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INEQUALITIES FOR HYPERBOLIC TYPE HARMONIC PREINVEX FUNCTION

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Abstract. In the present paper, we have introduced a new class of preinvexity namely hyperbolic type harmonic preinvex functions and to support this new definition, some of its algebraic properties are elaborated. By using this new class of preinvexity, we have established a few Hermite-Hadamard type integral inequalities. Some novel refinements of Hemite-Hadamard type inequalities for hyperbolic type harmonic preinvex functions are presented as well. Finally, the Riemann-Liouville fractional version of the Hermite-Hadamard Inequality is established.

1. Preliminaries

Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $p < q$ and $p, q \in \mathbb{K}$. Then the Hermite-Hadamard inequality is expressed as follows (see [1]):

$$\varphi\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q \varphi(x)dx \leq \frac{\varphi(p)+\varphi(q)}{2}. \tag{1.1}$$

The Hermite-Hadamard inequality which was proved separately by Hermite in 1883 and Hadamard in 1896 is extensively studied in the convex theory. The double inequality is known as Hermite-Hadamard integral inequality for convex function in the literature. It deals with a necessary and sufficient condition for a function to be convex. For some recent results associated with the inequality (1.1) we recommend interested readers to go through [2–5] and the references therein.

Key words and phrases. Preinvex function, Hyperbolic type convex function, fractional calculus, Hölder integral inequality, Hermite-Hadamard inequality.

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Recently, the concept of convexity has experienced very interesting developments. Many researchers generalised the classical concepts of convex sets and functions in different directions. A significant extension of convex function is invex function, introduced by Hanson [6]. Consequently, preinvex function is introduced by Ben Israel et al [7] and Weir et al. [8]. Rita Pini [9], introduced the concept of prequasi-invex as an extension of invex function.

Definition 1.1 ([10])**.** A function $\varphi : \mathbb{K} = [p, p + \eta(q, p)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic preinvex function if, the inequality

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq (1-k)\varphi(p) + k\varphi(q), \quad \text{for all } p, q \in K,\ k \in [0,1], \tag{1.2}$$

holds, where $\eta(\cdot,\cdot) : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is a bifunction.

For $\eta(q, p) = q - p$, (1.2) reduces to the inequality for harmonic convex function.

If the inequality is reversed in (1.2), then f is said to be harmonically preconcave function.

Condition C ([11]). Let $\mathbb{K} \subseteq \mathbb{R}$ be an invex set with respect to bi-function $\eta(\cdot,\cdot)$. Then for any $p, q \in \mathbb{K}$ and $k \in [0, 1]$

$$\begin{aligned} \eta\left(p, p + k\eta(q, p)\right) &= -k\eta(q, p), \\ \eta\left(q, p + k\eta(q, p)\right) &= (1-k)\eta(q, p), \end{aligned}$$

for every $p, q \in \mathbb{K}$, $k_1, k_2 \in [0, 1]$ and using Condition C, we get

$$\eta(p + k_2\eta(q, p), p + k_1\eta(q, p)) = (k_2 - k_1)\eta(q, p).$$

In [11], Mohan and Neogy proved that a differentiable function which is invex on $\mathbb{K}$, w.r.t η, is also preinvex under Condition C.

İşcan proved the Hermite-Hadamard type inequality for the harmonically convex function.

Theorem 1.1 ([12, Theorem 2.4])**.** *Let $\mathbb{K} \subseteq (0,\infty)$ be an interval and $\varphi : \mathbb{K} \to \mathbb{R}$ be a harmonically convex function with $p < q$ and $p, q \in \mathbb{K}$. Then the Hermite-Hadamard type inequality*

$$\varphi\left(\frac{2pq}{p+q}\right) \leq \frac{pq}{q-p}\int_p^q \frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(p)+\varphi(q)}{2} \tag{1.3}$$

holds.

Noor [10], has proved that a function φ is harmonic preinvex if and only if φ satisfies the following inequality

$$\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(p)+\varphi(q)}{2}.$$

Toplu [13], introduced the concept of Hyperbolic type convexity as follows.

Definition 1.2. A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type convex function if for every $p, q \in \mathbb{K}$ and $k \in [0,1]$, the inequality

$$\varphi(kp + (1-k)q) \leq \frac{\sinh k}{\sinh 1}\varphi(p) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(q)$$

holds.

Theorem 1.2 ([13, Theorem 3.1])**.** *Let $\varphi : [p,q] \to \mathbb{R}$ be a hyperbolic type convex function. If $p < q$ and $\varphi \in \mathcal{L}[p,q]$, then the following Hermite-Hadamard type inequality holds.*

$$\varphi\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q \varphi(x)dx \leq \frac{\cosh 1 - 1}{\sinh 1}\varphi(p) + \frac{e-1}{e\sinh 1}\varphi(q).$$

2. Main Result

In this section, we introduce new classes of hyperbolic type harmonic preinvex function. The main purpose of this paper is to introduce the concept of preinvexity for hyperbolic type harmonic convex functions and establish some results associated with the right hand side of the inequalities similar to (1.3) for the classes of hyperbolic type harmonic preinvex functions. For some recent results connected with preinvexity see [14–20] and the references therein.

Definition 2.1. A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type preinvex function if and only if for every $p, q \in K$ and $k \in [0,1]$

$$\varphi(p + k\eta(q,p)) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)$$

holds.

Definition 2.2. A function $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ is called hyperbolic type harmonic preinvex function if and only if for every $p, q \in \mathbb{K}$ and $k \in [0,1]$

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) \tag{2.1}$$

holds.

Definition 2.3. Let $h : (0,1) \subseteq \mathbb{K} \to \mathbb{R}$ be a non negative function, then a real valued function $\varphi : \mathbb{K} \subseteq [0,\infty] \to \mathbb{R}$ is called hyperbolic type h-harmonic preinvex function if and only if for every $p, q \in \mathbb{K}$ and $k \in [0,1]$

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq h\left(\frac{\sinh k}{\sinh 1}\right)\varphi(q) + h\left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p) \tag{2.2}$$

holds.

Remark 2.1. If $h(k) = k$, then (2.2) reduces to (2.1).

Theorem 2.1. *Consider φ and ψ be two real valued hyperbolic type harmonic preinvex functions, then*

(i) *$\varphi+\psi$ is hyperbolic type harmonic preinvex function;*

(ii) *for $c \in \mathbb{R},\ c \geq 0$, the function $c\varphi$ is hyperbolic type harmonic preinvex function.*

Proof. (i) Let φ and ψ be two hyperbolic type harmonic preinvex functions, then

$$\begin{aligned}
&(\varphi+\psi)\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\\
=&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)+\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\\
\leq&\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)+\frac{\sinh k}{\sinh 1}\psi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(p)\\
=&\frac{\sinh k}{\sinh 1}[\varphi(q)+\psi(q)]+\frac{\sinh 1-\sinh k}{\sinh 1}[\varphi(p)+\psi(p)]\\
=&\frac{\sinh k}{\sinh 1}(\varphi+\psi)(q)+\frac{\sinh 1-\sinh k}{\sinh 1}(\varphi+\psi)(p).
\end{aligned}$$

(ii) Let φ be hyperbolic type harmonic preinvex functions and $c \in \mathbb{R},\ c \geq 0$, then

$$\begin{aligned}
(c\varphi)\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &\leq c\left(\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right)\\
&=\frac{\sinh k}{\sinh 1}c\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}c\varphi(p)\\
&=\frac{\sinh k}{\sinh 1}(c\varphi)(q)+\frac{\sinh 1-\sinh k}{\sinh 1}(c\varphi)(p).
\end{aligned}$$ □

Theorem 2.2. *If $\varphi : \mathbb{K} \to \mathbb{K}$ is a hyperbolic type harmonic convex and $\psi : \mathbb{K} \to \mathbb{R}$ is a nondecreasing convex function, then $\psi \circ \varphi : \mathbb{K} \to \mathbb{R}$ is a hyperbolic type harmonic preinvex function.*

Proof. For $\alpha, \beta \in \mathbb{K}$ and $k \in [0,1]$

$$\begin{aligned}
\psi\circ\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &= \psi\left(\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right)\\
&\leq \frac{\sinh k}{\sinh 1}\psi(\varphi(q))+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(\varphi(p))\\
&\leq \frac{\sinh k}{\sinh 1}\psi\circ\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi\circ\varphi(p).
\end{aligned}$$ □

Theorem 2.3. *Let $\varphi : [p,q] \to \mathbb{R}$ be an arbitrary family of hyperbolic type harmonic preinvex functions and let $\varphi(x) = \sup_\alpha \varphi_\alpha(x)$. If $\mathbb{K} = \{v \in [p,q] : \varphi(v) < \infty\}$ is nonempty, then $\mathbb{K}$ is an interval and φ is a hyperbolic type harmonic preinvex function on $\mathbb{K}$.*

Proof. For $p, q \in \mathbb{K}$ and $k \in [0,1]$

$$\begin{aligned}
\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) &= \sup_{\alpha} \varphi_\alpha\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \\
&\leq \sup_{\alpha} \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) \\
&\leq \frac{\sinh k}{\sinh 1}\sup_{\alpha}\varphi_\alpha(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\sup_{\alpha}\varphi_\alpha(p) \\
&= \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p) < \infty.
\end{aligned}$$

□

Definition 2.4 ([10])**.** Two functions u and v are said to be of similar ordered if

$$(u(\alpha) - u(\beta))(v(\alpha) - v(\beta)) \geq 0, \quad \text{for all } \alpha, \beta \in \mathbb{R}.$$

Theorem 2.4. *Let φ and ψ be two similar ordered hyperbolic type harmonic preinvex function, then the product of two hyperbolic harmonic preinvex function is again a hyperbolic type harmonic preinvex function.*

Proof. Let φ and ψ be two hyperbolic type harmonic preinvex function, then

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \\
&\leq \left[\frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\right]\left[\frac{\sinh k}{\sinh 1}\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\psi(p)\right] \\
&= \left(\frac{\sinh k}{\sinh 1}\right)\varphi(q)\psi(q) + \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p)\psi(p) \\
&\quad + \left(\frac{\sinh k}{\sinh 1}\right)^2\varphi(q)\psi(q) + \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)^2\varphi(p)\psi(p) \\
&\quad + \frac{\sinh k}{\sinh 1}\cdot\frac{\sinh 1 - \sinh k}{\sinh 1}\left[\psi(q)\varphi(p) + \varphi(q)\psi(p)\right] \\
&\quad - \left(\frac{\sinh k}{\sinh 1}\right)\varphi(q)\psi(q) - \left(\frac{\sinh 1 - \sinh k}{\sinh 1}\right)\varphi(p)\psi(p) \\
&= \left[\frac{\sinh k}{\sinh 1}\varphi(q)\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\psi(p)\right] \\
&\quad - \left[\frac{\sinh k}{\sinh 1} + \frac{\sinh 1 - \sinh k}{\sinh 1}\right](\varphi(p)\psi(p) + \varphi(q)\psi(q) - \varphi(p)\psi(q) - \varphi(q)\psi(p)) \\
&= \frac{\sinh k}{\sinh 1}\varphi(q)\psi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\psi(p).
\end{aligned}$$

□

3. Hermite-Hadamard Type Inequalities for Hyperbolic Type Harmonic Preinvex Function

Theorem 3.1. *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a hyperbolic type harmonic preinvex function and $p, p+\eta(q,p) \in \mathbb{K}$. If condition C holds and $\varphi \in \mathcal{L}[p, p+\eta(q,p)]$, then the following inequality holds:*

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \\ \leq& \frac{\cosh 1 - 1}{\sinh 1} \varphi(p+\eta(q,p)) + \frac{e-1}{e \sinh 1} \varphi(p) \\ \leq& \frac{\cosh 1 - 1}{\sinh 1} \varphi(q) + \frac{e-1}{e \sinh 1} \varphi(p).\end{aligned}$$

Proof. Since φ is hyperbolic type harmonic preinvex function putting $k = \frac{1}{2}$ and choosing $x = \frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}$ and $y = \frac{p(p+\eta(q,p))}{p+k\eta(q,p)}$ in

$$\varphi\left(\frac{x(x+\eta(y,x))}{x+(1-k)\eta(y,x)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(y) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(x).$$

Using Condition C, we get

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \frac{\sinh \frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \\ &+ \frac{\sinh 1 - \sinh \frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right).\end{aligned}$$

Integrating with respect to k over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\frac{2p(p+\eta(q,p))}{2p+\eta(q,p)}\right) \leq& \left(\frac{\sinh \frac{1}{2}}{\sinh 1}\right) \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \\ &+ \left(\frac{\sinh 1 - \sinh \frac{1}{2}}{\sinh 1}\right) \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \\ =& \frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx.\end{aligned}$$

Using the property of Hyperbolic type harmonic preinvex function and let $x = \frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}$, we have

$$\begin{aligned}\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx =& \int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) dk \\ \leq& \int_0^1 \left[\frac{\sinh k}{\sinh 1}\varphi(p+\eta(q,p)) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)\right] dk \\ =& \left(\frac{\cosh 1 - 1}{\sinh 1}\right)\varphi(p+\eta(q,p)) + \frac{e-1}{e\sinh 1}\varphi(p)\end{aligned}$$

$$\leq \left(\frac{\cosh 1 - 1}{\sinh 1}\right)\varphi(q) + \frac{e-1}{e\sinh 1}\varphi(p). \qquad \square$$

Lemma 3.1. *Consider $p, q \in \mathbb{R}$, then*

$$\min(p,q) \leq \frac{p+q}{2}.$$

Theorem 3.2. *Consider $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a hyperbolic type harmonic preinvex function and $p, q \in \mathbb{K}$. If $\varphi \in \mathcal{L}[p,q]$, the following inequality holds:*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \\
\leq &\min\left\{\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\varphi(p), \left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\varphi(q)\right\} \\
\leq &\frac{1}{2}\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\left[\varphi(p) + \varphi(q)\right].
\end{aligned}$$

Proof. Let φ be a hyperbolic type harmonic preinvex function. Then

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(q) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(p)$$

and

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(p) + \frac{\sinh 1 - \sinh k}{\sinh 1}\varphi(q).$$

Adding both the above inequalities, we get

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right) + \varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) \\
\leq &\frac{\sinh k}{\sinh 1}[\varphi(p) + \varphi(q)] + \frac{\sinh 1 - \sinh k}{\sinh 1}[\varphi(p) + \varphi(q)].
\end{aligned} \tag{3.1}$$

Integrating (3.1) over the interval $[0, 1]$, one has

$$\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2} dx \leq \frac{1}{2}\left(\frac{\cosh 1 - 1}{\sinh 1} + \frac{e-1}{\sinh 1}\right)\left[\varphi(p) + \varphi(q)\right]. \tag{3.2}$$

From Lemma 3.1 and (3.2), we have the desired result. $\square$

Theorem 3.3. *Let φ and ψ be two real valued hyperbolic type harmonic preinvex function, then*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)} \int_p^{p+\eta(q,p)} \frac{\varphi(u)\psi(u)}{u^2} du \\
\leq &\left(\frac{e^4 - 4e^2 - 1}{8e^2\sinh^2 1}\right)\varphi(q)\psi(q) + \left(\frac{-e^4 + 8e^3 - 8e^2 - 8e + 5}{8e^2\sinh^2 1}\right)\varphi(p)\psi(p) \\
&+ \left(\frac{e^4 - 4e^3 + 4e^2 + 4e - 1}{8e^2\sinh^2 1}\right)\left[\varphi(p)\psi(q) + \varphi(q)\psi(p)\right].
\end{aligned}$$

Proof. Considering φ and ψ be two hyperbolic type harmonic preinvex function, then

$$\begin{aligned}
&\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\psi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\\
&\leq\left[\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)\right]\left[\frac{\sinh k}{\sinh 1}\psi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\psi(p)\right]\\
&=\left(\frac{\sinh k}{\sinh 1}\right)^2\varphi(q)\psi(q)+\left(\frac{\sinh 1-\sinh k}{\sinh 1}\right)^2\varphi(p)\psi(p)\\
&\quad+\frac{\sinh k}{\sinh 1}\cdot\frac{\sinh 1-\sinh k}{\sinh 1}\left[\psi(q)\varphi(p)+\varphi(q)\psi(p)\right].
\end{aligned}$$

Integrating both sides of the above inequality with respect to k over $[0,1]$, one has

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(u)\psi(u)}{u^2}du\\
\leq&\frac{\varphi(q)\psi(q)}{\sinh^2 1}\int_0^1(\sinh k)^2dk+\frac{\varphi(p)\psi(p)}{\sinh^2 1}\int_0^1(\sinh 1-\sinh k)^2dk\\
&+\frac{\varphi(p)\psi(q)+\varphi(q)\psi(p)}{\sinh^2 1}\int_0^1\sinh k(\sinh 1-\sinh k)dk\\
=&\frac{\varphi(q)\psi(q)}{\sinh^2 1}\cdot\frac{(e^4-4e^2-1)}{8e^2}+\frac{\varphi(p)\psi(p)}{\sinh^2 1}\\
&\times\left[\frac{8e^2\sinh^2 1-16e^2\cosh 1\sinh 1+e^4-4e^2-1}{8e^2}+2\sinh 1\right]\\
&+\frac{[\varphi(p)\psi(q)+\varphi(q)\psi(p)]}{\sinh^2 1}\left[\frac{8e^2\cosh 1\sinh 1-e^4+4e^2+1}{8e^2}-\sinh 1\right]\\
=&\varphi(q)\psi(q)\left(\frac{e^4-4e^2-1}{8e^2\sinh^2 1}\right)+\varphi(p)\psi(p)\left(\frac{-e^4+8e^3-8e^2-8e+5}{8e^2\sinh^2 1}\right)\\
&+[\varphi(p)\psi(q)+\varphi(q)\psi(p)]\left(\frac{e^4-4e^3+4e^2+4e-1}{8e^2\sinh^2 1}\right).
\end{aligned}$$

□

Theorem 3.4. *Let φ and ψ be two similarly ordered real valued hyperbolic type harmonic preinvex function, then*

$$\begin{aligned}
&\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(u)\psi(u)}{u^2}du\\
&\leq\left(\frac{e^4-2e^3+2e-1}{4e^2\sinh^2 1}\right)\varphi(q)\psi(q)+\left(\frac{e^3-e^2-e+1}{2e^2\sinh^2 1}\right)\varphi(p)\psi(p).
\end{aligned}$$

Proof. The proof can be done by direct calculation using similarly ordered property.

□

4. Refinements of Hermite-Hadamard Inequality via Hyperbolic Type Harmonic Preinvexity

We now present the following lemma, which is a generalization of a result in [12].

Lemma 4.1. *Let $\varphi : \mathbb{K} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $p, p+\eta(q,p) \in \mathbb{K}$ with $p+\eta(q,p) > p$. If $\varphi' \in \mathcal{L}[p, p+\eta(q,p)]$, then the following identity holds in the preinvex setting:*

$$\begin{aligned}&\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\\ =&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\varphi'\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)dk.\end{aligned}$$

Proof. Considering

$$I = \int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\varphi'\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)dk,$$

after integrating by parts and some suitable rearrangements, the result is obtained. ☐

Theorem 4.1. *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $\varphi' \in \mathcal{L}\left([p, p+\eta(q,p)]\right)$, where $[p, p+\eta(q,p)] \subseteq \mathbb{K}^\circ$. If $|\varphi'|$ is hyperbolic harmonic preinvex function on $[p, p+\eta(q,p)]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\right|\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{|\varphi'(p)|}{\sinh 1}S_2 + \frac{|\varphi'(q)|}{\sinh 1}S_3\right],\end{aligned}$$

where

$$\begin{aligned}S_1 &= \int_0^1 \frac{|1-2k|}{(p+k\eta(q,p))^2}dk,\\ S_2 &= \int_0^1 \frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk,\\ S_3 &= \int_0^1 \frac{|1-2k|(\sinh 1 - \sinh k)}{(p+k\eta(q,p))^2}dk.\end{aligned}$$

Proof. From Lemma 4.1 and using the concept of Hyperbolic harmonic preinvexity of φ', we get

$$\begin{aligned}&\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\right|\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\int_0^1 \frac{(1-2k)}{(p+k\eta(q,p))^2}\left[\frac{\sinh k}{\sinh 1}|\varphi'(p)| + \frac{\sinh 1 - \sinh k}{\sinh 1}|\varphi'(q)|\right]dk\\ \leq&\frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{\varphi'(p)}{\sinh 1}\int_0^1 \frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk\right.\end{aligned}$$

$$+\frac{\varphi'(q)}{\sinh 1}\int_0^1 \frac{|1-2k|\,(\sinh 1-\sinh k)}{(p+k\eta(q,p))^2}dk\Bigg]$$
$$\leq \frac{p(p+\eta(q,p))\eta(q,p)}{2}\left[\frac{|\varphi'(p)|}{\sinh 1}S_2+\frac{|\varphi'(q)|}{\sinh 1}S_3\right]. \qquad \square$$

Theorem 4.2. *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $\varphi' \in \mathcal{L}([p, p+\eta(q,p)])$, where $[p, p+\eta(q,p)] \subseteq \mathbb{K}^\circ$. If $|\varphi'|^s$ is hyperbolic harmonic preinvex function on $[p, p+\eta(q,p)]$ for $s \geq 1$, then the following inequality holds:*

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{pq(q-p)}{2}A^{1-\frac{1}{s}}\left[B|\varphi'(p)|^s+C|\varphi'(q)|^s\right]^{\frac{1}{s}},$$

where

$$A=\int_0^1 \frac{|1-2k|}{(p+k\eta(q,p))^2}dk,$$
$$B=\int_0^1 \frac{|1-2k|\sinh k}{(p+k\eta(q,p))^2}dk,$$
$$C=\int_0^1 \frac{|1-2k|\,(\sinh 1-\sinh k)}{(p+k\eta(q,p))^2}dk.$$

Proof. From Lemma 4.1 and using the Hölder's inequality, we get

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{pq(q-p)}{2}\left(\int_0^1\left|\frac{|1-2k|}{(p+k\eta(q,p))^2}dk\right|\right)^{1-\frac{1}{s}}$$
$$\times\left(\int_0^1\left|\frac{|1-2k|}{(p+k\eta(q,p))^2}\right|\cdot\left|\varphi'\left(\frac{pq}{(p+k\eta(q,p))}\right)\right|^s dk\right)^{\frac{1}{s}}$$
$$\leq \frac{pq(q-p)}{2}\left(\int_0^1 \frac{|1-2k|}{(p+k\eta(q,p))^2}dk\right)^{1-\frac{1}{s}}$$
$$\times\left(\int_0^1 \frac{|1-2k|}{(p+k\eta(q,p))^2}\left[\frac{\sinh k}{\sinh 1}|\varphi'(p)|^s+\frac{\sinh 1-\sinh k}{\sinh 1}|\varphi'(q)|^s\right]dk\right)^{\frac{1}{s}}$$
$$\leq \frac{pq(q-p)}{2}A^{1-\frac{1}{s}}\left[B|\varphi'(p)|^s+C|\varphi'(q)|^s\right]^{\frac{1}{s}}. \qquad \square$$

Theorem 4.3. *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{K}^\circ$ and $\varphi' \in \mathcal{L}([p, p+\eta(q,p)])$, where $[p, p+\eta(q,p)] \subseteq \mathbb{K}^\circ$. If $|\varphi'|^s$ is hyperbolic harmonic preinvex function on $[p, p+\eta(q,p)]$ for $s \geq 1$, then the following inequality*

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2}-\frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)}\frac{\varphi(x)}{x^2}dx\right|$$

$$\leq \frac{pq(q-p)}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \frac{|\varphi'(p)|^s - |\varphi'(q)|^s}{\sinh 1}\int_0^1 \frac{\sinh k}{(p+k\eta(q,p))^{2s}}dk$$
$$+ |\varphi'(q)|^s \int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}}dk$$

holds.

Proof. From Lemma 4.1 and using the Hölder's Integral inequality, we get

$$\left|\frac{\varphi(p)+\varphi(p+\eta(q,p))}{2} - \frac{p(p+\eta(q,p))}{\eta(q,p)}\int_p^{p+\eta(q,p)} \frac{\varphi(x)}{x^2}dx\right|$$
$$\leq \frac{p(p+\eta(q,p))}{2}\left(\int_0^1 |1-2k|^r\right)^{\frac{1}{r}}$$
$$\times \left(\int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}}\left|\varphi'\left(\frac{pq}{(p+k\eta(q,p))}\right)\right|^s dk\right)^{\frac{1}{s}}$$
$$\leq \frac{p(p+\eta(q,p))}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \times \left(\int_0^1 \frac{\left[\frac{\sinh k}{\sinh 1}|\varphi'(p)|^s + \frac{\sinh 1 - \sinh k}{\sinh 1}|\varphi'(q)|^s\right]}{(p+k\eta(q,p))^{2s}}dk\right)^{\frac{1}{s}}$$
$$= \frac{p(p+\eta(q,p))}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \frac{|\varphi'(p)|^s - |\varphi'(q)|^s}{\sinh 1}\int_0^1 \frac{\sinh k}{(p+k\eta(q,p))^{2s}}dk$$
$$+ |\varphi'(q)|^s \int_0^1 \frac{1}{(p+k\eta(q,p))^{2s}}dk. \qquad \square$$

5. Hermite-Hadamard Type Inequality via Fractional Integral

In this section, we have extended the above theorem 3.1 in the frame of Riemann-Liouville fractional operator. Recently, it is seen that integral inequalities using fractional operator has become an astonishing topic of research among mathematicians, for some recent papers and details (see [21, 22]).

Definition 5.1. Let $\varphi \in \mathcal{L}[p,q]$. The Riemann-Liouville operator $J^m_{q^-}\varphi$ and $J^m_{p^+}\varphi$ of order $m \geq 0$ are defined as

$$J^m_{p+}\varphi(x) = \frac{1}{\Gamma(m)}\int_p^x (x-k)^{m-1}\varphi(k)dk, \quad x \geq p,$$

and

$$J^m_{q-}\varphi(x) = \frac{1}{\Gamma(m)}\int_x^q (k-x)^{m-1}\varphi(k)dk, \quad x \leq q.$$

Theorem 5.1. *Let $\varphi : \mathbb{K} \subseteq \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi \in \mathcal{L}[p, p+\eta(q,p)]$, where $p, p+\eta(q,p) \in \mathbb{K}$ with $p < p+\eta(q,p)$. If φ is a hyperbolic type harmonic preinvex function on $[p, p+\eta(q,p)]$, then the following inequality for fractional integral holds*

$$\frac{\sinh 1}{\sinh\frac{1}{2}}\left\{\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right) - \left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\} \tag{5.1}$$

$$\leq \left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right) - J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}$$
$$\leq \varphi(p)+\varphi(q) - 2\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

where $g(x)=\frac{1}{x}$.

Proof. Since φ is hyperbolic type harmonic preinvex function putting $k=\frac{1}{2}$ and choosing

$$x=\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)} \quad \text{and} \quad y=\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}$$

in

$$\varphi\left(\frac{xy}{kx+(1-k)y}\right) \leq \frac{\sinh k}{\sinh 1}\varphi(y) + \frac{\sinh 1-\sinh k}{\sinh 1}\varphi(x),$$

we get

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right) + \frac{\sinh 1-\sinh\frac{1}{2}}{\sinh 1}\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right).$$

Multiplying both sides by k^{m-1} and integrating with respect to k over $[0,1]$, we get

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)\int_0^1 k^{m-1}dk$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)k^{m-1}dk$$
$$+\frac{\sinh 1-\sinh\frac{1}{2}}{\sinh 1}\int_0^1 \varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)k^{m-1}dk,$$

$$\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p)}\right)$$
$$\leq \frac{\sinh\frac{1}{2}}{\sinh 1}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)$$
$$+\frac{\sinh 1-\sinh\frac{1}{2}}{\sinh 1}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

$$\frac{\sinh 1}{\sinh\frac{1}{2}}\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)$$
$$\leq \left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)$$
$$+\frac{\sinh 1-\sinh\frac{1}{2}}{\sinh\frac{1}{2}}\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m \Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right),$$

$$\begin{aligned}(5.2)\quad &\frac{\sinh 1}{\sinh\frac{1}{2}}\left\{\varphi\left(\frac{2p(p+\eta(q,p))}{p+(p+\eta(q,p))}\right)-\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}\\ &\leq\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)-J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}.\end{aligned}$$

For the second part of the proof, let φ be a hyperbolic type harmonic preinvex function. Then

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)\leq\frac{\sinh k}{\sinh 1}\varphi(q)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(p)$$

and

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)\leq\frac{\sinh k}{\sinh 1}\varphi(p)+\frac{\sinh 1-\sinh k}{\sinh 1}\varphi(q).$$

Adding both the above inequalities,

$$\varphi\left(\frac{p(p+\eta(q,p))}{p+(1-k)\eta(q,p)}\right)+\varphi\left(\frac{p(p+\eta(q,p))}{p+k\eta(q,p)}\right)\leq\varphi(p)+\varphi(q).$$

Multiplying both the sides by k^{m-1} and integrating with respect to k over $[0,1]$, we get

$$\begin{aligned}(5.3)\quad &\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)\left\{J^m_{\frac{1}{q}+}(\varphi\circ g)\left(\frac{1}{p}\right)-J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right)\right\}\\ &\leq\varphi(p)+\varphi(q)-2\left(\frac{p(p+\eta(q,p))}{\eta(q,p)}\right)^m\Gamma(m+1)J^m_{\frac{1}{p}-}(\varphi\circ g)\left(\frac{1}{q}\right).\end{aligned}$$

Combining (5.2) and (5.3) we get (5.1). □

6. CONCLUSION

In this paper, we have introduced the generalizations of hyperbolic type convex functions as Hyperbolic type harmonic preinvex function. Applying this new class of preinvexity, we have presented few Hermite-Hadamard type inequalities (see Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4). Moreover, we have also presented some refinements of Hermite-Hadamard inequality (see Theorem 4.1, Theorem 4.2 and Theorem 4.3). At the end, we have also used fractional integral operator to generalize Theorem 3.1. The results, presented in this paper have the potential to establish more general inequalities involving fractional operators on different kinds of preinvexities.

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