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LOCAL K-CONVOLUTED C-GROUPS AND ABSTRACT CAUCHY PROBLEMS

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Abstract. We first present a new form of a local K-convoluted C-group on a Banach space X, and then deduce some basic properties of a nondegenerate local K-convoluted C-group on X and some generation theorems of local K-convoluted C-groups, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local K-convoluted C-group on X with subgenerator A and the unique existence of solutions of the abstract Cauchy problem ACP(A, f, x).

1. Introduction

Let X be a Banach space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|$, and let $L(X)$ denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\text{ACP}(A, f, x) \quad \begin{cases} u'(t) = Au(t) + f(t), & \text{for } t \in (-T_0, T_0), \\ u(0) = x, \end{cases}$$

where $x \in X$, A is a closed linear operator in X, and $f \in L^1_{loc}((-T_0, T_0), X)$ (the family of all locally integrable functions from $(-T_0, T_0)$ into X). A function u is called a solution of ACP(A, f, x) if $u \in \mathrm{C}((-T_0, T_0), X)$ satisfies ACP(A, f, x) (that is, $u(0) = x$ and for a.e. $t \in (-T_0, T_0)$, $u(t)$ is differentiable and $u(t) \in D(A)$, and $u'(t)$=$Au(t)$+$f(t)$ for a.e. $t \in (-T_0, T_0)$). For each $C \in L(X)$ and $K \in L^1_{loc}([0, T_0), \mathbb{F})$, a family $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ in $L(X)$ is called a local K-convoluted C-group on

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X if $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x$$
$$= \left(\operatorname{sgn} t \operatorname{sgn} s \ \operatorname{sgn}\ (t+s) \int_0^{t+s} - \operatorname{sgn} s \int_0^t - \operatorname{sgn} t \int_0^s \right) K(|t+s-r|)S(r)Cxdr,$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$. In particular, $S(\cdot)$ is called a local (0-times integrated) C-group on X if $K = j_{-1}$ (the Dirac measure at 0) or equivalently, $S(\cdot)$ is strongly continuous, $S(\cdot)C = CS(\cdot)$, and satisfies

$$S(t)S(s)x = S(t+s)Cx, \quad \text{for all } x \in X \text{ and } |t|, |s|, |t+s| < T_0,$$

(see [2]). Moreover, we say that $S(\cdot)$ is nondegenerate, if $x = 0$ whenever $S(t)x = 0$ for all $|t| < T_0$. The nondegeneracy of a local K-convoluted C-group $S(\cdot)$ on X implies that

$$S(0) = C \text{ if } K = j_{-1} \text{ and } S(0) = 0 \text{ (the zero operator on X) otherwise,}$$

and the (integral) generator $A : \mathrm{D}(A) \subset X \to X$ of $S(\cdot)$ is a closed linear operator in X defined by

$$\mathrm{D}(A) = \{x \in X \mid S(\cdot)x - K_0(|\cdot|)Cx = \widetilde{S}(\cdot)y_x \text{ on } (-T_0, T_0) \text{ for some } y_x \in X\}$$

and $Ax = y_x$ for all $x \in \mathrm{D}(A)$. Here $\widetilde{S}(t)z = \int_0^t S(s)zds$. In general, a local K-convoluted C-group on X is called a K-convoluted C-group on X if $T_0 = \infty$; a (local) K-convoluted C-group on X is called a (local) K-convoluted group on X if $C = I$ (the identity operator on X) or a (local) α-times integrated C-group on X if K is equal to the function $j_{\alpha-1}$ for some $\alpha \geq 0$, defined by $j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ (see [4, 7, 21]). Here $\Gamma(\cdot)$ denotes the Gamma function, a (local) α-times integrated C-group on X is called a (local) α-times integrated group on X if $C = I$; and a (local) C-group on X is called a c_0-group on X if $C = I$ (see [1, 5]). Some basic properites of a nondegenerate (local) α-times integrated C-semigroup on X have been established by many authors (in [2, 3, 26–28] for $\alpha = 0$, and in [6, 10, 17–20, 22, 23, 25, 29, 30] for $\alpha > 0$), which can be extended to the case of local K-convoluted C-semigroup just as results in [7–10, 13–16]. Some equivalence relations between the generation of a nondegenerate (local) K-convoluted C-semigroup on X with subgenerator A and the unique existence of solutions of the abstract Cauchy problem $\mathrm{ACP}(A, f, x)$ are also discussed in [2, 26, 27] for the case $K = j_{\alpha-1}$ with $\alpha = 0$ and in [11–13, 30, 31] with $\alpha > 0$, and in [8, 13, 16] for the general case. The purpose of this paper is to investigate the following basic properties of a nondegenerate local K-convoluted C-group $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ on X just as results in [13] concerning local K-convoluted C-semigroups on X when C is injective and some additional conditions are taken into consideration

(1.1) $C^{-1}AC = A$,

(1.2) $\widetilde{S}(t)x \in D(A)$ and $A\widetilde{S}(t)x = S(t)x - K_0(|t|)Cx, \quad$ for all $x \in X$ and $|t| < T_0$,

(1.3) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax, \quad$ for all $x \in D(A)$ and $|t| < T_0$;

and

$$S(t)S(s) = S(s)S(t), \quad \text{on } X, \text{ for all } |t|, |s| < T_0, \tag{1.4}$$

(see Theorems 2.5, 2.6 and 2.7 below), which have been established partially in [8] by another method, and then deduce some equivalence relations between the generation of a nondegenerate local K-convoluted C-group on X with subgenerator A and the unique existence of solutions of ACP(A, f, x), which are similar to some results in [13] concerning equivalence relations between the generation of a nondegenerate local K-convoluted C-semigroup on X with subgenerator A and the unique existence of solutions of ACP(A, f, x). To do these, we will first prove an important lemma which shows that a strongly continuous family $S(\cdot)$ in $\mathrm{L}(X)$ is a local K-convoluted C-group on X is equivalent to $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$ is a local K_0-convoluted C-group on X (see Lemma 2.1 below), and then show that a strongly continuous family $S(\cdot)$ in $L(X)$ which commutes with C on X is a local K-convoluted C-group on X is equivalent to $\widetilde{S}(t)[S(s)-K_0(|s|)C]=[S(t)-K_0(|t|)C]\widetilde{S}(s)$ for all $|t|, |s|, |t+s| < T_0$ (see Theorem 2.1 below). In order to show that $\mathrm{sgn}(\cdot)b*S(\cdot)$ is a local $a*K$-convoluted C-group on X if $S(\cdot)$ is a local K-convoluted C-group on X and $b(\cdot) = a(|\cdot|)$ for some $a \in \mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$. In particular, $\mathrm{sgn}(\cdot)J_\beta * S(\cdot)$ is a local K_β-convoluted C-group on X if $S(\cdot)$ is a local K-convoluted C-group on X and $\beta > -1$, which can be applied to show that its only if part is also true when β is a nonnegative integer (see Proposition 2.1 below). Here $K_\beta(t) = K * j_\beta(t)$ for $\beta > -1$, $J_\beta(\cdot) = j_\beta(|\cdot|)$, $f * S(t)x = \int_0^t f(t-s)S(s)x ds$ for all $x \in X$ and $f \in L^1_{loc}((-T_0, T_0), \mathbb{F})$. We also show that a strongly continuous family $S(\cdot)$ in $L(X)$ which commutes with C on X is a local K-convoluted C-group on X when it has a subgenerator (see Theorem 2.4 below). Moreover, $S(\cdot)$ is nondegenerate if C is injective and the generator of a nondegenerate local K-convoluted C-group $S(\cdot)$ on X is the unique subgenerator of $S(\cdot)$ which contains all its subgenerators, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$ when $S(\cdot)$ has a subgenerator (see Theorems 2.5, 2.6 and 2.7 below). This can be applied to show that $CA \subset AC$ and $S(\cdot)$ is a nondegenerate local K-convoluted C-group on X with generator $C^{-1}AC$ when C is injective, K_0 a kernel on $[0, T_0)$ (that is, $f = 0$ on $[0, T_0)$ whenever $f \in C([0, T_0), \mathbb{F})$ with $\int_0^t K_0(t-s)f(s)ds = 0$ for all $0 \leq t < T_0$) and $S(\cdot)$ a strongly continuous family in $\mathrm{L}(X)$ with closed subgenerator A. In this case, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$ (see Theorem 2.8 below). Some illustrative examples concerning these theorems are also presented in the final part of this paper.

2. Basic Properties of Local K-Convoluted C-Groups

In the following we will note some facts concerning local K-convoluted C-groups which can be expansively appled in this paper.

Remark 2.1. Let $S(\cdot)(= \{S(t) \,|\, |t| < T_0\})$ be a strongly continuous family in $L(X)$. Then the following are equivalent.

(i) $S(\cdot)$ is a local K-convoluted C-group on X.

(ii) (see [8]) $S_+(\cdot)$ and $S_-(\cdot)$ are local K-convoluted C-semigroups on X, $S(t)S(s)x = S(s)S(t)x$ on X for all $-T_0 < t \leq 0 \leq s < T_0$,

$$S(t)S(s)x = \int_{t+s}^{s} K(r-t-s)S(r)Cxdr + \int_{t}^{0} K(t+s-r)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$, and

$$S(t)S(s)x = \int_{t}^{t+s} K(t+s-r)S(r)Cxdr + \int_{0}^{s} K(r-t-s)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$.

(iii)

$$T(t)T(s)x = (\int_{0}^{t+s} - \int_{0}^{t} - \int_{0}^{s})K(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$.

Here $T(\cdot) = \operatorname{sgn}(\cdot)S(\cdot)$ on $(-T_0, T_0)$, $S_+(\cdot) = S(\cdot)$ and $S_-(\cdot) = S(-\cdot)$ on $[0, T_0)$.

Next we will deduce an important lemma which can be used to obtain a new equivalence relation between the generation of a local K-convoluted C-group $S(\cdot)$ on X and the equality of

$$\widetilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\widetilde{S}(s),$$

on X for all $|t|, |s|, |t+s| < T_0$ when $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ is a strongly continuous family in $L(X)$ commuting with C on X just as a result in [13] for the case of local K-convoluted C-semigroup and in [19] for the case of local α-times integrated C-semigroup.

Lemma 2.1. *Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $L(X)$. Then $S(\cdot)$ is a local K-convoluted C-group on X if and only if* $\operatorname{sgn}(\cdot)\widetilde{S}(\cdot)$ *is a local K_0-convoluted C-group on X. In this case,*

(i) *$S(\cdot)$ is nondegenerate if and only if $\widetilde{S}(\cdot)$ is;*

(ii) *A is the generator of $S(\cdot)$ if and only if it is the generator of* $\operatorname{sgn}(\cdot)\widetilde{S}(\cdot)$.

Proof. Let $x \in$ X be given. We set $T(\cdot) = \operatorname{sgn}(\cdot)\widetilde{S}(\cdot)$. Then

$$\frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \tag{2.1}$$
$$= -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(s)\widetilde{S}(t)Cx$$

and

$$\frac{d}{ds}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right] \tag{2.2}$$
$$= -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(|t|)\widetilde{S}(s)Cx,$$

for $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$. Using integration by parts to the right-hand sides of (2.1) and (2.2), we obtain

$$(2.3)\quad -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(s)\widetilde{S}(t)Cx$$
$$= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx$$

and

$$(2.4)\quad -\int_{t+s}^{s} K(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K(t+s-r)\widetilde{S}(r)Cxdr + K_0(|t|)\widetilde{S}(s)Cx$$
$$= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,$$

for $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$. Combining (2.1)–(2.4), we have

$$(2.5)\quad \frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right]$$
$$= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx$$

and

$$(2.6)\quad \frac{d}{ds}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right]$$
$$= \int_{t+s}^{s} K_0(r-t-s)S(r)Cxdr - \int_{t}^{0} K_0(t+s-r)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,$$

for $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$. Similarly, we can show that

$$(2.7)\quad \frac{d}{dt}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right]$$
$$= -\int_{t}^{t+s} K_0(t+s-r)S(r)Cxdr + \int_{0}^{s} K_0(r-t-s)S(r)Cxdr - K_0(|t|)\widetilde{S}(s)Cx$$

and

$$(2.8)\quad \frac{d}{ds}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right]$$
$$= -\int_{t}^{t+s} K_0(t+s-r)S(r)Cxdr + \int_{0}^{s} K_0(r-t-s)S(r)Cxdr - K_0(s)\widetilde{S}(t)Cx,$$

for $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$. By (2.6) and (2.8), we have

$$(2.9)\quad \frac{d}{ds}\frac{d}{dt}\left[\int_{t+s}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_{t}^{0} K_0(t+s-r)\widetilde{S}(r)Cxdr\right]$$
$$= -\int_{t+s}^{s} K(r-t-s)S(r)Cxdr + \int_{t}^{0} K(t+s-r)S(r)Cxdr,$$

for $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$ and

$$(2.10)\quad \frac{d}{dt}\frac{d}{ds}\left[-\int_{t}^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_{0}^{s} K_0(r-t-s)\widetilde{S}(r)Cxdr\right]$$

$$= -\int_t^{t+s} K(t+s-r)S(r)Cxdr + \int_0^s K(r-t-s)S(r)Cxdr,$$

for $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Suppose that $T(\cdot)$ is a local K_0-convoluted C-group on X. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local K_0-convoluted C-semigroups on X, $T_+(\cdot) = \widetilde{S_+}(\cdot)$ and $T_-(\cdot) = \widetilde{S_-}(\cdot)$ on $[0, T_0)$, $T(t)T(s) = T(s)T(t)$ on X for all $-T_0 < t \le 0 \le s < T_0$,

$$T(t)T(s)x = \int_{t+s}^s K_0(r-t-s)T(r)Cxdr + \int_t^0 K_0(t+s-r)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$ and

$$T(t)T(s)x = \int_t^{t+s} K_0(t+s-r)T(r)Cxdr + \int_0^s K_0(r-t-s)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$ or equivalently, $S_+(\cdot)$ and $S_-(\cdot)$ both are local K-convoluted C-semigroups on X, $S(t)S(s) = S(s)S(t)$ on X for all $-T_0 < t \le 0 \le s < T_0$

$$(2.11) \qquad -\widetilde{S}(t)\widetilde{S}(s)x = \int_{t+s}^s K_0(r-t-s)\widetilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\widetilde{S}(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$, and

$$(2.12) \qquad -\widetilde{S}(t)\widetilde{S}(s)x = -\int_t^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxdr + \int_0^s K_0(r-t-s)\widetilde{S}(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Combining (2.7)–(2.10), we have

$$(2.13) \qquad S(t)S(s)x = \int_{t+s}^s K(r-t-s)S(r)Cxdr + \int_t^0 K(t+s-r)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \ge 0$ and

$$(2.14) \qquad S(t)S(s)x = \int_t^{t+s} K(t+s-r)S(r)Cxdr + \int_0^s K(r-t-s)S(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \le 0 \le s < T_0$ with $t+s \le 0$. Consequently, $S(\cdot)$ is a local K-convoluted C-group on X. Conversely, suppose that $S(\cdot)$ is a local K-convoluted C-group on X. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local K_0-convoluted C-semigroups on X, $T(t)T(s) = T(s)T(t)$ on X for all $-T_0 < t \le 0 \le s < T_0$, and (2.13)–(2.14) both hold. By (2.9) and (2.10), we have (2.11) and (2.12) both hold. Consequently, $T(\cdot)$ is a local K_0-convoluted C-group on X. □

Theorem 2.1. *Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $L(X)$ which commutes with C on X. Then $S(\cdot)$ is a local K-convoluted C-group on X if and only if*

$$(2.15) \qquad \widetilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\widetilde{S}(s), \quad \text{on } X,$$

for all $|t|, |s|, |t+s| < T_0$.

Proof. We set $T(\cdot) = \mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$. Suppose that $S(\cdot)$ is a local K-convoluted C-group on X. Then $S_+(\cdot)$ and $S_-(\cdot)$ both are local K-convoluted C-semigroups on X. To show that (2.15) holds for all $|t|, |s|, |t+s| < T_0$, we observe from [13, Theorem 2.2] that we need only to show that $\widetilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $|t|, |s| < T_0$ with $ts \leq 0$. Let $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ be given with $t + s \geq 0$. By Lemma 2.1, (2.1) and (2.2), we have

$$\begin{aligned} -S(t)\widetilde{S}(s)x - K_0(|s|)\widetilde{S}(t)Cx &= \frac{d}{dt}T(t)T(s)x - K_0(|s|)\widetilde{S}(t)Cx \\ &= \frac{d}{ds}T(t)T(s)x - K_0(|t|)\widetilde{S}(s)Cx \\ &= -\widetilde{S}(t)S(s)x - K_0(|t|)\widetilde{S}(s)Cx, \end{aligned}$$

or equivalently, $\widetilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\widetilde{S}(s)x$. Similarly, we can show that $\widetilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t + s \leq 0$. Since $S(t)S(s) = S(s)S(t)$ on X for all $|t|, |s|, |t+s| < T_0$, we also have $\widetilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\widetilde{S}(s)x$ for all $x \in X$ and $-T_0 < s \leq 0 \leq t < T_0$. Consequently, (2.15) holds for all $|t|, |s|, |t+s| < T_0$. Conversely, suppose that (2.15) holds for all $|t|, |s|, |t+s| < T_0$. Then $T_+(\cdot)$ and $T_-(\cdot)$ both are local K_0-convoluted C-semigroups on X and $\widetilde{S}(t)S(s)x - S(t)\widetilde{S}(s)x = K_0(|s|)\widetilde{S}(t)Cx - K_0(|t|)\widetilde{S}(s)Cx$ for all $x \in X$ and $|t|, |s|, |t+s| < T_0$ with $t + s \geq 0$. Fix $x \in X$ and $-T_0 < t < 0 \leq s < T_0$ with $t + s \geq 0$, we have

$$\begin{aligned} &\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x \\ =&K_0(|r|)\widetilde{S}(t+s-r)Cx - K_0(|t+s-r|)\widetilde{S}(r)Cx, \end{aligned} \tag{2.16}$$

for all $t \leq r \leq 0$. Using integration by parts to the left-hand side of (2.16) over $[t, 0]$ and change of variables to the right-hand side of (2.16) over $[t, 0]$, we obtain

$$\begin{aligned} T(t)T(s)x &= -\widetilde{S}(t)\widetilde{S}(s)x \\ &= \int_t^0 [\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x]dr \\ &= \int_t^0 [K_0(|r|)\widetilde{S}(t+s-r)Cx - K_0(|t+s-r|)\widetilde{S}(r)Cx]dr \\ &= \int_s^{t+s} K_0(|t+s-r|)\widetilde{S}(r)Cxdr - \int_t^0 K_0(|t+s-r|)\widetilde{S}(r)Cxdr \\ &= \int_{t+s}^s K_0(|t+s-r|)T(r)Cxdr + \int_t^0 K_0(|t+s-r|)T(r)Cxdr. \end{aligned} \tag{2.17}$$

Using change of variables to the left-hand side of (2.16) over $[t, 0]$, we also have

$$T(s)T(t)x = -\widetilde{S}(s)\widetilde{S}(t)x = \int_t^0 [\widetilde{S}(t+s-r)S(r)x - S(t+s-r)\widetilde{S}(r)x]dr. \tag{2.18}$$

Combining (2.17) with (2.18), we have $T(t)T(s) = T(s)T(t)$ on X for all $|t|, |s|, |t+s| < T_0$ with $ts \leq 0$ and

$$T(t)T(s)x = \int_{t+s}^{s} K_0(|t+s-r|)T(r)Cxdr + \int_{t}^{0} K_0(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \geq 0$. Similarly, we can show that

$$T(t)T(s)x = \int_{t}^{t+s} K_0(|t+s-r|)T(r)Cxdr + \int_{0}^{s} K_0(|t+s-r|)T(r)Cxdr,$$

for all $x \in X$ and $-T_0 < t \leq 0 \leq s < T_0$ with $t+s \leq 0$ when the interval $[t,0]$ of the integration of (2.16) is replaced by $[t, t+s]$. Consequently, $T(\cdot)$ is a local K_0-convoluted C-group on X. Combining this with Lemma 2.1, we get that $S(\cdot)$ is a local K-convoluted C-group on X. □

Proposition 2.1. *Let $S(\cdot)$ be a local K-convoluted C-group on X, $a \in L^1_{loc}([0,T_0),\mathbb{F})$, and $b(\cdot) = a(|\cdot|)$. Then $\operatorname{sgn}(\cdot)b * S(\cdot)$ is a local $a * K$-convoluted C-group on X. In particular, for each $\beta > -1$, $\operatorname{sgn}(\cdot)J_\beta * S(\cdot)$ is a local K_β-convoluted C-group on X. Here $J_\beta(\cdot) = j_\beta(|\cdot|)$. Moreover, $S(\cdot)$ is a local K-convoluted C-group on X if it is a strongly continuous family in $L(X)$ such that $\operatorname{sgn}^k(\cdot)j_{k-1} * S(\cdot) = \operatorname{sgn}(\cdot)J_{k-1} * S(\cdot)$ is a local K_{k-1}-convoluted C-group on X for some nonnegative integer k.*

Proof. Clearly, $\operatorname{sgn}(\cdot)b * S(\cdot)$ is strongly continuous family in $L(X)$ which commutes with C on X. To show that $\operatorname{sgn}(\cdot)b * S(\cdot)$ is a local $a * K$-convoluted C-group on X, we remain only to show that

$$\begin{aligned}&[(\operatorname{sgn} t)b * S(t) - \widetilde{a * K}(|t|)C]j_0 * [\operatorname{sgn}(\cdot)b * S(\cdot)](s)\\ =&j_0 * [\operatorname{sgn}(\cdot)b * S(\cdot)](t)[(\operatorname{sgn} s)b * S(s) - \widetilde{a * K}(|s|)C],\end{aligned}$$

on X for all $|t|, |s|, |t+s| < T_0$. Here $\widetilde{a * K} = j_0 * (a * K)$. Clearly,

$$b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)j_0 * (b * K)(|t|),$$

on X for all $0 \leq t < T_0$. Next we will show that $b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)j_0 * b * K(|t|)$ on X for all $-T_0 < t \leq 0$. Let $-T_0 < t \leq 0$ be given, then

$$\begin{aligned}b * K_0(|\cdot|)(t) &= \int_0^t b(s)K_0(|t-s|)ds = \int_0^t b(s)K_0(s-t)ds\\ &= -\int_0^t a(-s)\int_s^t K(s-r)drds = -\int_t^0\int_s^t a(-s)K(s-r)drds\\ &= -\int_0^t\int_r^t a(-r)K(r-s)dsdr = \int_t^0\int_r^t a(-r)K(r-s)dsdr\end{aligned}$$

and

$$\begin{aligned}\int_t^0\int_r^t a(-r)K(r-s)dsdr &= -\int_t^0\int_s^0 a(-r)K(r-s)drds\\ &= -\int_0^t\int_0^s a(|r|)K(r-s)drds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_0^{-s} a(|r|)K(-r-s)drds \\
&= \int_0^t b * K(-s)ds = -\int_0^{-t} b * K(s)ds \\
&= (\operatorname{sgn} t) j_0 * (b * K)(|t|).
\end{aligned}
$$

Since $b * K(|t|) = a * K(|t|)$ for all $|t| < T_0$, we have $b * K_0(|\cdot|)(t) = (\operatorname{sgn} t)\widetilde{a * K}(|t|)$ for all $|t| < T_0$. Clearly, $b * \widetilde{S}(t) = j_0 * (b * S)(t)$ on X for all $|t| < T_0$. Since $j_0 * [\operatorname{sgn}(\cdot) b * S(\cdot)](t) = (\operatorname{sgn} t) j_0 * (b * S)(t) = (\operatorname{sgn} t) b * \widetilde{S}(t)$ on X for all $|t| < T_0$, we also have

$$
\begin{aligned}
&[(\operatorname{sgn} t)(b * S)(t) - \widetilde{a * K}(|t|)C](\operatorname{sgn} s)\widetilde{b * S}(s)x \\
=&[(\operatorname{sgn} t)(b * S)(t) - (\operatorname{sgn} t) b * K_0(|\cdot|)(t)C](\operatorname{sgn} s) b * \widetilde{S}(s)x \\
=&(\operatorname{sgn} t)[(b * S)(t) - b * K_0(|\cdot|)(t)C](\operatorname{sgn} s) b * \widetilde{S}(s)x \\
=&(\operatorname{sgn} t) \int_0^t b(t-s)[S(r) - K_0(|r|)C](\operatorname{sgn} s) b * \widetilde{S}(s)xdr \\
=&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S}\right](s)(\operatorname{sgn} s)xdr
\end{aligned}
$$

and

$$
\begin{aligned}
&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S}\right](s)(\operatorname{sgn} s)xdr \\
=&(\operatorname{sgn} t) b * \left[\int_0^t b(t-r)\widetilde{S}(r)(S(\cdot) - K_0(|\cdot|)C)\right](s)(\operatorname{sgn} s)xdr \\
=&(\operatorname{sgn} t) b * \widetilde{S}(t) b * [S(\cdot) - K_0(|\cdot|)C](s)(\operatorname{sgn} s)x \\
=&(\operatorname{sgn} t) b * \widetilde{S}(t)[b * S(s) - b * K_0(|\cdot|)(s)C](\operatorname{sgn} s)x \\
=&(\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s) b * S(s) - (\operatorname{sgn} s) b * K_0(|\cdot|)(s)C]x \\
=&(\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s) b * S(s) - \widetilde{a * K}(|s|)C]x,
\end{aligned}
$$

for all $x \in X$ and $|t|, |s|, |t+s| < T_0$. □

Definition 2.1. Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$. A linear operator A in X is called a subgenerator of $S(\cdot)$ if

$$
S(t)x - K_0(|t|)Cx = \int_0^t S(r)Axdr,
$$

for all $x \in D(A)$ and $|t| < T_0$, and

$$
\int_0^t S(r)xdr \in D(A) \quad \text{and} \quad A\int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx, \tag{2.19}
$$

for all $x \in X$ and $|t| < T_0$. A subgenerator A of $S(\cdot)$ is called the maximal subgenerator of $S(\cdot)$ if it is an extension of each subgenerator of $S(\cdot)$ to $D(A)$.

Remark 2.2. Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$, and A a linear operator in X. Then A is a subgenerator of $S(\cdot)$ if and only if A is a subgenerator of $S_+(\cdot)$ and $-A$ a subgenerator of $S_-(\cdot)$.

Remark 2.3. Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$, and A a (closed) linear operator in X. Then A is the maximal subgenerator of $S(\cdot)$ if A is the maximal subgenerator of $S_+(\cdot)$ and $-A$ the maximal subgenerator of $S_-(\cdot)$.

Theorem 2.2. *Let $S(\cdot)$ be a local K-convoluted C-group on X and K_0 not the zero function on $[0,T_0)$, or a K-convoluted C-group on X. Assume that C is injective. Then $S(\cdot)$ is nondegenerate if and only if $S_+(\cdot)$ and $S_-(\cdot)$ both are nondegenerate if and only if $S_+(\cdot)$ or $S_-(\cdot)$ is nondegenerate.*

Proof. Clearly, $S(\cdot)$ is nondegenerate if either $S_+(\cdot)$ or $S_-(\cdot)$ is nondegenerate. Conversely, suppose that $S(\cdot)$ is nondegenerate and $S_+(\cdot)x=0$ on $[0,T_0)$ for some $x\in X$. By Theorem 2.1, we have $\widetilde{S}(t)[S(s)-K_0(|s|)C]x=[S(t)-K_0(|t|)C]\widetilde{S}(s)x=0$ for all $-T_0<t\le 0\le s<T_0$, and so $\widetilde{S}(t)K_0(|s|)Cx=0$. Hence, $\widetilde{S}(t)x=0$. Since $-T_0<t\le 0$ is arbitrary, we have $S(\cdot)x=0$ on $(-T_0,0]$, which together with the nondegeneracy of $S(\cdot)$ implies that $x=0$. Consequently, $S_+(\cdot)$ is nondegenerate. Similarly, we can show that $S_-(\cdot)$ is nondegenerate when $S(\cdot)$ is nondegenerate. □

Theorem 2.3. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-group on X and K_0 not the zero function on $[0,T_0)$, or a K-convoluted C-group on X. Assume that C is injective. Then A is the generator of $S(\cdot)$ if and only if A is the generator of $S_+(\cdot)$ and $-A$ the generator of $S_-(\cdot)$ if and only if A is the generator of $S_+(\cdot)$ or $-A$ the generator of $S_-(\cdot)$.*

Proof. Suppose that A is the generator of $S_+(\cdot)$ and $-A$ is the generator of $S_-(\cdot)$. We set B to denote the generator of $S(\cdot)$. Then $S(\cdot)x-K_0(|\cdot|)Cx=\widetilde{S}(\cdot)Ax$ on $(-T_0,T_0)$ for all $x\in D(A)$ or equivalently, $A\subset B$. Since $S(\cdot)x-K_0(|\cdot|)Cx=\widetilde{S}(\cdot)Bx$ on $(-T_0,T_0)$ for all $x\in D(B)$, we have $B\subset A$. Consequently, $A=B$ is the generator of $S(\cdot)$. Suppose that A is the generator of $S(\cdot)$. We set B_+ and B_- to denote the generators of $S_+(\cdot)$ and $S_-(\cdot)$, respectively. To show that $B_+=A$ and $B_-=-A$, we observe from the preceding argument, we need only to show that $B_+=-B_-$. Let $x\in D(B_-)$ be given, then

$$\begin{aligned}\widetilde{S}(t)[S(s)-K_0(|s|)C]x&=[S(t)-K_0(|t|)C]\widetilde{S}(s)x=\widetilde{S}(s)[S(t)-K_0(|t|)C]x\\&=\widetilde{S}(s)[-\widetilde{S}(t)B_-x]=\widetilde{S}(s)[\widetilde{S}(t)(-B_-)x]\\&=\widetilde{S}(t)[\widetilde{S}(s)(-B_-)x],\end{aligned}$$

for all $-T_0<t\le 0\le s<T_0$. By the nondegeneracy of $S_-(\cdot)$, we have $[S(s)-K_0(|s|)C]=\widetilde{S}(s)[-B_-x]$ for all $0\le s<T_0$, and so $x\in D(B_+)$ and $B_+x=-B_-x$. Hence, $-B_-\subset B_+$. By symmetry, we also have $B_+\subset -B_-$. Consequently, $B_+=-B_-$. □

Theorem 2.4. *Let $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ be a strongly continuous family in $L(X)$ which commutes with C on X. Assume that $S(\cdot)$ has a subgenerator. Then $S(\cdot)$ is a local K-convoluted C-group on X. Moreover, $S(\cdot)$ is nondegenerate if the injectivity of C is added and K_0 is a nonzero function on $[0,T_0)$.*

Combining Remark 2.2 with [13, Lemma 2.8], the next lemma is also obtained.

Lemma 2.2. *Let A be a closed subgenerator of a strongly continuous family $S(\cdot)(=\{S(t)\,|\,|t|<T_0\})$ in $L(X)$, and K_0 a kernel on $[0,t_0)$ (or equivalently, K is a kernel on $[0,t_0)$) for some $0<t_0\leq T_0$. Assume that C is injective, and $u\in C((-t_0,t_0),X)$ satisfies $u(\cdot)=Aj_0*u(\cdot)$ on $(-t_0,t_0)$. Then $u=0$ on $(-t_0,t_0)$.*

By slightly modifying the proof of [13, Theorem 2.7], we can apply Lemma 2.2 to deduce the next theorem concerning nondegenerate K-convoluted C-groups, and so its proof is omitted.

Theorem 2.5. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-group on X with generator A. Assume that $S(\cdot)$ has a subgenerator. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover, if C is injective. Then (1.1)–(1.3) hold, and (1.4) also holds when K_0 is a kernel on $[0,T_0)$ or $T_0=\infty$.*

Lemma 2.3. *Let $S(\cdot)$ be a local K-convoluted C-group on X and $0\in\operatorname{supp}K_0$ (the support of K_0), or a K-convoluted C-group on X and K_0 not the zero function on $[0,\infty)$. Assume that $S(\cdot)x=0$ on $[0,t_0)$ or on $(-t_0,0]$ for some $x\in X$ and $0<t_0\leq T_0$. Then $CS(\cdot)x=0$ on $(-T_0,T_0)$. In particular, $S(t)x=0$ for all $|t|<T_0$ if the injectivity of C is added.*

Proof. Let $S(\cdot)x=0$ on $[0,t_0)$ and $|t|<T_0$ be given, then $|t|+s<T_0$ and $K_0(s)\neq 0$ for some $0<s<t_0$, so that $\widetilde{S}(s)S(t)x{=}S(t)\widetilde{S}(s)x=0$, $S(s)\widetilde{S}(t)x{=}\widetilde{S}(t)S(s)x=0$, and $\widetilde{S}(s)K_0(|t|)Cx=K_0(|t|)C\widetilde{S}(s)x=0$. By Theorem 2.3, we have $\widetilde{S}(s)[S(t)-K_0(|t|)C]x$ $=[S(s)-K_0(s)C]\widetilde{S}(t)x$. Hence, $K_0(s)\widetilde{S}(t)Cx=K_0(s)C\widetilde{S}(t)x=0$, which implies that $\widetilde{S}(t)Cx=0$. Since $|t|<T_0$ is arbitrary, we have $CS(t)x=S(t)Cx=0$ for all $|t|<T_0$. In particular, $S(t)x=0$ for all $|t|<T_0$ if the injectivity of C is added. □

Lemma 2.4. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-group on X with generator A and $0\in\operatorname{supp}K_0$. Assume that C is injective. Then A is a subgenerator of $S(\cdot)$.*

Proof. By Theorems 2.2 and 2.3, A is the generator of $S_+(\cdot)$ and $-A$ is the generator of $S_-(\cdot)$. It follows from [13, Theorem 2.9] that A is a subgenerator of $S_+(\cdot)$ and $-A$ is a subgenerator of $S_-(\cdot)$, which together with Remark 2.2 implies that A is a subgenerator of $S(\cdot)$. □

By slightly modifying the proof of Lemma 2.4, the next lemma concerning nondegenerate K-convoluted C-groups is also attained.

Lemma 2.5. *Let $S(\cdot)$ be a nondegenerate K-convoluted C-group on X with generator A. Then C is injective, and A is a subgenerator of $S(\cdot)$.*

Combining Theorem 2.5 with Lemma 2.5, the next theorem concerning nondegenerate K-convoluted C-groups is also obtained.

Theorem 2.6. *Let $S(\cdot)$ be a nondegenerate K-convoluted C-group on X with generator A. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover,* (1.1)–(1.4) *hold.*

Since $0 \in \mathrm{supp} K_0$ implies that K_0 is a kernel on $[0, T_0)$, we can apply Theorem 2.5 and Lemma 2.4 to obtain the next theorem.

Theorem 2.7. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-group on X with generator A and $0 \in \mathrm{supp}\, K_0$. Assume that C is injective. Then A is the maximal subgenerator of $S(\cdot)$, and each subgenerator of $S(\cdot)$ is closable and its closure is also a subgenerator of $S(\cdot)$. Moreover,* (1.1)–(1.4) *hold.*

Theorem 2.8. *Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $L(X)$ which has a subgenerator and K_0 a kernel on $[0, T_0)$. Assume that C is injective. Then $S(\cdot)$ is a nondegenerate local K-convoluted C-group on X, $CA \subset AC$ and $C^{-1}AC$ is the generator of $S(\cdot)$ for each closed subgenerator A of $S(\cdot)$. In particular, $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$.*

Proof. Suppose that A is a closed subgenerator of $S(\cdot)$. By Remark 2.2, A is a closed subgenerator of $S_+(\cdot)$. By [13], Theorem 2.13, we have $CA \subset AC$ and $C^{-1}AC$ is the generator of $S_+(\cdot)$. By Theorem 2.3, $C^{-1}AC$ is the generator of $S(\cdot)$. Similarly, we can show that $C^{-1}\overline{A_0}C$ is the generator of $S(\cdot)$ for each subgenerator A_0 of $S(\cdot)$. □

Corollary 2.1. *Let $S(\cdot)$ be a nondegenerate local K-convoluted C-group on X which has a subgenerator and K_0 a kernel on $[0, T_0)$. Assume that C is injective and $R(C)$ is dense in X. Then A is a closed subgenerator of $S_+(\cdot)$ if and only if $-A$ is a closed subgenerator of $S_-(\cdot)$.*

Proof. By Remark 2.2, we need only to show that A is a closed subgenerator of $S(\cdot)$ when A is a closed subgenerator of $S_+(\cdot)$. Since $\int_0^t S(r)Axdr = \int_0^t S(r)C^{-1}ACxdr = S(t)x - K_0(|t|)Cx$ for all $x \in D(A)$ and $|t| < T_0$, we remain only to show that (2.19) holds for all $x \in X$ and $|t| < T_0$. Suppose that $x \in X$ and $|t| < T_0$ are given. By [13], Theorem 2.13, $C^{-1}AC$ is the generator of $S_+(\cdot)$. By Theorem 2.3, $C^{-1}AC$ is the generator of $S(\cdot)$. By Theorems 2.5 and 2.8, $C^{-1}AC$ is the maximal subgenerator of $S(\cdot)$, and so $C^{-1}AC \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$. Hence, $AC \int_0^t S(r)xdr = A \int_0^t S(r)Cxdr = S(t)Cx - K_0(|t|)CCx$, which together with the denseness of $R(C)$ implies that $A \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$ for all $x \in X$ and $|t| < T_0$.

□

Remark 2.4. Let $S(\cdot)(= \{S(t) \mid |t| < T_0\})$ be a strongly continuous family in $\mathrm{L}(X)$. Then $S(\cdot)$ is a local K-convoluted C-group on X with closed subgenerator A if and only if $\mathrm{sgn}(\cdot)\widetilde{S}(\cdot)$ is a local K_0-convoluted C-group on X with closed subgenerator A.

3. Abstract Cauchy Problems

In the following, we always assume that $C \in L(X)$ is injective, K_0 a kernel on $[0, T_0)$, and A a closed linear operator in X such that $CA \subset AC$. We first note some basic properties concerning the solutions of ACP(A, f, x) just as results in [13] for the case of A is the generator of a nondegenerate local K_0-convoluted C-semigroup on X.

Proposition 3.1. *Let A be a subgenerator of a nondegenerate local K_0-convoluted C-group $S(\cdot)$ on X. Then for each $x \in D(A)$, $\operatorname{sgn}(\cdot)S(\cdot)x$ is the unique solution of ACP$(A, K_0(|\cdot|)Cx, 0)$ in $C((-T_0, T_0), [D(A)])$. Here $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$.*

Proposition 3.2. *Let A be a subgenerator of a nondegenerate local K-convoluted C-group $S(\cdot)$ on X and $C^1 = \{x \in X \mid S(\cdot)x$ is continuously differentiable on $(-T_0, T_0)\}$. Then*

(i) *for each $x \in C^1$, $S(t)x \in D(A)$ for a.e. $t \in (-T_0, T_0)$;*
(ii) *for each $x \in C^1$, $S(\cdot)x$ is the unique solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)Cx, 0)$;*
(iii) *for each $x \in D(A)$, $S(\cdot)x$ is the unique solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)Cx, 0)$ in $C((-T_0, T_0), [D(A)])$.*

Proposition 3.3. *Let A be the generator of a nondegenerate local K-convoluted C-group $S(\cdot)$ on X and $x \in X$. Assume that $S(t)x \in R(C)$ for all $|t| < T_0$ and $C^{-1}S(\cdot)x \in C((-T_0, T_0), X)$ is differentiable a.e. on $(-T_0, T_0)$. Then $C^{-1}S(t)x \in D(A)$ for a.e. $t \in (-T_0, T_0)$ and $C^{-1}S(\cdot)x$ is the unique solution of*

$$\text{ACP}(A, \operatorname{sgn}(\cdot)K(|\cdot|)x, 0).$$

Proof. Clearly, $S(\cdot)x = CC^{-1}S(\cdot)x$ is differentiable a.e. on $(-T_0, T_0)$. By (1.1)–(1.4), we have

$$\begin{aligned} C\frac{d}{dt}C^{-1}S(t)x =& \frac{d}{dt}S(t)x \\ =& AS(t)x + (\operatorname{sgn} t)K(|t|)Cx = ACC^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)Cx, \end{aligned}$$

for a.e. $t \in (-T_0, T_0)$. Hence, for a.e. $t \in (-T_0, T_0)$, $C^{-1}S(t)x \in \mathrm{D}(C^{-1}AC) = \mathrm{D}(A)$ and

$$\frac{d}{dt}\mathrm{C}^{-1}S(t)x = (C^{-1}AC)C^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)x = AC^{-1}S(t)x + (\operatorname{sgn} t)K(|t|)x,$$

which implies that $C^{-1}S(\cdot)x$ is a solution of ACP$(A, \operatorname{sgn}(\cdot)K(|\cdot|)x, 0)$. □

Applying Theorem 2.8, we can investigate an important result concerning the relation between the generation of a nondegenerate local K-convoluted C-group on X with subgenerator A and the unique existence of solutions of ACP(A, f, x), which extends some results in [13] for the case of local K-convoluted C-semigroup

Theorem 3.1. *The following statements are equivalent.*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-group $S(\cdot)$ on X.*

(ii) *For each* $x \in X$ *and* $g \in L^1_{loc}((-T_0, T_0), X)$, *ACP*$(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ *has a unique solution in* $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$.
(iii) *For each* $x \in X$ *the problem* *ACP*$(A, K_0(|\cdot|)Cx, 0)$ *has a unique solution in* $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$.
(iv) *For each* $x \in X$ *the integral equation* $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$ *has a unique solution* $v(\cdot; x)$ *in* $C((-T_0, T_0), X)$.

In this case, $\widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$ *is the unique solution of* *ACP*$(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ *and* $v(\cdot; x) = S(\cdot)x$.

Proof. We will first prove that (i)⇒(ii) holds. Let $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ be given. We set $u(\cdot) = \widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$, then $u \in \mathrm{C}^1((-T_0, T_0), X) \cap \mathrm{C}((-T_0, T_0), [\mathrm{D}(A)])$, $u(0) = 0$, and

$$\begin{aligned} Au(t) =& A\widetilde{S}(t)x + A\int_0^t \widetilde{S}(t-s)g(s)ds \\ =& S(t)x - K_0(|t|)Cx + \int_0^t [S(t-s) - K_0(|t-s|)C]g(s)ds \\ =& S(t)x + \int_0^t S(t-s)g(s)ds - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)] \\ =& u'(t) - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)], \end{aligned}$$

for all $0 \le t < T_0$. Hence, u is a solution of $\mathrm{ACP}(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ in $\mathrm{C}^1((-T_0, T_0), X) \cap \mathrm{C}((-T_0, T_0), [\mathrm{D}(A)])$. The uniqueness of solutions for $\mathrm{ACP}(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$ follows directly from the uniqueness of solutions for $\mathrm{ACP}(A, 0, 0)$.

Clearly, (ii)⇒(iii) holds, and (iii) and (iv) both are equivalent. We remain only to show that (iv)⇒(i) holds. The assumption of *(iv)* implies that for each $x \in X$, $v_+(\cdot) = v(\cdot; x)$ on $[0, T_0)$ is a unique solution of the integral equation $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$ on $[0, T_0)$, which together with [13, Theorem 3.4] implies that A is a subgenerator of a nondegenerate local K-convoluted C-semigroup on X. Similarly, we can show that $-A$ is a subgenerator of a nondegenerate local K-convoluted C-semigroup on X. It follows from Remark 2.2 and Theorem 2.2 that A is a subgenerator of a nondegenerate local K-convoluted C-group on X. □

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.5] to obtain the next result, and so its proof is omitted.

Theorem 3.2. *Assume that* $R(C) \subset R(\lambda - A)$ *for some* $\lambda \in \mathbb{F}$ *and*

$$ACP(A, \mathrm{sgn}(\cdot)K(|\cdot|)x, 0)$$

has a unique solution in $C((-T_0, T_0), [D(A)])$ *for each* $x \in D(A)$ *with* $(\lambda - A)x \in R(C)$. *Then* A *is a subgenerator of a nondegenerate local* K*-convoluted* C*-group on* X.

Since $C^{-1}AC = A$ and $R((\lambda - A)^{-1}C) = C(D(A))$ if $\rho(A) \neq \emptyset$, we can apply Theorem 3.2 to obtain the next corollary.

Corollary 3.1. *Assume that the resolvent set of $A : D(A) \to X$ is nonempty. Then A is the generator of a nondegenerate local K-convoluted C-group on X if and only if for each $x \in D(A)$ ACP$(A, \mathrm{sgn}(\cdot)K(|\cdot|)Cx, 0)$ has a unique solution in $C((-T_0, T_0), [D(A)])$.*

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.7] to obtain the next result, and so its proof is omitted.

Theorem 3.3. *Assume that A is densely defined. Then the following are equivalent.*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-group $S(\cdot)$ on X.*
(ii) *For each $x \in D(A)$ ACP$(A, \mathrm{sgn}(\cdot)K(|\cdot|)Cx, 0)$ has a unique solution $u(\cdot; Cx)$ in $C((-T_0, T_0), [D(A)])$ which depends continuously on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $(-T_0, T_0)$.*

We end this paper with several illustrative examples.

Example 3.1. Let $X = C_b(\mathbb{R})$, and $S(t)$ for $t \in \mathbb{R}$ be bounded linear operators on X defined by $S(t)f(x) = f(x+t)$ for all $x \in \mathbb{R}$. Then for each $K \in L^1_{loc}([0, T_0), \mathbb{F})$ and $\beta > -1$, $\mathrm{sgn}(\cdot)K_\beta(|\cdot|) * S(\cdot) = \{\mathrm{sgn}(t)K_\beta(||) * S(t) \mid |t| < T_0\}$ is local a K_β-convoluted group on X which is also nondegenerate with a closed subgenerator $\frac{d}{dx}$ when K_0 is not the zero function on $[0, T_0)$ (or equivalently, K is not the zero in $\mathrm{L}^1_{loc}([0, T_0), \mathbb{F})$), but $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$ may not be a local K-convoluted group on X except for $K \in L^1_{loc}([0, T_0), \mathbb{F})$ so that $K * S(\cdot)$ is a strongly continuous family in $L(X)$ for which $\frac{d}{dx}$ is a closed subgenerator of $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$ when K_0 is not the zero function on $[0, T_0)$. Moreover, (1.1)–(1.4) hold and $\frac{d}{dx}$ is its generator and maximal subgenerator when K_0 is a kernel on $[0, T_0)$. In this case, $\frac{d}{dx} = \overline{A_0}$ for each subgenerator A_0 of $\mathrm{sgn}(\cdot)K(|\cdot|) * S(\cdot)$.

Example 3.2. Let $X = \mathrm{C}_b(\mathbb{R})$(or $L^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum\limits_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $\mathrm{UC}_b(\mathbb{R})$ (or $\mathrm{C}_0(\mathbb{R})$) $= \overline{D(A)}$. Here $a_0, a_1, \ldots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2, 19] that $\{S(t) \mid |t| < T_0\}$ defined by

$$(S(t)f)(x) = \frac{1}{\sqrt{2\pi}} \mathrm{sgn}(t) \int_0^t \int_{-\infty}^{\infty} K(|t-s|)\widetilde{\phi_s}(x-y)f(y)dyds,$$

for all $f \in X$ and $|t| < T_0$, is a norm continuous local K_0-convoluted group on X with closed subgenerator A if the real-valued polynomial $p(x) = \sum_{j=0}^{k} a_j (ix)^j$ satisfies $\sup_{x \in \mathbb{R}} p(x) < \infty$, and $K \in L^1_{loc}([0, T_0), \mathbb{F})$ is not the zero function on $[0, T_0)$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of ϕ_t with $\phi_t(x) = \int_0^t e^{p(x)s} ds$ for all $t \geq 0$. Now if K_0 is a kernel on $[0, T_0)$, then A is its generator and maximal subgenerator. Applying Theorem 3.1, we get that for each $f \in X$ and continuous function g on $(-T_0, T_0) \times \mathbb{R}$ with $\int_{-t}^{t} \sup_{x \in \mathbb{R}} |g(s,x)| ds < \infty$ for all $0 \leq t < T_0$, the function u on

$(-T_0, T_0) \times \mathbb{R}$ defined by

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} K_0(|t-s|)\widetilde{\phi_s}(x-y)f(y)dyds + \frac{1}{\sqrt{2\pi}} \int_0^t \int_0^{t-r} \int_{-\infty}^{\infty} K_0(|t-r-s|)\widetilde{\phi_s}(x-y)g(r,y)dydsdr,$$

for all $|t| < T_0$ and $x \in \mathbb{R}$, is the unique solution of

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=0}^{k} a_j \left(\frac{\partial}{\partial x}\right)^j u(t,x) + K_1(|t|)f(x) + \int_0^t K_1(|t-s|)g(s,x)ds,$$
$$u(0,x) = 0, \quad \text{for } t \in (-T_0, T_0) \text{ and a.e. } x \in \mathbb{R},$$

in $\mathrm{C}^1((-T_0, T_0), \mathrm{X}) \cap \mathrm{C}((-T_0, T_0), [D(\mathrm{A})])$.

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