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THE NEW INEQUALITIES FOR tgs-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some Hadamard-Hadamard type inequalities for tgs-convex functions. Our results are the generalizations of some known results. The new generalized estimate of the midpoints product of two tgs-convex functions is also considered.

1. Introduction

Definition 1.1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

For convex functions, we have the following inequality which is known in the literature as Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is a positive concave function, then the inequality is reversed.

In 1906, Fejér [1] showed the following weighted generalization of inequality (1.2).

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Theorem 1.2. *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b q(t)dt \leq \frac{1}{b-a}\int_a^b f(t)q(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b q(t)dt, \tag{1.3}$$

where $q : [a, b] \to \mathbb{R}$ is positive, integrable, and symmetric with respect to $\frac{a+b}{2}$.

Some refinements, variations, generalizations and improvements of inequalities (1.2) and (1.3) can be seen [2, 3] and [4].

Definition 1.2 ([5])**.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. f is called a tgs-convex function on I if the inequality

$$f(tx + (1-t)y) \leq t(1-t)(f(x) + f(y)) \tag{1.4}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

For tgs-convex functions, the following results hold [5].

Theorem 1.3. *Assume that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a tgs-convex function and $a, b \in I$ with $a < b$, then we have*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{6}. \tag{1.5}$$

Theorem 1.4. *Assume that f and g are real valued, nonnegative tgs-convex functions on $[a, b]$, then we have*

$$8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)g(t)dt + \frac{1}{30}\left[M(a,b) + N(a,b)\right], \tag{1.6}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

The recent results on tgs-convex functions can be seen in [5, 6] and [7].

In this paper, we give the improvements of (1.5) and (1.6). The weighted generalization of inequality (1.5) are also established.

2. Main Results

The following result is an improvement of (1.5).

Theorem 2.1. *Assume that $f : [a, b] \to \mathbb{R}$ is a tgs-convex function, then we have*

$$\begin{aligned} 4f\left(\frac{a+b}{2}\right) \leq& f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \\ \leq& \frac{1}{b-a}\int_a^b f(t)dt \\ \leq& \frac{f(a)+f(b)}{12} + \frac{f(\frac{a+b}{2})}{6} \end{aligned} \tag{2.1}$$

$$\leq \frac{f(a)+f(b)}{8}.$$

Proof. Using (1.5) in $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$, we have

$$2f\left(\frac{3a+b}{4}\right) \leq \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt \leq \frac{f(a)+f(\frac{a+b}{2})}{6},$$
$$2f\left(\frac{a+3b}{4}\right) \leq \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(t)dt \leq \frac{f(b)+f(\frac{a+b}{2})}{6}.$$

Form the above inequalities, we have

$$f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)+2f(\frac{a+b}{2})}{12}.$$

A combination of the above inequality and the following results

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{(3a+b)/4+(a+3b)/4}{2}\right) \leq \frac{1}{4}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right),$$
$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{4},$$

deduces the desired inequality (2.1). ☐

The following Hadamard-Hadamard-Fejér type inequality for tgs-convex function holds.

Theorem 2.2. *Assume that $f : [a,b] \to \mathbb{R}$ is a tgs-convex function, then we have*

$$2f\left(\frac{a+b}{2}\right)\int_a^b q(x)dx \leq \int_a^b f(x)q(x)dx \leq (f(a)+f(b))\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}q(x)dx, \tag{2.2}$$

where $q : [a,b] \to \mathbb{R}$ is positive, integrable, and symmetric with respect to $\frac{a+b}{2}$.

Proof. Since $q(x) = q(a+b-x)$, we have

$$2f\left(\frac{a+b}{2}\right)\int_a^b q(x)dx \leq 2\int_a^b f\left(\frac{x}{2}+\frac{a+b-x}{2}\right)q(x)dx,$$
$$\leq \frac{1}{2}\int_a^b f(x)q(x)dx + \frac{1}{2}\int_a^b f(a+b-x)q(a+b-x)dx$$
$$= \int_a^b f(x)q(x)dx.$$

On the other hand,

$$\int_a^b f(x)q(x)dx = (b-a)\int_0^1 f(tb+(1-t)a)q(tb+(1-t)a)dt$$

$$\leq (b-a)(f(a)+f(b))\int_0^1 t(1-t)q(tb+(1-t)a)dt$$
$$=(f(a)+f(b))\int_a^b \frac{(b-x)(x-a)}{(b-a)^2}q(x)dx. \qquad \square$$

Remark 2.1. We get (1.5) by putting $q(x)=1$ in (2.2).

The following inequalities are improvements of (1.6).

Theorem 2.3. *Assume that f and g are real valued, nonnegative tgs-convex functions on $[a,b]$, then we have*

$$\begin{aligned}8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq & \frac{1}{b-a}\int_a^b f(t)q(t)dt \\ & + \frac{1}{60}(N((a+b)/2,(a+b)/2)+N(a,b) \\ & + N(a,(a+b)/2)+N((a+b)/2,b)] \\ \leq & \frac{1}{b-a}\int_a^b f(t)dt \\ & + \frac{1}{480}[5M(a,b)+13N(a,b)],\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined in Theorem 1.4.

Proof. For $\lambda \in [0,1]$, we have

$$\begin{aligned}& 8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ =& 8f\left(\frac{(1-\lambda)b+\lambda(a+b)/2}{2}+\frac{(1-\lambda)a+\lambda(a+b)/2}{2}\right) \\ & \times g\left(\frac{(1-\lambda)b+\lambda(a+b)/2}{2}+\frac{(1-\lambda)a+\lambda(a+b)/2}{2}\right) \\ \leq & \frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2) \\ & + \frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2) \\ & + \frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2) \\ & + \frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2) \\ \leq & \frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2) \\ & + \frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-\lambda)^2\lambda^2[f(a)+f((a+b)/2))(g((a+b)/2)+g(b))\\
&+(f((a+b)/2)+f(b))(g((a+b)/2)+g(a))]\\
=&\frac{1}{2}f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)\\
&+\frac{1}{2}f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)\\
&+\frac{1}{2}(1-\lambda)^2\lambda^2[N((a+b)/2,(a+b)/2))\\
&+N(a,b)+N(a,(a+b)/2)+N((a+b)/2,b)].
\end{aligned}
$$

Integrating both sides of the above inequality with respect to λ over $[0,1]$, we have

$$
\begin{aligned}
&8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
\le&\frac{1}{2}\int_0^1 f((1-\lambda)b+\lambda(a+b)/2)g((1-\lambda)b+\lambda(a+b)/2)d\lambda\\
&+\frac{1}{2}\int_0^1 f((1-\lambda)a+\lambda(a+b)/2)g((1-\lambda)a+\lambda(a+b)/2)d\lambda\\
&+\frac{1}{2}\int_0^1(1-\lambda)^2\lambda^2[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2))+N((a+b)/2,b)]d\lambda\\
=&\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b f(x)g(x)dx+\int_a^{\frac{a+b}{2}} f(x)g(x)dx\right]\\
&+\frac{1}{60}[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2)+N((a+b)/2,b)]\\
=&\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{60}[N((a+b)/2,(a+b)/2)+N(a,b)\\
&+N(a,(a+b)/2)+N((a+b)/2,b)].
\end{aligned}
$$

On the other hand, since

$$N((a+b)/2,(a+b)/2)\le\frac{1}{8}[M(a,b)+N(a,b)] \tag{2.3}$$

and

$$N(a,(a+b)/2)+N((a+b)/2,b)\le\frac{1}{2}[M(a,b)+N(a,b)], \tag{2.4}$$

we have

$$
\begin{aligned}
8f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le&\frac{1}{b-a}\int_a^b f(x)g(x)dx\\
&+\frac{1}{60}(N((a+b)/2,(a+b)/2)+N(a,b)
\end{aligned}
$$

$$
\begin{aligned}
&+N(a,(a+b)/2)+N((a+b)/2,b)] \\
\leq&\frac{1}{b-a}\int_a^b f(t)dt \\
&+\frac{1}{480}[5M(a,b)+13N(a,b)]. \qquad \square
\end{aligned}
$$

3. Applications to Probability Density Function

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=Pr(X\leq x)=\int_a^x f(t)dt$.

Theorem 3.1. *With the assumptions of Theorem* 2.1*, we have the inequality*

$$
\begin{aligned}
4F\left(\frac{a+b}{2}\right)\leq& F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right) \\
\leq&\frac{b-E(X)}{b-a} \\
\leq&\frac{F(a)+F(b)}{12}+\frac{F((a+b)/2)}{6} \\
\leq&\frac{F(a)+F(b)}{8}.
\end{aligned}
\tag{3.1}
$$

Proof. In the proof of Theorem 2.1, letting $f=F$, and taking into account that

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)dt,$$

we obtain (3.1). $\square$

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References

[1] L. Fejér, *Über die Fourierreihen II*, Anz Ungar. Akad. Wiss. **24** (1906), 369–360.

[2] S. Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, Rocky Mountain J. Math. **39**(5) (2009), 1741–1749. `https://doi.org10.1216/RMJ-2009-39-5-1741`

[3] F. X. Chen and X. F. Liu, *Refinements on the Hermite-Hadamard inequalities for r-convex functions*, J. Appl. Math. **2013** (2013), Article ID 978493, 5 pages. `https://doi.org/10.1155/2013/978493`

[4] Z. G. Xiao, Z. H. Zhang and Y. D. Wu, *On weighted Hermite-Hadamard inequalities*, Appl. Math. Comput. **218**(3) (2011), 1147–1152. `https://doi.org/10.1016/j.amc.2011.03.081`

[5] M. Tunç, E. Göv and U. Şanal, *On tgs-convex function and their inequalities*, Facta Univ. Ser. Math. Inform. **30**(5) (2015), 679–691.
[6] M. Tunç and U. Şanal, *Some perturbed trapezoid inequalities for convex, s-convex and tgs-convex functions and applications*, Tbilisi Math. J. **8**(2) (2015), 87–102. https://doi.org/10.1515/tmj-2015-0013
[7] M. A. Noor, M. U. Awan, K. I. Noor and F. Safdar, *Some new quantum inequalities via tgs-convex functions*, TWMS J. Pure Appl. Math. **9**(2) (2018), 135–146.

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