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LIPSCHITZ STABILITY FOR IMPULSIVE RIEMANN–LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

MARTIN BOHNER[1] AND SNEZHANA HRISTOVA[2]

Abstract. Initial and impulsive conditions for initial value problems of systems of nonlinear impulsive Riemann–Liouville fractional differential equations are introduced. The case when the lower limit of the fractional derivative is changed at each time point of the impulses is studied. In the case studied, the solution has a singularity at the initial time and at any point of the impulses. This leads to the need to appropriately generalize the classical concept of Lipschitz stability. Two derivative types of Lyapunov functions are utilized in order to deduce sufficient conditions for the new stability concept. Three examples are provided for illustration purpose of the theoretical results.

1. Introduction

Differential equations with impulses are intensively studied and applied in modeling various phenomena (see, e.g., the monograph of Lakshmikantham et al. [23]). Recently, fractional differential equations have proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. Actually, fractional differential equations are now considered as an alternative model to integer differential equations (for more details, see the monographs [16, 17, 26] and the references therein).

In addition, some modeling is done via impulsive fractional differential equations when these processes involve hereditary phenomena such as biological and social macrosystems and are subject to some impulsive perturbations. Note that the literature knows various types of fractional derivatives. To the best of our knowledge,

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impulsive fractional functional differential equations involving the Caputo fractional derivative have been studied in completeness. It is worth remarking that Feckan et al. [18] give a counterexample to show that some formula of solutions in previous papers is incorrect and reconsider a class of impulsive fractional differential equations and introduce a correct formula of solutions for an impulsive initial value problem with Caputo fractional derivative. The situation is not the same when the Riemann–Liouville (RL) fractional derivatives is used. The statement of the impulsive condition and the lower limit of the RL fractional derivative is presented in different ways by different authors. For example, in [13, 32], the impulsive conditions are related to the right and left limits of RL fractional integrals with fixed lower limit at the initial time. In [12, 31], the impulsive conditions are connected with the RL fractional integral on the intervals between two consecutive impulses. In [8], the impulses are RL fractional derivatives. In [24], the impulsive conditions contain RL fractional derivatives. Note that the formula for the exact solution of linear impulsive RL fractional differential equations is given recently in [5] and for scalar impulsive equation with delay in [6].

One of the most important properties of solutions is stability. Many stability concepts exist, describing various behavior of the solutions, e.g., Lipschitz stability defined for ODEs [15]. Later, this type of stability has been studied for various types of differential equations and problems such as, e.g., nonlinear differential systems [14, 19, 28], impulsive differential equations with delays [9], fractional differential systems [29], Caputo fractional differential equations with noninstantaneous impulses [4], a piecewise linear Schrödinger potential [7], a hyperbolic inverse problem [10], the electrical impedance tomography problem [11], and the radiative transport equation [25]. See also [2, 3, 8, 14, 19, 20, 27, 28] for related references. In the recent paper [21], a similar problem is considered without impulses.

In view of the above considerations, in this paper, in an appropriate way, we set up impulsive RL fractional differential equations and study Lipschitz stability properties of the zero solution. We will give some reasons for the defined impulsive conditions. Let an increasing sequence of nonnegative points $\{t_i\}_{i\in\mathbb{N}}$, $t_0 = 0$, be given such that $\lim_{i\to\infty} t_i = \infty$. When impulses are involved in fractional differential equations, there are mainly two interpretations of fractional derivatives:

- *fixed lower limit of the fractional derivative* – in this case, the lower limit of the fractional derivative is kept equal to the initial time on the whole interval of consideration.
- *changeable lower limit of the fractional derivative* – each time t_i, $i \in \mathbb{N}$, of the impulse is considered as a lower limit of the fractional derivative.

In this paper, we consider the case of changeable lower limit of the Riemann–Liouville fractional derivative. The presence of the Riemann–Liouville derivative leads to two specific types of initial conditions, which are equivalent (see the classical book [17]).

- *Integral form of the initial condition*

$$ {}_0I_t^{1-q}x(t)|_{t=0} = \lim_{t\to 0+} {}_0I_t^{1-q}x(t) = x_0. $$

- *Weighted form of the initial condition*

$$\lim_{t\to 0+}\left(t^{1-q}x(t)\right)=\frac{x_0}{\Gamma(q)}.$$

Here, the Riemann-Liouville (RL) fractional integral is defined by

$${}_aI_t^{1-q}x(t)=\frac{1}{\Gamma(1-q)}\int_a^t\frac{x(s)}{(t-s)^q}\mathrm{d}s,\quad t>a,$$

and Γ denotes the Gamma function. In the literature, when the RL derivative is applied, there are various types of statements for impulsive conditions. We will follow the ideas of impulses in ordinary differential equations, i.e., after the impulse, the differential equation is the same with a new initial condition. This will lead to two types of initial conditions (following [17]):

- *integral form of the impulsive conditions*

$${}_{t_i}I_t^{1-q}x(t)|_{t=t_i}=\lim_{t\to t_i}{}_{t_i}I_t^{1-q}x(t)=\Phi_i(x(t_i-0)),\quad i\in\mathbb{N}.$$

- *weighted form of the impulsive conditions*

$$\lim_{t\to t_i+}\left((t-t_i)^{1-q}x(t)\right)=\frac{\Phi_i(x(t_i-0))}{\Gamma(q)},\quad i\in\mathbb{N}.$$

In this paper, we will use the integral form of both, the initial condition and the impulsive conditions. Keeping in mind the above description, in this paper, we will study the initial value problem (IVP) for the following system of nonlinear RL fractional differential equations with impulses (IRLFDE) of fractional order $q\in(0,1)$:

$$\begin{cases}{}_{t_i}^{\mathrm{RL}}D_t^qx(t)=f(t,x(t)), & \text{for } t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0,\\ \lim\limits_{t\to t_i+}[(t-t_i)^{1-q}x(t)]=\dfrac{\Psi_i(x(t_i-0))}{\Gamma(q)}, & \text{for } i\in\mathbb{N},\\ \lim\limits_{t\to 0+}[t^{1-q}x(t)]=\dfrac{x_0}{\Gamma(q)},\end{cases}\tag{1.1}$$

where $x_0\in\mathbb{R}^n$, and the Riemann–Liouville fractional derivative of the function $x\in\mathrm{C}([a,T],\mathbb{R}^n)$, $T>a$, with lower limit $a\in\mathbb{R}$ and order $q\in(0,1)$ is defined by

$${}_a^{\mathrm{RL}}D_t^qx(t)=\frac{1}{\Gamma(1-q)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(t-s)^{-q}x(s)\mathrm{d}s,\quad t\in(a,T].\tag{1.2}$$

Remark 1.1. For $q\to 1$, the impulsive condition

$$\lim_{t\to t_i+}[(t-t_i)^{1-q}x(t)]=\frac{\Psi_i(x(t_i-0))}{\Gamma(q)}$$

in (1.1) is reduced to the well-known condition

$$x(t_i+)=\Psi_i(x(t_i-0))$$

for impulsive differential equations with ordinary derivative (see [23]).

In this paper, we study some stability properties of the zero solution of (1.1). Note that the solutions of the IVP for IRLFDE (1.1) have singularities at each point t_i, $i \in \mathbb{N}_0$. It requires a new definition of classical Lipschitz stability, introduced and studied in [15]. This is called generalized Lipschitz stability in time. It relates to singularity of the solution, and it is considered only on intervals excluding from the left both the initial time and the impulsive times. We use Lyapunov functions and two types of derivatives of these Lyapunov functions for the impulsive fractional equation under consideration. A number of conditions is presented that ensures generalized Lipschitz stability in time. Three examples are provided in order to illustrate the results.

2. Preliminary Results

In this paper, we will use the classical fractional derivatives (see, for example, [16, 17, 26]) such as RL fractional derivative (see the Definition 1.2) and Grünwald–Letnikov derivative defined by

$$ {}_a^{\mathrm{GL}}D_t^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{\left[\frac{t-a}{h}\right]} (-1)^r \binom{q}{r} m(t-rh), \quad t \in (a,T]. $$

Remark 2.1. If $m \in \mathrm{C}([a,T],\mathbb{R}^n)$, then ${}_a^{\mathrm{RL}}D_t^q m(t) = {}_a^{\mathrm{GL}}D_t^q m(t)$, see [17, Theorem 2.25].

For $a, T \in \mathbb{R}_+$, with $a < T$, we will use the sets

$$ \mathrm{C}_{1-q}([a,T],\mathbb{R}^n) = \Big\{ u : (a,T] \to \mathbb{R}^n : \ u \in \mathrm{C}((a,T],\mathbb{R}^n), \lim_{t\to a+} (t-a)^{1-q} u(t) < \infty \Big\}, $$

$$ \begin{aligned} \mathrm{PC}_{1-q}([0,\infty),\mathbb{R}^n) = \Bigg\{ & u : (0,\infty) \to \mathbb{R}^n : \ u \in \mathrm{C}\Bigg(\bigcup_{k\in\mathbb{N}_0} (t_k, t_{k+1}], \mathbb{R}^n \Bigg), \\ & u(t_k) = u(t_k - 0) = \lim_{\varepsilon\to 0+} u(t_k - \varepsilon) < \infty, \ k \in \mathbb{N}, \\ & \lim_{t\to t_k+} (t-t_k)^{1-q} u(t) < \infty, \ k \in \mathbb{N}_0 \Bigg\}. \end{aligned} $$

Remark 2.2. If $u \in \mathrm{PC}_{1-q}([0,\infty),\mathbb{R}^n)$, then $u \in \mathrm{C}_{1-q}([t_k,t_{k+1}],\mathbb{R}^n)$ for any $k \in \mathbb{N}_0$.

Now, we will state some known results, which will be applied in the proofs of our main results.

Proposition 2.1 ([30, Lemma 2.3])**.** *Let* $m \in \mathrm{C}_{1-q}([a,a+T),\mathbb{R})$, $t_1 \in (a,a+T)$*. If* $m(t_1) = 0$ *and* $m(t) < 0$ *for* $a \le t < t_1$*, then* ${}_a^{\mathrm{RL}}D_t^q m(t)|_{t=t_1} \ge 0$.

Remark 2.3. By Remark 2.1, Proposition 2.1 is also true with ${}_a^{\mathrm{GL}}D_t^q m(t)|_{t=t_1}$ in place of ${}_a^{\mathrm{RL}}D_t^q m(t)|_{t=t_1}$.

The next result is the basis for the practical definition of the initial condition and the impulsive conditions of (1.1).

Proposition 2.2 ([17]). *Let $m : [a, T] \to \mathbb{R}$ be Lebesgue measurable, $T > a > 0$, and $q \in (0, 1)$.*

(a) *If* $\lim_{t\to a+}[(t-a)^{1-q}m(t)] =: c \in \mathbb{R}$ *exists a.e., then*

$$\begin{aligned} {}_aI_t^{1-q}m(t)|_{t=a} :=& \lim_{t\to a+} \frac{1}{\Gamma(1-q)} \int_a^t \frac{m(s)}{(t-s)^q} \mathrm{d}s \\ =& c\Gamma(q) = \Gamma(q) \lim_{t\to a+}[(t-a)^{1-q}m(t)] \end{aligned}$$

is well defined.

(b) *If* ${}_aI_t^{1-q}m(t)|_{t=a} = c \in \mathbb{R}$ *exists a.e. and if* $\lim_{t\to a+}[(t-a)^{1-q}m(t)]$ *exists, then*

$$\lim_{t\to a+}[(t-a)^{1-q}m(t)] = \frac{c}{\Gamma(q)} = \frac{1}{\Gamma(q)} {}_aI_t^{1-q}m(t)|_{t=a}.$$

Remark 2.4. According to Proposition 2.2, both, the initial condition and the impulsive conditions in (1.1), could be replaced by the equalities

$${}_0I_t^{1-q}x(t)|_{t=0} = x_0$$

and

$${}_{t_i}I_t^{1-q}x(t)|_{t=t_i} = \Psi_i(x(t_i - 0)), \quad i \in \mathbb{N},$$

respectively.

We introduce the following assumptions:

(A_1) the increasing sequence $\{t_i\}_{i\in\mathbb{N}_0}$, $t_0 = 0$, is such that

$$\lim_{i\to\infty} t_i = \infty \quad \text{and} \quad \inf_{i\in\mathbb{N}_0}(t_{i+1} - t_i) = \lambda > 0;$$

(A_2) $f \in \mathrm{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, 0) = 0$ for $t \geq 0$;

(A_3) $\Psi_i \in \mathrm{C}(\mathbb{R}^n, \mathbb{R}^n)$ and $\Psi_i(0) = 0$ for $i \in \mathbb{N}$.

Let $\rho > 0$ and $\mathcal{J} \subset \mathbb{R}_+$, $0 \in \mathcal{J}$ be an interval. As in [21], we define $\mathcal{M}(\mathcal{J})$ to consist of all strictly increasing continuous functions a with $a(0) = 0$ and such that there exists q_a with $q_a(\alpha) \geq 1$ for $\alpha \geq 1$ and $a^{-1}(\alpha r) \leq r q_a(\alpha)$, $\mathcal{K}(\mathcal{J})$ to consist of all strictly increasing continuous functions a with $a(0) = 0$ and such that there exists a constant $K_a > 0$ with $a(r) \leq K_a r$ and

$$\mathcal{S}_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}.$$

Remark 2.5. If $a(u) = u$, then $a \in \mathcal{K}(\mathbb{R}_+) \cap \mathcal{M}(\mathbb{R}_+)$. If $a(u) = K_1 u$ for $K_1 > 0$, then $a \in \mathcal{K}(\mathbb{R}_+)$ and $K_a = K_1$. If $a(u) = K_2 u^2$ for $0 < K_2 \leq 1$, then $a \in \mathcal{M}([1, \infty))$ and $q(u) = \sqrt{\frac{u}{K_2}} \geq 1$ for $u \geq 1$.

From now on, we assume that the IVP for IRLFDE (1.1) possesses a solution, denoted by $x_{x_0} \in \mathrm{PC}_{1-q}([0, \infty), \mathbb{R}^n)$.

Example 2.1. Consider the IVP for the scalar linear IRLFDE

$$\begin{cases} {}_{t_i}^{\mathrm{RL}}D_t^q y(t) = ay(t), \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \lim\limits_{t \to t_i+} [(t-t_i)^{1-q} y(t)] = \dfrac{y(t_i - 0)}{t_i \Gamma(q)}, \quad \text{for } i \in \mathbb{N}, \\ \lim\limits_{t \to 0+} [t^{1-q} y(t)] = \dfrac{y_0}{\Gamma(q)}, \end{cases} \tag{2.1}$$

where $a, y_0 \in \mathbb{R}$. The solution of (2.1) is given by

$$y_{y_0}(t) = \begin{cases} y_0 t^{q-1} E_{q,q}(at^q), & \text{for } t \in (0, t_1], \\ y_0 \left(\prod_{i=0}^{k-1} \frac{(t_{i+1}-t_i)^{q-1} E_{q,q}(a(t_{i+1}-t_i)^q)}{t_{i+1}} \right) & \\ \times (t-t_k)^{q-1} E_{q,q}(a(t-t_k)^q) & \text{for } t \in (t_{k+1}, t_k],\ k \in \mathbb{N}. \end{cases}$$

It has singularities at the points t_k, $k \in \mathbb{N}_0$, which are the initial time and the impulsive times. In the particular case $a = 0.5$, $t_k = k$, $k \in \mathbb{N}_0$, $q = 0.3$, the graph of the solution y_{y_0} is given in Figure 1 for $y_0 = 1$ and in Figure 2 for $y_0 = -0.5$, respectively.

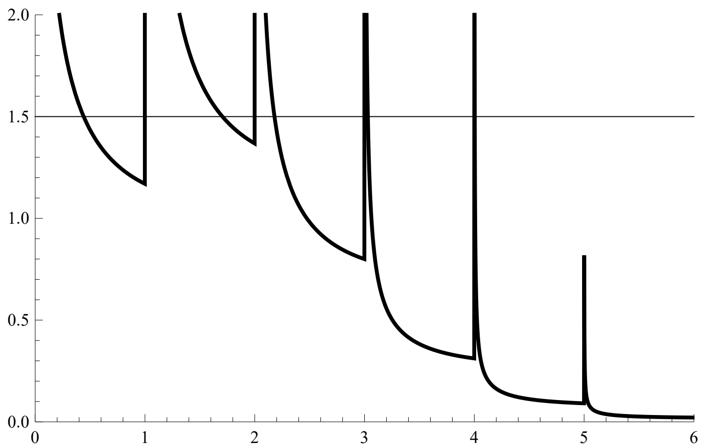

FIGURE 1. Graph of the solution of (2.1) for $a = 0.5$, $y_0 = 1$, and $q = 0.3$.

Example 2.1 illustrates that the stability of the solution for impulsive differential equations in the case of the RL fractional derivative has to be considered on intervals that exclude t_k, $k \in \mathbb{N}_0$, on the right ends.

There are some particular cases with zero initial value and zero impulsive functions with nonunique solutions without singularities at the initial time and the impulsive time points.

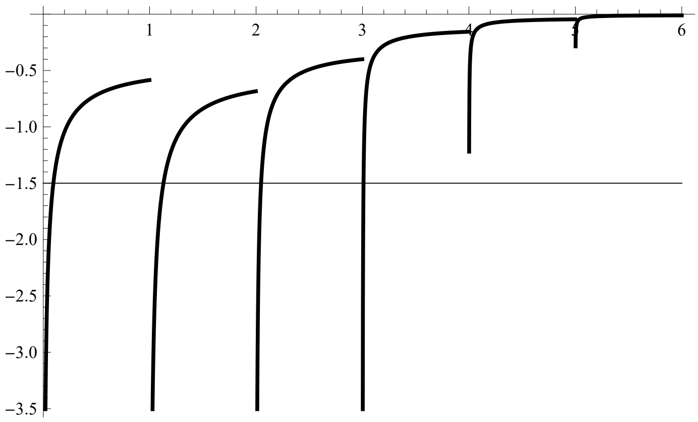

FIGURE 2. Graph of the solution of (2.1) for $a = 0.5$, $y_0 = -0.5$, and $q = 0.3$.

Example 2.2. Consider the IVP for the scalar linear IRLFDE

$$\begin{cases} {}^{\mathrm{RL}}_{t_i}D^q_t y(t) = a(t-t_i)^{\beta}\sqrt{y(t)}, \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \lim\limits_{t\to t_i+} \Big[(t-t_i)^{1-q} y(t)\Big] = 0, \quad \text{for } i \in \mathbb{N}, \\ \lim\limits_{t\to 0+} \Big[t^{1-q} y(t)\Big] = 0, \end{cases} \tag{2.2}$$

where $a \in \mathbb{R}$, $\beta = -0.5q$. Equation (2.2) has the zero solution, but it also has a nonzero solution. Using

$$2(q+\beta) = 1 + q > 0 > -1$$

and [22, Example 3.2], we obtain the solution of (2.2)

$$y_0(t) = \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 (t-t_i)^{2(q+\beta)}, \quad \text{for } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0.$$

It is easy to check that

$$\begin{aligned} \lim_{t\to t_i+} \Big[(t-t_i)^{1-q} y_0(t)\Big] &= \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 \lim_{t\to t_i+} \Big[(t-t_i)^{1-q}(t-t_i)^{2(q+\beta)}\Big] \\ &= \left(\frac{a\Gamma(q+2\beta+1)}{\Gamma(2q+2\beta+1)}\right)^2 \lim_{t\to t_i+} (t-t_i) = 0. \end{aligned}$$

The solution y_0 has no singularities at the points t_k, $k \in \mathbb{N}_0$, which are the initial time and the impulsive times. It is different from Example 2.1. In the particular case $a = 1$, $t_k = k$, $k \in \mathbb{N}_0$, $q = 0.4$, $\beta = -0.2$, the graph of the solution y is given in Figure 3.

In our further investigations in this paper, we will assume that the IVP for IRLFDE (1.1) has a unique solution x_{x_0} for any initial values $x_0 \in \mathbb{R}^n$ defined for $t > 0$. Lipschitz stability [15] will now be generalized to systems of impulsive RL differential equations.

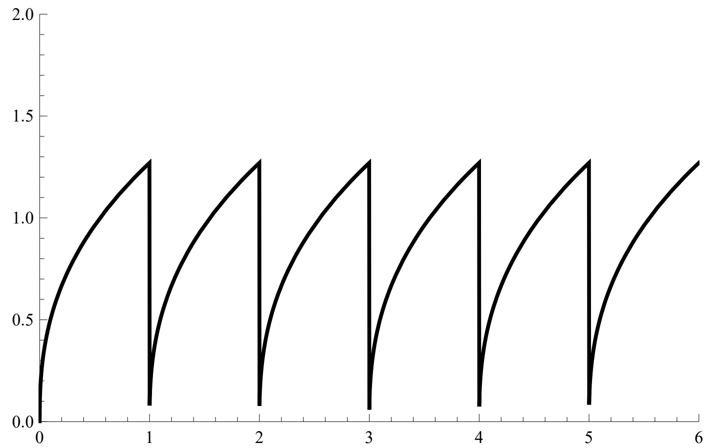

FIGURE 3. Graph of the solution of (2.2) for $a = 1$, $t_k = k$, $\beta = -0.2$, and $q = 0.4$.

Considering the phenomena described in Example 2.1 and 2.2, we now define a new stability type as follows.

Definition 2.1. We say that the trivial solution of the IVP for IRLFDE (1.1) is

- *generalized Lipschitz stable in time* if there are $N \in \mathbb{N}_0$, $M \geq 1$, $\delta > 0$, and $T_i \in (0, \lambda)$, $i \in \mathbb{N}_0$, such that for any initial value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$, we have
$$\|x_{x_0}(t)\| \leq M \|x_0\|, \quad \text{for all } t \in \bigcup_{i=N}^{\infty} [t_i + T_i, t_{i+1}];$$
- *globally generalized Lipschitz stable in time* if there exist $N \in \mathbb{N}_0$, $M \geq 1$, and $T_i \in (0, \lambda)$, $i \in \mathbb{N}_0$, such that for any for any initial value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \infty$, we have
$$\|x_{x_0}(t)\| \leq M \|x_0\|, \quad \text{for all } t \in \bigcup_{i=N}^{\infty} [t_i + T_i, t_{i+1}].$$

Now we define the class Λ of Lyapunov-like functions as follows.

Definition 2.2 ([1]). Let $0 \in J \subset \mathbb{R}_+$, $\mathcal{J} = J \cap \{\bigcup_{i \in \mathbb{N}_0} (t_i, t_{i+1}]\}$, and $\Delta \subset \mathbb{R}^n$. We will say that the function V belongs to the class $\Lambda(J, \Delta)$ if $V \in \mathrm{C}(\mathcal{J} \times \Delta, \mathbb{R}_+)$,
$$V(t_i, x) = V(t_i - 0, x) = \lim_{\varepsilon \to 0+} V(t_i - \varepsilon, x)$$
and
$$V(t_i + 0, x) = \lim_{\varepsilon \to 0+} V(t_i + \varepsilon, x),$$
for $i \in \mathbb{N}$, $x \in \Delta$, and it is locally Lipschitz with respect to its second argument.

We will use the following two types of fractional derivatives of Lyapunov functions among the system of nonlinear impulsive RL fractional differential equations (1.1).

- *RL derivative* of $V \in \Lambda(\mathbb{R}_+, \Delta)$ for IRLFDE (1.1) defined by
$$ {}_{t_k}^{\mathrm{RL}}D_t^q V(t, x_{x_0}(t)) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_k}^{t} (t-s)^{-q} V(s, x_{x_0}(s)) \mathrm{d}s, $$
for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, where $x_{x_0}(\cdot) \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \Delta)$ solves (1.1).
- *Dini derivative* of $V \in \Lambda(\mathbb{R}_+, \Delta)$ for IRLFDE (1.1) defined by
$$ D_{(1.1)}^{t_k} V(t,x) = \limsup_{h \to 0+} \frac{V(t,x) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t-rh, x - h^q f(t,x))}{h^q}, $$
for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, $x \in \Delta$.

Remark 2.6. Let x be a solution of (1.1). Then, for any $k \in \mathbb{N}_0$, the equality
$$ \begin{aligned} &D_{(1.1)}^{t_k} V(t, x(t)) \\ &= \limsup_{h \to 0+} \frac{V(t,x(t)) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t-rh, x(t) - h^q f(t,x(t)))}{h^q} \end{aligned} $$
holds for $t \in (t_k, t_{k+1}]$.

We consider the IVP
$$ \begin{cases} {}_{t_i}^{\mathrm{RL}}D_t^q u(t) = g(t, u(t)) \quad \text{for } t \in (t_i, t_{i+1}], \quad i \in \mathbb{N}_0, \\ \lim_{t \to t_i+} [(t-t_i)^{1-q} u(t)] = \frac{H_i(u(t_i - 0))}{\Gamma(q)}, \quad \text{for } i \in \mathbb{N}, \\ \lim_{t \to 0+} [t^{1-q} u(t)] = \frac{u_0}{\Gamma(q)}, \end{cases} \tag{2.3} $$
with $u_0 \in \mathbb{R}$. We will denote the solution of (2.3) by u_{u_0}. We will assume that the IVP for the scalar IRLFDE (2.3) has a unique solution u_{u_0} for any initial value $u_0 \in \mathbb{R}$ defined for $t > 0$. We also introduce the following conditions:

(A_4) $g \in \mathrm{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ decreases in the second variable, and $g(t,0) = 0$ for $t \in \mathbb{R}_+$;
(A_5) $H_k \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ are increasing w.r.t. the second argument and $H_k(0) = 0$, $k \in \mathbb{N}$.

In our main results, we will use some comparison results with both Dini and Riemann-Liouville derivatives.

Lemma 2.1. *Suppose:*
1. *conditions* (A_2)–(A_5) *hold;*
2. $x_{x_0}^* \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \mathbb{R}^n)$ *solves* (1.1)*;*
3. $u_{u_0} \in \mathrm{PC}_{1-q}(\mathbb{R}_+, \mathbb{R})$ *solves* (2.3)*;*
4. $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *satisfies:*
   (i) *the inequality*
$$ {}_{t_i}^{\mathrm{RL}}D_t^q V(t, x_{x_0}^*(t)) \le g(t, V(t, x_{x_0}^*(t))), \quad t \in (t_i, t_{i+1}], i \in \mathbb{N}_0 $$
   holds;

(ii) *for all* $i \in \mathbb{N}_0$*, the inequalities*

$$V(t_i, \Psi_i(x^*_{x_0}(t_i - 0))) \leq H_i(V(t_i, x^*_{x_0}(t_i - 0)))$$

hold;

(iii) *for all* $i \in \mathbb{N}$*, the inequalities*

$$\lim_{t \to t_i+} (t - t_i)^{1-q} V(t, x^*_{x_0}(t)) \leq \frac{V(t_i + 0, \Psi_i(x^*_{x_0}(t_i - 0)))}{\Gamma(q)}$$

hold.

If $\lim_{t\to 0+} t^{1-q} V(t, x^*_{x_0}(t)) \leq \frac{u_0}{\Gamma(q)}$*, then*

$$V(t, x^*_{x_0}(t)) \leq u_{u_0}(t), \quad \textit{for all } t > 0. \tag{2.4}$$

Proof. For $t > 0$, put $m(t) := V(t, x^*_{x_0}(t))$. We will prove (2.4) by induction w.r.t. the intervals $(t_i, t_{i+1}]$, $i \in \mathbb{N}_0$.

First, let $t \in (0, t_1]$. Let $\varepsilon > 0$ be arbitrary. We will prove

$$m(t) < u_{u_0}(t) + t^{q-1}\varepsilon, \quad t \in (0, t_1]. \tag{2.5}$$

We have

$$\begin{aligned}
\lim_{t \to 0+} t^{1-q} V(t, x^*_{x_0}(t)) &\leq \frac{u_0}{\Gamma(q)} < \frac{u_0}{\Gamma(q)} + \varepsilon \\
&= \lim_{t \to 0+} t^{1-q} u_{u_0}(t) + \lim_{t\to 0+} t^{1-q} t^{q-1}\varepsilon \\
&= \lim_{t \to 0+} t^{1-q} \left(u_{u_0}(t) + t^{q-1}\varepsilon \right).
\end{aligned} \tag{2.6}$$

From (2.6), there exists $\delta > 0$ such that

$$t^{1-q} V(t, x^*_{x_0}(t)) < t^{1-q} \left(u_{u_0}(t) + t^{q-1}\varepsilon \right), \quad \text{for } t \in (0, \delta),$$

that is, (2.5) is satisfied on $(0, \delta)$. If $\delta \geq t_1$, then (2.5) is proved. If $\delta < t_1$, then we assume that (2.5) is not true. Then there exists $t^* \in [\delta, t_1]$ such that

$$m(t^*) = u_{u_0}(t^*) + (t^*)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + t^{q-1}\varepsilon, \quad t \in (0, t^*).$$

From (A_4), ${}^{\mathrm{RL}}_{0}D^q_t t^{q-1} = 0$, and Proposition 2.1 with $t_1 = t^*$ together with

$$v(t) = m(t) - u_{u_0}(t) - t^{q-1}\varepsilon,$$

we obtain the inequality

$$\begin{aligned}
{}^{\mathrm{RL}}_{0}D^q_t m(t)|_{t=t^*} &\geq {}^{\mathrm{RL}}_{0}D^q_t \left(u_{u_0}(t) + t^{q-1}\varepsilon \right)|_{t=t^*} \\
&= {}^{\mathrm{RL}}_{0}D^q_t u_{u_0}(t)|_{t=t^*} = g(t^*, u_{u_0}(t^*)) \\
&= g\left(t^*, m(t^*) - (t^*)^{q-1}\varepsilon\right) > g(t^*, m(t^*)).
\end{aligned} \tag{2.7}$$

Inequality (2.7) contradicts assumption 4 (i). Therefore, (2.5) holds for any $\varepsilon > 0$, and hence, (2.4) holds for $t \in (0, t_1]$. From assumption 4 (ii), (A_5), and the inequality

$V(t_1, x^*_{x_0}(t_1-0)) \le u(t_1-0)$, we get

$$\begin{aligned} V(t_1+0, x^*_{x_0}(t_1+0)) &= V(t_1+0, \Psi_1(x^*_{x_0}(t_1-0))) \\ &\le H_1(V(t_1, x^*_{x_0}(t_1-0))) \\ &\le H_1(u_{u_0}(t_1-0)) = u_{u_0}(t_1+0). \end{aligned} \tag{2.8}$$

Let $t \in (t_1, t_2]$. Let $\varepsilon > 0$ be arbitrary. We will prove

$$m(t) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad t \in (t_1, t_2]. \tag{2.9}$$

From assumption 4 (iii) and (2.8), we obtain

$$\begin{aligned} \lim_{t\to t_1+} (t-t_1)^{1-q} V(t, x^*_{x_0}(t)) &\le \frac{V(t_1, x^*_{x_0}(t_1+0))}{\Gamma(q)} \le \frac{u_{u_0}(t_1+0))}{\Gamma(q)} \\ &< \frac{H_1(u_{u_0}(t_1-0))}{\Gamma(q)} + \varepsilon \\ &= \lim_{t\to t_1+} (t-t_1)^{1-q} \left(u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon\right). \end{aligned} \tag{2.10}$$

From (2.10), there exists $\delta_1 > 0$ such that

$$V(t, x^*_{x_0}(t)) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad \text{on } (t_1, t_1+\delta_1).$$

If $\delta_1 \ge t_2 - t_1$, then (2.9) is proved. If $\delta_1 < t_2 - t_1$, then we assume (2.9) is not true. Then there exists $t_1^* \in [t_1+\delta, t_2]$ such that

$$m(t_1^*) = u_{u_0}(t_1^*) + (t_1^*-t_1)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon, \quad t \in [t_1, t_1^*).$$

Now (A_4), ${}^{\mathrm{RL}}_{t_1}D^q_t (t-t_1)^{q-1} = 0$ and Proposition 2.1, with $t_1 = t_1^*$ together with

$$v(t) = m(t) - u_{u_0}(t) - (t-t_1)^{q-1}\varepsilon,$$

yield

$$\begin{aligned} {}^{\mathrm{RL}}_{t_1}D^q_t m(t)|_{t=t_1^*} &\ge {}^{\mathrm{RL}}_{t_1}D^q_t \left(u_{u_0}(t) + (t-t_1)^{q-1}\varepsilon\right)|_{t=t_1^*} \\ &= {}^{\mathrm{RL}}_{t_1}D^q_t u_{u_0}(t)|_{t=t_1^*} = g(t_1^*, u_{u_0}(t_1^*)) \\ &= g\left(t_1^*, m(t_1^*) - (t_1^*-t_1)^{q-1}\varepsilon\right) > g(t_1^*, m(t_1^*)). \end{aligned} \tag{2.11}$$

Inequality (2.11) contradicts assumption 4 (i). Therefore, (2.9) holds for any $\varepsilon > 0$, and hence, (2.4) holds for $t \in (t_1, t_2]$.

Following the above procedure, we complete the proof. $\square$

Lemma 2.2. *Suppose:*

1. *assumptions* 1, 2, 3, 4 (ii), 4 (iii) *from Lemma* 2.1 *hold;*
2. $V \in \Lambda(\mathbb{R}_+, \mathbb{R})$ *satisfies*

$$D^{t_k}_{(1.1)} V(t, x^*_{x_0}(t)) \le g(t, V(t, x^*_{x_0}(t)), \quad \textit{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0.$$

If $\lim_{t\to 0+} t^{1-q} V(t, x^*_{x_0}(t)) \le \frac{u_0}{\Gamma(q)}$, *then* (2.4) *holds.*

Proof. The proof is similar to the proof of Lemma 2.1, where instead of the RL fractional derivative of the Lyapunov function, we will use the Dini fractional derivative. The main difference between this proof and the proof of Lemma 2.1 is connected with inequalities (2.7) and (2.11) for $t^* \in (0, t_1]$ and $t_1^* \in (t_1, t_2]$.

Consider any of the intervals $(t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, and assume that for a fixed $k \in \mathbb{N}$, there exist $\delta_k \in (0, t_{k+1} - t_k)$ and $t_k^* \in (t_k + \delta_k, t_{k+1}]$ such that

$$m(t_k^*) = u_{u_0}(t_k^*) + (t_k^* - t_k)^{q-1}\varepsilon, \quad m(t) < u_{u_0}(t) + (t - t_k)^{q-1}\varepsilon, \quad t \in (t_k, t_k^*).$$

Thanks to Remark 2.3 with $\tau = t_k^*$, we obtain

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_+^q m(t)|_{t=t_k^*} &\ge {}_{t_k}^{\mathrm{GL}}D_+^q u_{u_0}(t)|_{t=t_k^*} + {}_{0}^{\mathrm{GL}}D_+^q ((t-t_k)^{q-1}\varepsilon)|_{t=t_k^*} \\ &= {}_{t_k}^{\mathrm{GL}}D_+^q u_{u_0}(t)|_{t=t_k^*} = g(t_k^*, u_{u_0}(t_k^*)) \\ &= g(t_k^*, m(t_k^*) - (t_k^* - t_k)^{q-1}\varepsilon) > g(t_k^*, m(t_k^*)). \end{aligned} \tag{2.12}$$

For any fixed $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_+^q m(t) &= \limsup_{h \to 0+} \frac{\sum_{r=0}^{\left[\frac{t-t_k}{h}\right]} (-1)^r \binom{q}{r} m(t - rh)}{h^q} \\ &= \limsup_{h \to 0+} \frac{m(t) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} m(t - rh)}{h^q} \\ &= \limsup_{h \to 0+} \left\{ \frac{m(t) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} V(t - rh, x_{x_0}^*(t) - h^q f(t, x_{x_0}^*(t)))}{h^q} \right. \\ &\quad \left. + \frac{\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} \left[V(t - rh, x_{x_0}^*(t) - h^q f(t, x_{x_0}^*(t))) - m(t - rh) \right]}{h^q} \right\}. \end{aligned} \tag{2.13}$$

Denote

$$F(t, x_{x_0}^*, t_k, h) = \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} x_{x_0}^*(t - rh).$$

From (1.1), we get

$$\begin{aligned} {}_{t_k}^{\mathrm{GL}}D_t^q x_{x_0}^*(t) &= \limsup_{h \to 0+} \frac{x_{x_0}^*(t) - F(t, x_{x_0}^*, t_k, h)}{h^q} \\ &= {}_{t_k}^{\mathrm{RL}}D_t^q x_{x_0}^*(t) = f(t, x_{x_0}^*(t)). \end{aligned}$$

Hence,

$$x^*_{x_0}(t) - h^q f(t, x^*_{x_0}(t)) = F(t, x^*_{x_0}, t_k, h) + \Omega(h^q),$$

with

$$\lim_{h\to 0+} \frac{\|\Omega(h^q)\|}{h^q} = 0.$$

Thus, for arbitrary $h > 0$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}
&V(t-rh, x^*_{x_0}(t) - h^q f(t, x^*_{x_0}(t))) - V(t-rh, x^*_{x_0}(t-rh))\\
\le & L\left\|F(t, x^*_{x_0}, t_k, h) + \Omega(h^q) - x^*_{x_0}(t-rh)\right\|\\
\le & L\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1}\binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh)\right\| + L\left\|\Omega(h^q)\right\|.
\end{aligned} \tag{2.14}$$

Hence, due to

$$(1+u)^\alpha = 1 + \sum_{k=1}^{\infty}\binom{\alpha}{k}u^k, \quad \text{i.e.,} \quad 1 = \sum_{k=1}^{\infty}(-1)^{k+1}\binom{\alpha}{k},$$

we get

$$\begin{aligned}
&\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1}\binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh)\right\|\\
=&\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1}\binom{q}{j} x^*_{x_0}(t-jh) - \left(\sum_{j=1}^{\infty}(-1)^{j+1}\binom{q}{j}\right) x^*_{x_0}(t-rh))\right\|\\
\le&\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1}\binom{q}{j} \left[x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh)\right]\right\|\\
&+\left\|\sum_{j=\left[\frac{t-t_k}{h}\right]}^{\infty} (-1)^{j+1}\binom{q}{j}\right\| \left\|x^*_{x_0}(t-rh)\right\|,
\end{aligned}$$

from which, together with (2.13), (2.14), and condition 2 in the statement, we get

$$\begin{aligned}
{}^{\mathrm{GL}}_{t_k}D^q_+ m(t) \le & D^{t_k}_{(1.1)} V(t, x^*_{x_0}(t)) + L \limsup_{h\to 0+} \frac{\|\Omega(h^q)\|}{h^q} \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1}\binom{q}{r}\\
&+ L \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1}\binom{q}{r}\\
&\times \left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{j+1}\binom{q}{j} x^*_{x_0}(t-jh) - x^*_{x_0}(t-rh)\right\|
\end{aligned}$$

$$
\begin{aligned}
=&D^{t_k}_{(1.1)}V(t,x^*_{x_0}(t))\\
&+L\limsup_{h\to 0+}\frac{1}{h^q}\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{r+1}\binom{q}{r}\\
&\times\left\|\sum_{j=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{j+1}\binom{q}{j}\left[x^*_{x_0}(t-jh)-x^*_{x_0}(t-rh)\right]\right\|\\
&+L\limsup_{h\to 0+}\left\|\sum_{j=\left[\frac{t-t_k}{h}\right]}^{\infty}(-1)^{j+1}\binom{q}{j}\right\|\\
&\times\frac{1}{h^q}\sum_{r=1}^{\left[\frac{t-t_k}{h}\right]}(-1)^{r+1}\binom{q}{r}\left\|x^*_{x_0}(t-rh)\right\|\\
=&D^{t_k}_{(1.1)}V(t,x^*_{x_0}(t))\le g(t,V(t,x^*_{x_0}(t))),
\end{aligned}
$$

contradicting (2.12) and completing the proof. □

3. Main Results

We now present the main results of this paper.

Theorem 3.1. *Suppose:*

1. *conditions (A_1)–(A_5) are fulfilled;*
2. *there exists $V\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ such that:*
   (i) *there exist $\tau_i\in(0,\lambda)$, $i\in\mathbb{N}_0$, satisfying*
   $$b(\|x\|)\le V(t,x),\quad \text{for all } x\in\mathbb{R}^n \text{ and } t\in\bigcup_{i\in\mathbb{N}_0}[t_i+\tau_i,t_{i+1}]$$
   where $b\in\mathcal{M}([0,\rho])$, with $\rho>0$;
   (ii) *for all $y\in \mathrm{C}_{1-q}([0,t_1],\mathbb{R}^n)$, with*
   $$\lim_{t\to 0+}\left(t^{1-q}y(t)\right)=y_0\in\mathcal{S}_\rho,$$
   we have
   $$t^{1-q}V(t,y(t))|_{t=0+}=\lim_{t\to 0+}t^{1-q}V(t,y(t))\le a(\|y_0\|),$$
   where $a\in\mathcal{K}([0,\rho])$;
   (iii) *for all $i\in\mathbb{N}_0$, the inequalities*
   $$V(t_i,\Psi_i(x))\le H_i(V(t_i,x))\quad\text{for all } x\in\mathbb{R}^n,$$
   hold;
   (iv) *for any $y\in \mathrm{C}_{1-q}([t_k,t_{k+1}],\mathbb{R}^n)$, with*
   $$\lim_{t\to t_k+}\left((t-t_k)^{1-q}y(t)\right)=\frac{y_k}{\Gamma(q)}<\infty,$$

the inequality

$$(t-t_k)^{1-q}V(t,y(t))|_{t=t_k+}=\lim_{t\to t_k+}(t-t_k)^{1-q}V(t,y(t))\le V(t_k+0,\Psi_k(y_k))$$

holds;

(v) *for any initial value $x_0\in\mathcal{S}_\rho$ and the corresponding solution x_{x_0} of* (1.1), *the inequality*

$${}_{t_k}^{\mathrm{RL}}D_t^q V(t,x_{x_0}(t))\le g(t,V(t,x_{x_0}(t))),\quad \textit{for all } t\in(t_k,t_{k+1}],k\in\mathbb{N}_0,$$

holds;

3. *the zero solution of the scalar comparison equation* (2.3) *is generalized Lipschitz stable in time.*

Then the zero solution of the IVP for IRLFDE (1.1) *is generalized Lipschitz stable in time.*

Proof. Suppose that the zero solution of (2.3) is generalized Lipschitz stable in time. Thus, there exist $N\in\mathbb{N}_0$, $\delta\in(0,\lambda)$, $\varsigma_i\in(0,\delta)$, $i\in\mathbb{N}_0$, $\delta_1>0$, and $M_1\ge 1$ such that for any $u_0\in\mathbb{R}^n$ with $|u_0|<\delta_1$, the inequality

$$|u_{u_0}(t)|\le M_1|u_0|,\quad \text{for } t\in\bigcup_{i=N}^{\infty}[t_i+\varsigma_i,t_{i+1}], \tag{3.1}$$

holds, where u_{u_0} solves (2.3) with the initial value u_0. Thanks to $a\in\mathcal{K}([0,\rho])$ and $b\in\mathcal{M}([0,\rho])$, there are $K_a>0$ and $q_b(u)\ge 1$ for $\alpha\ge 1$ with

$$\alpha r\le b(rq_a(\alpha)),\quad \text{for all } r\in[0,\rho], \tag{3.2}$$

and

$$a(r)\le K_a r,\quad \text{for all } r\in[0,\rho]. \tag{3.3}$$

We may assume $K_a\ge 1$. Pick $M_2,\delta>0$ satisfying

$$M_2>\max\{1,q_b(M_1K_a)\}\ge 1\quad \text{and}\quad \delta=\min\left\{\rho,\frac{\delta_1}{K_a}\right\}.$$

Pick $x_0\in\mathbb{R}^n$ such that $\|x_0\|<\delta$, and hence, $x_0\in\mathcal{S}_\rho$. Consider the solution x_{x_0} of (1.1) for the chosen initial value x_0. Thus, using $\Gamma(q)>1$ for $q\in(0,1)$, we get

$$\left\|\lim_{t\to 0+}t^{1-q}x_{x_0}(t)\right\|=\left\|\frac{x_0}{\Gamma(q)}\right\|<\frac{\delta}{\Gamma(q)}<\delta\le\rho,$$

that is, $\lim_{t\to 0+}t^{1-q}x_{x_0}(t)\in\mathcal{S}_\rho$, and employing assumption 2 (ii) with

$$y=x_{x_0}\in \mathrm{C}_{1-q}([0,t_1],\mathbb{R}^n),$$

we get

$$t^{1-q}V(t,x_{x_0}(t))|_{t=0+}<a\left(\frac{\|x_0\|}{\Gamma(q)}\right)<a(\|x_0\|). \tag{3.4}$$

Consider the solution $u_{u_0^*}$ of (2.3) with $u_0^* = \lim_{t\to 0+} t^{1-q}V(t, x_{x_0}(t))$. The choice of x_0, (3.3), (3.4), and assumption 2 (ii) yield

$$u_0^* = \lim_{t\to 0+} t^{1-q}V(t, x_{x_0}(t)) \le a\left(\frac{\|x_0\|}{\Gamma(q)}\right) < a(||x_0||) \le K_a \|x_0\| < K_a\delta \le \delta_1.$$

Hence, $u_{u_0^*}$ satisfies (3.1) for $\bigcup_{i=N}^{\infty}[t_i + \varsigma_i, t_{i+1}]$,with $u_0 = u_0^*$.

From conditions 2 (v), 2 (iii) with $x = x_{x_0}(t_{i+1} - 0)$ and 2 (iv), with

$$y \equiv x_{x_0} \in \mathrm{C}_{1-q}([t_k, t_{k+1}], \mathbb{R}^n) \quad \text{and} \quad y_k = x_{x_0}(t_i - 0),$$

we have conditions 4 (i), 4 (ii), 4 (iii) of Lemma 2.1, respectively. According to Lemma 2.1, we get

$$V(t, x_{x_0}(t)) \le u_{u_0^*}(t), \quad \text{for } t > 0. \tag{3.5}$$

Let $T_i = \max\{\tau_i, \varsigma_i\}$ for $i \in \mathbb{N}_0$. Then, for any $k \in \mathbb{N}_0$, the inclusions

$$[t_k + T_k, t_{k+1}] \subset [t_k + \tau_k, t_{k+1}] \quad \text{and} \quad [t_k + T_k, t_{k+1}] \subset [t_k + \varsigma_k, t_{k+1}]$$

hold. Let $k \ge N$. From 2 (i), 2 (ii), (3.1), (3.2), (3.3) with $r = \|x_0\|$, $\alpha = M_1K_a > 1$, and (3.4), (3.5), we obtain for $t \in [t_k + T_k, t_{k+1}]$

$$\begin{aligned} b(\|x_{x_0}(t)\|) \le& V(t, x_{x_0}(t)) \le u_{u_0^*}(t) < M_1|u_0^*| \\ =& M_1 t^{1-q}V(t, x_{x_0}(t))|_{t=0+} < M_1 a(\|x_0\|) \\ \le& M_1K_a \|x_0\| \le b\left(q_b(M_1K_a)\|x_0\|\right) \le b(M_2\|x_0\|), \end{aligned}$$

completing the proof. □

Theorem 3.2. *Let conditions* 1. *and* 2. *of Theorem* 3.1 *be satisfied, where conditions* 2 (ii) *and* 2 (v) *are fulfilled for all* $y_0 \in \mathbb{R}^n$ *and* $x_0 \in \mathbb{R}^n$*, respectively. If the zero solution of* (2.3) *is globally generalized Lipschitz stable in time, then the zero solution of IVP for IRLFDE* (1.1) *is globally generalized Lipschitz stable in time.*

Proof. The proof follows the proof of Theorem 3.1, with an arbitrary initial value $x_0 \in \mathbb{R}^n$, and so we omit it. □

Theorem 3.3. *Let the conditions of Theorem* 3.1 *be satisfied, where* $a(s) = A_2 s^p$, $A_2 > 0$, $p \ge 1$ *in condition* 2 (ii)*, and condition*2 (i) *is replaced by*

2*(i) *there exist* $\tau_i \in (0, \lambda)$*,* $i \in \mathbb{N}_0$*, satisfying*

$$\mu(t)\|x\|^p \le V(t, x), \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } t \in \bigcup_{i\in\mathbb{N}_0} [t_i + \tau_i, t_{i+1}],$$

holds, where $\mu(t) \ge A_1$, $t \in \bigcup_{i\in\mathbb{N}_0}[t_i + \tau_i, t_{i+1}]$ *and* $A_1 > 0$.

Then, the trivial solution of the IVP for IRLFDE (1.1) *is generalized Lipschitz stable in time.*

Proof. The proof follows the proof of Theorem 3.1, with

$$M_2 = \sqrt[p]{\frac{M_1 A_2}{A_1}} \quad \text{and} \quad \delta = \min\left\{\lambda, \sqrt[p]{\frac{\delta_1}{A_2}}\right\},$$

and so we omit it. □

In the case when the Dini fractional derivative of Lyapunov functions is used instead of RL fractional derivative of Lyapunov functions, we obtain some sufficient conditions for the introduced generalized Lipschitz stability in time. Since the proofs are similar to the already presented proofs, we omit them, and we will only state the results.

Theorem 3.4. *Let the conditions of Theorem* 3.1 *be satisfied, where condition* 2 (v) *is replaced by*

2(v*) *the inequality*

$$D^{t_k}_{(1.1)} V(t,x) \leq g(t, V(t,x)), \quad \textit{for all } x \in \mathbb{R}^n,\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}_0,$$

holds.

Then the trivial solution of the IVP for IRLFDE (1.1) *is generalized Lipschitz stable in time.*

Proof. The proof follows the proof of Theorem 3.1, with Lemma 2.2 applied in place of Lemma 2.1. □

Theorem 3.5. *Let conditions* 1 *and* 2 *of Theorem* 3.1 *be satisfied, where condition* 2 (v) *is replaced by* 2 (v*) *and condition* 2 (ii) *is fulfilled for all* $y_0 \in \mathbb{R}^n$. *If the trivial solution of* (2.3) *is globally generalized Lipschitz stable in time, then the trivial solution of the IVP for IRLFDE* (1.1) *is globally generalized Lipschitz stable in time.*

Theorem 3.6. *Let the conditions of Theorem* 3.1 *be satisfied, where* $a(s) = A_2 s^p$, $A_2 > 0$, $p \geq 1$ *in condition* 2 (ii), *condition* 2 (i) *is replaced by condition* 2*(i) *of Theorem* 3.3, *and condition* 2 (v) *is replaced by condition* 2 (v*) *of Theorem* 3.4. *Then the trivial solution of the IVP for IRLFDE* (1.1) *is generalized Lipschitz stable in time.*

4. Applications

We now illustrate the application of our obtained sufficient conditions and the practical use of the fractional derivatives of Lyapunov functions.

Example 4.1. Let the sequence $\{t_i\}_{i\in\mathbb{N}_0}$, $t_0 = 0$, be given such that

$$L = \sup_{k\in\mathbb{N}_0} (t_{k+1} - t_k) \geq 1 \quad \text{and} \quad \lambda = \inf_{k\in\mathbb{N}_0} (t_{k+1} - t_k) > 0.$$

Consider now the IVP for the system of impulsive Riemann-Liouville equations

$$
\begin{cases}
{}_{t_k}^{\mathrm{RL}}D_t^q x_1(t) = -\left(0.5t^{q-1} + t^{-q}\dfrac{\Gamma(2-q)}{\Gamma(2-2q)} + x_2^2(t)\right) x_1(t),\\
{}_{t_k}^{\mathrm{RL}}D_t^q x_2(t) = -\left(0.5t^{q-1} + t^{-q}\dfrac{\Gamma(2-q)}{\Gamma(2-2q)} - \dfrac{x_1^2(t)}{1+x_2^2(t)}\right) x_2(t), \quad \text{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0,\\
\lim\limits_{t\to t_k+} \left[(t-t_k)^{1-q}x_1(t)\right] = \dfrac{\Psi_k^1(x_1(t_k-0), x_2(t_k-0))}{\Gamma(q)},\\
\lim\limits_{t\to t_k+} \left[(t-t_k)^{1-q}x_2(t)\right] = \dfrac{\Psi_k^2(x_1(t_k-0), x_2(t_k-0))}{\Gamma(q)}, \quad \text{for } k \in \mathbb{N},\\
\lim\limits_{t\to t_0+} \left[t^{1-q}x_1(t)\right] = \dfrac{x_{0,1}}{\Gamma(q)}, \quad \lim\limits_{t\to t_0+} \left[t^{1-q}x_2(t)\right] = \dfrac{x_{0,2}}{\Gamma(q)},
\end{cases} \tag{4.1}
$$

where $x_0 = (x_{0,1}, x_{0,2}) \in \mathbb{R}^2$,

$$\Psi_k^1(t, x_1, x_2) = \frac{x_1}{t} \quad \text{and} \quad \Psi_k^2(t, x_1, x_2) = \frac{x_2}{t},$$

for $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, $x_1, x_2 \in \mathbb{R}$. Consider the Lyapunov function

$$V(t,x) = (t-t_k)^{1-q}(x_1^2 + x_2^2), \quad \text{for } t \in (t_k, t_{k+1}], k \in \mathbb{N}_0, x_1, x_2 \in \mathbb{R},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. The function $V \in \Lambda([0,\infty), \mathbb{R}^2)$ is locally Lipschitz with constant L. Thus, assumption 2*(i) of Theorem 3.3 holds with

$$p = 2, \quad \mu(t) = (t-t_k)^{1-q}, \quad t \in (t_k, t_{k+1}), \quad \tau_k = t_k + \sqrt[1-q]{0.1}, \quad A_1 = \sqrt[1-q]{0.1}.$$

Let $y \in \mathrm{C}_{1-q}([0, t_1], \mathbb{R}^2)$, $y = (y_1, y_2)$, be such that

$$\lim_{t\to 0+} \left(t^{1-q}y_k(t)\right) = y_{0,k}, \quad k = 1,2, \quad y_0 = (y_{0,1}, y_{0,2}).$$

Then, because of

$$\lim_{t\to 0+} t^{1-q}V(t, y(t)) = \left(\lim_{t\to 0+} t^{1-q}y_1(t)\right)^2 + \left(\lim_{t\to 0+} t^{1-q}y_2(t)\right)^2 = \|y_0\|^2,$$

condition 2 (ii) of Theorem 3.1 holds with

$$a(s) = A_2 s^p, \quad A_2 = 1, \quad p = 2.$$

Let $y \in \mathrm{C}_{1-q}([t_k, t_{k+1}], \mathbb{R}^2)$ satisfies

$$\lim_{t\to t_k+} \left((t-t_k)^{1-q}y(t)\right) = \frac{y_k}{\Gamma(q)} < \infty, \quad y_k = (y_{1,k}, y_{2,k}).$$

Then, for $t \in (t_k, t_{k+1}]$, we get the inequality

$$
\begin{aligned}
(t-t_k)^{1-q}V(t, y(t))|_{t=t_k+} &= \lim_{t\to t_k+} (t-t_k)^{1-q}V(t, y(t))\\
&= \lim_{t\to t_k+} (t-t_k)^{1-q}(t-t_k)^{1-q}(y_1(t)^2 + y_2(t)^2)\\
&= \left(\lim_{t\to t_k+} (t-t_k)^{1-q}y_1(t)\right)^2 + \left(\lim_{t\to t_k+} (t-t_k)^{1-q}y_2(t)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= y_{1,k}^2 + y_{2,k}^2 \\
&\leq (t_k - t_{k-1})^{1-q}(y_{1,k}^2 + y_{2,k}^2) = V(t_k, y_k),
\end{aligned}
$$

and therefore, condition 2 (iv) of Theorem 3.1 is satisfied. Let $k \in \mathbb{N}_0$ and $t \in [t_k, t_{k+1}]$, $x = (x_1, x_2) \in \mathbb{R}^2$. Then, we get the inequalities

$$
V(t, \Psi_k(t, x^*(t_k - 0))) = (t - t_k)^{1-q} \left(\frac{x_1^2}{t^2} + \frac{x_2^2}{t^2} \right) = H_k(t, V(t_k, x^*(t_k - 0))),
$$

and therefore, condition 2 (iii) of Theorem 3.1 is satisfied with

$$
H_i(t, u) = \frac{u}{t^2}.
$$

The RL fractional derivative of the Lyapunov function, i.e.,

$$
{}_{t_k}^{\mathrm{RL}}D_t^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_k}^{t} (t-s)^{-q}(s - t_k)^{1-q} \left(x_1^2(s) + x_2^2(s) \right) \mathrm{d}s,
$$

where $x = (x_1, x_2)$, being the solution of (4.1), is rather difficult to obtain, so we cannot apply the results with RL derivative. Instead, we apply the results with Dini derivative of the function V among (4.1). Let $k \in \mathbb{N}_0$, $t \in (t_k, t_{k+1}]$, $x_1, x_2 \in \mathbb{R}$. Then, for $1 - 2q \geq 0$, i.e., $q \in (0, 0.5]$, we get

$$
\begin{aligned}
& D_{(4.1)}^{t_k}(t - t_k)^{1-q} \left(x_1^2 + x_2^2 \right) \\
&= \limsup_{h \to 0+} \frac{1}{h^q} \Big\{ (t - t_k)^{1-q} \left(x_1^2 + x_2^2 \right) - \sum_{r=1}^{\left[\frac{t-t_k}{h}\right]} (-1)^{r+1} \binom{q}{r} (t - rh - t_k)^{1-q} \\
&\quad \times \left[(x_1 - h^q f_1(t, x))^2 + (x_2 - h^q f_2(t, x))^2 \right] \Big\} \\
&= \limsup_{h \to 0+} \frac{1}{h^q} (t - t_k)^{1-q} \left[x_1^2 - (x_1 - h^q f_1(t, x))^2 + x_2^2 - (x_2 - h^q f_2(t, x))^2 \right] \\
&\quad + \limsup_{h \to 0+} \frac{1}{h^q} \left[(x_1 - h^q f_1(t, x))^2 + (x_2 - h^q f_2(t, x))^2 \right] \\
&\quad \times \sum_{r=0}^{\left[\frac{t-t_k}{h}\right]} (-1)^r \binom{q}{r} (t - t_k - rh)^{1-q} \\
&= \limsup_{h \to 0+} \frac{1}{h^q} (t - t_k)^{1-q} \left[(2x_1 - h^q f_1(t, x)) \, h^q f_1(t, x) + (2x_2 - h^q f_2(t, x)) \, h^q f_2(t, x) \right] \\
&\quad + \left[x_1^2 + x_2^2 \right] {}_{t_k}^{\mathrm{RL}}D_t^q (t - t_k)^{1-q} \\
&= 2(t - t_k)^{1-q} x_1 f_1(t, x) + 2(t - t_k)^{1-q} x_2 f_2(t, x) \\
&\quad + \left[x_1^2 + x_2^2 \right] \frac{\Gamma(2-q)}{\Gamma(2-2q)} (t - t_k)^{1-2q} \\
&= 2(t - t_k)^{1-q} x_1 \left(-0.5 t^{q-1} x_1 - t^{-q} \frac{\Gamma(2-q)}{\Gamma(2-2q)} x_1 - x_2^2 x_1 \right)
\end{aligned}
$$

$$
\begin{aligned}
&+2(t-t_k)^{1-q}x_2\left(-0.5t^{q-1}x_2-t^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_2+\frac{x_2x_1^2}{1+x_2^2}\right)\\
&+\left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
\leq&2(t-t_k)^{1-q}x_1\left(-0.5t^{q-1}x_1-(t-t_k)^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_1-x_2^2x_1\right)\\
&+2(t-t_k)^{1-q}x_2\left(-0.5t^{q-1}x_2-t^{-q}\frac{\Gamma(2-q)}{\Gamma(2-2q)}x_2+\frac{x_2x_1^2}{1+x_2^2}\right)\\
&+\left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
=&-0.5V(t,x)-\left[x_1^2+x_2^2\right]\frac{\Gamma(2-q)}{\Gamma(2-2q)}(t-t_k)^{1-2q}\\
\leq&-0.5V(t,x),\quad \text{for } x\in\mathbb{R}^2 \text{ and } t>0,
\end{aligned}
$$

and therefore, assumption 2 (v*) of Theorem 3.4 holds with

$$g(t,u)\equiv -0.5u,\quad u\in\mathbb{R} \text{ and } q\in(0,0.5].$$

Consider the scalar comparison linear RL fractional equation with noninstantaneous impulses

$$
\begin{cases}
{}_{t_i}^{\mathrm{RL}}D_t^q u(t)=-0.5u(t), & \text{for } t\in(t_i,t_{i+1}], i\in\mathbb{N}_0,\\
\lim\limits_{t\to t_i+}\left[(t-t_i)^{1-q}u(t)\right]=\dfrac{u(t_i-0)}{t_i\Gamma(q)}, & \text{for } i\in\mathbb{N},\\
\lim\limits_{t\to 0+}\left[t^{1-q}u(t)\right]=\dfrac{u_0}{\Gamma(q)},
\end{cases}
\tag{4.2}
$$

where $u_0\in\mathbb{R}$. Similar to Example 2.1, the solution of (4.2) is given by

$$
y(t)=\begin{cases}
u_0t^{q-1}E_{q,q}(-0.5t^q), & \text{for } t\in(0,t_1],\\
u_0\left(\prod_{i=0}^{k-1}E_{q,q}(-0.5(t_{i+1}-t_i)^q)t_i\right) & \\
\times(t-t_k)^qE_{q,q}(-0.5(t-t_k)^q), & \text{for } t\in(t_{k+1},t_k],\ k\in\mathbb{N}.
\end{cases}
$$

In the case $q\in(0,0.5]$, the trivial solution of (4.2) is generalized Lipschitz stable in time (for particular values $q=0.3$, $t_k=k$, $k\in\mathbb{N}$, and $u_0=1$, $u_0=2$, the graphs of the corresponding solutions u_{u_0} are given in Figure 4, and in the partial case of $q=0.5$, $t_k=k$, $k\in\mathbb{N}$ and $u_0=1$, $u_0=2$, the graphs of the solutions u_{u_0} are given in Figure 5). In the case $q\in(0.5,1)$, the zero solution of (4.2) is not generalized Lipschitz stable in time (for particular values $q=0.8$, $t_k=k$, $k\in\mathbb{N}$ and $u_0=1$, $u_0=2$, the graphs of the solutions u_{u_0} are given in Figure 6). Due to Theorem 3.6, the zero solution of (4.1) is generalized Lipschitz stable in time for $q\in(0,0.5)$.

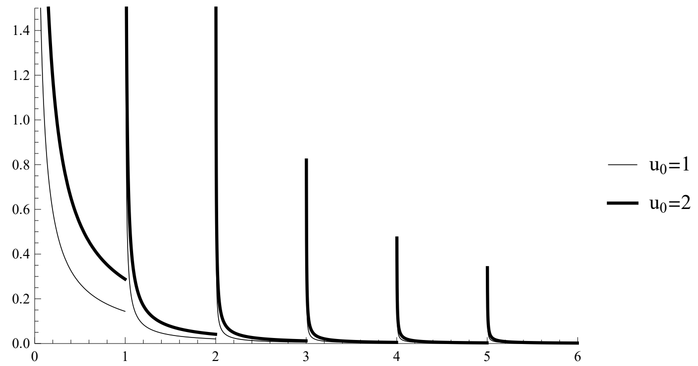

FIGURE 4. Graph of the solutions of (4.2) for $q = 0.3$, $u_0 = 1$ and for $u_0 = 2$.

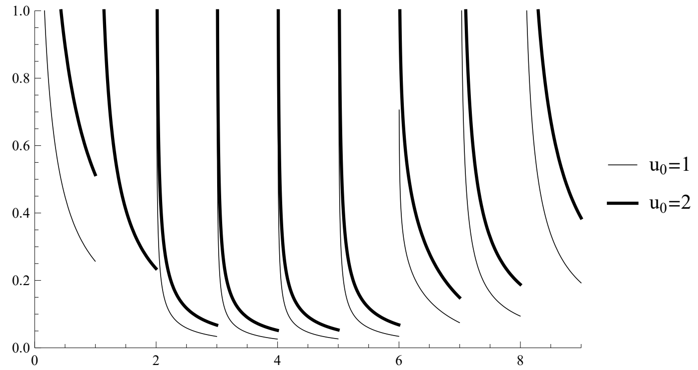

FIGURE 5. Graph of the solution of (4.2) for $q = 0.5$, $u_0 = 1$ and for $u_0 = 2$.

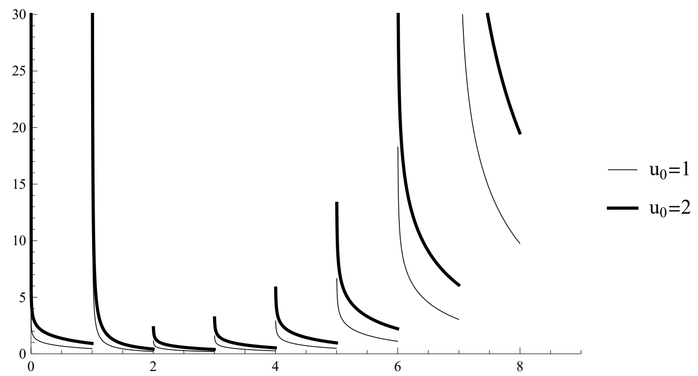

FIGURE 6. Graph of the solution of (4.2) for $q = 0.8$ with $u_0 = 1$ and $u_0 = 2$.

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[1]Missouri S&T,
Rolla, MO 65409, USA
Email address: bohner@mst.edu

[2]Faculty of Mathematics and Informatics,
University of Plovdiv "Paisii Hilendarski",
Plovdiv, Bulgaria
Email address: snehri@gmail.com