup_\alpha \varphi_\alpha(x)$. If $\mathbb{K} = \{v \in [p,q] : \varphi(v) < \infty\}$ is nonempty, then $\mathbb{K}$ is an interval and $\varphi$ is a hyperbolic type harmonic preinvex function on $\mathbb{K}$.*

*Proof.* For $p, q \in \mathbb{K}$ and $k \in [0,1]$

$$\begin{aligned}
\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &= \sup_{\alpha} \varphi_\alpha\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \\
&\leq \sup_{\alpha} \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) \\
&\leq \frac{\sinh k}{\sinh 1}\sup_{\alpha}\varphi_\alpha(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\sup_{\alpha}\varphi_\alpha(p) \\
&= \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) < \infty. \qquad \square
\end{aligned}$$

**Definition 2.4** ([10])**.** Two functions $u$ and $v$ are said to be of similar ordered if

$$(u(\alpha) - u(\beta))(v(\alpha) - v(\beta)) \geq 0, \quad \text{for all } \alpha, \beta \in \mathbb{R}.$$

**Theorem 2.4.** *Let $\varphi$ and $\psi$ be two similar ordered hyperbolic type harmonic preinvex function, then the product of two hyperbolic harmonic preinvex function is again a hyperbolic type harmonic preinvex function.*

*Proof.* Let $\varphi$ and $\psi$ be two hyperbolic type harmonic preinvex function, then

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \\
\leq& \left[\frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\right]\left[\frac{\sinh k}{\sinh 1}\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\psi(p)\right] \\
=& \left(\frac{\sinh k}{\sinh 1}\right)\varphi(q)\psi(q) + \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p)\psi(p) \\
&+ \left(\frac{\sinh k}{\sinh 1}\right)^2\varphi(q)\psi(q) + \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)^2\varphi(p)\psi(p) \\
&+ \frac{\sinh k}{\sinh 1}\cdot\frac{\sinh 1 - \sinh k}{\sinh 1}\left[\psi(q)\varphi(p) + \varphi(q)\psi(p)\right] \\
&- \left(\frac{\sinh k}{\sinh 1}\right)\varphi(q)\psi(q) - \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p)\psi(p) \\
=& \left[\frac{\sinh k}{\sinh 1}\varphi(q)\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\psi(p)\right] \\
&- \left[\frac{\sinh k}{\sinh 1} + \frac{\sinh 1 - \sinh k}{\sinh 1}\right](\varphi(p)\psi(p) + \varphi(q)\psi(q) - \varphi(p)\psi(q) - \varphi(q)\psi(p)) \\
=& \frac{\sinh k}{\sinh 1}\varphi(q)\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\psi(p). \qquad \square
\end{aligned}$$

## 3. Hermite-Hadamard Type Inequalities for Hyperbolic Type Harmonic Preinvex Function

**Theorem 3.1.** *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a hyperbolic type harmonic preinvex function and $p, p+\eta(q,p) \in \mathbb{K}$. If condition C holds and $\varphi \in \mathcal{L}[p, p+\eta(q,p)]$, then the following inequality holds:*

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx \\ \leq& \frac{\cosh 1 - 1}{\sinh 1}\varphi(p+\eta(q,p)) + \frac{e-1}{e\sinh 1}\varphi(p) \\ \leq& \frac{\cosh 1 - 1}{\sinh 1}\varphi(q) + \frac{e-1}{e\sinh 1}\varphi(p).\end{aligned}$$

*Proof.* Since $\varphi$ is hyperbolic type harmonic preinvex function putting $k = \frac{1}{2}$ and choosing $x = \frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}$ and $y = \frac{p(p+\eta(q,p))}{p+k\eta(q,p)}$ in

$$\varphi\left(\frac{x(x+\eta(y,x))}{x+(1-k)\eta(y,x)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(y) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(x).$$

Using Condition C, we get

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \frac{\sinh \frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \\ &+ \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right).\end{aligned}$$

Integrating with respect to $k$ over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \left(\frac{\sinh \frac{1}{2}}{\sinh 1}\right)\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx \\ &+ \left(\frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\right)\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx \\ =& \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx.\end{aligned}$$

Using the property of Hyperbolic type harmonic preinvex function and let $x = \frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}$, we have

$$\begin{aligned}\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx =& \int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)dk \\ \leq& \int_0^1 \left[\frac{\sinh k}{\sinh 1}\varphi(p+\eta(q,p)) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\right]dk \\ =& \left(\frac{\cosh 1 - 1}{\sinh 1}\right)\varphi(p+\eta(q,p)) + \frac{e-1}{e\sinh 1}\varphi(p)\end{aligned}$$

$$\leq \left(\frac{\cosh 1 - 1}{\sinh 1}\right)\varphi(q) + \frac{e-1}{e\sinh 1}\varphi(p). \qquad \square$$

**Lemma 3.1.** *Consider* $p, q \in \mathbb{R}$, *then*

$$\min(p, q) \leq \frac{p+q}{2}.$$

**Theorem 3.2.** *Consider* $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ *be a hyperbolic type harmonic preinvex function and* $p, q \in \mathbb{K}$. *If* $\varphi \in \mathcal{L}[p, q]$, *the following inequality holds:*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \\
&\leq \min\left\{\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\varphi(p), \left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\varphi(q)\right\} \\
&\leq \frac{1}{2}\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)[\varphi(p) + \varphi(q)].
\end{aligned}$$

*Proof.* Let $\varphi$ be a hyperbolic type harmonic preinvex function. Then

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)$$

and

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(p) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(q).$$

Adding both the above inequalities, we get

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) + \varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \\
&\leq \frac{\sinh k}{\sinh 1}[\varphi(p) + \varphi(q)] + \frac{\sinh 1 - \sinh k}{\sinh 1}[\varphi(p) + \varphi(q)].
\end{aligned} \tag{3.1}$$

Integrating (3.1) over the interval $[0, 1]$, one has

$$\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \leq \frac{1}{2}\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)[\varphi(p) + \varphi(q)]. \tag{3.2}$$

From Lemma 3.1 and (3.2), we have the desired result. $\square$

**Theorem 3.3.** *Let* $\varphi$ *and* $\psi$ *be two real valued hyperbolic type harmonic preinvex function, then*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(u)\psi(u)}{u^2} du \\
&\leq \left(\frac{e^4 - 4e^2 - 1}{8e^2 \sinh^2 1}\right)\varphi(q)\psi(q) + \left(\frac{-e^4 + 8e^3 - 8e^2 - 8e + 5}{8e^2 \sinh^2 1}\right)\varphi(p)\psi(p) \\
&\quad + \left(\frac{e^4 - 4e^3 + 4e^2 + 4e - 1}{8e^2 \sinh^2 1}\right)[\varphi(p)\psi(q) + \varphi(q)\psi(p)].
\end{aligned}$$

*Proof.* Considering $\varphi$ and $\psi$ be two hyperbolic type harmonic preinvex function, then

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \\
\leq&\left[\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right]\left[\frac{\sinh k}{\sinh 1}\psi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(p)\right] \\
=&\left(\frac{\sinh k}{\sinh 1}\right)^2\varphi(q)\psi(q)+\left(\frac{\sinh 1-\sinh k}{\sinh 1}\right)^2\varphi(p)\psi(p) \\
&+\frac{\sinh k}{\sinh 1}\cdot\frac{\sinh 1-\sinh k}{\sinh 1}\left[\psi(q)\varphi(p)+\varphi(q)\psi(p)\right].
\end{aligned}$$

Integrating both sides of the above inequality with respect to $k$ over $[0,1]$, one has

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(u)\psi(u)}{u^2}du \\
\leq&\frac{\varphi(q)\psi(q)}{\sinh^2 1}\int_0^1(\sinh k)^2dk+\frac{\varphi(p)\psi(p)}{\sinh^2 1}\int_0^1(\sinh 1-\sinh k)^2dk \\
&+\frac{\varphi(p)\psi(q)+\varphi(q)\psi(p)}{\sinh^2 1}\int_0^1\sinh k(\sinh 1-\sinh k)dk \\
=&\frac{\varphi(q)\psi(q)}{\sinh^2 1}\cdot\frac{(e^4-4e^2-1)}{8e^2}+\frac{\varphi(p)\psi(p)}{\sinh^2 1} \\
&\times\left[\frac{8e^2\sinh^2 1-16e^2\cosh 1\sinh 1+e^4-4e^2-1}{8e^2}+2\sinh 1\right] \\
&+\frac{[\varphi(p)\psi(q)+\varphi(q)\psi(p)]}{\sinh^2 1}\left[\frac{8e^2\cosh 1\sinh 1-e^4+4e^2+1}{8e^2}-\sinh 1\right] \\
=&\varphi(q)\psi(q)\left(\frac{e^4-4e^2-1}{8e^2\sinh^2 1}\right)+\varphi(p)\psi(p)\left(\frac{-e^4+8e^3-8e^2-8e+5}{8e^2\sinh^2 1}\right) \\
&+[\varphi(p)\psi(q)+\varphi(q)\psi(p)]\left(\frac{e^4-4e^3+4e^2+4e-1}{8e^2\sinh^2 1}\right).
\end{aligned}$$

□

**Theorem 3.4.** *Let $\varphi$ and $\psi$ be two similarly ordered real valued hyperbolic type harmonic preinvex function, then*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(u)\psi(u)}{u^2}du \\
\leq&\left(\frac{e^4-2e^3+2e-1}{4e^2\sinh^2 1}\right)\varphi(q)\psi(q)+\left(\frac{e^3-e^2-e+1}{2e^2\sinh^2 1}\right)\varphi(p)\psi(p).
\end{aligned}$$

*Proof.* The proof can be done by direct calculation using similarly ordered property.

□

## 4. Refinements of Hermite-Hadamard Inequality via Hyperbolic Type Harmonic Preinvexity

We now present the following lemma, which is a generalization of a result in [12].

**Lemma 4.1.** *Let* $\varphi : \mathbb{K} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{K}^\circ$ *and* $p, p+\eta(q,p) \in \mathbb{K}$ *with* $p+\eta(q,p) > p$. *If* $\varphi' \in \mathcal{L}[p, p+\eta(q,p)]$, *then the following identity holds in the preinvex setting:*

$$\begin{aligned}&\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\\ =&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\varphi'\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)dk.\end{aligned}$$

*Proof.* Considering

$$I = \int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\varphi'\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)dk,$$

after integrating by parts and some suitable rearrangements, the result is obtained. □

**Theorem 4.1.** *Let* $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{K}^\circ$ *and* $\varphi' \in \mathcal{L}([p, p+\eta(q,p)])$, *where* $[p, p+\eta(q,p)] \subseteq \mathbb{K}^\circ$. *If* $|\varphi'|$ *is hyperbolic harmonic preinvex function on* $[p, p+\eta(q,p)]$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\right|\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{|\varphi'(p)|}{\sinh 1}S_2 + \frac{|\varphi'(q)|}{\sinh 1}S_3\right],\end{aligned}$$

*where*

$$\begin{aligned}S_1 &= \int_0^1 \frac{|1-2k|}{(p+k\eta(q,p))^2}dk,\\ S_2 &= \int_0^1 \frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk,\\ S_3 &= \int_0^1 \frac{|1-2k|(\sinh 1-\sinh k)}{(p+k\eta(q,p))^2}dk.\end{aligned}$$

*Proof.* From Lemma 4.1 and using the concept of Hyperbolic harmonic preinvexity of $\varphi'$, we get

$$\begin{aligned}&\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\right|\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\left[\frac{\sinh k}{\sinh 1}|\varphi'(p)| + \frac{\sinh 1-\sinh k}{\sinh 1}|\varphi'(q)|\right]dk\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{\varphi'(p)}{\sinh 1}\int_0^1 \frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk\right.\end{aligned}$$

$$+\frac{\varphi'(q)}{\sinh 1}\int_0^1 \frac{|1-2k|\,(\sinh 1-\sinh k)}{(p+k\eta(q,p))^2}dk\Bigg]$$
$$\le\frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{|\varphi'(p)|}{\sinh 1}S_2+\frac{|\varphi'(q)|}{\sinh 1}S_3\right]. \qquad \square$$

**Theorem 4.2.** *Let $\varphi:\mathbb{K}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $\varphi'\in\mathcal{L}([p,p+\eta(q,p)])$, where $[p,p+\eta(q,p)]\subseteq\mathbb{K}^\circ$. If $|\varphi'|^s$ is hyperbolic harmonic preinvex function on $[p,p+\eta(q,p)]$ for $s\ge 1$, then the following inequality holds:*

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$
$$\le\frac{pq(q-p)}{2}A^{1-\frac{1}{s}}\left[B|\varphi'(p)|^s+C|\varphi'(q)|^s\right]^{\frac{1}{s}},$$

*where*

$$A=\int_0^1\frac{|1-2k|}{(p+k\eta(q,p))^2}dk,$$
$$B=\int_0^1\frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk,$$
$$C=\int_0^1\frac{|1-2k|\,(\sinh 1-\sinh k)}{(p+k\eta(q,p))^2}dk.$$

*Proof.* From Lemma 4.1 and using the Hölder's inequality, we get

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$
$$\le\frac{pq(q-p)}{2}\left(\int_0^1\left|\frac{|1-2k|}{(p+k\eta(q,p))^2}dk\right|\right)^{1-\frac{1}{s}}$$
$$\times\left(\int_0^1\left|\frac{|1-2k|}{(p+k\eta(q,p))^2}\right|\cdot\left|\varphi'\left(\frac{pq}{(p+k\eta(q,p))}\right)\right|^s dk\right)^{\frac{1}{s}}$$
$$\le\frac{pq(q-p)}{2}\left(\int_0^1\frac{|1-2k|}{(p+k\eta(q,p))^2}dk\right)^{1-\frac{1}{s}}$$
$$\times\left(\int_0^1\frac{|1-2k|}{(p+k\eta(q,p))^2}\left[\frac{\sinh k}{\sinh 1}|\varphi'(p)|^s+\frac{\sinh 1-\sinh k}{\sinh 1}|\varphi'(q)|^s\right]dk\right)^{\frac{1}{s}}$$
$$\le\frac{pq(q-p)}{2}A^{1-\frac{1}{s}}\left[B|\varphi'(p)|^s+C|\varphi'(q)|^s\right]^{\frac{1}{s}}. \qquad \square$$

**Theorem 4.3.** *Let $\varphi:\mathbb{K}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $\varphi'\in\mathcal{L}([p,p+\eta(q,p)])$, where $[p,p+\eta(q,p)]\subseteq\mathbb{K}^\circ$. If $|\varphi'|^s$ is hyperbolic harmonic preinvex function on $[p,p+\eta(q,p)]$ for $s\ge 1$, then the following inequality*

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$

$$\leq \frac{pq(q-p)}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \frac{|\varphi'(p)|^s - |\varphi'(q)|^s}{\sinh 1} \int_0^1 \frac{\sinh k}{(p+k\eta(q,p))^{2s}} dk$$
$$+ |\varphi'(q)|^s \int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}} dk$$

*holds.*

*Proof.* From Lemma 4.1 and using the Hölder's Integral inequality, we get

$$\left| \frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \right|$$
$$\leq \frac{p(p+\eta(q,p))}{2} \left( \int_0^1 |1-2k|^r \right)^{\frac{1}{r}}$$
$$\times \left( \int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}} \left| \varphi'\left( \frac{pq}{(p+k\eta(q,p))} \right) \right|^s dk \right)^{\frac{1}{s}}$$
$$\leq \frac{p(p+\eta(q,p))}{2} \left( \frac{1}{r+1} \right)^{\frac{1}{r}} \times \left( \int_0^1 \frac{\left[ \frac{\sinh k}{\sinh 1} |\varphi'(p)|^s + \frac{\sinh 1 - \sinh k}{\sinh 1} |\varphi'(q)|^s \right]}{(p+k\eta(q,p))^{2s}} dk \right)^{\frac{1}{s}}$$
$$= \frac{p(p+\eta(q,p))}{2} \left( \frac{1}{r+1} \right)^{\frac{1}{r}} \frac{|\varphi'(p)|^s - |\varphi'(q)|^s}{\sinh 1} \int_0^1 \frac{\sinh k}{(p+k\eta(q,p))^{2s}} dk$$
$$+ |\varphi'(q)|^s \int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}} dk. \qquad \square$$

## 5. Hermite-Hadamard Type Inequality via Fractional Integral

In this section, we have extended the above theorem 3.1 in the frame of Riemann-Liouville fractional operator. Recently, it is seen that integral inequalities using fractional operator has become an astonishing topic of research among mathematicians, for some recent papers and details (see [21, 22]).

**Definition 5.1.** Let $\varphi \in \mathcal{L}[p,q]$. The Riemann-Liouville operator $J^m_{q^-}\varphi$ and $J^m_{p^+}\varphi$ of order $m \geq 0$ are defined as

$$J^m_{p+}\varphi(x) = \frac{1}{\Gamma(m)} \int_p^x (x-k)^{m-1} \varphi(k) dk, \quad x \geq p,$$

and

$$J^m_{q-}\varphi(x) = \frac{1}{\Gamma(m)} \int_x^q (k-x)^{m-1} \varphi(k) dk, \quad x \leq q.$$

**Theorem 5.1.** *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi \in \mathcal{L}[p, p+\eta(q,p)]$, where $p, p+\eta(q,p) \in \mathbb{K}$ with $p < p+\eta(q,p)$. If $\varphi$ is a hyperbolic type harmonic preinvex function on $[p, p+\eta(q,p)]$, then the following inequality for fractional integral holds*

$$\frac{\sinh 1}{\sinh \frac{1}{2}} \left\{ \varphi\left( \frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))} \right) - \left( \frac{p(p+\eta(q,p))}{\eta(q,p)} \right)^m \Gamma(m+1) J^m_{\frac{1}{p}-} (\varphi \circ g) \left( \frac{1}{q} \right) \right\} \tag{5.1}$$

$$\leq \left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right) - J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}$$
$$\leq \varphi(p)+\varphi(q) - 2\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1) J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

*where* $g(x) = \frac{1}{x}$.

*Proof.* Since $\varphi$ is hyperbolic type harmonic preinvex function putting $k=\frac{1}{2}$ and choosing

$$x = \frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)} \quad \text{and} \quad y = \frac{p(p+\eta(q,p))}{p+k\eta(q,p)}$$

in

$$\varphi\left(\frac{xy}{kx+(1-k)y}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(y) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(x),$$

we get

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) + \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right).$$

Multiplying both sides by $k^{m-1}$ and integrating with respect to $k$ over $[0,1]$, we get

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)\int_0^1 k^{m-1}dk$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)k^{m-1}dk$$
$$+ \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)k^{m-1}dk,$$

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p)}\right)$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1) J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)$$
$$+ \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh 1}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1) J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

$$\frac{\sinh 1}{\sinh\frac{1}{2}}\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)$$
$$\leq \left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1) J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)$$
$$+ \frac{\sinh 1 - \sinh\frac{1}{2}}{\sinh\frac{1}{2}}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1) J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

$$\frac{\sinh 1}{\sinh\frac{1}{2}}\left\{\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)-\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}$$
$$\leq\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)-J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}. \tag{5.2}$$

For the second part of the proof, let $\varphi$ be a hyperbolic type harmonic preinvex function. Then

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\leq\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)$$

and

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)\leq\frac{\sinh k}{\sinh 1}\varphi(p)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(q).$$

Adding both the above inequalities,

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)+\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)\leq\varphi(p)+\varphi(q).$$

Multiplying both the sides by $k^{m-1}$ and integrating with respect to $k$ over $[0,1]$, we get

$$\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)-J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}$$
$$\leq\varphi(p)+\varphi(q)-2\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right). \tag{5.3}$$

Combining (5.2) and (5.3) we get (5.1). □

## 6. CONCLUSION

In this paper, we have introduced the generalizations of hyperbolic type convex functions as Hyperbolic type harmonic preinvex function. Applying this new class of preinvexity, we have presented few Hermite-Hadamard type inequalities (see Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4). Moreover, we have also presented some refinements of Hermite-Hadamard inequality (see Theorem 4.1, Theorem 4.2 and Theorem 4.3). At the end, we have also used fractional integral operator to generalize Theorem 3.1. The results, presented in this paper have the potential to establish more general inequalities involving fractional operators on different kinds of preinvexities.

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# ON COMMUTATIVITY DEGREE OF CROSSED MODULES

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ABSTRACT. In this paper, we define and study the notion of commutativity degree of finite crossed modules. We shall state some results concerning commutativity degree of crossed modules and obtain some upper and lower bounds for commutativity degree of finite crossed modules. Finally we show that, if two crossed modules are isoclinic, then they have the same commutativity degree.

## 1. INTRODUCTION

In 1968, Erdös and Turán [3], introduced the concept of commutativity degree of groups, when they worked on symmetric groups. Let $G$ be a finite group, the commutativity degree of $G$, denoted by $d(G)$ is defined as

$$d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Note that $d(G) > 0$ and $d(G) = 1$ if and only if $G$ is abelian. In 1973, Gustafson [5] obtained an upper bound for $d(G)$, when $G$ is a non-abelian finite group. Few years later, Rusin [14] computed the value of $d(G)$, when $G' \subseteq Z(G)$ and $G' \cap Z(G)$ is trivial and classified all finite groups $G$ for which $d(G)$ is greater than $\frac{11}{32}$. In 1995, all finite groups $G$, where $d(G) \geq \frac{1}{2}$ are classified, up to isoclinism, by Lescot [7]. Furthermore, Lescot [8] has also classified, up to isomorphism, all finite groups whose commutativity degrees lie in the interval $[\frac{1}{2}, 1]$. In 2006, Barry et al. [1] have shown that if $G$ is a finite group with odd order and $d(G) \geq \frac{11}{75}$, then $G$ is supersolvable. In addition, it had been proved that if $d(G) > \frac{1}{3}$, then $G$ is supersolvable. In 2007, Erfanian et al. [4] studied relative commutativity degree $d(H, G)$, the probability that

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elements of a given subgroup $H$ of a finite group $G$ commute with elements of $G$. In 2008, Pournaki and Sobhani [11] studied $d_g(G)$, the probability that the commutator of an arbitrarily chosen pair of elements in a finite group $G$ equals a given element $g$. In 2018, Sepehrizadeh and Rismanchian [16] introduced and studied the concept of characteristic degree of a subgroup in a finite group and determined the upper and lower bounds for this probability.

A crossed module $(T, G, \delta)$ is a group homomorphism $\delta : T \to G$ together with an action of $G$ on $T$ satisfying certain conditions. This notion is an algebraic model for homotopy 3-types was already introduced by Whitehead [17] in 1948. In [10], [13] and [15] the concepts of isoclinism and $n$-isoclinism have been generalized for crossed modules (see also [6]). In 2019, Yavari and Salemkar [18] presented a generalized crossed module and investigated the category of generalized crossed modules. Also, in [2] and [12] the notions of stem cover and universal central extension have been extended for lie crossed modules.

In this paper, we generalize the concept of commutativity degree for the finite crossed modules and show that two isoclinic crossed modules have the same commutativity degree.

## 2. Definitions and Preliminaries on Crossed Modules

In this section, we state some basic definitions, notions and elementary results.

A crossed module $(T, G, \delta)$ is a pair of groups $T$ and $G$ together with an action of $G$ on $T$ and a homomorphism $\delta : T \to G$ called the boundary map, satisfying the following axioms:

*i)* $\delta({}^{g}t) = g\delta(t)g^{-1}$ for all $g \in G$, $t \in T$;

*ii)* ${}^{\delta(t)}s = tst^{-1}$ for all $t, s \in T$.

We will denote such a crossed module by $T \overset{\delta}{\to} G$. A crossed module $(T, G, \delta)$ is said to be finite, if the groups $T$ and $G$ are both finite. A crossed module $(S, H, \delta')$ is a subcrossed module of $(T, G, \delta)$, when

*i)* $S$ is a subgroup of $T$ and $H$ is a subgroup of $G$;

*ii)* $\delta' = \delta|_S$, the restriction of $\delta$ to $S$;

*iii)* the action of $H$ on $S$ is induced by the action of $G$ on $T$.

In this case, we write $(S, H, \delta') \leqslant (T, G, \delta)$. A subcrossed module $(S, H, \delta)$ of $(T, G, \delta)$ is a normal subcrossed module, if

*i)* $H$ is a normal subgroup of $G$;

*ii)* ${}^{g}s \in S$ for all $g \in G$, $s \in S$;

*iii)* ${}^{h}tt^{-1} \in S$ for all $h \in H$, $t \in T$.

This is denoted by $(S, H, \delta) \trianglelefteq (T, G, \delta)$.

Let $(S, H, \delta)$ be a normal subcrossed module of $(T, G, \delta)$. Consider the triple $(\frac{T}{S}, \frac{G}{H}, \bar{\delta})$, where $\bar{\delta} : \frac{T}{S} \to \frac{G}{H}$ is induced by $\delta$. There is the action of $\frac{G}{H}$ on $\frac{T}{S}$ given by ${}^{gH}(tS) = ({}^{g}t)S$. It is called the quotient crossed module of $(T, G, \delta)$ by $(S, H, \delta)$ and denoted by $\frac{(T,G,\delta)}{(S,H,\delta)}$.

Let $(T, G, \delta)$ be a crossed module. The center of $(T, G, \delta)$ is the crossed module $Z(T, G, \delta) : T^G \to St_G(T) \cap Z(G)$, where $T^G = \{t \in T : {}^g t = t \text{ for all } g \in G\}$ and $St_G(T) = \{g \in G : {}^g t = t \text{ for all } t \in T\}$. A crossed module $(T, G, \delta)$ is abelian, if $(T, G, \delta) = Z(T, G, \delta)$. In addition, the commutator subcrossed module $[(T, G, \delta), (T, G, \delta)]$ of $(T, G, \delta)$ is $[(T, G, \delta), (T, G, \delta)] : D_G(T) \to [G, G]$, where $D_G(T)$ is the subgroup generated by $\{{}^g tt^{-1} : t \in T,\ g \in G\}$ and $[G, G]$ is the commutator subgroup of $G$. A crossed module $(T, G, \delta)$ is faithful, if the action of $G$ on $T$ is faithful, that is $St_G(T) = 1$.

If $(S, H, \delta')$ and $(R, K, \delta'')$ are two crossed modules, then consider the triple $(S \times R, H \times K, \delta' \times \delta'')$, where $S \times R$ and $H \times K$ are direct products of groups and $\delta' \times \delta'' : S \times R \to H \times K$ is defined by $(\delta' \times \delta'')(s, r) = (\delta'(s), \delta''(r))$ for all $(s, r) \in S \times R$. There is a componentwise action of $H \times K$ on $S \times R$, induced by the actions of two crossed modules. The crossed module $(S \times R, H \times K, \delta' \times \delta'')$ is called the direct product of $(S, H, \delta')$ and $(R, K, \delta'')$ and denoted by $(S, H, \delta') \times (R, K, \delta'')$.

Let $(T, G, \delta)$ and $(T', G', \delta')$ be crossed modules. A crossed module morphism $\langle \alpha, \phi \rangle : (T, G, \delta) \to (T', G', \delta')$ is a pair of homomorphism $\alpha : T \to T'$, $\phi : G \to G'$ such that

*i)* $\delta'(\alpha(t)) = \phi(\delta(t))$ for all $t \in T$;

*ii)* $\alpha({}^g t) = {}^{\phi(g)}\alpha(t)$ for all $t \in T$, $g \in G$.

If $\langle \alpha, \phi \rangle : (S, H, \delta') \to (T, G, \delta)$ is a crossed module morphism such that $\alpha$ and $\phi$ are both group isomorphisms, then $\langle \alpha, \phi \rangle$ is called an isomorphism.

**Lemma 2.1** ([9])**.** *The $(T, G, \delta)$ is abelian if and only if $G$ is abelian and the action of the crossed module is trivial.*

*Remark* 2.1. Let $(T, G, \delta)$ be a crossed module. We denote $\frac{(T,G,\delta)}{Z(T,G,\delta)}$ by $\bar{T} \xrightarrow{\bar{\delta}} \bar{G}$, where $\bar{T} = \frac{T}{T^G}$ and $\bar{G} = \frac{G}{St_G(T) \cap Z(G)}$, for shortness.

**Lemma 2.2** ([10])**.** *Let $(T, G, \delta)$ be a crossed module. Define the maps $c_1 : \bar{T} \times \bar{G} \to D_G(T)$, where $(tT^G, g(St_G(T) \cap Z(G))) \mapsto {}^g tt^{-1}$ and $c_0 : \bar{G} \times \bar{G} \to [G, G]$, where $(g(St_G(T) \cap Z(G)), g'(St_G(T) \cap Z(G))) \mapsto [g, g']$ for all $t \in T$, $g, g' \in G$. Then the maps $c_1$ and $c_0$ are well-defined.*

**Definition 2.1** ([15])**.** The crossed modules $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ are isoclinic, if there exist isomorphisms

$$(\eta_1, \eta_0) : (\bar{T_1}, \bar{G_1}, \bar{\delta_1}) \to (\bar{T_2}, \bar{G_2}, \bar{\delta_2})$$

and

$$(\epsilon_1, \epsilon_0) : (D_{G_1}(T_1) \to [G_1, G_1]) \to (D_{G_2}(T_2) \to [G_2, G_2])$$

such that the diagrams

$$\begin{array}{ccc} \bar{T_1} \times \bar{G_1} & \xrightarrow{c_1} & D_{G_1}(T_1) \\ \Big\downarrow{\eta_1\times\eta_0} & & \Big\downarrow{\epsilon_1} \\ \bar{T_2} \times \bar{G_2} & \xrightarrow{c_1'} & D_{G_1}(T_1) \end{array}$$

and

$$\begin{array}{ccc} \bar{G_1} \times \bar{G_1} & \xrightarrow{c_0} & [G_1, G_1] \\ \Big\downarrow{\eta_0\times\eta_0} & & \Big\downarrow{\epsilon_0} \\ \bar{G_2} \times \bar{G_2} & \xrightarrow{c_0'} & [G_2, G_2] \end{array}$$

are commutative, where $(c_1, c_0)$ and $(c_1', c_0')$ are commutator maps of crossed modules $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$, that introduced in Lemma 2.2. The pair $((\eta_1, \eta_0), (\epsilon_1, \epsilon_0))$ will be called an isoclinism from $(T_1, G_1, \delta_1)$ to $(T_2, G_2, \delta_2)$ and this situation will be denoted by $((\eta_1, \eta_0), (\epsilon_1, \epsilon_0)) : (T_1, G_1, \delta_1) \sim (T_2, G_2, \delta_2)$.

## 3. Commutativity Degree of Crossed Modules

In the section, we generalize the notion of commutativity degree for crossed modules and state the main results of this paper.

**Definition 3.1.** Let $(T, G, \delta)$ be a finite crossed module. The commutativity degree $d(T, G, \delta)$ of $(T, G, \delta)$ is defined by

$$d(T, G, \delta) = \frac{|\{(x, y) \in G \times G : xy = yx, x, y \in St_G(T)\}|}{|G|^2}.$$

Let $(T, G, \delta)$ be a finite crossed module, then we set $cs(G) = \{(x, y) \in G \times G : xy = yx \text{ and } x, y \in St_G(T)\}$. Now, the commutativity degree $(T, G, \delta)$ is $d(T, G, \delta) = \frac{|cs(G)|}{|G\times G|}$. If $d(T, G, \delta)$ is abelian and the action of $G$ on $T$ is trivial, then $|cs(G)| = |G \times G|$ and $d(T, G, \delta) = 1$ and vice-versa. Therefore, $(T, G, \delta)$ is abelian if and only if $d(T, G, \delta) = 1$. In addition, if the action of $G$ on $T$ is faithful, then $d(T, G, \delta) = \frac{1}{|G|^2}$.

**Proposition 3.1.** *Let $(S, H, \delta')$ and $(R, K, \delta'')$ be two crossed modules and $(T, G, \delta) = (S, H, \delta') \times (R, K, \delta'')$. Then $d(T, G, \delta) = d(S, H, \delta') \times d(R, K, \delta'')$.*

*Proof.* By the definition of commutativity degree, we have

$$\begin{aligned} d(T, G, \delta) =& \frac{1}{|G|^2} |\{((h_1, k_1), (h_2, k_2)) \in G^2 : (h_1, k_1)(h_2, k_2) = (h_2, k_2)(h_1, k_1) \\ & \text{and } (h_1, k_1), (h_2, k_2) \in St_G(T)\}| \\ =& \frac{1}{|G|^2} |\{((h_1, k_1), (h_2, k_2)) \in G^2 : (h_1h_2, k_1k_2) = (h_2h_1, k_2k_1) \\ & \text{and } (h_1, k_1), (h_2, k_2) \in St_G(T)\}| \\ =& \left( \frac{1}{|H|^2} |\{(h_1, h_2) \in H^2 : h_1h_2 = h_2h_1 \text{ and } h_1, h_2 \in St_H(S)\}| \right) \end{aligned}$$

$$\times\left(\frac{1}{|K|^2}|\{(k_1,k_2)\in K^2 : k_1k_2=k_2k_1 \text{ and } k_1,k_2\in St_K(R)\}|\right)$$
$$=d(S,H,\delta')\times d(R,K,\delta''). \qquad \square$$

**Theorem 3.1.** *Let $(T,G,\delta)$ be a crossed module. Then $d(T,G,\delta)\leq \frac{K(G)}{|G|}$, where $K(G)$ is the number of conjugacy classes of $G$.*

*Proof.* Let $r$ be the number of conjugacy classes of $G$ and $C_1,C_2,C_3,\ldots,C_r$ be the conjugacy classes of $G$. For $i\in\{1,2,3,\ldots,r\}$, let $x_i\in C_i$. If $y\in C_i$, then $y=x_i^g$ for some $g\in G$. Thus, $C_G(y)=C_G(x_i^g)=C_G(x_i)^g$ and $|C_G(y)|=|C_G(x_i)|$. Now

$$\begin{aligned}|G|^2d(T,G,\delta)=&|\{(x,y)\in G\times G : xy=yx \text{ and } x,y\in St_G(T)\}|\\ =&|cs(G)|\leq\sum_{x\in G}|C_G(x)|=\sum_{i=1}^{r}\sum_{x\in C_i}|C_G(x)|\\ =&\sum_{i=1}^{r}[G:C_G(x_i)]|C_G(x_i)|=|G|r=|G|K(G).\end{aligned}$$

Therefore, $d(T,G,\delta)\leq\frac{K(G)}{|G|}$. $\square$

**Corollary 3.1.** *If $(T,G,\delta)$ is a crossed module and the action of $G$ on $T$ is trivial, then $d(T,G,\delta)=\frac{K(G)}{|G|}$.*

**Corollary 3.2.** *Let $(T,G,\delta)$ be a crossed module. If the action of $G$ on $T$ is trivial, then $\frac{1}{|G'|}\leq d(T,G,\delta)$.*

*Proof.* Since $[G:G']$ count irreducible characters of degree one, $[G:G']<K(G)$. Then $\frac{|G|}{|G||G'|}\leq\frac{K(G)}{|G|}=d(T,G,\delta)$ so $\frac{1}{|G'|}\leq d(T,G,\delta)$. $\square$

**Theorem 3.2.** *Let $(T,G,\delta)$ be a crossed module. Then $d(T,G,\delta)\leq\frac{1}{4}(1+\frac{3}{|G'|})$.*

*Proof.* Let $l$ be the number of non-equivalent irreducible representation of degree $1,n_2,\ldots,n_l$. Consider the degree equation

$$|G|=[G:G']+\sum_{i=[G:G']+1}^{K(G)}(n_i)^2,$$

for each $n_i\geq 2$. Hence, $|G|\geq[G:G']+4(K(G)-[G:G'])$. Solving for $K(G)$ yield $K(G)\leq\frac{1}{4}(|G|+3[G:G'])$. Therefore, $d(T,G,\delta)\leq\frac{K(G)}{|G|}\leq\frac{1}{4}(1+\frac{3}{|G'|})$. $\square$

**Theorem 3.3.** *Let $(T,G,\delta)$ be a crossed module. If $G$ is a non-abelian finite group, then $d(T,G,\delta)\leq\frac{5}{8}$.*

*Proof.* Consider the class equation $|G|=|Z(G)|+\sum_{i=|Z(G)|+1}^{K(G)}|[x_i]|$, where for each $i$, $|[x_i]|\geq 2$. So $|G|\geq|Z(G)|+2(K(G)-|Z(G)|)\geq|Z(G)\cap St_G(T)|+2(K(G)-|Z(G)|)$.

Since $G$ is not abelian, $\frac{G}{Z(G)}$ is not cyclic. Then $|\frac{G}{Z(G)}| \geq 4$, hence

$$\left|\frac{G}{Z(G) \cap St_G(T)}\right| \geq \left|\frac{G}{Z(G)}\right| \geq 4$$

and $|Z(G) \cap St_G(T)| \leq \frac{|G|}{4}$. Therefore, $K(G) \leq \frac{1}{2}\left(|G| + \frac{|G|}{4}\right) = \frac{5|G|}{8}$. Then $d(T, G, \delta) \leq \frac{K(G)}{|G|} \leq \frac{5}{8}$. □

**Theorem 3.4.** *Let $(T, G, \delta)$ be a crossed module. If $G$ be a nonablian $p$-group, then $d(T, G, \delta) \leq \frac{p^2+p-1}{p^3}$.*

*Proof.* Let $p$ be the prime number, $|G| = p^n$ and $|Z(G)| = P^m$. If $|G/Z(G)| = 1$ or $|G/Z(G)| = p$, then $G/Z(G)$ is cyclic and $G$ will be abelian. Thus, $m \leq n-2$ and we have

$$\begin{aligned} |G|^2 d(T, G, \delta) \leq & |G|K(G) \\ = & \sum_{x \in G} |C_G(x)| \\ = & \sum_{x \in Z(G)} |C_G(x)| + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\ \leq & p^{m+n} + p^{n-1}(p^n - p^m) \\ = & p^{m+n} + p^{n-1}(|G| - |Z(G)|) \\ \leq & p^{2n-3}(p^2 + p - 1). \end{aligned}$$

Hence, $d(T, G, \delta) \leq \dfrac{p^2 + p - 1}{p^3}$. □

The following corollary obtains from previous theorem.

**Corollary 3.3.** *Let $(T, G, \delta)$ be a crossed module and the action of $G$ on $T$ is trivial. If $p$ is a prime number and $G$ is non-abelian with $|G| = p^3$, then $d(T, G, \delta) = \frac{p^2+p-1}{p^3}$.*

*Proof.* Let $|G| = p^3$. By Corollary 3.1 and Theorem 3.4,

$$\begin{aligned} p^6 d(T, G, \delta) = & |G|^2 d(T, G, \delta) \\ = & |G|K(G) \\ = & \sum_{x \in G} |C_G(x)| \\ = & \sum_{x \in Z(G)} |C_G(x)| + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\ = & p^4 + p^2(|G| - |Z(G)|) \\ = & p^3(p^2 + p - 1). \end{aligned}$$

Hence, $d(T, G, \delta) = \frac{p^2+p-1}{p^3}$. □

**Proposition 3.2.** *Let $(T, G, \delta)$ be a crossed module and the action of $G$ on $T$ be trivial. If $|\frac{G}{Z(G)}| = p^k$, then $d(T, G, \delta) \geq \frac{p^{k-1}+p^k-1}{p^{2k-1}}$.*

*Proof.* Let $|\frac{G}{Z(G)}| = p^k$ and $x \in G$ such that $x \notin Z(G)$. Then, since $x \in C_G(x)$ and $x \notin Z(G)$, we have $C_G(x) \subsetneq G$. Also $Z(G) \subsetneq C_G(x)$. Thus $|Z(G)| < |C_G(x)| < |G|$ and then $p|Z(G)| \leq |C_G(x)| \leq p^{k-1}|Z(G)|$, where $|C_G(x)|$ dividing $|G|$. So $|[x]| = [G : C_G(x)]$ and $p^{k-1} \geq |[x]| \geq p$. From the class equation we have $|G| = |Z(G)| + \sum_{x\in G} |[x]| \leq |Z(G)|+p^{k-1}(K(G)-|Z(G)|) \leq |Z(G)|+p^{k-1}(K(G)-|Z(G)\cap St_G(T)|)$, therefore

$$K(G) \geq \frac{|G| + (p^{k-1})|Z(G) \cap St_G(T)| - |Z(G)|}{p^{k-1}}.$$

Now solving for $d(T, G, \delta)$ and Corollary 3.1 yield $d(T, G, \delta) \geq \frac{p^k+p^{k-1}-1}{p^{2k-1}}$. □

*Example* 3.1. Let $D_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$ such that $p$ is prime, $q|p-1$ and $r$ has order $q$ mod $p$. This type of group is called a generalized dihedral group. Conjugacy classes type are $[e]$, $[a^u]$ and $[b^w]$ so that no classes are 1, $\frac{p-1}{q}$ and $q-1$, respectively and $Z(D_{pq}) = \{e\}$. Consider the map $i : D_{pq} \to D_{pq}$. If the action of $D_{pq}$ on $D_{pq}$ is conjugacy, then $St_{D_{pq}}(D_{pq}) = Z(D_{pq})$ and $d(D_{pq}, D_{pq}, i) = \frac{|Z(D_{pq})|^2}{|D_{pq}|^2} = \frac{1}{(pq)^2}$. If the action of $D_{pq}$ on $D_{pq}$ is trivial, then

$$d(D_{pq}, D_{pq}, i) = \frac{K(D_{pq})}{|D_{pq}|} = \frac{1 + \frac{p-1}{q} + q - 1}{pq} = \frac{q^2 + p - 1}{pq^2}.$$

If the action of $D_{pq}$ on $D_{pq}$ is faithful, then $d(D_{pq}, D_{pq}, i) = \frac{1}{|D_{pq}|^2} = \frac{1}{(pq)^2}$.

**Lemma 3.1.** *Let $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ be two crossed modules and $\varphi = \langle\alpha, \beta\rangle : (T_1, G_1, \delta_1) \to (T_2, G_2, \delta_2)$ be an isomorphism. Then $\varphi$ induced isomorphisms*

$$\varphi_1 : \frac{G_1}{St_{G_1}(T_1) \cap Z(G_1)} \to \frac{G_2}{St_{G_2}(T_2) \cap Z(G_2)},$$

*where*

$$\varphi_1(g_1(St_{G_1}(T_1) \cap Z(G_1))) = \beta(g_1)(St_{G_2}(T_2) \cap Z(G_2)),$$

*$\varphi_2 : \frac{T_1}{T_1^{G_1}} \to \frac{T_2}{T_2^{G_2}}$, where $\varphi_2(t_1T_1^{G_1}) = \alpha(t_1)T_2^{G_2}$, $\varphi_3 : [G_1, G_1] \to [G_2, G_2]$, where $\varphi_3[g_1, g_1] = [g_2, g_2]$ or $\varphi_3(g_1) = \beta(g_1)$, $\varphi_4 : D_{G_1}(T_1) \to D_{G_2}(T_2)$, where $\varphi_4({}^{g_1}t_1t_1^{-1}) = {}^{\beta(g_1)}\alpha(t_1)\alpha(t_1{}^{-1})$. In addition the following diagrams commute:*

$$\begin{array}{ccc} \bar{G_1} \times \bar{G_1} & \xrightarrow{\varphi_1\times\varphi_1} & \bar{G_2} \times \bar{G_2} \\ \downarrow c_0 & & \downarrow c_0' \\ G_1' & \xrightarrow{\varphi_3} & G_2' \end{array}$$

*and*

$$\begin{array}{ccc}\bar{T_1}\times\bar{G_1} & \xrightarrow{\varphi_2\times\varphi_1} & \bar{T_2}\times\bar{G_2}\\ \Big\downarrow{c_1} & & \Big\downarrow{c_1'}\\ D_{G_1}(T_1) & \xrightarrow{\varphi_4} & D_{G_1}(T_1).\end{array}$$

*Proof.* Let $(g_1,g_1')\in G_1\times G_1$. Then $c_0'(\varphi_1\times\varphi_1)(\bar{g_1},\bar{g_1'}) = c_0'(\varphi_1(\bar{g_1}),\varphi_1(\bar{g_1'})) = c_0'(\overline{\beta(g_1)},\overline{\beta(g_1')}) = [\beta(g_1),\beta(g_1')] = \beta[g_1,g_1'] = \varphi_3[g_1,g_1'] = \varphi_3 c_0(\bar{g_1},\bar{g_1'})$. Now, let $(\bar{t_1},\bar{g_1})\in\bar{T_1}\times\bar{G_1}$. Then

$$\begin{aligned}c_1'(\varphi_2\times\varphi_1)(\bar{t_1},\bar{g_1}) =& c_1'(\varphi_2(\bar{t_1}),\varphi_1(\bar{g_1}))\\ =& c_1'(\alpha(t_1)T_2^{G_2},\beta(g_1)(St_{G_2}(T_2)\cap Z(G_2)))\\ =& {}^{\beta(g_1)}\alpha(t_1)\alpha(t_1)^{-1} = \varphi_4({}^{g_1}t_1t_1^{-1})\\ =& \varphi_4 c_1(\bar{t_1},\bar{g_1}).\end{aligned}$$

□

**Theorem 3.5.** *Let $(T_1,G_1,\delta_1)$,$(T_2,G_2,\delta_2)$ be two isoclinic finite crossed modules. Then $d(T_1,G_1,\delta_1) = d(T_2,G_2,\delta_2)$.*

*Proof.* Consider the following relation:

$$\begin{aligned}\frac{|G_1|^2 d(T_1,G_1,\delta_1)}{|St_{G_1}(T_1)\cap Z(G_1)|^2} =& \frac{|G_1|^2}{|St_{G_1}(T_1)\cap Z(G_1)|^2}\\ &\times\frac{|\{(x,y)\in G_1\times G_1 : xy=yx \text{ and } x,y\in St_{G_1}(T_1)\}|}{|G_1|^2}\\ =& \left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2\\ &\times|\{(x,y)\in G_1\times G_1 : xy=yx \text{ and } x,y\in St_{G_1}(T_1)\}|\\ =& \left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2\\ &\times|\{(x,y)\in G_1\times G_1 : [x,y]=1 \text{ and } x,y\in St_{G_1}(T_1)\}|\\ =& \left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2 |\{(x,y)\in G_1\times G_1 : x,y\in St_{G_1}(T_1)\\ &\text{and } c_0(x(St_{G_1}(T_1)\cap Z(G_1)),y(St_{G_1}(T_1)\cap Z(G_1)))=1\}|\\ =& \left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2 : c_0(\alpha,\beta)=1\right\}\right|\\ =& \left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2 : \varphi_3 c_0(\alpha,\beta)=1\right\}\right|\\ =& \left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2 : c_0'(\varphi_1,\varphi_1)(\alpha,\beta)=1\right\}\right|\end{aligned}$$

$$= \left| \left\{ (\alpha, \beta) \in \left( \frac{G_1}{St_{G_1}(T_1) \cap Z(G_1)} \right)^2 : c'_0(\varphi_1(\alpha), \varphi_1(\beta)) = 1 \right\} \right|$$
$$= \left| \left\{ (\gamma, \sigma) \in \left( \frac{G_2}{St_{G_2}(T_2) \cap Z(G_2)} \right)^2 : c'_0(\gamma, \sigma) = 1 \right\} \right|.$$

By the above reasoning applied to $(T_2, G_2, \delta_2)$ in place of $(T_1, G_1, \delta_1)$, this expression equals to $\left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|^2 d(T_2, G_2, \delta_2)$. That is $\left|\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2 d(T_1, G_1, \delta_1) = \left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|^2 d(T_2, G_2, \delta_2)$. But $\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}$ and $\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}$ are isomorphic, hence $\left|\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right| = \left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|$. Now the equality $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$ follows. □

**Corollary 3.4.** *Let $(T_2, G_2, \delta_2)$ be a subcrossed module of crossed module $(T_1, G_1, \delta_1)$ and $(T_1, G_1, \delta_1) = (T_2, G_2, \delta_2)Z(T_1, G_1, \delta_1)$, where $T_1 = G_2 T_1^{G_1}$ and $G_1 = G_2(St_{G_1}(T_1) \cap Z(G_1))$. Then $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$.*

*Proof.* By Proposition 4 of [10], $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ are isoclinic, therefore $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$. □

## 4. CONCLUSION

In this paper, we extended the concept of commutativity degree in group theory to finite crossed modules and derived some properties of this new concept. All our previous results show that the notion of commutativity degree, which was introduced in this paper, can be used to classify finite crossed modules. It is clear that this study which started here, can be successfully extended to calculating commutativity degree of some specific crossed modules. This will surely be the subject of further research.

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# LIPSCHITZ STABILITY FOR IMPULSIVE RIEMANN–LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Initial and impulsive conditions for initial value problems of systems of nonlinear impulsive Riemann–Liouville fractional differential equations are introduced. The case when the lower limit of the fractional derivative is changed at each time point of the impulses is studied. In the case studied, the solution has a singularity at the initial time and at any point of the impulses. This leads to the need to appropriately generalize the classical concept of Lipschitz stability. Two derivative types of Lyapunov functions are utilized in order to deduce sufficient conditions for the new stability concept. Three examples are provided for illustration purpose of the theoretical results.

## 1. Introduction

Differential equations with impulses are intensively studied and applied in modeling various phenomena (see, e.g., the monograph of Lakshmikantham et al. [23]). Recently, fractional differential equations have proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. Actually, fractional differential equations are now considered as an alternative model to integer differential equations (for more details, see the monographs [16, 17, 26] and the references therein).

In addition, some modeling is done via impulsive fractional differential equations when these processes involve hereditary phenomena such as biological and social macrosystems and are subject to some impulsive perturbations. Note that the literature knows various types of fractional derivatives. To the best of our knowledge,

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impulsive fractional functional differential equations involving the Caputo fractional derivative have been studied in completeness. It is worth remarking that Feckan et al. [18] give a counterexample to show that some formula of solutions in previous papers is incorrect and reconsider a class of impulsive fractional differential equations and introduce a correct formula of solutions for an impulsive initial value problem with Caputo fractional derivative. The situation is not the same when the Riemann–Liouville (RL) fractional derivatives is used. The statement of the impulsive condition and the lower limit of the RL fractional derivative is presented in different ways by different authors. For example, in [13, 32], the impulsive conditions are related to the right and left limits of RL fractional integrals with fixed lower limit at the initial time. In [12, 31], the impulsive conditions are connected with the RL fractional integral on the intervals between two consecutive impulses. In [8], the impulses are RL fractional derivatives. In [24], the impulsive conditions contain RL fractional derivatives. Note that the formula for the exact solution of linear impulsive RL fractional differential equations is given recently in [5] and for scalar impulsive equation with delay in [6].

One of the most important properties of solutions is stability. Many stability concepts exist, describing various behavior of the solutions, e.g., Lipschitz stability defined for ODEs [15]. Later, this type of stability has been studied for various types of differential equations and problems such as, e.g., nonlinear differential systems [14, 19, 28], impulsive differential equations with delays [9], fractional differential systems [29], Caputo fractional differential equations with noninstantaneous impulses [4], a piecewise linear Schrödinger potential [7], a hyperbolic inverse problem [10], the electrical impedance tomography problem [11], and the radiative transport equation [25]. See also [2, 3, 8, 14, 19, 20, 27, 28] for related references. In the recent paper [21], a similar problem is considered without impulses.

In view of the above considerations, in this paper, in an appropriate way, we set up impulsive RL fractional differential equations and study Lipschitz stability properties of the zero solution. We will give some reasons for the defined impulsive conditions. Let an increasing sequence of nonnegative points $\{t_i\}_{i\in\mathbb{N}}$, $t_0 = 0$, be given such that $\lim_{i\to\infty} t_i = \infty$. When impulses are involved in fractional differential equations, there are mainly two interpretations of fractional derivatives:

- *fixed lower limit of the fractional derivative* – in this case, the lower limit of the fractional derivative is kept equal to the initial time on the whole interval of consideration.
- *changeable lower limit of the fractional derivative* – each time $t_i$, $i \in \mathbb{N}$, of the impulse is considered as a lower limit of the fractional derivative.

In this paper, we consider the case of changeable lower limit of the Riemann–Liouville fractional derivative. The presence of the Riemann–Liouville derivative leads to two specific types of initial conditions, which are equivalent (see the classical book [17]).

- *Integral form of the initial condition*

$$ {}_0I_t^{1-q}x(t)|_{t=0} = \lim_{t\to 0+} {}_0I_t^{1-q}x(t) = x_0. $$

- *Weighted form of the initial condition*

$$\lim_{t\to 0+}\left(t^{1-q}x(t)\right)=\frac{x_0}{\Gamma(q)}.$$

Here, the Riemann-Liouville (RL) fractional integral is defined by

$${}_aI_t^{1-q}x(t)=\frac{1}{\Gamma(1-q)}\int_a^t\frac{x(s)}{(t-s)^q}\mathrm{d}s,\quad t>a,$$

and $\Gamma$ denotes the Gamma function. In the literature, when the RL derivative is applied, there are various types of statements for impulsive conditions. We will follow the ideas of impulses in ordinary differential equations, i.e., after the impulse, the differential equation is the same with a new initial condition. This will lead to two types of initial conditions (following [17]):

- *integral form of the impulsive conditions*

$${}_{t_i}I_t^{1-q}x(t)|_{t=t_i}=\lim_{t\to t_i}{}_{t_i}I_t^{1-q}x(t)=\Phi_i(x(t_i-0)),\quad i\in\mathbb{N}.$$

- *weighted form of the impulsive conditions*

$$\lim_{t\to t_i+}\left((t-t_i)^{1-q}x(t)\right)=\frac{\Phi_i(x(t_i-0))}{\Gamma(q)},\quad i\in\mathbb{N}.$$

In this paper, we will use the integral form of both, the initial condition and the impulsive conditions. Keeping in mind the above description, in this paper, we will study the initial value problem (IVP) for the following system of nonlinear RL fractional differential equations with impulses (IRLFDE) of fractional order $q\in(0,1)$:

$$\begin{cases}{}_{t_i}^{\mathrm{RL}}D_t^qx(t)=f(t,x(t)), & \text{for } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0,\\ \lim\limits_{t\to t_i+}[(t-t_i)^{1-q}x(t)]=\dfrac{\Psi_i(x(t_i-0))}{\Gamma(q)}, & \text{for } i\in\mathbb{N},\\ \lim\limits_{t\to 0+}[t^{1-q}x(t)]=\dfrac{x_0}{\Gamma(q)}, & \end{cases}\tag{1.1}$$

where $x_0\in\mathbb{R}^n$, and the Riemann–Liouville fractional derivative of the function $x\in\mathrm{C}([a,T],\mathbb{R}^n)$, $T>a$, with lower limit $a\in\mathbb{R}$ and order $q\in(0,1)$ is defined by

$${}_a^{\mathrm{RL}}D_t^qx(t)=\frac{1}{\Gamma(1-q)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(t-s)^{-q}x(s)\mathrm{d}s,\quad t\in(a,T].\tag{1.2}$$

*Remark* 1.1. For $q\to 1$, the impulsive condition

$$\lim_{t\to t_i+}[(t-t_i)^{1-q}x(t)]=\frac{\Psi_i(x(t_i-0))}{\Gamma(q)}$$

in (1.1) is reduced to the well-known condition

$$x(t_i+)=\Psi_i(x(t_i-0))$$

for impulsive differential equations with ordinary derivative (see [23]).

In this paper, we study some stability properties of the zero solution of (1.1). Note that the solutions of the IVP for IRLFDE (1.1) have singularities at each point $t_i$, $i \in \mathbb{N}_0$. It requires a new definition of classical Lipschitz stability, introduced and studied in [15]. This is called generalized Lipschitz stability in time. It relates to singularity of the solution, and it is considered only on intervals excluding from the left both the initial time and the impulsive times. We use Lyapunov functions and two types of derivatives of these Lyapunov functions for the impulsive fractional equation under consideration. A number of conditions is presented that ensures generalized Lipschitz stability in time. Three examples are provided in order to illustrate the results.

## 2. Preliminary Results

In this paper, we will use the classical fractional derivatives (see, for example, [16, 17, 26]) such as RL fractional derivative (see the Definition 1.2) and Grünwald–Letnikov derivative defined by

$$ {}_{a}^{\mathrm{GL}}D_t^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t-a}{h}\right]} (-1)^r \binom{q}{r} m(t-rh), \quad t \in (a,T]. $$

*Remark* 2.1. If $m \in \mathrm{C}([a,T],\mathbb{R}^n)$, then ${}_{a}^{\mathrm{RL}}D_t^q m(t) = {}_{a}^{\mathrm{GL}}D_t^q m(t)$, see [17, Theorem 2.25].

For $a, T \in \mathbb{R}_+$, with $a < T$, we will use the sets

$$ \mathrm{C}_{1-q}([a,T],\mathbb{R}^n) = \Big\{ u : (a,T] \to \mathbb{R}^n : \; u \in \mathrm{C}((a,T],\mathbb{R}^n), \lim_{t\to a+} (t-a)^{1-q} u(t) < \infty \Big\}, $$

$$ \begin{aligned} \mathrm{PC}_{1-q}([0,\infty),\mathbb{R}^n) = \Bigg\{ & u : (0,\infty) \to \mathbb{R}^n : \; u \in \mathrm{C}\Bigg( \bigcup_{k\in\mathbb{N}_0} (t_k, t_{k+1}], \mathbb{R}^n \Bigg), \\ & u(t_k) = u(t_k - 0) = \lim_{\varepsilon\to 0+} u(t_k-\varepsilon) < \infty, \; k \in \mathbb{N}, \\ & \lim_{t\to t_k+} (t-t_k)^{1-q} u(t) < \infty, \; k \in \mathbb{N}_0 \Bigg\}. \end{aligned} $$

*Remark* 2.2. If $u \in \mathrm{PC}_{1-q}([0,\infty),\mathbb{R}^n)$, then $u \in \mathrm{C}_{1-q}([t_k,t_{k+1}],\mathbb{R}^n)$ for any $k \in \mathbb{N}_0$.

Now, we will state some known results, which will be applied in the proofs of our main results.

**Proposition 2.1** ([30, Lemma 2.3])**.** *Let* $m \in \mathrm{C}_{1-q}([a,a+T),\mathbb{R})$*,* $t_1 \in (a,a+T)$*. If* $m(t_1) = 0$ *and* $m(t) < 0$ *for* $a \le t < t_1$*, then* ${}_{a}^{\mathrm{RL}}D_t^q m(t)|_{t=t_1} \ge 0$.

*Remark* 2.3. By Remark 2.1, Proposition 2.1 is also true with ${}_{a}^{\mathrm{GL}}D_t^q m(t)|_{t=t_1}$ in place of ${}_{a}^{\mathrm{RL}}D_t^q m(t)|_{t=t_1}$.

The next result is the basis for the practical definition of the initial condition and the impulsive conditions of (1.1).

**Proposition 2.2** ([17]). *Let $m : [a,T] \to \mathbb{R}$ be Lebesgue measurable, $T > a > 0$, and $q \in (0,1)$.*

(a) *If $\lim_{t\to a+}[(t-a)^{1-q}m(t)] =: c \in \mathbb{R}$ exists a.e., then*

$$\begin{aligned} {}_aI_t^{1-q}m(t)|_{t=a} :=& \lim_{t\to a+} \frac{1}{\Gamma(1-q)} \int_a^t \frac{m(s)}{(t-s)^q}\mathrm{d}s \\ =& c\Gamma(q) = \Gamma(q) \lim_{t\to a+}[(t-a)^{1-q}m(t)] \end{aligned}$$

*is well defined.*

(b) *If ${}_aI_t^{1-q}m(t)|_{t=a} = c \in \mathbb{R}$ exists a.e. and if $\lim_{t\to a+}[(t-a)^{1-q}m(t)]$ exists, then*

$$\lim_{t\to a+}[(t-a)^{1-q}m(t)] = \frac{c}{\Gamma(q)} = \frac{1}{\Gamma(q)}\,{}_aI_t^{1-q}m(t)|_{t=a}.$$

*Remark* 2.4. According to Proposition 2.2, both, the initial condition and the impulsive conditions in (1.1), could be replaced by the equalities

$${}_0I_t^{1-q}x(t)|_{t=0} = x_0$$

and

$${}_{t_i}I_t^{1-q}x(t)|_{t=t_i} = \Psi_i(x(t_i-0)), \quad i \in \mathbb{N},$$

respectively.

We introduce the following assumptions:

($A_1$) the increasing sequence $\{t_i\}_{i\in\mathbb{N}_0}$, $t_0 = 0$, is such that

$$\lim_{i\to\infty} t_i = \infty \quad \text{and} \quad \inf_{i\in\mathbb{N}_0}(t_{i+1}-t_i) = \lambda > 0;$$

($A_2$) $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t,0) = 0$ for $t \geq 0$;

($A_3$) $\Psi_i \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\Psi_i(0) = 0$ for $i \in \mathbb{N}$.

Let $\rho > 0$ and $\mathcal{J} \subset \mathbb{R}_+$, $0 \in \mathcal{J}$ be an interval. As in [21], we define $\mathcal{M}(\mathcal{J})$ to consist of all strictly increasing continuous functions $a$ with $a(0) = 0$ and such that there exists $q_a$ with $q_a(\alpha) \geq 1$ for $\alpha \geq 1$ and $a^{-1}(\alpha r) \leq r q_a(\alpha)$, $\mathcal{K}(\mathcal{J})$ to consist of all strictly increasing continuous functions $a$ with $a(0) = 0$ and such that there exists a constant $K_a > 0$ with $a(r) \leq K_a r$ and

$$\mathcal{S}_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}.$$

*Remark* 2.5. If $a(u) = u$, then $a \in \mathcal{K}(\mathbb{R}_+) \cap \mathcal{M}(\mathbb{R}_+)$. If $a(u) = K_1u$ for $K_1 > 0$, then $a \in \mathcal{K}(\mathbb{R}_+)$ and $K_a = K_1$. If $a(u) = K_2u^2$ for $0 < K_2 \leq 1$, then $a \in \mathcal{M}([1,\infty))$ and $q(u) = \sqrt{\frac{u}{K_2}} \geq 1$ for $u \geq 1$.

From now on, we assume that the IVP for IRLFDE (1.1) possesses a solution, denoted by $x_{x_0} \in PC_{1-q}([0,\infty), \mathbb{R}^n)$.

*Example* 2.1. Consider the IVP for the scalar linear IRLFDE

$$
\begin{cases}
{}^{\mathrm{RL}}_{t_i}D^q_t y(t) = ay(t), \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0,\\
\lim\limits_{t\to t_i+}[(t-t_i)^{1-q}y(t)] = \dfrac{y(t_i-0)}{t_i\Gamma(q)}, \quad \text{for } i \in \mathbb{N},\\
\lim\limits_{t\to 0+}[t^{1-q}y(t)] = \dfrac{y_0}{\Gamma(q)},
\end{cases}
\tag{2.1}
$$

where $a, y_0 \in \mathbb{R}$. The solution of (2.1) is given by

$$
y_{y_0}(t) = \begin{cases}
y_0 t^{q-1}E_{q,q}(at^q), & \text{for } t \in (0, t_1],\\
y_0\left(\prod_{i=0}^{k-1}\frac{(t_{i+1}-t_i)^{q-1}E_{q,q}(a(t_{i+1}-t_i)^q)}{t_{i+1}}\right) & \\
\times (t-t_k)^{q-1}E_{q,q}(a(t-t_k)^q) & \text{for } t \in (t_{k+1}, t_k],\ k \in \mathbb{N}.
\end{cases}
$$

It has singularities at the points $t_k$, $k \in \mathbb{N}_0$, which are the initial time and the impulsive times. In the particular case $a = 0.5$, $t_k = k$, $k \in \mathbb{N}_0$, $q = 0.3$, the graph of the solution $y_{y_0}$ is given in Figure 1 for $y_0 = 1$ and in Figure 2 for $y_0 = -0.5$, respectively.

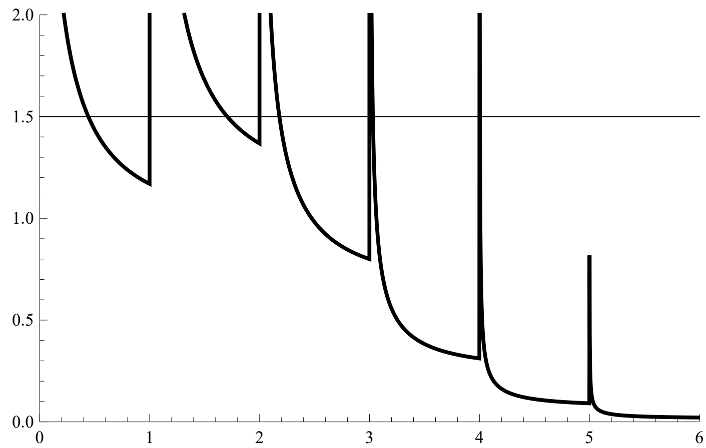


FIGURE 1. Graph of the solution of (2.1) for $a = 0.5$, $y_0 = 1$, and $q = 0.3$.

Example 2.1 illustrates that the stability of the solution for impulsive differential equations in the case of the RL fractional derivative has to be considered on intervals that exclude $t_k$, $k \in \mathbb{N}_0$, on the right ends.

There are some particular cases with zero initial value and zero impulsive functions with nonunique solutions without singularities at the initial time and the impulsive time points.

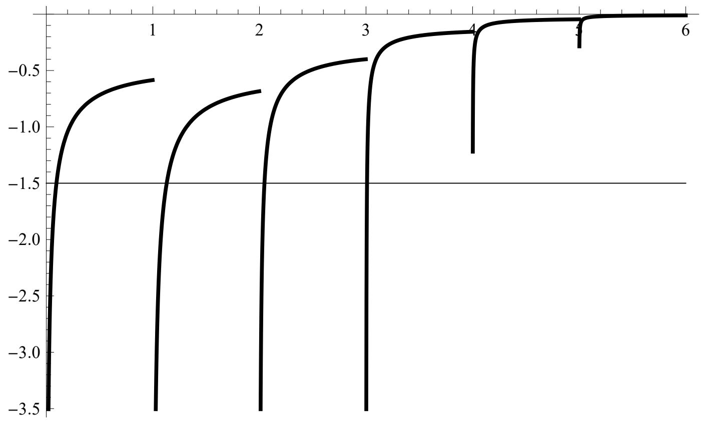


FIGURE 2. Graph of the solution of (2.1) for $a = 0.5$, $y_0 = -0.5$, and $q = 0.3$.

*Example* 2.2. Consider the IVP for the scalar linear IRLFDE

$$\begin{cases} {}^{\mathrm{RL}}_{t_i}D^q_t y(t) = a(t-t_i)^{\beta}\sqrt{y(t)}, \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \lim\limits_{t\to t_i+} \left[(t-t_i)^{1-q} y(t)\right] = 0, \quad \text{for } i \in \mathbb{N}, \\ \lim\limits_{t\to 0+} \left[t^{1-q} y(t)\right] = 0, \end{cases} \tag{2.2}$$

where $a \in \mathbb{R}$, $\beta = -0.5q$. Equation (2.2) has the zero solution, but it also has a nonzero solution. Using

$$2(q+\beta) = 1 + q > 0 > -1$$

and [22, Example 3.2], we obtain the solution of (2.2)

$$y_0(t) = \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 (t-t_i)^{2(q+\beta)}, \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0.$$

It is easy to check that

$$\begin{aligned} \lim_{t\to t_i+} \left[(t-t_i)^{1-q} y_0(t)\right] &= \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 \lim_{t\to t_i+} \left[(t-t_i)^{1-q}(t-t_i)^{2(q+\beta)}\right] \\ &= \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 \lim_{t\to t_i+} (t-t_i) = 0. \end{aligned}$$

The solution $y_0$ has no singularities at the points $t_k$, $k \in \mathbb{N}_0$, which are the initial time and the impulsive times. It is different from Example 2.1. In the particular case $a = 1$, $t_k = k$, $k \in \mathbb{N}_0$, $q = 0.4$, $\beta = -0.2$, the graph of the solution $y$ is given in Figure 3.

In our further investigations in this paper, we will assume that the IVP for IRLFDE (1.1) has a unique solution $x_{x_0}$ for any initial values $x_0 \in \mathbb{R}^n$ defined for $t > 0$. Lipschitz stability [15] will now be generalized to systems of impulsive RL differential equations.

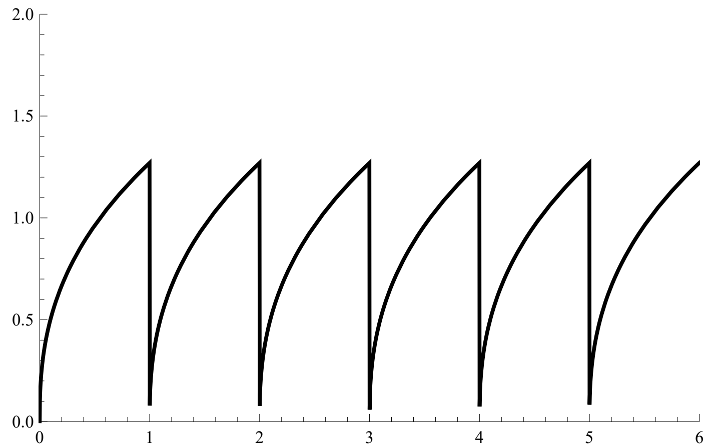


FIGURE 3. Graph of the solution of (2.2) for $a = 1$, $t_k = k$, $\beta = -0.2$, and $q = 0.4$.

Considering the phenomena described in Example 2.1 and 2.2, we now define a new stability type as follows.

**Definition 2.1.** We say that the trivial solution of the IVP for IRLFDE (1.1) is

- *generalized Lipschitz stable in time* if there are $N \in \mathbb{N}_0$, $M \geq 1$, $\delta > 0$, and $T_i \in (0, \lambda)$, $i \in \mathbb{N}_0$, such that for any initial value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$, we have
$$\|x_{x_0}(t)\| \leq M \|x_0\|, \quad \text{for all } t \in \bigcup_{i=N}^{\infty} [t_i + T_i, t_{i+1}];$$
- *globally generalized Lipschitz stable in time* if there exist $N \in \mathbb{N}_0$, $M \geq 1$, and $T_i \in (0, \lambda)$, $i \in \mathbb{N}_0$, such that for any for any initial value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \infty$, we have
$$\|x_{x_0}(t)\| \leq M \|x_0\|, \quad \text{for all } t \in \bigcup_{i=N}^{\infty} [t_i + T_i, t_{i+1}].$$

Now we define the class $\Lambda$ of Lyapunov-like functions as follows.

**Definition 2.2** ([1])**.** Let $0 \in J \subset \mathbb{R}_+$, $\mathcal{J} = J \cap \{\bigcup_{i \in \mathbb{N}_0} (t_i, t_{i+1}]\}$, and $\Delta \subset \mathbb{R}^n$. We will say that the function $V$ belongs to the class $\Lambda(J, \Delta)$ if $V \in C(\mathcal{J} \times \Delta, \mathbb{R}_+)$,
$$V(t_i, x) = V(t_i - 0, x) = \lim_{\varepsilon \to 0+} V(t_i - \varepsilon, x)$$
and
$$V(t_i + 0, x) = \lim_{\varepsilon \to 0+} V(t_i + \varepsilon, x),$$
for $i \in \mathbb{N}$, $x \in \Delta$, and it is locally Lipschitz with respect to its second argument.

We will use the following two types of fractional derivatives of Lyapunov functions among the system of nonlinear impulsive RL fractional differential equations (1.1).

- *RL derivative* of $V \in \Lambda(\mathbb{R}_+, \Delta)$ for IRLFDE (1.1) defined by
$$ {}_{t_k}^{\mathrm{RL}}D_t^q V(t, x_{x_0}(t)) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_k}^t (t-s)^{-q} V(s, x_{x_0}(s))\mathrm{d}s, $$
for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, where $x_{x_0}(\cdot) \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \Delta)$ solves (1.1).
- *Dini derivative* of $V \in \Lambda(\mathbb{R}_+, \Delta)$ for IRLFDE (1.1) defined by
$$ D_{(1.1)}^{t_k} V(t,x) = \limsup_{h \to 0+} \frac{V(t,x) - \sum\limits_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t - rh, x - h^q f(t,x))}{h^q}, $$
for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, $x \in \Delta$.

*Remark* 2.6. Let $x$ be a solution of (1.1). Then, for any $k \in \mathbb{N}_0$, the equality
$$ D_{(1.1)}^{t_k} V(t, x(t)) = \limsup_{h \to 0+} \frac{V(t,x(t)) - \sum\limits_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t - rh, x(t) - h^q f(t,x(t)))}{h^q} $$
holds for $t \in (t_k, t_{k+1}]$.

We consider the IVP
$$ \begin{cases} {}_{t_i}^{\mathrm{RL}}D_t^q u(t) = g(t, u(t)) \quad \text{for } t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0, \\ \lim\limits_{t \to t_i+} [(t-t_i)^{1-q} u(t)] = \dfrac{H_i(u(t_i - 0))}{\Gamma(q)}, \quad \text{for } i \in \mathbb{N}, \\ \lim\limits_{t \to 0+} [t^{1-q} u(t)] = \dfrac{u_0}{\Gamma(q)}, \end{cases} \tag{2.3} $$
with $u_0 \in \mathbb{R}$. We will denote the solution of (2.3) by $u_{u_0}$. We will assume that the IVP for the scalar IRLFDE (2.3) has a unique solution $u_{u_0}$ for any initial value $u_0 \in \mathbb{R}$ defined for $t > 0$. We also introduce the following conditions:

($\mathrm{A}_4$) $g \in \mathrm{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ decreases in the second variable, and $g(t, 0) = 0$ for $t \in \mathbb{R}_+$;

($\mathrm{A}_5$) $H_k \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ are increasing w.r.t. the second argument and $H_k(0) = 0$, $k \in \mathbb{N}$.

In our main results, we will use some comparison results with both Dini and Riemann-Liouville derivatives.

**Lemma 2.1.** *Suppose:*

1. *conditions* $(A_2)$–$(A_5)$ *hold;*
2. $x^*_{x_0} \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \mathbb{R}^n)$ *solves* (1.1)*;*
3. $u_{u_0} \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \mathbb{R})$ *solves* (2.3)*;*
4. $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *satisfies:*
   (i) *the inequality*
$$ {}_{t_i}^{\mathrm{RL}}D_t^q V(t, x^*_{x_0}(t)) \le g(t, V(t, x^*_{x_0}(t))), \quad t \in (t_i, t_{i+1}], i \in \mathbb{N}_0 $$
   *holds;*

(ii) *for all* $i \in \mathbb{N}_0$*, the inequalities*

$$V(t_i, \Psi_i(x^*_{x_0}(t_i - 0))) \leq H_i(V(t_i, x^*_{x_0}(t_i - 0)))$$

*hold;*

(iii) *for all* $i \in \mathbb{N}$*, the inequalities*

$$\lim_{t \to t_i+} (t - t_i)^{1-q} V(t, x^*_{x_0}(t)) \leq \frac{V(t_i + 0, \Psi_i(x^*_{x_0}(t_i - 0)))}{\Gamma(q)}$$

*hold.*

*If* $\lim_{t\to 0+} t^{1-q} V(t, x^*_{x_0}(t)) \leq \frac{u_0}{\Gamma(q)}$*, then*

$$V(t, x^*_{x_0}(t)) \leq u_{u_0}(t), \quad \textit{for all } t > 0. \tag{2.4}$$

*Proof.* For $t > 0$, put $m(t) := V(t, x^*_{x_0}(t))$. We will prove (2.4) by induction w.r.t. the intervals $(t_i, t_{i+1}]$, $i \in \mathbb{N}_0$.

First, let $t \in (0, t_1]$. Let $\varepsilon > 0$ be arbitrary. We will prove

$$m(t) < u_{u_0}(t) + t^{q-1}\varepsilon, \quad t \in (0, t_1]. \tag{2.5}$$

We have

$$\begin{aligned} \lim_{t \to 0+} t^{1-q} V(t, x^*_{x_0}(t)) &\leq \frac{u_0}{\Gamma(q)} < \frac{u_0}{\Gamma(q)} + \varepsilon \\ &= \lim_{t\to 0+} t^{1-q} u_{u_0}(t) + \lim_{t \to 0+} t^{1-q} t^{q-1}\varepsilon \\ &= \lim_{t \to 0+} t^{1-q} \left( u_{u_0}(t) + t^{q-1}\varepsilon \right). \end{aligned} \tag{2.6}$$

From (2.6), there exists $\delta > 0$ such that

$$t^{1-q} V(t, x^*_{x_0}(t)) < t^{1-q} \left( u_{u_0}(t) + t^{q-1}\varepsilon \right), \quad \text{for } t \in (0, \delta),$$

that is, (2.5) is satisfied on $(0, \delta)$. If $\delta \geq t_1$, then (2.5) is proved. If $\delta < t_1$, then we assume that (2.5) is not true. Then there exists $t^* \in [\delta, t_1]$ such that

$$m(t^*) = u_{u_0}(t^*) + (t^*)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + t^{q-1}\varepsilon, \quad t \in (0, t^*).$$

From ($A_4$), ${}^{\mathrm{RL}}_{0}D^q_t t^{q-1} = 0$, and Proposition 2.1 with $t_1 = t^*$ together with

$$v(t) = m(t) - u_{u_0}(t) - t^{q-1}\varepsilon,$$

we obtain the inequality

$$\begin{aligned} {}^{\mathrm{RL}}_{0}D^q_t m(t)|_{t=t^*} &\geq {}^{\mathrm{RL}}_{0}D^q_t \left( u_{u_0}(t) + t^{q-1}\varepsilon \right)|_{t=t^*} \\ &= {}^{\mathrm{RL}}_{0}D^q_t u_{u_0}(t)|_{t=t^*} = g(t^*, u_{u_0}(t^*)) \\ &= g\left(t^*, m(t^*) - (t^*)^{q-1}\varepsilon\right) > g(t^*, m(t^*)). \end{aligned} \tag{2.7}$$

Inequality (2.7) contradicts assumption 4 (i). Therefore, (2.5) holds for any $\varepsilon > 0$, and hence, (2.4) holds for $t \in (0, t_1]$. From assumption 4 (ii), ($A_5$), and the inequality

$V(t_1, x^*_{x_0}(t_1 - 0)) \le u(t_1 - 0)$, we get

$$\begin{aligned} V(t_1+0, x^*_{x_0}(t_1+0)) &= V(t_1+0, \Psi_1(x^*_{x_0}(t_1-0))) \\ &\le H_1(V(t_1, x^*_{x_0}(t_1-0))) \\ &\le H_1(u_{u_0}(t_1-0)) = u_{u_0}(t_1+0). \end{aligned} \tag{2.8}$$

Let $t \in (t_1, t_2]$. Let $\varepsilon > 0$ be arbitrary. We will prove

$$m(t) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad t \in (t_1, t_2]. \tag{2.9}$$

From assumption 4 (iii) and (2.8), we obtain

$$\begin{aligned} \lim_{t\to t_1+}(t-t_1)^{1-q}V(t, x^*_{x_0}(t)) &\le \frac{V(t_1, x^*_{x_0}(t_1+0))}{\Gamma(q)} \le \frac{u_{u_0}(t_1+0))}{\Gamma(q)} \\ &< \frac{H_1(u_{u_0}(t_1-0))}{\Gamma(q)} + \varepsilon \\ &= \lim_{t\to t_1+}(t-t_1)^{1-q}\left(u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon\right). \end{aligned} \tag{2.10}$$

From (2.10), there exists $\delta_1 > 0$ such that

$$V(t, x^*_{x_0}(t)) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad \text{on } (t_1, t_1+\delta_1).$$

If $\delta_1 \ge t_2 - t_1$, then (2.9) is proved. If $\delta_1 < t_2 - t_1$, then we assume (2.9) is not true. Then there exists $t_1^* \in [t_1 + \delta, t_2]$ such that

$$m(t_1^*) = u_{u_0}(t_1^*) + (t_1^* - t_1)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad t \in [t_1, t_1^*).$$

Now $(A_4)$, ${}^{\mathrm{RL}}_{t_1}D^q_t(t-t_1)^{q-1} = 0$ and Proposition 2.1, with $t_1 = t_1^*$ together with

$$v(t) = m(t) - u_{u_0}(t) - (t-t_1)^{q-1}\varepsilon,$$

yield

$$\begin{aligned} {}^{\mathrm{RL}}_{t_1}D^q_t m(t)|_{t=t_1^*} &\ge {}^{\mathrm{RL}}_{t_1}D^q_t\left(u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon\right)|_{t=t_1^*} \\ &= {}^{\mathrm{RL}}_{t_1}D^q_t u_{u_0}(t)|_{t=t_1^*} = g(t_1^*, u_{u_0}(t_1^*)) \\ &= g\left(t_1^*, m(t_1^*) - (t_1^*-t_1)^{q-1}\varepsilon\right) > g(t_1^*, m(t_1^*)). \end{aligned} \tag{2.11}$$

Inequality (2.11) contradicts assumption 4 (i). Therefore, (2.9) holds for any $\varepsilon > 0$, and hence, (2.4) holds for $t \in (t_1, t_2]$.

Following the above procedure, we complete the proof. □

**Lemma 2.2.** *Suppose:*

1. *assumptions* 1, 2, 3, 4 (ii), 4 (iii) *from Lemma* 2.1 *hold;*
2. $V \in \Lambda(\mathbb{R}_+, \mathbb{R})$ *satisfies*

$$D^{t_k}_{(1.1)}V(t, x^*_{x_0}(t)) \le g(t, V(t, x^*_{x_0}(t)), \quad \textit{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0.$$

*If* $\lim_{t\to 0+} t^{1-q}V(t, x^*_{x_0}(t)) \le \frac{u_0}{\Gamma(q)}$, *then* (2.4) *holds.*

*Proof.* The proof is similar to the proof of Lemma 2.1, where instead of the RL fractional derivative of the Lyapunov function, we will use the Dini fractional derivative. The main difference between this proof and the proof of Lemma 2.1 is connected with inequalities (2.7) and (2.11) for $t^* \in (0, t_1]$ and $t_1^* \in (t_1, t_2]$.

Consider any of the intervals $(t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, and assume that for a fixed $k \in \mathbb{N}$, there exist $\delta_k \in (0, t_{k+1} - t_k)$ and $t_k^* \in (t_k + \delta_k, t_{k+1}]$ such that

$$m(t_k^*) = u_{u_0}(t_k^*) + (t_k^* - t_k)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + (t - t_k)^{q-1}\varepsilon, \quad t \in (t_k, t_k^*).$$

Thanks to Remark 2.3 with $\tau = t_k^*$, we obtain

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_+^q m(t)|_{t=t_k^*} &\geq {}_{t_k}^{\mathrm{GL}}D_+^q u_{u_0}(t)|_{t=t_k^*} + {}_0^{\mathrm{GL}}D_+^q((t-t_k)^{q-1}\varepsilon)|_{t=t_k^*} \\ &= {}_{t_k}^{\mathrm{GL}}D_+^q u_{u_0}(t)|_{t=t_k^*} = g(t_k^*, u_{u_0}(t_k^*)) \\ &= g(t_k^*, m(t_k^*) - (t_k^* - t_k)^{q-1}\varepsilon) > g(t_k^*, m(t_k^*)). \end{aligned} \tag{2.12}$$

For any fixed $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_+^q m(t) &= \limsup_{h\to 0+} \frac{\sum_{r=0}^{\left[\frac{t-t_k}{h}\right]} (-1)^r \binom{q}{r} m(t-rh)}{h^q} \\ &= \limsup_{h\to 0+} \frac{m(t) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} m(t-rh)}{h^q} \\ &= \limsup_{h\to 0+} \left\{ \frac{m(t) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t-rh, x_{x_0}^*(t) - h^q f(t, x_{x_0}^*(t)))}{h^q} \right. \\ &\quad \left. + \frac{\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} \left[ V(t-rh, x_{x_0}^*(t) - h^q f(t, x_{x_0}^*(t))) - m(t-rh) \right]}{h^q} \right\}. \end{aligned} \tag{2.13}$$

Denote

$$F(t, x_{x_0}^*, t_k, h) = \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} x_{x_0}^*(t-rh).$$

From (1.1), we get

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_t^q x_{x_0}^*(t) &= \limsup_{h\to 0+} \frac{x_{x_0}^*(t) - F(t, x_{x_0}^*, t_k, h)}{h^q} \\ &= {}_{t_k}^{\mathrm{RL}}D_t^q x_{x_0}^*(t) = f(t, x_{x_0}^*(t)). \end{aligned}$$

Hence,

$$x^*_{x_0}(t) - h^q f(t, x^*_{x_0}(t)) = F(t, x^*_{x_0}, t_k, h) + \Omega(h^q),$$

with

$$\lim_{h\to 0+} \frac{\|\Omega(h^q)\|}{h^q} = 0.$$

Thus, for arbitrary $h > 0$ and $r \in \mathbb{N}$, we have

$$\begin{aligned} & V(t-rh, x^*_{x_0}(t) - h^q f(t, x^*_{x_0}(t))) - V(t-rh, x^*_{x_0}(t-rh)) \\ \leq & L \left\| F(t, x^*_{x_0}, t_k, h) + \Omega(h^q) - x^*_{x_0}(t-rh) \right\| \\ \leq & L \left\| \sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1} \binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh) \right\| + L \left\| \Omega(h^q) \right\|. \end{aligned} \tag{2.14}$$

Hence, due to

$$(1+u)^\alpha = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} u^k, \quad \text{i.e.,} \quad 1 = \sum_{k=1}^{\infty} (-1)^{k+1} \binom{\alpha}{k},$$

we get

$$\begin{aligned} & \left\| \sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1} \binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh) \right\| \\ = & \left\| \sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1} \binom{q}{j} x^*_{x_0}(t-jh) - \left( \sum_{j=1}^{\infty} (-1)^{j+1} \binom{q}{j} \right) x^*_{x_0}(t-rh)) \right\| \\ \leq & \left\| \sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1} \binom{q}{j} \left[ x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh) \right] \right\| \\ & + \left\| \sum_{j=\left[\frac{t-t_k}{h}\right]}^{\infty} (-1)^{j+1} \binom{q}{j} \right\| \left\| x^*_{x_0}(t-rh) \right\|, \end{aligned}$$

from which, together with (2.13), (2.14), and condition 2 in the statement, we get

$$\begin{aligned} {}^{\mathrm{GL}}_{t_k}D^q_+ m(t) \leq & D^{t_k}_{(1.1)} V(t, x^*_{x_0}(t)) + L \limsup_{h\to 0+} \frac{\|\Omega(h^q)\|}{h^q} \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} \\ & + L \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} \\ & \times \left\| \sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1} \binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh) \right\| \end{aligned}$$

$$
\begin{aligned}
=&D^{t_k}_{(1.1)}V(t,x^*_{x_0}(t))\\
&+L\limsup_{h\to 0+}\frac{1}{h^q}\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{r+1}\binom{q}{r}\\
&\times\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{j+1}\binom{q}{j}\left[x^*_{x_0}(t-jh)-x^*_{x_0}(t-rh)\right]\right\|\\
&+L\limsup_{h\to 0+}\left\|\sum_{j=\left[\frac{t-t_k}{h}\right]}^{\infty}(-1)^{j+1}\binom{q}{j}\right\|\\
&\times\frac{1}{h^q}\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{r+1}\binom{q}{r}\left\|x^*_{x_0}(t-rh)\right\|\\
=&D^{t_k}_{(1.1)}V(t,x^*_{x_0}(t))\le g(t,V(t,x^*_{x_0}(t))),
\end{aligned}
$$

contradicting (2.12) and completing the proof. □

## 3. Main Results

We now present the main results of this paper.

**Theorem 3.1.** *Suppose:*

1. *conditions* $(A_1)$*–*$(A_5)$ *are fulfilled;*
2. *there exists* $V\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ *such that:*
   (i) *there exist* $\tau_i\in(0,\lambda)$*,* $i\in\mathbb{N}_0$*, satisfying*
   $$b(\|x\|)\le V(t,x),\quad \text{for all } x\in\mathbb{R}^n \text{ and } t\in\bigcup_{i\in\mathbb{N}_0}[t_i+\tau_i,t_{i+1}]$$
   *where* $b\in\mathcal{M}([0,\rho])$*, with* $\rho>0$*;*
   (ii) *for all* $y\in C_{1-q}([0,t_1],\mathbb{R}^n)$*, with*
   $$\lim_{t\to 0+}\left(t^{1-q}y(t)\right)=y_0\in\mathcal{S}_\rho,$$
   *we have*
   $$t^{1-q}V(t,y(t))|_{t=0+}=\lim_{t\to 0+}t^{1-q}V(t,y(t))\le a(\|y_0\|),$$
   *where* $a\in\mathcal{K}([0,\rho])$*;*
   (iii) *for all* $i\in\mathbb{N}_0$*, the inequalities*
   $$V(t_i,\Psi_i(x))\le H_i(V(t_i,x))\quad \text{for all } x\in\mathbb{R}^n,$$
   *hold;*
   (iv) *for any* $y\in C_{1-q}([t_k,t_{k+1}],\mathbb{R}^n)$*, with*
   $$\lim_{t\to t_k+}\left((t-t_k)^{1-q}y(t)\right)=\frac{y_k}{\Gamma(q)}<\infty,$$

*the inequality*

$$(t-t_k)^{1-q}V(t,y(t))|_{t=t_k+} = \lim_{t\to t_k+}(t-t_k)^{1-q}V(t,y(t)) \le V(t_k+0,\Psi_k(y_k))$$

*holds;*

(v) *for any initial value* $x_0 \in \mathcal{S}_\rho$ *and the corresponding solution* $x_{x_0}$ *of* (1.1), *the inequality*

$$^{\mathrm{RL}}_{t_k}D^q_t V(t,x_{x_0}(t)) \le g(t,V(t,x_{x_0}(t))), \quad \textit{for all } t\in(t_k,t_{k+1}], k\in\mathbb{N}_0,$$

*holds;*

3. *the zero solution of the scalar comparison equation* (2.3) *is generalized Lipschitz stable in time.*

*Then the zero solution of the IVP for IRLFDE* (1.1) *is generalized Lipschitz stable in time.*

*Proof.* Suppose that the zero solution of (2.3) is generalized Lipschitz stable in time. Thus, there exist $N\in\mathbb{N}_0$, $\delta\in(0,\lambda)$, $\varsigma_i\in(0,\delta)$, $i\in\mathbb{N}_0$, $\delta_1>0$, and $M_1\ge 1$ such that for any $u_0\in\mathbb{R}^n$ with $|u_0|<\delta_1$, the inequality

$$|u_{u_0}(t)| \le M_1|u_0|, \quad \text{for } t\in\bigcup_{i=N}^{\infty}[t_i+\varsigma_i,t_{i+1}], \tag{3.1}$$

holds, where $u_{u_0}$ solves (2.3) with the initial value $u_0$. Thanks to $a\in\mathcal{K}([0,\rho])$ and $b\in\mathcal{M}([0,\rho])$, there are $K_a>0$ and $q_b(u)\ge 1$ for $\alpha\ge 1$ with

$$\alpha r \le b(rq_a(\alpha)), \quad \text{for all } r\in[0,\rho], \tag{3.2}$$

and

$$a(r)\le K_a r, \quad \text{for all } r\in[0,\rho]. \tag{3.3}$$

We may assume $K_a\ge 1$. Pick $M_2,\delta>0$ satisfying

$$M_2 > \max\{1,q_b(M_1K_a)\}\ge 1 \quad \text{and} \quad \delta=\min\left\{\rho,\frac{\delta_1}{K_a}\right\}.$$

Pick $x_0\in\mathbb{R}^n$ such that $\|x_0\|<\delta$, and hence, $x_0\in\mathcal{S}_\rho$. Consider the solution $x_{x_0}$ of (1.1) for the chosen initial value $x_0$. Thus, using $\Gamma(q)>1$ for $q\in(0,1)$, we get

$$\left\|\lim_{t\to 0+}t^{1-q}x_{x_0}(t)\right\| = \left\|\frac{x_0}{\Gamma(q)}\right\| < \frac{\delta}{\Gamma(q)} < \delta \le \rho,$$

that is, $\lim_{t\to 0+}t^{1-q}x_{x_0}(t)\in\mathcal{S}_\rho$, and employing assumption 2 (ii) with

$$y = x_{x_0}\in \mathrm{C}_{1-q}([0,t_1],\mathbb{R}^n),$$

we get

$$t^{1-q}V(t,x_{x_0}(t))|_{t=0+} < a\left(\frac{\|x_0\|}{\Gamma(q)}\right) < a(\|x_0\|). \tag{3.4}$$

Consider the solution $u_{u_0^*}$ of (2.3) with $u_0^* = \lim_{t\to 0+} t^{1-q}V(t, x_{x_0}(t))$. The choice of $x_0$, (3.3), (3.4), and assumption 2 (ii) yield

$$u_0^* = \lim_{t\to 0+} t^{1-q}V(t, x_{x_0}(t)) \le a\left(\frac{\|x_0\|}{\Gamma(q)}\right) < a(||x_0||) \le K_a \|x_0\| < K_a\delta \le \delta_1.$$

Hence, $u_{u_0^*}$ satisfies (3.1) for $\bigcup_{i=N}^{\infty}[t_i + \varsigma_i, t_{i+1}]$ ,with $u_0 = u_0^*$.

From conditions 2 (v), 2 (iii) with $x = x_{x_0}(t_{i+1} - 0)$ and 2 (iv), with

$$y \equiv x_{x_0} \in \mathrm{C}_{1-q}([t_k, t_{k+1}], \mathbb{R}^n) \quad \text{and} \quad y_k = x_{x_0}(t_i - 0),$$

we have conditions 4 (i), 4 (ii), 4 (iii) of Lemma 2.1, respectively. According to Lemma 2.1, we get

$$V(t, x_{x_0}(t)) \le u_{u_0^*}(t), \quad \text{for } t > 0. \tag{3.5}$$

Let $T_i = \max\{\tau_i, \varsigma_i\}$ for $i \in \mathbb{N}_0$. Then, for any $k \in \mathbb{N}_0$, the inclusions

$$[t_k + T_k, t_{k+1}] \subset [t_k + \tau_k, t_{k+1}] \quad \text{and} \quad [t_k + T_k, t_{k+1}] \subset [t_k + \varsigma_k, t_{k+1}]$$

hold. Let $k \ge N$. From 2 (i), 2 (ii), (3.1), (3.2), (3.3) with $r = \|x_0\|$, $\alpha = M_1 K_a > 1$, and (3.4), (3.5), we obtain for $t \in [t_k + T_k, t_{k+1}]$

$$\begin{aligned} b(\|x_{x_0}(t)\|) \le& V(t, x_{x_0}(t)) \le u_{u_0^*}(t) < M_1|u_0^*| \\ =& M_1 t^{1-q}V(t, x_{x_0}(t))|_{t=0+} < M_1 a(\|x_0\|) \\ \le& M_1 K_a \|x_0\| \le b\left(q_b(M_1K_a)\|x_0\|\right) \le b(M_2\|x_0\|), \end{aligned}$$

completing the proof. □

**Theorem 3.2.** *Let conditions* 1. *and* 2. *of Theorem* 3.1 *be satisfied, where conditions* 2 (ii) *and* 2 (v) *are fulfilled for all* $y_0 \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$*, respectively. If the zero solution of* (2.3) *is globally generalized Lipschitz stable in time, then the zero solution of IVP for IRLFDE* (1.1) *is globally generalized Lipschitz stable in time.*

*Proof.* The proof follows the proof of Theorem 3.1, with an arbitrary initial value $x_0 \in \mathbb{R}^n$, and so we omit it. □

**Theorem 3.3.** *Let the conditions of Theorem* 3.1 *be satisfied, where* $a(s) = A_2 s^p$*,* $A_2 > 0$*,* $p \ge 1$ *in condition* 2 (ii)*, and condition*2 (i) *is replaced by*

2*(i) *there exist* $\tau_i \in (0, \lambda)$*,* $i \in \mathbb{N}_0$*, satisfying*

$$\mu(t)\|x\|^p \le V(t, x), \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } t \in \bigcup_{i\in\mathbb{N}_0}[t_i + \tau_i, t_{i+1}],$$

*holds, where* $\mu(t) \ge A_1$*,* $t \in \bigcup_{i\in\mathbb{N}_0}[t_i + \tau_i, t_{i+1}]$ *and* $A_1 > 0$*.*

*Then, the trivial solution of the IVP for IRLFDE* (1.1) *is generalized Lipschitz stable in time.*

*Proof.* The proof follows the proof of Theorem 3.1, with

$$M_2 = \sqrt[p]{\frac{M_1 A_2}{A_1}} \quad \text{and} \quad \delta = \min\left\{\lambda, \sqrt[p]{\frac{\delta_1}{A_2}}\right\},$$

and so we omit it. ☐

In the case when the Dini fractional derivative of Lyapunov functions is used instead of RL fractional derivative of Lyapunov functions, we obtain some sufficient conditions for the introduced generalized Lipschitz stability in time. Since the proofs are similar to the already presented proofs, we omit them, and we will only state the results.

**Theorem 3.4.** *Let the conditions of Theorem* 3.1 *be satisfied, where condition* 2 (v) *is replaced by*

2(v*) *the inequality*

$$D^{t_k}_{(1.1)}V(t,x) \le g(t, V(t,x)), \quad \textit{for all } x \in \mathbb{R}^n,\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}_0,$$

*holds.*

*Then the trivial solution of the IVP for IRLFDE* (1.1) *is generalized Lipschitz stable in time.*

*Proof.* The proof follows the proof of Theorem 3.1, with Lemma 2.2 applied in place of Lemma 2.1. ☐

**Theorem 3.5.** *Let conditions* 1 *and* 2 *of Theorem* 3.1 *be satisfied, where condition* 2 (v) *is replaced by* 2 (v*) *and condition* 2 (ii) *is fulfilled for all* $y_0 \in \mathbb{R}^n$. *If the trivial solution of* (2.3) *is globally generalized Lipschitz stable in time, then the trivial solution of the IVP for IRLFDE* (1.1) *is globally generalized Lipschitz stable in time.*

**Theorem 3.6.** *Let the conditions of Theorem* 3.1 *be satisfied, where* $a(s) = A_2 s^p$, $A_2 > 0$, $p \ge 1$ *in condition* 2 (ii), *condition* 2 (i) *is replaced by condition* 2*(i) *of Theorem* 3.3, *and condition* 2 (v) *is replaced by condition* 2 (v*) *of Theorem* 3.4. *Then the trivial solution of the IVP for IRLFDE* (1.1) *is generalized Lipschitz stable in time.*

## 4. Applications

We now illustrate the application of our obtained sufficient conditions and the practical use of the fractional derivatives of Lyapunov functions.

*Example* 4.1. Let the sequence $\{t_i\}_{i\in\mathbb{N}_0}$, $t_0 = 0$, be given such that

$$L = \sup_{k\in\mathbb{N}_0} (t_{k+1} - t_k) \ge 1 \quad \text{and} \quad \lambda = \inf_{k\in\mathbb{N}_0} (t_{k+1} - t_k) > 0.$$

Consider now the IVP for the system of impulsive Riemann-Liouville equations

(4.1)
$$\begin{cases} {}^{\mathrm{RL}}_{t_k}D^q_t x_1(t) = -\left(0.5t^{q-1} + t^{-q}\dfrac{\Gamma(2-q)}{\Gamma(2-2q)} + x_2^2(t)\right) x_1(t), \\ {}^{\mathrm{RL}}_{t_k}D^q_t x_2(t) = -\left(0.5t^{q-1} + t^{-q}\dfrac{\Gamma(2-q)}{\Gamma(2-2q)} - \dfrac{x_1^2(t)}{1+x_2^2(t)}\right) x_2(t), \quad \text{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0, \\ \lim\limits_{t\to t_k+} \left[(t-t_k)^{1-q} x_1(t)\right] = \dfrac{\Psi_k^1(x_1(t_k-0), x_2(t_k-0))}{\Gamma(q)}, \\ \lim\limits_{t\to t_k+} \left[(t-t_k)^{1-q} x_2(t)\right] = \dfrac{\Psi_k^2(x_1(t_k-0), x_2(t_k-0))}{\Gamma(q)}, \quad \text{for } k \in \mathbb{N}, \\ \lim\limits_{t\to t_0+} \left[t^{1-q} x_1(t)\right] = \dfrac{x_{0,1}}{\Gamma(q)}, \quad \lim\limits_{t\to t_0+} \left[t^{1-q} x_2(t)\right] = \dfrac{x_{0,2}}{\Gamma(q)}, \end{cases}$$

where $x_0 = (x_{0,1}, x_{0,2}) \in \mathbb{R}^2$,

$$\Psi_k^1(t, x_1, x_2) = \frac{x_1}{t} \quad \text{and} \quad \Psi_k^2(t, x_1, x_2) = \frac{x_2}{t},$$

for $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, $x_1, x_2 \in \mathbb{R}$. Consider the Lyapunov function

$$V(t,x) = (t-t_k)^{1-q}(x_1^2 + x_2^2), \quad \text{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0, x_1, x_2 \in \mathbb{R},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. The function $V \in \Lambda([0,\infty), \mathbb{R}^2)$ is locally Lipschitz with constant $L$. Thus, assumption 2*(i) of Theorem 3.3 holds with

$$p = 2, \quad \mu(t) = (t-t_k)^{1-q}, \quad t \in (t_k, t_{k+1}), \quad \tau_k = t_k + \sqrt[1-q]{0.1}, \quad A_1 = \sqrt[1-q]{0.1}.$$

Let $y \in \mathrm{C}_{1-q}([0, t_1], \mathbb{R}^2)$, $y = (y_1, y_2)$, be such that

$$\lim_{t\to 0+} \left(t^{1-q} y_k(t)\right) = y_{0,k}, \quad k = 1, 2, \quad y_0 = (y_{0,1}, y_{0,2}).$$

Then, because of

$$\lim_{t\to 0+} t^{1-q} V(t, y(t)) = \left(\lim_{t\to 0+} t^{1-q} y_1(t)\right)^2 + \left(\lim_{t\to 0+} t^{1-q} y_2(t)\right)^2 = \|y_0\|^2,$$

condition 2 (ii) of Theorem 3.1 holds with

$$a(s) = A_2 s^p, \quad A_2 = 1, \quad p = 2.$$

Let $y \in \mathrm{C}_{1-q}([t_k, t_{k+1}], \mathbb{R}^2)$ satisfies

$$\lim_{t\to t_k+} \left((t-t_k)^{1-q} y(t)\right) = \frac{y_k}{\Gamma(q)} < \infty, \quad y_k = (y_{1,k}, y_{2,k}).$$

Then, for $t \in (t_k, t_{k+1}]$, we get the inequality

$$\begin{aligned} (t-t_k)^{1-q} V(t, y(t))|_{t=t_k+} &= \lim_{t\to t_k+} (t-t_k)^{1-q} V(t, y(t)) \\ &= \lim_{t\to t_k+} (t-t_k)^{1-q} (t-t_k)^{1-q} (y_1(t)^2 + y_2(t)^2) \\ &= \left(\lim_{t\to t_k+} (t-t_k)^{1-q} y_1(t)\right)^2 + \left(\lim_{t\to t_k+} (t-t_k)^{1-q} y_2(t)\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= y_{1,k}^2 + y_{2,k}^2 \\
&\leq (t_k - t_{k-1})^{1-q}(y_{1,k}^2 + y_{2,k}^2) = V(t_k, y_k),
\end{aligned}
$$

and therefore, condition 2 (iv) of Theorem 3.1 is satisfied. Let $k \in \mathbb{N}_0$ and $t \in [t_k, t_{k+1}]$, $x = (x_1, x_2) \in \mathbb{R}^2$. Then, we get the inequalities

$$
V(t, \Psi_k(t, x^*(t_k - 0))) = (t - t_k)^{1-q} \left( \frac{x_1^2}{t^2} + \frac{x_2^2}{t^2} \right) = H_k(t, V(t_k, x^*(t_k - 0))),
$$

and therefore, condition 2 (iii) of Theorem 3.1 is satisfied with

$$
H_i(t, u) = \frac{u}{t^2}.
$$

The RL fractional derivative of the Lyapunov function, i.e.,

$$
{}_{t_k}^{\mathrm{RL}}D_t^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_k}^t (t-s)^{-q}(s-t_k)^{1-q} \left( x_1^2(s) + x_2^2(s) \right) \mathrm{d}s,
$$

where $x = (x_1, x_2)$, being the solution of (4.1), is rather difficult to obtain, so we cannot apply the results with RL derivative. Instead, we apply the results with Dini derivative of the function $V$ among (4.1). Let $k \in \mathbb{N}_0$, $t \in (t_k, t_{k+1}]$, $x_1, x_2 \in \mathbb{R}$. Then, for $1 - 2q \geq 0$, i.e., $q \in (0, 0.5]$, we get

$$
\begin{aligned}
& D_{(4.1)}^{t_k} (t - t_k)^{1-q} \left( x_1^2 + x_2^2 \right) \\
&= \limsup_{h \to 0+} \frac{1}{h^q} \Bigg\{ (t - t_k)^{1-q} \left( x_1^2 + x_2^2 \right) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} (t - rh - t_k)^{1-q} \\
&\quad \times \left[ (x_1 - h^q f_1(t, x))^2 + (x_2 - h^q f_2(t, x))^2 \right] \Bigg\} \\
&= \limsup_{h \to 0+} \frac{1}{h^q} (t - t_k)^{1-q} \left[ x_1^2 - (x_1 - h^q f_1(t, x))^2 + x_2^2 - (x_2 - h^q f_2(t, x))^2 \right] \\
&\quad + \limsup_{h \to 0+} \frac{1}{h^q} \left[ (x_1 - h^q f_1(t, x))^2 + (x_2 - h^q f_2(t, x))^2 \right] \\
&\quad \times \sum_{r=0}^{\left[\frac{t-t_k}{h}\right]} (-1)^r \binom{q}{r} (t - t_k - rh)^{1-q} \\
&= \limsup_{h \to 0+} \frac{1}{h^q} (t - t_k)^{1-q} \left[ (2x_1 - h^q f_1(t, x))\, h^q f_1(t, x) + (2x_2 - h^q f_2(t, x))\, h^q f_2(t, x) \right] \\
&\quad + \left[ x_1^2 + x_2^2 \right] {}_{t_k}^{\mathrm{RL}}D_t^q (t - t_k)^{1-q} \\
&= 2(t - t_k)^{1-q} x_1 f_1(t, x) + 2(t - t_k)^{1-q} x_2 f_2(t, x) \\
&\quad + \left[ x_1^2 + x_2^2 \right] \frac{\Gamma(2-q)}{\Gamma(2-2q)} (t - t_k)^{1-2q} \\
&= 2(t - t_k)^{1-q} x_1 \left( -0.5 t^{q-1} x_1 - t^{-q} \frac{\Gamma(2-q)}{\Gamma(2-2q)} x_1 - x_2^2 x_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(t-t_k)^{1-q}x_2\left(-0.5t^{q-1}x_2 - t^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_2 + \frac{x_2x_1^2}{1+x_2^2}\right)\\
&+ \left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
\le& 2(t-t_k)^{1-q}x_1\left(-0.5t^{q-1}x_1 - (t-t_k)^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_1 - x_2^2x_1\right)\\
&+ 2(t-t_k)^{1-q}x_2\left(-0.5t^{q-1}x_2 - t^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_2 + \frac{x_2x_1^2}{1+x_2^2}\right)\\
&+ \left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
=& -0.5V(t,x) - \left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
\le& -0.5V(t,x), \quad \text{for } x\in\mathbb{R}^2 \text{ and } t>0,
\end{aligned}
$$

and therefore, assumption 2 (v*) of Theorem 3.4 holds with

$$g(t,u)\equiv -0.5u, \quad u\in\mathbb{R} \text{ and } q\in(0,0.5].$$

Consider the scalar comparison linear RL fractional equation with noninstantaneous impulses

$$
\begin{cases}
{}^{\mathrm{RL}}_{t_i}D^q_t u(t) = -0.5u(t), \quad \text{for } t\in(t_i,t_{i+1}], i\in\mathbb{N}_0,\\
\lim\limits_{t\to t_i+}\left[(t-t_i)^{1-q}u(t)\right] = \dfrac{u(t_i-0)}{t_i\Gamma(q)}, \quad \text{for } i\in\mathbb{N},\\
\lim\limits_{t\to 0+}\left[t^{1-q}u(t)\right] = \dfrac{u_0}{\Gamma(q)},
\end{cases}
\tag{4.2}
$$

where $u_0\in\mathbb{R}$. Similar to Example 2.1, the solution of (4.2) is given by

$$
y(t) = \begin{cases}
u_0t^{q-1}E_{q,q}(-0.5t^q), & \text{for } t\in(0,t_1],\\
u_0\left(\prod_{i=0}^{k-1}E_{q,q}(-0.5(t_{i+1}-t_i)^q)t_i\right) & \\
\times(t-t_k)^qE_{q,q}(-0.5(t-t_k)^q), & \text{for } t\in(t_{k+1},t_k],\ k\in\mathbb{N}.
\end{cases}
$$

In the case $q\in(0,0.5]$, the trivial solution of (4.2) is generalized Lipschitz stable in time (for particular values $q=0.3$, $t_k=k$, $k\in\mathbb{N}$, and $u_0=1$, $u_0=2$, the graphs of the corresponding solutions $u_{u_0}$ are given in Figure 4, and in the partial case of $q=0.5$, $t_k=k$, $k\in\mathbb{N}$ and $u_0=1$, $u_0=2$, the graphs of the solutions $u_{u_0}$ are given in Figure 5). In the case $q\in(0.5,1)$, the zero solution of (4.2) is not generalized Lipschitz stable in time (for particular values $q=0.8$, $t_k=k$, $k\in\mathbb{N}$ and $u_0=1$, $u_0=2$, the graphs of the solutions $u_{u_0}$ are given in Figure 6). Due to Theorem 3.6, the zero solution of (4.1) is generalized Lipschitz stable in time for $q\in(0,0.5)$.

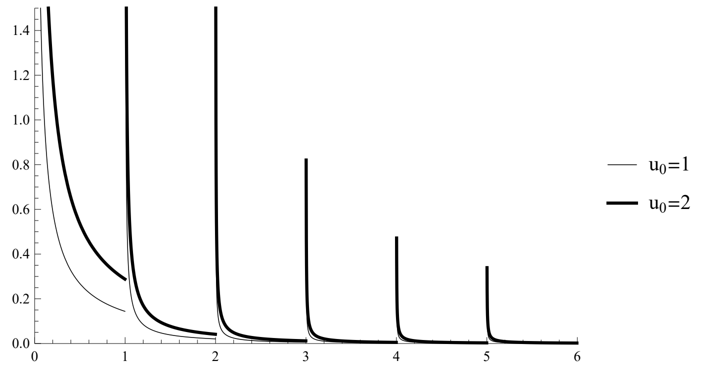


FIGURE 4. Graph of the solutions of (4.2) for $q = 0.3$, $u_0 = 1$ and for $u_0 = 2$.

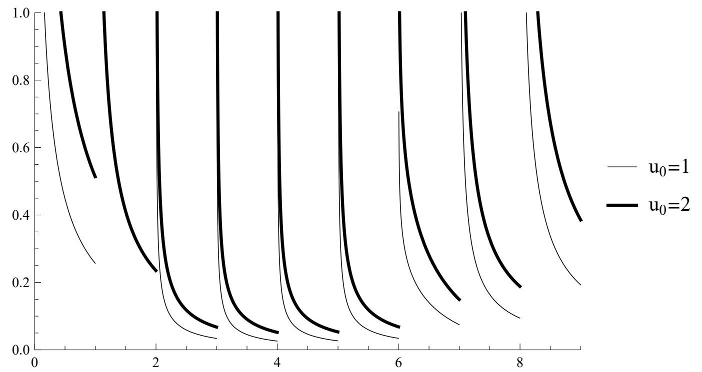


FIGURE 5. Graph of the solution of (4.2) for $q = 0.5$, $u_0 = 1$ and for $u_0 = 2$.

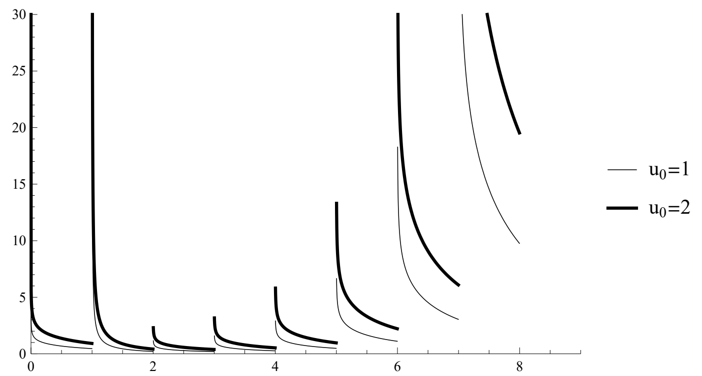


FIGURE 6. Graph of the solution of (4.2) for $q = 0.8$ with $u_0 = 1$ and $u_0 = 2$.

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# ORESME HYBRID NUMBERS AND HYBRATIONALS

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Abstract. In this paper we introduce and study Oresme hybrid numbers and hybrationals based on the known Oresme sequence. The main aim is to present these new concepts and to give some properties of Oresme hybrid numbers.

## 1. Introduction

Let $p, q, n$ be integers. For $n \geq 0$ Horadam (see [2]) defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation

$$W_{n+2} = p \cdot W_{n+1} - q \cdot W_n, \tag{1.1}$$

with fixed real numbers $W_0$, $W_1$. For the historical reasons these numbers were later called Horadam numbers.

For special $W_0, W_1, p, q$ the equation (1.1) defines selected numbers of the Fibonacci type, e.g. Fibonacci numbers $F_n = W_n(0, 1; 1, -1)$, Pell numbers $P_n = W_n(0, 1; 2, -1)$, Jacobsthal numbers $J_n = W_n(0, 1; 1, -2)$.

In [3] Horadam extended the equation (1.1) considering values of $p$, $q$ to be arbitrary rational numbers. Then taking $W_0 = 0$, $W_1 = \frac{1}{2}$, $p = 1$ and $q = \frac{1}{4}$ the equation (1.1) gives the known Oresme sequence $\{O_n\} = \{W_n(0, \frac{1}{2}; 1, \frac{1}{4})\}$, where $O_n$ is the $n$th Oresme number. Consequently,

$$O_n = O_{n-1} - \frac{1}{4} O_{n-2}, \tag{1.2}$$

for $n \geq 2$ with $O_0 = 0$, $O_1 = \frac{1}{2}$.

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Solving the above recurrence equation we obtain Binet formula for Oresme numbers of the form

$$O_n = \frac{n}{2^n}. \tag{1.3}$$

Then Oresme sequence has the form $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \ldots$

For Oresme numbers some identities can be found in [3], we recall some of them

$$\begin{aligned} O_{n+3} =& \frac{3}{4}O_{n+1} - \frac{1}{4}O_n, \\ O_{n+3} =& \frac{3}{4}O_{n+2} - \frac{1}{16}O_n, \\ \sum_{j=0}^{n} O_j =& 4\left(\frac{1}{2} - O_{n+2}\right). \end{aligned}$$

In some mathematical sources we can find that the Oresme sequence has a biological applications, see [3].

Oresme numbers were generalized by Cook in [1]. We use this concept for our future investigations.

Let $k \geq 2$, $n \geq 0$, be integers. Then $k$-Oresme numbers $\left\{O_n^{(k)}\right\}$ are defined by

$$O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k^2}O_{n-2}^{(k)}, \tag{1.4}$$

for $n \geq 2$ with $O_0^{(k)} = 0$, $O_1^{(k)} = \frac{1}{k}$.

Clearly $W_n(0, \frac{1}{k}; 1, \frac{1}{k^2}) = O_n^{(k)}$ and $O_n^{(2)} = O_n$.

Although the equation (1.4) works for $k \geq 2$ Binet formulas for $O_n^{(k)}$ have to be given separately for $k = 2$ and $k \geq 3$. It follows from roots of the characteristic equation of (1.4). If $k \geq 3$ then Binet formula for $k$-Oresme numbers has the form

$$O_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{k+\sqrt{k^2-4}}{2k}\right)^n - \left(\frac{k-\sqrt{k^2-4}}{2k}\right)^n\right], \tag{1.5}$$

if $k^2 - 4 > 0$.

In [1] identities provided by Horadam in [3] are extended for some of $k$-Oresme numbers. We recall some of them for future investigations

$$O_{n+3}^{(k)} = \frac{k^2-1}{k^2}O_{n+1}^{(k)} - \frac{1}{k^2}O_n^{(k)}, \tag{1.6}$$

$$O_{n+3}^{(k)} = \frac{k^2-1}{k^2}O_{n+2}^{(k)} - \frac{1}{k^4}O_n^{(k)}, \tag{1.7}$$

$$\sum_{j=0}^{n} O_j^{(k)} = k^2\left(\frac{1}{k} - O_{n+2}^{(k)}\right) = k - k^2 O_{n+2}^{(k)}. \tag{1.8}$$

## 2. Oresme Hybrid Numbers

In [4] Özdemir introduced a new non-commutative number system called hybrid numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined by

$$\mathbb{K} = \{\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \colon a, b, c, d \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = -1, \quad \varepsilon^2 = 0, \quad \mathbf{h}^2 = 1, \quad \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{2.1}$$

Two hybrid numbers

$$\mathbf{z_1} = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}, \quad \mathbf{z_2} = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h},$$

are equal if

$$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2, \quad d_1 = d_2.$$

The sum of two hybrid numbers is defined by

$$\mathbf{z_1} + \mathbf{z_2} = a_1 + a_2 + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}.$$

Addition operation is commutative and associative, zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is $-\mathbf{z} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. Hence, $(\mathbb{K}, +)$ is Abelian group.

Using (2.1), we get the multiplication table (see Table 1).

Table 1.

| $\cdot$ | $\mathbf{i}$ | $\varepsilon$ | $\mathbf{h}$ |
|---|---|---|---|
| $\mathbf{i}$ | $-1$ | $1 - \mathbf{h}$ | $\varepsilon + \mathbf{i}$ |
| $\varepsilon$ | $1 + \mathbf{h}$ | $0$ | $-\varepsilon$ |
| $\mathbf{h}$ | $-(\varepsilon + \mathbf{i})$ | $\varepsilon$ | $1$ |

The conjugate of a hybrid number $\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, denoted by $\overline{\mathbf{z}}$, is defined as $\overline{\mathbf{z}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. The real number

$$\mathcal{C}(\mathbf{z}) = \mathbf{z}\overline{\mathbf{z}} = \overline{\mathbf{z}}\mathbf{z} = a^2 + (b - c)^2 - c^2 - d^2$$

is called the character of the hybrid number $\mathbf{z}$.

Some interesting results for Horadam hybrid numbers, i.e., numbers defined in the following way

$$H_n = W_n + W_{n+1}\mathbf{i} + W_{n+2}\varepsilon + W_{n+3}\mathbf{h}, \tag{2.2}$$

were obtained in [5]. Tan and Ait-Amrane in *On a new generalization of Fibonacci hybrid numbers* (`https://arxiv.org/abs/2006.09727`) introduced the bi-periodic Horadam hybrid numbers which generalize the classical Horadam hybrid numbers. In this paper we define and study Oresme hybrid numbers and $k$-Oresme hybrid numbers.

Let $n \geq 0$ be an integer. Oresme hybrid sequence $\{OH_n\}$ we define by the following recurrence

$$OH_n = O_n + O_{n+1}\mathbf{i} + O_{n+2}\varepsilon + O_{n+3}\mathbf{h}, \tag{2.3}$$

where $O_n$ denotes the $n$th Oresme number.

Using (2.3) we get

$$\begin{aligned} OH_0 =& \frac{1}{2}\mathbf{i} + \frac{2}{4}\varepsilon + \frac{3}{8}\mathbf{h}, \\ OH_1 =& \frac{1}{2} + \frac{2}{4}\mathbf{i} + \frac{3}{8}\varepsilon + \frac{4}{16}\mathbf{h}, \\ OH_2 =& \frac{2}{4} + \frac{3}{8}\mathbf{i} + \frac{4}{16}\varepsilon + \frac{5}{32}\mathbf{h}, \\ OH_3 =& \frac{3}{8} + \frac{4}{16}\mathbf{i} + \frac{5}{32}\varepsilon + \frac{6}{64}\mathbf{h}. \end{aligned} \tag{2.4}$$

In [5] it was determined the character of the $n$th Horadam hybrid number $H_n$.

**Theorem 2.1** ([5])**.** *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} \mathcal{C}(H_n) =& W_n^2(1 - p^2q^2) + W_nW_{n+1}(2q + 2p^3q - 2pq^2) + \\ &+ W_{n+1}^2(1 - 2p - p^4 + 2p^2q - q^2). \end{aligned} \tag{2.5}$$

By (2.5) we get the following.

**Corollary 2.1.** *Let $n \geq 0$ be an integer. Then*

$$\mathcal{C}(OH_n) = \frac{15}{16}O_n^2 + \frac{14}{16}O_nO_{n+1} - \frac{25}{16}O_{n+1}^2 \tag{2.6}$$

*and using* (1.3) *we obtain*

$$\mathcal{C}(OH_n) = \frac{63n^2 - 22n - 25}{64 \cdot 2^{2n}}.$$

**Theorem 2.2** (Binet formula for Oresme hybrid numbers)**.** *Let $n \geq 0$ be an integer. Then*

$$OH_n = \frac{n}{2^n} + \frac{n+1}{2^{n+1}}\mathbf{i} + \frac{n+2}{2^{n+2}}\varepsilon + \frac{n+3}{2^{n+3}}\mathbf{h}. \tag{2.7}$$

*Proof.* Using (2.3) and (1.3) we obtain the desired formula. □

**Theorem 2.3** (Catalan identity for Oresme hybrid numbers)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$OH_{n+r} \cdot OH_{n-r} - (OH_n)^2 = \frac{-65r^2}{64 \cdot 4^n} + \frac{-4r^2 + r}{4 \cdot 4^n}\mathbf{i} + \frac{-8r^2 + 3r}{16 \cdot 4^n}\varepsilon + \frac{-r^2 - r}{4 \cdot 4^n}\mathbf{h}.$$

*Proof.* For integers $n \geq 0$, $r \geq 0$ and $n \geq r$, using Binet formula for Oresme hybrid numbers, we have

$$OH_{n+r} = \frac{n+r}{2^{n+r}} + \frac{n+r+1}{2^{n+r+1}}\mathbf{i} + \frac{n+r+2}{2^{n+r+2}}\varepsilon + \frac{n+r+3}{2^{n+r+3}}\mathbf{h}$$

and

$$OH_{n-r} = \frac{n-r}{2^{n-r}} + \frac{n-r+1}{2^{n-r+1}}\mathbf{i} + \frac{n-r+2}{2^{n-r+2}}\varepsilon + \frac{n-r+3}{2^{n-r+3}}\mathbf{h}.$$

So, after calculations the result follows. □

Note that for $r = 1$ we obtain Cassini type identity for Oresme hybrid numbers.

**Corollary 2.2** (Cassini identity for Oresme hybrid numbers)**.** *Let $n \geq 1$ be an integer. Then*

$$OH_{n+1} \cdot OH_{n-1} - (OH_n)^2 = \frac{-65}{64 \cdot 4^n} + \frac{-3}{4 \cdot 4^n}\mathbf{i} + \frac{-5}{16 \cdot 4^n}\varepsilon + \frac{-2}{4 \cdot 4^n}\mathbf{h}.$$

Let $k \geq 2$, $n \geq 0$, be integers. Then $k$-Oresme hybrid sequence $\left\{OH_n^{(k)}\right\}$ we define by the following recurrence

$$OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}, \tag{2.8}$$

where $O_n^{(k)}$ denotes the $n$th $k$-Oresme number.

**Theorem 2.4.** *Let $n \geq 0$, $k \geq 2$, be integers. Then*

(i) $OH_n^{(k)} + \overline{OH_n^{(k)}} = 2O_n^{(k)}$;

(ii) $\mathcal{C}(OH_n^{(k)}) = 2O_n^{(k)} \cdot OH_n^{(k)} - \left(OH_n^{(k)}\right)^2$;

(iii) $OH_{n+3}^{(k)} = \frac{k^2-1}{k^2}OH_{n+1}^{(k)} - \frac{1}{k^2}OH_n^{(k)}$;

(iv) $OH_{n+3}^{(k)} = \frac{k^2-1}{k^2}OH_{n+2}^{(k)} - \frac{1}{k^4}OH_n^{(k)}$;

(v) $\sum\limits_{j=0}^{n} OH_j^{(k)} = k^2\left(OH_1^{(k)} - OH_{n+2}^{(k)}\right)$.

*Proof.* (i) By the definition of the conjugate of a hybrid number we obtain

$$\begin{aligned} OH_n^{(k)} + \overline{OH_n^{(k)}} =& O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h} + O_n^{(k)} - O_{n+1}^{(k)}\mathbf{i} - O_{n+2}^{(k)}\varepsilon - O_{n+3}^{(k)}\mathbf{h} \\ =& 2O_n^{(k)}. \end{aligned}$$

(ii) By formula (2.3) and Table 1 we have

$$\begin{aligned} \left(OH_n^{(k)}\right)^2 =& \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + \left(O_{n+3}^{(k)}\right)^2 \\ &+ 2O_n^{(k)}O_{n+1}^{(k)}\mathbf{i} + 2O_n^{(k)}O_{n+2}^{(k)}\varepsilon + 2O_n^{(k)}O_{n+3}^{(k)}\mathbf{h} \\ &+ O_{n+1}^{(k)}O_{n+2}^{(k)}(\mathbf{i}\varepsilon + \varepsilon\mathbf{i}) + O_{n+1}^{(k)}O_{n+3}^{(k)}(\mathbf{ih} + \mathbf{hi}) + O_{n+2}^{(k)}O_{n+3}^{(k)}(\varepsilon\mathbf{h} + \mathbf{h}\varepsilon) \\ =& \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + \left(O_{n+3}^{(k)}\right)^2 + 2O_{n+1}^{(k)}O_{n+2}^{(k)} \\ &+ 2\left(O_n^{(k)}O_{n+1}^{(k)}\mathbf{i} + O_n^{(k)}O_{n+2}^{(k)}\varepsilon + O_n^{(k)}O_{n+3}^{(k)}\mathbf{h}\right) \\ =& 2O_n^{(k)} \cdot OH_n^{(k)} - \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + 2O_{n+1}^{(k)}O_{n+2}^{(k)} + \left(O_{n+3}^{(k)}\right)^2 \\ =& 2O_n^{(k)} \cdot OH_n^{(k)} - \mathcal{C}(OH_n^{(k)}). \end{aligned}$$

Hence, we get the result.

(iii) Using (1.6) we have

$$\begin{aligned}OH_{n+3}^{(k)} =& O_{n+3}^{(k)} + O_{n+4}^{(k)}\mathbf{i} + O_{n+5}^{(k)}\varepsilon + O_{n+6}^{(k)}\mathbf{h}\\ =& \left(\frac{k^2-1}{k^2}O_{n+1}^{(k)} - \frac{1}{k^2}O_n^{(k)}\right) + \left(\frac{k^2-1}{k^2}O_{n+2}^{(k)} - \frac{1}{k^2}O_{n+1}^{(k)}\right)\mathbf{i}\\ &+ \left(\frac{k^2-1}{k^2}O_{n+3}^{(k)} - \frac{1}{k^2}O_{n+2}^{(k)}\right)\varepsilon + \left(\frac{k^2-1}{k^2}O_{n+4}^{(k)} - \frac{1}{k^2}O_{n+3}^{(k)}\right)\mathbf{h}\\ =& \frac{k^2-1}{k^2}OH_{n+1}^{(k)} - \frac{1}{k^2}OH_n^{(k)}.\end{aligned}$$

(iv) Using (1.7) and proceeding analogously as in (iii) we obtain (iv).

(v) We have

$$\begin{aligned}\sum_{j=0}^{n} OH_j^{(k)} =& OH_0^{(k)} + OH_1^{(k)} + \cdots + OH_n^{(k)}\\ =& O_0^{(k)} + O_1^{(k)}\mathbf{i} + O_2^{(k)}\varepsilon + O_3^{(k)}\mathbf{h} + O_1^{(k)} + O_2^{(k)}\mathbf{i} + O_3^{(k)}\varepsilon + O_4^{(k)}\mathbf{h}\\ &+ \cdots + O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}\\ =& O_0^{(k)} + O_1^{(k)} + \cdots + O_n^{(k)}\\ &+ \left(O_1^{(k)} + O_2^{(k)} + \cdots + O_{n+1}^{(k)} + O_0^{(k)} - O_0^{(k)}\right)\mathbf{i}\\ &+ \left(O_2^{(k)} + O_3^{(k)} + \cdots + O_{n+2}^{(k)} + O_0^{(k)} + O_1^{(k)} - O_0^{(k)} - O_1^{(k)}\right)\varepsilon\\ &+ \left(O_3^{(k)} + O_4^{(k)} + \cdots + O_{n+3}^{(k)} + O_0^{(k)} + O_1^{(k)} + O_2^{(k)}\right.\\ &\left.-O_0^{(k)} - O_1^{(k)} - O_2^{(k)}\right)\mathbf{h}.\end{aligned}$$

Using (1.8) we obtain

$$\begin{aligned}\sum_{j=0}^{n} OH_j^{(k)} &= k - k^2 O_{n+2}^{(k)} + \left(k - k^2 O_{n+3}^{(k)} - 0\right)\mathbf{i}\\ &\quad + \left(k - k^2 O_{n+4}^{(k)} - 0 - \frac{1}{k}\right)\varepsilon + \left(k - k^2 O_{n+5}^{(k)} - 0 - \frac{1}{k} - \frac{1}{k}\right)\mathbf{h}\\ &= \left(k + k\mathbf{i} + \frac{k^2-1}{k}\varepsilon + \frac{k^2-2}{k}\mathbf{h}\right) - k^2 OH_{n+2}^{(k)}\\ &= k^2\left(\frac{1}{k} + \frac{1}{k}\mathbf{i} + \frac{k^2-1}{k^3}\varepsilon + \frac{k^2-2}{k^3}\mathbf{h}\right) - k^2 OH_{n+2}^{(k)}\\ &= k^2\left(OH_1^{(k)} - OH_{n+2}^{(k)}\right).\end{aligned}$$

□

## 3. Oresme Hybrationals

Cerda-Morales in *Oresme polynomials and their derivatives* (`https://arxiv.org/abs/1904.01165v1`) extended the sequence of $k$-Oresme numbers to the sequence of rational functions $\{O_n(x)\}$ by replacing $k$ with a real variable $x$. These rational functions were named as Oresme polynomials.

Let $x$ be a nonzero real variable. The sequence of Oresme polynomials is recursively defined as follows:

$$O_n(x) = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{x}, & \text{if } n = 1 \\ O_{n-1}(x) - \frac{1}{x^2}O_{n-2}(x), & \text{if } n \geq 2. \end{cases} \tag{3.1}$$

Solving the characteristic equation of Oresme polynomials recurrence relation

$$r^2 - r + \frac{1}{x^2} = 0,$$

we obtain Binet formula

$$O_n(x) = \frac{1}{\sqrt{x^2-4}}\left[\left(\frac{x+\sqrt{x^2-4}}{2x}\right)^n - \left(\frac{x-\sqrt{x^2-4}}{2x}\right)^n\right],$$

for $x^2 - 4 > 0$, $x \neq 0$, and

$$O_n(x) = \frac{\mathbf{i}}{\sqrt{4-x^2}}\left[\left(\frac{x-\sqrt{4-x^2}\,\mathbf{i}}{2x}\right)^n - \left(\frac{x+\sqrt{4-x^2}\,\mathbf{i}}{2x}\right)^n\right],$$

for $x^2 - 4 < 0$. Moreover, $O_n(2) = O_n$ and $O_n(-2) = -O_n$.

For $n \geq 0$ and nonzero real variable $x$ Oresme hybrationals are defined by

$$OH_n(x) = O_n(x) + O_{n+1}(x)\mathbf{i} + O_{n+2}(x)\varepsilon + O_{n+3}(x)\mathbf{h},$$

where $O_n(x)$ is the $n$th Oresme polynomial and $\mathbf{i}$, $\varepsilon$, $\mathbf{h}$ are hybrid units.

For $x = k$ we obtain $k$-Oresme hybrid numbers.

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# EXISTENCE RESULT FOR FRACTIONAL DIFFERENTIAL EQUATION ON UNBOUNDED DOMAIN

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Abstract. In this article, we establish certain sufficient conditions to show the existence of solutions of boundary value problem for fractional differential equations on the half-line in a Fréchet space. The main result is based on Tykhonoff fixed point theorem combining with a suitable measure of non-compactness. An example is given to illustrate our approach.

## 1. Introduction

The theoretical study of fractional differential equations has recently acquired great importance in applied mathematics and the modeling of many phenomena in various sciences, let us quote for example [11, 12, 15, 17]. The monographs [14, 16, 18, 20] contain basic concepts and theory in fractional differential equations and fractional calculus.

Very recently, excellent works have been done to study fractional differential equations with various conditions which resides in the existence and uniqueness theorem by utilizing some analytical and numerical methods and certain basic tools from functional analysis, we refer the reader to [1, 4–9].

Several results existence of these problems were obtained on unbounded domains like $[0, +\infty)$ involving classical methods, for example, Xinwei Su discussed in the work [19] the existence of solutions of the following problem

$$\begin{cases} D^{\alpha}_{0^+}y(t) = f(t, y(t)), & t \in J = (0, +\infty), 1 < \alpha \leq 2, \\ y(0) = 0, & D^{\alpha-1}_{0^+}y(\infty) = y_\infty, \end{cases}$$

*Key words and phrases.* Boundary value problem, measure of non-compactness of Kuratowski, Tykhonoff fixed point theorem, Riemann-Liouville fractional derivative.

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where $D^{\alpha}_{0+}$ is the Riemann-Liouville fractional derivative of order $\alpha$, the state $y(\cdot)$ takes value in a Banach space $E$, $f : J \times E \to E$ is a continuous function and $y_\infty \in E$. The main approach is based on Dardo's fixed point theorem. The contents of this article are an extension of their work for a class generally.

This article studies the existence of solutions of boundary value problem for fractional differential equations on unbounded interval. We consider the following problem

$$D^{\alpha}_{0+}y(t) = f(t, y(t)), \quad t \in J = (0, +\infty), \tag{1.1}$$

$$I^{2-\alpha}_{0+}y(0^+) = y_0, \tag{1.2}$$

$$D^{\alpha-1}_{0+}y(\infty) = y_\infty, \tag{1.3}$$

where $D^{\delta}_{0+}$ denotes Riemann-Liouville fractional derivative for $\delta \in \{\alpha, \alpha - 1\}$ with $1 < \alpha \leq 2$, $I^{2-\alpha}_{0+}$ denotes the left-sided Riemann-Liouville fractional integral, $E$ is a real Banach space with the norm $\| \cdot \|$, $y_0, y_\infty \in E$ and $f : (0, \infty) \times E \to E$ a function satisfying some specified conditions (see Section 3).

The present work is organized in the following way. In Section 2, we give some general results and preliminaries and in Section 3, we show the existence solution for the problem (1.1)–(1.3) by using the Tykhonoff fixed point theorem combined with the technique of measure of non-compactness of Kuratowski. Finally an illustrative example will be presented in the last section.

## 2. Backgrounds

We introduce in this section some notation and technical results which are used throughout this paper. Let $I \subset J$ be a compact interval and denote by $C(I, E)$ the Banach space of continuous functions $y : I \to E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\| \mid t \in I\}.$$

$L^1(I, E)$ denotes the space of $E$-valued Bochner integrable functions on $I$ with the norm

$$\|f\|_{L^1} = \int_I \|f(t)\| dt.$$

We consider the following Fréchet space

$$C_\alpha([0, \infty), E) = \left\{ y \in C(J, E) \mid \lim_{t \to 0^+} t^{2-\alpha} y(t) \text{ exists and is finite} \right\},$$

equipped with the family of seminorms

$$\|y\|_T = \sup_{t \in [0,T]} \left\{ \frac{t^{2-\alpha}}{1 + t^\alpha} \|y(t)\| \mid T \geq 0 \right\}.$$

For $y \in C_\alpha((0, \infty), E)$, we define $y_\alpha$ by

$$y_\alpha(t) = \begin{cases} \dfrac{t^{2-\alpha}}{1 + t^\alpha} y(t), & \text{if } t \in (0, \infty), \\ \lim\limits_{t \longrightarrow 0} t^{2-\alpha} y(t), & \text{if } t = 0. \end{cases}$$

It is clear that $y_\alpha \in C([0,\infty), E)$.

We begin with the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For more details, we refer the reader [2, 3, 13].

**Definition 2.1.** The Kuratowski measure of non-compactness $\gamma$ is defined on each bounded subset $\Omega$ of $E$ by

$$\gamma(\Omega) = \inf\{\varepsilon > 0 \mid \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}.$$

**Lemma 2.1** ([3]). *Let $\{D_n\}_0^\infty$ be a sequence of nonempty, bounded and closed subsets of $E$, such that for all positive integer $n$, $D_{n+1} \subset D_n$. If $\lim_{n\to\infty} \gamma(D_n) = 0$, then the set*

$$D_\infty = \bigcap_{n=0}^{\infty} D_n$$

*is nonempty and compact.*

**Lemma 2.2** ([2]). *Let $E$ be a Banach space and $A, B$ be two bounded subsets of $E$. The following properties hold:*

- $(i_1)$ *$\gamma(A) = 0$ if and only if $A$ is relatively compact;*
- $(i_2)$ *$\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ denotes the closure of $A$;*
- $(i_3)$ *$\gamma(A + B) \leq \gamma(A) + \gamma(B)$;*
- $(i_4)$ *$A \subset B$ implies $\gamma(A) \leq \gamma(B)$;*
- $(i_5)$ *$\gamma(a.A) = |a|.\gamma(A)$ for all $a \in E$;*
- $(i_6)$ *$\gamma(\{a\} \cup A) = \gamma(A)$ for all $a \in E$;*
- $(i_7)$ *$\gamma(A) = \gamma(Conv(A))$, where $Conv(A)$ denotes the convex hull of $A$.*

**Lemma 2.3** ([2]). *If $\Omega$ is a bounded and equicontinuous subset of $C(I, E)$, then $\gamma(\Omega(t))$ is continuous on $I$ and*

$$\gamma_C(\Omega) = \max_{t\in I} \gamma(\Omega(t)), \quad \gamma\left(\left\{\int_I x(t)dt : x \in \Omega\right\}\right) \leq \int_I \gamma(\Omega(t))dt,$$

*where $\Omega(t) = \{x(t) \mid x \in \Omega\}$ and $\gamma_C$ is the non-compactness measure on the space $C(I, E)$.*

The following theorem is due to Tykhonoff.

**Theorem 2.1** ([10]). *Let $F$ be a locally convex space, $K$ a compact convex subset of $F$ and $N : K \to K$ a continuous map. Then $N$ has at least one fixed point in $K$.*

Let us now give some definitions from the theory of fractional calculus.

**Definition 2.2** ([14]). Let $\Gamma$ be the gamma function, $\alpha$ a non-negative real number and $h \in C(J, E)$.

(1) The Riemann-Liouville fractional integral of the function $h$ of order $\alpha$ is defined by

$$I_{0+}^\alpha h(t) = g_\alpha(t) * h(t) = \int_0^t g_\alpha(t-s)h(s)ds, \quad t > 0,$$

where $*$ denotes convolution and $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$.

(2) The Riemann-Liouville fractional derivative of the function $h$ of order $\alpha$ is defined by

$$D_{0+}^{\alpha}h(t) = \frac{d^n}{dt^n}(g_{n-\alpha}(t) * h(t)),$$

for all $t > 0$, where $n$ is the least integer greater than or equal to $\alpha$.

*Remark* 2.1. For $\alpha > 0,\ k > -1$, we have

$$I_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}t^{\alpha+k} \quad \text{and} \quad D_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha},\ t > 0,$$

giving in particular $D_{0+}^{\alpha}t^{\alpha-m} = 0,\ m = 1,\dots,n$, where $n$ is the smallest integer greater than or equal to $\alpha$.

*Remark* 2.2. If $h$ is suitabe function (see for instance [14, 16, 18]), we have the composition relations $D_{0+}^{\alpha}I_{0+}^{\alpha}h(t) = h(t),\ \alpha > 0$, and $D_{0+}^{\alpha}I_{0+}^{k}h(t) = I_{0+}^{k-\alpha}h(t),\ k > \alpha > 0,\ t > 0$.

## 3. Main Result

We need to introduce the following four hypotheses to present our main result at the end of this section.

$(H_1)$ $f : J \times E \to E$ is a Carathéodory function.

$(H_2)$ There exists nonnegative functions $a, b \in C(J, \mathbb{R}^+)$ such that

$$\|f(t,u)\| \le a(t) + t^{2-\alpha}b(t)\|u\|, \quad \text{for all } t \in J \text{ and } u \in E,$$

where

$$\int_0^{\infty}(1+t^{\alpha})b(t)dt < \Gamma(\alpha), \quad \int_0^{\infty} a(t)dt < \infty.$$

$(H_3)$ There exists a locally integrable function $\ell \in L^1(J, \mathbb{R}^+)$ such that, for each nonempty, bounded set, we have $\Omega \subset C_{\alpha}(J, E)$

$$\gamma(f(t,\Omega(t))) \le \ell(t)\gamma(t^{2-\alpha}\Omega(t)), \quad \text{for all } t \in J,$$

where

$$\int_0^{\infty}(1+s^{\alpha})\ell(s)ds < \Gamma(\alpha). \tag{3.1}$$

$(H_4)$ There exists $R > 0$ such that

$$R > \frac{\|y_{\infty}\| + (\alpha-1)\|y_0\| + \int_0^{\infty} a(t)dt}{\Gamma(\alpha) - \int_0^{\infty}(1+t^{\alpha})b(t)dt}.$$

**Definition 3.1.** A function $y \in C_{\alpha}([0, +\infty))$ is said to be a solution of the problem (1.1)–(1.3) if $y$ satisfies the equation $D_{0+}^{\alpha}y(t) = f(t, y(t))$ and the conditions (1.2)–(1.3).

**Lemma 3.1.** *Let* $1 < \alpha < 2$ *and let* $h : J \to E$ *be continuous. If* $y$ *is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)}\left[y_\infty - \int_0^\infty h(t)dt\right]t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds, \tag{3.2}$$

*then it is also a solution of the problem*

$$D^\alpha_{0+}y(t) = h(t), \quad t \in J = (0, +\infty), \tag{3.3}$$

$$I^{2-\alpha}_{0+}y(0^+) = y_0, \tag{3.4}$$

$$D^{\alpha-1}_{0+}y(\infty) = y_\infty. \tag{3.5}$$

*Proof.* Suppose that $y$ is a solution of the integral equation (3.2). Applying $I^{2-\alpha}_{0+}$ to both sides of (3.2) and using Remark 2.1, we obtain

$$I^{2-\alpha}_{0+}y(t) = \left(y_\infty - \int_0^\infty h(t))dt\right)t + y_0 + I^2_{0+}h(t).$$

As $t \to 0$, we get

$$I^{2-\alpha}_{0+}y(0^+) = y_0.$$

Now, by applying $D^{\alpha-1}_{0+}$ to both sides of (3.2) and by using Remark 2.1, Remark 2.2, we have

$$D^{\alpha-1}_{0+}y(t) = y_\infty - \int_0^\infty h(t)dt + I^1_{0+}h(t).$$

As $t \to \infty$, we get

$$D^{\alpha-1}_{0+}y(\infty) = y_\infty.$$

Next, by applying $D^\alpha_{0+}$ to both sides of (3.2) and by using Remark 2.1, Remark 2.2, we obtain $D^\alpha_{0+}y(t) = h(t)$. The results are proved completely. □

Consider the operator $N : C_\alpha([0,\infty), E) \to C_\alpha([0,\infty), E)$ defined by

$$Ny(t) = \frac{y_\infty - \int_0^\infty f(t, y(t))dt}{\Gamma(\alpha)}t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, y(s))ds.$$

Let

$$B = \{y \in C_\alpha([0,\infty), E) \mid \|y\|_T \le R\}.$$

*Remark* 3.1. **(1)** Clearly the operator $N$ is well defined.

**(2)** There exists a positive real number $M$ such that

$$\int_0^\infty \|f(t, y(t))\|dt \le M, \quad \text{for any } y \in B.$$

**Lemma 3.2.** *If the conditions* $(H_1)$ *and* $(H_2)$ *are valid, then*

**(1)** *the operator* $N$ *is bounded and continuous on the subset* $B$*;*

**(2)** *the subset* $(NB)_\alpha = \{(Ny)_\alpha \mid y \in B\}$ *is equicontinuous on the compact interval* $[0,T]$, $T > 0$*;*

**(3)** *for given* $\varepsilon > 0$, *there exists a constant* $N_1 > 0$ *such that*

$$\left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1+t_2^\alpha} \right\| < \varepsilon, \quad \text{for any } t_1, t_2 \geq N_1 \text{ and } y(\cdot) \in B.$$

*Proof.* In order to prove (**1**), let $y \in B$ and $t \in [0,T],\ T > 0$, from $(H_2)$, we have

$$\begin{aligned} \frac{t^{2-\alpha}\|N(y)(t)\|}{1+t^\alpha} \leq & \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \int_0^\infty \|f(s,y(s))\| ds \\ \leq & \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{R}{\Gamma(\alpha)} \int_0^\infty (1+t^\alpha) b(t) dt + \frac{1}{\Gamma(\alpha)} \int_0^\infty a(t) dt. \end{aligned}$$

Hence, $N$ is bounded on the subset $B$. Next, we will prove that $N$ is continuous. We have

$$Ny(t) = \frac{y_\infty - \int_0^\infty f(t,y(t))dt}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s)) ds.$$

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $B$, such that $y_n \to y$ in $B$. Let $T > 0$ and $\varepsilon > 0$, from $(H_1)$ and $(H_2)$, there exists $L > T$ such that

$$\int_L^\infty a(t) dt < \frac{\Gamma(\alpha)}{6} \varepsilon, \quad \int_L^\infty (1+t^\alpha) b(t) dt < \frac{\Gamma(\alpha)}{6M} \varepsilon,$$

and there exists $\widetilde{N} \in \mathbb{N}$ such that, for all $n \geq \widetilde{N}$, we have

$$\int_0^\infty \|f(s,y_n(s)) - f(s,y(s))\| ds < \frac{\Gamma(\alpha)}{3} \varepsilon.$$

Therefore, for $t \in [0,T],\ T > 0$ and $n > \widetilde{N}$, we have

$$\begin{aligned} & \frac{t^{2-\alpha}}{1+t^\alpha} \|N(y_n)(t) - N(y)(t)\| \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^t \|f(s,y_n(s)) - f(s,y(s))\| ds + \frac{1}{\Gamma(\alpha)} \int_t^\infty \|f(s,y_n(s)) - f(s,y(s))\| ds \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \|f(s,y_n(s)) - f(s,y(s))\| ds \\ & + \frac{1}{\Gamma(\alpha)} \left[ \int_t^L \|f(s,y_n(s)) - f(s,y(s))\| ds + \int_L^\infty \|f(s,y_n(s)) - f(s,y(s))\| ds \right] \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^L \|f(s,y_n(s)) - f(s,y(s))\| ds + \frac{2M}{\Gamma(\alpha)} \int_L^\infty (1+s^\alpha) b(s) ds + \frac{2}{\Gamma(\alpha)} \int_L^\infty a(s) ds \\ \leq & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then,

$$\|Ny_n - Ny\|_T \to 0 \quad \text{as} \quad n \to \infty.$$

We will prove (**2**). Let $y \in B$ and $t_1, t_2 \in [0,T],\ T > 0$ where $t_1 > t_2$. Then

$$\left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1+t_2^\alpha} \right\|$$

$$
\begin{aligned}
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\
& + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s)) ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) ds \right\| \\
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| \|f(s, y(s))\| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, y(s))\| ds \\
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| a(s) ds \\
& + \frac{r}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| (1 + s^\alpha) b(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} a(s) ds \\
& + \frac{r}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} (1 + s^\alpha) b(s) ds \\
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\
& + \frac{a^* + b^* r}{\Gamma(\alpha)} \int_0^{t_1} \Big( (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} ds \Big) + \frac{a^* + b^* r}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
& + \frac{2b^* r}{\Gamma(\alpha)} \Big( \int_0^{t_2} (t_2 - s)^{\alpha-1} s^\alpha ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} s^\alpha ds \Big) \\
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \\
& + \frac{a^* + b^* r}{\Gamma(1+\alpha)} (t_2^\alpha - t_1^\alpha - (t_2 - t_1)^\alpha) + \frac{a^* + b^* r}{\Gamma(1+\alpha)} (t_2 - t_1)^\alpha \\
& + \frac{2b^* r \mathcal{B}(\alpha, \alpha+1)}{\Gamma(\alpha)} (t_2^{2\alpha} - t_1^{2\alpha}),
\end{aligned}
$$

where $a^* = \max_{t\in[0,T]} a(t)$ and $b^* = \max_{t\in[0,T]} b(t)$. As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. Then $(NB)_\alpha$ is equicontinuous on $[0, T]$.

Next, we verify assertion (**3**). Let $\varepsilon > 0$, we have

$$
\begin{aligned}
& \left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1+t_2^\alpha} \right\| \\
\leq & \frac{\|y_\infty\| + M}{\Gamma(\alpha)} \left| \frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha} \right| + \frac{\|y_0\|}{\Gamma(\alpha-1)} \left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right|
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|.$$

It is sufficient to prove that

$$\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\leq\varepsilon.$$

Remark 3.1 yields that, there exists $N_0>0$ such that

$$\int_{N_0}^{\infty}\|f(t,y(t))\|dt\leq\frac{\varepsilon}{3},\quad\text{for all } y\in B. \tag{3.6}$$

On the other hand, since $\lim_{t\to\infty}\frac{t^{2-\alpha}(t-N_0)^{\alpha-1}}{1+t^\alpha}=0$, there exists $N_1>N_0$ such that, for all $t_1,t_2>N_1$ and $s\in[0,N_1]$, we have

$$\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|<\frac{\varepsilon}{3M}. \tag{3.7}$$

Now taking $t_1,t_2\geq N_1$, from (3.6), (3.7), we can arrive at

$$\begin{aligned}&\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\\
\leq&\int_0^{N_1}\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\|f(s,y(s))\|ds\\
&+\int_{N_1}^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\|f(s,y(s))\|ds+\int_{N_1}^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\|f(s,y(s))\|ds\\
<&\frac{\varepsilon}{3M}\int_0^\infty\|f(s,y(s))\|ds+2\int_{N_1}^\infty\|f(s,y(s))\|ds<\varepsilon.\end{aligned}$$

□

**Theorem 3.1.** *Suppose that the conditions* $(H_1)$, $(H_2)$, $(H_3)$ *and* $(H_4)$ *are valid. Then the problem* (1.1)–(1.3) *has at least one solution.*

*Proof.* We shall prove that $N$ satisfies the conditions of Tykhonoff fixed point theorem 2.1. From Lemma 3.2, the operator $N$ is continuous on $B$. We can derive that $N:B\to B$. Indeed, for any $y\in B$ and $t\in[0,T]$, $T>0$, and by condition $(H_1)$ and $(H_4)$, we get

$$\begin{aligned}\|t^{2-\alpha}N(y)(t)\|\leq&\frac{\|y_\infty\|}{\Gamma(\alpha)}+\frac{\|y_0\|}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha)}\int_0^\infty\|f(t,y(t))\|dt\\
\leq&\frac{(1}{\Gamma(\alpha)}\left(\|y_\infty\|+(\alpha-1)\|y_0\|+\int_0^\infty a(t)dt+R\int_0^\infty(1+t^\alpha))b(t)dt\right)\\
<&R.\end{aligned}$$

Let $\gamma_\alpha$ be the measure of non-compacteness of Kuratowski defined on the family of bounded subsets of the space $C_\alpha(J,E)$. We have

$$\gamma_\alpha(NB)=\sup_{T>0}\left\{\sup_{t\in[0,T]}\gamma\left(\frac{t^{\alpha-2}}{1+t^\alpha}N(B)(t)\right)\right\}.$$

For demostrations, see [19, Lemma 3.4].

We define the sequence of sets $\{D_n\}_{n=0}^{\infty}$ by

$$\begin{cases} D_0 = B, \\ D_{n+1} = Conv((N(D_n))), \quad n = 0, 1, \dots, \\ D_\infty = \cap_{n=0}^{\infty} D_n. \end{cases}$$

We have $D_{n+1} \subset D_n$, for each $n$. Finally, we need to prove the following relation

$$\lim_{n\to\infty} \gamma_\alpha(D_n) = 0.$$

Suppose that $T$ is sufficiently large. For each $y \in B$, we consider

$$\begin{aligned} N_T(y)(t) = & \frac{y_\infty}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1} - (t-s)^{\alpha-1}] f(s, y(s)) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_t^T (t-s)^{\alpha-1} f(s, y(s)) ds. \end{aligned}$$

Then, from $(H_2)$, we obtain that

$$\begin{aligned} \frac{t^{2-\alpha}}{1+t^\alpha} \|N_T(y)(t) - N(y)(t)\| & \le \frac{1}{\Gamma(\alpha)} \int_T^\infty \|f(t, y(t))\| dt \\ & \le \frac{1}{\Gamma(\alpha)} \left( \int_T^\infty a(t) dt + R \int_T^\infty (1+t^\alpha) b(t) dt \right), \end{aligned}$$

this shows that

$$H_d \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha}, \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right) \to 0 \quad \text{as} \quad T \to \infty, t \in J.$$

Where $H_d$ denotes the Hausdorff metric in space $E$. By Property of non-compactness measure, we get

$$\lim_{T\to\infty} \gamma \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha} \right) = \gamma \left( \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right). \tag{3.8}$$

Let $\varepsilon > 0$, from (3.8), then exists $\widetilde{T} > 0$ such that, for $T \ge \widetilde{T}$, we have

$$\gamma \left( \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right) < \varepsilon + \gamma \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha} \right).$$

Using Lemma 2.3, Lemma 3.2 and assumption $(H_3)$, for each $n \in \mathbb{N}$ and $t \in [0, T]$, $T > \widetilde{T}$, we get

$$\begin{aligned} \gamma \left( \frac{t^{2-\alpha} N(D_{n+1})(t)}{1+t^\alpha} \right) & \le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^T (1+s^\alpha) \ell(s) \gamma \left( \frac{s^{2-\alpha} D_{n+1}(s)}{1+s^\alpha} \right) ds \\ & \le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^T (1+s^\alpha) \ell(s) \sup_{s\in[0,T]} \gamma \left( \frac{s^{2-\alpha} N(D_n)(s)}{1+s^\alpha} \right) ds. \end{aligned}$$

Then

$$\sup_{s\in[0,T]} \gamma\left(\frac{t^{2-\alpha}N(D_{n+1})(t)}{1+t^{\alpha}}\right) \leq \varepsilon + \frac{1}{\Gamma(\alpha)}\int_0^T (1+s^{\alpha})\ell(s) \sup_{s\in[0,T]} \gamma\left(\frac{s^{2-\alpha}N(D_n)(s)}{1+s^{\alpha}}\right) ds.$$

$$\leq \varepsilon + \frac{\tau}{\Gamma(\alpha)} \sup_{s\in[0,T]} \gamma\left(\frac{s^{2-\alpha}N(D_n)(s)}{1+s^{\alpha}}\right),$$

where

$$\tau = \int_0^{\infty} (1+s^{\alpha})\ell(s)ds.$$

Consequently, since $\varepsilon$ is arbitrary, we obtain

$$\gamma_\alpha(D_{n+1}) \leq \frac{\tau\gamma_\alpha(D_n)}{\Gamma(\alpha)}, \quad \text{for each } n \in \mathbb{N}.$$

By induction, we can show that

$$\gamma_\alpha(D_{n+1}) \leq \left(\frac{\tau}{\Gamma(\alpha}\right)^{n+1} \gamma_\alpha(D_0), \quad \text{for each } n \in \mathbb{N}.$$

Hence, by (3.1), we get

$$\lim_{n\to\infty} \gamma_\alpha(D_n) = 0.$$

Taking into account Lemma 2.1, we infer that $D_\infty = \bigcap_{n=0}^{\infty} D_n$ is nonempty, convex and compact. From Theorem 2.1, we conclude that $N : D_\infty \to D_\infty$ has a fixed point $y \in D_\infty$, which is a solution of problem (1.1)–(1.3). □

## 4. Example

As an application of our results, we consider the following fractional differential equation

$$D^{\frac{3}{2}}y(t) = \left(\frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{5t}} + \frac{\sin(t)}{1+t^2}\right)_{n=1}^{\infty}, \quad t \in J = (0,+\infty), \tag{4.1}$$

$$I_{0^+}^{\frac{1}{2}} y(t) = y_0, \tag{4.2}$$

$$D_{0^+}^{\frac{1}{2}} y(\infty) = y_\infty. \tag{4.3}$$

Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) \mid \sup |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $E$ is a Banach space and problem (4.1)–(4.3) can be regarded as a problem of the form (1.1)–(1.3), with

$$\alpha = \frac{3}{2} \quad \text{and} \quad f(t, y(t)) = (f(t, y_1(t)), \ldots, f(t, y_n(t)), \ldots),$$

where

$$f(t, y_n(t)) = \frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{5t}} + \frac{\sin(t)}{1+t^2}, \quad n \in \mathbb{N}^*.$$

We shall verify the conditions $(H_2)$–$(H_4)$. Evidently, $f$ is Carathéodory function in $J \times E$ and

$$\|f(t, y(t))\| \leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}\|y(t)\| + \frac{1}{1+t^2}.$$

With the aid of simple computation, we find that

$$\int_0^\infty e^{-5t}dt = \frac{1}{5} < \Gamma\left(\frac{3}{2}\right) \quad \text{and} \quad \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2} < \infty.$$

Finally, we verify the condition $(H_3)$. For any bounded set $B \subset E$, we have

$$f(t, B(t)) = \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}B(t) + \left\{\frac{\sin(t)}{1+t^2}\right\}.$$

Then

$$\gamma(f(t, B(t)) \leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}\gamma(B(t)).$$

Since $\int_0^\infty e^{-5t}dt = \frac{1}{5} < \Gamma(\frac{3}{2})$, we conclude that the condition $(H_3)$ is satisfied. Therefore, Theorem 3.1 ensures that the Problem (4.1)–(4.3) has a solution.

## 5. Conclusion

We hope that we have given some result as far as we know not existing in the literature concerning existence solution for Riemann-Liouville fractional differential equation on the half line involving the discontinuity of the state $y$ at $0^+$, to overcome this obstruction we have defined a special weight space of continuous function $C_\alpha([0, +\infty))$. The constructed space is in a natural way. In this work we have assumed a more general growth condition $(H_1)$ unlike what is in the literature, condition $(H_2)$ being supposed to overcome the equiconvergence at infinity, condition $(H_3)$ ensure the proof of Tykhonoff fixed point theorem, these conditions are optimal in the sense that no condition implies the other. We make use in our approach Tykhonoff fixed point theorem combining with analysis functional tools and a suitable measure of non-compactness. The paper is ended by an example to illustrate the main result.

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# FOLDING THEORY APPLIED TO FUZZY (POSITIVE) IMPLICATIVE (PRE)FILTERS IN EQ-ALGEBRAS

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Abstract. In this paper, the concepts of fuzzy $n$-fold positive implicative and fuzzy $n$-fold implicative (pre)filters in $EQ$-algebras are introduced and several properties of them are provided. Moreover, the relationship between fuzzy $n$-fold positive implicative (pre)filter and fuzzy $n$-fold implicative (pre)filter is considered. Using the level subset of a fuzzy set in $EQ$-algebras, some characterizations of fuzzy $n$-fold (positive) implicative (pre)filters in $EQ$-algebras are given. Also, we investigate under what conditions the fuzzy $n$-fold positive implicative (pre)filter is a fuzzy $n$-fold implicative (pre)filter in $EQ$-algebras.

## 1. Introduction

EQ-algebras were proposed by Novák and De Baets [6, 8]. One of the motivations was to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory ($FTT$) [7] that generalizes the system of classical type theory [1] in which the sole basic connective is equality. Analogously, the basic connective in ($FTT$) should be fuzzy equality. Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\odot$ and fuzzy equality $\sim$. The implication operation $\to$ is the derived of the fuzzy equality $\sim$ and it together with $\odot$ no longer strictly form the adjoint pair in general. $EQ$-algebras are interesting and important for studying and researching and residuated lattices are particular cases of $EQ$-algebras. In fact, $EQ$-algebras generalize non-commutative residuated lattices [3]. From the point of view of logic, the main diference between residuated lattices and $EQ$-algebras lies in the way the implication operation is obtained. While in residuated

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lattices it is obtained from (strong) conjunction, in $EQ$-algebras it is obtained from equivalence. Consequently, the two kinds of algebras differ in several essential points despite their many similar or identical properties. Since residuated lattices ($BL$-algebras, $MV$-algebras, $MTL$-algebras, and $R_0$-algebras) are particular types of $EQ$-algebras, it is natural and meaningful to extend some notions of residuated lattices to $EQ$-algebras. Filter theory plays an important role in studying these algebras because the properties of filters have a strong influence on the structure properties of algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Up to now, some types of (fuzzy) filters on ordered algebras based logical algebras have been widely studied [2, 4, 5, 10] and some important results have been obtained. Fuzzy algebra is an important branch of fuzzy mathematics and Rosenfeld [12] started the study of fuzzy algebraic structures with the introduction of the concept of fuzzy sub-groups in 1971. Since then these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. So generalization existing results in BL-algebras and residuated lattices, to EQ-algebras is important tool for studying various algebraic and logical systems in special case EQ-algebras.

In BL-algebras, residuated lattices, MTL-algebra fuzzy ($n$-fold) implicative filters and fuzzy ($n$-fold) positive implicative filters were provided. In EQ-algebras, the notions of implicative filters and positive implicative filters were introduced by Liu and Zhang. Moreover, Paad and et al. [11] extended this filters to $n$-fold implicative filters and $n$-fold positive implicative filters in $EQ$-algebras and fuzzy implicative filters and fuzzy positive implicative filters were provided by Xin and et al. [13]. This motivates us to extend different types of fuzzy (implicative, positive implicative) (pre)filters of EQ-algebras. Hence, in this paper, we introduce the notions fuzzy $n$-fold implicative and fuzzy $n$-fold positive implicative (pre)filters in EQ-algebras and investigate the properties and characterized them as it have done in residuated lattices. Moreover, we study the relationship between fuzzy $n$-fold positive implicative (pre)filters and fuzzy $n$-fold implicative (pre)filters. In the follow, by using the level subset of a fuzzy set in $EQ$-algebras, we give some characterizations of fuzzy $n$-fold (positive) implicative (pre)filters in $EQ$-algebras. Also, we investigate under what conditions the fuzzy $n$-fold positive implicative (pre)filter is fuzzy $n$-fold implicative (pre)filter in $EQ$-algebras.

## 2. Preliminaries

**Definition 2.1** ([3, 6])**.** An $EQ$-algebra is an algebra $(L, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms.

($E1$) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leq y$ if and only if $x \wedge y = x$.

($E2$) $(L, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leq$.

($E3$) $x \sim x = 1$ (reflexivity axiom).

($E4$) $((x \wedge y) \sim z) \odot (s \sim x) \leq z \sim (s \wedge y)$ (substitution axiom).

$(E5)$ $(x \sim y) \odot (s \sim t) \leq (x \sim s) \sim (y \sim t)$ (congruence axiom).
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ (monotonicity axiom).
$(E7)$ $x \odot y \leq x \sim y$ (boundedness axiom),
for all $s, t, x, y, z \in L$.

Let $L$ be an $EQ$-algebra. Then for all $x, y \in L$, we put

$$x \to y = (x \wedge y) \sim x, \quad \tilde{x} = x \sim 1.$$

The derived operation $\to$ is called implication and if $L$ contains an element 0 such that $0 \leq x$, for any $x \in L$, then 0 is called bottom element and we may define the unary operation $\neg$ on $L$ by $\neg x = x \sim 0$.

**Definition 2.2** ([6])**.** Let $L$ be an $EQ$-algebra. Then $L$ is called:
$(i)$ separated, if $x \sim y = 1$ implies $x = y$ for all $x, y \in L$;
$(ii)$ good, if $\tilde{x} = x$ for all $x \in L$;
$(iii)$ residuated, if $(x \odot y) \wedge z = x \odot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$ for all $x, y, z \in L$.

**Lemma 2.1** ([3,6])**.** *Let $L$ be an $EQ$-algebra. Then the following properties hold for any $x, y, z \in L$:*
$(i)$ *$x \sim y = y \sim x$, $x \sim y \leq x \to y$, $x \odot y \leq x \wedge y \leq x, y$;*
$(ii)$ *$x \leq 1 \sim x = 1 \to x \leq y \to x$;*
$(iii)$ *$x \to y \leq (y \to z) \to (x \to z)$;*
$(iv)$ *$x \to y \leq (z \to x) \to (z \to y)$;*
$(v)$ *if $x \leq y$, then $x \to y = 1$;*
$(vi)$ *if $x \leq y$, then $z \to x \leq z \to x$, $y \to z \leq x \to z$;*
$(vii)$ *if $L$ contains a bottom element 0, then $\neg 0 = 1$, $\neg x = x \to 0$.*

In general, the identity $x \to (y \to z) = y \to (x \to z)$ may not be true in $EQ$-algebras. We call that $EQ$-algebra $L$ has exchange principle if $x \to (y \to z) = y \to (x \to z)$ for any $x, y, z \in L$.

**Theorem 2.1** ([6])**.** *Let $L$ be an $EQ$-algebra. Then the following are equivalent:*
$(i)$ *$L$ is good;*
$(ii)$ *$L$ is separated and satisfies exchange principle;*
$(iii)$ *$L$ is separated and satisfies $x \leq (x \to y) \to y$ for any $x, y \in L$.*

**Theorem 2.2** ([3])**.** *Let $L$ be a residuated $EQ$-algebra. Then for any $x, y, z \in L$:*
$(i)$ *$x \odot y \leq z$ if and only if $x \leq y \to z$;*
$(ii)$ *$(x \odot y) \to z = x \to (y \to z)$.*

**Definition 2.3** ([3])**.** Let $L$ be an $EQ$-algebra and $\emptyset \neq F \subseteq L$. Then $F$ is called a prefilter of $L$ if it satisfies for any $x, y \in L$
$(F1)$ $1 \in F$;
$(F2)$ if $x \in F$, $x \to y \in F$, then $y \in F$.
A prefilter $F$ is said to be a filter if it satisfies:
$(F3)$ if $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for any $x, y, z \in L$.

Note that prefilter and filters are coincide in residuated $EQ$-algebras.

**Definition 2.4** ([5])**.** Let $F$ be a prefilter of $EQ$-algebra $L$. Then we say that $F$ has weak exchange principle, if it satisfies for any $x, y, z \in L$

$$x \to (y \to z) \in F \text{ implies } y \to (x \to z) \in F.$$

**Definition 2.5** ([5,11])**.** Let $F$ be a prefilter of $EQ$-algebra $L$. Then $F$ is called an $n$-fold positive implicative prefilter if it satisfies:

$(F5)$ $x^n \to (y \to z) \in F$, $x^n \to y \in F$ imply $x^n \to z \in F$ for all $x, y, z \in L$.

If $F$ is a filter and satisfies $(F5)$, then $F$ is called an $n$-fold positive implicative filter and 1-fold positive implicative (pre)filter is called positive implicative (pre)filter.

**Theorem 2.3** ([5])**.** *Let $F$ be a prefilter of $EQ$-algebra $L$. Then the following are equivalent.*

$(i)$ *$F$ is a positive implicative prefilter.*

$(ii)$ $(x \wedge (x \to y)) \to y \in F$ *for any* $x, y \in L$.

**Definition 2.6** ([5,11])**.** Let $L$ be an $EQ$-algebra and $\emptyset \neq F \subseteq L$. Then $F$ is called an $n$-fold implicative prefilter if

$(i)$ $1 \in F$;

$(ii)$ $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$ for any $x, y, z \in L$.

1-fold implicative (pre)filter is called implicative (pre)filter.

**Theorem 2.4** ([11])**.** *Let $F \subseteq Q$ be two prefilters of $EQ$-algebra $L$ and $L$ has exchange principle. If $F$ is an $n$-fold positive implicative prefilter, then so is $Q$.*

**Theorem 2.5** ([11])**.** *Let $F$ and $G$ be two prefilters of $EQ$-algebra $L$ such that $F \subseteq G$. If $F$ is an $n$-fold implicative prefilter with the weak exchange principle, then $G$ is an $n$-fold implictive prefilter.*

**Theorem 2.6** ([11])**.** *Let $F$ be an $n$-fold implicative filter of residuated $EQ$-algebra $L$. Then $F$ is an $n$-fold positive implicative filter of $L$.*

A fuzzy set of $L$ is a mapping $\mu : L \to [0, 1]$ and for all $t \in [0, 1]$, the set $\mu_t = \{x \in L \mid \mu(x) \geq t\}$ is called a level subset of $\mu$.

**Definition 2.7** ([13])**.** Let $\mu$ be a fuzzy set of $EQ$-algebra $L$. Then $\mu$ is called a fuzzy prefilter of $L$ if it satisfies for all $x, y \in L$:

$(FF1)$ $\mu(x) \leq \mu(1)$;

$(FF2)$ $\mu(x) \wedge \mu(x \to y) \leq \mu(y)$.

A fuzzy prefilter $\mu$ is called a fuzzy filter if it satisfies:

$(FF3)$ $\mu(x \to y) \leq \mu((x \odot z) \to (y \odot z))$ for all $x, y, z \in L$.

**Definition 2.8** ([13])**.** Let $\mu$ be a fuzzy prefilter of $EQ$-algebra $L$. Then $\mu$ is called a fuzzy positive implicative prefilter of $L$ if it satisfies for all $x, y, z \in L$:

$(FF4)$ $\mu(x \to (y \to z)) \wedge \mu(x \to y) \leq \mu(x \to z)$.

A fuzzy filter $\mu$ of $L$ is called a fuzzy positive implicative filter if it satisfies $(FF4)$.

**Definition 2.9** ([13])**.** Let $\mu$ be a fuzzy set of $EQ$-algebra $L$. Then $\mu$ is called a fuzzy implicative prefilter of $L$ if it satisfies for all $x, y, z \in L$:
$(FF1)$ $\mu(x) \leq \mu(1)$;
$(FF5)$ $\mu(z \to ((x \to y) \to x)) \wedge \mu(z) \leq \mu(x)$.

**Proposition 2.1** ([13])**.** *Let $\mu$ be a fuzzy prefilter of $EQ$-algebra $L$. Then for any $x, y, z \in L$:*
*$(i)$ if $x \leq y$, then $\mu(x) \leq \mu(y)$;*
*$(ii)$ if $\mu$ is a fuzzy filter, then $\mu(x \to y) \wedge \mu(y \to z) \leq \mu(x \to z)$.*

**Theorem 2.7** ([13])**.** *Let $\mu$ be a fuzzy filter of $EQ$-algebra $L$. Then $\mu$ is a fuzzy positive implicative filter of $L$ if and only if $\mu((x \wedge (x \to y)) \to y) = \mu(1)$ for all $x, y \in L$.*

**Note.** From now on, in this paper, $L$ will denote a $EQ$-algebra, unless otherwise stated.

## 3. Fuzzy $n$-Fold Positive Implicative (Pre)filters in EQ-Algebras

In this section, we introduce the concept of fuzzy $n$-fold positive implicative (pre)filters in $EQ$-algebras and we give some relate results.

**Definition 3.1.** Let $\mu$ be a fuzzy prefilter of $L$. Then $\mu$ is called a *fuzzy $n$-fold positive implicative prefilter* of $L$ if for all $x, y, z \in L$, it satisfies
$(FF6)$ $\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z)$.

A fuzzy $n$-fold positive implicative prefilter $\mu$ is called a *fuzzy $n$-fold positive implicative filter* of $L$ if it satisfies $(FF3)$.

*Example* 3.1 ([13]). Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $a$ | $a$ |
| $b$ | 0 | $a$ | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

| $\sim$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $a$ | $a$ |
| $b$ | 0 | $a$ | 1 | 1 |
| 1 | 0 | $a$ | 1 | 1 |

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | 0 | 1 | 1 | 1 |
| $b$ | 0 | $a$ | 1 | 1 |
| 1 | 0 | $a$ | 1 | 1 |

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an $EQ$-algebra. Define fuzzy set $\mu$ in $L$ as follows: $\mu(1) = 0.8$, $\mu(b) = 0.6$ and $\mu(0) = \mu(a) = 0.4$. One can check that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$ for any natural number $n$.

**Theorem 3.1.** *Let $\mu$ be a fuzzy prefilter of $L$. Then $\mu$ is a fuzzy $n$-fold positive implicative prefilter if and only if for any $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an $n$-fold positive implicative prefilter of $L$.*

*Proof.* The proof is straightforward. □

**Theorem 3.2.** *Let $\mu$ be a fuzzy prefilter of $L$. Then $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$ if and only if for any $a \in L$, $\mu^{a,n} : L \to [0, 1]$ is a fuzzy prefilter of $L$, where $\mu^{a,n}(x) = \mu(a^n \to x)$ for any $x \in L$.*

*Proof.* Suppose that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. Since $a^n \to 1 = 1$, we get that $\mu(a^n \to 1) = \mu(1)$ and so $\mu^{a,n}(1) = \mu(a^n \to 1) = \mu(1)$. From $a^n \to x \leq 1$, we have $\mu(a^n \to x) \leq \mu(1)$, that is $\mu^{a,n}(x) \leq \mu(1)$. Therefore, $\mu^{a,n}(x) \leq \mu^{a,n}(1)$, for any $x \in L$. On the other hand, since $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$, we conclude that $\mu(a^n \to (x \to y)) \wedge \mu(a^n \to x) \leq \mu(a^n \to y)$, for any $a, x, y \in L$, that is $\mu^{a,n}(x \to y) \wedge \mu^{a,n}(x) \leq \mu^{a,n}(y)$. Therefore, $\mu^{a,n}$ is a fuzzy prefilter in $L$.

Conversely, let for any $a \in L$, $\mu^{a,n}$ is a fuzzy prefilter of $L$. Then $\mu^{x,n}$ is a fuzzy prefilter of $L$ and so it follows that $\mu^{x,n}(y \to z) \wedge \mu^{x,n}(y) \leq \mu^{x,n}(z)$, for any $y, z \in L$. Hence, $\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z)$ for any $x, y, z \in L$. Therefore, $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. □

**Proposition 3.1.** *Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. Then for any $a \in L$, $\mu^{a,n}$ is the fuzzy prefilter containing $\mu$.*

*Proof.* Assume that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$, then by Theorem 3.2, $\mu^{a,n}$ is a fuzzy prefilter of $L$. Since by Lemma 2.1 $(ii)$, $x \leq a^n \to x$, by Proposition 2.1 $(i)$, we get that $\mu(x) \leq \mu(a^n \to x)$ and so $\mu(x) \leq \mu^{a,n}(x)$. Therefore, $\mu^{a,n}$ is the fuzzy prefilter containing $\mu$. □

**Proposition 3.2.** *Let $\mu$ and $\nu$ be two fuzzy prefilters of $L$. Then for any $a, b \in L$ and natural number $n$, the following statements hold:*

*$(i)$ $\mu^{a,n} = \mu$ if and only if $\mu(a^n) = \mu(1)$;*

*$(ii)$ $a \leq b$ implies that $\mu^{b,n} \subseteq \mu^{a,n}$;*

*$(iii)$ $\mu \subseteq \nu$ implies that $\mu^{a,n} \subseteq \nu^{a,n}$;*

*$(iv)$ $(\mu \cap \nu)^{a,n} = \mu^{a,n} \cap \nu^{a,n}$, $(\mu \cup \nu)^{a,n} = \mu^{a,n} \cup \nu^{a,n}$.*

*Proof.* $(i)$ Let $\mu^{a,n} = \mu$, for $a \in L$ and natural number $n$. Then $\mu(a^n) = \mu^{a,n}(a^n) = \mu(a^n \to a^n) = \mu(1)$. Conversely, assume that $\mu(a^n) = \mu(1)$, since by Lemma 2.1 $(ii)$, $x \leq a^n \to x$, for any $x \in L$ and since $\mu$ is a fuzzy prefilter, we get that $\mu(x) \leq \mu(a^n \to x) = \mu^{a,n}(x)$. Hence, $\mu \subseteq \mu^{a,n}$. On the other hand, since $\mu$ is a fuzzy prefilter, we have $\mu(a^n \to x) = \mu(a^n \to x) \wedge \mu(1) = \mu(a^n \to x) \wedge \mu(a^n) \leq \mu(x)$, for all $x \in L$. Hence, $\mu^{a,n}(a) \leq \mu(x)$ and so $\mu^{a,n} \subseteq \mu$. Therefore, $\mu^{a,n} = \mu$.

$(ii)$ Suppose that $a, b, x \in L$ and $a \leq b$. Then by $(EQ2)$, $a^n \leq b^n$ and so by Lemma 2.1 $(iv)$, $b^n \to x \leq a^n \to x$ and since $\mu$ is a fuzzy prefilter, we get that $\mu(b^n \to x) \leq \mu(a^n \to x)$. Hence, $\mu^{b,n}(x) \leq \mu^{a,n}(x)$ and so $\mu^{b,n} \subseteq \mu^{a,n}$.

$(iii)$ Suppose that $\mu \subseteq \nu$ and $x \in L$, then $\mu(a^n \to x) \leq \nu(a^n \to x)$ and so $\mu^{a,n} \subseteq \nu^{a,n}$.

$(iv)$ For any $x \in L$, we have

$$(\mu \cup \nu)^{a,n}(x) = (\mu \cup \nu)(a^n \to x) = \mu(a^n \to x) \vee \nu(a^n \to x) = \mu^{a,n}(x) \vee \nu^{a,n}(x).$$

Thus, $(\mu \cup \nu)^{a,n} = \mu^{a,n} \cup \nu^{a,n}$. Similarly, $(\mu \cap \nu)^{a,n} = \mu^{a,n} \cap \nu^{a,n}$. □

**Theorem 3.3.** *Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. Then for any $x, y \in L$*

$$\mu(x^n \odot (x \to y)^n \to y) = \mu(1).$$

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$ and $x, y \in L$. Then $\mu(x^n \odot (x \to y)^n \to (x \to y)) \wedge \mu(x^n \odot (x \to y)^n \to x) \leq \mu(x^n \odot (x \to y)^n \to y)$. Since by Lemma 2.1 $(i)$, $x^n \odot (x \to y)^n \leq (x \to y)^n \leq x \to y$ and $x^n \odot (x \to y)^n \leq x^n \leq x$, we conclude that $x^n \odot (x \to y)^n \to (x \to y) = 1$ and $x^n \odot (x \to y)^n \to x = 1$. Hence, $\mu(x^n \odot (x \to y)^n \to (x \to y)) = \mu(1)$ and $\mu(x^n \odot (x \to y)^n \to x) = \mu(1)$ and so $\mu(x^n \odot (x \to y)^n \to y) = \mu(1)$. □

**Theorem 3.4.** *Let $\mu$ be a fuzzy prefilter of $L$. Then the following statements are equivalent:*
($i$) *$\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$;*
($ii$) *$\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for all $x, y \in L$;*
($ii$) *$\mu(x^n \to (x^n \to y)) = \mu(x^n \to y)$ for all $x, y \in L$.*

*Proof.* $(i) \Rightarrow (ii)$ Suppose that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. Then by Definition 3.1, we have

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \to (x^n \to y)) \wedge \mu(1)\\ &= \mu(x^n \to (x^n \to y)) \wedge \mu(x^n \to x^n)\\ &\leq \mu(x^n \to y).\end{aligned}$$

Therefore, $\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$.

$(ii) \Rightarrow (iii)$ Since by Lemma 2.1 $(i)$, $x^n \to y \leq x^n \to (x^n \to y)$, by Proposition 2.1 $(i)$, we get that $\mu(x^n \to y) \leq \mu(x^n \to (x^n \to y))$ and so by $(ii)$, we that conclude that $\mu(x^n \to (x^n \to y)) = \mu(x^n \to y)$.

$(iii) \Rightarrow (i)$ Since by Lemma 2.1 $(iii)$, we have $x^n \to (y \to z) \leq ((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z))$ and $x^n \to y \leq (y \to z) \to (x^n \to z)$, we get that $\mu(x^n \to (y \to z)) \leq \mu(((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)))$ and $\mu(x^n \to y) \leq \mu((y \to z) \to (x^n \to z))$, by Proposition 2.1 $(i)$. Hence, by $(iii)$ we conclude that

$$\begin{aligned}\mu(x^n \to y) \wedge \mu(x^n \to (y \to z)) \leq& \mu((y \to z) \to (x^n \to z))\\ &\wedge \mu(((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)))\\ \leq& \mu(x^n \to (x^n \to z))\\ =& \mu(x^n \to z).\end{aligned}$$

Therefore, $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. □

**Proposition 3.3** ([9])**.** *The following properties are equivalent:*
($i$) *an EQ-algebra $L$ is residuated;*
($ii$) *the EQ-algebra $L$ is good, and*

$$(x \odot y) \to z \leq x \to (y \to z),$$

*holds for all $x, y, z \in L$.*

**Definition 3.2.** Let $L$ be an $EQ$-algebra. Then we say that $L$ has condition $(*)$, if for any $x, y, z \in L$

$$(x \odot y) \to z \leq x \to (y \to z). \quad (*)$$

Note that by Proposition 3.3, every residuated $EQ$-algebra satisfying in the condition $(*)$ and the following example shows that the $EQ$-algebra satisfying the condition $(*)$ may not be a residuated $EQ$-algebra.

*Example* 3.2 ([5]). Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | $a$ | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

,

| $\sim$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | $a$ | $a$ | $a$ |
| $a$ | $a$ | 1 | $b$ | $b$ |
| $b$ | $a$ | $b$ | 1 | 1 |
| 1 | $a$ | $b$ | 1 | 1 |

,

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | 1 | 1 | 1 |
| $b$ | $a$ | $b$ | 1 | 1 |
| 1 | $a$ | $b$ | 1 | 1 |

.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an $EQ$-algebra. It is easily checked that $L$ satisfies $(*)$ for any $x, y, z \in L$ and $L$ is not residuated because $1 \leq 1 \to b$, but $1 \odot 1 \nleq b$.

**Theorem 3.5.** *Let $L$ be an $EQ$-algebra by condition $(*)$ and $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. Then for any $x \in L$ $\mu(x^n \to x^{2n}) = \mu(1)$.*

*Proof.* Assume that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$ and $x \in L$. Since $x^{2n} \to x^{2n} = x^n \odot x^n \to x^n \odot x^n = 1$, we get that $\mu(x^n \odot x^n \to x^n \odot x^n) = \mu(1)$ and since by $(*)$, $x^n \odot x^n \to x^n \odot x^n \leq x^n \to (x^n \to (x^n \odot x^n))$, we conclude that $\mu(x^n \odot x^n \to x^n \odot x^n) \leq \mu(x^n \to (x^n \to (x^n \odot x^n)))$. Hence, $\mu(x^n \to (x^n \to (x^n \odot x^n))) = \mu(1)$ and so by Theorem 3.4, we conclude that $\mu(x^n \to x^n \odot x^n) = \mu(x^n \to x^{2n}) = \mu(1)$. □

**Theorem 3.6.** *Let $\mu$ be a fuzzy prefilter of good $EQ$-algebra $L$. Then the following statements are equivalent:*
*($i$) $\mu$ is a fuzzy $n$-fold positive implicative prefilter;*
*($ii$) $\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
*($iii$) $\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$.*

*Proof.* ($i$) $\Rightarrow$ ($ii$) Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$ and $x, y \in L$. Then by Lemma 2.1 ($i$) we have:

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \to (x^n \to y)) \wedge \mu(1)\\ &= \mu(x^n \to (x \to y)) \wedge \mu(x^n \to x^n)\\ &\leq \mu(x^n \to y).\end{aligned}$$

($ii$) $\Rightarrow$ ($iii$) Let $x, y, z \in L$. Then by Lemma 2.1 ($iv$) $y \to z \leq (x^n \to y) \to (x^n \to z)$ and so by Lemma 2.1 ($vi$), we conclude that $x^n \to (y \to z) \leq x^n \to ((x^n \to y) \to (x^n \to z))$. Hence, by Theorem 2.1, we have

$$x^n \to ((x^n \to y) \to (x^n \to z)) = x^n \to (x^n \to ((x^n \to y) \to z)),$$

and so by Proposition 2.1 $(i)$, we obtain

$$\mu(x^n \to (y \to z)) \leq \mu(x^n \to (x^n \to ((x^n \to y) \to z)).$$

Now, by $(ii)$ we have

$$\mu(x^n \to (x^n \to ((x^n \to y) \to z)) \leq \mu(x^n \to ((x^n \to y) \to z)),$$

and since $\mu(x^n \to ((x^n \to y) \to z)) = \mu((x^n \to y) \to (x^n \to z))$, we conclude that

$$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z)).$$

$(iii) \Rightarrow (i)$ Let $x, y, z \in L$. Then, by $(iii)$, we have

$$\begin{aligned}\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) &\leq \mu((x^n \to y) \to (x^n \to z)) \wedge \mu(x^n \to y)\\ &\leq \mu(x^n \to z).\end{aligned}$$

Therefore, $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. □

**Theorem 3.7.** *Let $\mu$ be a fuzzy prefilter of residuated EQ-algebra $L$. Then the following statements are equivalent:*
$(i)$ *$\mu$ is a fuzzy $n$-fold positive implicative filter of $L$;*
$(ii)$ *$\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
$(ii)$ *$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$.*

*Proof.* The proof is similar to the proof of Theorem 3.6. □

**Definition 3.3** ([13])**.** Let $\mu$ be a fuzzy prefilter of $L$. Then we say that $\mu$ has weak exchange principle, if it satisfies $\mu(x \to (y \to z)) = \mu(y \to (x \to z))$ for all $x, y, z \in L$.

**Theorem 3.8.** *Let $\mu, \nu$ be two fuzzy prefilters of $L$ and satisfy weak exchange principle such that $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If $\mu$ is a fuzzy $n$-fold positive implicative prefilter, then so is $\nu$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. Then, by Theorem 3.1, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an $n$-fold positive implicative prefilter of $L$ and satisfies weak exchange principle because if $x \to (y \to z) \in \mu_t$, then $\mu(y \to (x \to z)) = \mu(x \to (y \to z)) \geq t$ and so $y \to (x \to z) \in \mu_t$. Hence, $\mu_t$ satisfies weak exchange principle and by similar way $\nu_t$ satisfies weak exchange principle. Now, since $\mu \subseteq \nu$, we get that $\mu_t \subseteq \nu_t$, for each $t \in [0,1]$ and so, by Theorem 2.4, for each $t \in [0,1]$, $\emptyset \neq \nu_t$ is an $n$-fold positive implicative prefilter of $L$. Thus, by Theorem 3.1, $\nu$ is a fuzzy $n$-fold positive implicative prefilters of $L$. □

**Theorem 3.9.** *Let $\mu$ be a fuzzy filter of $L$ and satisfy weak exchange principle such that $\mu(x^n \to x^n \odot x^n) = \mu(1)$ and $\mu((x^n \odot (x^n \to y)) \to y) = \mu(1)$ for any $x, y \in L$. Then $\mu$ is a fuzzy $n$-fold positive implicative filter of $L$.*

*Proof.* Let $\mu$ be a fuzzy filter of $L$ such that satisfy weak exchange principle and $x, y, z \in L$. Then by Proposition 2.1 $(ii)$

$$\begin{aligned}\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) &= \mu(y \to (x^n \to z)) \wedge \mu(x^n \to y)\\ &\leq \mu((x^n \odot y) \to (x^n \odot (x^n \to z)))\\ &\wedge \mu((x^n \odot x^n) \to (x^n \odot y))\\ &\leq \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z)))\\ &= \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z))) \wedge \mu(1)\\ &= \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z)))\\ &\wedge \mu(x^n \to (x^n \odot x^n))\\ &\leq \mu(x^n \to (x^n \odot (x^n \to z))\\ &= \mu(x^n \to (x^n \odot (x^n \to z)) \wedge \mu(1)\\ &= \mu(x^n \to (x^n \odot (x^n \to z)))\\ &\wedge \mu((x^n \odot (x^n \to z)) \to z)\\ &\leq \mu(x^n \to z).\end{aligned}$$

Therefore, $\mu$ is a fuzzy $n$-fold positive implicative filter of $L$. ☐

**Proposition 3.4.** *If $\mu$ be a fuzzy positive implicative filter of $L$, then $\mu((x \odot (x \to y)) \to y) = \mu(1)$ for any $x, y \in L$.*

*Proof.* If $\mu$ be a fuzzy positive implicative filter of $L$, then by Theorem 2.7, $\mu((x \wedge (x \to y)) \to y) = \mu(1)$, for any $x, y \in L$. From $x \odot (x \to y) \leq x \wedge (x \to y)$, we have $(x \wedge (x \to y)) \to y \leq (x \odot (x \to y)) \to y$. Hence, $\mu((x \wedge (x \to y)) \to y) \leq \mu((x \odot (x \to y)) \to y)$. Therefore, $\mu((x \odot (x \to y)) \to y) = \mu(1)$. ☐

**Theorem 3.10.** *Let $L$ be an $EQ$-algebra with condition $(*)$ and $\mu$ be a fuzzy filter of $L$ such that satisfy weak exchange principle. Then the following statements are equivalent:*

$(i)$ *$\mu$ is a fuzzy positive implicative filter;*

$(ii)$ *$\mu(x \to x^2) = \mu(1)$ and $\mu((x \odot (x \to y)) \to y) = \mu(1)$ for any $x, y \in L$.*

*Proof.* It follows from Theorem 3.5, Theorem 3.9 and Proposition 3.4, whenever $n = 1$. ☐

**Theorem 3.11.** *Let $\mu$ be a fuzzy filter of residuated $EQ$-algebra $L$. Then $\mu$ is a fuzzy $n$-fold positive implicative filter of $L$ if and only if $\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative filter of residuated $EQ$-algebra $L$. Then $L$ satisfies in condition $(*)$ and so, by Theorem 3.5, $\mu(x^n \to x^{2n}) = \mu(1)$, for any $x \in L$. Conversely, let $\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$. Then, by Theorem

2.2 $(ii)$ and Proposition 2.1 $(ii)$ for $x, y \in L$ we have

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \odot x^n \to y)\\ &= \mu(x^{2n} \to y)\\ &= \mu(x^{2n} \to y) \wedge \mu(1)\\ &= \mu(x^{2n} \to y) \wedge \mu(x^n \to x^{2n})\\ &\leq \mu(x^n \to y).\end{aligned}$$

Therefore, by Theorem 2.2, $\mu$ is a fuzzy $n$-fold positive implicative filter of $L$. □

**Theorem 3.12.** *Let $L$ be an EQ-algebra with condition $(*)$ and $\mu$ be a fuzzy $n$-fold positive implicative filter of $L$. Then $\mu$ is a fuzzy $(n+1)$-fold positive implicative filter of $L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative filter of $L$ and $x, y, z \in L$. Then by Theorem 3.5, $\mu(x^n \to x^{2n}) = \mu(1)$. By Lemma 2.1 $(i)$ and $(vi)$, we have $x^{2n} = x^{n+n} \leq x^{n+1}$ and so $x^{n+1} \to (y \to z) \leq x^{n+n} \to (y \to z)$ and $x^{n+1} \to y \leq x^{n+n} \to y$ and since $x^{n+n} \to (y \to z) = (x^2)^n \to (y \to z)$ and $x^{n+n} \to y = (x^2)^n \to y$, by Proposition 2.1 $(i)$, we get that $\mu(x^{n+1} \to (y \to z)) \leq \mu((x^2)^n \to (y \to z))$ and $\mu(x^{n+1} \to y) \leq \mu((x^2)^n \to y)$. Now, by Proposition 2.1 $(ii)$, we have

$$\begin{aligned}\mu(x^{n+1} \to (y \to z)) \wedge \mu(x^{n+1} \to y) &\leq \mu((x^2)^n \to (y \to z)) \wedge \mu((x^2)^n \to y)\\ &\leq \mu((x^2)^n \to z)\\ &= \mu((x^2)^n \to z) \wedge \mu(1)\\ &\leq \mu((x^2)^n \to z) \wedge \mu(x^n \to x^{2n})\\ &\leq \mu(x^n \to z)\\ &\leq \mu(x^{n+1} \to z).\end{aligned}$$

Therefore, $\mu$ is a fuzzy $(n+1)$-fold positive implicative filter of $L$. □

**Theorem 3.13.** *Let $L$ be a residuated EQ-algebra and $\mu$ be a fuzzy filter of $L$. Then the following statements are equivalent:*

$(i)$ *$\mu$ is a fuzzy $n$-fold positive implicative filter;*
$(ii)$ *$\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
$(iii)$ *$\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
$(iv)$ *$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$;*
$(v)$ *$\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$;*
$(vi)$ *$\mu((x^n \odot y) \to z) \leq \mu((x \wedge y)^n \to z)$ for any $x, y, z \in L$.*

*Proof.* Let $L$ be a residuated $EQ$-algebra and $\mu$ be a fuzzy filter of $L$. Then by Theorem 3.7, Theorem 3.6 and Theorem 3.11, the parts $(i)$, $(ii)$, $(iii)$, $(iv)$ and $(v)$ are equivalent.

$(i) \Rightarrow (vi)$ Let $\mu$ be a fuzzy $n$-fold positive implicative filter of $L$ and $x, y, z \in L$. Then by Lemma 2.1 $(vi)$ and Proposition 2.1 $(i)$, we have

$$\begin{aligned}\mu((x^n \odot y) \to z) &= \mu(x^n \to (y \to z))\\ &= \mu(y \to (x^n \to z))\\ &\leq \mu((x \wedge y) \to (x^n \to z))\\ &= \mu(x^n \to ((x \wedge y) \to z)\\ &\leq \mu((x \wedge y)^n \to ((x \wedge y) \to z))\\ &= \mu((x \wedge y)^n \to ((x \wedge y) \to z)) \wedge \mu(1)\\ &= \mu((x \wedge y)^n \to ((x \wedge y) \to z) \wedge \mu((x \wedge y)^n \to (x \wedge y))\\ &\leq \mu((x \wedge y)^n \to z).\end{aligned}$$

$(vi) \Rightarrow (v)$ Let $x, y \in L$. Then by $(vi)$, we have

$$\begin{aligned}\mu(x^{n+1} \to y) &= \mu((x^n \odot x) \to y)\\ &\leq \mu((x \wedge x)^n \to y)\\ &= \mu(x^n \to y),\end{aligned}$$

and since $(v)$ and $(i)$ are equivalent, we conclude that $(vi)$ and $(i)$ are equivalent and the proof is complete. □

## 4. Fuzzy $n$-Fold Implicative (Pre)filters in EQ-Algebras

In this section we introduce the concept of fuzzy $n$-fold implicative (pre)filters in $EQ$-algebras and we give some related results.

**Definition 4.1.** Let $\mu$ be a fuzzy set of $L$. Then $\mu$ is called a *fuzzy n-fold implicative prefilter* of $L$ if it satisfies for all $x, y, z \in L$

$(FF1)$ $\mu(x) \leq \mu(1)$;

$(FF7)$ $\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu(x)$.

A fuzzy $n$-fold implicative prefilter $\mu$ is called a *fuzzy n-fold implicative filter* of $L$ if it satisfies $(FF3)$.

*Example* 4.1 ([13]). Let $L = \{0, a, b, c, 1\}$ be a chain with Cayley tables as follows:

| $\odot$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | 0 | 0 | $b$ |
| $c$ | 0 | 0 | 0 | 0 | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

,

| $\sim$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $b$ | $b$ | $b$ |
| $b$ | 0 | $b$ | 1 | $c$ | $c$ |
| $c$ | 0 | $b$ | $c$ | 1 | 1 |
| 1 | 0 | $b$ | $c$ | 1 | 1 |

,

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 0 | 1 | 1 | 1 | 1 |
| $b$ | 0 | $b$ | 1 | 1 | 1 |
| $c$ | 0 | $b$ | $c$ | 1 | 1 |
| 1 | 0 | $b$ | $c$ | 1 | 1 |

.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an $EQ$-algebra. Define a fuzzy set $\mu$ in $L$ as follows: $\mu(1) = \mu(a) = \mu(b) = \mu(c) = t_1$ and $\mu(0) = t_2$, where $0 \leq t_2 < t_1 \leq 1$. We can see that $\mu$ is a fuzzy 2-fold implicative prefilter of $L$.

**Theorem 4.1.** *Let $\mu$ be a fuzzy set of $L$. Then $\mu$ is a fuzzy $n$-fold implicative prefilter if and only if for any $t \in [0,1]$, $\mu_t$ is an $n$-fold implicative prefilter of $L$.*

*Proof.* The proof is straightforward. □

**Theorem 4.2.** *Every fuzzy $n$-fold implicative prefilter of $L$ is a fuzzy prefilter.*

*Proof.* Suppose that $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$. Then $\mu(x) \leq \mu(1)$, for all $x \in L$. Firstly, we prove if $x \leq y$, then $\mu(x) \leq \mu(y)$. Let $x, y \in L$ such that $x \leq y$. Then by Lemma 2.1 $(i)$ and $(vi)$, we have $y \leq (y^n \to y) \to y$ and so $x \to y \leq x \to ((y^n \to y) \to y)$ and since by Lemma 2.1 $(v)$, $x \to y = 1$, we get that $x \to ((y^n \to y) \to y) = 1$. Hence, $\mu(x \to ((y^n \to y) \to y)) = \mu(1)$ and since $\mu$ is a fuzzy $n$-fold implicative prefilter, we have

$$\mu(x) = \mu(x) \wedge \mu(1) = \mu(x) \wedge \mu(x \to ((y^n \to y) \to y)) \leq \mu(y).$$

Now, by $y \leq 1 \to y$, we get that $x \to y \leq x \to (1 \to y)$ and so $\mu(x \to y) \leq \mu(x \to (1 \to y))$. Hence, by Definition 4.1, we have

$$\begin{aligned} \mu(x \to y) \wedge \mu(x) &\leq \mu(x \to (1 \to y)) \wedge \mu(x) \\ &= \mu(x \to ((y^n \to 1) \to y)) \wedge \mu(x) \\ &\leq \mu(y). \end{aligned}$$

Therefore, $\mu$ is a fuzzy prefilter of $L$. □

**Theorem 4.3.** *Let $\mu$ be a fuzzy (pre)filter of $L$. The following statements are equivalent:*

$(i)$ *$\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$;*
$(ii)$ *$\mu((x^n \to y) \to x) \leq \mu(x)$ for all $x, y \in L$;*
$(iii)$ *$\mu((x^n \to y) \to x) = \mu(x)$ for all $x, y \in L$.*

*Proof.* $(i) \Rightarrow (ii)$ Suppose that $\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$. Then

$$\mu(1 \to ((x^n \to y) \to x)) = \mu(1 \to ((x^n \to y) \to x) \wedge \mu(1) \leq \mu(x).$$

Since by Lemma 2.1 $(ii)$, $(x^n \to y) \to x \leq 1 \to ((x^n \to y) \to x)$, by Proposition 2.1 $(i)$, we conclude that $\mu((x^n \to y) \to x) \leq \mu(1 \to ((x^n \to y) \to x))$. Consequently, we have $\mu((x^n \to y) \to x) \leq \mu(x)$.

$(ii) \Rightarrow (iii)$ Since by Lemma 2.1 $(ii)$, $x \leq (x^n \to y) \to x$, it follows that by Proposition 2.1 $(i)$, $\mu(x) \leq \mu((x^n \to y) \to x)$. Combining $(ii)$, we get $\mu((x^n \to y) \to x) = \mu(x)$.

$(iii) \Rightarrow (i)$ Let $\mu$ be a fuzzy prefilter of $L$. Then for $x, y, z \in L$, $\mu(x) \leq \mu(1)$ and by $(iii)$ we have

$$\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu((x^n \to y) \to x) = \mu(x).$$

Therefore, $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$. □

**Theorem 4.4.** *Let $\mu$ be a fuzzy $n$-fold implicative prefilter of residuated EQ-algebra $L$. Then $\mu$ is a fuzzy $n$-fold positive implicative prefilter.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold implicative prefilter of residuated $EQ$-algebra $L$. Then by Theorem 4.1, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an $n$-fold implicative prefilter of $L$ and so by Theorem 2.6, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a $n$-fold positive implicative prefilter of $L$. Therefore, by Theorem 3.1, $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$. □

**Theorem 4.5.** *Let $L$ be an $EQ$-algebra with bottom element $0$ and $\mu$ be a fuzzy (pre)filter of $L$. The following statements are equivalent:*
(*i*) *$\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$;*
(*ii*) *$\mu(\neg x^n \to x) \leq \mu(x)$ for all $x \in L$;*
(*iii*) *$\mu(\neg x^n \to x) = \mu(x)$ for all $x \in L$.*

*Proof.* $(i) \Rightarrow (ii)$ Assume that $\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$. Then by Theorem 4.3, for all $x \in L$,

$$\mu(\neg x^n \to x) = \mu((x^n \to 0) \to x) \leq \mu(x).$$

$(ii) \Rightarrow (iii)$ Since by Lemma 2.1 $(ii)$, $x \leq \neg x^n \to x$, we get that $\mu(x) \leq \mu(\neg x^n \to x)$ as $\mu$ is a fuzzy prefilter of $L$. Combining $(ii)$, we get $\mu(\neg x^n \to x) = \mu(x)$.

$(iii) \Rightarrow (i)$ Let $\mu$ be a fuzzy prefilter of $L$. Then by Lemma 2.1 $(vi)$ and by $0 \leq y$, we have $\neg x^n = x^n \to 0 \leq x^n \to y$ and so $(x^n \to y) \to x \leq \neg x^n \to x$. Hence, by Proposition 2.1 $(i)$, $\mu((x^n \to y) \to x) \leq \mu(\neg x^n \to x)$. Combining $(iii)$, we get that $\mu((x^n \to y) \to x) \leq \mu(x)$. Therefore, by Theorem 4.3, $\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$. □

**Theorem 4.6.** *Let $\mu$ and $\nu$ be two fuzzy (pre)filters of $L$ such that $\mu \subseteq \nu$. If $\mu$ is a fuzzy $n$-fold implicative (pre)filter with weak exchange principle of $L$, then $\nu$ is a fuzzy $n$-fold implicative (pre)filter of $L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold implicative (pre)filter of $L$. Then by Theorem 4.1, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an $n$-fold implicative (pre)filter of $L$ and since $\mu \subseteq \nu$, we get that $\mu_t \subseteq \nu_t$, for each $t \in [0,1]$. Now, since for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an $n$-fold implicative (pre)filter of $L$, by Theorem 2.5, we conclude that for each $t \in [0,1]$, $\emptyset \neq \nu_t$ is an $n$-fold implicative (pre)filter of $L$. Therefore, by Theorem 3.1, $\nu$ is a fuzzy $n$-fold implicative (pre)filters of $L$. □

**Theorem 4.7.** *Let $\mu$ be a fuzzy (pre)filter of $EQ$-algebra $L$. If $\mu$ is a fuzzy $n$-fold implicative (pre)filter of $L$, then $\mu$ is a fuzzy $(n+1)$-fold implicative (pre)filter of $L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative (pre)filter of $L$. Since $x^{n+1} \leq x^n$, we get that by Lemma 2.1 $(vi)$, $x^n \to y \leq x^{n+1} \to y$ and so $(x^{n+1} \to y) \to x \leq (x^n \to y) \to x$. Hence, by Proposition 2.1 $(i)$, we have $\mu((x^{n+1} \to y) \to x) \leq \mu((x^n \to y) \to x)$ and since by Theorem 4.5, $\mu((x^n \to y) \to x) \leq \mu(x)$, we conclude that $\mu((x^{n+1} \to y) \to x) \leq \mu(x)$. Therefore, by Theorem 4.5, $\mu$ is a fuzzy $(n+1)$-fold implicative (pre)filter of $L$. □

**Theorem 4.8.** *Let $L$ be an EQ-algebra with a bottom element $0$ and $\mu$ be a fuzzy prefilter of $L$ with weak exchange principle. Then the following statements are equivalent:*

($i$) *$\mu$ is a fuzzy $n$-fold implicative prefilter of $L$;*

($ii$) *$\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z)$ for all $x, y, z \in L$;*

($iii$) *$\mu(x \to (\neg z^n \to z)) \leq \mu(x \to z)$ for all $x, z \in L$;*

($iv$) *$\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$ for all $x, z \in L$.*

*Proof.* ($i$) $\Rightarrow$ ($ii$) Let $\mu$ be a fuzzy $n$-fold implicative prefilter of $L$ and $x, y, z \in L$. Then by Lemma 2.1 ($iii$) and ($iv$), $y \to z \leq (x \to y) \to (x \to z)$ and $\neg z^n \to (x \to y) \leq ((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z))$, and so by Proposition 2.1 ($i$), $\mu(y \to z) \leq \mu((x \to y) \to (x \to z))$ and $\mu(\neg z^n \to (x \to y)) \leq \mu(((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z)))$. Now, by weak exchange principle we have

$$\begin{aligned}\mu(y \to z) \wedge \mu(x \to (\neg z^n \to y)) &= \mu(y \to z) \wedge \mu(\neg z^n \to (x \to y)) \\ &\leq \mu((x \to y) \to (x \to z)) \\ &\wedge \mu(((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z))) \\ &\leq \mu(\neg z^n \to (x \to z)),\end{aligned}$$

and since by Lemma 2.1 ($ii$), $z \leq x \to z$, by ($EQ2$) we get that $z^n \leq (x \to z)^n$ and so $\neg(x \to z)^n \leq \neg z^n$. Hence, $\neg z^n \to (x \to z) \leq \neg(x \to z)^n \to (x \to z)$ and so by Proposition 2.1 ($i$), $\mu(\neg z^n \to (x \to z)) \leq \mu(\neg(x \to z)^n \to (x \to z))$ and since $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$, by Theorem 4.5 we conclude that $\mu(\neg(x \to z)^n \to (x \to z)) \leq \mu(x \to z)$. Consequently, we obtain

$$\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z).$$

($ii$) $\Rightarrow$ ($i$) Suppose that $\mu$ satisfies $\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z)$, for all $x, y, z \in L$. Then

$$\begin{aligned}\mu(\neg x^n \to x) &= \mu(\neg x^n \to x) \wedge \mu(1) \\ &\leq \mu(1 \to (\neg x^n \to x)) \wedge \mu(x \to x) \\ &\leq \mu(1 \to x) \\ &= \mu(1 \to x) \wedge \mu(1) \\ &\leq \mu(x).\end{aligned}$$

Therefore, by Theorem 4.5, $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$.

($ii$) $\Rightarrow$ ($iii$) Let $x, z \in L$. Then by ($ii$), we have:

$$\begin{aligned}\mu(x \to (\neg z^n \to z)) &= \mu(x \to (\neg z^n \to z)) \wedge \mu(1) \\ &= \mu(x \to (\neg z^n \to z)) \wedge \mu(z \to z) \\ &\leq \mu(x \to z).\end{aligned}$$

$(iii) \Rightarrow (iv)$ From $z \leq \neg z^n \to z$, it follows that $x \to z \leq x \to (\neg z^n \to z)$. Then $\mu(x \to z) \leq \mu(x \to (\neg z^n \to z))$ as $\mu$ is a fuzzy prefilter. Combining $(iii)$, we get $\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$.

$(iv) \Rightarrow (i)$ Let $\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$, for all $x, z \in L$. Then $\mu(1 \to (\neg x^n \to x)) = \mu(1 \to x)$ and since $\mu$ is a fuzzy prefilter, we get that $\mu(1 \to x) = \mu(1 \to x) \wedge \mu(1) \leq \mu(x)$ and so $\mu(1 \to (\neg x^n \to x)) \leq \mu(x)$. Moreover, from $\neg x^n \to x \leq 1 \to (\neg x^n \to x)$, it follows that by Proposition 2.1 $(i)$, $\mu(\neg x^n \to x) \leq \mu(1 \to (\neg x^n \to x))$. Consequently, we obtain $\mu(\neg x^n \to x) \leq \mu(x)$. Therefore, by Theorem 4.5 $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$. □

**Theorem 4.9.** *Let $\mu$ be a fuzzy $n$-fold implicative prefilter with the weak exchange principle. Then for any $x, y \in L$*

$$\mu((x^n \to y) \to y) \leq \mu((y \to x) \to x).$$

*Proof.* Let $\mu$ be a fuzzy $n$-fold implicative prefilter of $L$ and put $u = (y \to x) \to x$. Then, by Lemma 2.1 $(iii)$, $(x^n \to y) \to y \leq (y \to x) \to ((x^n \to y) \to x)$ and so, by Proposition 2.1 $(i)$,

$$\begin{aligned} \mu((x^n \to y) \to y) &\leq \mu((y \to x) \to ((x^n \to y) \to x)) \\ &= \mu((x^n \to y) \to ((y \to x) \to x)) \\ &= \mu((x^n \to y) \to u), \end{aligned}$$

by Lemma 2.1 $(ii)$, we have $x \leq (y \to x) \to x = u$ and so by $(EQ2)$ we get that $x^n \leq u^n$. Hence, by Lemma 2.1 $(vi)$, we have $u^n \to y \leq x^n \to y$ and so $(x^n \to y) \to u \leq (u^n \to y) \to u$. Hence, by 2.1 $(i)$ and Theorem 4.3, $\mu((x^n \to y) \to u) \leq \mu((u^n \to y) \to u) \leq \mu(u)$. Consequently, we obtain

$$\mu((x^n \to y) \to y) \leq \mu((y \to x) \to x).$$ □

**Theorem 4.10.** *Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. If $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$ for any $x, y \in L$, then $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$.*

*Proof.* Suppose that $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$ and $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$, for any $x, y \in L$. Then by Lemma 2.1 $(ii)$ and $(vi)$, we have $y \leq x^n \to y$ and $(x^n \to y) \to x \leq y \to x$ and so by Proposition 2.1 $(i)$, we get that

$$\mu((x^n \to y) \to x) \leq \mu(y \to x). \tag{4.1}$$

Moreover, since by Lemma 2.1 $(iii)$, $(x^n \to y) \to x \leq (x \to y) \to ((x^n \to y) \to y)$, by Proposition 2.1 $(i)$, we get that $\mu((x^n \to y) \to x) \leq \mu((x \to y) \to ((x^n \to y) \to y))$ and so by (4.1), we have

$$\mu((x^n \to y) \to x) \leq \mu((x \to y) \to ((x^n \to y) \to y)) \wedge \mu(y \to x). \tag{4.2}$$

Now, since by Lemma 2.1 $(i)$ and $(vi)$, $(x \to y)^n \leq x \to y$ and $(x \to y) \to ((x^n \to y) \to y) \leq ((x \to y)^n \to ((x^n \to y) \to y))$, we conclude that

$$\mu((x \to y) \to ((x^n \to y) \to y)) \leq \mu((x \to y)^n \to ((x^n \to y) \to y)),$$

and since $x^n \leq x$ and $(x \to y)^n \leq x \to y \leq x^n \to y$, we get that $(x \to y)^n \to (x^n \to y) = 1$ and so $\mu((x \to y)^n \to (x^n \to y)) = \mu(1)$ and since $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$ and since $\mu$ is a fuzzy $n$-fold positive implicative prefilter of $L$, we get that

$$\begin{aligned}\mu((x \to y)^n \to ((x^n \to y) \to y)) &= \mu((x \to y)^n \to ((x^n \to y) \to y)) \wedge \mu(1)\\ &= \mu((x \to y)^n \to [(x^n \to y) \to y])\\ &\wedge \mu((x \to y)^n \to (x^n \to y))\\ &\leq \mu((x \to y)^n \to y)\\ &\leq \mu((y \to x) \to x).\end{aligned}$$

Hence, by (4.2), we conclude that

$$\mu((x^n \to y) \to x) \leq \mu((y \to x) \to x) \wedge \mu(y \to x) \leq \mu(x).$$

Therefore, by Theorem 4.3, $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$. □

**Theorem 4.11.** *Let $\mu$ be fuzzy positive implicative prefilter of $L$ with the weak exchange principle. Then the following are equivalent:*
*$(i)$ $\mu$ is a fuzzy implicative prefilter of $L$;*
*$(ii)$ $\mu((x \to y) \to y) \leq \mu((y \to x) \to x)$ for all $x, y \in L$.*

*Proof.* It follows from Theorem 4.9 and Theorem 4.10, whenever $n = 1$. □

**Theorem 4.12.** *Let $L$ be an EQ-algebra with a bottom element 0 and $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. If $\mu(\neg(\neg x)^n) \leq \mu(x)$ for any $x \in L$, then $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold positive implicative prefilter of $L$. Then for any $x \in L$, by Lemma 2.1 $(iii)$, $\neg x^n \to x \leq (x \to 0) \to (\neg x^n \to 0) = \neg x \to (\neg x^n \to 0)$. Hence, by Proposition 2.1 $(i)$, we have $\mu(\neg x^n \to x) \leq \mu(\neg x \to (\neg x^n \to 0))$ and since $(\neg x)^n \leq \neg x$, by Lemma 2.1 $(vi)$, we get that $\neg x \to (\neg x^n \to 0) \leq (\neg x)^n \to (\neg x^n \to 0)$ and so $\mu(\neg x \to (\neg x^n \to 0)) \leq \mu((\neg x)^n \to (\neg x^n \to 0))$. Hence, $\mu(\neg x^n \to x) \leq \mu((\neg x)^n \to (\neg x^n \to 0))$. Now, since $x^n \leq x$, we have $\neg x \leq \neg x^n$ and so $(\neg x)^n \leq \neg x$, we conclude that $(\neg x)^n \leq \neg x^n$ and so $(\neg x)^n \to \neg x^n = 1$ and since $\mu$ is a fuzzy $n$-fold positive implicative prefilter, we conclude that

$$\begin{aligned}\mu((\neg x)^n \to (\neg x^n \to 0)) &= \mu((\neg x)^n \to (\neg x^n \to 0)) \wedge \mu(1)\\ &= \mu((\neg x)^n \to (\neg x^n \to 0)) \wedge \mu((\neg x)^n \to \neg x^n)\\ &\leq \mu((\neg x)^n \to 0),\end{aligned}$$

and since by hypothesis $\mu(\neg(\neg x)^n) = \mu((\neg x)^n \to 0) \leq \mu(x)$, we get that $\mu(\neg x^n \to x) \leq \mu(x)$. Therefore, by Theorem 4.5, $\mu$ is a fuzzy $n$-fold implicative prefilter of $L$. □

**Theorem 4.13.** *Let $L$ be an EQ-algebra with a bottom element $0$ and $\mu$ be a fuzzy $n$-fold implicative prefilter of $L$. Then $\mu(\neg\neg x^n) \leq \mu(x)$ for all $x \in L$.*

*Proof.* Let $\mu$ be a fuzzy $n$-fold implicative prefilter of $L$ and $x \in L$. Then by Lemma 2.1 $(vi)$, $\neg\neg x^n = \neg x^n \to 0 \leq \neg x^n \to x$ and so by Proposition 2.1 $(i)$, $\mu(\neg\neg x^n) \leq \mu(\neg x^n \to x)$ and since by Theorem 4.5, $\mu(\neg x^n \to x) \leq \mu(x)$, we conclude that $\mu(\neg\neg x^n) \leq \mu(x)$. □

## 5. Conclusion

In this paper, the notion of fuzzy $n$-fold positive implicative and fuzzy $n$-fold implicative (pre)filters in *EQ*-algebras are introduced and several properties of them are stated. Using the concept of level subsets, some characterizations of fuzzy $n$-fold (positive) implicative (pre)filters are proved. Furthermore, we discussed the relationship between fuzzy $n$-fold positive implicative (pre)filters and fuzzy $n$-fold implicative (pre)filters and verified that under what conditions the fuzzy $n$-fold positive implicative (pre)filters and fuzzy $n$-fold implicative (pre)filters are equivalent in *EQ*-algebras. In this article, there are theorems and propositions that have been proved by adding some conditions to an EQ-algebra. One of the important questions for future research is how we can prove these theorems without these conditions or with less conditions. Also, how to define the notions of fuzzy $n$-fold fantastic filters in *EQ*-algebras? What is the relation between fuzzy $n$-fold fantastic filters and other types fuzzy filters?

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# NEW TAUBERIAN THEOREMS FOR CESÀRO SUMMABLE TRIPLE SEQUENCES OF FUZZY NUMBERS

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Abstract. The purpose of this paper is to establish new results on Tauberian theorem for Cesàro summability of triple sequences of fuzzy numbers. Besides, we extend and unify several results in the available literature. Furthermore, a huge number of special cases, theorems and their implications are proved. We show some illustrative examples in support of the results obtained in this paper.

## 1. Introduction

The notion of the fuzzy set was originally introduced by Zadeh [23]. Later, Matloka [11] established bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence is bounded. Then, Nanda [12] studied the spaces of bounded and convergent sequences of fuzzy numbers and proved that every Cauchy sequence of fuzzy numbers is convergent. Subrahmanyam [14] presented the notion of Cesàro summability of sequences of fuzzy numbers and established Tauberian hypotheses identified with the Cesàro summability method. Talo and Çanak [15] introduced the necessary and sufficient Tauberian conditions, under which convergence follows from Cesàro convergence of sequences of fuzzy numbers. Altın et al. [1] studied the concept of statistical summability by $(C, 1)$-mean for sequences of fuzzy numbers and obtained a Tauberian theorem on that basis. Talo and Başar [16] introduced the concept of slow decreasing sequence for fuzzy numbers and have proved that Cesàro summable sequence $(X_n)$ is convergent, if $(X_n)$ is slowly decreasing. Çanak [3] established the concept of the slow oscillation (that is, both slowly decreasing and

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slowly increasing) sequences for fuzzy numbers and proved that Cesàro summable sequence $(X_n)$ is convergent if $(X_n)$ is slowly oscillating. Later on, many researchers have investigated on sequences and sequences of fuzzy numbers for proving Tauberian theorems. Different classes of sequences and sequences of fuzzy numbers have been presented and studied by Tripathy et al. [22], Dutta [4], Dutta [5], Dutta and Bilgin [6], Tripathy and Debnath [21], Dutta and Başar [7], Jena et al. [9], Jena et al. [10] and many others. Çanak [3] introduced Tauberian theorem for Cesàro summability of sequences of fuzzy numbers. Later on, Jena et al. [8] proved some Tauberian theorems on Cesàro summable double sequences of fuzzy numbers and proved some interesting results. The reader can refer to the monograph [2] and the papers [17–19] and [20] on the classical sequence spaces and related topics. Motivated by the above-mentioned works, in this paper we present the notion of $((C,1,1,1)X)$-summability of a triple sequences of fuzzy numbers defined in Definition 2.11. This paper is organized in two principal parts. In the first one, we provide the necessary definitions which are useful for the development of this paper, and the second one, we show theorems, lemmas and corollaries that we obtained.

## 2. Notations and Definitions

In this section, we recall some well-know notions which are useful for the developing of this paper. Besides, we define some new notions on Cesàro means $(C,1,1,1)$ of triple sequences $(X_{mng})$ of fuzzy numbers.

**Definition 2.1.** Let $D$ denote the set of all closed and bounded intervals $X=[x_1,x_2]$ on the real line $\mathbb{R}$. For $X,Y\in D$, we define

$$d(X,Y)=\max\{|x_1-y_1|,|x_2-y_2|\},$$

where $X=[x_1,x_2]$ and $Y=[y_1,y_2]$

*Remark* 2.1. It is known that $(D,d)$ is a complete metric space.

**Definition 2.2.** A fuzzy number $X$ is a fuzzy set on $\mathbb{R}$ and is a mapping $X:\mathbb{R}\to[0,1]$ associating each number $t$ with its grade of membership $X(t)$.

**Definition 2.3.** A fuzzy number $X$ is said to be convex if,

$$X(t)=\min\{X(s),X(r)\},\quad s<t<r.$$

**Definition 2.4.** If there exists $t_0\in\mathbb{R}$, such that $X(t_0)=1$, then the fuzzy number $X$ is called normal. Besides, a fuzzy number $X$ is said to be upper semi-continuous if, for each $X^{-1}([0,x+\varepsilon])$ for all $x\in(0,1)$, is open in the usual topology of $\mathbb{R}$. The set of all upper semicontinuous, normal, convex fuzzy numbers is denoted by $\mathbb{R}([0,1])$. For $\alpha\in(0,1]$, $\alpha$-level set $X^\alpha$ of fuzzy number $X$ is defined by

$$X^\alpha=\{t\in\mathbb{R}:X(t)\geq\alpha\}.$$

**Definition 2.5.** The set $X^0$ is defined as the closure of the following set $\{t \in \mathbb{R} : X(t) > 0\}$. We define $\bar{d} : \mathbb{R}([0,1]) \times \mathbb{R}([0,1]) \to \mathbb{R}_+ \cup \{0\}$, by

$$\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

**Definition 2.6.** A triple sequence $(X_{mng})$ of fuzzy numbers is a function $X : \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \to \mathbb{R}([0,1])$ and is said to be convergent to a fuzzy number $X_0$ if, for every $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$\bar{d}(X_{mng}, X_0) < \varepsilon, \quad \text{as } m, n, g \ge n_0.$$

*Remark* 2.2. We will denote

$$\begin{aligned} \Delta_n X_{mng} =& \bar{d}(X_{mng}, X_{m,n-1,g}), \\ \Delta_m X_{mng} =& \bar{d}(X_{mng}, X_{m-1,n,g}), \\ \Delta_g X_{mng} =& \bar{d}(X_{mng}, X_{m,n,g-1}) \end{aligned}$$

and

$$\Delta_{m,n,g} X_{mng} = \bar{d}(X_{mng}, X_{m-1,n,g}) - \bar{d}(X_{m,n-1,g-1}, X_{m-1,n-1,g-1}), \quad X_{-1} = 0.$$

**Definition 2.7.** A triple sequence $(X_{mng})$ of fuzzy numbers is said to be bounded, if there exists a positive number $K > 0$ such that

$$\bar{d}(X_{mng}, X_0) \le K, \quad \text{as } m, n, g \in \mathbb{N} \cup \{0\}.$$

**Definition 2.8.** The Cesàro transform $(C,1,1,1)X$ of triple sequences $(X_{mng})$ of fuzzy numbers is defined by

$$\begin{aligned} ((C,1,1,1)X)_{mng} &= \frac{1}{(m+1)(n+1)(g+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \sum_{h=0}^{g} X_{pqh} \\ &= \sum_{p=0}^{m} \sum_{q=0}^{n} \sum_{h=0}^{g} \frac{Y_{pqh}^{(1,1,1)}}{pqh} + X_{000}. \end{aligned} \tag{2.1}$$

Analogous to (2.1), we can define the $(C,1,0,0)$-, $(C,0,0,1)$- and $(C,0,1,0)$- transforms of a sequence $(X_{mng})$ as follows

$$\begin{aligned} ((C,1,0,0)X)_{mng} =& \frac{1}{m+1} \sum_{p=0}^{m} X_{png}, \\ ((C,0,1,0)X)_{mng} =& \frac{1}{n+1} \sum_{q=0}^{n} X_{mqg}, \\ ((C,0,0,1)X)_{mng} =& \frac{1}{g+1} \sum_{h=0}^{g} X_{mnh}, \end{aligned} \tag{2.2}$$

respectively. Additionally, analogues to (2.1) and (2.2), we can define the $(C,1,1,0)$-, $(C,0,1,1)$- and $(C,1,0,1)$-transforms of a sequence $(X_{mng})$ as follows

$$\begin{aligned}((C,1,1,0)X)_{mng} =&\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}X_{pqg},\\ ((C,0,1,1)X)_{mng} =&\frac{1}{(n+1)(q+1)}\sum_{q=0}^{n}\sum_{h=0}^{g}X_{mqh},\\ ((C,1,0,1)X)_{mng} =&\frac{1}{(m+1)(q+1)}\sum_{p=0}^{m}\sum_{h=0}^{g}X_{pnh},\end{aligned}$$

respectively.

*Remark* 2.3. A triple sequence $X=(X_{mng})$ of fuzzy numbers is $(C,1,1,1)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(((C,1,1,1)X)_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

Similarly, we say that it is $(C,1,0,0)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(((C,1,0,0)X)_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty,$$

$(C,0,0,1)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(((C,0,0,1)X)_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty,$$

and $(C,0,1,0)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(((C,0,1,0)X)_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

We say that it is $(C,1,1,0)$-summable, $(C,0,1,1)$-summable and $(C,1,0,1)$- summable to a fuzzy number $L$ if for every $\varepsilon>0$ we have $\bar{d}(((C,1,1,0)X)_{mng},L)<\varepsilon$, $\bar{d}(((C,0,1,1)X)_{mng},L)<\varepsilon$ and $\bar{d}(((C,1,0,1)X)_{mng},L)<\varepsilon$ as $m,n,g\to\infty$, respectively.

**Definition 2.9.** For each non-negative integers $k$, $r$ and $j$, we define $((C,k,j,r)X)_{mng}$ as follows:

$$((C,k,j,r)X)_{mng} =\begin{cases}\dfrac{1}{(m+1)(n+1)(g+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g}((C,k-1,j-1,r-1)X)_{phq}, & k,r,j\geq 1,\\ X_{mng}, & k,r,j=0.\end{cases}$$

**Definition 2.10.** A triple sequence $X=(X_{mng})$ of fuzzy numbers is said to be $(C,k,r,j)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(((C,k,j,r)X)_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

*Remark* 2.4. If $k = 1$, $r = 1$ and $j = 1$, then $(C, k, r, j)$-summability reduces to $(C, 1, 1, 1)$-summability. Moreover, if $k \neq 0$, $r = 0$ and $j = 0$, then $(C, r, k, j)$-summability reduces to $(C, k, 0, 0)$-summability. If $k = 0$, $r \neq 0$ and $j = 0$, then $(C, r, k, j)$-summability reduces to $(C, 0, r, 0)$-summability. Finally, if $k = 0$, $r = 0$ and $j \neq 0$, then $(C, r, k, j)$-summability reduces to $(C, 0, 0, j)$-summability.

*Remark* 2.5. Note that, Cesàro summability of $X = (X_{mng})$ refers $(C, 1, 1, 1)$ and $(C, k, r, j)$-summability of $X = (X_{mng})$.

*Remark* 2.6. It can also be noted that, the convergence of a triple sequence $X = (X_{mng})$ of fuzzy numbers implies the Cesàro summability of $X = (X_{mng})$, but the converse is not generally true as can be seen in the following example.

*Example* 2.1. Consider a function $f(a, b, c) = e^{7a} \sin(11b)$. The sequence $(X_{mng})$ of fuzzy numbers which is the sequence of coefficients in the Taylor's series expansion of the function $f(a, b, c)$ about origin is Cesàro summable but not convergent. For the proof of converse part, certain conditions are presented in terms of oscillatory behavior of triple sequence $X = (X_{mng})$ of fuzzy numbers.

**Definition 2.11.** Let us define $(X_{mng})$ as

$$X_{mng} = Y_{mng}^{(1,1,1)} + \sum_{p=1}^{m} \sum_{q=1}^{n} \sum_{h=1}^{g} \frac{Y_{pqh}^{(1,1,1)}}{pqh} + X_{000}, \quad m, n, g \in \mathbb{N},$$

where

$$X_{mng} - ((C, 1, 1, 1)X)_{mng} = Y_{mng}^{(1,1,1)}(\Delta X) = \frac{1}{(m+1)(n+1)(g+1)} \sum_{p=1}^{m} \sum_{q=1}^{n} \sum_{h=1}^{g} pqh(\Delta_{p,q,h} X_{pqh}). \tag{2.3}$$

Further, in analogy to Kronecker identity for a single sequence of fuzzy numbers, we can write

$$Y_{mng}^{(1,0,0)}(\Delta X) = \frac{1}{(m+1)} \sum_{p=1}^{m} p(\Delta_p, X_{png}), \tag{2.4}$$

$$Y_{mng}^{(0,1,0)}(\Delta X) = \frac{1}{(n+1)} \sum_{q=1}^{n} q(\Delta_q, X_{mqg}), \tag{2.5}$$

$$Y_{mng}^{(0,0,1)}(\Delta X) = \frac{1}{(g+1)} \sum_{h=1}^{g} h(\Delta_h X_{mnh}), \tag{2.6}$$

as the $(C, 1, 0, 0)$-mean of the sequence $(m\Delta_m X_{mng})$ of fuzzy numbers, $(C, 0, 1, 0)$-mean of the sequence $(n\Delta_n X_{mng})$ of fuzzy numbers and $(C, 0, 0, 1)$-mean of the sequence $(g\Delta_g X_{mng})$ of fuzzy numbers, respectively.

We define $Y_{mng}^{(1,1,0)}$, $Y_{mng}^{(1,0,1)}$ and $Y_{mng}^{(0,1,1)}$ in the similar manner to (2.4), (2.5) and (2.6)

*Remark* 2.7. Since the sequence $Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng})$ of fuzzy numbers is the $(C,1,1,1)$-mean of the sequence $mng(\Delta_{mng}X_{mng})$ of fuzzy number, the sequence $mng(\Delta_{mng}X_{mng})$ is $(C,1,1,1)$-summable to a fuzzy number $L$, whenever

$$\bar{d}(Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng}),L)<\varepsilon, \quad \text{as } m,n,g\to\infty.$$

**Definition 2.12.** For each non-negative integers $k$, $r$ and $j$, we define $Y^{(k,r,j)}_{mng}$ as follows:

$$Y^{(k,r,j)}_{mng}=\begin{cases}\dfrac{1}{(m+1)(n+1)(g+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g}Y^{(k-1,r-1,j-1)}_{pqh}, & k,r,j\geq 1,\\ \\ mng(\Delta_{mng}X_{mng}), & k,r,j=0.\end{cases}$$

**Definition 2.13.** The sequence $mng(\Delta_{mng}X_{mng})$ of fuzzy numbers is said to be $(C,k,r,j)$-summable to a fuzzy number $L$ if for every $\varepsilon>0$, we have

$$\bar{d}(Y^{krj}_{mng}(\Delta_{mng}X_{mng}),L)<\varepsilon, \quad \text{as } m,n,g\to\infty.$$

*Remark* 2.8. If $k=1$, $r=1$ and $j=1$, then $(C,k,r,j)$-summability reduces to $(C,1,1,1)$-summability. Moreover, if $k\neq 0$, $r=0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0$, $r\neq 0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,0,r,0)$-summability. For $k=0$, $r=0$ and $j\neq 0$, $(C,r,k,j)$-summability reduces to $(C,0,0,j)$-summability.

Now, we define the De la Vallée Poussin transform of triple sequence $(X_{mng})$ of fuzzy numbers for sufficiently large non-negative integers $m,n,g$ for $\lambda>1$ and $0<\lambda<1$

$$\tau_{mng}(X)=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}X_{ipu}$$

and

$$\tau_{mng}(X)=\frac{1}{(m-[\lambda m])(n-[\lambda n])(g-[\lambda g])}\sum_{i=\lambda m+1}^{m}\sum_{p=\lambda n+1}^{n}\sum_{u=\lambda g+1}^{g}X_{ipu},$$

respectively.

**Definition 2.14** ([13])**.** A single sequence $X=(X_n)$ of fuzzy numbers is slowly oscillating (in the sense of Stanojevic) if

$$\lim_{\lambda\to 1^+}\limsup_{n}\max_{\underline{n+1\leq k\leq[\lambda n]}}\bar{d}(X_k,X_n)=0.$$

Similar to Definition 2.14, we will define a triple sequence $X=(X_{mng})$ of fuzzy numbers.

**Definition 2.15.** A triple sequence $X=(X_{mng})$ of fuzzy numbers is slowly oscillating (in the sense of Stanojević) if

$$\lim_{\lambda\to 1^+}\limsup_{m,n,g}\max_{\underline{m+1,n+1,q+1\leq i,p,u\leq[\lambda m],[\lambda n],[\lambda g]}}\bar{d}\left(\sum_{a=m+1}^{i}\sum_{b=m+1}^{p}\sum_{v=+1}^{u}\Delta_{a,b,v}X_{a,b,v},0\right)\leq\varepsilon.$$

## 3. Main Results

**Lemma 3.1.** *A triple sequence $X = (X_{mng})$ of fuzzy numbers is slowly oscillating if and only if $(Y^{(1,1,1)}_{mng})$ is slowly oscillating and bounded.*

*Proof.* Let $X = (X_{mng})$ be a slowly oscillation triple sequence. First of all, let us show that $\bar{d}(V^{(1,1,1)}_{mng}, 0) = O(1)$.

We have by definition of slow oscillation, for $\lambda > 1$,

$$\lim_{\lambda\to 1^+} \limsup_{m,n,g} \max_{\underline{m+1,n+1,g+1\leq i,p,u\leq[\lambda m,\lambda n,\lambda g]}} \bar{d}\left(\sum_{a=m+1}^{i}\sum_{b=m+1}^{p}\sum_{v=+1}^{u}\Delta_{a,b,v}X_{a,b,v}, 0\right) \leq \varepsilon$$

and let us rewrite the finite sum $\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{u=1}^{g} ipu\Delta X_{ipu}$ as the series

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{v=0}^{\infty}\sum_{\underline{\frac{m}{2^a+1},\frac{n}{2^b+1},\frac{g}{2^v+1}\leq i,p,u\leq\frac{m}{2^a},\frac{n}{2^b},\frac{g}{2^v}}} ipu\Delta X_{ipu}.$$

Clearly,

$$\begin{aligned}
&\bar{d}\left(\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{u=1}^{g} ipu\Delta X_{ipu}, 0\right)\\
\leq&\bar{d}\left(\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{v=0}^{\infty}\sum_{\underline{\frac{m}{2^a+1},\frac{n}{2^b+1},\frac{g}{2^v+1}\leq i,p,u\leq\frac{m}{2^a},\frac{n}{2^b},\frac{g}{2^v}}} ipu\Delta X_{ipu}, 0\right)\\
\leq&\bar{d}\left(\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{v=0}^{\infty}\frac{mng}{2^{a+n+v}}, 0\right)
\end{aligned}$$

and

$$\begin{aligned}
&\max_{\underline{\frac{m}{2^a+1}+1,\frac{n}{2^b+1}+1,\frac{g}{2^v+1}+1\leq i,p,u\leq\frac{\lambda m}{2^{i+1}},\frac{\lambda n}{2^{p+1}},\frac{\lambda g}{2^{u+1}}}}\\
&\bar{d}\left(\sum_{a=\frac{m}{2^{a+1}}+1}^{i}\sum_{b=\frac{n}{2^{b+1}}+1}^{p}\sum_{v=\frac{g}{2^{v+1}}+1}^{u}\Delta_{a,b,v}X_{a,b,v}, 0\right)\\
\leq&mngC\left(\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{v=0}^{\infty}\frac{1}{2^{a+b+v}}\right) = mngC^*, \quad C^* > 0.
\end{aligned}$$

Therefore, we have

$$\bar{d}(Y^{(1,1,1)}_{mng}(\Delta X),0) = \bar{d}\left(\frac{1}{(m+1)(n+1)(g+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g} pqh(\Delta_{p,q,h}X_{p,q,h}),0\right)$$
$$=O(1), \quad \text{as } m,n,g\to\infty.$$

Since

$$\left\{((C,1,1,1)X)_{mng} = \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{h=1}^{g}\frac{Y^{(1,1,1)}_{pqh}}{pqh} + X_{000}\right\}$$

is slowly oscillating, hence, $(Y^{(1,1,1)}_{mng})$ is slowly oscillating.

Now, to prove the converse part, consider $(Y^{(1,1,1)}_{mng})$ is bounded and slowly oscillating. Thus, the boundedness of $(Y^{(1,1,1)}_{mng})$ implies that $((C,1,1,1)X)_{mng}$ is slowly oscillating. Moreover, $(Y^{(1,1,1)}_{mng})$ being oscillating slowly, so by Kronecker identity (2.3), $(X_{mng})$ is oscillating slowly. □

**Lemma 3.2.** *Let $X = (X_{mng})$ be a triple sequence of fuzzy numbers with $m$, $n$ sufficiently large, then the following statements hold.*

(a) *For $\lambda > 1$*

(3.1)

$$\begin{aligned}
&\bar{d}(X_{mng},((C,1,1,1)X)_{mng})\\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]},((C,1,1,1)X)_{[\lambda m],n,g})\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]},((C,1,1,1)X)_{mng})\}\\
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g},((C,1,1,1)X)_{m,n,g})\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g},((C,1,1,1)X)_{m,n,g})\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]},((C,1,1,1)X)_{m,n,g}(X))\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)(\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu},X_{mng})\right).
\end{aligned}$$

(b) *For $0 < \lambda < 1$*

(3.2) $$\begin{aligned}
&\bar{d}(X_{mng},((C,1,1,1)X)_{mng})\\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{(m-[\lambda m])(n-[\lambda n])(g-[\lambda g])}\{\bar{d}(((C,1,1,1)X)_{mng},((C,1,1,1)X)_{[\lambda m],n,g})\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]},((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]})\}
\end{aligned}$$

$$+\frac{[\lambda m]+1}{(m-[\lambda m])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{[\lambda m],n,g})$$
$$+\frac{[\lambda n]+1}{(n-[\lambda n])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{m,[\lambda n],g})$$
$$+\frac{[\lambda g]+1}{([g-\lambda g])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{m,n,[\lambda g]})$$
$$-\frac{1}{(m-[\lambda m])(n-[\lambda n])(g-\lambda g])}\bar{d}\left(\sum_{i=[\lambda m]+1}^{m}\sum_{p=[\lambda n]+1}^{n}\sum_{u=[\lambda g]+1}^{g}(X_{mng},X_{ipu})\right).$$

*Proof.* We just prove (3.1), (3.2) by the similar way.

We have by De la Vallée Poussin mean of triple sequence $(X_{mng})$ of fuzzy numbers

$$\tau_{mng}(X)$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}X_{ipu}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]},\sum_{i=0}^{[m]}\right)\bar{d}\left(\sum_{p=0}^{[\lambda n]},\sum_{p=0}^{[n]}\right)\bar{d}\left(\sum_{u=0}^{[\lambda g]},\sum_{u=0}^{[g]}\right)\right\}X_{ipu}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]}\sum_{p=0}^{[\lambda n]}\sum_{u=0}^{[\lambda u]},\sum_{i=0}^{[\lambda m]}\sum_{p=0}^{[n]}\sum_{u=0}^{[u]}\right)X_{ipu}\right\}$$
$$-\left\{\bar{d}\left(\sum_{i=0}^{[m]}\sum_{p=0}^{[\lambda n]}\sum_{u=0}^{[\lambda g]},\sum_{i=0}^{m}\sum_{p=0}^{n}\sum_{u=0}^{g}\right)X_{ipu}\right\}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}$$
$$\{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}$$
$$-([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)((C,1,1,1)X)_{[\lambda m],n,g}\}$$
$$-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{(m+1)([\lambda n]+1)([\lambda g]+1)$$
$$((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}-(m+1)(n+1)(g+1)((C,1,1,1)X)_{mng}\}$$
$$=\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}(C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}$$
$$-\left\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{[\lambda m],n,g}\right.$$
$$\left.-\frac{([\lambda m]+1)}{([\lambda m]-m)}((C,1,1,1)X)_{[\lambda m],n,g}\right\}$$
$$-\left\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{m,[\lambda n],g}\right.$$

$$- \frac{([\lambda n]+1)}{([\lambda n]-n)}((C,1,1,1)X)_{m,[\lambda n],g}\Bigg\}$$
$$- \left\{ \frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{m,n,[\lambda g]}\right.$$
$$\left. - \frac{([\lambda g]+1)}{([\lambda g]-g)}((C,1,1,1)X)_{m,n,[\lambda g]}\right\}$$
$$+ \left\{ \frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{mng}\right.$$
$$- \frac{([\lambda m]+1)}{([\lambda m]-m)}((C,1,1,1)X)_{mng} - \frac{([\lambda n]+1)}{([\lambda n]-n)}((C,1,1,1)X)_{mng}$$
$$\left. - \frac{([\lambda g]+1)}{([\lambda g]-g)}((C,1,1,1)X)_{mng} + ((C,1,1,1)X)_{mng}\right\},$$

which implies

$$\begin{aligned}
&\tau_{mng} - ((C,1,1,1)X)_{mng} \\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}, \\
&((C,1,1,1)X)_{[\lambda m],n,g}) - \bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{m,n,g})\} \\
&+ \frac{([\lambda m]+1)}{([\lambda m]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}, (C,1,1,1)X)_{mng}) \\
&+ \frac{([\lambda n]+1)}{([\lambda n]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}, ((C,1,1,1),X)_{mng}) \\
&+ \frac{([\lambda g]+1)}{([\lambda g]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]}, ((C,1,1,1)X)_{mng}).
\end{aligned}$$

Besides,

$$X_{mng} = \tau_{mng} - \frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu}, X_{mng})\right).$$

On subtracting $(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]})$ from the above identity, we have

$$\begin{aligned}
&\bar{d}(X_{mng}, ((C,1,1,1)X)_{mng}) \\
=&\bar{d}(\tau_{mng}(X), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]} \\
&- \frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu}, X_{mng})\right) \\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{[\lambda m],n,g}) \\
&- \bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{mng})\}
\end{aligned}$$

$$+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g},((C,1,1,1)X)_{m,n,g})$$
$$+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g},((C,1,1,1)X)_{m,n,g})$$
$$+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]},((C,1,1,1))_{m,n,g})$$
$$-\frac{1}{([\lambda m]-m)([\lambda n]-n)(\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu},X_{mng})\right). \qquad \square$$

**Theorem 3.1.** *If a triple sequence* $(X_{mng})$ *of fuzzy number is* $(C,1,1,1)$*-summable to a fuzzy number* $L$ *and* $(X_{mng})$ *is slowly oscillating* (*in the sense of Stanojević*), *then*

$$\bar{d}(X_{mng},L)<\varepsilon, \quad \textit{as } m,n,g\to\infty.$$

*Proof.* Let $(X_{mng})$ be $(C,1,1,1)$-summable to a fuzzy number $L$, this implies $\sigma^{(1,1,1)}_{mng}$ is $(C,1,1,1)$-summable to a fuzzy number $L$. Now, from (2.3), we have $(Y^{(1,1,1)}_{mng})$ is $(C,1,1,1)$-summable to zero. Hence, by Lemma 3.1, $(Y^{(1,1,1)}_{mng})$ is slowly oscillating. Additionally, by Lemma 3.2 part (a), we obtain

$$\begin{aligned}(3.3)\quad &\bar{d}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))\\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y^{(1,1,1)}_{mng}),\\
&((C,1,1,1)X)_{[\lambda m],n,g}(Y^{(1,1,1)}_{mng}))\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))\}\\
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y^{(1,1,1)}_{ipu},Y^{(1,1,1)}_{mng})\right).\end{aligned}$$

It is easy to verify that for $\lambda>1$ and sufficiently large $n$ and $g$

$$\begin{aligned}\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}<&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)([\lambda g]-1-g)}\\
<&\frac{9\lambda^3}{(\lambda-1)^3}.\end{aligned}$$

Now, by (3.3),

$$\bar{d}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))$$

$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\bar{d}(\tau_{mng}(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}))$$
$$- \max_{\underline{m+1,n+1,g+1\leq i,p,u\leq[\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

Taking lim sup on both sides in the above inequality, we have

$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)}))$$
$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\limsup_{m,n,g}\bar{d}(\tau_{mng}(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}^{(1,1,1)}))$$
$$- \limsup_{m,n,g}\max_{\underline{m+1,n+1,g+1\leq i,p,u\leq[\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

Moreover, $((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}^{(1,1,1)}) \to 0$ as $m,n,g\to\infty$, so first term on the right hand side of above inequality, must vanish. This implies,

$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)})) \tag{3.4}$$
$$\leq \limsup_{m,n,g}\max_{\underline{m+1,n+1,g+1\leq i,p,u\leq[\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

As $\lambda\to 1^+$ in (3.4), thus we have

$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)})) \leq 0.$$

This implies $\bar{d}(Y_{mng}^{(1,1,1)}, 0) < \varepsilon$ as $m,n,g\to\infty$. Since $(X_{mng})$ is summable to a fuzzy number $L$ by $(C,1,1,1)$ mean and $\bar{d}(Y_{mng}^{(1,1,1)}, 0) < \varepsilon$ as $m,n,g\to\infty$, thus $\bar{d}(X_{mng}, L) < \varepsilon$, $m,n,g\to\infty$. □

**Corollary 3.1.** *If $(X_{mng})$ is $(C,k,r,j)$-summable to a fuzzy number $L$ and $(X_{mng})$ is slowly oscillating (in the sense of Stanojević), then*

$$\bar{d}(X_{mng}, L) < \varepsilon, \quad \text{as } m,n,g\to\infty.$$

*Proof.* Let $X=(X_{mng})$ be slowly oscillating. Then, $((C,k,r,j)X)$ is slowly oscillating by Lemma 3.1. Furthermore, since $X=(X_{mng})$ is $(C,k,r,j)$-summable to a fuzzy number $L$, we have by Theorem 3.1 that

$$\bar{d}(((C,k,r,j)X)_{mng}, L) < \varepsilon \quad \text{as} \quad m,n,g\to\infty. \tag{3.5}$$

Now, from the definition,

$$((C,k,r,j)X)_{mng} = ((C,1,1,1)X)_{mng}(((C,k-1,r-1,j-1)X)_{mng}). \tag{3.6}$$

It is clear that (3.5) and (3.6) imply $X=(X_{mng})$ is $(C,k-1,r-1,j-1)$-summable to a fuzzy number $L$. Thus, $(((C,k-1,r-1,j-1)X)_{mng})$ is slowly oscillating by Lemma

3.1. Therefore, by Theorem 3.1, we have $\bar{d}(((C,k-1,r-1,j-1)X)_{mng},L)<\varepsilon$ as $m,n,g\to\infty$. Continuing in this way, we get $\bar{d}(X_{mng},L)<\varepsilon$ as $m,n,g\to\infty$. □

*Remark* 3.1. If $k\neq 0, r=0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0, r\neq 0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,0,r,0)$-summability. Finally, if $k=0$, $r=0$ and $j\neq 0$, then $(C,r,k,j)$-summability reduces to $(C,0,0,j)$-summability.

**Theorem 3.2.** *If a triple sequence $(X_{mng})$ of fuzzy number is $(C,1,1,1)$-summable to a fuzzy number $L$ and $Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng})$ is slowly oscillating, then*

$$\bar{d}(X_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

*Proof.* Since $(X_{mng})$ is $(C,1,1,1)$-summable to a fuzzy number $L$, so $((C,1,1,1)X)_{mng}$ is $(C,1,1,1)$-summable to a fuzzy number $L$. Hence, $(Y^{(1,1,1)}_{mng})$ is $(C,1,1,1)$-summable to zero by (2.3). Using identity (2.3) to $(Y^{(1,1,1)}_{mng})$ we have $Y(Y^{(1,1,1)}_{mng})$ is Cesàro summable to zero. This means that $Y(Y^{(1,1,1)}_{mng})$ is slowly oscillating by Lemma 3.1. Now, by Lemma 3.2 part (a), we get

$$\begin{aligned}(3.7)\quad &\bar{d}(Y(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))\\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y^{(1,1,1)}_{mng}),\\
&((C,1,1,1)X)_{[\lambda m],n,g}Y(Y^{(1,1,1)}_{mng}))\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}Y(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))\}\\
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(\sigma^{(1,1,1)}_{m,n,[\lambda g]}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}Y(Y^{(1,1,1)}_{ipu},Y^{(1,1,1)}_{mng})\right).\end{aligned}$$

It is easy to verify that for $\lambda>1$ and sufficiently large $n$ and $g$

$$\begin{aligned}\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}<&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)([\lambda g]-1-g)}\\
<&\frac{9\lambda^3}{(\lambda-1)^3}.\end{aligned}$$

Now, by (3.7),

$$\bar{d}(Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))$$

$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\bar{d}(\tau_{mng}Y(Y_{mng}^{(1,1,1)}),((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}))$$
$$-\max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}),Y(Y_{mng}^{(1,1,1)}))\right).$$

Taking lim sup on both sides in the above inequality, we have

$$\limsup_{m,n,g}\bar{d}(Y(Y_{mng}^{(1,1,1)}),((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)})))$$
$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\limsup_{m,n,g}\bar{d}(\tau_{mng}Y(Y_{mng}^{(1,1,1)}),((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}^{(1,1,1)}))$$
$$-\limsup_{m,n,g}\max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}),Y(Y_{mng}^{(1,1,1)}))\right).$$

Further, $((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}^{(1,1,1)})\to 0$ as $m,n,g\to\infty$, so first term in the right hand side of above inequality, must vanish. This implies

(3.8)

$$\limsup_{m,n,g}\bar{d}(Y_{mng}^{(1,1,1)},((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)}))$$
$$\leq \limsup_{m,n,g}\max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}),Y(Y_{mng}^{(1,1,1)}))\right).$$

Taking $\lambda\to 1^+$ in (3.8), we have

$$\limsup_{m,n,g}\bar{d}(Y(Y_{mng}^{(1,1,1)}),((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)})))\leq 0,$$

which implies, $\bar{d}(Y(Y_{mng}^{(1,1,1)}),0))<\varepsilon$ as $m,n,g\to\infty$. Since $(X_{mng})$ is summable to a fuzzy number $L$ by $(C,1,1,1)$ mean and $\bar{d}(Y(Y_{mng}^{(1,1,1)}),0)<\varepsilon$ as $m,n,g\to\infty$, thus $\bar{d}(X_{mng},L)<\varepsilon$, as $m,n,g\to\infty$. □

**Corollary 3.2.** *If $(X_{mng})$ is $(C,k,r,j)$-summable to a fuzzy number $L$ and $Y_{mng}^{(1,1,1)}(\Delta X)$ is slowly oscillating, then*

$$\bar{d}(X_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

*Proof.* As $Y_{mng}^{(1,1,1)}(\Delta X)$ is slowly oscillating, setting $X=(X_{mng})$ instead of $Y_{mng}^{(1,1,1)}(\Delta X)$, then $((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))$ is slowly oscillating by Lemma 3.1. Moreover, as $Y_{mng}^{(1,1,1)}(\Delta X)$ is $(C,k,r,j)$-summable to a fuzzy number $L$, so by Theorem 3.2,

$$\bar{d}(((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X)),L)<\varepsilon,\quad \text{as } m,n,g\to\infty. \tag{3.9}$$

By definition,

(3.10)

$$((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))$$

$$=((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))[((C,k-1,r-1,j-1)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))].$$

From (3.9) and (3.10) we have $Y_{mng}^{(1,1,1)}(\Delta X)$ is $(C,k-1,r-1,j-1)$-summable to a fuzzy number $L$. Thus, $((C,k-1,r-1,j-1)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))$ is slowly oscillating by Lemma 3.1. Therefore, by Theorem 3.1, we have

$$\bar{d}(((C,k-1,r-1,j-1)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X)),L)<\varepsilon, \quad \text{as } m,n,g\to\infty.$$

Continuing this way, we get $\bar{d}((Y_{mng}^{(1,1,1)}(\Delta X)),L)<\varepsilon$ as $m,n,g\to\infty$. □

*Remark* 3.2. If $k\neq 0$, $r=0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0$, $r\neq 0$ and $j=0$, then $(C,r,k,j)$-summability reduces to $(C,0,r,0)$-summability. Finally, if $k=0$, $r=0$ and $j\neq 0$, then $(C,r,k,j)$-summability reduces to $(C,0,0,j)$-summability and consequently more corollaries can be generated from the main results of this paper.

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