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Radoja Domanovića 12
34000 Kragujevac
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Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Email: krag_j_math@kg.ac.rs
Website: http://kjm.pmf.kg.ac.rs

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SPECTRAL EXPANSION FOR CONFORMABLE FRACTIONAL STURM-LIOUVILLE PROBLEM ON THE WHOLE LINE

BİLENDER P. ALLAHVERDİEV[1], HÜSEYİN TUNA[2], AND YÜKSEL YALÇINKAYA[1]

Abstract. In this article, we discuss a conformable fractional Sturm-Liouville boundary-value problem on the whole line. We prove the existence of a spectral function for the singular conformable fractional Sturm-Lioville problem. Further, we establish a Parseval equality and spectral expansion formula by terms of the spectral function for conformable fractional Sturm-Liouville problem on the whole line.

1. Introduction

Fractional order differential equations first appeared towards the end of the 17th century with a letter of L'Hospital to Leibnitz in which he asked the meaning of "fractional order derivative". Up to the present time, many mathematicians such as Liouville, Riemann, Weyl, Fourier, Lagrange, Grönwald, Letnikov, Abel, and Caputo have made research in this field [2]. Fractional differential equations are used today in many fields such as transmission line theory, signal processing, chemical analysis, heat transfer, hydraulics of dams, material science, temperature field problems oil strata, diffusion problems, waves in liquids and gases, Schrödinger equation, and fractal equation [2–7]. Recently, based on the definition of the classical derivative, a new fractional derivative is put forward by Khalil et al. and named as conformable fractional derivative [1]. In this study Khalil et al. provided the linearity property for his new definition of the fractional derivative, they proved the product rule, the quotient rule, the fractional Rolle theorem, and the fractional mean value theorem. Later in [8],

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Abdeljawad defined the right and left conformable fractional derivatives, the fractional chain rule, and fractional integrals of higher orders. Conformable fractional derivative aims to expand the definition of the classical derivative by providing the natural characteristics of the classical derivative and gain new perspectives for the theory of differential equations [9]. Examples of these perspectives are [10–16, 31, 34]. In [10], the form of the Wronskian for conformable fractional linear differential equations with variable coefficients is discussed and Abel's formula for fractional differential equations with variable coefficients is proven. In [11], the exact solution of the heat conformable fractional differential equation is given. In [12], being existent and uniqueness theories of consecutive linear conformable fractional differential equations are demonstrated. In [13], some of the conformable fractional partial equations including the wave equation are solved. In [14], linear second-order conformable differential equations using a proportional derivative are shown to be formally self-adjoint equations with respect to a certain inner product and the associated self-adjoint boundary conditions. Defining a Wronskian, a Lagrange identity, and Abel's formula are established. Several reduction-of-order theorems are given. Solutions of the conformable second-order self-adjoint equation are then shown to be related to corresponding solutions of a first-order Riccati equation and a related quadratic functional and a conformable Picone identity. A Lyapunov inequality, factorizations of the second-order equation are established. Boundary value problems and Green's functions are studied. In [31], a regular fractional generalization of the Sturm-Liouville eigenvalue problems is suggested and some fundamental results of this suggested model are established. In [34], a general notion of fractional derivative for functions defined on arbitrary time scales is introduced. The basic tools for the time-scale fractional calculus are then developed. In [15], a conformable fractional Dirac system with separated boundary conditions on an arbitrary time scale is studied, some basic spectral properties of the classical Dirac system are extended to the conformable fractional case. In [16], a conformable fractional Sturm-Liouville equation with boundary conditions on an arbitrary time scale is analyzed, basic spectral properties of the classical Sturm-Liouville equation are extended to the conformable fractional case, some sufficient conditions are established to guarantee the existence of a solution for this conformable fractional Sturm-Liouville problem on $\mathbb{T}$ by using certain fixed point theorems.

Today, it is widely accepted that spectral expansion theorems are beneficial in science and engineering. If, for example, a partial differential equation is solved by the method of separation of variables (i.e., the Fourier method) then the problems of expanding an arbitrary function to a series of eigenfunctions and showing that the eigenfunctions form a complete system occur. The first study for the spectral expansion problem is constructed by Weyl [17] (see [18–29, 32, 33, 35, 36]).

The primary aim of this study is to prove the existence of a spectral function for singular conformable fractional (CF) Sturm-Liouville equation of the form

$$-T_\alpha^2 y(t) + v(t)y(t) = \lambda y(t), \quad -\infty < t < \infty,$$

where λ is a complex parameter, $v(\cdot)$ is a real-valued conformable fractional locally integrable function on $(-\infty, \infty)$. The article is structured as follows. In Section 2, necessary concepts and properties are reviewed. In Section 3, we construct resolvent in view of Green's function. We show that the regular CF-Sturm-Liouville operator has a compact resolvent, so it has a purely discrete spectrum. Finally, in Section 4, we establish a Parseval equality and spectral expansion formula by terms of the spectral function for the CF-Sturm-Liouville problem on the whole line.

2. Preliminaries

In this section, our goal is to present some basic definitions and properties of conformable fractional calculus and operator theory. For more details, the reader may want to consult [1–8, 30, 37]. Throughout this paper, we will fix $\alpha \in (0, 1)$.

Definition 2.1. Assume α be a positive number with $0 < \alpha < 1$. A function $f : \mathbb{R} \to \mathbb{R} =(-\infty, \infty)$ the conformable fractional derivative of order α of f at $t > 0$ is defined by

$$T_\alpha f(t) = \lim_{\varepsilon\to\infty} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f\left(t\right)}{\varepsilon} \tag{2.1}$$

and the fractional derivative at 0 is defined by

$$\left(T_\alpha f\right)(0) = \lim_{t\to 0^+} T_\alpha f(t).$$

Definition 2.2. The left conformable fractional derivative starting from a of a function $f : [a, \infty) \to \mathbb{R}$ of order α is defined by

$$(T_\alpha^a\ f)(t) = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon\left(t-a\right)^{1-\alpha}) - f\left(t\right)}{\varepsilon}, \quad 0 < \alpha \le 1.$$

Definition 2.3. The right conformable fractional derivative of order $0 < \alpha \le 1$ of f is defined by

$$({}_\alpha^b T f)(t) = -\lim_{\varepsilon\to 0} \frac{f(t + \varepsilon(b-t)^{1-\alpha}) - f(t)}{\varepsilon},$$

where f is terminating at b and $({}^b T_\alpha f)(t) = \lim_{t\to b^-} ({}^b T_\alpha f)(t)$.

In the next lemma, we consider some properties of conformable derivatives.

Lemma 2.1. *Let f, g be conformable differentiable of order α, $0 < \alpha \le 1$, at a point t. Then*
(i) $T_\alpha\left(\lambda f + \delta g\right) = \lambda T_\alpha\left(f\right) + \delta T_\alpha\left(g\right)$, $\lambda, \delta \in \mathbb{R}$;
(ii) $T_\alpha\left(fg\right) = fT_\alpha\left(g\right) + gT_\alpha\left(f\right)$;
(iii) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$;
(iv) f *is differentiable then* $T_\alpha^a\left(f\right)(t) = \left(t-a\right)^{1-\alpha} f'\left(t\right)$.
(v) $T_\alpha\left(t^n\right) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$.

Next, we present the conformable fractional integral and some of its properties.

Definition 2.4. The conformable fractional integral starting from a of a function f of order $0 < \alpha \leq 1$ is defined by

$$\left(I_\alpha^a f\right)(t) = \int\limits_a^t f(x) d\alpha\left(x, a\right) = \int\limits_a^t (x-a)^{\alpha-1} f(x) dx.$$

Similarly, in the right case, we have

$$\left({}^b I_\alpha f\right)(t) = \int\limits_t^b f(x) d\alpha\left(b, x\right) = \int\limits_t^b (b-x)^{\alpha-1} f(x) dx.$$

Lemma 2.2. *Assume that f is a continuous function on (a, ∞) and $0 < \alpha < 1$. Then we have*

$$T_\alpha^a I_\alpha^a f\left(t\right) = f\left(t\right),$$

for all $t > a$.

Theorem 2.1. *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that f and g are conformable fractional differentiable. So, we have*

$$\int_a^b f(t) T_\alpha^a\left(g\right)\left(t\right) d\alpha\left(t, a\right) + \int_a^b g\left(t\right) T_\alpha^a\left(f\right)\left(t\right) d\alpha\left(t, a\right) = f(b)g(b) - f(a)g(a).$$

Let $L_\alpha^2(-b, b)$, $-\infty \leq -b < b \leq \infty$, is the space of all complex-valued functions defined on $(-b, b)$ as

$$\|f\| := \left(\int_{-b}^b \left|f\left(t\right)\right|^2 d_\alpha\left(t\right)\right)^{1/2} = \left(\int_{-b}^b \left|f\left(t\right)\right|^2 t^{\alpha-1} dt\right)^{1/2} < \infty.$$

The space $L_\alpha^2(-b, b)$ is a Hilbert space with the inner product

$$(f, g) := \int_{-b}^b f\left(t\right) \overline{g\left(t\right)} d_\alpha\left(t\right), \quad f, g \in L_\alpha^2(-b, b).$$

Let us define the conformable α-Wronskian of x and y by

$$W_\alpha(x, y)(t) = x(t) T_\alpha y(t) - y(t) T_\alpha x(t), \quad t \in (-b, b).$$

Definition 2.5. A function $M(t, x)$ in $\mathbb{C}^2$ of two variables with $-b < t$, $x < b$ is called the α-Hilbert-Schmidt kernel if

$$\int_{-b}^b \int_{-b}^b |M(t, x)|^2 d_\alpha(t) d_\alpha(x) < \infty.$$

Theorem 2.2. *If*

$$\sum_{i,k=1}^\infty \left|a_{ik}\right|^2 < \infty, \tag{2.2}$$

then the operator A defined by the formula

$$A\left\{x_i\right\} = \left\{y_i\right\}, \quad i \in \mathbb{N} := \left\{1, 2, 3, \dots\right\},$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \quad i \in \mathbb{N}, \tag{2.3}$$

is compact in the sequence space ℓ^2 [30].

Theorem 2.3 ([37])**.** *Let* $(w_n)_{n\in\mathbb{N}}$ *be a uniformly bounded sequence of real non-decreasing functions on a finite interval* $[c, d]$ *. Then there exists a subsequence* $(w_{n_k})_{k\in\mathbb{N}}$ *and a non-decreasing function* w *such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),$$

where $\lambda \in [c, d]$ *.*

Theorem 2.4 ([37])**.** *Assume* $(w_n)_{n\in\mathbb{N}}$ *is a real, uniformly bounded, sequence of non-decreasing functions on a finite interval* $[c, d]$ *, and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),$$

where $\lambda \in [c, d]$ *. If* f *is any continuous function on* $[c, d]$ *, then*

$$\lim_{n\to\infty} \int_c^d f(\lambda)\, dw_n(\lambda) = \int_c^d f(\lambda)\, dw(\lambda).$$

3. Regular CF-Sturm-Liouville Problem

In this part, we construct Green's function and prove that the regular CF-Sturm-Liouville operator has a compact resolvent, so it has a purely discrete spectrum.

We consider the regular CF-Sturm-Liouville equation defined by

$$-T_\alpha^2 y(t) + v(t)y(t) = \lambda y(t), \quad -\infty < -b < x < b < \infty. \tag{3.1}$$

Let γ and β be arbitrary real numbers and let $y(t, \lambda)$ satisfies the boundary conditions

$$y(-b, \lambda)\cos\beta + T_\alpha y(-b, \lambda)\sin\beta = 0, \tag{3.2}$$

$$y(b, \lambda)\cos\gamma + T_\alpha y(b, \lambda)\sin\gamma = 0, \tag{3.3}$$

in which λ is a complex eigenvalue parameter, $v(t)$ is a real-valued continuous function defined on $\mathbb{R}$ and $v \in L^1_{\alpha,loc}(\mathbb{R})$, where

$$L^1_{\alpha,loc}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{-b}^{b} |f(t)|\, d_\alpha(t) < \infty \text{ for all } b \in \mathbb{R} \right\}.$$

Denote by $\theta_1(t, \lambda)$, and $\theta_2(t, \lambda)$ the linearly independent solutions of the (3.1) subject to the initial conditions

$$\theta_1(-b, \lambda) = \sin\beta, \quad T_\alpha\theta_1(-b, \lambda) = -\cos\beta, \tag{3.4}$$

$$\theta_2(b, \lambda) = \sin\gamma, \quad T_\alpha\theta_2(b, \lambda) = -\cos\gamma. \tag{3.5}$$

In this way, the *Green's function* of the problem is defined by (3.1)–(3.4) (see [23])

$$G(t,x,\lambda)=\begin{cases}\frac{\theta_1(t,\lambda)\theta_2(x,\lambda)}{W_\alpha(\theta_1,\theta_2)}, & -b\le x<t,\\ \frac{\theta_2(t,\lambda)\theta_1(x,\lambda)}{W_\alpha(\theta_1,\theta_2)}, & t<x\le b.\end{cases} \tag{3.6}$$

In the next results, without restriction of generality, we assume that $\lambda=0$ is not an eigenvalue of the problem (3.1)–(3.3).

Theorem 3.1. *$G(t,x)$ defined by* (3.6) *is a α-Hilbert-Schmidt kernel.*

Proof. By the upper half of the formula (3.6), we obtain

$$\int_{-b}^{b}d_\alpha(t)\int_{-b}^{t}|G(t,x)|^2d_\alpha(x)<\infty,$$

and by the lower half of (3.6), we have

$$\int_{-b}^{b}d_\alpha(t)\int_{t}^{b}|G(t,x)|^2d_\alpha(x)<\infty,$$

because the inner integral exists and is products $\theta_1(x)\,\theta_2(t)$, and these products belong to $L^2_\alpha(-b,b)\times L^2_\alpha(-b,b)$ because each of the factors belongs to $L^2_\alpha(-b,b)$. Then, we obtain

$$\int_{-b}^{b}\int_{-b}^{b}|G(t,x)|^2d_\alpha(t)d_\alpha(x)<\infty. \tag{3.7}$$

□

Theorem 3.2. *The operator* $\mathbf{S}$ *defined by the formula*

$$(\mathbf{S}f)(t)=\int_{-b}^{b}G(t,x)f(x)d_\alpha(x)$$

is compact and self-adjoint on $L^2_\alpha(-b,b)$.

Proof. Let $\phi_i=\phi_i(x)$, $i\in\mathbb{N}$, is an orthonormal basis of $L^2_\alpha(-b,b)$. Because $G(t,x)$ is a $\alpha-$Hilbert-Schmidt kernel, it can be defined as

$$t_i=(f,\phi_i)=\int_{-b}^{b}f(x)\overline{\phi_i(x)}d_\alpha(x),$$

$$y_i=(g,\phi_i)=\int_{-b}^{b}g(x)\overline{\phi_i(x)}d_\alpha(x),$$

$$a_{ik}=\int_{-b}^{b}\int_{-b}^{b}G(t,x)\phi_i(t)\overline{\phi_k(x)}d_\alpha(t)d_\alpha(x),\quad i,k\in\mathbb{N}.$$

Then, $L^2_\alpha(-b,b)$ is mapped isometrically ℓ^2. As a consequence, the integral operator turns into the operator which is defined by the formula (2.3) in the space ℓ^2 by this mapping, and the condition (3.7) is translated into the condition (2.2). So, the original operator is compact.

Let $f,g\in L^2_\alpha(-b,b)$. As $G(t,x)=G(x,t)$ and we have

$$(\mathbf{S}f,g)=\int_{-b}^{b}(\mathbf{S}f)(t)\overline{g(t)}d_\alpha(t)$$

$$= \int_{-b}^{b}\int_{-b}^{b} G(t,x)f(x)d_\alpha(x)\overline{g(t)}d_\alpha(t)$$
$$= \int_{-b}^{b} f(x)\Big(\overline{\int_0^b G(x,t)g(t)d_\alpha(t)}\Big)d_\alpha(x) = (f, \mathbf{S}g).$$

Thus, we have proved that the operator $\mathbf{S}$ is self-adjoint. □

4. Parseval Equality and Spectral Expansion In The Case of The Whole Line

In this part, the existence of a spectral function for singular Sturm-Liouville problem (3.1)–(3.2) will be proven. A Parseval equality and spectral expansion formula by terms of the spectral function is set up.

Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues and $y_1, y_1, \ldots$ the corresponding eigenfunctions of the problem (3.1)–(3.3). Let $\theta_1(t,\lambda)$ and $\theta_2(t,\lambda)$ be solutions of the problem (3.1)–(3.2) satisfying the initial conditions

$$\theta_1(0,\lambda) = 0, \quad T_\alpha\theta_1(0,\lambda) = 1, \quad \theta_2(0,\lambda) = 1, \quad T_\alpha\theta_2(0,\lambda) = 0,$$

and let

$$y_n(t) = c_n\theta_1(t,\lambda_n) + d_n\theta_2(0,\lambda_n).$$

Let f be a real-valued function and $f \in L^2_\alpha(-b,b)$. Then it follows from Theorem 3.2 and the Hilbert-Schmidt theorem that

$$\begin{aligned}\int_{-b}^{b} f^2(t)d_\alpha(t) &= \sum_{n=1}^{\infty}\left\{\int_{-b}^{b} f(t)y_n(t)d_\alpha(t)\right\}^2\\
&= \sum_{n=1}^{\infty}\left\{\int_{-b}^{b} f(t)\{c_n\theta_1(t,\lambda_n) + d_n\theta_2(t,\lambda_n)\}d_\alpha(t)\right\}^2\\
&= \sum_{n=1}^{\infty} c_n^2\left\{\int_{-b}^{b} f(t)\theta_1(t,\lambda_n)d_\alpha(t)\right\}^2\\
&\quad + 2\sum_{n=1}^{\infty} c_nd_n\left\{\int_{-b}^{b} f(t)\theta_1(t,\lambda_n)d_\alpha(t)\int_{-b}^{b} f(t)\theta_2(t,\lambda_n)d_\alpha(t)\right\}\\
&\quad + \sum_{n=1}^{\infty} d_n^2\left\{\int_{-b}^{b} f(t)\theta_2(t,\lambda_n)d_\alpha(t)\right\}^2.\end{aligned} \tag{4.1}$$

Now, we will define the step functions by

$$\xi_{-b,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} c_n^2, & \text{for } \lambda \le 0,\\ \sum\limits_{0\le\lambda_n<\lambda} c_n^2 & \text{for } \lambda > 0,\end{cases}$$

$$\zeta_{-b,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} c_nd_n, & \text{for } \lambda \le 0,\\ \sum\limits_{0\le\lambda_n<\lambda} c_nd_n & \text{for } \lambda > 0,\end{cases}$$

$$\varsigma_{-b,b}(\lambda)=\begin{cases}-\sum\limits_{\lambda<\lambda_n<0} d_n^2, & \text{for } \lambda\leq 0,\\ \sum\limits_{0\leq\lambda_n<\lambda} d_n^2 & \text{for } \lambda>0.\end{cases}$$

Then the Parseval equality (4.1) can be stated as

$$\begin{aligned}\int_{-b}^{b} f^2(t)d_\alpha(t) &= \int_{-\infty}^{\infty}\left\{\int_{-b}^{b} f(t)c_n\theta_1(t,\lambda_n)d_\alpha(t)\right\}^2 d\xi_{-b,b}(\lambda)\\ &+2\int_{-\infty}^{\infty}\left\{\int_{-b}^{b} f(t)c_n\theta_1(t,\lambda_n)d_\alpha(t)\right\}\left\{\int_{-b}^{b} f(t)d_n\theta_2(t,\lambda_n)d_\alpha(t)\right\}d\zeta_{-b,b}(\lambda)\\ &+\int_{-\infty}^{\infty}\left\{\int_{-b}^{b} f(t)d_n\theta_2(t,\lambda_n)d_\alpha(t)\right\}^2 d\varsigma_{-b,b}(\lambda).\end{aligned}\tag{4.2}$$

In the sequel, we shall present a lemma.

Lemma 4.1. *For any $s>0$, there exists a positive constant $M=M(S)$ not depending on b such that*

$$\overset{S}{\underset{-S}{V}}\{\varrho_{ij,b}(\lambda)\}<M,\quad i,j=1,2,\tag{4.3}$$

where

$$\varrho_{11,b}(\lambda)=\xi_{-b,b}(\lambda),\quad \varrho_{12,b}(\lambda)=\varrho_{21,b}(\lambda)=\zeta_{-b,b}(\lambda),\quad \varrho_{22,b}(\lambda)=\varsigma_{-b,b}(\lambda).$$

Proof. To see the validity of (4.3), it suffices to put $i=j$, because

$$\overset{S}{\underset{-S}{V}}\{\varrho_{12,b}(\lambda)\}\leq\frac{1}{2}\{\varrho_{11,b}(S)-\varrho_{11,b}(-S)+\varrho_{22,b}(S)-\varrho_{22,b}(S)\}.$$

The Parseval equality (4.2) then takes the form

$$\int_{-b}^{b} f^2(t)d_\alpha(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)F_j(\lambda)d\varrho_{ij,b}(\lambda),\tag{4.4}$$

where

$$F_i(\lambda)=\int_{-b}^{b} f(t)\theta_i(t,\lambda)d_\alpha(t),\quad i=1,2.$$

If follows from (4.4) that

$$T_\alpha^{(j-1)}\theta_i(0,\lambda)=\delta_{i,j}\quad i,j=1,2,$$

where $\delta_{i,j}$ is the Kronecker delta. Thus, for any $\varepsilon>0$ and given $S>0$, there exists a $r>0$ such that

$$|T_\alpha^{(j-1)}\theta_i(t,\lambda)-\delta_{i,j}|<\varepsilon,\tag{4.5}$$

where $|\lambda|\leq S,\ t\in[0,r]$. □

Let $f_r(t)$ be a non-negative twice continuously differentiable function such that $f_r(t)$ vanishes outside the interval $[0, r]$, with

$$\int_0^r f_r(t) d_\alpha(t) = 1. \tag{4.6}$$

Now, if we apply the Parseval equality (4.4) to the functions $T_\alpha^{(j-1)} f_r(t)$, $j = 1, 2$, then we get

$$\int_0^r \left| T_\alpha^{(j-1)} f_r(t) \right|^2 d_\alpha(t) \geq \int_{-S}^{S} \sum_{i,k=1}^{2} F_{i,j}(\lambda) F_{kj}(\lambda) d\varrho_{ik,b}(\lambda), \tag{4.7}$$

where

$$F_{ij}(\lambda) = \int_0^r T_\alpha^{(j-1)} f_r(t) \theta_i(t,\lambda) d_\alpha(t) = \pm \int_0^r f_r(t) T_\alpha^{(j-1)} \theta_i(t,\lambda) d_\alpha(t).$$

Using (4.5) and (4.6), we obtain

$$|F_{ij}(\lambda) - \delta_{i,j}| < \varepsilon, \quad i, j = 1, 2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\int_0^r \left| T_\alpha^{(j-1)} f_r(t) \right|^2 d_\alpha(t) \geq \int_{-S}^{S} \sum_{i,k=1}^{2} (\delta_{ij} - \varepsilon)(\delta_{kj} - \varepsilon) |d\varrho_{ik,b}(\lambda)|. \tag{4.9}$$

If we take $j = 1$ in (4.8), we have

$$\begin{aligned} \int_0^r f_r^2(t) d_\alpha(t) &> (1-\varepsilon)^2 \int_{-S}^{S} |d\varrho_{11,b}(\lambda)| - \varepsilon(1+\varepsilon) \int_{-S}^{S} |d\varrho_{12,b}(\lambda)| \\ &\quad - \varepsilon(1+\varepsilon) \int_{-S}^{S} |d\varrho_{21,b}(\lambda)| + \varepsilon^2 \int_{-S}^{S} |d\varrho_{22,b}(\lambda)| \\ &> (1-\varepsilon)^2 (\varrho_{11,b}(S) - \varrho_{11,b}(-S)) - 2\varepsilon(1+\varepsilon) \bigvee_{-S}^{S} \{\varrho_{12,b}(\lambda)\}. \end{aligned}$$

Since

$$\bigvee_{-S}^{S} \{\varrho_{12,b}(\lambda)\} \leq \frac{1}{2} [\varrho_{11,b}(S) - \varrho_{11,b}(-S) + \varrho_{22,b}(S) - \varrho_{22,b}(-S)], \tag{4.10}$$

we obtain

$$\begin{aligned} &\int_0^r f_r^2(t) d_\alpha(t) \\ &> \left(\varepsilon^2 - 2\varepsilon + 1\right) \{\varrho_{11,b}(S) - \varrho_{11,b}(-S)\} \\ &\quad - \varepsilon(1+\varepsilon) \{\varrho_{11,b}(S) - \varrho_{11,b}(-S) + \varrho_{22,b}(S) - \varrho_{22,b}(-S)\} \\ &= (1 - 3\varepsilon) \{\varrho_{11,b}(S) - \varrho_{11,b}(-S)\} - \varepsilon(1+\varepsilon) \{\varrho_{22,b}(S) - \varrho_{22,b}(-S)\}. \end{aligned} \tag{4.11}$$

Putting $j = 2$ in (4.9), we see that

$$\begin{aligned} \int_0^r |T_\alpha f_r(t)|^2 d_\alpha(t) &\geq (1 - 3\varepsilon) \{\varrho_{22,b}(S) - \varrho_{22,b}(-S)\} \\ &\quad - \varepsilon(1+\varepsilon) \{\varrho_{11,b}(S) - \varrho_{11,b}(-S)\}. \end{aligned} \tag{4.12}$$

If we add the inequalities (4.11) and (4.12), then we deduce that

$$\int_0^r f_r^2(t)\, d_\alpha(t) + \int_0^r |T_\alpha f_r(t)|^2\, d_\alpha(t)$$
$$\geq (2\varepsilon - 1)^2 \left(\varrho_{11,b}(S) - \varrho_{11,b}(-S) + \varrho_{22,b}(S) - \varrho_{22,b}(-S)\right).$$

If we choose $\varepsilon > 0$ such that $1 - 4\varepsilon - \varepsilon^2 > 0$, then we have the assertion of the lemma for the functions $\varrho_{11,b}(\lambda)$ and $\varrho_{22,b}(\lambda)$ relying on their monotonicity. From (4.10), we have the assertion of the lemma for the function $\varrho_{12,b}(\lambda)$.

Let ϱ be any non-decreasing function on $-\infty < \lambda < \infty$. Denote by $L_\varrho^2(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\, d\varrho(\lambda) < \infty,$$

with the inner product

$$(f, g)_\varrho := \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, d\varrho(\lambda).$$

The main results of this paper are the following three theorems.

Theorem 4.1. *Let f is a real-valued function and $f \in L_\alpha^2(\mathbb{R})$. Then there exist monotonic functions $\varrho_{11}(\lambda)$ and $\varrho_{22}(\lambda)$ which are bounded over every finite interval, and a function $\varrho_{12}(\lambda)$ which is of bounded variation over every finite interval with the property*

$$\int_{-\infty}^{\infty} f^2(t)\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, F_j(\lambda)\, d\varrho_{ij}(\lambda),$$

where

$$F_i(\lambda) = \lim_{b\to\infty} \int_{-b}^{b} f(t)\, \theta_i(t, \lambda)\, d_\alpha(t).$$

We note that the matrix-valued function $\varrho = (\varrho_{ij})_{i,j=1}^2$ ($\varrho_{12} = \varrho_{21}$) is called a *spectral function* for (3.1).

Proof. Assume that the real-valued function $f_n(t)$ satisfies the following conditions.

1) $f_n(t)$ vanishes outside the interval $[-n, n]$, where $n < b$.

2) The functions $f_n(t)$ and $T_\alpha f_n(t)$ are continuous.

If we apply the Parseval equality to $f_n(t)$, we get

$$\int_{-n}^{n} f_n^2(t)\, d_\alpha(t) = \sum_{k=1}^{\infty} \left\{ \int_{-b}^{b} f_n(t)\, y_k(t)\, d_\alpha(t) \right\}^2. \tag{4.13}$$

Then, by integrating by parts, we obtain

$$\int_{-n}^{n} f_n(t) y_k(t) d_\alpha(t) = \frac{1}{\lambda_k} \int_{-b}^{b} f_n(t) \left[-T_\alpha^2 y_k(t) + v(t) y_k(t)\right] d_\alpha(t),$$
$$= \frac{1}{\lambda_k} \int_{-b}^{b} \left[-T_\alpha^2 f_n(t) + v(t) f_n(t)\right] y_k(t) d_\alpha(t).$$

Thus we have

$$\begin{aligned}&\sum_{|\lambda_k|\geq\mu}\left\{\int_{-b}^{b} f_n(t)\,y_k(t)\,d_\alpha(t)\right\}^2\\ \leq&\frac{1}{\mu^2}\sum_{|\lambda_k|\geq\mu}\left\{\int_{-b}^{b}\left[-T_\alpha^2 f_\xi(t)+v(t)\,f_n(t)\right]y_k(t)\,d_\alpha(t)\right\}^2\\ \leq&\frac{1}{\mu^2}\sum_{k=1}^{\infty}\left\{\int_{-b}^{b}\left[-T_\alpha^2 f_n(t)+v(t)\,f_n(t)\right]y_k(t)\,d_\alpha(t)\right\}^2\\ =&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n(t)+v(t)\,f_n(t)\right]^2 d_\alpha(t).\end{aligned}$$

Using (4.13), we conclude that

$$\begin{aligned}&\left|\int_{-n}^{n} f_n(t)\,y_k(t)\,d_\alpha(t)-\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n(t)\,y_k(t)\,d_\alpha(t)\right\}^2\right|\\ \leq&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n(t)+v(t)\,f_n(t)\right]^2 d_\alpha(t).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n(t)\,y_k(t)\,d_\alpha(t)\right\}^2\\ =&\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n(t)\left\{c_n\theta_1(t,\lambda_k)+d_n\theta_2(t,\lambda_k)\right\}d_\alpha(t)\right\}^2\\ =&\int_{-\mu}^{\mu}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\varrho_{ij,b}(\lambda),\end{aligned}$$

where

$$F_{in}(\lambda)=\int_{-b}^{b} f_n(t)\,\theta_i(t,\lambda)\,d_\alpha(t),\quad i=1,2.$$

Consequently, we get

$$\begin{aligned}&\left|\int_{-n}^{n} f_n^2(t)\,d_\alpha(t)-\int_{-\mu}^{\mu}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\varrho_{ij,b}(\lambda)\right|\\ \leq&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n(t)+v(t)\,f_n(t)\right]^2 d_\alpha(t).\end{aligned}\tag{4.14}$$

By Lemma 4.1 and Theorems 2.3 and 2.4, we can find sequences $\{-b_k\}$ and $\{b_k\}$ $(b_k\to\infty)$ such that the function $\varrho_{ij,b_k}(\lambda)$ converges to a monotone function $\varrho_{ij}(\lambda)$. Passing to the limit with respect to $\{-b_k\}$ and $\{b_k\}$ in (4.14), we have

$$\left|\int_{-n}^{n} f_n^2(t)\,d_\alpha(t)-\int_{-\mu}^{\mu}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\varrho_{ij}(\lambda)\right|$$

$$\leq \frac{1}{\mu^2}\int_{-n}^{n}\left[-T_{\alpha}^{2}f_n(t)+v(t)f_n(t)\right]^2 d_{\alpha}(t).$$

As $\mu \to \infty$, we get

$$\int_{-n}^{n} f_n^2(t)\,d_{\alpha}(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,d\varrho_{ij}(\lambda).$$

Let $f(\cdot)\in L_{\alpha}^{2}(\mathbb{R})$. Choose functions $\{f_{\xi}(t)\}$ satisfying the conditions 1)–2) and such that

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty}(f(t)-f_{\xi}(t))^2\,d_{\alpha}(t)=0.$$

Let

$$F_{i\xi}(\lambda)=\int_{-\infty}^{\infty} f_{\xi}(t)\,\theta_i(t,\lambda)\,d_{\alpha}(t),\quad i=1,2.$$

Then we have

$$\int_{-\infty}^{\infty} f_{\xi}^2(t)\,d_{\alpha}(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_{i\xi}(\lambda)\,F_{j\xi}(\lambda)\,d\varrho_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{\infty}(f_{\xi_1}(t)-f_{\xi_2}(t))^2\,d_{\alpha}(t)\to 0 \quad \text{as} \quad \xi_1,\xi_2\to\infty,$$

we have

$$\int_{-\infty}^{\infty}\sum_{i=1}^{2}(F_{i\xi_1}(\lambda)\,F_{j\xi_1}(\lambda)-F_{i\xi_2}(\lambda)\,F_{j\xi_2}(\lambda))\,d\varrho_{ij}(\lambda)$$
$$=\int_{-\infty}^{\infty}(f_{\xi_1}(t)-f_{\xi_2}(t))^2\,d_{\alpha}(t)\to 0,$$

as $\xi_1,\xi_2\to\infty$. Therefore, there is a limit function F_i, $i=1,2$, that satisfies

$$\int_{-\infty}^{\infty} f^2(t)\,d_{\alpha}(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)\,F_j(\lambda)\,d\varrho_{ij}(\lambda),$$

by the completeness of the space $L_{\varrho}^{2}(\mathbb{R})$.

Now we will show that the sequence $(K_{i\xi})$ defined by

$$K_{i\xi}(\lambda)=\int_{-\xi}^{\xi} f(t)\,\theta_i(t,\lambda)\,d_{\alpha}(t),\quad i=1,2,$$

converges as $\xi\to\infty$ to $F_i(\lambda)$, $i=1,2$, in the metric of space $L_{\varrho}^{2}(\mathbb{R})$. Let g be another function in $L_{\alpha}^{2}(\mathbb{R})$. By a similar argument, $G_i(\lambda)$, $i=1,2$, be defined by g. It is obvious that

$$\int_{0}^{\infty}(f(t)-g(t))^2\,d_{\alpha}(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\{(F_i(\lambda)-G_i(\lambda))\,(F_j(\lambda)-G_j(\lambda))\}\,d\varrho_{ij}(\lambda).$$

Let

$$g(t)=\begin{cases} f(t), & t\in[-\xi,\xi],\\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{\left(F_i\left(\lambda\right)-K_{i\xi}\left(\lambda\right)\right)\left(F_j\left(\lambda\right)-K_{i\xi}\left(\lambda\right)\right)\right\}d\varrho_{ij}\left(\lambda\right)$$
$$=\int_{-\infty}^{-\xi}f^2\left(t\right)d_\alpha(t)+\int_{\xi}^{\infty}f^2\left(t\right)d_\alpha(t)\to 0,\quad \xi\to\infty,$$

which proves that (K_ξ) converges to F in $L^2_\varrho(\mathbb{R})$ as $\xi\to\infty$. □

Theorem 4.2. *Suppose that the real-valued functions $f(\cdot)$ and $g(\cdot)$ are in $L^2_\alpha(\mathbb{R})$, and $F_i(\lambda)$ and $G_i(\lambda)$, $i=1,2$, are their Fourier transforms. Then we have*

$$\int_{-\infty}^{\infty}f\left(t\right)g\left(t\right)d_\alpha(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i\left(\lambda\right)G_j\left(\lambda\right)d\varrho_{ij}\left(\lambda\right),$$

which is called the generalized Parseval equality.

Proof. It is clear that $F_i\mp G_i$, $i=1,2$, are transforms of $f\mp g$. Therefore, we have

$$\int_{-\infty}^{\infty}\left(f\left(t\right)+g\left(t\right)\right)^2d_\alpha(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(F_i\left(\lambda\right)+G_i\left(\lambda\right)\right)\left(F_j\left(\lambda\right)+G_j\left(\lambda\right)\right)d\varrho_{ij}\left(\lambda\right),$$
$$\int_{-\infty}^{\infty}\left(f\left(t\right)-g\left(t\right)\right)^2d_\alpha(t)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(F_i\left(\lambda\right)-G_i\left(\lambda\right)\right)\left(F_j\left(\lambda\right)-G_j\left(\lambda\right)\right)d\varrho_{ij}\left(\lambda\right).$$

Subtracting one of these equalities from the other one, we get the desired result. □

Theorem 4.3. *Let f be a real-valued function and $f\in L^2_\alpha(\mathbb{R})$. Then, the integrals*

$$\int_{-\infty}^{\infty}F_i\left(\lambda\right)\theta_j\left(t,\lambda\right)d\varrho_{ij}\left(\lambda\right),\quad i,j=1,2,$$

converge in $L^2_\alpha(\mathbb{R})$. Consequently, we have

$$f\left(t\right)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i\left(\lambda\right)\theta_j\left(t,\lambda\right)d\varrho_{ij}\left(\lambda\right),$$

which is called the spectral expansion formula.

Proof. Take any function $f_\xi\in L^2_\alpha(\mathbb{R})$ and any positive number ξ, and set

$$f_\xi\left(t\right)=\int_{-\xi}^{\xi}\sum_{i,j=1}^{2}F_i\left(\lambda\right)\theta_j\left(t,\lambda\right)d\varrho_{ij}\left(\lambda\right).$$

Let $g(\cdot)\in L^2_\alpha(\mathbb{R})$ be a real-valued function which equals zero outside the finite interval $[-\tau,\tau]$, where $\tau>0$. Thus, we obtain

$$\int_{-\tau}^{\tau}f_\xi\left(t\right)g\left(t\right)d_\alpha(t)$$
$$=\int_{-\tau}^{\tau}\left(\int_{-\xi}^{\xi}\sum_{i,j=1}^{2}F_i\left(\lambda\right)\theta_j\left(t,\lambda\right)d\varrho_{ij}\left(\lambda\right)\right)g\left(t\right)d_\alpha(t)$$

$$= \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda) \left\{ \int_{-\tau}^{-\tau} g(t)\, \theta_j(t,\lambda)\, d_\alpha(t) \right\} d\varrho_{ij}(\lambda)$$

$$= \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\varrho_{ij}(\lambda). \tag{4.15}$$

From Theorem 4.2, we get

$$\int_{-\infty}^{\infty} f(t)\, g(t)\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\varrho_{ij}(\lambda). \tag{4.16}$$

By (4.15) and, (4.16) we have

$$\int_{-\infty}^{\infty} (f(t) - f_\xi(t))\, g(t)\, d_\alpha(t) = \int_{|\lambda|>\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\alpha_{ij}(\lambda).$$

If we apply this equality to the function

$$g(t) = \begin{cases} f(t) - f_\xi(t), & t \in [-\xi, \xi], \\ 0, & \text{otherwise}, \end{cases}$$

then we get

$$\int_{-\infty}^{\infty} (f(t) - f_\xi(t))^2\, d_\alpha(t) \le \sum_{i,j=1}^{2} \int_{|\lambda|>\xi} F_i(\lambda)\, F_j(\lambda)\, d\varrho_{ij}(\lambda).$$

Letting $\xi \to \infty$ yields the expansion result. □

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[1]Department of Mathematics,
University of Süleyman Demirel, Isparta, Turkey
Email address: bilenderpasaoglu@sdu.edu.tr
ORCID iD: https://orcid.org/:0000-00002-9315-4652

[2]Department of Mathematics,
University of Burdur Mehmet Akif Ersoy, Burdur, Turkey
Email address: hustuna@gmail.com
ORCID iD: https://orcid.org/0000-0001-7240-8687

Email address: matyuksel@hotmail.com
ORCID iD: https://orcid.org/0000-0002-1633-8343

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ENERGY LANDSCAPES AND NON-ARCHIMEDEAN PSEUDO-DIFFERENTIAL OPERATORS AS TOOLS FOR STUDYING THE SPREADING OF INFECTIOUS DISEASES IN A SITUATION OF EXTREME SOCIAL ISOLATION

VICTOR A. AGUILAR-ARTEAGA[1], I. S. GUTIÉRREZ[2],
AND ANSELMO TORRESBLANCA-BADILLO[3]

Abstract. In this article, we introduce a new type of pseudo-differential equations naturally connected with non-archimedean pseudo-differential operators and whose symbols are new classes of negative definite functions in the p-adic context and in arbitrary dimension. These equations are proposed as a mathematical models to study the spreading of infectious diseases (say COVID-19) through a random walk on a complex energy landscape and taking into account social clusters in a situation of extreme social isolation.

1. Introduction

In the archimedean setting the nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)dy - u(x,t) \tag{1.1}$$

have been widely used to model diffusion processes. Here, $J : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, radial, continuous function with

$$\int_{\mathbb{R}^n} J(z)dz = 1.$$

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The model (1.1) can be interpreted as follows: if $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then

$$\int_{\mathbb{R}^n} J(y-x)u(y,t)dy = (J*u)(x,t)$$

is the rate at which individuals are arriving at position x from all other places. In the same way, $-u(x,t) = -\int_{\mathbb{R}^n} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1). For further details the reader may consult [12, 16], and the references therein. In [6, 7] Avetisov et al. developed a class of p-adic pseudo-differential equations in dimension one with the aim of studying the dynamics of a large class of complex systems such as macromolecules, glasses and proteins. In these models, the time-evolution of the system is controlled by a master equation of the form

$$\frac{\partial u\,(x,t)}{\partial t} = \int_{\mathbb{Q}_p} j\left(|x-y|_p\right)\left\{u\,(y,t) - u\,(x,t)\right\} dy, \quad t \geq 0,$$

where $j : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_+$ is the probability of transition from state y to the state x per unit time, and the function $u(x,t) : \mathbb{Q}_p \times \mathbb{R}_+ \to \mathbb{R}_+$ is a probability density distribution.

In the latest years, p-adic nonlocal evolution equations and variations of it have been studied extensively. For example, modeling of geological processes (such as petroleum reservoir dynamics, fluid flows in porous media such as rock); the dynamics of myoglobin (myoglobin is a monomeric protein that gives muscle its red color); the relaxation of spin glasses, etc., see e.g. [1, 2, 5, 11, 21, 22, 24–29, 37] and the references therein. In this models the dynamics of complex systems is described by a random walk on a complex energy landscape. An energy landscape (or simply a landscape) is a continuous function $\mathbb{U} : X \to \mathbb{R}$ that assigns to each physical state of a system its energy. The term complex landscape means that function $\mathbb{U}$ has many local minima. In this case the method of *interbasin kinetics* is applied, in this approach, the study of a random walk on a complex landscape is based on a description of the kinetics generated by transitions between groups of states (basins). This concept can be outlined as follows. A complex system is assumed to have a large number of metastable configurations which realize local minima on the potential energy surface. The local minima are clustered in hierarchically nested basins of minima, namely, each large basin consists of smaller basins, each of these consisting of even smaller ones, and so on. Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. The transition rate between basins depends on the energy barrier between these basins. By using these methods, the configuration space of the system is approximated by an ultrametric space (a rooted tree) and by a function on the tree which describes by stochastic motions the distribution of the activation energies. Procedures for constructing hierarchies of basins kinetics from any energy landscapes have been studied extensively, see e.g. [6–8, 22, 27, 31, 32, 37] and the references therein.

In [7] an Arrhenius type relation was used, that is,

$$j\,(x \mid y) \sim A(T) \exp\left\{-\frac{\mathbb{U}\,(x \mid y)}{kT}\right\},$$

where $\mathbb{U}\,(x \mid y)$ is the height of the activation barrier for the transition from the state y to state x, k is the Boltzmann constant and T is the temperature. This formula establishes a relation between the structure of the energy landscape $\mathbb{U}\,(x \mid y)$ and the transition function $j\,(x \mid y)$.

In this paper, we introduce new classes of p-adic pseudo-differential equations naturally connected with certain types of non-archimedean pseudo-differential operators whose symbols are associated with new classes of negative definite functions on the p-adic numbers. This type of pseudo-differential equations may be seen as a generalization of the equations studied in [6, 7, 30, 36–38] and the references therein.

We establish rigorously that such equations are ultradiffusion equations, i.e., we show that the fundamental solutions of the Cauchy problems naturally associated to these equations are transition density functions of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$, see Theorem 3.2, Theorem 4.2 and Theorem 5.1.

Given the non-archimedean topology of $\mathbb{Q}_p^n$ we have that two balls in $\mathbb{Q}_p^n$ have nonempty intersection if and only if one is contained in the other. Moreover, any ball can be represented as disjoint union of balls of smaller radius, each of the latter can be represented in the same way with even smaller radius and so on, see e.g. [4, 38]. The above implies that every ball in $\mathbb{Q}_p^n$ can be identified with a rooted tree. For this reason, a particular population group in a human society can be represented as a system of hierarchically coupled disjoint clusters. Any cluster is slit into disjoint sub-cluster, each of the latter is split into its own disjoint) sub-clusters and so on. Therefore, the ultrametric spaces (in particular the p-adic numbers) are proposed as a natural, necessary and essential structure to study the spreading of infectious diseases (say COVID-19) through a random walk on a complex energy landscape and taking into account social clusters in a situation of extreme social isolation. For more details, the reader can consult [3, 19, 20, 23, 34].

From a mathematical, physical and computational point of view, we consider that the spread of infectious diseases (such as the COVID-19 epidemic and new variants with very high rates of contagion) on social clusters in a situation of extreme social isolation, can be modeled as a random walk in an complex energy landscape. Therefore, an interesting open problem consists in determine if our p-adic pseudo-differential equations (in combination with the method of interbasin kinetics) can be applied, among other things, to study the dynamics of the spread infectious diseases through in a random walk in a complex energy landscape.

It should also be noted that, recently in [36] and [37], the authors Torresblanca-Badillo and Zúñiga-Galindo introduce a large class of non-archimedean pseudo-differential operators whose symbols are negative definite functions. Since then, in the last four years, the first author and his collaborators have been studied new classes of non-archimedean pseudo-differential operators whose symbols are associated with negative definite functions on the p-adic numbers, see [9, 13–15, 34, 35].

This article is organized as follows. In Section 2, we will collect some basic results on the p-adic analysis and fix the notation that we will use through the article. In Section 3, we introduce a large class of negative definite functions of the semi-smooth and elliptic types, see Theorem 3.1 and Corollary 3.1, respectively. These functions are the symbols of a large class of non-archimedean pseudo-differential operators (denoted by $\mathcal{A}$) which determine certain ultradiffusion equations on $\mathbb{Q}_p^n$, see Theorem 3.2. In Section 4 we also introduced new classes of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions (in the p-adic context) associated with logarithmic functions, see Theorem 4.1 and Corollary 4.1. This operators determine certain Lévy process $\mathfrak{X}(t, \omega)$ with state space $\mathbb{Q}_p^n$, see Theorem 4.2. In Section 5 we will study a new class of non-archimedean operators (denoted by $\mathcal{A}_{\boldsymbol{\psi}}$) associated with a non-archimedean negative definite function $\boldsymbol{\psi}$. Imposing certain conditions to the function $\boldsymbol{\psi}$ we obtain that $\mathcal{A}_{\boldsymbol{\psi}}$ is a pseudo-differential operator which also determine ultradiffusion equations, see Theorem 5.1.

2. Fourier Analysis on $\mathbb{Q}_p^n$: Essential Ideas

2.1. **The field of p-adic numbers.** Along this article p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if } x = 0, \\ p^{-\gamma}, & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of x.*

Any p-adic number $x \neq 0$ has a unique expansion of the form $x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if } x = 0 \text{ or } ord(x) \geq 0, \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j, & \text{if } ord(x) < 0. \end{cases}$$

We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n \mid ||x - a||_p \leq p^r\}$ *the ball of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$ and take $B_r^n(0) =: B_r^n$.

Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p \mid |x_i - a_i|_p \le p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of p-adic integers of* $\mathbb{Q}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n \mid ||x - a||_p = p^r\}$ *the sphere of radius* p^r *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) =: S_r^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. The group of invertible elements in $\mathbb{Z}_p$ constitutes the set $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid |x|_p = 1\}$. As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [38, Section 1.3], or [4, Section 1.8]. The balls and spheres are compact subsets. Thus, $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

We will use $\Omega\left(p^{-r}||x - a||_p\right)$ to denote the characteristic function of the ball $B_r^n(a)$. We will use the notation 1_A for the characteristic function of a set A. Along the article $d^n x$ will denote a Haar measure on $\mathbb{Q}_p^n$ normalized such that $\int_{\mathbb{Z}_p^n} d^n x = 1$.

2.2. **Some function spaces.** A complex-valued function φ defined on $\mathbb{Q}_p^n$ is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x), \quad \text{for } x' \in B_{l(x)}^n.$$

Denote by $\mathcal{E}(\mathbb{Q}_p^n)$ the linear space of locally constant $\mathbb{C}$-value functions on $\mathbb{Q}_p^n$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function* (*or a test function*) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n) =: \mathcal{D}$. Let $\mathcal{D}'(\mathbb{Q}_p^n) =: \mathcal{D}'$ denote the set of all continuous functional (distributions) on $\mathcal{D}$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^n) \times \mathcal{D}(\mathbb{Q}_p^n) \to \mathbb{C}$ is denoted as $\langle T, \varphi \rangle$ for $T \in \mathcal{D}'(\mathbb{Q}_p^n)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [4, Section 4.4].

Denote by $L^1_{loc}(\mathbb{Q}_p^n) := L^1_{loc}$ the set of functions $f : \mathbb{Q}_p^n \to \mathbb{C}$ such that $f \in L^1(K)$ for any compact $K \subset \mathbb{Q}_p^n$. Every $f \in L^1_{loc}$ defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ by the formula

$$\langle f, \varphi \rangle = \int_{\mathbb{Q}_p^n} f(x)\, \varphi(x)\, d^n x.$$

Such distributions are called *regular distributions.*

Given $\rho \in [0, \infty)$, we denote by $L^\rho\left(\mathbb{Q}_p^n, d^n x\right) = L^\rho\left(\mathbb{Q}_p^n\right) := L^\rho$, the $\mathbb{C}-$vector space of all the complex valued functions g satisfying $\int_{\mathbb{Q}_p^n} |g(x)|^\rho\, d^n x < \infty$, $L^\infty := L^\infty\left(\mathbb{Q}_p^n\right) = L^\infty\left(\mathbb{Q}_p^n, d^n x\right)$ denotes the $\mathbb{C}-$vector space of all the complex valued functions g such that the essential supremum of $|g|$ is bounded.

Let denote by $C(\mathbb{Q}_p^n, \mathbb{C}) =: C_{\mathbb{C}}$ the $\mathbb{C}-$vector space of all the complex valued functions which are continuous, by $C(\mathbb{Q}_p^n, \mathbb{R}) =: C_{\mathbb{R}}$ the $\mathbb{R}-$vector space of continuous functions. Set

$$C_0(\mathbb{Q}_p^n, \mathbb{C}) := C_0(\mathbb{Q}_p^n) = \left\{ f : \mathbb{Q}_p^n \to \mathbb{C} \mid \ f \text{ is continuous and } \lim_{||x||_p \to \infty} f(x) = 0 \right\},$$

where $\lim_{||x||_p\to\infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in \mathbb{Q}_p^n \backslash B(\epsilon)$. We recall that $(C_0(\mathbb{Q}_p^n, \mathbb{C}), ||\cdot||_{L^\infty})$ is a Banach space.

2.3. **Fourier transform.** Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \mapsto \chi_p(\xi x)$, see e.g. [4, Section 2.3].

Given $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, we set $x \cdot \xi := \sum_{j=1}^n x_j \xi_j$. If $f \in L^1(\mathbb{Q}_p^n)$, its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \mathcal{F}_{x\to\xi}(f) = \widehat{f}(\xi) := \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) f(x) d^n x, \quad \text{for } \xi \in \mathbb{Q}_p^n.$$

The inverse Fourier transform of a function $f \in L^1(\mathbb{Q}_p^n)$ is

$$(\mathcal{F}^{-1}f)(x) = \mathcal{F}^{-1}_{\xi\to x}(f) = \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) f(\xi) d^n \xi, \quad \text{for } x \in \mathbb{Q}_p^n.$$

The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}f))(\xi) = f(-\xi),$$

for every $f \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [4, Section 4.8].

The set $L^2(\mathbb{Q}_p^n)$ is the Hilbert space with the scalar product

$$(f, g) = \int_{\mathbb{Q}_p^n} f(x)\overline{g}(x) d^n x, \quad f, g \in L^2(\mathbb{Q}_p^n),$$

so that $||f||_{L^2} = \sqrt{(f, f)}$.

If $f \in L^2(\mathbb{Q}_p^n)$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||\le p^k} \chi_p(\xi \cdot x) f(x) d^n x, \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in $L^2(\mathbb{Q}_p^n)$. We recall that the Fourier transform is unitary on $L^2(\mathbb{Q}_p^n)$, i.e., $||f||_{L^2} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2(\mathbb{Q}_p^n)$ and that (2.3) is also valid in $L^2(\mathbb{Q}_p^n)$, see e.g. [33, Chapter III, Section 2].

3. Non-Archimedean Pseudo-Differential Operators with Semi-Smooth and Elliptic Symbols

In this section we introduce a large class of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions on p-adic numbers. Moreover, we introduce a new class of non-archimedean ultradiffusion equations. From now on denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of (positive) natural numbers and by $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \ge 0\}$ the set of non-negative real numbers.

Definition 3.1. A function $\psi : \mathbb{Q}_p^n \to \mathbb{C}$ is called negative definite, if

$$\sum_{i,j=1}^{m} \left(\psi(\xi_i) + \overline{\psi(\xi_j)} - \psi(\xi_i - \xi_j)\right) \lambda_i \overline{\lambda_j} \geq 0,$$

for all $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_m \in \mathbb{Q}_p^n$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Remark 3.1. We denote by $\mathcal{N}(\mathbb{Q}_p^n)$ the set of negative definite functions on $\mathbb{Q}_p^n$ and by $\mathcal{CN}(\mathbb{Q}_p^n)$ the set of continuous negative definite functions on $\mathbb{Q}_p^n$. The following assertions hold:

(i) $\mathcal{N}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$;
(ii) the non-negative constant functions belong to $\mathcal{N}(\mathbb{Q}_p^n)$;
(iii) $\mathcal{CN}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of compact convergence on $\mathbb{Q}_p^n$;
(iv) if $\psi : \mathbb{Q}_p^n \to \mathbb{R}$ is negative definite function, then $\psi(-x) = \psi(x)$ and $\psi(x) \geq \psi(0) \geq 0$ for all $x \in \mathbb{Q}_p^n$.

For the basic results on negative definite functions the reader may consult [10].

Remark 3.2. (i) It is relevant to mention that for any locally bounded negative definite function $\psi \in \mathcal{N}(\mathbb{R}^n)$ there exists a constant $C_\psi > 0$ such that $|\psi(\xi)|_{\mathbb{R}^n} \leq C_\psi(1+|\xi|_{\mathbb{R}^n}^2)$, for all $\xi \in \mathbb{R}^n$, see e.g. [17, Lemma 3.6.22]. However, in the p-adic context this is not always the case, see e.g. [36].

(ii) Another aspect to be highlighted is the fact that the function $y \mapsto ||y||^\alpha$ is continuous and negative definite on $\mathbb{R}^n$ for all $\alpha \in (0, 2]$, see [10, 10.5, page 74]. However, in the p-adic context for all fixed $\alpha > 0$ and $\beta > 0$, the function $y \mapsto \alpha||y||_p^\beta$ is continuous and negative definite on $\mathbb{Q}_p^n$, see [36, Example 3.5].

Definition 3.2. We say that a function $\mathfrak{a} : \mathbb{Q}_p^n \to \mathbb{R}_+$ is a semi-smooth symbol, if it satisfies the following properties.

(i) $\mathfrak{a}$ is a continuous function.
(ii) $\mathfrak{a}$ is a increasing function with respect to $||\cdot||_p$.
(iii) $\mathfrak{a}$ is a *radial function* on $\mathbb{Q}_p^n$, i.e. $\mathfrak{a}(x) = g(||x||_p)$ for some $g : \mathbb{R}_+ \to \mathbb{R}_+$. To make the radial dependence clear we use the notation $\mathfrak{a}(x) = \mathfrak{a}(||x||_p)$ for all $x \in \mathbb{Q}_p^n$.
(iv) $\mathfrak{a}(||x||_p) = 0 \Leftrightarrow x = 0$.
(v) There exist positive constants $C_0 := C_0(\mathfrak{a})$ and $d := d(\mathfrak{a})$ such that

$$C_0||x||_p^d \leq \mathfrak{a}(||x||_p), \quad \text{for every } x \in \mathbb{Q}_p^n.$$

Example 3.1. (i) The simplest example of semi-smooth symbols is the elliptic polynomial of degree d (in particular the p-adic norm $||\cdot||_p$). For more details the reader may consult [27].

(ii) Taking $C_0 = d = 1$ in the above definition, we have that the function $\mathfrak{a}(x) := e^{||x||_p} - 1$, $x \in \mathbb{Q}_p^n$, is a semi-smooth symbol.

Remark 3.3. (i) For $t > 0$ note that

$$\begin{aligned}\int_{\mathbb{Q}_p^n} e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi &= \sum_{j=0}^{\infty} e^{-t\mathfrak{a}(p^{-j})} \int_{||\xi||_p = p^{-j}} d^n\xi + \sum_{j=1}^{\infty} e^{-t\mathfrak{a}(p^j)} \int_{||\xi||_p = p^j} d^n\xi \\ &= (1 - p^{-n}) \left(\sum_{j=0}^{\infty} e^{-t\mathfrak{a}(p^{-j})} p^{-nj} + \sum_{j=1}^{\infty} e^{-t\mathfrak{a}(p^j)} p^{nj} \right) \\ &\le (1 - p^{-n}) \left(\sum_{j=0}^{\infty} p^{-nj} + \sum_{j=1}^{\infty} e^{-tC_0 p^{jd}} p^{nj} \right) < \infty,\end{aligned}$$

i.e., $e^{-t\mathfrak{a}} \in L^1(\mathbb{Q}_p^n)$.

Consider the operator non-archimedean pseudo-differential operator $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A}(\varphi)(x) :&= \mathcal{F}^{-1}_{\xi \to x} \left(\mathfrak{a}(||\xi||_p) \widehat{\varphi}(\xi) \right) \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \mathfrak{a}(||\xi||_p) \widehat{\varphi}(\xi) d^n\xi,\end{aligned}$$

where $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and $\mathfrak{a}$ is a semi-smooth symbol, and the Cauchy problem (or p-adic heat equation)

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \mathcal{A}u(x,t), \quad t \in [0,\infty),\, x \in \mathbb{Q}_p^n, \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n). \end{cases} \tag{3.1}$$

Then, the fundamental solution (or heat Kernel) of the Cauchy problem (3.1) is defined as

$$Z_t(x) = Z(x,t) := \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi)\, e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi, \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t > 0. \tag{3.2}$$

Therefore, by [33, (1.6), page 118] we have that $Z_t(x) \in C_0(\mathbb{Q}_p^n)$.

(ii) For all $t > 0$ we have that $Z_t(\cdot) \in L^1_{loc}$, i.e., $Z_t(\cdot)$ is a regular distribution on $\mathbb{Q}_p^n$. Therefore, for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and [4, Section 4.9] we have that

$$\langle \mathcal{F}(Z(x,t)), \varphi \rangle = \langle Z(x,t), \widehat{\varphi} \rangle = \left\langle e^{-t\mathfrak{a}(||x||_p)}, \varphi \right\rangle.$$

The above implies that $\mathcal{F}(Z(x,t)) = e^{-t\mathfrak{a}(||x||_p)}$.

Lemma 3.1. *The fundamental solution $Z_t(x)$ has the following properties:*

(i) $Z_t(x) \ge 0$ *for any* $t > 0$;
(ii) $\int_{\mathbb{Q}_p^n} Z_t(x) d^n x = 1$ *for any* $t > 0$;
(iii) $Z_{t+s}(x) = \int_{\mathbb{Q}_p^n} Z_t(x-y) Z_s(y) d^n y$ *for all* $t, s > 0$;
(iv) $Z_t(x) \le t||x||_p^{-n}$ *for all* $t > 0$ *and* $x \in \mathbb{Q}_p^n \setminus \{0\}$.

Proof. (i) If $x = 0$, the assertion is clear. Then, for $x \in \mathbb{Q}_p^n \setminus \{0\}$ with $||x||_p = p^{-\gamma}$, $\gamma \in \mathbb{Z}$, $t > 0$ and making the change of variable $w = p^j \xi$, we have that

$$Z_t(x) = \sum_{-\infty < j < \infty} e^{-t\mathfrak{a}(p^j)} \int_{||\xi||_p = p^j} \chi_p(-x \cdot \xi)\, d^n\xi$$

$$
\begin{aligned}
&= \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(p^j)} \int_{||p^j\xi||_p=1} \chi_p\left(-x\cdot\xi\right) d^n\xi \\
&= \sum_{-\infty<j<\infty} p^{nj} e^{-t\mathfrak{a}(p^j)} \int_{||w||_p=1} \chi_p\left(-p^{-j}x\cdot w\right) d^n w.
\end{aligned}
$$

By using the formula

$$
\int_{||w||_p=1} \chi_p\left(-p^{-j}x\cdot w\right) d^n w = \begin{cases} 1-p^{-n}, & \text{if } j\le\gamma, \\ -p^{-n}, & \text{if } j=\gamma+1, \\ 0, & \text{if } j\ge\gamma+2, \end{cases}
$$

we have that

$$
\begin{aligned}
Z_t(x) &= (1-p^{-n}) \sum_{j=-\gamma}^{\infty} p^{-nj} e^{-t\mathfrak{a}(p^{-j})} - p^{n\gamma} e^{-t\mathfrak{a}(p^{\gamma+1})} \\
&\ge e^{-t\mathfrak{a}(p^{\gamma})} \sum_{j=-\gamma}^{\infty} (1-p^{-n}) p^{-nj} - p^{n\gamma} e^{-t\mathfrak{a}(p^{\gamma+1})} \\
&= p^{n\gamma}\left(e^{-t\mathfrak{a}(p^{\gamma})} - e^{-t\mathfrak{a}(p^{\gamma+1})}\right) \\
&\ge 0.
\end{aligned}
$$

(ii) As a direct consequence of Remark 3.3 (ii) we have that $\mathcal{F}(Z(0,t))=1$. On the other hand, $\mathcal{F}(Z(x,t))=\int_{\mathbb{Q}_p^n}\chi_p(\xi\cdot x)Z(x,t)d^nx$ and $\mathcal{F}(Z(0,t))=\int_{\mathbb{Q}_p^n}Z(x,t)d^nx$. Therefore, $\int_{\mathbb{Q}_p^n}Z(x,t)d^nx=1$ for all $t>0$.

(iii) For $t,s>0$, we have by (3.2) that

$$
\begin{aligned}
Z_{t+s}(x) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-x\cdot\xi\right) e^{-t\mathfrak{a}(||\xi||_p)} e^{-s\mathfrak{a}(||\xi||_p)} d^n\xi \\
&= Z_t(x) * Z_s(x) \\
&= \int_{\mathbb{Q}_p^n} Z_t(x-y) Z_s(y) d^n y.
\end{aligned}
$$

(iv) For $t>0$, $x=p^{\gamma}x_0\neq 0$ such that $\gamma\in\mathbb{Z}$ and $||x_0||_p=1$, and making the change of variable $z=p^{\gamma}\xi$, we have that

$$
\begin{aligned}
Z(x,t) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-p^{\gamma}\xi\cdot x_0\right) e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi \\
&= ||x||_p^{-n} \int_{\mathbb{Q}_p^n} \chi_p\left(-x_0\cdot z\right) e^{-t\mathfrak{a}(p^{\gamma}||z||_p)} d^n z \\
&= ||x||_p^{-n} \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(||x||_p^{-1}p^j)} \int_{||p^j z||_p=1} \chi_p(-x_0\cdot z) d^n z \\
&= ||x||_p^{-n} \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(||x||_p^{-1}p^j)} p^{nj} \int_{||z||_p=1} \chi_p(-x_0 p^{-j}\cdot z) d^n z.
\end{aligned}
$$

By using the formula

$$\int_{||z||_p=1} \chi_p\left(-x_0 p^{-j}\cdot z\right) d^n z = \begin{cases} 1-p^{-n}, & \text{if } j\le 0,\\ -p^{-n}, & \text{if } j=1,\\ 0, & \text{if } j\ge 2,\end{cases}$$

we get

$$\begin{aligned} Z(x,t) &= ||x||_p^{-n}\left\{(1-p^{-n})\sum_{j=0}^{\infty} p^{-nj} e^{-t\mathfrak{a}(||x||_p^{-1}p^{-j})} - e^{-t\mathfrak{a}(||x||_p^{-1}p)}\right\}\\ &\le ||x||_p^{-n}\left\{1-e^{-t\mathfrak{a}(||x||_p^{-1}p)}\right\}. \end{aligned}$$

By applying the Mean value theorem to the real function $g(v)=e^{-v\mathfrak{a}(||x||_p^{-1}p)}$ on $[0,t]$, $t>0$, we have that

$$1-e^{-t\mathfrak{a}(||x||_p^{-1}p)} = te^{-\tau\mathfrak{a}(||x||_p^{-1}p)},$$

for some $\tau\in(0,t)$. So that,

$$Z(x,t)\le t||x||_p^{-n}.$$

□

Theorem 3.1. *If $\mathfrak{a}$ is a semi-smooth symbol, then $\mathfrak{a}$ is a negative definite function.*

Proof. Due to Lemma 3.1 the proof of this theorem is completely similar to the proof given in [14, Theorem 3]. □

The converse of the previous theorem generally does not hold. For example, the non-negative constant functions are negative definite functions but they are not semi-smooth symbol.

Definition 3.3. A function $f:\mathbb{Q}_p^n\to\mathbb{R}_+$ is called an elliptic symbol, if it satisfies the following properties:

(i) f is a continuous and radial function on $\mathbb{Q}_p^n$;
(ii) $f(||x||_p)=0\Leftrightarrow x=0$;
(iii) f is a increasing functions with respect to $||\cdot||_p$ and there exist positive constants $C_0:=C_0(f)$, $C_1:=C_1(f)$ and $d:=d(f)$ such that

$$C_0||x||_p^d\le f(||x||_p)\le C_1||x||_p^d,$$

for every $x\in\mathbb{Q}_p^n$.

Example 3.2. (i) For any $d>0$ and $\beta>0$, the function $f(x)=\beta||x||_p^d$, $x\in\mathbb{Q}_p^n$, is an elliptic symbol.

(ii) Let $h(x)\in\mathbb{Z}_p[x_1,\dots,x_n]$ with $h(0)=0$ be a non constant homogeneous polynomial of degree d with coefficients in $\mathbb{Z}_p^\times$ such that $h(x)$ is strongly elliptic modulo p, see [30, Definition 3]. Defining $f(x)=|h(x)|_p$ with $x\in\mathbb{Q}_p^n$, by [30, Lemma 15] we have that f is a elliptic symbol.

(iii) [27] For any $n \in \mathbb{N}$ and $p \neq 2$, there exists an elliptic polynomial $h(\xi_1, \dots, \xi_n)$ with coefficients in $\mathbb{Z}_p^\times$ and degree $2d(n) := 2d$ such that

$$|h(\xi_1, \dots, \xi_n)|_p = ||(\xi_1, \dots, \xi_n)||_p^{2d}.$$

Therefore, proceeding analogously to the previous case, we can obtain infinitely many elliptic symbols.

Since every elliptic symbol is a semi-smooth symbol, then as a direct consequence of Theorem 3.1 we obtain the following result.

Corollary 3.1. *If f is an elliptic symbol, then f is a negative definite function.*

Next we will show that the heat Kernel Z_t associated with the non-archimedean pseudo-differential operator $\mathcal{A}$ determine a transition function of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$.
Let $\mathcal{B}(\mathbb{Q}_p^n)$ denote the σ-algebra of the Borel sets of $(\mathbb{Q}_p^n)$. For the basic results on positive bounded measure and Markov processes the reader may consult, respectively, [10] and [18].

Definition 3.4. A function $p_t(x, E)$, defined for all $t \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$, is called a Markov transition function on $\mathbb{Q}_p^n$ if it satisfies the following four conditions:

(i) $p_t(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$ and $p_t(x, \mathbb{Q}_p^n) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{Q}_p^n$;
(ii) $p_t(\cdot, E)$ is a Borel measurable function for all $t \geq 0$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$;
(iii) $p_0(x, \{x\}) = 1$ for all $x \in \mathbb{Q}_p^n$;
(iv) **(The Chapman-Kolmogorov equation)** for all $t, s \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we have the equations

$$p_{t+s}(x, E) = \int_{\mathbb{Q}_p^n} p_t(x, d^n y) p_s(y, E).$$

Definition 3.5. For $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we define

$$p_t(x, E) = \begin{cases} Z_t(x) * 1_E(x), & \text{for } t > 0, x \in \mathbb{Q}_p^n, \\ 1_E(x), & \text{for } t = 0, x \in \mathbb{Q}_p^n, \end{cases}$$

where $Z_t(x)$ is the fundamental solution defined in (3.2).

Theorem 3.2. *$p_t(x, \cdot)$ is a transition function of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$ whose paths are right continuous and have no discontinuities other than jumps.*

Proof. The result follows from Lemma 3.1 by using the argument given in the proof of [14, Theorem 2]. □

4. Non-Archimedean Pseudo-Differential Operators with Negative Definite Logarithmic Symbols

In this section we introduce a large class of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions (in the p-adic context) associated with logarithmic functions. Moreover, we introduce a new class of non-archimedean ultradiffusion equations.

Definition 4.1. (i) A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called positive definite, if

$$\sum\nolimits_{i,j=1}^{m} \varphi(x_i - x_j)\lambda_i\overline{\lambda_j} \geq 0,$$

for all $m \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_m \in \mathbb{Q}_p^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

(ii) A C^∞-function $f : (0, \infty) \to \mathbb{R}$ is said to be a Bernstein function, if

$$f \geq 0 \quad \text{and} \quad (-1)^m D^m f \leq 0, \quad \text{for all integers } m \geq 1.$$

The set of positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{P}(\mathbb{Q}_p^n)$ and the subset of $\mathcal{P}(\mathbb{Q}_p^n)$ consisting of the continuous positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{CP}(\mathbb{Q}_p^n)$. For a more detailed discussion of positive definite functions and its properties the reader may consult [10].

Remark 4.1. The following assertions hold:

(i) $\mathcal{P}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$;

(ii) if φ_1, $\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$, then $\varphi_1\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$; the non-negative constant functions belong to $\mathcal{P}(\mathbb{Q}_p^n)$;

(iii) $\mathcal{CP}(\mathbb{Q}_p^n)$ is a convex cone which is a closed subset of the set of continuous complex-valued functions in the topology of compact convergence.

Theorem 4.1. *Let $\psi : \mathbb{Q}_p^n \to [1, \infty)$ be a continuous negative definite function. Then, the function $\ln(\psi) : \mathbb{Q}_p^n \to \mathbb{R}_+$ is negative definite.*

Proof. First, for fixed $t \in [0, 1]$ we consider the function $f_t : (0, \infty) \to \mathbb{R}_+$ given by $f_t(x) = x^t$. Clearly, $f \geq 0$ and by a direct calculation one verifies that

$$(-1)^m D^m(f) = (-1)^m \prod_{i=0}^{m-1} (t - i)x^{t-m} \leq 0,$$

i.e., f_t is a Bernstein function. Then, for fixed $t \in [0, 1]$ and by [10, 9.20, page 69] we have that

$$(f_t \circ \psi)(\xi) = \psi^t(\xi) : \mathbb{Q}_p^n \to [1, \infty) \tag{4.1}$$

is a continuous negative definite function. Moreover, by [10, Corollary 7.9] we have that $\frac{1}{\psi^t}$ is a positive definite function for fixed $t \in [0, 1]$.

On the other hand, for fixed $t_1, t_2 \in [0, 1]$ we have that the product $\frac{1}{\psi^{t_1}} \cdot \frac{1}{\psi^{t_2}}$ is a continuous positive definite function on $\mathbb{Q}_p^n$, see [10, Proposition 3.6]. Therefore,

$\frac{1}{\psi^t} = e^{-t\ln(\psi)}$ is a continuous positive definite function on $\mathbb{Q}_p^n$ for all $t > 0$, and by [10, Theorem 7.8] we have that the function $\ln(\psi)$ is negative definite. □

As an immediate consequence of the theorem above and Remark 3.1, the following corollary is obtained.

Corollary 4.1. *Let $\psi_j : \mathbb{Q}_p^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, be radial, continuous, negative definite functions such that at least one function ψ_j satisfies $\psi_j : \mathbb{Q}_p^n \to [1, \infty)$. Then the function $\ln\left(\sum_{j=1}^m \psi_j(||\xi||_p)\right)$ is negative definite.*

Example 4.1. For every fixed $k > 1$, $\alpha, \beta > 0$, the function $\psi : \mathbb{Q}_p^n \to \mathbb{R}_+$ given by $\psi(\xi) = \ln\left(k + \alpha||\xi||_p^\beta\right)$ is negative definite. By Remark 3.2-(*ii*), we have that in the real context the function $\psi' : \mathbb{R}^n \to \mathbb{R}_+$ given by $\psi'(\xi) = \ln\left(k + ||\xi||_{\mathbb{R}^n}^\beta\right)$ is not a negative definite function for $\beta > 2$.

Corollary 4.2. *Let $\psi : \mathbb{Q}_p^n \to [1, \infty)$ be a continuous negative definite function. Then the function $\ln^\alpha(\psi) : \mathbb{Q}_p^n \to [1, \infty)$, $\alpha > 1$, is negative definite.*

Proof. The result follows from Theorem 4.1, (4.1) and Remark 4.1 (ii). □

Example 4.2. By Remark 3.1 and Remark 3.2 we have that the function $f(x) = 1 + ||x||_p$, $x \in \mathbb{Q}_p^n$, is negative definite and $f(x) \geq 1$ for all $x \in \mathbb{Q}_p^n$. Then, by above corollary we have that $\ln^\alpha(1 + ||x||_p) : \mathbb{Q}_p^n \to [1, \infty)$, $\alpha > 1$, is a negative definite function. Moreover, by [10, Corollary 7.9] we have that $\frac{1}{\ln^\alpha(1+||x||_p)}$, $\alpha > 1$, is a positive definite function.

Consider the operator non-archimedean pseudo-differential operator $\widetilde{\mathcal{A}}$ given by

$$\begin{aligned}\widetilde{\mathcal{A}}(\varphi)(x) &:= \mathcal{F}^{-1}_{\xi\to x}\left\{\ln(\psi(\xi))\widehat{\varphi}(\xi)\right\} \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi)\ln(\psi(\xi))\widehat{\varphi}(\xi)d^n\xi, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n),\end{aligned}$$

and the Cauchy problem (or p-adic heat equation)

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \widetilde{\mathcal{A}}u(x,t), \quad t \in [0,\infty),\, x \in \mathbb{Q}_p^n, \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n), \end{cases} \tag{4.2}$$

where ψ is a continuous negative definite function satisfies hypothesis of Theorem 4.1. Then the fundamental solution (or heat Kernel) of the Cauchy problem (4.2) is defined as

$$\widetilde{Z}_t(x) = \widetilde{Z}(x,t) := \int_{\mathbb{Q}_p^n} \chi_p\left(-x\cdot\xi\right)e^{-t\ln(\psi(\xi))}d^n\xi = \int_{\mathbb{Q}_p^n} \chi_p\left(-x\cdot\xi\right)\frac{1}{\psi^t(\xi)}d^n\xi,$$

for $x \in \mathbb{Q}_p^n$ and $t > 0$.

Lemma 4.1. *The family $\left(\widetilde{Z}_t\right)_{t>0}$ determine a convolution semigroup on $\mathbb{Q}_p^n$, i.e., $\left(\widetilde{Z}_t\right)_{t>0}$ satisfies the following properties:*

(i) *for all* $t > 0$, $\widetilde{Z}_t$ *is a positive bounded measure on* $\mathbb{Q}_p^n$*;*
(ii) *for all* $t > 0$, $\widetilde{Z}_t(\mathbb{Q}_p^n) \leq 1$*;*
(iii) *for all* $t, s > 0$, *we have that* $\widetilde{Z}_t * \widetilde{Z}_s = \widetilde{Z}_{t+s}$*;*
(iv) $\lim_{t\to 0} \widetilde{Z}_t = \delta$, *where* δ *is the Dirac delta function.*

Proof. Following the proof of Theorem 4.1, we have that $\frac{1}{\psi^t}$, $t > 0$, is a continuous positive definite function on $\mathbb{Q}_p^n$. Moreover, by [10, Theorem 3.12] we have that $\widetilde{Z}_t$, $t > 0$, is a positive bounded measure on $\mathbb{Q}_p^n$. The desired result follows by application of [10, Theorem 8.3]. □

Theorem 4.2. *There exists a Lévy process* $\mathfrak{X}(t, \omega)$ *with state space* $\mathbb{Q}_p^n$ *and transition function* $\widetilde{p}_t(x, \cdot)$ *given by*

$$\widetilde{p}_t(x, E) = \begin{cases} \widetilde{Z}_t(x) * 1_E(x), & \text{for } t > 0 x \in \mathbb{Q}_p^n, \\ 1_E(x), & \text{for } t = 0, x \in \mathbb{Q}_p^n, \end{cases}$$

for $E \in \mathcal{B}(\mathbb{Q}_p^n)$*,*

Proof. Due to Lemma 4.1 the proof of this Theorem is completely similar to the proof given in [37, Theorem 2]. □

5. Other Classes of Non-Archimedean Pseudo-Differential Operators Associated with Certain Types of Negative Definite Functions

In this section we will study a new class of non-archimedean operators (denoted by $\mathcal{A}_{\psi}$) associated with a non-archimedean negative definite function $\boldsymbol{\psi}$. Imposing certain conditions to the function $\boldsymbol{\psi}$ we obtain that $\mathcal{A}_{\psi}$ is a pseudo-differential operator which also determine ultradiffusion equations.

Along this section $\boldsymbol{\psi} : \mathbb{Q}_p^n \to \mathbb{R}_+ \setminus \{0\}$ will denote a radial, continuous and negative definite function such that there exist positive real constants C_2 and β, $\beta > n$, such that $\boldsymbol{\psi}(||x||_p) \geq C_2 ||x||_p^{\beta}$ for all $x \in \mathbb{Q}_p^n$.

By Remark 3.1 (iv) note that $\boldsymbol{\psi}(||x||_p) \geq \boldsymbol{\psi}(0) > 0$ for all $x \in \mathbb{Q}_p^n$. For examples of this type of functions, the reader can consult [36].

We now note that

$$\int_{\mathbb{Q}_p^n} \frac{d^n x}{\boldsymbol{\psi}(||x||_p)} = \int_{\mathbb{Z}_p^n} \frac{d^n x}{\boldsymbol{\psi}(||x||_p)} + \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n x}{\boldsymbol{\psi}(||x||_p)} = I_1 + I_2.$$

Now, since $\boldsymbol{\psi}$ is a continuous function on $\mathbb{Z}_p^n$ and given the normalization of the norm $|| \cdot ||_p$, we have that $I_1 < \infty$.

On the other hand, note that

$$I_2 \leq \frac{1}{C_2} \sum_{j=1}^{\infty} \frac{1}{p^{j\beta}} \int_{||x||_p = p^j} d^n x = \frac{1 - p^{-n}}{C_2} \sum_{j=1}^{\infty} p^{j(n-\beta)} < \infty.$$

Therefore, $\frac{1}{\boldsymbol{\psi}} \in L^1(\mathbb{Q}_p^n)$ and consequently there is a positive real constant C such that

$$C \int_{\mathbb{Q}_p^n} \frac{d^n x}{\boldsymbol{\psi}(||x||_p)} = 1. \tag{5.1}$$

We define the operator

$$\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) = C \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{\boldsymbol{\psi}(||y||_p)} d^n y, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

where C is the constant given by (5.1).

Lemma 5.1. *The application*

$$\mathcal{D}\left(\mathbb{Q}_p^n\right) \to \mathcal{D}\left(\mathbb{Q}_p^n\right),$$
$$\varphi \to \mathcal{A}_{\boldsymbol{\psi}}(\varphi),$$

is a well-defined non-archimedean pseudo-differential operator.

Proof. The condition (5.1) implies that

$$\begin{aligned}
\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) &= \left(\frac{C}{\boldsymbol{\psi}} * \varphi\right)(x) - \varphi(x) \\
&= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p) \widehat{\varphi}(\xi) d^n \xi - \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \widehat{\varphi}(\xi) d^n \xi \\
&= -\int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi) d^n \xi \\
&= -\mathcal{F}^{-1}_{\xi \to x}\left(\left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi)\right),
\end{aligned}$$

i.e., $\mathcal{A}_{\boldsymbol{\psi}}$ is a pseudo-differential operator with symbol $1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}$.

On the other hand, since $\frac{C}{\boldsymbol{\psi}}$ is a radial function, then by [34, Lemma 1] and the n-dimensional version of [38, Example 8, page 43] we have that

$$\left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi) \in \mathcal{D}\left(\mathbb{Q}_p^n\right).$$

Therefore, by [38, VII, Section 2] we have that $\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) \in \mathcal{D}\left(\mathbb{Q}_p^n\right)$. □

Due to the condition (5.1) we have that $\frac{C}{\boldsymbol{\psi}}$ codify the structure of the function J given in [9] and [37]. Therefore, by Lemma 5.1 and proceeding analogous to these references, we can prove the following theorem.

Theorem 5.1. *There exists a Lévy process $\mathfrak{X}(t,\omega)$ with state space $\mathbb{Q}_p^n$ and transition function $q_t(x,\cdot)$ given by*

$$q_t(x,E) = \begin{cases} Z_{\psi}(t,x) * 1_E(x), & \text{for } t>0, x\in\mathbb{Q}_p^n, \\ 1_E(x), & \text{for } t=0, x\in\mathbb{Q}_p^n, \end{cases}$$

where $Z_{\psi}(t,x)$ is the fundamental solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \mathcal{A}_{\psi}u(x,t), & t\in[0,\infty), x\in\mathbb{Q}_p^n, \\ u(x,0)=u_0(x)\in\mathcal{D}(\mathbb{Q}_p^n). \end{cases}$$

Remark 5.1. Note that the above Cauchy problem corresponds to the p-adic non local evolution equation

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p^n}\left(\frac{C}{\boldsymbol{\psi}}\right)(x-y)u(y,t)dy - u(x,t) = \left(\left(\frac{C}{\boldsymbol{\psi}}\right)*u-u\right)(x,t).$$

For further details the reader may consult [7] and [6].

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[1]Facultad de Ingeniería,
Universidad Autónoma de Querétaro,
Cerro de las Campanas S/N, C. P. 76010 Querétaro, Qro., México
Email address: victor.aguilara@uaq.mx

[2]Departamento de Matemáticas y Estadística,
Universidad del Norte,
Km. 5 Vía Puerto Colombia, Barranquilla, Colombia
Email address: isgutier@uninorte.edu.co

[3]Universidad del Norte, Departamento de Matemáticas y Estadística,
Km 5 Vía Pto. Colombia,
Barranquilla, Colombia
Email address: atorresblanca@uninorte.edu.co

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ON THE PROXIMAL POINT ALGORITHM OF HYBRID-TYPE IN FLAT HADAMARD SPACES WITH APPLICATIONS

LAWAL YUSUF HARUNA[1], GODWIN CHIDI UGWUNNADI[2,3], AND BASHIR ALI[4]

Abstract. In this paper, we introduce a hybrid-type proximal point algorithm for approximating zero of monotone operator in Hadamard-type spaces. We then prove that a sequence generated by the algorithm involving Mann-type iteration converges strongly to a zero of the said operator in the setting of flat Hadamard spaces. To the best of our knowledge, this result presents the first hybrid-type proximal point algorithm in the space. The result is applied to convex minimization and fixed point problems.

1. Introduction

Let (X, d) be a metric space, an isometry $c : [0, d(x, y)] \to X$ satisfying $c(0) = x$ and $c(d(x, y)) = y$ is called a geodesic path joining x to y for any $x, y \in X$. A geodesic segment between x and y is the image of a geodesic path joining x to y and is denoted by $[x, y]$ when it is unique. A geodesic space is a metric space (X, d) in which every two points of X are joined by a geodesic segment. It is said to be uniquely geodesic space if every two points of X are joined by only one geodesic segment. Let X be a uniquely geodesic space and $(1-t)x \oplus ty$ denote the unique point z of the geodesic segment joining x to y for each $x, y \in X$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$. Set $[x, y] := \{(1-t)x \oplus ty : t \in [0, 1]\}$, then a subset $C \subset X$ is said to be convex if $[x, y] \subset C$ for all $x, y \in C$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of points (the edges

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of Δ). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. A geodesic space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle Δ is said to satisfy the CAT(0) inequality if $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$ for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. A complete CAT(0) space is called a Hadamard space.

Definition 1.1. Let X be a Hadamard space and $g : X \to (-\infty, \infty)$ be a function with domain $\text{dom}(g) = \{x \in X : g(x) < +\infty\}$. Then g is said to be

(i) proper, if $\text{dom}(g) \neq \emptyset$;
(ii) convex, if $g(\alpha x \oplus (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$ for all $x, y \in X$ and $\alpha \in (0, 1)$;
(iii) lower semicontinuous at a point $x \in \text{dom}(g)$, if for each sequence $\{x_n\}$ in $\text{dom}(g)$ with $x_n \to x$ implies $g(x) \leq \liminf_{n\to\infty} g(x_n)$;
(iv) lower semicontinuous on $\text{dom}(g)$, if it is lower semicontinuouos at every point in $\text{dom}(g)$.

The concept of quasilinearisation was introduced by Berg and Nicolev [4] in a complete CAT(0) space. They denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. A quasilinearisation is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)),$$

for every $a, b, c, d \in X$. From the definition, it is easy to see that for all $a, b, c, d, e \in X$, $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$. The space X is said to satisfy Cauchy Schwartz inequality if for all $a, b, c, d \in X$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$. It is known from [4] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

Definition 1.2 ([16])**.** A Hadamard space is called flat if and only if for all $x, y, z \in X$ and $t \in [0, 1]$

$$d^2((1-t)x \oplus ty, z) = (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y).$$

It is worth mentioning that every Hilbert space is flat Hadamard space but the converse is not always true (see [16, Theorem 3.3]) for details. It is not hard to see that in a flat Hadamard space X, for each $x, y, z, w \in X$ and $t \in [0, 1]$

$$\langle \overrightarrow{xy}, \overrightarrow{x(tz \oplus (1-t)w)} \rangle = t\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle + (1-t)\langle \overrightarrow{xy}, \overrightarrow{xw} \rangle. \tag{1.1}$$

The necessary and sufficient conditions for nonemptyness of the subdifferential set (see Definition 1.3 below) with respect to convexity happened to be the basis for introducing the concept of flat Hadamard spaces. The authors [16] observed that various important results about subdifferentials which hold under topological vector spaces are not valid on Hadamard spaces in general. As such, they establish some basic properties of subdifferentials under the setting of flat Hadamard spaces.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and for $x \in X$ $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n)$, the asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. In a complete CAT(0) space, it is generally known that $A(\{x_n\})$ consists of exactly one point, see [6] for details. A sequence $\{x_n\}$ is said to be Δ-convergent to a point $x \in X$ if for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$. In this case x is called Δ-limit of $\{x_n\}$ and it is written as $\Delta - \lim_{n\to\infty} x_n = x$.

Kakavandi and Amini [9] introduced the concept of dual space in a complete CAT(0) space X as follow. Let $C(X, \mathbb{R})$ be the space of all continuous real-valued functions on X. Consider a map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax}\rangle, \quad t \in \mathbb{R}, a, b, x \in X.$$

The Cauchy-Schwartz inequality implies $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$, $t \in \mathbb{R}$, $a, b \in X$, where the Lipschitz semi-norm $L(\phi)$ of any function $\phi : X \to \mathbb{R}$ is given by $L(\phi) = \sup\left\{\frac{\phi(x)-\phi(y)}{d(x,y)} : x, y \in X, x \neq y\right\}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b),(s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)),$$

for $t, s \in \mathbb{R}$ and $a, b, c, d \in X$. In a complete CAT(0) space, it is shown [9] that $D((t, a, b),(s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$. Thus D induces an equivalence relation on $\mathbb{R} \times X \times X$ with equivalence class defined by

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : D((t, a, b),(s, c, d)) = 0\}.$$

The pair (X^*, D) is called the dual space of the metric space (X, d), where $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ and the function D on X^* is a metric.

Definition 1.3 ([9])**.** Let X^* be a dual of a Hadamard space X and $g : X \to (-\infty, +\infty]$ be a proper function with effective domain $\mathrm{dom}(f) := \{x \in X : g(x) < +\infty\}$. A subdifferential of g is a multi-valued mapping $\delta g : X \to 2^{X^*}$ defined by

$$\delta g(x) = \{x^* \in X^* : g(y) - g(x) \geq \langle x^*, \overrightarrow{xy}\rangle \text{ for all } y \in X\},$$

for $x \in \mathrm{dom}(g)$ and $\delta g(x) = \emptyset$, otherwise.

Let X^* be a dual of the Hadamard space X and $A : X \to 2^{X^*}$ be a multivalued operator. Let the domain and range of A be respectively denoted by $D(A) := \{x \in X : Ax \neq \emptyset\}$ and $R(A) := \cup_{x\in X} Ax$, $A^{-1}x^* := \{x \in X : x^* \in Ax\}$. The multivalued operator $A : X \to 2^{X^*}$ is said to be monotone if and only if, for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$, $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$. The monotone inclusion problem (MIP) is to find a point

$$x \in D(A) \quad \text{such that} \quad 0 \in Ax, \tag{1.2}$$

where 0 is the zero element of the dual space X^*. We say that A satisfies the range condition if for every $z \in X$ and $\alpha > 0$, there exists an element $x \in X$ such $[\alpha\overrightarrow{xz}] \in Ax$. In a Hilbert space H, it is known that if A is a maximal monotone operator, then $R(I + \lambda A) = H$ for $\lambda > 0$. If A is monotone, then there exists a nonexpansive single-valued mapping $J_\lambda^A : R(I+\lambda A) \to \text{dom}(A)$ defined by $J_\lambda^A = (I+\lambda A)^{-1}$, which is called the resolvent of A. A monotone operator A is said to satisfy the range condition in H if $\overline{\text{dom}}(A) \subset R(I + \lambda A)$ for all $\lambda > 0$, where $Dom(A)$ denotes the closure of the domain of A. We know that in H, a monotone operator A which satisfy the range condition, $A^{-1}0 = F(J_\lambda^A)$ and every maximal monotone operator in H has range condition. Also the subdifferential function δg satisfies the range condition, whenever g is a proper, lower semicontinuous and convex function on a Hadamard space. However, it is not yet known whether every maximal monotone operator in Hadamard spaces satisfy the range condition. This could be seen as one of the significant issue of MIP in Hadamard spaces. But every maximal monotone operator has the range condition in a flat Hadamard space see [11, 18].

The said problem (MIP) is one of the most important problems in nonlinear and convex analysis due to its application in optimization and other related mathematical problems such as variational inequality problems (VIPs), and convex feasibility feasibility problems. Let the solution set of problem (1.2) be denoted by $A^{-1}(0)$. It is known (see [19]) that the set $A^{-1}(0)$ is closed and convex. The proximal point algorithm (PPA) which was introduced by Martinet [15] and further studied by Rockafellar [20] in Hilbert spaces, is a well-known method for approximating solutions of the MIPs. The said algorithm generates a sequence $\{x_n\}$ iteratively by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_{\lambda_n}^A x_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

where $J_{\lambda_n}^A = (I + \lambda_n A)^{-1}$ is the resolvent of the monotone operator A and (λ_n) is a sequence of positive real numbers. It is a fact proved by Rockafellar [20] that the sequence generated by the PPA converges weakly to a zero of the monotone operator A provided $\lambda_n \geq \lambda > 0$, for each $n \geq 1$. To get strong convergence, Solodov and Svaiter [21] modified the proximal point algorithm with resolvent $R_{\lambda_n}A := (I+\lambda_n A)^{-1}$ of A and generate a sequence $\{x_n\}$ iteratively by

$$\begin{cases} x_0 \in H, \\ y_n = R_{\lambda_n}A(x_n), \\ C_n = \{z \in H : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \leq 0\}, \\ x_{n+1} = \mathrm{P}_{C_n \cap Q_n}(x_0), \quad n \geq 0. \end{cases}$$

In 2013, Bacak [3] proved Δ-convergence of the PPA in Hadamard spaces by considering the operator A to be a subdifferential of a convex, proper and lower semicontinuous. Khatibzadeh and Ranjbar [11] studied PPA in Hadamard spaces when the operator A is monotone.

Remark 1.1. The well definedness of the PPA (1.3) (as a well known and most important method for solving the MIP) requires among others, the monotone operator A to satisfy the range condition and it is not known yet, whether every maximal monotone operators satisfy the range condition in Hadamard spaces as in the case of Hilbert spaces and Hadamard manifolds.

Ranjbar and Khatibzadeh [19] proved that the sequence $\{x_n\}$ defined by the following Mann-type PPA Δ-converges to zero of the monotone operator (see [19] for Halpern-type PPA that converges strongly to zero of the monotone operator A).

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \quad n \geq 0. \end{cases} \tag{1.4}$$

For more recent and related PPA results, the reader may consult [10, 12, 14, 22].

To the best of our knowledge, it appears in the literature that the only PPA that guaranteed the strong convergence in Hadamard spaces are the one generated by Halpern and viscosity-type algorithms (see for example,[2, 5, 7, 17, 23–25]) unlike in the setting of Hilbert spaces where the PPA of hybrid-type is proved to be among. As such, there arises a question: Can we establish a strong convergence of the Mann-type PPA (1.4) by hybrid method in Hadamard spaces?

In this paper, an affirmative answer is given to such question by introducing a hybrid-type PPA involving Mann-type iteration in the setting of flat Hadamard spaces. We also prove that the sequence generated by the said algorithm converges strongly to the zero of the monotone operator in the space.

Remark 1.2. The proposed method guaranteed the strong convergence of the Mann-type PPA rather than the Δ-convergence of the corresponding algorithm as in [19]. Also, the method does not require monotonicity assumption on the sequence as it can be proved. Hence, the two cases approach in proving the strong convergence is not required unlike in the existing methods. The result established generalized the corresponding ones in Hilbert spaces.

2. Preliminaries

Throughout this section, the symbols "$\to$" and "$\rightharpoonup$" represent the strong and Δ-convergence, respectively. The following results will play vital roles in establishing our main result.

Lemma 2.1 ([6])**.** *Let X be a CAT(0) space and $x, y \in X$, $t \in [0,1]$. Then*

(i) $d(z, tx \oplus (1-t)y) \leq td(z,x) + (1-t)d(z,y)$;

(ii) $d^2(z, tx \oplus (1-t)y) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y)$.

Lemma 2.2. *Let C be a nonempty convex subset of a CAT(0) space X. For $x \in X$ and $u \in C$, then $u = P_C x$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \geq 0, \quad \textit{for all } y \in C.$$

Lemma 2.3 ([13])**.** *Let X be a complete CAT(0) space. Then every bounded sequence in X has a Δ-convergence subsequence.*

Lemma 2.4 ([8])**.** *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle\overrightarrow{x_nx}, \overrightarrow{yx}\rangle \leq 0$ for all $y \in X$.*

Lemma 2.5 ([9])**.** *Let X^* be a dual of the Hadamard space X and $g : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then*

(i) *g attains its minimum at $x \in X$ if and only if $0 \in \delta g(x)$;*
(ii) *$\delta g : X \to 2^{X^*}$ is a monotone operator;*
(iii) *for any $x \in X$ and $\alpha > 0$ there exists a unique point $y \in X$ such that $[\alpha\overrightarrow{xy}] \in \delta g(y)$, that is $\mathrm{dom}(J_\lambda^{\delta g}) = X$ for all $\lambda > 0$.*

Lemma 2.6 ([11])**.** *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*. Then*

$$J_\lambda^{\delta g}(x) = \underset{y\in X}{\operatorname{argmin}}\{g(y) + \frac{1}{2\lambda}d^2(y, x)\},$$

for all $\lambda > 0$ and $x \in X$.

Lemma 2.7 ([11])**.** *Let X be a CAT(0) space and J_λ^A be the resolvent of the monotone operator A with order λ. Then*

(i) *for any $\lambda > 0$, $R(J_\lambda^A) \subset \mathrm{dom}(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A,*
(ii) *if A is monotone, then J_λ^A is single-valued and firmly nonexpansive and hence nonexpansive,*
(iii) *if A is monotone and $\mu \geq \lambda > 0$, then $d(x, J_\lambda^A x) \leq d(x, J_\mu^A x)$.*

3. Main Results

In this section, C is considered to be nonempty closed convex subset of a flat Hadamard space X. We introduce a hybrid-type proximal point algorithm involving Mann-type iteration for approximating zero of monotone operator in flat Hadamard spaces.

Theorem 3.1. *Let X be a flat Hadamard space with its dual X^*. Let A be a multi-valued monotone operator of X into 2^{X^*} satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n)J_{\lambda_n}^A v_n, \\ D_n = \{v \in C : d(v, u_n) \leq d(v, v_n)\}, \\ E_n = \{v \in C : \langle\overrightarrow{v_0v_n}, \overrightarrow{vv_n}\rangle \leq 0\}, \\ v_{n+1} = P_{D_n\cap E_n}(v_0), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\lambda_n \in (0, \infty)$ with $\lambda_n \geq \lambda > 0$ and $\{\alpha_n\} \subset [0, 1]$. Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}$ is the metric projection from X onto $A^{-1}(0)$.

Proof. We divide the proof into the following steps.

Step 1. We show that the set $D_n \cap E_n$ is closed and convex. From D_n and definition of quasilinearization, we see $d(v, u_n) \leq d(v, v_n)$ if and only if $-d^2(v_n, u_n) + \langle \overrightarrow{vu_n}, \overrightarrow{v_n u_n} \rangle \leq 0$. Thus, it is an evident from [1] that $D_n \cap E_n$ is closed and convex. For completeness sake, we give the proof here. Let $y_m \in D_n$ such that $\lim_{m\to\infty} y_m = y$ then we show that $y \in D_n$. But

$$\begin{aligned} -d^2(v_n, u_n) + \langle \overrightarrow{yu_n}, \overrightarrow{v_n u_n} \rangle &= -d^2(v_n, u_n) + \langle \overrightarrow{\lim_{m\to\infty} y_m u_n}, \overrightarrow{v_n u_n} \rangle \\ &= -d^2(v_n, u_n) + \lim_{m\to\infty} \langle \overrightarrow{y_m u_n}, \overrightarrow{v_n u_n} \rangle \\ &= \lim_{m\to\infty} (-d^2(v_n, u_n) + \langle \overrightarrow{y_m u_n}, \overrightarrow{v_n u_n} \rangle) \leq 0. \end{aligned}$$

Thus, D_n is closed. For convexity, let $y_1, y_2 \in D_n$ then we show that $y = ry_1 \oplus (1 - r)y_2 \in D_n$ for $r \in [0, 1]$. Using equation (1.1), we get

$$\begin{aligned} -d^2(v_n, u_n) + \langle \overrightarrow{yu_n}, \overrightarrow{v_n u_n} \rangle =& - d^2(v_n, u_n) + \langle \overrightarrow{(ry_1 \oplus (1-r)y_2)u_n}, \overrightarrow{v_n u_n} \rangle \\ =& - d^2(v_n, u_n) + r\langle \overrightarrow{y_1 u_n}, \overrightarrow{v_n u_n} \rangle + (1-r)\langle \overrightarrow{y_2 u_n}, \overrightarrow{v_n u_n} \rangle \\ \leq& 0. \end{aligned}$$

Thus, D_n is convex. Therefore, D_n is closed and convex. Similarly, for the set E_n, we take $y_m \in E_n$ with $\lim_{m\to\infty} y_m = y$ and by continuity of quasilinearization, we get

$$\langle \overrightarrow{v_0 v_n}, \overrightarrow{yv_n} \rangle = \langle \overrightarrow{v_0 v_n}, \overrightarrow{\lim_{m\to\infty} y_m v_n} \rangle = \lim_{m\to\infty} \langle \overrightarrow{v_0 v_n}, \overrightarrow{y_m v_n} \rangle \leq 0.$$

Thus, E_n is closed. Also, for $y = ry_1 \oplus (1 - r)y_2$ where $y_1, y_2 \in E_n$, we see that

$$\begin{aligned} \langle \overrightarrow{v_0 v_n}, \overrightarrow{yv_n} \rangle &= \langle \overrightarrow{v_0 v_n}, \overrightarrow{(ry_1 \oplus (1-r)y_2)v_n} \rangle \\ &= r\langle \overrightarrow{v_0 v_n}, \overrightarrow{y_1 v_n} \rangle + (1-r)\langle \overrightarrow{v_0 v_n}, \overrightarrow{y_2 v_n} \rangle \\ &\leq 0. \end{aligned}$$

Thus, E_n is convex. Therefore, E_n is closed and convex. Hence $D_n \cap E_n$ is closed and convex.

Step 2. We show that the sequence $\{v_n\}$ is well-defined. The well-definedness of $\mathrm{P}_{A^{-1}(0)}$ follows from the fact that $A^{-1}(0)$ is closed and convex. Now let $w_n = J^A_{\lambda_n} v_n$ and $A^{-1}(0) \neq \emptyset$. Then, we can take $u = \mathrm{P}_{A^{-1}(0)} \subset A^{-1}(0)$ so that $J^A_{\lambda_n} u = u$. It follows from (3.1) and nonexpansivity of $J^A_{\lambda_n}$ that

$$d(u, w_n) = d(J^A_{\lambda_n} u, J^A_{\lambda_n} v_n) \leq d(u, v_n). \tag{3.2}$$

Also, using (3.1) and (3.2) we get

$$\begin{aligned} d(u, u_n) &= d(u, \alpha_n v_n \oplus (1-\alpha_n)w_n) \\ &\leq \alpha_n d(u, v_n) + (1-\alpha_n)d(u, w_n) \leq \alpha_n d(u, v_n) + (1-\alpha_n)d(u, v_n) \\ &= d(u, v_n). \end{aligned}$$

Thus, $u \in D_n$ and therefore $A^{-1}(0) \subset D_n$. Next we show that $A^{-1}(0) \subset D_n \cap E_n$, for all $n \in \mathbb{N}$. We do this by induction. Now let $n = 1$, we see that $\mathcal{F} \subset D_1 = E_1 = C$ and so $A^{-1}(0) \subset D_1 \cap E_1$. Suppose that $A^{-1}(0) \subset D_k \cap E_k$ for some $k > 1$. Since $v_{k+1} = P_{D_k \cap E_k}(v_0)$, then using Lemma 2.2, we get

$$\langle \overrightarrow{v_0 v_{k+1}}, \overrightarrow{p v_{k+1}} \rangle \leq 0,$$

for all $p \in D_k \cap E_k$. Also, since $A^{-1}(0) \subset D_k \cap E_k$, we have

$$\langle \overrightarrow{v_0 v_{k+1}}, \overrightarrow{u v_{k+1}} \rangle \leq 0,$$

for all $u \in A^{-1}(0)$. This implies $A^{-1}(0) \subset D_{k+1} \cap E_{k+1}$. Therefore, $A^{-1}(0) \subset D_n \cap E_n$ for all $n \in \mathbb{N}$. Hence, the sequence $\{v_n\}$ is well defined.

Step 3. The $\lim_{n\to\infty} d(v_n, v_0)$ exists. First we show that the sequence $\{v_n\}$ is bounded. Using the property of metric projection and the fact that $v_n = P_{E_n}(v_0)$, we get

$$d(v_n, v_0) = d(P_{E_n}(v_0), v_0) \leq d(u, v_0) - d(u, P_{E_n}(v_0)) = d(u, v_0).$$

This implies that the sequence $\{d(v_n, v_0)\}$ is bounded. Thus, the sequence $\{v_n\}$ is bounded too. Since $v_n = P_{E_n}(v_0)$ and $v_{n+1} \in E_n$, then using Lemma 2.2 and quasilinearization definition, we have

$$\begin{aligned} 0 &\leq \langle \overrightarrow{v_0 v_n}, \overrightarrow{v_n v_{n+1}} \rangle \\ &= d^2(v_0, v_{n+1}) + d^2(v_n, v_n) - d^2(v_0, v_n) - d^2(v_n, v_{n+1}) \\ &\leq d^2(v_0, v_{n+1}) - d^2(v_0, v_n). \end{aligned} \tag{3.3}$$

This implies $d(v_0, v_n) \leq d(v_0, v_{n+1})$. Thus, the sequence $\{d(v_0, v_n)\}$ is monotone increasing. Since it is bounded, then $\lim_{n\to\infty} d(v_n, v_0)$ exists.

Step 4. We show that $\lim_{n\to\infty} d(v_n, J_\lambda^A v_n) = 0$. From equation (3.3), we see that

$$d^2(v_n, v_{n+1}) \leq d^2(v_0, v_{n+1}) - d^2(v_0, v_n). \tag{3.4}$$

Using the fact that $\lim_{n\to\infty} d(v_n, v_0)$ exists, it follows from (3.4) that

$$\lim_{n\to\infty} d(v_n, v_{n+1}) = 0. \tag{3.5}$$

Since $v_{n+1} \in D_n$, then $d(v_{n+1}, u_n) \leq d(v_{n+1}, v_n)$. Thus, it follows from (3.5) that

$$\lim_{n\to\infty} d(v_{n+1}, u_n) = 0. \tag{3.6}$$

With the use of (3.5), (3.6) and the property of metric distance, we get

$$\lim_{n\to\infty} d(v_n, u_n) = 0. \tag{3.7}$$

On the other hand,

$$\begin{aligned} d^2(u, u_n) =& d^2(u, \alpha_n v_n \oplus (1-\alpha_n) w_n) \\ =& \alpha_n d^2(u, v_n) + (1-\alpha_n) d^2(u, w_n) - \alpha_n(1-\alpha_n) d^2(u_n, w_n) \\ \leq& d^2(u, v_n) - \alpha_n(1-\alpha_n) d^2(v_n, w_n). \end{aligned}$$

Thus, using quasilinearization definition, Cauchy-Schwartz inequality and (3.7) we get

$$\begin{aligned}\alpha_n(1-\alpha_n)d^2(v_n,w_n) &\leq d^2(u,v_n)-d^2(u,u_n)\\ &= d^2(v_n,v_n)-d^2(v_n,u_n)+2\langle\overrightarrow{uv_n},\overrightarrow{u_nv_n}\rangle\\ &\leq 2\langle\overrightarrow{uv_n},\overrightarrow{u_nv_n}\rangle \leq 2d(u,v_n)d(u_n,v_n)\\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Using the fact that $\alpha_n \in (0,1)$, we get

$$\lim_{n\to\infty} d(v_n, J^A_{\lambda_n}v_n) = \lim_{n\to\infty} d(v_n,w_n) = 0.$$

Since $\lambda_n \geq \lambda$, then by Lemma 2.7 (iii) we get

$$\lim_{n\to\infty} d(v_n, J^A_{\lambda}v_n) \leq 2\lim_{n\to\infty} d(v_n, J^A_{\lambda_n}v_n) = 0.$$

Since the sequence $\{v_n\}$ is bounded and the space X is Hadamard, then from Lemma 2.3 there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\Delta - \lim_{k\to\infty} v_{n_k} = w$. Since J_λ is nonexpansive then by demiclosedness of J_λ, we get $w \in F(J_\lambda) = A^{-1}(0)$.

Since $v_{n+1} = P_{D_n\cap E_n}(v_0)$, then by letting $q = P_{A^{-1}(0)}(v_0) \in D_n \cap E_n$, we get $d(v_{n+1},v_0) \leq d(q,v_0)$. Also, $v_{n_k} \rightharpoonup w$ and $d(\cdot,\cdot)$ is convex and lower semicontinuous hence Δ-lower semicontinuous (see [3]), we get

$$d(w,v_0) \leq \liminf_{k\to\infty} d(v_{n_k},v_0) \leq d(q,v_0).$$

From the definition of q, we can conclude that $w = q$ and so $v_n \rightharpoonup q$. It follows from Lemma 2.4 that the $\limsup_{n\to\infty}\langle\overrightarrow{qs},\overrightarrow{qv_n}\rangle \leq 0$ for all $s \in X$. Thus, it holds for $v_0 \in X$, i.e.,

$$\limsup_{n\to\infty}\langle\overrightarrow{qv_0},\overrightarrow{qv_n}\rangle \leq 0. \tag{3.8}$$

We now show that $v_n \to q$. Using quasilinearization definition, we see that

$$\begin{aligned}d^2(v_n,q) &= (d^2(q,v_0)+d^2(v_n,v_0)-2\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle)\\ &\leq (d^2(q,v_0)+d^2(q,v_0)-2\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle)\\ &= 2(d^2(q,v_0)-\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle) = 2(\langle\overrightarrow{qv_0},\overrightarrow{qv_0}\rangle+\langle\overrightarrow{qv_0},\overrightarrow{v_0v_n}\rangle)\\ &= 2\langle\overrightarrow{qv_0},\overrightarrow{qv_n}\rangle. \end{aligned}\tag{3.9}$$

Taking lim sup of the inequality (3.9) as $n\to\infty$ together with the use of (3.8), we see that the $\limsup_{n\to\infty} d^2(v_n,q) = 0$. Thus, $\lim_{n\to\infty} d^2(v_n,q) = 0$ and hence the sequence $v_n \to q$. This completes the prove. □

In view of the fact that every closed convex subset of Hilbert spaces is flat Hadamard space, the following results can be obtained from Theorem 3.1 as corollaries.

Corollary 3.1. *Let X be a Hilbert space and A be a multi-valued monotone operator of X into 2^X satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n + (1-\alpha_n) J^A_{\lambda_n} v_n, \\ D_n = \{v \in C : \|v - u_n\| \leq \|v - v_n\|\}, \\ E_n = \{v \in C : \langle v_0 - v_n, v - v_n \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0, \infty)$ with $\lambda_n \geq \lambda > 0$ and $\{\alpha_n\} \subset [0,1]$. Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}$ is the metric projection from X onto $A^{-1}(0)$.

Proof. Since every closed convex subset of a Hilbert space is flat Hadamard space then by Theorem 3.1, we get the desired result. This completes the proof. ☐

Corollary 3.2 ([21])**.** *Let X be a Hilbert space and A be a multi-valued monotone operator of X into 2^X satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = J^A_{\lambda_n} v_n, \\ D_n = \{v \in C : \|v - u_n\| \leq \|v - v_n\|\}, \\ E_n = \{v \in C : \langle v_0 - v_n, v - v_n \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0. \end{cases}$$

Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}(v_0)$ is the metric projection from X onto $A^{-1}(0)$.

4. Application to Convex Minimization Problem

In this section, we consider an application to convex minimization problem. Recall that the minimization problem is a problem of finding a point

$$u \in X \quad \text{such that} \quad g(u) = \min_{v \in X} f(v).$$

In view of Theorem 2.5 (i) this problem can be formulated as follow: find $u \in X$ such that $0 \in \delta g(u)$. Thus, by setting $A = \delta g$ in Theorem 3.1 together with the use of Lemma 2.6, the following result can easily be obtained.

Theorem 4.1. *Let X be a flat Hadamard space with its dual X^* and $g : X \to (-\infty, +\infty]$ be proper, lower semicontinuous function such that $(\delta g)^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be defined by*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n) J^{\delta g}_{\lambda_n} v_n, \\ D_n = \{v \in C : d(v, u_n) \leq d(v, v_n)\}, \\ E_n = \{v \in C : \langle \overrightarrow{v_0 v_n}, \overrightarrow{v v_n} \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0,\infty)$ *with* $\lambda_n \geq \lambda > 0$ *and* $\{\alpha_n\} \subset [0,1]$. *Then* $\{v_n\}$ *converges strongly to* $u = P_{(\delta g)^{-1}(0)}(v_0)$, *where* $P_{(\delta g)^{-1}(0)}$ *is the metric projection from* X *onto* $(\delta g)^{-1}(0)$.

5. Application to Fixed Point Problem

In this section, we consider application to fixed point of nonexpansive mapping. Recall that a mapping T of a metric space X into itself is called nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$. If X is a Hilbert space, the operator $I-T$ is known to be maximal monotone and hence satisfies the range condition, where I is the identity mapping. For the operator $I-T$, the maximal monotonicity and the range condition are considered in Hadamard spaces by Khatibzadeh and Ranjbar [11] as can be seen from the following results.

Proposition 5.1 ([11])**.** *Let* X *be a Hadamard space and* $T : X \to X$ *be an arbitrary nonexpansive mapping. If the monotone operator* $Az = [\overrightarrow{Tzz}]$ *is maximal, then* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition.*

Proposition 5.2 ([11])**.** *Let* X *be a Hadamard space. For every nonexpansive mapping* $T : X \to X$, *the operator* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition if and only if for all* $x,y,z \in X$

$$d^2(\alpha x \oplus (1-\alpha)y, z) = \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y).$$

This is equivalent to saying that, for every nonexpansive mapping $T : X \to X$, *the operator* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition if and only if the Hadamard space* X *is flat.*

Thus, $F(T) = A^{-1}(0)$ (see Ranjbar and Khatibzadeh [19]), where $F(T) := \{x \in X : Tx = x\}$ and $Az = [\overrightarrow{Tzz}]$. Hence the following result follows from Theorem 3.1.

Theorem 5.1. *Let* X *be a flat Hadamard space with its dual* X^* *and* $T : X \to X$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$ *with the operator* $Az = [\overrightarrow{Tzz}]$. *Let the sequence* $\{v_n\} \subset C$ *be iteratively defined by*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n) J^A_{\lambda_n} v_n, \\ D_n = \{v \in C : d(v,u_n) \leq d(v,v_n)\}, \\ E_n = \{v \in C : \langle \overrightarrow{v_0 v_n}, \overrightarrow{v v_n} \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0,\infty)$ *with* $\lambda_n \geq \lambda > 0$ *and* $\{\alpha_n\} \subset [0,1]$. *Then* $\{v_n\}$ *converges strongly to* $u = P_{T^{-1}(0)}(v_0)$, *where* $P_{T^{-1}(0)}$ *is the metric projection from* X *onto* $T^{-1}(0)$.

Proof. Since X is flat Hadamard space then by proposition 5.2, the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition and $F(T) = A^{-1}(0)$. Thus, by Theorem 3.1, we get the desired result. □

6. Conclusion

In this article, a new Mann hybrid-type proximal point algorithm for solving monotone inclusion problem is presented in Hadamard-type spaces. It is shown that our algorithm converges strongly to a zero solution of the said operator in the setting of flat Hadamard spaces. To the best of our knowledge, this result presents the first hybrid-type proximal point algorithm in Hadamard-type spaces.

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[1]Department of Mathematical Sciences,
Kaduna State University,
Tafawa Balewa Way, Kaduna-Nigeria
Email address: yulah121@gmail.com
ORCID iD: https://orcid.org/0000-0003-0693-1263

[2]Department of Mathematics,
University of Eswatini,
Kwaluseni, Eswatini

[3]Department of Mathematics and Applied Mathematics,
Sefako Makgato Health Science University,
P.O. Box 94, Pretoria 0204, South Africa
Email address: gc.ugwunnadi@mouau.edu.ng
Email address: ugwunnadi4u@yahoo.com
ORCID iD: https://orcid.org/0000-0002-2711-7888

[4]Department of Mathematical Sciences,
Bayero University, Kano,
Kano-Nigeria
Email address: bashiralik@yahoo.com
ORCID iD: https://orcid.org/0000-0003-4379-2418

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PSEUDO GE-ALGEBRAS AS THE EXTENSION OF GE-ALGEBRAS

RAVIKUMAR BANDARU[1], AKBAR REZAEI[2], ARSHAM BORUMAND SAEID[3], AND YOUNG BAE JUN[4]

Abstract. In this paper, the notion of a pseudo GE-algebra as an extension of a GE-algebra is introduced. Basic properties of pseudo GE-algebras are described. The concepts of strong pseudo BE-algebra, good pseudo BE-algebra, good pseudo GE-algebra, and the relationship between them are established. We provide a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1]). In 1966, Y. Imai and K. Iseki [10, 12] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Pseudo-valuations were introduced and studied by Y. B. Jun [13]. Georgescu and Iorgulescu [9] introduced an extension of BCK-algebra called pseudo BCK-algebra. Di Nola et al. presented pseudo BL-algebras, which are non-commutative BL-algebras [5, 6]. Moreover, they gave the connection of pseudo BCK-algebra with pseudo MV-algebra and with pseudo BL-algebra. Pseudo BCI-algebras were introduced and studied by W. A. Dudek and Y.

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B Jun (see [7]), as generalizations of pseudo BCK-algebras and BCI-algebras, and they form an important tool for an algebraic axiomatization of implicational fragment of non-classical logic (see [8]). A. Walendziak [16] gave a system of axioms defining pseudo BCK-algebras. Pseudo BCK-algebras were intensively studied in [3, 11, 14]. R. A. Borzooei et al. [2] applied pseudo structure to BE-algebras and investigated its properties. They studied the concepts of pseudo-subalgebra, pseudo-filter and pseudo-upper-set and proved that every pseudo-filter is a union of pseudo-upper-sets. Later on, in 2019, Rezaei et al. defined pseudo CI-algebras, which are a generalization of the pseudo BE-algebras, pseudo BCK-algebras and pseudo MV-algebras [15].

In this paper, we introduce the notion of pseudo GE-algebra as a non-commutative generalization of GE-algebra and study its properties. We define the notion of $(\circledast, \boxplus)$-pseudo GE-algebra, $(\boxplus, \circledast)$-pseudo GE-algebra and investigate its properties. We define the concept of strong pseudo BE-algebra, good pseudo GE-algebra and study relation between them. Finally, we give a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

2. Preliminaries

Definition 2.1 ([1]). By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $u * u = 1$;
(GE2) $1 * u = u$;
(GE3) $u * (v * w) = u * (v * (u * w))$,

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \Leftrightarrow x * y = 1)\,.$$

Definition 2.2 ([1]). A GE-algebra X is said to be *transitive* if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.1}$$

Proposition 2.1 ([1]). *Every GE-algebra X satisfies the following items*

$$\begin{aligned}
&(\forall u \in X)\,(u * 1 = 1)\,,\\
&(\forall u, v \in X)\,(u * (u * v) = u * v)\,,\\
&(\forall u, v \in X)\,(u \leq v * u)\,,\\
&(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,,\\
&(\forall u \in X)\,(1 \leq u \Rightarrow u = 1)\,,\\
&(\forall u, v \in X)\,(u \leq (v * u) * u)\,,\\
&(\forall u, v \in X)\,(u \leq (u * v) * v)\,,\\
&(\forall u, v, w \in X)\,(u \leq v * w \Leftrightarrow v \leq u * w)\,.
\end{aligned}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w)\,,$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,.$$

Lemma 2.1 ([1])**.** *In a GE-algebra X, the following facts are equivalent to each other*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,,$$
$$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,.$$

3. Pseudo GE-Algebras

We consider the notion of a pseudo GE-algebra as a generalization of a GE-algebra. Let X be a set with two binary operations "$\circledast$" and "$\boxplus$". Then we can consider the following two cases:

$$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = x \boxplus (y \circledast (x \boxplus z)) \text{ and } x \boxplus (y \circledast z) = x \circledast (y \boxplus (x \circledast z))), \tag{3.1}$$

$$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \circledast z)) \text{ and } x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \boxplus z))). \tag{3.2}$$

Hence we can think of two types of pseudo GE-algebra so called type A and type B.

Definition 3.1. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo GE-algebra of type* A if it satisfies (3.1) and the following conditions:

$$(\forall x \in X)\,(x \circledast x = 1 \text{ and } x \boxplus x = 1)\,, \tag{3.3}$$

$$(\forall x \in X)\,(1 \circledast x = x \text{ and } 1 \boxplus x = x)\,. \tag{3.4}$$

Definition 3.2. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo GE-algebra of type* B if it satisfies (3.2), (3.3) and (3.4).

The pseudo GE-algebra $(X, \circledast, \boxplus, 1)$ of type A or type B is sometimes only shown as X. It is clear that if a pseudo GE-algebra X of type A or type B satisfies $x \circledast y = x \boxplus y$ for all $x, y \in X$, then X is a GE-algebra.

As you can see above, we have defined two types of pseudo GE-algebra. In considering the pseudo theory as a generalization for a given algebraic system, it is not desirable to have multiple types in the development of theory.

The following theorem shows that one of the two types has no meaning.

Theorem 3.1. *If X is a pseudo GE-algebra of type A, then it is just a GE-algebra.*

Proof. Let X be a pseudo GE-algebra of type A and let $x, y \in X$. Then $1 \circledast (x \boxplus y) = 1 \boxplus (x \circledast (1 \boxplus y))$ by (3.1), and so $x \boxplus y = x \circledast y$ by (3.4). Therefore, X is a GE-algebra. □

So in thinking about pseudo theory, which is the generalization of GE-algebra, we can see that Definition 3.2 is the only definition expressed. Based on these discussions, we can call pseudo GE-algebra of type B just pseudo GE-algebra.

Now, we give examples of a pseudo GE-algebra.

Example 3.1. Let $X = \{1, a, b, c, d, e\}$ and define binary operations $\circledast$ and $\boxplus$ as follows:

$\circledast$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	d	d	d
b	1	a	1	1	d	d
c	1	a	1	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

,

$\boxplus$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	1	1	1
b	1	a	1	c	e	e
c	1	a	1	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

.

It is routine to verify that $(X, \circledast, \boxplus, 1)$ is a pseudo-GE-algebra.

Definition 3.3. A $(\circledast, \boxplus)$-*pseudo GE-algebra* is a structure $(X, \circledast, \boxplus, 1)$ in which X is set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$" satisfying the conditions (3.3), (3.4) and

$$(\forall x, y, z \in X)\,(x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \circledast z)))\,. \tag{3.5}$$

Example 3.2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	c	1
b	1	a	1	a
c	a	1	a	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra.

Definition 3.4. A $(\boxplus, \circledast)$-*pseudo GE-algebra* is a structure $(X, \circledast, \boxplus, 1)$ in which X is set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$" satisfying the conditions (3.3), (3.4) and

$$(\forall x, y, z \in X)\,(x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \boxplus z)))\,. \tag{3.6}$$

Example 3.3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	1
b	a	a	1	a
c	a	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra.

It is clear that if a structure $(X, \circledast, \boxplus, 1)$ is both a $(\circledast, \boxplus)$-pseudo GE-algebra and a $(\boxplus, \circledast)$-pseudo GE-algebra, then it is a pseudo GE-algebra.

Every $(\circledast, \boxplus)$-pseudo GE-algebra need not be a $(\boxplus, \circledast)$-pseudo GE-algebra. In Example 3.2, X is $(\circledast, \boxplus)$-pseudo GE-algebra. But X is not a $(\boxplus, \circledast)$-pseudo GE-algebra, since

$$a \boxplus (a \circledast b) = a \boxplus b = c \neq a = a \boxplus 1 = a \boxplus (a \circledast c) = a \boxplus (a \circledast (a \boxplus b)).$$

Every $(\boxplus, \circledast)$-pseudo GE-algebra need not be a $(\circledast, \boxplus)$-pseudo GE-algebra. In Example 3.3, X is $(\boxplus, \circledast)$-pseudo GE-algebra. But X is not a $(\circledast, \boxplus)$-pseudo GE-algebra, since

$$a \circledast (a \boxplus b) = a \circledast c = 1 \neq a = a \circledast 1 = a \circledast (a \boxplus a) = a \circledast (a \boxplus (a \circledast b)).$$

In a $(\circledast, \boxplus)$-pseudo GE-algebra or a $(\boxplus, \circledast)$-pseudo GE-algebra $(X, \circledast, \boxplus, 1)$, we define two binary operations "$\ll_{\circledast}$" and "$\ll_{\boxplus}$" as follows:

$$(\forall x, y \in X)(x \ll_{\circledast} y \Leftrightarrow x \circledast y = 1),$$
$$(\forall x, y \in X)(x \ll_{\boxplus} y \Leftrightarrow x \boxplus y = 1),$$

respectively. For every elements x and y of a pseudo GE-algebra X, if $x \ll_{\circledast} y$ and $x \ll_{\boxplus} y$ are formed at the same time, it is represented as $x \ll y$.

Proposition 3.1. *Every $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x \in X)(x \circledast 1 = 1), \tag{3.7}$$
$$(\forall x, y \in X)(x \circledast (x \circledast y) = x \circledast y), \tag{3.8}$$
$$(\forall x, y \in X)(x \ll_{\circledast} (x \circledast y) \boxplus y). \tag{3.9}$$

Proof. Let $x, y, z \in X$. Then

$$1 = x \circledast x = x \circledast (1 \boxplus x) = x \circledast ((x \circledast x) \boxplus x)) = x \circledast ((x \circledast x) \boxplus (x \circledast x)) = x \circledast 1$$

which proves (3.7). Using (3.4) and (3.5), we have $x \circledast y = x \circledast (1 \boxplus y) = x \circledast (1 \boxplus (x \circledast y)) = x \circledast (x \circledast y)$ which shows (3.8). Using (3.3), (3.5) and (3.7), we obtain

$$x \circledast ((x \circledast y) \boxplus y) = x \circledast ((x \circledast y) \boxplus (x \circledast y)) = x \circledast 1 = 1.$$ □

Proposition 3.2. *Every $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies:*

$$(\forall x \in X)(x \ll_{\boxplus} 1), \tag{3.10}$$
$$(\forall x, y \in X)(x \boxplus (x \boxplus y) = x \boxplus y), \tag{3.11}$$
$$(\forall x, y \in X)(x \ll_{\boxplus} (x \boxplus y) \circledast y). \tag{3.12}$$

Proof. Let $x, y, z \in X$. Then

$$1 = x \boxplus x = x \boxplus (1 \circledast x) = x \boxplus (1 \circledast (x \boxplus x)) = x \boxplus (1 \circledast 1) = x \boxplus 1,$$

by (3.3), (3.4) and (3.6), i.e., $x \ll_{\boxplus} 1$. Using (3.4) and (3.6) induces

$$x \boxplus (x \boxplus y) = x \boxplus (1 \circledast (x \boxplus y)) = x \boxplus (1 \circledast y) = x \boxplus y.$$

Using (3.3), (3.6) and (3.10), we obtain

$$x \boxplus ((x \boxplus y) \circledast y) = x \boxplus ((x \boxplus y) \circledast (x \boxplus y)) = x \boxplus 1 = 1,$$

and so (3.12) is valid. □

Lemma 3.1. *Every* $(\circledast, \boxplus)$*-pseudo GE-algebra or* $(\boxplus, \circledast)$*-pseudo GE-algebra* X *satisfies:*

$$(\forall x \in X)(1 \ll_{\circledast} x \Rightarrow x = 1),$$
$$(\forall x \in X)(1 \ll_{\boxplus} x \Rightarrow x = 1).$$

Proof. Straightforward. □

Proposition 3.3. *Every pseudo GE-algebra* X *satisfies:*

$$(\forall x \in X)(1 \ll x \Rightarrow x = 1). \tag{3.13}$$

Proof. Lemma 3.1 induces (3.13). □

The combination of Propositions 3.1 and 3.2 induces the next proposition.

Proposition 3.4. *Every pseudo GE-algebra* X *satisfies for all* $x, y, z \in X$

(1) $x \circledast 1 = 1$ *and* $x \boxplus 1 = 1$;
(2) $x \circledast (x \circledast y) = x \circledast y$ *and* $x \boxplus (x \boxplus y) = x \boxplus y$;
(3) $x \ll_{\circledast} (x \circledast y) \boxplus y$ *and* $x \ll_{\boxplus} (x \boxplus y) \circledast y$;
(4) $x \ll_{\boxplus} y \circledast x$ *and* $x \ll_{\circledast} y \boxplus x$;
(5) $x \ll_{\circledast} (y \circledast x) \boxplus x$ *and* $x \ll_{\boxplus} (y \boxplus x) \circledast x$;
(6) $x \ll_{\circledast} (x \circledast y) \boxplus x$ *and* $x \ll_{\boxplus} (x \boxplus y) \circledast x$;
(7) $y \ll_{\circledast} y \boxplus x \Rightarrow x \ll_{\boxplus} y \circledast (y \boxplus x)$;
(8) $y \ll_{\boxplus} y \circledast x \Rightarrow x \ll_{\circledast} y \boxplus (y \circledast x)$;
(9) $x \circledast (y \boxplus z) \ll_{\circledast} y \boxplus (x \circledast z)$ *and* $x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$;
(10) $y \ll_{\boxplus} x \circledast (y \boxplus z) \Rightarrow y \ll_{\boxplus} (x \circledast z)$;
(11) $y \ll_{\circledast} x \boxplus (y \circledast z) \Rightarrow y \ll_{\circledast} (x \boxplus z)$;
(12) $x \ll_{\boxplus} y \circledast z \Leftrightarrow y \ll_{\circledast} x \boxplus z$.

Proof. Propositions 3.1 and 3.2 prove (1), (2), (3). Now,

$$x \boxplus (y \circledast x) = x \boxplus (y \circledast (x \boxplus x)) = x \boxplus (y \circledast 1) = x \boxplus 1 = 1$$

and

$$x \circledast (y \boxplus x) = x \circledast (y \boxplus (x \circledast x)) = x \circledast (y \boxplus 1) = x \circledast 1 = 1.$$

Hence $x \ll_{\boxplus} y \circledast x$ and $x \ll_{\circledast} y \boxplus x$. Hence, (4) follows. (5) and (6) follow from (4). Assume that $y \ll_{\circledast} y \boxplus x$. Then $y \circledast (y \boxplus x) = 1$, which implies that

$$\begin{aligned} x \boxplus (y \circledast (y \boxplus x)) &= x \boxplus (y \circledast (x \boxplus (y \boxplus x))) \\ &= x \boxplus (y \circledast (x \boxplus (y \circledast (y \boxplus x)))) \\ &= x \boxplus (y \circledast (x \boxplus 1)) \\ &= x \boxplus (y \circledast 1) = x \boxplus 1 = 1. \end{aligned}$$

Thus $x \ll_{\boxplus} y \circledast (y \boxplus x)$. If $y \ll_{\boxplus} y \circledast x$, then $y \boxplus (y \circledast x) = 1$. Hence,

$$\begin{aligned} x \circledast (y \boxplus (y \circledast x)) &= x \circledast (y \boxplus (x \circledast (y \circledast x))) \\ &= x \circledast (y \boxplus (x \circledast (y \boxplus (y \circledast x)))) \\ &= x \circledast (y \boxplus (x \circledast 1)) \\ &= x \circledast (y \boxplus 1) = x \circledast 1 = 1, \end{aligned}$$

and so, $x \ll_{\circledast} y \boxplus (y \circledast x)$. The combination of (3.2), (3.3) and (1) induces

$$(x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast z)) = (x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast (y \boxplus z))) = 1, \quad \text{by (4)}$$

and

$$(x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus z)) = (x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus (y \circledast z))) = 1, \quad \text{by (4).}$$

Hence, $x \circledast (y \boxplus z) \ll_{\circledast} y \boxplus (x \circledast z)$ and $x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$, that is, (9) is true. If $y \ll_{\boxplus} x \circledast (y \boxplus z)$, then $y \boxplus (x \circledast (y \boxplus z)) = 1$ and hence $y \boxplus (x \circledast z) = 1$. Therefore $y \ll_{\boxplus} (x \circledast z)$. Thus (10) follows. (11) is similar to (10). If $x \ll_{\boxplus} y \circledast z$, then $1 = x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$ by (9) and so $y \circledast (x \boxplus z) = 1$ by Lemma 3.1, i.e., $y \ll_{\circledast} x \boxplus z$. Similarly, if $y \ll_{\circledast} x \boxplus z$, then $x \ll_{\boxplus} y \circledast z$. Therefore, (12) holds. □

We consider the following four items

(3.14) $(\forall x, y, z \in X)\, (x \circledast y \ll_{\circledast} (z \circledast x) \boxplus (z \circledast y))\,,$

(3.15) $(\forall x, y, z \in X)\, (x \circledast y \ll_{\boxplus} (y \circledast z) \circledast (x \circledast z))\,,$

(3.16) $(\forall x, y, z \in X)\, (x \boxplus y \ll_{\boxplus} (z \boxplus x) \circledast (z \boxplus y))\,,$

(3.17) $(\forall x, y, z \in X)\, (x \boxplus y \ll_{\circledast} (y \boxplus z) \boxplus (x \boxplus z))\,.$

Example 3.4. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra. But it does not satisfy (3.14), since

$$(a \circledast 1) \circledast ((1 \circledast a) \boxplus (1 \circledast 1)) = a \circledast (a \boxplus 1) = a \circledast 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra. But it does not satisfy (3.15), since

$$(a \circledast 1) \boxplus ((1 \circledast b) \circledast (a \circledast b)) = 1 \boxplus (b \circledast c) = 1 \boxplus c = c \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	1	a	1	1
c	1	a	1	1

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra. But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \circledast (1 \boxplus 1)) = a \boxplus (a \circledast 1) = a \boxplus 1 = a \neq 1.$$

4. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	1
c	a	1	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	b	1

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra. But it does not satisfy (3.17), since

$$(a \boxplus 1) \circledast ((1 \boxplus b) \boxplus (a \boxplus b)) = 1 \circledast (b \boxplus c) = 1 \circledast c = c \neq 1.$$

Proposition 3.5. *Let X be a $(\boxplus, \circledast)$-pseudo GE-algebra. If X meets condition* (3.14)*, it also meets condition* (3.15)*.*

Proof. Assume that a $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies (3.14). Then $(x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1$ for all $x, y, z \in X$, and so

$$\begin{aligned}(x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) =&(x \circledast y) \boxplus ((y \circledast z) \circledast ((x \circledast y) \boxplus (x \circledast z)))\\ =&(x \circledast y) \boxplus 1 = 1,\end{aligned}$$

by (3.6) and (3.10). Therefore $x \circledast y \ll_{\boxplus} (y \circledast z) \circledast (x \circledast z)$ for all $x, y, z \in X$. □

Proposition 3.6. *Let X be a $(\circledast, \boxplus)$-pseudo GE-algebra. If X meets condition* (3.15)*, it also meets condition* (3.14)*.*

Proof. Assume that a $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies (3.15). Then $(x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1$ for all $x, y, z \in X$, which implies from (3.5) and (3.7) that

$$\begin{aligned}(x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) =&(x \circledast y) \circledast ((z \circledast x) \boxplus ((x \circledast y) \circledast (z \circledast y)))\\ =&(x \circledast y) \circledast 1 = 1.\end{aligned}$$

Thus, $x \circledast y \ll (z \circledast x) \boxplus (z \circledast y)$ for all $x, y, z \in X$. □

Corollary 3.1. *In a pseudo GE-algebra X,* (3.14) *and* (3.15) *are equivalent with each other.*

The following example shows that the converse of Propositions 3.5 and 3.6 is not true in general.

Example 3.5. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra satisfying (3.15). But it does not satisfy (3.14), since

$$(a \circledast 1) \circledast ((1 \circledast a) \boxplus (1 \circledast 1)) = a \circledast (a \boxplus 1) = a \circledast 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	a
c	1	1	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	a
c	1	1	1	1

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra satisfying (3.14). But it does not satisfy (3.15), since

$$(a \circledast b) \boxplus ((b \circledast c) \circledast (a \circledast c)) = b \boxplus (a \circledast c) = b \boxplus c = a \neq 1.$$

Proposition 3.7. *Let X be a $(\circledast, \boxplus)$-pseudo GE-algebra. If X meets condition* (3.16)*, it also meets condition* (3.17)*.*

Proof. If a $(\circledast, \boxplus)$-pseudo GE-algebra X meets condition (3.16), then $(x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1$ for all $x, y, z \in X$. Hence,

$$\begin{aligned}(x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) =&(x \boxplus y) \circledast ((y \boxplus z) \boxplus ((x \boxplus y) \circledast (x \boxplus z))) \\ =&(x \boxplus y) \circledast 1 = 1,\end{aligned}$$

by (3.5) and (3.7), that is, $x \boxplus y \ll_{\circledast} (y \boxplus z) \boxplus (x \boxplus z)$ for all $x, y, z \in X$. □

Proposition 3.8. *Let X be a $(\boxplus, \circledast)$-pseudo GE-algebra. If X meets condition* (3.17)*, it also meets condition* (3.16)*.*

Proof. Suppose that the condition (3.17) holds in a $(\boxplus, \circledast)$-pseudo GE-algebra X. Then

$$(x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

for all $x, y, z \in X$. Using (3.6) and (3.10) induces

$$\begin{aligned}(x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) =& (x \boxplus y) \boxplus ((z \boxplus x) \circledast ((x \boxplus y) \boxplus (z \boxplus y))) \\ =& (x \boxplus y) \boxplus 1 = 1,\end{aligned}$$

that is, $x \boxplus y \ll_{\boxplus} (z \boxplus x) \circledast (z \boxplus y)$ for all $x, y, z \in X$. □

Corollary 3.2. *In a pseudo GE-algebra X,* (3.16) *and* (3.17) *are equivalent with each other.*

The following example shows that the converse of Propositions 3.7 and 3.8 is not true in general.

Example 3.6. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	1	a	1	1
c	1	a	1	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra satisfying (3.17). But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \circledast (1 \boxplus 1)) = a \boxplus (a \circledast 1) = a \boxplus 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	a
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra satisfying (3.16). But it does not satisfy (3.17), since

$$(a \boxplus b) \circledast ((b \boxplus c) \boxplus (a \boxplus c)) = b \circledast (a \boxplus c) = b \circledast c = a \neq 1.$$

Definition 3.5. A $(\circledast, \boxplus)$-pseudo GE-algebra X is said to be

- $\circledast$-*transitive* if it satisfies (3.15);
- $\boxplus$-*transitive* if it satisfies (3.16).

If a $(\circledast, \boxplus)$-pseudo GE-algebra X is both $\circledast$-transitive and $\boxplus$-transitive, we say X is a *transitive* $(\circledast, \boxplus)$-*pseudo GE-algebra*.

Example 3.7. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

.

Then X is a $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. But it is not $\boxplus$-transitive since

$$(a \boxplus b) \boxplus ((1 \boxplus a) \circledast (1 \boxplus b)) = a \boxplus (a \circledast b) = a \boxplus b = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	c
c	1	a	1	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra which is $\boxplus$-transitive. But it is not $\circledast$-transitive since

$$(a \circledast b) \boxplus ((b \circledast c) \circledast (a \circledast c)) = 1 \boxplus (1 \circledast c) = 1 \boxplus c = c \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	b
b	1	a	1	1
c	1	a	1	1

.

Then X is a transitive $(\circledast, \boxplus)$-pseudo GE-algebra.

Proposition 3.9. *Every $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies*

(3.18) $(\forall x, y, z \in X)(x \ll_{\circledast} y \Rightarrow y \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast y)$,

(3.19) $(\forall x, y, z \in X)(((x \circledast y) \boxplus y) \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast ((x \circledast y) \boxplus y))$,

(3.20) $(\forall x, y, z \in X)(((y \circledast x) \boxplus x) \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast ((y \circledast x) \boxplus x))$.

Proof. Let X be a $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. If $x \ll_{\circledast} y$, then $x \circledast y = 1$ and thus

$$(y \circledast z) \circledast (x \circledast z) = 1 \boxplus ((y \circledast z) \circledast (x \circledast z)) = (x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1,$$

that is, $y \circledast z \ll_{\circledast} x \circledast z$. Since X satisfies (3.14) by Proposition 3.6, we have

$$(z \circledast x) \boxplus (z \circledast y) = 1 \circledast ((z \circledast x) \boxplus (z \circledast y)) = (x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1$$

and so, $z \circledast x \ll_{\boxplus} z \circledast y$. This proves (3.18). The combination of (3.9) and (3.18) induces (3.19). The combination of Proposition 3.4 (5) and (3.18) induces (3.20). □

Proposition 3.10. *Every $\boxplus$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow z \boxplus x \ll_{\circledast} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z).$$

Proof. Let X be a $\boxplus$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. If $x \ll_{\boxplus} y$, then $x \boxplus y = 1$ and thus

$$(z \boxplus x) \circledast (z \boxplus y) = 1 \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1.$$

Thus, $z \boxplus x \ll_{\circledast} z \boxplus y$. By Proposition 3.7, we know that X satisfies (3.17). Hence,

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

and so $y \boxplus z \ll_{\boxplus} x \boxplus z$. □

Corollary 3.3. *Every transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)\left(x \ll y \Rightarrow \begin{cases} y \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast y \\ z \boxplus x \ll_{\circledast} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z \end{cases} \right).$$

Definition 3.6. A $(\boxplus, \circledast)$-pseudo GE-algebra X is said to be

- $\circledast$-*transitive* if it satisfies (3.14);
- $\boxplus$-*transitive* if it satisfies (3.17).

If a $(\boxplus, \circledast)$-pseudo GE-algebra X is both $\circledast$-transitive and $\boxplus$-transitive, we say X is a *transitive* $(\boxplus, \circledast)$*-pseudo GE-algebra.*

Example 3.8. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	b
b	1	1	1	1
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra which is $\circledast$-transitive. But it is not $\boxplus$-transitive since

$$(a \boxplus b) \circledast ((b \boxplus c) \boxplus (a \boxplus c)) = 1 \circledast (1 \boxplus c) = 1 \circledast c = c \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra which is $\boxplus$-transitive. But it is not $\circledast$-transitive since

$$(a \circledast b) \circledast ((1 \circledast a) \boxplus (1 \circledast b)) = a \circledast (a \boxplus b) = a \circledast b = a \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	1
c	1	a	1	1

.

Then X is a transitive $(\boxplus, \circledast)$-pseudo GE-algebra.

Proposition 3.11. *Every $\boxplus$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies*

$$(3.21)\qquad (\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus y),$$

$$(3.22)\qquad (\forall x, y, z \in X)(((x \boxplus y) \circledast y) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus ((x \boxplus y) \circledast y)),$$

$$(3.23)\qquad (\forall x, y, z \in X)(((y \boxplus x) \circledast x) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus ((y \boxplus x) \circledast x)).$$

Proof. Let X be a $\boxplus$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra. Let $x, y \in X$ be such that $x \ll_{\boxplus} y$, Then $x \boxplus y = 1$ and so

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

by (3.4) and (3.17). Hence, $y \boxplus z \ll_{\boxplus} x \boxplus z$. We know that X satisfies (3.16) by Proposition 3.8. Thus,

$$(z \boxplus x) \circledast (z \boxplus y) = 1 \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1$$

and so $z \boxplus x \ll_{\circledast} z \boxplus y$, which proves (3.21). If we combine (3.21) and (3.12), then we have (3.22). The result (3.23) follows from the combination of (3.21) and Proposition 3.4 (5). □

Proposition 3.12. *Every $\circledast$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\circledast} y \Rightarrow z \circledast x \ll_{\boxplus} z \circledast y, y \circledast z \ll_{\circledast} x \circledast z).$$

Proof. Let X be a $\circledast$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra. If $x \ll_{\circledast} y$, then $x \circledast y = 1$ and so

$$(z \circledast x) \boxplus (z \circledast y) = 1 \circledast ((z \circledast x) \boxplus (z \circledast y)) = (x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1,$$

which shows that $z \circledast x \ll_{\boxplus} z \circledast y$. Using Proposition 3.5, we know that X satisfies condition (3.15). Thus,

$$(y \circledast z) \circledast (x \circledast z) = 1 \boxplus ((y \circledast z) \circledast (x \circledast z)) = (x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1,$$

and therefore $y \circledast z \ll_{\circledast} x \circledast z$. □

Corollary 3.4. *Every transitive* $(\boxplus, \circledast)$*-pseudo GE-algebra* X *satisfies:*

$$(\forall x, y, z \in X)\left(x \ll y \Rightarrow \begin{cases} y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus y \\ z \circledast x \ll_{\boxplus} z \circledast y, y \circledast z \ll_{\circledast} x \circledast z \end{cases} \right).$$

4. Relations Between Pseudo BE-Algebras and Pseudo GE-Algebras

As an extension of BE-algebras, Borzooei et al. introduced the notion of pseudo BE-algebras, and investigated its properties.

Definition 4.1 ([2])**.** Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo BE-algebra* if it satisfies (3.3), (3.4) and

(4.1) $$(\forall x \in X)(x \circledast 1 = 1, x \boxplus 1 = 1),$$

(4.2) $$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = y \boxplus (x \circledast z)),$$

(4.3) $$(\forall x, y \in X)(x \circledast y = 1 \Leftrightarrow x \boxplus y = 1).$$

Pseudo GE-algebra and pseudo BE-algebra basically form no relationship. In other words, the pseudo GE-algebra may not be the pseudo BE-algebra, and vice versa as seen in the following example.

Example 4.1. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	1
c	1	a	1	1

.

Then X is a pseudo GE-algebra. But X is not a pseudo BE-algebra since

$$a \circledast (a \boxplus b) = a \circledast c = b \neq c = a \boxplus b = a \boxplus (a \circledast b).$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	c	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	1
c	1	a	1	1

.

Then X is a pseudo BE-algebra. But X is not a pseudo GE-algebra since

$$a \circledast (a \boxplus b) = a \circledast b = c \neq b = a \circledast c = a \circledast (a \boxplus c) = a \circledast (a \boxplus (a \circledast b)).$$

The following example shows that when $(X, \circledast, \boxplus, 1)$ is a pseudo BE-algebra, $(X, \circledast, 1)$ or $(X, \boxplus, 1)$ does not need to be BE-algebra.

Example 4.2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

Then X is a pseudo BE-algebra. But $(X, \circledast, 1)$ is not a BE-algebra since

$$a \circledast (b \circledast c) = a \circledast a = 1 \neq a = b \circledast a = b \circledast (a \circledast c).$$

Also, $(X, \boxplus, 1)$ is not a BE-algebra since

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

Definition 4.2. A pseudo BE-algebra $(X, \circledast, \boxplus, 1)$ is said to be *strong* if $(X, \circledast, 1)$ and $(X, \boxplus, 1)$ are BE-algebras.

Example 4.3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	1	a	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	1	a	1

Then X is a strong pseudo BE-algebra.

We have the following question.

Question 4.1. Does pseudo GE-algebra X satisfy the following conditions

$$(\forall x, y, z \in X) \begin{pmatrix} x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \boxplus z)) \\ x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \circledast z)) \end{pmatrix}? \tag{4.4}$$

The following example shows that the answer to Question 4.1 is negative.

Example 4.4. Let $X = \{1, a, b, c, d\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following table:

$\circledast$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	1

$\boxplus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	d	1	d
b	1	a	1	1	a
c	1	a	a	1	a
d	1	a	a	1	1

Then X is a pseudo GE-algebra. But it does not satisfy (4.4) since

$$a \circledast (1 \boxplus b) = a \circledast b = c \neq 1 = a \circledast d = a \circledast (1 \boxplus d) = a \circledast (1 \boxplus (a \boxplus b)),$$

$$a \boxplus (1 \circledast b) = a \boxplus b = d \neq 1 = a \boxplus c = a \boxplus (1 \circledast c) = a \boxplus (1 \circledast (a \circledast b)).$$

Definition 4.3. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *good pseudo GE-algebra* if it satisfies (3.3), (3.4) and (4.4).

Example 4.5. Let $X = \{1, a, b, c, d\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following table:

$\circledast$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	c	c
c	1	a	1	1	1
d	1	a	1	1	1

,

$\boxplus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

.

Then X is a good pseudo GE-algebra.

Theorem 4.1. *Every good pseudo GE-algebra is a pseudo GE-algebra. But the converse is not true.*

Proof. Example 4.4 shows that any pseudo GE-algebra may not be a good pseudo GE-algebra. Let X be a good pseudo GE-algebra and let $x, y, z \in X$. Then

$$x \circledast y = x \circledast (1 \boxplus y) = x \circledast (1 \boxplus (x \boxplus y)) = x \circledast (x \boxplus y)$$

and

$$x \boxplus y = x \boxplus (1 \circledast y) = x \boxplus (1 \circledast (x \circledast y)) = x \boxplus (x \circledast y).$$

It follows from (4.4) that

$$x \circledast (y \boxplus (x \circledast z)) = x \circledast (y \boxplus (x \boxplus (x \circledast z))) = x \circledast (y \boxplus (x \boxplus z)) = x \circledast (y \boxplus z)$$

and

$$x \boxplus (y \circledast (x \boxplus z)) = x \boxplus (y \circledast (x \circledast (x \boxplus z))) = x \boxplus (y \circledast (x \circledast z)) = x \boxplus (y \circledast z).$$

Therefore, X is a pseudo GE-algebra. □

The following example shows that any pseudo BE-algebra X does not satisfy the condition

$$(\forall x, y, z \in X) \left(\begin{array}{c} x \circledast (y \boxplus z) = (x \circledast y) \boxplus (x \circledast z) \\ x \boxplus (y \circledast z) = (x \boxplus y) \circledast (x \boxplus z) \end{array} \right). \tag{4.5}$$

Example 4.6. Let $X = \{1, a, b, c, d\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	a	a	c	1

,

$\boxplus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	d	c	1
b	1	1	1	1	1
c	1	a	b	1	1
d	1	a	b	c	1

.

Then X is a pseudo BE-algebra. But X does not satisfy (4.5) since

$$a \circledast (b \boxplus c) = a \circledast 1 = 1 \neq c = a \boxplus c = (a \circledast b) \boxplus (a \circledast c)$$

and

$$a \boxplus (b \circledast c) = a \boxplus 1 = 1 \neq c = d \circledast c = (a \boxplus b) \circledast (a \boxplus c).$$

Question 4.2. Does pseudo BE-algebra X satisfy the following conditions

$$(\forall x, y, z \in X) \left(\begin{array}{c} x \circledast (y \circledast z) = y \circledast (x \circledast z)) \\ x \boxplus (y \boxplus z) = y \boxplus (x \boxplus z)) \end{array} \right)? \tag{4.6}$$

The answer to Question 4.2 is negative as seen in the following example.

Example 4.7. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

.

Then X is a pseudo BE-algebra. But it does not satisfy (4.6) since

$$a \circledast (b \circledast c) = a \circledast a = 1 \neq a = b \circledast a = b \circledast (a \circledast c)$$

and

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

Definition 4.4. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *good pseudo BE-algebra* if it satisfies (3.3), (3.4), (4.1), (4.3) and (4.6).

Example 4.8. Let $X = \{1, a, b, c\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	b	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

.

Then X is a good pseudo BE-algebra. But X is not pseudo BE-algebra since

$$a \circledast (a \boxplus b) = a \circledast b = a \neq 1 = a \boxplus a = a \boxplus (a \circledast b).$$

We now consider conditions for a pseudo BE-algebra to be a pseudo GE-algebra.

Theorem 4.2. *If a good pseudo BE-algebra X satisfies the condition* (4.5)*, then it is a pseudo GE-algebra.*

Proof. Let X be a good pseudo BE-algebra that satisfies the conditions (4.5). It is sufficient to show that X satisfies the condition (4.4). Let $x, y, z \in X$. Then $x \circledast (x \boxplus y) = (x \circledast x) \boxplus (x \circledast y) = 1 \boxplus (x \circledast y) = x \circledast y$ and $x \boxplus (x \circledast y) = (x \boxplus x) \circledast (x \boxplus y) = 1 \circledast (x \boxplus y) = x \boxplus y$ by (3.3), (3.4) and (4.5). It follows that $x \circledast (y \boxplus z) = x \circledast (x \boxplus (y \boxplus z)) = x \circledast (y \boxplus (x \boxplus z))$ and $x \boxplus (y \circledast z) = x \boxplus (x \circledast (y \circledast z)) = x \boxplus (y \circledast (x \circledast z))$. Hence X is a good pseudo GE-algebra, and therefore it is a pseudo GE-algebra by Theorem 4.1. □

Corollary 4.1. *Every strong pseudo BE-algebra X satisfying the condition* (4.5) *is a* (*good*) *pseudo GE-algebra.*

We finally pose the following question.

Question 4.3. What conditions will be required to make pseudo GE-algebra into pseudo BE-algebra?

5. Concluding Remarks

In this paper, we generalized GE-algebras to the case of pseudo GE-algebras and studied some basic of those properties. In the last section we investigated among relation between pseudo BE-algebras and pseudo GE-algebras. Starting from these notions, one can define and investigate commutative pseudo GE-algebras, involutive pseudo-GE algebras and Smarandache pseudo GE-algebras. Another topic of research could be to define and investigate state and monadic on pseudo GE-algebras.

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[1]Department of Mathematics,
School of Advanced Sciences, VIT-AP University,
Vijayawada-522237, Andhra Pradesh, India
Email address: ravimaths83@gmail.com
ORCID iD: https://orcid.org/0000-0001-8661-7914

[2]Department of Mathematics,
Payame Noor University,
P.O. Box 19395-3697, Tehran, Iran
Email address: rezaei@pnu.ac.ir
ORCID iD: https://orcid.org/0000-0002-6003-3993

[3] Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran
Email address: arsham@uk.ac.ir
ORCID iD: https://orcid.org/0000-0001-9495-6027

[4]Department of Mathematics Education,
Gyeongsang National University,
Jinju 52828, Korea
Email address: skywine@gmail.com
ORCID iD: http://orcid.org/0000-0002-0181-8969

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ON THE EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED NONLINEAR COUPLED PETROVSKY-WAVE SYSTEM

MOHAMED SAADAOUI[1], MOUNIR BAHLIL[2,3], AND MAMA ABDELLI[3]

Abstract. In this paper, we consider the initial-boundary value problem for a class of nonlinear coupled wave equation and Petrovesky system in a bounded domain. The strong damping is nonlinear. First, we prove the existence of global weak solutions by using the energy method combined with Faedo-Galarkin method and the multiplier method.

In addition, under suitable conditions on functions $g_i(\cdot)$, $i = 1, 2$ and $a(\cdot)$, we obtain both exponential and polynomial decay estimates. The method of proofs is direct and based on the energy method combined with the multipliers technique, on some integral inequalities due to Haraux and Komornik.

1. Introduction

The study of nonlinear wave phenomena was performed by certain eminent scientists. The theory of nonlinear waves, on the other hand, emerged as a coherent science in the late 1960s and early 1970s, which were the years of its rapid growth. While study in this area was undertaken only recently, the theory of nonlinear damped waves is still an emerging theme. In this paper, we study the existence and decay properties of solutions for the initial boundary value problem of the Petrovsky-wave system of

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the type

$$(1.1)\qquad \begin{cases} y_1'' + \Delta^2 y_1 - a(x)\Delta y_2 - g_1(\Delta y_1') = 0, & x\in\Omega, t\geq 0,\\ y_2'' - \Delta y_2 - a(x)\Delta y_1 - g_2(\Delta y_2') = 0, & x\in\Omega, t\geq 0,\\ \Delta y_1 = y_1 = y_2 = 0, & x\in\Gamma, t\geq 0,\\ y_i(x,0) = y_i^0(x),\quad y_i'(x,0) = y_i^1(x), & x\in\Omega, i=1,2, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with regular boundary Γ and $g_i:\mathbb{R}\to\mathbb{R}$ is a nondecreasing continuous function with $g_i(0)=0$, $i=1,2$.

When $a(x)=0$, the Petrovsky equation has been investigated in [7] by Komornik. The author has used the semigroup approach to present the existence and uniqueness of a global solution y_1 for (1.1). Then, using a multiplier technique, he directly proved exponential and polynomial decay estimates for the associated energy.

Bahlil et al. [4], studied the system:

$$(1.2)\qquad \begin{cases} y_1'' + a(x)y_2 + \Delta^2 y_1 - g_1(y_1'(x,t)) = f_1(y_1,y_2), & \text{in } \Omega\times\mathbb{R}^+,\\ y_2'' + a(x)y_1 - \Delta y_2 - g_2(y_2'(x,t)) = f_2(y_1,y_2), & \text{in } \Omega\times\mathbb{R}^+,\\ \partial_\nu y_1 = y_1 = v = y_2 = 0, & \text{on } \Gamma\times\mathbb{R}^+, \end{cases}$$

under suitable assumptions on the weight of the damping, they proved the global existence of solutions by use of the potential well method due to Payne and Sattinger [13] and Sattinger [14] combined with the Faedo-Galerkin method.

Also they proved general stability estimates using some properties of convex functions and the multiplier method.

In [5] Guesmia studied problem (1.2) with $f_i(y_1,y_2)=0$. He proved the existence of a global weak solution and uniform decay of solutions.

Motivated by previous works, it is interesting to investigate the global existence and decay of solutions to problem (1.1). Firstly, we show that, under suitable conditions on the functions g_i and a, the solutions are global in time. After that, we establish the rate of decay of solutions by the multiplier method. Precisely, we show that the decay rate of energy function is exponential or polynomial.

This article is organized as follows: in the next section, we give some preliminaries. In Section 3, we study the existence of global solutions of the problem (1.1). Then in Section 4, we are devoted to the proof of decay estimate.

2. Preliminaries and Main Results

In this section, we present some material for the proof of our result.

We first introduce the following spaces: $H = L^2(\Omega)\times L^2(\Omega)$, $W = H_0^1(\Omega)\times H_0^1(\Omega)$, $H^3_\Delta(\Omega) = \{v\in H^3(\Omega):\ v = \Delta v = 0 \text{ on } \Gamma\}$ and $\|u\|^2_{H^3_\Delta(\Omega)} = \int_\Omega |\nabla\Delta v|^2\,dx$, and

$$V = \left(H^3_\Delta(\Omega)\cap H^2(\Omega)\right)\times H^2(\Omega),\quad \tilde V = \left(H^3_\Delta(\Omega)\cap H^4(\Omega)\right)\times\left(H^3_\Delta(\Omega)\cap H^2(\Omega)\right).$$

Let H', V', $\tilde V'$, W' the dual spaces of H, V, $\tilde V$, W, respectively. We have

$$\tilde V\subset V\subset W\subset H = H'\subset W'\subset \tilde V'\subset V.$$

For the relaxation function g and a we assume the following.

(**H0**) Let $a : \Omega \to \mathbb{R}$ be non-increasing differentiable function bounded such that

$$a(x) \in W^{1,\infty}(\Omega), \quad \|a\|_{L^\infty(\Omega)} = \min\left\{\frac{1}{c'}, 1\right\}, \tag{2.1}$$

where $c' > 0$ is the constant $\|\nabla\Delta v\| \leq c'\|\Delta v\|$.

(**H1**) $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are non-increasing differentiable functions such that g_i is a C^1 and globally lipschitz with $g_i(0) = 0$ and there exists $p \geq 1$, c_j, $j = 1, \ldots, 4$, τ_0, τ_1 are strictly positive constants for all $s \in \mathbb{R}$ satisfying

$$c_1|s|^p \leq g_i(s) \leq c_2|s|^{\frac{1}{p}}, \quad \text{if } |s| \leq 1, \tag{2.2}$$

$$c_3|s| \leq g_i(s) \leq c_4|s|, \quad \text{if } = |s| > 1, \tag{2.3}$$

$$\text{exists } \tau_0, \quad \tau_1 > 0, \quad \tau_0 \leq g_i'(s) \leq \tau_1, \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

Now inspired by Komornik [7], we define the energy associated with the solution of system (1.1).

Lemma 2.1. *The energy associated with the solution of the problem* (1.1) *by the following formula*

$$E(t) = \frac{1}{2}\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + |\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx + \int_\Omega a(x)\Delta y_1 \Delta y_2 dx \tag{2.5}$$

is a nonnegative function and satisfies $E'(t) \leq 0$.

Proof. Multiplying the first equation in (1.1) by $-\Delta y_1'$ and the second equation by $-\Delta y_2'$, integrating over Ω using integration by part and Green's formula, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + |\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx + 2\int_\Omega a(x)\Delta y_1 \Delta y_2 \, dx\right] \\
&= -\int_\Omega \Delta y_1' g_1(\Delta y_1') + \Delta y_2' g_2(\Delta y_2') \, dx.
\end{aligned}$$

Using Hölder's inequality, Sobolev embedding and condition (2.1), we get

$$\begin{aligned}
\int_\Omega a(x)\Delta y_1 \Delta y_2 dx &\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)} \frac{\sqrt{c'}}{\sqrt{c'}} \int_\Omega |\Delta y_1 \Delta y_2| \, dx \\
&\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)} \int_\Omega \left(\frac{1}{c'}|\Delta y_1|^2 + c'|\Delta y_2|^2\right) dx \\
&\geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)} \int_\Omega \left(\frac{c'^2}{c'}|\nabla\Delta y_1|^2 + c'|\Delta y_2|^2\right) dx \\
&\geq -\frac{c'}{2}\|a\|_{L^\infty(\Omega)} \int_\Omega \left(|\nabla\Delta y_1|^2 + |\Delta y_2|^2\right) dx.
\end{aligned}$$

Then

$$E(t) \geq \frac{1}{2}\int_\Omega \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 + (1 - c'\|a\|_{L^\infty(\Omega)})(|\nabla\Delta y_1|^2 + |\Delta y_2|^2)\right) dx \geq 0.$$

Now, E is a nonnegative function

$$E'(t) = -\int_\Omega (\Delta y_1' g_1(\Delta y_1') + \Delta y_2' g_2(\Delta y_2'))\, dx. \tag{2.6}$$ □

3. Global Existence

In this section, we use the Faedo-Galerkin approximation to construct an approximate solutions of (1.1). We are now in the position to state our results.

Theorem 3.1. *Let* $(y_1^0, y_2^0) \in \widetilde{V}$ *and* $(y_1^1, y_2^1) \in V$*, arbitrarily. Assume that* (2.1) *and* (2.2)–(2.4) *hold. Then system* (1.1) *has a unique weak solution satisfying*

$$(y_1, y_2) \in L^\infty(\mathbb{R}_+, \widetilde{V}), \quad (y_1', y_2') \in L^\infty(\mathbb{R}_+, V)$$

and

$$(y_1'', y_2'') \in L^\infty(\mathbb{R}_+, W).$$

Proof. We use the Faedo-Galerkin method to prove the existence of global solutions. Let $T > 0$ be fixed and denoted by V^k the space generated by $\{w_i^1, w_i^2, \dots, w_i^k\}$, where the set $\{w_i^k,\ k \in \mathbb{N}\}$ is a basis of $\widetilde{V}$.

We construct approximate solution y_i^k, $k = 1, 2, 3, \dots$, in the form

$$y_i^k(x,t) = \sum_{j=1}^{k} c^{jk}(t) w_i^j(x),$$

where c^{jk}, $j = 1, 2, \dots, k$, are determined by the following ordinary differential equations

$$\begin{cases} (\ddot{y}_1^k + \Delta^2 y_1^k - a(x)\Delta y_2^k - g_1(\Delta \dot{y}_1^k), w_1^j) = 0, & \text{for all } w_1^j \in V^k, \\ (\ddot{y}_2^k - \Delta u_2^k - a(x)\Delta y_1^k - g_2(\Delta \dot{y}_2^k), w_2^j) = 0, & \text{for all } w_2^j \in V^k, \\ y_i^k(0) = y_i^{0k}, \quad \dot{y}_i^k(0) = y_i^{1k}, & x \in \Omega, i = 1, 2, \end{cases} \tag{3.1}$$

with initial conditions

$$y_1^k(0) = y_1^{0k} = \sum_{j=1}^{k} \langle y_1^0, w_1^j \rangle w_1^j \to y_1^0, \text{ in } H^4(\Omega) \cap H^3_\Delta(\Omega) \text{ as } k \to +\infty, \tag{3.2}$$

$$y_2^k(0) = y_2^{0k} = \sum_{j=1}^{k} \langle y_2^0, w_2^j \rangle w_2^j \to y_2^0, \quad \text{in } H^3_\Delta(\Omega) \cap H^2(\Omega) \text{ as } k \to +\infty, \tag{3.3}$$

$$\dot{y}_1^k(0) = y_1^{1k} = \sum_{j=1}^{k} \langle y_1^1, w_1^j \rangle w_1^j \to y_1^1, \quad \text{in } H^3_\Delta(\Omega) \cap H^2(\Omega) \text{ as } k \to +\infty, \tag{3.4}$$

$$\dot{y}_2^k(0) = y_2^{1k} = \sum_{j=1}^{k} \langle y_2^1, w_2^j \rangle w_2^j \to y_2^1, \quad \text{in } H^2(\Omega) \text{ as } k \to +\infty, \tag{3.5}$$

and

$$-\Delta^2 y_1^{0k} + a(x)\Delta y_2^{0k} + g_1(\Delta y_1^{1k}) \to -\Delta^2 y_1^0 + a(x)\Delta y_2^0 + g_1(\Delta y_1^1), \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty, \tag{3.6}$$

$$\Delta y_2^{0k} + a(x)\Delta y_1^{0k} + g_2(\Delta y_2^{1k}) \to \Delta y_2^0 + a(x)\Delta y_1^0 + g_2(\Delta y_2^1), \quad \text{in } H_0^1(\Omega) \text{ as } k \to +\infty. \tag{3.7}$$

By using some a priori estimates to show that $t_k = \infty$. Then, we show that the sequence of solutions to (3.1) converges to a solution of (1.1) with the claimed smoothness.

The first estimate. Taking $w_i^j = -2\Delta \dot{y}_i^k$ in (3.1), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(|\nabla \dot{y}_1^k|^2 + |\nabla \dot{y}_2^k|^2 + |\nabla\Delta y_1^k|^2 + |\Delta y_2^k|^2\right) dx + 2a(x)\Delta y_1^k \Delta y_2^k\, dx \\ &+ 2\int_\Omega \Delta \dot{y}_1^k g_1(\Delta \dot{y}_1^k)\, dx + 2\int_\Omega \Delta \dot{y}_2^k g_2(\Delta \dot{y}_2^k)\, dx = 0. \end{aligned} \tag{3.8}$$

Integrating it over $(0,t)$, we obtain

$$\begin{aligned} &\int_\Omega \left(|\nabla \dot{y}_1^k(t)|^2 + |\nabla \dot{y}_2^k(t)|^2\right) dx + \left(1 - c'\|a\|_{L^\infty(\Omega)}\right)\int_\Omega \left(|\nabla\Delta y_1^k(t)|^2 + |\Delta y_2^k(t)|^2\right) dx \\ &+ 2\int_0^t\int_\Omega \Delta \dot{y}_1^k(s) g_1(\Delta \dot{y}_1^k(s))\, dx\, ds + 2\int_0^t\int_\Omega \Delta \dot{y}_2^k(s) g_2(\Delta \dot{y}_2^k(s))\, dx\, ds \\ \le& A^k(0) \le C_1, \end{aligned} \tag{3.9}$$

where

$$A^k(0) = \int_\Omega \left(|\nabla \dot{y}_1^k(t)|^2 + |\nabla \dot{y}_2^k(t)|^2\right) dx + (1 + c'\|a\|_{L^\infty(\Omega)})\int_\Omega \left(|\nabla\Delta y_1^k(t)|^2 + |\Delta y_2^k(t)|^2\right) dx,$$

for some C_1 independent of k. These estimates imply that the solutions y_i^k exist globally in $]0,+\infty[$. Estimate (3.9) yields

$$y_1^k \text{ is bounded in } L^\infty(0,T; H^3_\Delta(\Omega)), \tag{3.10}$$

$$y_2^k \text{ is bounded in } L^\infty(0,T; H^2(\Omega)), \tag{3.11}$$

$$\dot{y}_1^k \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)), \tag{3.12}$$

$$\dot{y}_2^k \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)), \tag{3.13}$$

$$\Delta \dot{y}_i^k g_i(\Delta \dot{y}_i^k) \text{ is bounded in } L^1(\mathcal{A}), \tag{3.14}$$

where $\mathcal{A} = \Omega \times (0,T)$.

The second estimate. Taking $w_i^j = \Delta^2 \dot{y}_i^k$ in (3.1), implies

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(|\Delta \dot{y}_1^k|^2 + |\Delta \dot{y}_2^k|^2 + |\Delta^2 y_1^k|^2 + |\nabla\Delta y_2^k|^2 + 2a(x)\nabla\Delta y_1^k \nabla\Delta y_2^k\right) dx \\ &+ 2\int_\Omega \nabla a(x)\Delta y_2^k \nabla\Delta \dot{y}_1^k\, dx + 2\int_\Omega \nabla a(x)\Delta y_1^k \nabla\Delta \dot{y}_2^k\, dx \\ &+ 2\int_\Omega |\nabla\Delta \dot{y}_1^k|^2 g_1'(\Delta \dot{y}_1^k)\, dx + 2\int_\Omega |\nabla\Delta \dot{y}_2^k|^2 g_2'(\Delta \dot{y}_2^k)\, dx = 0 \end{aligned} \tag{3.15}$$

By Using Hölder's inequality and Sobolev embedding, (3.10) and condition (2.2), we have

$$\begin{aligned}\left|2\int_\Omega a(x)\nabla\Delta y_1^k\nabla\Delta y_2^k\,dx\right| &\le 2\|a\|\int_\Omega |\nabla\Delta y_1^k||\nabla\Delta y_2^k|\,dx\\ &\le 2\|a\|^2\int_\Omega |\nabla\Delta y_1^k|^2\,dx+\frac{1}{2}\int_\Omega|\nabla\Delta y_2^k|^2\,dx\\ &\le 2\|a\|^2C'+\frac{1}{2}\int_\Omega|\nabla\Delta y_2^k|^2\,dx\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}2\left|\int_\Omega \nabla a(x)\Delta y_2^k\nabla\Delta\dot y_1^k\,dx\right| &\le 2\int_\Omega|\nabla a(x)||\Delta y_2^k||\nabla\Delta\dot y_1^k|\,dx\\ &\le \frac{2}{\sqrt{\tau_0}}\int_\Omega|\nabla a(x)||\Delta y_2^k||\nabla\Delta\dot y_1^k|\sqrt{g_1'(\Delta\dot y_1^k)}\,dx\\ &\le \int_\Omega|\nabla\Delta\dot y_1^k|^2g_1'(\Delta\dot y_1^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2\int_\Omega|\Delta y_2^k|^2\,dx\\ &\le \int_\Omega|\nabla\Delta\dot y_1^k|^2g_1'(\Delta\dot y_1^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2C'.\end{aligned}\tag{3.17}$$

Similarly, we have

$$\begin{aligned}2\left|\int_\Omega \nabla a(x)\Delta y_1^k\nabla\Delta\dot y_2^k\,dx\right| &\le \int_\Omega|\nabla\Delta\dot y_2^k|^2g_2'(\Delta\dot y_2^k)\,dx+\frac{1}{\tau_0}\|\nabla a\|^2\int_\Omega|\Delta y_1^k|^2\,dx\\ &\le \int_\Omega|\nabla\Delta\dot y_2^k|^2g_2'(\Delta\dot y_2^k)\,dx+\frac{c'}{\tau_0}\|\nabla a\|^2\int_\Omega|\nabla\Delta y_1^k|^2\,dx\\ &\le \int_\Omega|\nabla\Delta\dot y_2^k|^2g_2'(\Delta\dot y_2^k)\,dx+\frac{c'}{\tau_0}\|\nabla a\|^2C'.\end{aligned}\tag{3.18}$$

Combining (3.16)–(3.18), into (3.15) and integrating over $(0,t)$, we obtain

$$\begin{aligned}&F^k(t)+\int_0^t\int_\Omega|\nabla\Delta\dot y_1^k(s)|^2g_1'(\Delta\dot y_1^k(s))\,dx\,dt+\int_0^t\int_\Omega|\nabla\Delta\dot y_2^k(s)|^2g_2'(\Delta\dot y_2^k(s))\,dx\,dt\\ &\le B^k(0)\le C_2,\quad \text{for all } t\in[0,t_k),\end{aligned}$$

where C_2 independent of k and

$$\begin{aligned}F^k(t)&=\int_\Omega\left(|\Delta\dot y_1^k|^2+|\Delta\dot y_2^k|^2+|\Delta^2y_1^k|^2\right)dx+\frac{1}{2}\int_\Omega|\nabla\Delta y_2^k|^2\,dx,\\ B^k(0)&=\int_\Omega\left(|\Delta y_1^{1k}|^2+|\Delta y_2^{1k}|^2+|\Delta^2y_1^{0k}|^2\right)dx+2\|a\|^2C'+\frac{1}{2}\int_\Omega|\nabla\Delta y_2^{0k}|^2\,dx\\ &\quad+\left(\frac{c'}{\tau_0}\|\nabla a\|^2C'+\frac{c'}{\tau_0}\|\nabla a\|^2C'\right)T.\end{aligned}$$

Therefore, we conclude that

$$y_1^k \text{ is bounded in } L^\infty(0,T;H^4(\Omega)),\tag{3.19}$$

$$y_2^k \text{ is bounded in } L^\infty(0,T;H^3_\Delta(\Omega)),\tag{3.20}$$

$$\dot y_1^k \text{ is bounded in } L^\infty(0,T;H^2(\Omega)),\tag{3.21}$$

$$\dot y_2^k \text{ is bounded in } L^\infty(0,T;H^2(\Omega)).\tag{3.22}$$

The third estimate. Assume that $t < T$ and let $0 < \xi < T - t$ and

$$y_i^{k\xi}(x,t) = y_i^k(x, t+\xi), \quad i = 1, 2.$$

So, $U_1^{k,\xi}(x,t) = y_1^k(x,t+\xi) - y_1^k(x,t)$, solves the differential equation

$$\left(\ddot{U}_1^{k,\xi} + \Delta^2 U_1^{k,\xi} - a(x)\Delta y_2^{k,\xi} - (g_1(\Delta \dot{y}_1^{k\xi}) - g_1(\Delta \dot{y}_1^k)), w_1^j\right) = 0, \quad \text{for all } w_1^j \in V^k, \tag{3.23}$$

and the set

$$U_2^{k,\xi}(x,t) = y_2^k(x,t+\xi) - y_2^k(x,t).$$

$U_2^{k,\xi}$ solves the differential equation

$$\left(\ddot{U}_2^{k,\xi} - \Delta U_2^{k,\xi} - a(x)\Delta U_1^{k,\xi} - (g_2(\Delta \dot{y}_2^{k\xi}) - g_2(\Delta \dot{y}_2^k)), w_2^j\right) = 0, \quad \text{for all } w_2^j \in V^k. \tag{3.24}$$

Choosing $w_1^j = -\Delta \dot{y_1}^{k\xi}$ in (3.23) and $w_2^j = \Delta \dot{U_2}^{k\xi}$ in (3.24), and using the fact that g_i is nondecreasing, we obtain

$$\frac{d}{dt}\int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(x,t)|^2 + |\nabla \dot{U_2}^{k\xi}(x,t)|^2 + |\nabla \Delta U_1^{k\xi}(x,t)|^2 + |\Delta U_2^{k\xi}(x,t)|^2\right) dx$$
$$+ 2\frac{d}{dt}\int_\Omega a(x)\Delta U_2^{k\xi}(x,t)\Delta U_1^{k\xi}(x,t)\, dx \le 0, \quad \text{for all } t \ge 0.$$

Integrating over $[0,t]$, we get

$$\int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(t)|^2 + |\nabla \dot{U_2}^{k\xi}(t)|^2\right) dx + (1 - c'\|a\|)\int_\Omega \left(|\nabla \Delta U_1^{k\xi}(t)|^2 + |\Delta U_2^{k\xi}(t)|^2\right) dx$$
$$\le C_2 \int_\Omega \left(|\nabla \dot{U_1}^{k\xi}(0)|^2 + |\nabla \dot{U_2}^{k\xi}(0)|^2 + |\nabla \Delta U_1^{k\xi}(0)|^2 + |\Delta U_2^{k\xi}(0)|^2\right) dx,$$

where C_2 is a positive constant depending only on $\|a\|$ and c'. By dividing by ξ^2, and pass to the limit when $\xi \to 0$, we have

$$\int_\Omega \left(|\nabla \ddot{y}_1^k(t)|^2 + |\nabla \ddot{y}_2^k(t)|^2 + |\nabla \Delta \dot{y}_1^k(t)|^2 + |\Delta \dot{y}_2^k(t)|^2\right) dx$$
$$\le C_2' \int_\Omega \left(|\nabla \ddot{y}_1^k(0)|^2 + |\nabla \ddot{y}_2^k(0)|^2 + |\nabla \Delta y_1^{1k}|^2 + |\Delta y_2^{1k}|^2\right) dx.$$

Now we estimate $\|\nabla \ddot{y}_i^k(0)\|$. Choosing $v = -\Delta \ddot{y}_i^k$ in (3.1) and substitute $t = 0$, we obtain

$$\|\nabla \ddot{y}_1^k(0)\|^2 = \int_\Omega \nabla \ddot{y}_1^k(0)\nabla\left(-\Delta^2 y_1^{0k} - a(x)y_2^{0k} + g_1(\Delta y_1^{1k})\right) dx$$

and

$$\|\nabla \ddot{y}_2^k(0)\|^2 = \int_\Omega \nabla \ddot{y}_2^k(0)\nabla\left(\Delta y_2^{0k} - a(x)y_1^{0k} + g_2(\Delta y_2^{1k})\right) dx.$$

By Cauchy-Schwarz inequality, we obtain

$$\|\nabla \ddot{y}_1^k(0)\| \le \left(\int_\Omega \left|\nabla\left(-\Delta^2 y_1^{0k} - a(x)y_2^{0k} + g_1(\Delta y_1^{1k})\right)\right|^2 dx\right)^{\frac{1}{2}}$$

and

$$\|\nabla \ddot{y}_2^k(0)\| \le \left(\int_\Omega \left|\nabla\left(\Delta y_2^{0k} - a(x)y_1^{0k} + g_2(\Delta y_2^{1k})\right)\right|^2 dx\right)^{\frac{1}{2}}.$$

(3.6) and (3.7) yields

$$(\ddot{y}_1^k(0), \ddot{y}_2^k(0)) \text{ are bounded in } W \times W. \tag{3.25}$$

And by (3.4), (3.5) and (3.25) we deduce

$$\int_\Omega \Big(|\nabla \ddot{y}_1^k(t)|^2 + |\nabla \ddot{y}_2^k(t)|^2 + |\nabla\Delta \dot{y}_1^k(t)|^2 + |\Delta \dot{y}_2^k(t)|^2\Big)\, dx \le C_3, \quad \text{for all } t \ge 0,$$

where C_3 is a positive constant independent of $k \in \mathbb{N}$. Therefore, we deduce

$$\dot{y}_1^k \text{ is bounded in } L^\infty(0,T; H^3_\Delta(\Omega)), \tag{3.26}$$

$$\dot{y}_2^k \text{ is bounded in } L^\infty(0,T; H^2(\Omega)), \tag{3.27}$$

$$\ddot{y}_1^k \text{ is bounded in } L^\infty(0,T; H^1_0(\Omega)), \tag{3.28}$$

$$\ddot{y}_2^k \text{ is bounded in } L^\infty(0,T; H^1_0(\Omega)). \tag{3.29}$$

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (3.10)–(3.14), (3.19)–(3.22) and (3.26)–(3.29) that there exists a subsequence $\{y_i^m\}$ of $\{y_i^k\}$ such that

$$(y_1^m, y_2^m) \rightharpoonup (y_1, y_2), \quad \text{weak-star in } L^\infty(0,T;\widetilde{V}), \tag{3.30}$$

$$(\dot{y}_1^m, \dot{y}_2^m) \rightharpoonup (y_1', y_2'), \quad \text{weak-star in } L^\infty(0,T;V), \tag{3.31}$$

$$(\ddot{y}_1^m, \ddot{y}_2^m) \rightharpoonup (y_1'', y_2''), \quad \text{weak-star in } L^\infty(0,T;W), \tag{3.32}$$

$$(\dot{y}_1^m, \dot{y}_2^m) \rightarrow (y_1', y_2'), \quad \text{almost everywhere in } \Omega \times [0,+\infty), \tag{3.33}$$

$$g_i(\Delta \dot{y}_i^m) \rightharpoonup \chi_i, \quad \text{weak-star in } L^2(\mathcal{A}). \tag{3.34}$$

As (y_1^m, y_2^m) is bounded in $L^\infty(0,T;\widetilde{V})$ by (3.30) and the injection of $\widetilde{V}$ in H is compact, we have

$$(y_1^m, y_2^m) \rightarrow (y_1, y_2), \quad \text{strong in } L^2(0,T;H). \tag{3.35}$$

On the other hand, using (3.30), (3.32) and (3.35), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(\ddot{y}_1^m(x,t) + \Delta^2 y_1^m(x,t) - a(x)\Delta y_2^m(x,t)\Big)\, w\, dxdt \\
\rightarrow &\int_0^T \int_\Omega \Big(y_1''(x,t) + \Delta^2 y_1(x,t) - a(x)\Delta y_2(x,t)\Big)\, w\, dxdt
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(\ddot{y}_2^m(x,t) - \Delta y_2^m(x,t) - a(x)\Delta y_1^m(x,t)\Big) w\, dxdt \\
\rightarrow &\int_0^T \int_\Omega \Big(y_2''(x,t) - \Delta y_2(x,t) - a(x)\Delta y_1(x,t)\Big) w\, dxdt,
\end{aligned} \tag{3.37}$$

for all $w \in L^2(0,T;L^2(\Omega))$.

It remains to prove the convergence

$$\int_0^T \int_\Omega g_i(\Delta \dot{y}_i^m)\; w\, dxdt \rightarrow \int_0^T \int_\Omega g_i(\Delta y_i')\; w\, dxdt,$$

when $m \rightarrow +\infty$. To finish the proof we shall use the following lemma.

Lemma 3.1. *Let $g_i(\Delta y_i') \in L^1(\mathcal{A})$ and $\|g_i(\Delta y_i')\|_{L^1(\mathcal{A})} \le K$, where K is a constant independent of t. Then $g_i(\Delta \dot{y}_i^m) \rightarrow g_i(\Delta y_i')$ in $L^1(\mathcal{A})$.*

Proof. Let $g(\Delta y') \in L^1(\mathcal{A})$. Since g_i is continuous, we deduce from (3.33)

$$g_i(\Delta \dot{y}_i^k) \to g_i(\Delta y_i'), \quad \text{almost everywhere in } \mathcal{A}, \tag{3.38}$$
$$\Delta \dot{y}_i^m g_i(\Delta \dot{y}_i^m) \to \Delta y_i' g_i(\Delta y_i'), \quad \text{almost everywhere in } \mathcal{A}.$$

Also, by (3.13) and Fatou's lemma, we have

$$\int_0^T \int_\Omega \Delta y_i'(x,t) g_i(\Delta y_i'(x,t))\, dxdt \le K_1, \quad \text{for } T > 0. \tag{3.39}$$

Now, we can estimate $\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dxdt$. By using Cauchy-Schwarz inequality and (2.3), we have the following.

1. If $|\Delta y_i'| \ge 1$, then

$$\begin{aligned}\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dx\, dt &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^2\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega \Delta y_i' g_i(\Delta y_i'(x,t))\, dx\, dt \right)^{1/2} \\ &\le K_2.\end{aligned}$$

2. If $|\Delta y_i'| < 1$, then

$$\begin{aligned}\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dx\, dt &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^2\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{1/2} \left(\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|^{\frac{2}{p+1}}\, dx\, dt \right)^{1/2} \\ &\le c|\mathcal{A}|^{(3p+1)/2(p+1)} \left(\int_0^T \int_\Omega \Delta y_i' g_i(\Delta y_i'(x,t))\, dx\, dt \right)^{1/(p+1)} \\ &\le K_3, \quad \text{for } T > 0.\end{aligned}$$

Then

$$\int_0^T \int_\Omega |g_i(\Delta y_i'(x,t))|\, dxdt \le K, \quad \text{for } T > 0.$$

And let $E \subset \Omega \times [0,T]$ and $|E|$ is the measure of E and the set

$$E_1 = \left\{ (x,t) \in E : |g_i(\Delta \dot{y}_i^m(x,t))| \le \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \backslash E_1.$$

If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g_i(s)| \ge r\}$, then

$$\int_E |g_i(\Delta \dot{y}_i^m)|\, dxdt \le c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\Delta \dot{y}_i^m g_i(\Delta \dot{y}_i^m)|\, dxdt.$$

By applying (3.13), we deduce

$$\sup_m \int_E g_i(\Delta \dot{y}_i^m)\, dxdt \to 0, \quad \text{when } |E| \to 0.$$

From Vitali's convergence theory, we deduce

$$g_i(\Delta \dot{y}_i^m) \to g_i(\Delta y_i'), \quad \text{in } L^1(\mathcal{A}).$$

Proof of lemma is completed. □

End of proof of Theorem 3.1**.** Now (3.34) implies that

$$g_i(\Delta \dot{y}_i^m) \rightharpoonup g_i(\Delta y_i'), \quad \text{weak-star in } L^2([0,T]\times\Omega).$$

We deduce, for all $v \in L^2([0,T]\times L^2(\Omega)$, that

$$\int_0^T \int_\Omega g_i(\Delta \dot{y}_i^m) w \, dxdt \to \int_0^T \int_\Omega g_i(\Delta y_i') w \, dxdt.$$

Finally, for all $w \in L^2([0,T]\times L^2(\Omega))$:

$$\int_0^T \int_\Omega \Big(y_1''(x,t) + \Delta^2 y_1(x,t) - a(x)\Delta y_2(x,t) - g_1(\Delta y_1'(x,t))\Big) w \, dxdt = 0$$

and

$$\int_0^T \int_\Omega \Big(y_2''(x,t) - \Delta y_2(x,t) - a(x)\Delta y_1(x,t) - g_2(\Delta y_2'(x,t))\Big) w \, dxdt = 0.$$

Therefore, (y_1, y_2) are a solutions for the problem (1.1).

This concludes the proof of Theorem 3.1. □

4. Asymptotic Behavior

In this section, we prove stability result for the energy of the solution of system (1.1), by using the multiplier technique.

Theorem 4.1. *Let* $(y_1^0, y_2^0) \in \widetilde{V}$ *and* $(y_1^1, y_2^1) \in V$*. Assume that* (2.1)–(2.4) *hold. The energy of system* (1.1)*, given by* (2.5) *decay estimate:*

$$E(t) \le C t^{-2/(p-1)}, \quad \textit{for all } t > 0 \textit{ if } p > 1, \tag{4.1}$$

and

$$E(t) \le C' E(0) e^{-wt}, \quad \textit{for all } t > 0 \textit{ if } p = 1, \tag{4.2}$$

where C is a positive constant only depending on $E(0)$ and C', w are positive constants independent of the initial data.

Proof. This proof is established in two steps.

Step 1. Multiplying the first equation of (1.1) by $-E^\mu \Delta y_1$, we obtain

$$\begin{aligned}
0 &= \int_S^T -E^\mu \int_\Omega \Delta y_1 \Big(y_1'' + \Delta^2 y_1 - a(x)\Delta y_2 + g_1(\Delta y_1')\Big) dxdt \\
&= -\Big[E^\mu \int_\Omega y_1' \Delta y_1 \, dx\Big]_S^T + \mu \int_S^T E' E^{\mu-1} \int_\Omega \Delta y_1 y_1' \, dxdt \\
&\quad - 2\int_S^T E^\mu \int_\Omega |\nabla y_1'|^2 \, dxdt + \int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla \Delta y_1|^2\Big) \, dxdt \\
&\quad + \int_S^T E^\mu \int_\Omega a(x)\Delta y_1 \Delta y_2 \, dxdt + \int_S^T E^\mu \int_\Omega \Delta y_1 g_1(\Delta y_1') \, dxdt.
\end{aligned}$$

Step 2. Multiplying the second equation of (1.1) by $-E^\mu \Delta y_2$, we obtain

$$\begin{aligned}
0 = & \int_S^T -E^\mu \int_\Omega \Delta y_2 \Big(y_2'' + \Delta y_2 - a(x)\Delta y_1 + g_2(\Delta y_2')\Big)\, dxdt \\
= & -\Big[E^\mu \int_\Omega y_2' \Delta y_2 \, dx\Big]_S^T + \mu \int_S^T E' E^{\mu-1} \int_\Omega \Delta y_2 y_2' \, dxdt \\
& - 2\int_S^T E^\mu \int_\Omega |\nabla y_2'|^2 \, dxdt + \int_S^T E^\mu \int_\Omega \Big(|\nabla y_2'|^2 + |\Delta y_2|^2) \Big)\, dxdt \\
& + \int_S^T E^\mu \int_\Omega a(x)\Delta y_2 \Delta y_1 \, dxdt + \int_S^T E^\mu \int_\Omega \Delta y_2 g_2(\Delta y_2')\, dxdt.
\end{aligned}$$

By their sum, we obtain

$$\begin{aligned}
\int_S^T E^{\mu+1}\, dt \le & \Big[E^\mu \int_\Omega \Big(y_1' \Delta y_1 + y_2' \Delta y_2 \Big) dx \Big]_S^T \\
& - \mu \int_S^T E' E^{\mu-1} \int_\Omega \Big(\Delta y_1 y_1' + \Delta y_2 y_2' \Big) dxdt \\
& + 2 \int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2 \Big) dxdt \\
& - \int_S^T E^\mu \int_\Omega \Big(\Delta y_1 g_1(\Delta y_1') + \Delta y_2 g_2(\Delta y_2') \Big) dxdt.
\end{aligned} \tag{4.3}$$

Since E is non-increasing, we find that

$$\begin{aligned}
\Big[E^\mu \int_\Omega (y_1' \Delta y_1 + y_2' \Delta y_2)\, dx \Big]_S^T \le & c E^{\mu+1}(S), \\
\mu \Big| \int_S^T E' E^{\mu-1} \int_\Omega (\Delta y_1 y_1' + \Delta y_2 y_2')\, dxdt \Big| \le & c E^{\mu+1}(S).
\end{aligned}$$

Using these estimates, we conclude from (4.3) that

$$\begin{aligned}
\int_S^T E^{\mu+1}\, dt \le & C E^{\mu+1}(S) + 2\int_S^T E^\mu \int_\Omega \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2 \Big)\, dxdt \\
& + \int_S^T E^\mu \int_\Omega \Big(|\Delta y_1| |g_1(\Delta y_1')| + |\Delta y_2| |g_2(\Delta y_2')| \Big) dxdt.
\end{aligned} \tag{4.4}$$

Now, we estimate the terms of the right-hand side of the inequality (4.4), see Komornik [7].

We consider the following partition of Ω

$$\Omega^+ = \{x \in \Omega : |\Delta y_i'| \ge 1\}, \quad \Omega^- = \{x \in \Omega : |\Delta y_i'| < 1\}.$$

By using Sobolev embedding and Young's inequality, we obtain

(4.5)

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} |\Delta y_1||g_1(\Delta y_1')| \, dxdt + \int_S^T E^\mu \int_{\Omega^+} |\nabla y_1'|^2 \, dxdt \\
\leq& \varepsilon \int_S^T E^\mu \int_{\Omega^+} |\Delta y_1|^2 \, dxdt + C(\varepsilon) \int_S^T E^\mu \int_{\Omega^+} |g_1(\Delta y_1')|^2 \, dx \, dt + c \int_S^T E^\mu \int_{\Omega^+} |\Delta y_1'|^2 \\
\leq& \varepsilon c' \int_S^T E^\mu \int_\Omega |\nabla \Delta y_1|^2 \, dxdt + \left(C(\varepsilon) c_2 + \frac{c}{c_1} \right) \int_S^T E^\mu \int_\Omega \Delta y_1' g_1(\Delta y_1') \, dxdt \\
\leq& \varepsilon C \int_S^T E^{\mu+1} \, dt + C_1(\varepsilon) \int_S^T E^\mu (-E') \, dt \\
\leq& \varepsilon C \int_S^T E^{\mu+1} \, dt + C_1(\varepsilon, \mu) E^{\mu+1}(S).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} |\Delta y_2||g_2(\Delta y_2')| \, dxdt + \int_S^T E^\mu \int_{\Omega^+} |\nabla y_2'|^2 \, dxdt \\
\leq& \varepsilon C \int_S^T E^{\mu+1} \, dt + C_2(\varepsilon, \mu) E^{\mu+1}(S).
\end{aligned} \tag{4.6}$$

Summing (4.5) and (4.6), we obtain

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^+} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\Delta y_2||g_2(\Delta y_2')| \Big) dxdt \\
&+ \int_S^T E^\mu \int_{\Omega^+} \left(|\nabla y_1'|^2 + |\nabla y_2'|^2 \right) dxdt \\
\leq& \varepsilon C \int_S^T E^{\mu+1} \, dt + C(\varepsilon, \mu) E^{\mu+1}(S)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\nabla y_1'|^2 \Big) dxdt \\
\leq& \varepsilon' c' \int_S^T E^\mu \int_\Omega |\nabla \Delta y_1|^2 \, dxdt + C(\varepsilon') \int_S^T E^\mu \int_\Omega \Big(|\Delta y_1'|^2 + |g_1(\Delta y_1')|^2 \Big) dxdt \\
\leq& \varepsilon' c' \int_S^T E^{\mu+1} \, dt + C(\varepsilon') \int_S^T E^\mu \int_\Omega (\Delta y_1' g_1(\Delta y_1'))^{\frac{2}{p+1}} \, dxdt \\
\leq& \varepsilon' C \int_S^T E^{\mu+1} \, dt + C(\varepsilon', p) \int_S^T E^\mu \left(\int_\Omega \Delta y_1' g_1(\Delta y_1') \, dx \right)^{\frac{2}{p+1}} \, dt.
\end{aligned} \tag{4.8}$$

Similarly, we have

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_2||g_2(\Delta y_2')| + |\nabla y_2'|^2 \Big) dxdt \\
\leq& \varepsilon' C \int_S^T E^{\mu+1} \, dt + C(\varepsilon', p) \int_S^T E^\mu \left(\int_\Omega \Delta y_2' g_2(\Delta y_2') \, dx \right)^{\frac{2}{p+1}} \, dt.
\end{aligned} \tag{4.9}$$

Summing (4.8) and (4.9), we obtain

$$\begin{aligned}
&\int_S^T E^\mu \int_{\Omega^-} \Big(|\Delta y_1||g_1(\Delta y_1')| + |\Delta y_2||g_2(\Delta y_2')|\Big)dxdt \\
&+ \int_S^T E^\mu \int_{\Omega^-} \Big(|\nabla y_1'|^2 + |\nabla y_2'|^2\Big)dxdt \\
\leq & \varepsilon_0 C \int_S^T E^{\mu+1}\, dt + C(\varepsilon_0, p) \int_S^T E^\mu \,(-E')^{\frac{2}{p+1}}\, dt \\
&\leq \varepsilon_0 C \int_S^T E^{\mu+1}\, dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}}\, dt + C\,(\varepsilon_1, p)\, E(S).
\end{aligned} \tag{4.10}$$

Comblining (4.7) and (4.10) in (4.4), we find

$$\int_\Omega E^{\mu+1}\, dt \leq CE(S) + C'E^{\mu+1}(S) + \varepsilon C \int_S^T E^{\mu+1}\, dt + \varepsilon_1 \int_S^T E^{\mu\frac{p+1}{p-1}}\, dt.$$

We choose μ such that $\mu\frac{p+1}{p-1} = \mu+1$, so, $\mu = \frac{p-1}{2}$, and choosing ε and ε_1 small enough, we obtain

$$\int_\Omega E^{\mu+1}\, dt \leq C'E(S) + C'E^\mu(0)E(S),$$

where C' is positive constant independent of $E(0)$. Hence, the estimates (4.1) and (4.2) follow by applying the following result of Martinez.

Lemma 4.1. *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and assume that there are two constants $\mu \geq 0$, $\omega > 0$ such that*

$$\int_t^{+\infty} E(s)^{\mu+1}\, ds \leq \omega E(0)^\mu E(t), \quad \textit{for all } t \geq 0.$$

Then, we have for every $t > 0$

$$\begin{cases} E(t) \leq E(0)\left(\dfrac{1+\mu}{1+\omega\mu t}\right)^{-\frac{1}{\mu}}, & \textit{if } \mu > 0, \\ E(t) \leq E(0)e^{1-\omega t}, & \textit{if } \mu = 0. \end{cases}$$

For a short proof of this lemma we refer to [12].

This completes the proof of Theorem 4.1. □

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[1]Faculty of Technology,
Amaar Tlaidji University,
P.O. Box 37, Laghouat 03000, Algeria
Email address: saadaouik16@gmail.com
ORCID iD: https://orcid.org/0000-0002-0906-7186

[2]Department of Mathematics,
Faculty of Exact Sciences,
Mascara Mustapha Stambouli University,
P.O. Box 305, Mascara 29000, Algeria
Email address: mounir.bahlil@univ-mascara.dz
ORCID iD: https://orcid.org/0000-0002-9688-8897
Email address: bahlilmounir@yahoo.fr

[3]Laboratory of Analysis and Control of Partial Differential Equations,
University Djillali Liabes,
PO Box 89, Sidi Bel Abbes, 22000, Algeria
Email address: abdelli.mama@gmail.com
ORCID iD: https://orcid.org/0000-0003-2641-5223

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AN APPROACH TO LAGRANGE'S THEOREM IN PYTHAGOREAN FUZZY SUBGROUPS

SUPRIYA BHUNIA[1] AND GANESH GHORAI[1]

Abstract. The Pythagorean fuzzy environment is a modern way of depicting uncertainty. The concept of Pythagorean fuzzy semi-level subgroups of any group is described in this paper. The Pythagorean fuzzy order of an element in a Pythagorean fuzzy subgroup is introduced and established various algebraic attributes. The relation between the Pythagorean fuzzy order of an element of a group and the order of that group is established. The Pythagorean fuzzy normalizer and Pythagorean fuzzy centralizer of Pythagorean fuzzy subgroups are discussed. Further, the concept of Pythagorean fuzzy quotient group and the index of a Pythagorean fuzzy subgroup are defined. Finally, a framework is developed for proving Lagrange's theorem in Pythagorean fuzzy subgroups.

1. Introduction

One of the most important theorems in Abstract algebra is Lagrange's theorem. This theorem is very crucial in case of finite groups because it provides an overview of subgroup size. Lagrange's theorem has various applications in number theory. For further details, we refer to [16].

Uncertainty is an unavoidable element of our lives. This universe isn't built on assumptions or precise measures. It is not always feasible to make straightforward decisions. We face a significant problem in dealing with errors in decision-making situations. In 1965, Zadeh [19] proposed the fuzzy set as a way to deal with ambiguity in real-world problems. Following that, fuzzy sets become a worldwide study trend. Rosenfeld [15] was the first to examine the concept of fuzzy subgroup and its features

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in 1971. The concepts of fuzzy coset and fuzzy normal subgroup were introduced by Ajmal and Prajapati [2]. Dixit et al. [10] addressed fuzzy level subgroups and the union of fuzzy subgroups in 1990. Biswas[8] was the first to suggest the concept of an anti-fuzzy subgroup. The concepts of fuzzy normal subgroup, fuzzy coset, and fuzzy quotient subgroup were presented by Ajmal and Prajapati [2] in 1992. Chakraborty and Khare [9] investigated a variety of fuzzy homomorphism features. Ajmal [3] also looked into homomorphisms of fuzzy subgroups. Kim [11] established the order of fuzzy subgroups and fuzzy p-subgroups in 1994. In 1999, Ray [14] proposed the product of fuzzy subgroups. Many researchers have been studying the features of fuzzy groups in recent years. In 2015, Tarnauceanu [17] developed fuzzy normal subgroups of finite groups. Addis [1] proposed fuzzy homomorphism theorems for groups in 2018. In 2021, Bhunia [5] and Ghorai [7] presented the concept of (α, β)-Pythagorean fuzzy sets and characterized (α, β)-Pythagorean fuzzy subgroups.

When it comes to decision-making, assigning membership values isn't always adequate. Atanassov [4] established the intuitionistic fuzzy set in 1986 by attributing non-membership degrees to membership degrees. Yager [18] defined Pythagorean fuzzy set (PFS) in 2013 using this approach. In comparison to intuitionistic fuzzy sets, this set provides a modern technique to model vagueness and uncertainty with high precision and accuracy. Peng [13] and Yang presented some results relating to it. Bhunia et al. [6] started exploring Pythagorean fuzzy subgroups (PFSG) in 2021. Pythagorean fuzzy subgroup was shown to be a larger class of Intuitionistic fuzzy subgroup. The major goal of this study is to prove Lagrange's theorem in Pythagorean fuzzy subgroups. This article is designed in such a way that we can approach Lagrange's theorem.

This paper's outline is as follows: in Section 2, we review several key definitions and ideas. In Section 3, we define Pythagorean fuzzy order of elements of groups and go over some of its features. Section 4 discusses the algebraic properties of the Pythagorean fuzzy subgroup. We introduce the concept of a Pythagorean fuzzy quotient group and prove Lagrange's theorem in Section 5. In Section 6, we come to a conclusion.

2. Preliminaries

This section covers some definitions and concepts that are crucial for the development of subsequent sections.

Definition 2.1 ([18])**.** A PFS ψ on a set C is defined by $\psi = \{(m, \mu(m), \nu(m)) \mid m \in C\}$ where $\mu(m) \in [0,1]$ and $\nu(m) \in [0,1]$ are the degree of membership and non membership of $m \in C$, respectively, which fulfill the condition $0 \leq \mu^2(m) + \nu^2(m) \leq 1$ for all $m \in C$.

PFS will be denoted as $\psi = (\mu, \nu)$ rather than $\psi = \{(m, \mu(m), \nu(m)) \mid m \in C\}$.

Definition 2.2 ([6])**.** Let $\psi = (\mu, \nu)$ be a PFS on a group $(C, \circ)$. Then ψ is a PFSG of C if:

(i) $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$;
(ii) $\mu^2(m^{-1}) \geq \mu^2(m)$ and $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m \in C$.

Here, $\mu^2(m) = \{\mu(m)\}^2$ and $\nu^2(m) = \{\nu(m)\}^2$ for all $m \in C$.

Proposition 2.1 ([6]). *Let $\psi = (\mu, \nu)$ be a PFS on a group $(C, \circ)$. Then ψ is a PFSG of $(C, \circ)$ if and only if $\mu^2(m \circ n^{-1}) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n^{-1}) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$.*

Definition 2.3 ([6]). Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then for $m \in C$, the PFLC of ψ is the PFS $m\psi = (m\mu, m\nu)$, defined by $(m\mu)^2(u) = \mu^2(m^{-1} \circ u)$, $(m\nu)^2(u) = \nu^2(m^{-1} \circ u)$ and the PFRC of ψ is the PFS $\psi m = (\mu m, \nu m)$, defined by $(\mu m)^2(u) = \mu^2(u \circ m^{-1})$, $(\nu m)^2(u) = \nu^2(u \circ m^{-1})$ for all $u \in C$.

Definition 2.4 ([6]). Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG on the group $(C, \circ)$ if every PFLC of ψ is a PFRC of ψ on C.

Equivalently, $m\psi = \psi m$ for all $m \in C$.

Proposition 2.2 ([6]). *Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG on C if and only if $\mu^2(m \circ n) = \mu^2(n \circ m)$ and $\nu^2(m \circ n) = \nu^2(n \circ m)$ for all $m, n \in C$.*

Proposition 2.3 ([6]). *Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG of C if and only if $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ for all $u, k \in C$.*

3. Pythagorean Fuzzy Order of Elements in PFSG

This section establishes the Pythagorean fuzzy order of elements in PFSGs and introduce the concept of Pythagorean fuzzy semi-level subgroups of any group. We also compare the fuzzy order of elements in fuzzy subgroups with the Pythagorean fuzzy order of elements in PFSGs. We also go over some of the algebraic features of Pythagorean fuzzy order of elements in PFSGs.

Theorem 3.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$ is a subgroup of C.*

Proof. We have $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$, where $m \in C$. So, $\Gamma(m) \subset C$ as $m \in \Gamma(m)$. Also, $e \in \Gamma(m)$ as $\mu^2(e) \geq \mu^2(m)$ and $\nu^2(e) \leq \nu^2(m)$. Let $p, q \in \Gamma(m)$. Then

$$\mu^2(pq^{-1}) \geq \mu^2(p) \wedge \mu^2(q^{-1}) = \mu^2(p) \wedge \mu^2(q) \geq \mu^2(m).$$

In the same way, we can prove that $\nu^2(pq^{-1}) \leq \nu^2(m)$. Thus, $pq^{-1} \in \Gamma(m)$. Therefore, $\Gamma(m)$ is a subgroup of C. □

Definition 3.1. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then the subgroup $\Gamma(m)$ is a Pythagorean fuzzy semi-level subgroup of C corresponding to m.

Definition 3.2. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then the Pythagorean fuzzy order (PFO) of m in ψ is denoted by $PFO(m)_\psi$ and defined by the order of the Pythagorean fuzzy semi-level subgroup of m in C.

Therefore, $PFO(m)_\psi = O(\Gamma(m))$ for all $m \in C$.

Example 3.1. Consider the group $(\mathbb{Z}_4, +_4)$.

Assign membership and non-membership degree of the elements of $\mathbb{Z}_4$ by

$$\mu(0) = 0.95, \quad \mu(1) = 0.65, \quad \mu(2) = 0.65, \quad \mu(3) = 0.85,$$
$$\nu(0) = 0.25, \quad \nu(1) = 0.75, \quad \nu(2) = 0.75, \quad \nu(3) = 0.45.$$

Clearly, $\psi = (\mu, \nu)$ is a PFSG on $\mathbb{Z}_4$. Then PFO of the elements of $\mathbb{Z}_4$ in ψ is presented by

$$PFO(0)_\psi = O(\Gamma(0)) = 2, \quad PFO(1)_\psi = O(\Gamma(1)) = 4,$$
$$PFO(2)_\psi = O(\Gamma(2)) = 4, \quad PFO(3)_\psi = O(\Gamma(3)) = 2.$$

From above example, we see that $PFO(0)_\psi \neq O(0)$ and $PFO(0)_\psi = PFO(3)_\psi = 2$.

Remark 3.1. The PFO of an element in PFSG may not always be same to the element's order in the group.

Proposition 3.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $PFO(e)_\psi \leq PFO(m)_\psi$ for all $m \in C$, where e is group's identity.*

Proof. Let $PFO(e)_\psi = s$, where $s \in \mathbb{Z}^+$. Assume that $\Gamma(e) = \{m_1, m_2, \dots, m_s\}$, where $m_i \neq m_j$ for all i, j.

Then $\mu^2(m_1) = \mu^2(m_2) = \cdots = \mu^2(m_s) = \mu^2(e)$ and $\nu^2(m_1) = \nu^2(m_2) = \cdots = \nu^2(m_s) = \nu^2(e)$.

As $\psi = (\mu, \nu)$ is a PFSG on C, $\mu^2(e) \geq \mu^2(m)$ and $\nu^2(e) \leq \nu^2(m)$ for all $m \in C$. So, $m_1, m_2, \dots, m_s \in \Gamma(u)$. Then $\Gamma(e) \subseteq \Gamma(u)$. Thus, $O(\Gamma(e)) \leq O(\Gamma(m))$ for all $m \in C$. Therefore, $PFO(e)_\psi \leq PFO(m)_\psi$ for all $m \in C$. □

The next result represents a relation between the order and PFO of an element in a group.

Theorem 3.2. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $O(m)$ divides $PFO(m)_\psi$ for all $m \in C$.*

Proof. Let $m \in C$ and $O(m) = k$, where $k \in \mathbb{Z}^+$. Then $m^k = e$. Consider $D = \langle m \rangle$ as a subgroup of C.

Now, $\mu^2(m^2) \geq \mu^2(m) \wedge \mu^2(m) = \mu^2(m)$ and $\nu^2(m^2) \leq \nu^2(m) \vee \nu^2(m) = \nu^2(m)$. Therefore, by induction, $\mu^2(m^p) \geq \mu^2(m)$ and $\nu^2(m^p) \leq \nu^2(m)$ for all $p \in \mathbb{Z}^+$.

So, $m, m^2, \dots, m^k \in \Gamma(m)$. Consequently, $D \subseteq \Gamma(m)$. Therefore, D is a subgroup of $\Gamma(m)$.

Thus, by Lagrange's theorem, $O(D)|O(\Gamma(m))$. Therefore, $O(m)|PFO(m)_\psi$. Since m is a random element of C, $O(m)|PFO(m)_\psi$ for all $m \in C$. □

We will now construct a relationship between the PFO of an element of a group in PFSG and the group's order.

Theorem 3.3. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFO of each element of C in ψ divides the order of C.*

Proof. According to the definition, $PFO(m)_\psi = O(\Gamma(m))$ for all $m \in C$.

From Theorem 3.1, $\Gamma(m)$ is a subgroup of C. Therefore, by Lagrange's theorem, the order of $\Gamma(m)$ divides the order of C. That is $O(\Gamma(m))|O(C)$.

This represent that $PFO(m)_\psi|O(C)$ for all $m \in C$. Hence, the PFO of each element of C in ψ divides the order of C. □

Theorem 3.4. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $PFO(m)_\psi = PFO(m^{-1})_\psi$ for all $m \in C$.*

Proof. Let $m \in C$. Then $PFO(m)_\psi = O(\Gamma(m))$.

As $\psi = (\mu, \nu)$ is a PFSG on C, then $\mu^2(m) = \mu^2(m^{-1})$ and $\nu^2(m) = \nu^\beta(m^{-1})$. Therefore, $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m^{-1}),\ \nu^2(n) \leq \nu^2(m^{-1})\} = \Gamma(m^{-1})$.

This proves that, $O(\Gamma(m)) = O(\Gamma(m^{-1}))$. That is $PFO(m)_\psi = PFO(m^{-1})_\psi$. Therefore, $PFO(m)_\psi = PFO(m^{-1})_\psi$ for all $m \in C$. □

Now, we will introduce the PFO of a PFSG on a group.

Definition 3.3. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then the PFO of the PFSG ψ is denoted by $PFO(\psi)$ and is defined by $PFO(\psi) = \bigvee\{PFO(m)_\psi \mid m \in C\}$.

Example 3.2. Consider the PFSG ψ on $\mathbb{Z}_4$ in Example 3.1.

The PFO of the elements of $\mathbb{Z}_4$ in ψ is presented by $PFO(0)_\psi = 2$, $PFO(1)_\psi = 4$, $PFO(2)_\psi = 4$ and $PFO(3)_\psi = 2$. Therefore, $PFO(\psi) = \bigvee\{PFO(m)_\psi \mid m \in \mathbb{Z}_4\}$=4.

Theorem 3.5. *The PFO of each PFSG on a group is the same as the group's order.*

Proof. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$.

Without sacrificing generality, we assume that $\mu^2(n) \geq \mu^2(m)$ and $\nu^2(n) \leq \nu^2(m)$ for all $n \in C$. Since $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$, then $\Gamma(m) = C$. Also, $|\Gamma(m)| \geq |\Gamma(n)|$ for all $n \in C$. Consequently, $PFO(\psi) = PFO(m)_\psi$.

Again $PFO(m)_\psi = O(\Gamma(m))$. Therefore, $PFO(\psi) = O(C)$.

Hence, the PFO of an PFSG on a group is the same as group's order. □

Remark 3.2. For a PFSG on a group C, the PFO of an element of C divides the PFO of that PFSG.

Theorem 3.6. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$ such that $PFO(m)_\psi = s$. If $\gcd(s, t) = 1$, then $\mu^2(m^t) = \mu^2(m)$ and $\nu^2(m^t) = \nu^2(m)$.*

Proof. Since $PFO(m)_\psi = s$, then $m^s = e$. Also $\psi = (\mu, \nu)$ is a PFSG on C, then $\mu^2(m^t) \geq \mu^2(m)$ and $\nu^2(m^t) \leq \nu^2(m)$.

As $\gcd(s,t)=1$, then there exist a and b such that $as+bt=1$. Now

$$\mu^2(m) = \mu^2(m^{as+bt}) \geq \mu^2(m^{as}) \wedge \mu^2(m^{bt}) \geq \mu^2(e) \wedge \mu^2(m^t) = \mu^2(m^t).$$

Therefore, $\mu^2(m) \geq \mu^2(m^t)$. Same way we can prove that $\nu^2(m) \leq \nu^2(m^t)$. Hence, $\mu^2(m^t) = \mu^2(m)$ and $\nu^2(m^t) = \nu^2(m)$. □

Theorem 3.7. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $\mu^2(m^t) = \mu^2(e)$ and $\nu^2(m^t) = \nu^2(e)$ then $t|PFO(m)_\psi$, where $t \in \mathbb{Z}$.*

Proof. Let $PFO(m)_\psi = s$. We can suppose that q is the smallest integer for which $\mu^2(m^q) = \mu^2(e)$ and $\nu^2(m^q) = \nu^2(e)$ holds.

By division algorithm, there exist $a, b \in \mathbb{Z}$ such that $s = at + b$ where $0 \leq b < t$. Now

$$\begin{aligned}
\mu^2(m^b) =& \mu^2(m^{s-at}) \\
\geq& \mu^2(m^s) \wedge \mu^2((m^{-1})^{at}) \\
=& \mu^2(m^s) \wedge \mu^2(m^{at}) = \mu^2(e) \wedge \mu^2((m^t)^a) \\
\geq& \mu^2(e) \wedge \mu^2(m^t) \\
=& \mu^2(e).
\end{aligned}$$

Similarly, $\nu^2(m^b) \leq \nu^2(e)$. Thus, $\mu^2(m^b) = \mu^2(e)$ and $\nu^2(m^b) = \nu^2(e)$. This contradicts q's minimality as $0 \leq b < t$.

Therefore, $b = 0$, so $s = at$. Hence, $t|PFO(m)_\psi$. □

Theorem 3.8. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $PFO(m)_\psi = s$, then $PFO(m^v)_\psi = \frac{s}{\gcd(s,v)}$, where $v \in \mathbb{Z}$.*

Proof. Let $PFO(m^v)_\psi = a$ and $\gcd(s,v) = g$.

As $PFO(m)_\psi = s$ then by Theorem 3.7, $m^s = e$. Now

$$\mu^2((m^v)^{\frac{s}{g}}) = \mu^2((m^s)^{\frac{v}{g}}) = \mu^2(e^{\frac{v}{g}}) = \mu^2(e).$$

Similarly, $\nu^2((m^v)^{\frac{s}{g}}) = \nu^2(e)$. As a result of the Theorem 3.7, $\frac{s}{g}$ divides a.

Also, $\gcd(s,v) = g$, then there exist $p, q \in \mathbb{Z}$ such that $sp + vq = g$. Therefore,

$$\begin{aligned}
\mu^2(m^{ga}) =& \mu^2(m^{(ps+vq)a}) \\
=& \mu^2(m^{psa} m^{vqa}) \\
\geq& \mu^2((m^s)^{pa}) \wedge \mu^2((m^{va})^q) \\
\geq& \mu^2(m^s) \wedge \mu^2((m^v)^a) \\
=& \mu^2(e) \wedge \mu^2(e) \\
=& \mu^2(e).
\end{aligned}$$

As a result, the only option is $\mu^2(m^{ga}) = \mu^2(e)$. Similarly, $\nu^2(m^{ga}) = \nu^2(e)$.

Thus, by Theorem 3.7, $ga|s$, that is $a|\frac{s}{g}$. Therefore, $a = \frac{s}{g}$. Hence, $PFO(m^v)_\psi = \frac{s}{\gcd(s,v)}$. □

Theorem 3.9. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $PFO(m)_\psi = z$ and $g \cong h \pmod{z}$, then $PFO(m^g)_\psi = PFO(m^h)_\psi$, where $g, h, z \in \mathbb{Z}$.*

Proof. Let $PFO(m^g)_\psi = l_1$ and $PFO(m^h)_\psi = l_2$.

As $g \cong h \pmod{z}$, then $g = wz + h$, where $w \in \mathbb{Z}$. Then

$$\begin{aligned}\mu^2((m^g)^{l_2}) =& \mu^2((m^{zw+h})^{l_2}) = \mu^2(m^{wzl_2}m^{hl_2})\\ \geq& \mu^2((m^z)^{wl_2}) \wedge \mu^2((m^h)^{l_2})\\ =& \mu^2(e) \wedge \mu^2(e)\\ =& \mu^2(e).\end{aligned}$$

As a result, the only option is $\mu^2((m^g)^{l_2}) = \mu^2(e)$. Similarly, $\nu^2((m^g)^{l_2}) = \nu^2(e)$. Thus by Theorem 3.7, $l_2|l_1$. In the same manner, we can prove that $l_1|l_2$. Thus $l_1 = l_2$.

Hence $PFO(m^g)_\psi = PFO(m^h)_\psi$, where $g, h \in \mathbb{Z}$. □

Theorem 3.10. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a group C and $m \in C$. Then $PFO(m)_\psi = PFO(nmn^{-1})_\psi$ for all $n \in C$.*

Proof. Let n be any element of C.

As ψ is a PFNSG on the group C, then $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$. Therefore the Pythagorean fuzzy semi-level subgroup corresponding to m is equal to nmn^{-1}.

This implies that $\Gamma(m) = \Gamma(nmn^{-1})$. Consequently, $O(\Gamma(m)) = O(\Gamma(nmn^{-1}))$. Since n is a random element of C, hence $PFO(m)_\psi = PFO(nmn^{-1})_\psi$ for all $n \in C$. □

Theorem 3.11. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a group C. Then $PFO(mn)_\psi = PFO(nm)_\psi$ for all $m, n \in C$.*

Proof. Assume m and n are elements of C.

Then we have $\mu^2(mn) = \mu^2((n^{-1}n)(mn)) = \mu^2(n^{-1}(nm)n)$. Similarly, $\nu^2(mn) = \nu^2(n^{-1}(nm)n)$. Therefore, $\Gamma(mn) = \Gamma(n^{-1}(mn)(n^{-1})^{-1})$. Consequently, $PFO(mn)_\psi = PFO(n^{-1}(nm)(n^{-1})^{-1})_\psi$.

Using Theorem 3.10, we get $PFO(n(nm)n^{-1})_\psi = PFO(nm)_\psi$. As m and n are random elements of C, hence $PFO(mn)_\psi = PFO(nm)_\psi$ for all $m, n \in G$. □

Theorem 3.12. *Assume $\psi = (\mu, \nu)$ is a PFSG on a commutative group C and m, n are two elements of C such that $\gcd(PFO(m)_\psi, PFO(n)_\psi) = 1$. If $\mu^2(mn) = \mu^2(e)$ and $\nu^2(mn) = \nu^2(e)$, then $PFO(m)_\psi = PFO(n)_\psi = 1$.*

Proof. Assume $PFO(m)_\psi = p$ and $PFO(n)_\psi = q$. So, we get $\gcd(p, q) = 1$. Now

$$\mu^2(m^q n^q) = \mu^2((mn)^q) \geq \mu^2(mn) = \mu^2(e).$$

As a result, the only option is $\mu^2(m^q n^q) = \mu^2(e)$. Also,

$$\begin{aligned}\mu^2(m^q) =&\mu^2(m^q n^q v^{-q})\\ \geq&\mu^2(m^q n^q) \wedge \mu^2((n^{-1})^q)\\ =&\mu^2(e) \wedge \mu^2(e)\\ =&\mu^2(e).\end{aligned}$$

So, we get $\mu^2(m^q) = \mu^2(e)$. Similarly, anyone can verify that $\nu^2(m^q) = \nu^2(e)$.

Using Theorem 3.15, we get $q|p$. Again $\gcd(p, q) = 1$, thus $q = 1$. Similarly, we can present that $p = 1$.

Hence, $PFO(m)_\psi = PFO(n)_\psi = 1$. □

Theorem 3.13. *Generators of a cyclic group have same PFO in a PFSG.*

Proof. Assume C is a cyclic group of order k.

Let m, n are any two generators of C. Then $m^k = e = n^k$.

As m is a generator, then $n = m^p$ for some $p \in \mathbb{Z}^+$. Therefore, k and p are co-prime, so $\gcd(k, p) = 1$. Thus, by Theorem 3.6, we get $PFO(m)_\psi = PFO(m^p)_\psi = PFO(n)_\psi$. For an infinite cyclic group it has only two generator. If m is a generator of C, then m^{-1} is the only other generator. Thus, by Theorem 3.4, we get $PFO(m)_\psi = PFO(m^{-1})_\psi$.

Hence, any generators of a cyclic group have same PFO in a PFSG. □

4. Some Algebraic Attributes of PFSG

The concepts of Pythagorean fuzzy normalizer (PFNL) and Pythagorean fuzzy centralizer (PFCL) are developed in this section. We also look into a number of algebraic properties of it.

Definition 4.1. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFNL of ψ is denoted by $\delta(\psi)$ and defined by $\delta(\psi) = \{m \mid m \in C, \mu^2(x) = \mu^2(mxm^{-1})$ and $\nu^2(m) = \nu^2(mxm^{-1})\}$ for all $x \in G$.

Example 4.1. Consider the group $C = (\mathbb{Z}, +)$.

Assume $\psi = (\mu, \nu)$ is a PFS on $\mathbb{Z}$, which is presented by

$$\mu(m) = \begin{cases} 0.87, & \text{where } m \in 2\mathbb{Z}, \\ 0.62, & \text{elsewhere}, \end{cases}$$

$$\nu(m) = \begin{cases} 0.31, & \text{where } m \in 2\mathbb{Z}, \\ 0.68, & \text{elsewhere}. \end{cases}$$

We can clearly verify that $\psi = (\mu, \nu)$ is a PFSG on $\mathbb{Z}$. Then the PFNL of ψ is $\delta(\psi) = \mathbb{Z}$.

Theorem 4.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a finite group C. Then the PFNL $\delta(\psi)$ forms a subgroup of C.*

Proof. Let $m, n \in \delta(\psi)$. Then

$$\mu^2(p) = \mu^2(mpm^{-1}), \quad \nu^2(p) = \nu^2(mpm^{-1}), \quad \text{for all } p \in C, \tag{4.1}$$

and

$$\mu^2(q) = \mu^2(nqn^{-1}), \quad \nu^2(q) = \nu^2(nqn^{-1}), \quad \text{for all } q \in C. \tag{4.2}$$

Clearly, $e \in \delta(\psi)$, so $\delta(\psi)$ is a non-empty finite subset of C.

To show $\delta(\psi)$ is a subgroup of C, we need to show $mn \in \delta(\psi)$. Put $p = nqn^{-1}$ in (4.1), we get

$$\mu^2(nqn^{-1}) = \mu^2(mnqn^{-1}m^{-1}) \quad \text{and} \quad \nu^2(nqn^{-1}) = \nu^2(mnqn^{-1}m^{-1}). \tag{4.3}$$

Then applying (4.2) in (4.3), we have $\mu^2(q) = \mu^2(mnqn^{-1}m^{-1})$ and $\nu^2(q) = \nu^2(mnqn^{-1}m^{-1})$.

This shows that $\mu^2(q) = \mu^2((mn)q(mn)^{-1})$ and $\nu^\beta(q) = \nu^\beta((mn)q(mn)^{-1})$. Therefore, $mn \in \delta(\psi)$. Hence, $\delta(\psi)$ forms a subgroup of C. □

Proposition 4.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $\psi = (\mu, \nu)$ is a PFNSG of C if and only if $\delta(\psi) = C$.*

Proof. We have $\delta(\psi) = \{m \mid m \in C, \mu^2(p) = \mu^2(mpm^{-1})$ and $\nu^2(m) = \nu^2(mpm^{-1})$ for all $p \in C$. Therefore, $\delta(\psi) \subseteq C$.

Assume $\psi = (\mu, \nu)$ is a PFNSG on C. Then we get $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$ for all $m, n \in C$.

This presents that $C \subseteq \delta(\psi)$. Hence, $\delta(\psi) = C$.

Conversely, let $\delta(\psi) = C$. Then $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$ for all $m, n \in C$. Hence, $\psi = (\mu, \nu)$ forms a PFNSG on C. □

Theorem 4.2. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then ψ forms a PFNSG on the group $\delta(\psi)$.*

Proof. Let $m, n \in \delta(\psi)$. Then $\mu^2(w) = \mu^2(mwm^{-1})$ and $\nu^2(w) = \nu^2(mwm^{-1})$ for all $w \in C$. As $\delta(\psi)$ forms a subgroup of C, then $nm \in \delta(\psi)$. Putting $w = nm$ in above relation we have $\mu^2(nm) = \mu^2(mnmm^{-1})$ and $\nu^2(nm) = \nu^2(mnmm^{-1})$. This presents that $\mu^2(nm) = \mu^2(mn)$ and $\nu^2(nm) = \nu^2(mn)$. Hence, ψ forms a PFNSG on the group $\delta(\psi)$. □

Definition 4.2. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFCL of ψ is denoted by $\omega(\psi)$ and defined by $\omega(\psi) = \{m \mid m \in C, \mu^2(mn) = \mu^2(nm)$ and $\nu^2(mn) = \nu^2(nm)\}$ for all $n \in C$.

Example 4.2. From Example 3.1, consider the PFSG ψ on the group $\mathbb{Z}_4$. Then the PFCL of ψ is $\omega(\psi) = \mathbb{Z}_4$.

Theorem 4.3. *The PFCL of a PFSG on a group forms a subgroup of the group.*

Proof. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then the PFCL of ψ is presented by $\omega(\psi) = \{m \mid m \in C,\ \mu^2(mn) = \mu^2(nm) \text{ and } \nu^2(mn) = \nu^2(nm)\}$ for all $n \in C$.

Let $s, t \in \omega(\psi)$. Then for all $r \in C$, we get

$$\mu^2((st)r) = \mu^2(s(tr)) = \mu^2((tr)s) = \mu^2(t(rs)) = \mu^2((rs)t) = \mu^2(r(st)).$$

Thus, $\mu^2((st)r) = \mu^2(r(st))$ for all $r \in C$.

Similarly, we get $\nu^2((st)r) = \nu^2(r(st))$ for all $r \in C$. This presents that $st \in \omega(\psi)$. Also, for all $g \in C$, we get

$$\mu^2(s^{-1}g) = \mu^2((g^{-1}s)^{-1}) = \mu^2(g^{-1}s) = \mu^2(sg^{-1}) = \mu^2((gs^{-1})^{-1}) = \mu^2(gs^{-1}).$$

Thus, $\mu^2(s^{-1}g) = \mu^2(gs^{-1})$ for all $g \in C$.

Similarly, we get $\nu^2(s^{-1}g) = \nu^2(gs^{-1})$ for all $g \in C$. This presents that for $s \in \omega(\psi)$, we have $s^{-1} \in \omega(\psi)$. Hence, $\omega(\psi)$ forms a subgroup of C. □

5. Lagrange's Theorem in PFSG

This section revolves around the development of theories for Lagrange's theorem fuzzification in PFSG.

Theorem 5.1. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C and Λ is the set of all PFCs of ψ on C. Then Λ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$.*

Proof. To prove $(\Lambda, \circ)$ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$, we need to verify that $\circ$ is well defined.

Let $m, n, p, q \in C$ such that $m\psi = p\psi$ and $n\psi = q\psi$.

Therefore, $m\mu(x) = p\mu(x)$, $m\nu(x) = p\nu(x)$ and $n\mu(x) = q\mu(x)$, $n\nu(x) = q\nu(x)$ for all $x \in C$. This presents that for all $x \in C$

$$\mu^2(m^{-1}x) = \mu^2(p^{-1}x), \quad \nu^2(m^{-1}x) = \nu^2(p^{-1}x) \tag{5.1}$$

and

$$\mu^2(n^{-1}x) = \mu^2(q^{-1}x), \quad \nu^2(n^{-1}x) = \nu^2(q^{-1}x). \tag{5.2}$$

We need to verify that $m\psi \circ n\psi = p\psi \circ q\psi$. So, $(mn)\psi = (pq)\psi$. We get $(mn)\mu(x) = \mu^2(n^{-1}m^{-1}x)$ and $(pq)\mu(x) = \mu^2(q^{-1}p^{-1}x)$ for all $x \in C$. Then

$$\begin{aligned}\mu^2(n^{-1}m^{-1}x) =& \mu^2(n^{-1}m^{-1}pp^{-1}x)\\ =& \mu^2(n^{-1}m^{-1}pqq^{-1}p^{-1}x)\\ \geq& \mu^2(n^{-1}m^{-1}pq) \wedge \mu^2(q^{-1}p^{-1}x).\end{aligned}$$

So,

$$\mu^2(n^{-1}m^{-1}x) \geq \mu^2(n^{-1}m^{-1}pq) \wedge \mu^2(q^{-1}p^{-1}x). \tag{5.3}$$

Replace x with $m^{-1}pq$ in (5.2), then

$$\mu^2(n^{-1}m^{-1}pq) = \mu^2(q^{-1}m^{-1}pq).$$

As $\psi = (\mu, \nu)$ is a PFNSG on C, then $\mu^2(q^{-1}m^{-1}pq) = \mu^2(m^{-1}p)$. Replace x with p in (5.1), we get

$$\mu^2(m^{-1}p) = \mu^2(p^{-1}p) = \mu^2(e).$$

Consequently, $\mu^2(n^{-1}m^{-1}pq) = \mu^2(e)$.

From (5.3), we get $\mu^2(n^{-1}m^{-1}x) \geq \mu^2(q^{-1}p^{-1}x)$.

Similarly, $\mu^2(q^{-1}p^{-1}x) \geq \mu^2(n^{-1}m^{-1}x)$. Therefore, $\mu^2(n^{-1}m^{-1}x) = \mu^2(q^{-1}p^{-1}x)$, for all $x \in C$. Also, we can verify that $\nu^2(n^{-1}m^{-1}x) = \nu^2(q^{-1}p^{-1}x)$ for all $x \in C$. This presents that $(mn)\mu(x) = (pq)\mu(x)$ and $(mn)\nu(x) = (pq)\nu(x)$ for all $x \in C$. Consequently, $(mn)\psi = (pq)\psi$. Hence, $\circ$ is well defined on Λ. Clearly, Λ's identity element is $e\psi$. Also, $m^{-1}\psi \in \Lambda$ is the inverse of $m\psi$ in Λ. That is $(m\psi)\circ(m^{-1}\psi) = e\psi$. Therefore, $(\Lambda, \circ)$ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$. □

Definition 5.1. The index of ψ is denoted by $[C : \psi]$ and defined by $[C : \psi] = O(\Lambda)$.

Example 5.1. Consider the group $C = (\mathbb{Z}_4, +_4)$. From Example 3.1, take the PFSG ψ on $\mathbb{Z}_4$. We can clearly show that ψ is a PFNSG on the group $C = (\mathbb{Z}_4, +_4)$. Then the set of all PFCs of ψ is $\Lambda = \{0\psi, 1\psi, 2\psi, 3\psi\}$.

Now $(1\mu)^2(1) = \mu^2(1^{-1} +_4 1) = \mu^2(3 +_4 1) = \mu^2(0) = 0.9025$, $(2\mu)^2(1) = \mu^2(2^{-1} +_4 1) = \mu^2(2 +_4 1) = \mu^2(3) = 0.7225$ and $(3\mu)^2(1) = \mu^2(3^{-1} +_4 1) = \mu^\alpha(1 +_4 1) = \mu^2(2) = 0.4225$.

Thus, $(1\mu)^2(1) \neq (2\mu)^2(1) \neq (3\mu)^2(1)$. This presents that $1\psi \neq 2\psi \neq 3\psi$. Therefore, the index of ψ is $[C : \psi] = O(\Lambda) = 4$.

Theorem 5.2. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C. Then a PFS $\Psi = (\mu_*, \nu_*)$ on Λ defined by $\mu_*(m\mu) = \mu(m)$ and $\nu_*(m\nu) = \nu(m)$ constructs a PFSG on $(\Lambda, \circ)$ for all $m \in C$.*

Proof. Let $m\psi, n\psi \in \Lambda$, where $m, n \in C$. Then

$$\begin{aligned}\mu_*^2((m\mu) \circ (n\mu)) =& \mu_*^2((mn)\mu) = \mu^2(mn)\\ \geq& \mu^2(m) \wedge \mu^2(n)\\ =& \mu_*^2(m\mu) \wedge \mu_*^2(n\mu).\end{aligned}$$

Therefore, $\mu_*^2((m\mu) \circ (n\mu)) \geq \mu_*^2(m\mu) \wedge \mu_*^2(n\mu)$.

Similarly, we get $\nu_*^2((m\nu) \circ (n\nu)) \leq \nu_*^2(m\nu) \vee \nu_*^2(n\nu)$. Also, $\mu_*^2(m^{-1}\mu) = \mu^2(m^{-1}) = \mu^2(m) = \mu_*^2(m\mu)$. Similarly, $\nu_*^2(m^{-1}\nu) = \nu_*^2(m\nu)$. Therefore, $\Psi = (\mu_*, \nu_*)$ constructs a PFSG on $(\Lambda, \circ)$. □

Definition 5.2. The PFSG $\Psi = (\mu_*, \nu_*)$ on the group $(\Lambda, \circ)$ is referred to as Pythagorean fuzzy quotient group (PFQG) on ψ.

Example 5.2. From Example 5.1, consider the PFNSG ψ on the group $(\mathbb{Z}_4, +_4)$. Then the set of all PFCs of ψ is $\Lambda = \{0\psi, 1\psi, 2\psi, 3\psi\}$. We create a PFS $\Psi = (\mu_*, \nu_*)$ on Λ by $\mu_*(m\mu) = \mu(m)$ and $\nu_((m\nu) = \nu(m)$.

Then $\mu_*(0\mu) = \mu(0) = 0.95$, $\mu_*(1\mu) = \mu(1) = 0.65$, $\mu_*(2\mu) = \mu(2) = 0.65$, $\mu_*(3\mu) = \mu(3) = 0.85$ and $\nu_*(0\nu) = \nu(0) = 0.25$, $\nu_*(1\nu) = \nu(1) = 0.75$, $\nu_*(2\nu) = \nu(2) = 0.75$, $\nu_*(3\nu) = \nu(3) = 0.45$.

We can clearly verify that $\Psi = (\mu_*, \nu_*)$ is a PFSG on Λ. Hence, $\Psi = (\mu_*, \nu_*)$ is the PFQG on ψ.

Theorem 5.3. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C and constructs a function $\kappa : C \to \Lambda$ by $\kappa(m) = m\psi$ for all $m \in C$. Then κ forms a group homomorphism with kernel* $\ker(\kappa) = \{m \in C \mid \mu^2(m) = \mu^2(e),\ \nu^2(m) = \nu^2(e)\}$.

Proof. Let $m, n \in C$.

Then $\kappa(mn) = (mn)\psi = (m\psi) \circ (m\psi) = \kappa(m) \circ \kappa(n)$. This presents that $\kappa : C \to \Lambda$ forms a group homomorphism. The kernel of κ is presented by

$$\begin{aligned}
\ker(\kappa) =& \{m \in C \mid \kappa(m) = e\psi\}\\
=& \{m \in C \mid m\psi = e\psi\}\\
=& \{m \in C \mid m\psi(n) = e\psi(n) \text{ for all } n \in C\}\\
=& \{m \in C \mid m\mu(n) = e\mu(n), m\nu(n) = e\nu(n) \text{ for all } n \in C\}\\
=& \{m \in C \mid \mu^2(m^{-1}n) = \mu^2(n), \nu^2(m^{-1}n) = \nu^2(n) \text{ for all } n \in C\}\\
=& \{m \in C \mid \mu^2(m) = \mu^2(e), \nu^2(m) = \nu^2(e)\}.
\end{aligned}$$

Hence, $\ker(\kappa) = \{m \in C \mid \mu^2(m) = \mu^2(e), \nu^2(m) = \nu^2(e)\}$. □

Remark 5.1. $\ker(\kappa)$ forms a subgroup of C.

Theorem 5.4. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C. Then $[C : \psi]$ divides $O(C)$.*

Proof. $\Lambda = \{m\psi \mid m \in C\}$, the set of all PFC of ψ on C is finite as C is finite.

Theorem 5.3 proves that $\kappa : C \to \Lambda$ defined by $\kappa(m) = m\psi$ for all $m \in C$ is a group homomorphism.

We define $H = \{m \in C \mid m\psi = e\psi\} = \ker(\kappa)$, which is a subgroup of C. C is now decomposed as union of left cosets modulo p by

$$C = m_1H \cup m_2H \cup m_3H \cup \cdots \cup m_pH,$$

where $m_pH = H$.

We need to verify that there exists a one-one relation between Λ's elements and cosets m_iH of C.

We consider an element $p \in H$ and coset m_iH of C. Then we get $\kappa(m_ip) = m_ip\psi = (m_i\psi) \circ (p\psi) = (m_i\psi) \circ (e\psi) = (m_i\psi)$. This represents that κ maps elements of m_iH to $m_i\psi$.

We now create a mapping $\underline{\kappa}$ between $\{m_iH \mid 1 \leq i \leq p\}$ and Λ by $\underline{\kappa}(m_iH) = m_i\psi$. Let $m_a\psi = m_b\psi$. Then $m_b^{-1}m_a\psi = e\psi$. Therefore, $m_b^{-1}m_a \in H$. This presents that $m_aH = m_bH$.

Hence, $\underline{\kappa}(m_iH) = m_i\psi$ is a one-one map. As a result, we can establish that the number of distinct cosets is same as Λ's cardinality. That is $[C : H] = [C : \psi]$.

Since $[C : H]$ divides $O(C)$, then $[C : \psi]$ must divide $O(C)$. □

6. Conclusion

Various fuzzy algebraic structures have significant importance in decision making problems. This paper explores the study of PFSGs. We have arranged the sections of this paper in such a way that we can approach Lagrange's theorem at the end. In this paper, we have defined Pythagorean fuzzy semi-level subgroups. We have introduced the notion of PFO of an element in PFSG and discussed various algebraic properties of it. Further, We have developed the concept of PFNL and PFCL of a PFSG. Moreover, we have introduced PFQG and the index of a PFSG. Finally, we have presented Lagrange's theorem in the form of PFS. In future work, we will implement direct product of groups and group actions in PFSG.

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[1]Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University,
Midnapore - 721 102, India
Email address: supriya1995.sb@gmail.com
ORCID iD: https://orcid.org/0000-0001-6353-7137
Email address: math.ganesh@mail.vidyasagar.ac.in
ORCID iD: https://orcid.org/0000-0002-3877-8059

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INDECOMPOSABLE MODULES IN THE GRASSMANNIAN CLUSTER CATEGORY $\mathrm{CM}(B_{5,10})$

DUŠKO BOGDANIĆ[1] AND IVAN-VANJA BOROJA[2]

Abstract. In this paper, we study indecomposable rank 2 modules in the Grassmannian cluster category $\mathrm{CM}(B_{5,10})$. This is the smallest wild case containing modules whose profile layers are 5-interlacing. We construct all rank 2 indecomposable modules with a specific natural filtration, classify them up to isomorphism, and parameterize all infinite families of non-isomorphic rank 2 modules.

1. Introduction and Preliminaries

In their seminal work [7], Fomin and Zelevinsky used the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(2,n)]$ of the Grassmannian of 2-dimensional subspaces of $\mathbb{C}^n$ as one of the first examples of the theory of cluster algebras. Scott proved in [17] that this cluster structure can be generalized to the coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$. These results initiated a lot of research activities in cluster theory, e.g. [5, 8, 10, 12–16, 18]. Geiss, Leclerc, and Schroer [9, 11] gave an additive categorification of the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian variety of k-dimensional subspaces in $\mathbb{C}^n$ in terms of a subcategory of the category of finite dimensional modules over the preprojective algebra of type A_{n-1}, called the boundary algebra. Jensen, King, and Su [14] introduced a new additive categorification of this cluster structure using the maximal Cohen-Macaulay modules over the completion of an algebra $B_{k,n}$ which is a quotient of the preprojective algebra of type A_{n-1}. In the category $\mathrm{CM}(B_{k,n})$ of Cohen-Macaulay modules over $B_{k,n}$, among the indecomposable modules are the rank 1 modules which are known to be in

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bijection with k-subsets of $\{1, 2, \dots, n\}$, and their explicit construction has been given in [14] (a k-subset I corresponds to a rank 1 module denoted L_I). Rank 1 modules are the building blocks of the category as any module in $\mathrm{CM}(B_{k,n})$ can be filtered by rank 1 modules (the filtration is noted in the profile of a module, [14, Corollary 6.7]). The number of rank 1 modules appearing in the filtration of a given module is called the rank of that module.

The aim of this paper is to explicitly construct all rank 2 indecomposable Cohen-Macaulay $B_{k,n}$-modules in the case when $k = 5$ and $n = 10$. All indecomposable $B_{k,n}$-modules of rank 2 whose rank 1 filtration layers L_I and L_J satisfy the condition $|I \cap J| \geq k - 4$ have been constructed in [4]. This covers all tame cases and the wild case $(k, n) = (4, 9)$. The case $(k, n) = (5, 10)$ is the smallest wild case that contains rank 2 indecomposable modules whose layers are 5-interlacing. In this case, the only profiles with 5-interlacing layers are of the form $\{i, i+2, i+4, i+6, i+8\} \mid \{i+1, i+3, i+5, i+7, i+9\}$, where $i = 1, 2$. We construct all indecomposable modules with the profile $\{i, i+2, i+4, i+6, i+8\} \mid \{i+1, i+3, i+5, i+7, i+9\}$, classify them up to isomorphism, and parameterize all infinite families of non-isomorphic rank 2 modules. It is important to remark that even though we only treat the case $(5, 10)$ in this paper, all arguments and results are also valid for the general case (k, n) for all rank 2 modules with tightly 5-interlacing layers.

We follow the exposition from [2, 3, 14] in order to introduce notation and background results. Here, the central combinatorial notion is that of r-interlacing.

Definition 1.1 (r-interlacing)**.** Let I and J be two k-subsets of $\{1, \dots, n\}$. The sets I and J are said to be *r-interlacing* if there exist subsets $\{i_1, i_3, \dots, i_{2r-1}\} \subset I \setminus J$ and $\{i_2, i_4, \dots, i_{2r}\} \subset J \setminus I$ such that $i_1 < i_2 < i_3 < \cdots < i_{2r} < i_1$ (cyclically) and if there exist no larger subsets of I and of J with this property. We say that I and J are *tightly r-interlacing* if they are r-interlacing and $|I \cap J| = k - r$.

Let Γ_n be the quiver of the boundary algebra, with vertices $C_0 = \mathbb{Z}_n$ on a cycle and arrows $x_i : i-1 \to i$, $y_i : i \to i-1$, $i \in C_0$. We write $\mathrm{CM}(B_{k,n})$ for the category of maximal Cohen-Macaulay modules for the completed path algebra $B_{k,n}$ of Γ_n, with relations $xy - yx$ and $x^k - y^{n-k}$ (at every vertex). The centre of $B_{k,n}$ is $Z := \mathbb{C}[|t|]$, where $t = \sum_i x_i y_i$. For example, in the following figure we have the quiver Γ_n for $n = 5$. We view the completed path algebra of Γ_n as a topological algebra via the m-adic topology, where m is the two-sided ideal generated by the arrows of the quiver, see [6, Section 1]. The algebra $B_{k,n}$ was introduced in [14, Section 3]. Observe that $B_{k,n}$ is isomorphic to $B_{n-k,n}$, so we will always take $k \leq \frac{n}{2}$. Moreover, throughout this paper, we will be working with the algebra $B_{5,10}$. For background results on algebras given by quivers and relations and their representations we recommend [1].

The (maximal) Cohen-Macaulay $B_{k,n}$-modules are precisely those which are free as Z-modules. Such a module M is given by a representation $\{M_i : i \in C_0\}$ of the quiver with each M_i a free Z-module of the same rank (which is the rank of M).

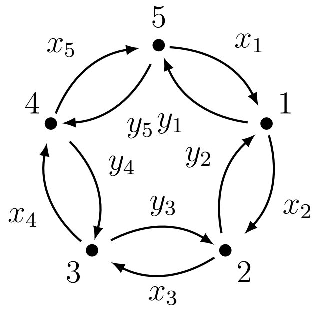

Figure 1. The quiver Γ_5

Definition 1.2 ([14, Definition 3.5])**.** For any $B_{k,n}$-module M and K the field of fractions of Z, the **rank** of M, denoted by $\mathrm{rk}(M)$, is defined to be the length of $M \otimes_Z K$, $\mathrm{rk}(M) := \mathrm{len}(M \otimes_Z K)$.

Note that $B_{k,n} \otimes_Z K \cong M_n(K)$, which is a simple algebra. It is easy to check that the rank is additive on short exact sequences and $\mathrm{rk}(M) = 0$ for any finite-dimensional $B_{k,n}$-module (because these are torsion over Z). Also, for any Cohen-Macaulay $B_{k,n}$-module M and every idempotent e_j, $j \in C_0$, $\mathrm{rk}_Z(e_j M) = \mathrm{rk}(M)$, so that, in particular, $\mathrm{rk}_Z(M) = n\mathrm{rk}(M)$.

Definition 1.3 ([14, Definition 5.1])**.** For any k-subset I of C_0, we define a rank 1 $B_{k,n}$-module

$$L_I = (U_i,\ i \in C_0\,;\, x_i, y_i,\ i \in C_0)$$

as follows. For each vertex $i \in C_0$, set $U_i = \mathbb{C}[[t]]$, e_i acts as the identity on U_i and $e_i U_j = 0$, for $i \neq j$. For each $i \in C_0$, set

$x_i \colon U_{i-1} \to U_i$ to be multiplication by 1 if $i \in I$, and by t if $i \notin I$;
$y_i \colon U_i \to U_{i-1}$ to be multiplication by t if $i \in I$, and by 1 if $i \notin I$.

The module L_I can be represented by a lattice diagram $\mathcal{L}_I$ in which $U_0, U_1, U_2, \ldots, U_n$ are represented by columns of vertices (dots) from left to right (with U_0 and U_n to be identified), going down infinitely. The vertices in each column correspond to the natural monomial $\mathbb{C}$-basis of $\mathbb{C}[t]$. The column corresponding to U_{i+1} is displaced half a step vertically downwards (respectively, upwards) in relation to U_i if $i+1 \in I$ (respectively, $i+1 \notin I$), and the actions of x_i and y_i are shown as diagonal arrows. Note that the k-subset I can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram. For example, the lattice diagram $\mathcal{L}_{\{1,4,5\}}$ in the case $k = 3$, $n = 8$, is shown in Figure 2.

We see from Figure 2 that the module L_I is determined by its upper boundary, denoted by the thick lines, which we refer to as the *rim* of the module L_I (this is why we call the k-subset I the rim of L_I). Throughout this paper we will identify a rank 1 module L_I with its rim. Moreover, most of the time we will omit the arrows in the rim of L_I and represent it as an undirected graph.

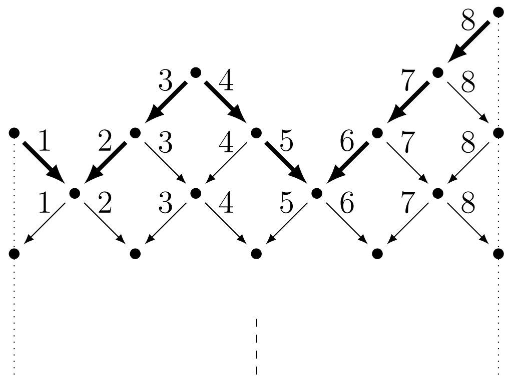

FIGURE 2. Lattice diagram of the module $L_{\{1,4,5\}}$

Proposition 1.1 ([14], Proposition 5.2)**.** *Every rank* 1 *Cohen-Macaulay $B_{k,n}$-module is isomorphic to L_I for some unique k-subset I of C_0.*

Every $B_{k,n}$-module has a canonical endomorphism given by multiplication by $t \in Z$. For L_I this corresponds to shifting $\mathcal{L}_I$ one step downwards. Since Z is central, $\mathrm{Hom}_{B_{k,n}}(M, N)$ is a Z-module for arbitrary $B_{k,n}$-modules M and N. If M, N are free Z-modules, then so is $\mathrm{Hom}_{B_{k,n}}(M, N)$. In particular, for any two rank 1 Cohen-Macaulay $B_{k,n}$-modules L_I and L_J, $\mathrm{Hom}_{B_{k,n}}(L_I, L_J)$ is a free module of rank 1 over $Z = \mathbb{C}[[t]]$, generated by the canonical map given by placing the lattice of L_I inside the lattice of L_J as far up as possible so that no part of the rim of L_I is strictly above the rim of L_J [14, Section 6].

Every indecomposable module M of rank n in $\mathrm{CM}(B_{k,n})$ has a filtration having factors $L_{I_1}, L_{I_2}, \dots, L_{I_n}$ of rank 1. A specific filtration given by the dimension vector of a module is noted in its *profile*, $\mathrm{pr}(M) = I_1 \mid I_2 \mid \dots \mid I_n$, [14, Corollary 6.7]. In the case of a rank 2 module M with filtration $L_I \mid L_J$ (i.e. with profile $I \mid J$), we picture the profile of this module by drawing the rim J below the rim I, in such a way that J is placed as far up as possible so that no part of the rim J is strictly above the rim I. Note that there is at least one point where the rims I and J meet (see Figure 3 for an example).

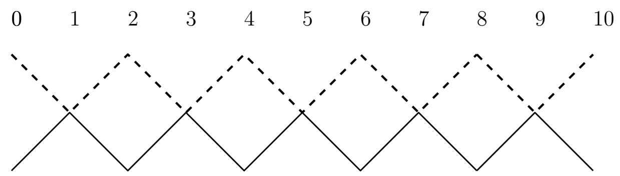

FIGURE 3. The profile $\{1, 3, 5, 7, 9\} \mid \{2, 4, 6, 8, 10\}$ in $\mathrm{CM}(B_{5,10})$.

For background on the poset and dimension vector associated with an indecomposable module or to its profile, we refer to [14, Section 6].

2. Tight 5-Interlacing

In this section we construct all rank 2 indecomposable modules with the profile $I \mid J$ in the case when I and J are tightly 5-interlacing 5-subsets, i.e., when $|I \setminus J| = |J \setminus I| = 5$ and non-common elements of I and J interlace, that is, $|I \cap J| = 0$. Rank 2 indecomposable modules with 3-interlacing and 4-interlacing layers have been constructed and parameterized in [4].

In the case $(5, 10)$, there are only two profiles with 5-interlacing layers, namely $I \mid J$ and $J \mid I$, where $I = \{1, 3, 5, 7, 9\}$ and $J = \{2, 4, 6, 8, 10\}$. We will work with the profile $I \mid J$, the arguments are the same for $J \mid I$.

In [4], we defined a rank 2 module $\mathbb{M}(I, J)$ with filtration $L_I \mid L_J$ in a similar way as rank 1 modules are defined in $\mathrm{CM}(B_{k,n})$. We recall the construction here. Let $V_i := \mathbb{C}[|t|] \oplus \mathbb{C}[|t|]$, $i = 1, \dots, n$. The module $\mathbb{M}(I, J)$ has V_i at each vertex $1, 2, \dots, n$ of Γ_n. In order to have a module structure for $B_{k,n}$, for every i we need to define $x_i \colon V_{i-1} \to V_i$ and $y_i \colon V_i \to V_{i-1}$ in such a way that $x_i y_i = t \cdot \mathrm{id}$ and $x^k = y^{n-k}$.

Define

$$x_{2i+1} = \begin{pmatrix} t & b_{2i+1} \\ 0 & 1 \end{pmatrix}, \qquad x_{2i} = \begin{pmatrix} 1 & b_{2i} \\ 0 & t \end{pmatrix},$$
$$y_{2i+1} = \begin{pmatrix} 1 & -b_{2i+1} \\ 0 & t \end{pmatrix}, \qquad y_{2i} = \begin{pmatrix} t & -b_{2i} \\ 0 & 1 \end{pmatrix},$$

for $i = 0, 1, 2, 3, 4$. Also, we assume that $\sum_{i=0}^{9} b_i = 0$. By construction it holds that $xy = yx$ and $x^5 = y^{10-5}$ at all vertices and that $\mathbb{M}(I, J)$ is free over the centre of $B_{5,10}$. Hence, $\mathbb{M}(I, J)$ is in $\mathrm{CM}(B_{5,10})$.

It was shown in [4] that $\mathbb{M}(I, J)$ is isomorphic to $L_I \oplus L_J$ if and only if $t \mid b_i + b_{i+1}$, for i odd.

Our aim is to study the structure of the module $\mathbb{M}(I, J)$ in terms of the divisibility conditions the coefficients b_i satisfy. Since I and J are fixed, $\mathbb{M}(I, J)$ will be denoted by $\mathbb{M}$.

We distinguish between different cases depending on whether the sums $b_1 + b_2$, $b_3 + b_4$, $b_5 + b_6$, $b_7 + b_8$, and $b_9 + b_{10}$ are divisible by t or not. We will call these the five *divisibility conditions* $t \mid b_1 + b_2$, $t \mid b_3 + b_4$, $t \mid b_5 + b_6$, $t \mid b_7 + b_8$, and $t \mid b_9 + b_{10}$, and write (div) to abbreviate. Also, we write $B_i = b_i + b_{i+1}$ for odd i. There are four base cases: one of the sums B_i is divisible by t and four are not, two are divisible by t and three are not, three are divisible by t and two are not, and none of the sums is divisible by t. Note that it is not possible that four of the sums are divisible by t and one is not because they sum up to 0.

Theorem 2.1. *The module $\mathbb{M}(I, J)$ is indecomposable if and only if there exist odd indices i_{l_1} and i_{l_2} such that $t \mid b_i + b_{i+1}$, for $i_{l_1} < i < i_{l_2}$, i odd, $t \nmid b_{i_{l_1}} + b_{i_{l_1}+1}$, $t \nmid b_{i_{l_2}} + b_{i_{l_2}+1}$, and $t \nmid b_{i_{l_1}} + b_{i_{l_1}+1} + b_{i_{l_2}} + b_{i_{l_2}+1}$.*

Throughout the paper, in all the cases we consider, we will assume that the assumptions of the previous theorem are fulfilled, i.e. that there are odd indices i_{l_1} and i_{l_2} such that $t \mid b_i+b_{i+1}$, for $i_{l_1} < i < i_{l_2}$, i odd, $t \nmid b_{i_{l_1}}+b_{i_{l_1}+1}$, $t \nmid b_{i_{l_2}}+b_{i_{l_2}+1}$, and $t \nmid b_{i_{l_1}}+b_{i_{l_1}+1}+b_{i_{l_2}}+b_{i_{l_2}+1}$. This means that one of the base cases, the case where two of the sums B_i are not divisible by t and three are divisible by t, will not be considered, because in this case the assumptions of the previous theorem are not fulfilled. More precisely, the sum of the only two B_i's that are not divisible by t is divisible by t, because $\sum_{i=1}^{10} b_i = 0$. Therefore, there are only three base cases to consider.

We will show that there are infinitely many non-isomorphic modules with the same filtration for the cases when none of the sums is divisible by t and when four of the sums are not divisible by t.

Let $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10})$ be another 10-tuple such that $\sum_{i=1}^{10} c_i = 0$ and that the module defined by this tuple is indecomposable. Denote this module by $\mathbb{M}'$ and by C_i the sum $c_i + c_{i+1}$, for odd i. We say that the modules $\mathbb{M}$ and $\mathbb{M}'$ satisfy the same divisibility conditions if the following holds: $t \mid B_i$ if and only if $t \mid C_i$, and $t \mid B_i + B_j$ if and only if $t \mid C_i + C_j$.

For the rest of the paper, if $t^d v = w$, for a positive integer d, then $t^{-d}w$ denotes v. If there is an isomorphism $\varphi = (\varphi_i)$ between the modules $\mathbb{M}$ and $\mathbb{M}'$, then the following holds.

Let us assume that $\varphi_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then from $\varphi_i x_i = x_i \varphi_{i-1}$ we get that φ_{2i+1} is

$$\begin{pmatrix} \alpha + (c_1 + \cdots + c_{2i+1})t^{-1}\gamma & \beta t - \alpha \sum_{j=1}^{2i+1} b_j + \delta \sum_{j=1}^{2i+1} c_j - (\sum_{j=1}^{2i+1} b_j)(\sum_{j=1}^{2i+1} c_j)t^{-1}\gamma \\ t^{-1}\gamma & \delta - (b_1 + \dots b_{2i+1})t^{-1}\gamma_0 \end{pmatrix},$$

and that φ_{2i} is equal to

$$\begin{pmatrix} \alpha + (c_1 + \cdots + c_{2i})t^{-1}\gamma & \beta + t^{-1}(-\alpha \sum_{j=1}^{2i} b_j + \delta \sum_{j=1}^{2i} c_j - t^{-1}\gamma \sum_{j=1}^{2i} b_j \sum_{j=1}^{2i} c_j) \\ \gamma & \delta - (b_1 + \cdots + b_{2i})t^{-1}\gamma \end{pmatrix},$$

where $t \mid \gamma$ and

$$\begin{aligned} & t \mid -\alpha(b_1 + b_2) + \delta(c_1 + c_2) - (b_1 + b_2)(c_1 + c_2)t^{-1}\gamma, \\ & t \mid -\alpha \cdot \sum_{i=1}^{4} b_i + \delta \cdot \sum_{i=1}^{4} c_i - t^{-1}\gamma \sum_{i=1}^{4} b_i \sum_{i=1}^{4} c_i, \\ & t \mid -\alpha \cdot \sum_{i=1}^{6} b_i + \delta \cdot \sum_{i=1}^{6} c_i - t^{-1}\gamma \sum_{i=1}^{6} b_i \sum_{i=1}^{6} c_i, \\ & t \mid -\alpha \sum_{i=1}^{8} b_i + \delta \sum_{i=1}^{8} b_i - t^{-1}\gamma \sum_{i=1}^{8} b_i \sum_{i=1}^{8} c_i. \end{aligned} \tag{2.1}$$

Since $t \mid \gamma$ and we would like φ to be invertible, then it must be that $t \nmid \alpha$ and $t \nmid \delta$. Then the inverse of φ_0 is $\frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$. Thus, in order to construct an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$, we have to make sure that the divisibility conditions (2.1) are met for the coefficients of φ_0. This will be used repeatedly throughout the paper.

Before considering base cases, in the next theorem we show that if the modules $\mathbb{M}$ and $\mathbb{M}'$ do not satisfy the same divisibility conditions, then they are not isomorphic.

Theorem 2.2. *The above defined modules* $\mathbb{M}$ *and* $\mathbb{M}'$ *are not isomorphic if they do not satisfy the same divisibility conditions.*

Proof. Let us assume that there is an odd index, say i_1, such that $t \nmid b_{i_1} + b_{i_1+1}$ and $t \mid c_{i_1} + c_{i_1+1}$. If $\varphi = (\varphi_i)$ is an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$, let $\varphi_{i_1-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then the coefficients of φ_{i_1-1} have to satisfy divisibility conditions (2.1). Since $t \mid c_{i_1} + c_{i_1+1}$, the first condition from (2.1), $t \mid -\alpha(b_{i_1} + b_{i_1+1}) + \delta(c_{i_1} + c_{i_1+1}) - (b_{i_1} + b_{i_1+1})(c_{i_1} + c_{i_1+1})t^{-1}\gamma$, reduces to $t \mid \alpha(b_{i_1} + b_{i_1+1})$. But $t \nmid \alpha$ and $t \nmid b_{i_1} + b_{i_1+1}$ which is a contradiction. Hence, $\mathbb{M}$ and $\mathbb{M}'$ are not isomorphic in this case.

Assume that, for every odd i, $t \nmid b_i + b_{i+1}$ if and only if $t \nmid c_i + c_{i+1}$. Since $\mathbb{M}$ and $\mathbb{M}'$ do not satisfy the same divisibility conditions, there is an index, say i_1, such that $t \nmid B_{i_1} + B_{i_2}$ and $t \mid C_{i_1} + C_{i_2}$. Then the second divisibility condition from (2.1), $t \mid -\alpha(B_{i_1} + B_{i_2}) + \delta(C_{i_1} + C_{i_2}) - (B_{i_1} + B_{i_2})(C_{i_1} + C_{i_2})t^{-1}\gamma$, reduces to $t \mid -\alpha(B_{i_1} + B_{i_2})$. But, $t \nmid -\alpha$ and $t \nmid B_{i_1} + B_{i_2}$ which is a contradiction. Hence, $\mathbb{M}$ and $\mathbb{M}'$ are not isomorphic in this case as well. □

For the remainder of the paper, when we investigate if the modules $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic, we will implicitly assume that they satisfy the same divisibility conditions.

2.1. **Three of the sums B_i are not divisible by t.** Assume that $t \nmid b_{i_l} + b_{i_l+1}$, $l = 1, 2, 3$, and $t \mid b_{i_l} + b_{i_l+1}$, $l = 4, 5$, where $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 3, 5, 7, 9\}$. Since $\sum_{i=1}^{10} b_i = 0$, it follows that $t \nmid B_{i_l} + B_{i_s}$, for all $l, s \leq 3$. By Theorem 2.1, the constructed module is indecomposable. Denote this module by $\mathbb{M}_{i_1,i_2,i_3}$. Let (c_i) be another tuple giving rise to the module $\mathbb{M}_{j_1,j_2,j_3}$. The following theorem says that the modules $\mathbb{M}_{i_1,i_2,i_3}$ and $\mathbb{M}_{j_1,j_2,j_3}$ are isomorphic if and only if they satisfy the same divisibility conditions.

Theorem 2.3. *The above-defined modules* $\mathbb{M}_{i_1,i_2,i_3}$ *and* $\mathbb{M}_{j_1,j_2,j_3}$ *are isomorphic if and only if* $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\}$.

Proof. Let $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\}$ and $\varphi_{i_1-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. The divisibility conditions (2.1) reduce to the following two conditions (recall that we write B_i for $b_i + b_{i+1}$):

$$t \mid -\alpha B_{i_1} + \delta C_{i_1} - B_{i_1} C_{i_1} t^{-1}\gamma,$$
$$t \mid -\alpha(B_{i_1} + B_{i_2}) + \delta(C_{i_1} + C_{i_2}) - (B_{i_1} + B_{i_2})(C_{i_1} + C_{i_2})t^{-1}\gamma.$$

Here, we assume that we started numbering from i_1, and that $i_1 < i_2 < i_3$.

Since there are no conditions attached to β, we set it to be 0. If we set

$$-\alpha B_{i_1} + \delta C_{i_1} - B_{i_1} C_{i_1} t^{-1}\gamma = 0,$$

$$-\alpha(B_{i_1}+B_{i_2})+\delta(C_{i_1}+C_{i_2})-(B_{i_1}+B_{i_2})(C_{i_1}+C_{i_2})t^{-1}\gamma=0,$$

then we get

$$\alpha(B_{i_1}+B_{i_2})[C_{i_1}^{-1}(C_{i_1}+C_{i_2})-1]+\delta(C_{i_1}+C_{i_2})[B_{i_1}^{-1}(B_{i_1}+B_{i_2})-1]=0.$$

If $t\mid C_{i_1}^{-1}(C_{i_1}+C_{i_2})-1$, then $t\mid C_{i_2}$, which is not true. It follows that $C_{i_1}^{-1}(C_{i_1}+C_{i_2})-1$ is invertible. The same holds for $B_{i_1}^{-1}(B_{i_1}+B_{i_2})-1$. Thus, if we set $\delta=1$, then we get

$$\alpha=-(C_{i_1}+C_{i_2})(B_{i_1}+B_{i_2})^{-1}[B_{i_1}^{-1}(B_{i_1}+B_{i_2})-1][C_{i_1}^{-1}(C_{i_1}+C_{i_2})-1]^{-1}$$

and

$$\gamma=t(-\alpha C_{i_1}^{-1}+B_{i_1}^{-1}).$$

Hence,

$$\varphi_0=\begin{pmatrix}-(C_{i_1}+C_{i_2})(B_{i_1}+B_{i_2})^{-1}[B_{i_1}^{-1}(B_{i_1}+B_{i_2})-1][C_{i_1}^{-1}(C_{i_1}+C_{i_2})-1]^{-1} & 0\\ t(-\alpha C_{i_1}^{-1}+B_{i_1}^{-1}) & 1\end{pmatrix}.$$

The other invertible matrices φ_i are now determined from $\varphi_i x_i = x_i\varphi_{i-1}$. Note that all of them are invertible because their determinant is equal to $\alpha\delta-\beta\gamma$ which is an invertible element.

If $\{i_1,i_2,i_3\}\neq\{j_1,j_2,j_3\}$, then $\mathbb{M}$ and $\mathbb{M}'$ do not satisfy the same divisibility conditions. It follows by Theorem 2.2 that $\mathbb{M}$ and $\mathbb{M}'$ are not isomorphic. □

The previous theorem tells us that the module $\mathbb{M}_{i_1,i_2,i_3}$ only depends on the divisibility conditions of the coefficients b_i, so if we have two different tuples satisfying the same divisibility conditions, then they give rise to isomorphic modules. In total, there are $\binom{5}{3}$ non-isomorphic indecomposable modules that arise this way, one for each subset of $\{1,2,3,4,5\}$ with three elements.

2.2. **Four of the sums B_i are not divisible by t.** Assume that $t\nmid b_{i_l}+b_{i_l+1}$, $l=1,2,3,4$, and $t\mid b_j+b_{j+1}$, where $\{i_1,i_2,i_3,i_4\}\cup\{j\}=\{1,3,5,7,9\}$. Since $\sum_{i=1}^{10}b_i=0$, it follows that $t\mid\sum_{l=1}^{4}B_{i_l}$. Recall that we assume that the divisibility conditions from Theorem 2.1 hold so that the constructed module is indecomposable. Denote this module by $\mathbb{M}$. It means that $t\nmid B_{i_l}+B_{i_{l+1}}$ for at least one index l. If $t\nmid B_{i_l}+B_{i_{l+1}}$, then $t\nmid B_{i_{l+2}}+B_{i_{l+3}}$. For the remaining two sums $B_{i_{l+1}}+B_{i_{l+2}}$ and $B_{i_{l+3}}+B_{i_l}$, either both of them are divisible by t or none of them is. Thus, we have to distinguish between these subcases.

Before we start considering these subcases, we recall that two modules that do not satisfy the same divisibility conditions are not isomorphic. Let (c_i) be another tuple giving rise to another indecomposable module $\mathbb{M}'$. Here, we assume that $t\nmid c_{j_l}+c_{j_l+1}$, $l=1,2,3,4$, and $t\mid c_i+c_{i+1}$, where $\{j_1,j_2,j_3,j_4\}\cup\{i\}=\{1,3,5,7,9\}$. Since $\sum_{i=1}^{10}c_i=0$, it follows that $t\mid\sum_{l=1}^{4}C_{j_l}$. Also, we assume that $t\nmid C_{j_l}+C_{j_{l+1}}$ for at least one index l.

Now we examine if the modules $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic when they satisfy the same divisibility conditions. Let $\{i_1, i_2, i_3, i_4\} = \{j_1, j_2, j_3, j_4\}$. Here, we can assume that these odd numbers are consecutive.

The first subcase is when two of the sums $B_{i_1} + B_{i_2}$ are divisible by t, and two are not. Thus, we assume that $t \nmid B_{i_1} + B_{i_2}$, $t \nmid B_{i_3} + B_{i_4}$, $t \mid B_{i_2} + B_{i_3}$, and $t \mid B_{i_4} + B_{i_1}$. The same conditions hold for $\mathbb{M}'$, so $t \nmid C_{i_1} + C_{i_2}$, $t \nmid C_{i_3} + C_{i_4}$, $t \mid C_{i_2} + C_{i_3}$, and $t \mid C_{i_4} + C_{i_1}$. Note that if $t \nmid B_{i_1} + B_{i_2}$, then $t \nmid B_{i_3} + B_{i_4}$ because $t \mid \sum_{l=1}^{4} B_{i_l}$. Analogously, if $t \mid B_{i_2} + B_{i_3}$, then $t \mid B_{i_4} + B_{i_1}$.

Theorem 2.4. *If $\mathbb{M}$ and $\mathbb{M}'$ are such that $t \nmid B_{i_1} + B_{i_2}$, $t \mid B_{i_2} + B_{i_3}$, $t \nmid C_{i_1} + C_{i_2}$, and $t \mid C_{i_2} + C_{i_3}$, then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic.*

Proof. Keeping the same notation as before when constructing isomorphisms, the divisibility conditions (2.1) reduce to:

$$t \mid -\alpha B_{i_1} + \delta C_{i_1} - B_{i_1} C_{i_1} t^{-1}\gamma,$$
$$t \mid -\alpha(B_{i_1} + B_{i_2}) + \delta(C_{i_1} + C_{i_2}) - (B_{i_1} + B_{i_2})(C_{i_1} + C_{i_2})t^{-1}\gamma,$$

because $t \mid B_{i_2} + B_{i_3}$, $t \mid C_{i_2} + C_{i_3}$, $t \mid \sum_{l=1}^{4} B_{i_l}$, and $t \mid \sum_{l=1}^{4} C_{i_l}$. Now, we proceed as in the proof of Theorem 2.3 in order to construct an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$. ☐

In total, this subcase gives $2\binom{5}{4}$ non-isomorphic indecomposable modules. There are two modules for every choice of a four-element subset of $\{1, 3, 5, 7, 9\}$.

The second subcase is when none of the sums $B_{i_1} + B_{i_2}$ is divisible by t. Thus, we assume that $t \nmid B_{i_l} + B_{i_{l+1}}$ and $t \nmid C_{i_l} + C_{i_l+1}$, for $l = 1, 2, 3, 4$.

Theorem 2.5. *If $t \nmid B_{i_l} + B_{i_l+1}$ and $t \nmid C_{i_l} + C_{i_l+1}$, for $l = 1, 2, 3, 4$, then the modules $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if*

$$t \mid B_{i_1} C_{i_2} B_{i_3} C_{i_4} - C_{i_1} B_{i_2} C_{i_3} B_{i_4}.$$

Proof. As before, if there were an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$, its coefficients would have to satisfy the following conditions that we obtain from (2.1):

$$t \mid -\alpha B_{i_1} + \delta C_{i_1} - B_{i_1} C_{i_1} t^{-1}\gamma,$$
$$t \mid -\alpha(B_{i_1} + B_{i_2}) + \delta(C_{i_1} + C_{i_2}) - (B_{i_1} + B_{i_2})(C_{i_1} + C_{i_2})t^{-1}\gamma,$$
$$t \mid \alpha B_{i_4} - \delta C_{i_4} - B_{i_4} C_{i_4} t^{-1}\gamma,$$

because $t \mid \sum_{l=1}^{4} B_{i_l}$, and $t \mid \sum_{l=1}^{4} C_{i_l}$.

From these we get that

$$t \mid \alpha C_{i_2}[C_{i_1}(C_{i_1} + C_{i_2})]^{-1} - \delta B_{i_2}[B_{i_1}(B_{i_1} + B_{i_2})]^{-1},$$
$$t \mid \alpha C_{i_3}[B_{i_4}(C_{i_1} + C_{i_2})]^{-1} - \delta B_{i_3}[B_{i_4}(B_{i_1} + B_{i_2})]^{-1}.$$

Finally, from the last two relations we get

$$t \mid \alpha[B_{i_1} C_{i_2} B_{i_3} C_{i_4} - C_{i_1} B_{i_2} C_{i_3} B_{i_4}].$$

If $t \nmid B_{i_1}C_{i_2}B_{i_3}C_{i_4} - C_{i_1}B_{i_2}C_{i_3}B_{i_4}$, then there is no isomorphism between $\mathbb{M}'$ and $\mathbb{M}$. If $t \mid B_{i_1}C_{i_2}B_{i_3}C_{i_4} - C_{i_1}B_{i_2}C_{i_3}B_{i_4}$, then we simply set $\alpha = 1$, and compute δ and γ from the above relations (as before, we set $\beta = 0$). □

Remark 2.1. To classify all non-isomorphic indecomposable modules given by the previous theorem, we use exactly the same arguments as in Section 5 in [4]. To each $\beta \in \mathbb{C}\backslash\{-1,0,1\}$ corresponds an indecomposable module M_β defined by $B_{i_1} = 1$, $B_{i_2} = \beta$, $B_{i_3} = -1$, $B_{i_4} = -\beta$, and $B_{i_5} = 0$. Here, $i_j < i_{j+1}$. It was proved in [4] that $M_\beta \cong M_\gamma$ if and only if $\beta = \pm\gamma$, and that for a given indecomposable module $\mathbb{M}$ there exists β such that $\mathbb{M} \cong \mathbb{M}_\beta$. This means that all indecomposable modules in this case are parameterized by a single parameter β. Obviously, there are five different families (each in bijection with $\mathbb{C}$), depending on which i_j is set to be divisible by t.

2.3. **None of the five sums B_i is divisible by t.** Since $t \nmid B_i + B_{i+2}$ for at least one odd index i, there are three subcases we have to consider. The first subcase is when $t \mid B_i + B_{i+2}$ and $t \mid B_{i+2} + B_{i+4}$ for a unique odd index i. The second subcase is when $t \mid B_i + B_{i+2}$ for a unique odd index i. The third subcase is when $t \nmid B_i + B_{i+2}$ for all odd i. In the last two cases, we get infinitely many non-isomorphic indecomposable modules as we will show.

As before, let us assume that $(c_i)_1^{10}$ is another 10-tuple giving rise to a module $\mathbb{M}'$ satisfying the same divisibility conditions as the module $\mathbb{M}$.

Assume that there is a unique odd index i such that $t \mid B_i + B_{i+2}$ and $t \mid B_{i+2} + B_{i+4}$. Recall that whenever we state the divisibility conditions for the b_i's, we assume that the same conditions hold for the c_i's.

Theorem 2.6. *If l is odd such that $t \mid B_l + B_{l+2}$, $t \mid B_{l+2} + B_{l+4}$, $t \nmid B_i + B_{i+2}$, for $i \neq l, l+2$, and $t \mid C_l + C_{l+2}$, $t \mid C_{l+2} + C_{l+4}$, $t \nmid C_i + C_{i+2}$, for $i \neq l, l+2$, then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic.*

Proof. Without loss of generality we can assume that $l = 3$. Keeping the same notation as before when constructing isomorphisms, because $t \mid B_3 + B_5$, $t \mid B_5 + B_7$, the divisibility conditions (2.1) reduce to:

$$t \mid -\alpha B_1 + \delta C_1 - B_1C_1t^{-1}\gamma,$$
$$t \mid -\alpha(B_1 + B_3) + \delta(C_1 + C_3) - (B_1 + B_3)(C_1 + C_3)t^{-1}\gamma.$$

Now, we proceed as in the proof of Theorem 2.3 in order to construct an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$. □

There are five non-isomorphic indecomposable modules arising in this subcase, one for each index $l \in \{1, 3, 5, 7, 9\}$.

Assume that there is a unique odd index i such that $t \mid B_i + B_{i+2}$ and $t \nmid B_j + B_{j+2}$, for $j \neq i$.

Theorem 2.7. *If l is odd such that $t \mid B_l + B_{l+2}$, $t \nmid B_i + B_{i+2}$, for $i \neq l$, and $t \mid C_l + C_{l+2}$, $t \nmid C_i + C_{i+2}$, for $i \neq l$, then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if*

$$t \mid (B_{l-2} + B_l)C_lB_{l+4}C_{l+6} - (C_{l-2} + C_l)B_lC_{l+4}B_{l+6}.$$

Proof. Without loss of generality, assume that $l = 3$. As before, if there were an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$, its coefficients would have to satisfy the following conditions that we obtain from (2.1):

$$t \mid -\alpha B_1 + \delta C_1 - B_1 C_1 t^{-1}\gamma,$$
$$t \mid -\alpha(B_1 + B_3) + \delta(C_1 + C_3) - (B_1 + B_3)(C_1 + C_3)t^{-1}\gamma,$$
$$t \mid \alpha B_9 - \delta C_9 - B_9 C_9 t^{-1}\gamma.$$

Now we proceed as in the proof of Theorem 2.5. □

Let us parameterize the indecomposable modules from the previous theorem. Let $\beta \in \mathbb{C}$ and denote by $\mathbb{M}_\beta$ the indecomposable module whose coefficients b_i satisfy $B_1 = \beta$, $B_3 = 1$, $B_5 = -1$, $B_7 = -\beta - 1$, and $B_9 = 1$. Also, $\beta \neq 0, -1, -2$.

Proposition 2.1. *Let $\mathbb{M}'$ be a module such that $t \mid C_l + C_{l+2}$, $t \nmid C_i + C_{i+2}$, for $i \neq l$. There exists $\beta \in \mathbb{C} \setminus \{0, -1, -2\}$ such that $\mathbb{M}' \cong \mathbb{M}_\beta$.*

Proof. Assume again that $l = 3$. By the previous theorem, if $\mathbb{M}$ and $\mathbb{M}_\beta$ were isomorphic, then $t \mid (C_1 + C_3)C_7\beta + C_3 C_9(\beta + 1)^2$. If γ_i is the constant term of C_i, then we set β to be a solution of the equation

$$(\beta + 1)^2 = -\gamma_3^{-1}\gamma_9^{-1}(\gamma_1 + \gamma_3)\gamma_7.$$

Since the right-hand side of the previous equation is invertible, $\beta \neq -1$. If $\beta = 0$ or $\beta = -2$, then $-\gamma_3\gamma_9 = (\gamma_1 + \gamma_3)\gamma_7$, and subsequently, $-\gamma_3\gamma_9 - \gamma_3\gamma_7 = \gamma_1\gamma_7$. From $\gamma_1 + \gamma_7 + \gamma_9 = 0$ (this follows from $t \mid C_1 + C_7 + C_9$), we get $\gamma_3\gamma_1 = \gamma_1\gamma_7$. Thus, $\gamma_3 = \gamma_7$. This means that $\gamma_7 + \gamma_5 = \gamma_3 + \gamma_5 = 0$, which is not possible since $t \nmid C_5 + C_7$. Hence, $\beta \neq 0, -1, -2$. □

It is clear that $\mathbb{M}_\beta \cong \mathbb{M}_\gamma$ if and only if $(1+\beta)^2 = (1+\gamma)^2$. This means that either $\beta = \gamma$ or $\beta + \gamma = -2$. This means that the non-isomorphic indecomposable modules given in this subcase are parameterized by the set $\mathbb{C} \setminus \{0, -1, -2\}$, where we identify two points if they sum up to -2.

There are five different families (each in bijection with $\mathbb{C}$) of indecomposable modules arising in this subcase, one for each $l \in \{1, 3, 5, 7, 9\}$.

Assume that $t \nmid B_i + B_{i+2}$, for all odd i.

Theorem 2.8. *If $t \nmid B_i + B_{i+2}$ and $t \nmid C_i + C_{i+2}$, for all odd i, then the modules $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if the following conditions hold:*

$$t \mid C_1 B_3(C_5 + C_7)B_9 - B_1 C_3(B_5 + B_7)C_9,$$
$$t \mid C_1 B_3 C_5(B_7 + B_9) - B_1 C_3 B_5(C_7 + C_9).$$

Proof. As before, if there were an isomorphism $\varphi = (\varphi_i)$ between $\mathbb{M}$ and $\mathbb{M}'$, the coefficients of $\varphi_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ would have to satisfy the following conditions that we obtain from (2.1):

$$t \mid -\alpha B_1 + \delta C_1 - B_1 C_1 t^{-1}\gamma,$$

$$t \mid -\alpha(B_1+B_3)+\delta(C_1+C_3)-(B_1+B_3)(C_1+C_3)t^{-1}\gamma,$$
$$t \mid \alpha(B_7+B_9)-\delta(C_7+C_9)-(B_7+B_9)(C_7+C_9)t^{-1}\gamma,$$
$$t \mid \alpha B_9-\delta C_9-B_9C_9t^{-1}\gamma.$$

Now, we use the same calculations as in the proof of Theorem 2.5 in order to obtain the desired divisibility conditions. The trick is to use any three of the above divisibility conditions and treat them as in the proof of Theorem 2.5. For example, we use the first two and the last condition, and treat B_5+B_7 as B_5 in the proof of Theorem 2.5. This gives us that $t \mid C_1B_3(C_5+C_7)B_9-B_1C_3(B_5+B_7)C_9$. Analogously, use the first three conditions and treat B_7+B_9 as B_7 in the proof of Theorem 2.5 to obtain $t \mid C_1B_3C_5(B_7+B_9)-B_1C_3B_5(C_7+C_9)$.

Conversely, if the given conditions hold, by setting $\alpha=1$, one easily computes δ and γ from the above relations: $\delta = B_1C_3B_3^{-1}C_1^{-1}(B_1+B_3)(C_1+C_3)^{-1}$, $\gamma = t(-C_1^{-1}+\delta B_1^{-1})$. We set $\beta=0$. □

We are left to parameterize the indecomposable modules from the previous theorem.

Denote by $\mathbb{M}$ the indecomposable module corresponding to the coefficients $B_i=b_i+b_{i+1}$, for odd i. Since $\sum B_i=0$, we can rescale so that one of the B_i's is equal to 1, say B_7, because from the previous theorem it holds that the module $\mathbb{M}$ is isomorphic to the module corresponding to the coefficients $C_i=B_iB_7^{-1}$, for $i\neq 7$, and $C_7=1$.

Let $\mathbb{M}'$ denote another indecomposable module determined by the coefficients $C_1=\alpha$, $C_3=\beta$, $C_5=\gamma$, $C_7=1$, and $C_9=\delta$, all of them being complex numbers such that $\alpha+\beta+\gamma+1+\delta=0$. Also, $\alpha,\beta,\gamma,\delta\neq 0$, $\gamma,\delta\neq -1$, $\alpha+\beta\neq 0$, $\alpha+\delta\neq 0$, $\gamma+\beta\neq 0$. Under the assumption that $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic, we will express α,β, and δ as a function of γ and coefficients B_i. This will help us to find an appropriate parameterization of indecomposable modules in this subcase.

By the previous theorem, it must hold

$$t \mid \alpha(1+\gamma)B_3B_9-\beta\delta B_1(B_5+1),$$
$$t \mid \alpha\gamma B_3(1+B_9)-\beta(1+\delta)B_1B_5.$$

These two relations imply that

$$t \mid (1+\delta)(1+\gamma)B_5B_9-\gamma\delta(1+B_9)(B_5+1),$$
$$t \mid B_1^{-1}B_5^{-1}[\alpha\gamma B_3(1+B_9)-\alpha(1+\gamma)B_3B_5B_9(1+B_5)^{-1}]-\beta.$$

The first of the last two relations is equivalent to

$$t \mid B_5B_9(\alpha+\beta)-\gamma\delta(B_1+B_3).$$

Since, $\alpha+\beta=-1-\gamma-\delta$, this yields

$$t \mid -(1+\gamma)[1+\gamma B_5^{-1}B_9^{-1}(B_1+B_3)]^{-1}-\delta.$$

The last divisibility condition is under the assumption that $1+\gamma B_5^{-1}B_9^{-1}(B_1+B_3)$ is invertible, i.e., that $t \nmid 1+\gamma B_5B_9(B_1+B_3)^{-1}$. If this condition holds, then $\delta\neq 0$. If

$B_5^{-1}B_9^{-1}(B_1+B_3)=1$, then from $B_1+B_3+B_5+1+B_9=0$ we get $(B_5+1)(B_9+1)=0$, which is not possible. Thus, $1+\gamma B_5^{-1}B_9^{-1}(B_1+B_3)\neq 1+\gamma$ and $\delta\neq -1$. Also, from $t\mid B_5B_9(\alpha+\beta)-\gamma\delta(B_1+B_3)$ follows that $\alpha+\beta\neq 0$ because $\gamma\delta(B_1+B_3)$ is invertible.

From $t\mid B_5B_9(\alpha+\beta)-\gamma\delta(B_1+B_3)$ and $t\mid B_1^{-1}B_5^{-1}[\alpha\gamma B_3(1+B_9)-\alpha(1+\gamma)B_3B_5B_9(1+B_5)^{-1}]-\beta$, we get $t\mid \alpha[(1-B_1^{-1}(1+B_5)^{-1}B_3B_9)-\gamma B_1^{-1}(1+B_5)^{-1}B_3B_5^{-1}(B_1+B_3))]-\delta\gamma B_5^{-1}B_9^{-1}(B_1+B_3)$. If $t\nmid B_5B_3^{-1}(B_1+B_3)^{-1}(B_1(1+B_5)-B_3B_9)-\gamma$, then $(1-B_1^{-1}(1+B_5)^{-1}B_3B_9)-\gamma B_1^{-1}(1+B_5)^{-1}B_3B_5^{-1}(B_1+B_3))$ is invertible, and so $\alpha\neq 0$. If $t\mid\alpha+\delta$, then from $t\mid\alpha[(1-B_1^{-1}(1+B_5)^{-1}B_3B_9)-\gamma B_1^{-1}(1+B_5)^{-1}B_3B_5^{-1}(B_1+B_3))]-\delta\gamma B_5^{-1}B_9^{-1}(B_1+B_3)$ direct computation yields that $t\mid 1+\gamma B_5B_9(B_1+B_3)^{-1}$ which we already assumed is not true. Hence, $t\nmid\alpha+\delta$ and $\alpha+\delta\neq 0$. It is shown in a similar fashion that $\beta+\gamma\neq 0$.

Therefore, if we define δ to be the constant term of $-(1+\gamma)[1+\gamma B_5^{-1}B_9^{-1}(B_1+B_3)]^{-1}$, α to be the constant term of $-\delta\gamma[(1-B_1^{-1}(1+B_5)^{-1}B_3B_9)-\gamma B_1^{-1}(1+B_5)^{-1}B_3B_5^{-1}(B_1+B_3))]^{-1}B_5^{-1}B_9^{-1}(B_1+B_3)$, and β to be the constant term of $-\alpha B_1^{-1}(1+B_5)^{-1}B_3B_9[1+\gamma B_5^{-1}B_9^{-1}(B_1+B_3)]$, we get a parameterization of the coefficients of $\mathbb{M}$ with only one complex parameter γ involved. The other parameters, α, β, and δ, are expressed as a function of γ and the coefficients B_i. If we want to fix a value of one of the parameters α, β, γ, and δ, then the sum of the remaining three is fixed. Thus, one of them is determined by the remaining two, so we end up with a parameterization of the form, e.g., $\alpha,-2-\alpha-\gamma,\gamma,1,1$, with two parameters. Two such modules corresponding to different 5-tuples of parameters are isomorphic if and only if the divisibility conditions from Theorem 2.8 are satisfied. Thus, we identify two 5-tuples if and only if they satisfy the divisibility conditions from Theorem 2.8.

In this subcase there is only one family of indecomposable modules.

3. Conclusion

We explicitly constructed all rank 2 indecomposable Cohen-Macaulay $B_{k,n}$-modules in the case when $k=5$ and $n=10$. This is the smallest wild case containing modules whose profile layers are 5-interlacing. In this case, the only profiles with 5-interlacing layers are of the form $\{i,i+2,i+4,i+6,i+8\}\mid\{i+1,i+3,i+5,i+7,i+9\}$, where $i=1,2$. We constructed all indecomposable modules with such a profile, classified them up to isomorphism, and parameterized all infinite families of non-isomorphic rank 2 modules. All arguments and results are also valid for the general case (k,n) for all rank 2 modules with tightly 5-interlacing layers.

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[1]Faculty of Natural Sciences and Mathematics,
University of Banja Luka,
Address Mladena Stojanovica 2, 78000 Banja Luka, Bosnia and Herzegovina
Email address: dusko.bogdanic@pmf.unibl.org
ORCID iD: https://orcid.org/0000-0002-6511-7475

[2]Faculty of Electrical Engineering,
University of Banja Luka,
Address Patre 5, 78000 Banja Luka, Bosnia and Herzegovina
Email address: ivan-vanja.boroja@etf.unibl.org
ORCID iD: https://orcid.org/0000-0003-1836-0381

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SUFFICIENT CONDITIONS OF SUBCLASSES OF SPIRAL-LIKE FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTIONS

GANGADHARAN MURUGUSUNDARAMOORTHY[1] AND TEODOR BULBOACĂ[2]

Abstract. The purpose of the present paper is to find the sufficient conditions for some subclasses of analytic functions associated with Mittag-Leffler functions to be in subclasses of spiral-like univalent functions. Further, we discuss geometric properties of an integral operator related to Mittag-Leffler functions.

1. Introduction and Definitions

Let $\mathbf{E}_\alpha$ be the function defined by

$$\mathbf{E}_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C},\ \alpha \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0,$$

that was introduced by Mittag-Leffler [14] and commonly known as the *Mittag-Leffler function*. Wiman [25] defined a more general function $\mathbf{E}_{\alpha,\beta}$ generalizing the $\mathbf{E}_\alpha$ Mittag-Leffler function, that is

$$\mathbf{E}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},\ \alpha, \beta \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0.$$

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When $\beta = 1$, it is abbreviated as $\mathbf{E}_\alpha(z) = \mathbf{E}_{\alpha,1}(z)$. Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\mathbf{E}_{1,1}(z) = e^z, \quad \mathbf{E}_{1,2}(z) = \frac{e^z - 1}{z}, \quad \mathbf{E}_{2,1}\left(z^2\right) = \cosh z,$$
$$\mathbf{E}_{2,1}\left(-z^2\right) = \cos z, \quad \mathbf{E}_{2,2}\left(z^2\right) = \frac{\sinh z}{z}, \quad \mathbf{E}_{2,2}\left(-z^2\right) = \frac{\sin z}{z},$$
$$\mathbf{E}_4(z) = \frac{1}{2}\left(\cos z^{1/4} + \cosh z^{1/4}\right), \quad \mathbf{E}_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2} z^{1/3}\right)\right].$$

We recall the error function erf given by [1, p. 297]

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

the complement of the error function erfc defined by

$$\operatorname{erfc}(z) := 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

and the normalized form of the error function erf denoted by Erf (normalized with the condition $\operatorname{Erf}'(0) = 1$) is given by

$$\operatorname{Erf}(z) := \frac{\sqrt{\pi z}}{2} \operatorname{erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n.$$

It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2}\left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1}\right).$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found for example in [2, 3, 8, 9, 11, 12]. We note that the above generalized (Mittag-Leffler) function $\mathbf{E}_{\alpha,\beta}$ does not belongs to the family $\mathcal{A}$, where $\mathcal{A}$ represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$. Thus, it is expected to define the following normalization of Mittag-Leffler functions as below, due to Bansal and Prajapat [3]:

$$E_{\alpha,\beta}(z) := z\Gamma(\beta)\,\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \tag{1.2}$$

that holds for the parameters $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $z \in \mathbb{C}$. In this paper we shall confine our attention to the case of real-valued parameters α and β, and we will consider that $z \in \mathbb{D}$.

For functions $f \in \mathcal{A}$ be given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$, we define the *Hadamard product* (*or convolution*) of f and g by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

The two well known subclasses of $\mathcal{S}$ are namely the class of starlike and convex functions (for details see Robertson [20]). Thus, a function $f \in \mathcal{A}$ given by (1.1) is said to be *starlike of order* γ, $0 \le \gamma < 1$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{D},$$

and this function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be *convex of order* γ, $0 \le \gamma < 1$, if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{D},$$

and this class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} := \mathcal{K}(0)$ represents the well-known standard class of convex functions. By Alexander's duality relation (see [6]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

A function $f \in \mathcal{A}$ is said to be *spiral-like* if

$$\operatorname{Re}\left(e^{-i\xi}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some $\xi \in \mathbb{C}$ with $|\xi| < \frac{\pi}{2}$, and the class of spiral-like functions was introduced in [23]. Also, the function f is said to be convex spiral-like if $zf'(z)$ is spiral-like. Due to Murugusundramoorthy [15, 16], we consider the following subclasses of spiral-like functions as below.

Definition 1.1. For $0 \le \rho < 1$, $0 \le \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{S}(\xi, \gamma, \rho)$ by

$$\mathcal{S}(\xi,\gamma,\rho) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\xi}\frac{zf'(z)}{(1-\rho)f(z) + \rho zf'(z)}\right) > \gamma\cos\xi,\ z \in \mathbb{D} \right\}.$$

By virtue of Alexander's relation (see [6]) we define the following subclass $\mathcal{K}(\xi,\gamma,\rho)$.

Definition 1.2. For $0 \le \rho < 1$, $0 \le \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{K}(\xi,\gamma,\rho)$ by

$$\mathcal{K}(\xi,\gamma,\rho) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\xi} \frac{zf''(z) + f'(z)}{f'(z) + \rho z f''(z)} \right) > \gamma \cos\xi,\ z \in \mathbb{D} \right\}.$$

By specializing the parameter $\rho = 0$ in the above two definitions we obtain the subclasses $\mathcal{S}(\xi,\gamma) := \mathcal{S}(\xi,\gamma,0)$ and $\mathcal{K}(\xi,\gamma) := \mathcal{K}(\xi,\gamma,0)$, respectively.

Now we state a sufficient conditions for the function f to be in the above classes.

Lemma 1.1 ([15,16])**.** *A function f given by* (1.1) *is a member of* $\mathcal{S}(\xi,\gamma,\rho)$ *if*

$$\sum_{n=2}^{\infty} \Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \Big] |a_n| \le 1-\gamma,$$

where $|\xi| < \dfrac{\pi}{2}$, $0 \le \rho < 1$, $0 \le \gamma < 1$.

Since $f \in \mathcal{K}(\xi,\gamma,\rho)$ if and only if $zf'(z) \in \mathcal{S}(\xi,\gamma,\rho)$, and from Lemma 1.1 we get the next result.

Lemma 1.2. *A function f given by* (1.1) *is a member of* $\mathcal{K}(\xi,\gamma,\rho)$ *if*

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \Big] |a_n| \le 1-\gamma,$$

where $|\xi| < \frac{\pi}{2}$, $0 \le \rho < 1$, $0 \le \gamma < 1$.

The next class $\mathcal{R}^{\tau}(\vartheta,\delta)$ was introduced earlier by Swaminathan [24], and for special cases see the references cited there in.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(\vartheta,\delta)$, where $\tau \in \mathbb{C}\setminus\{0\}$, $0 < \vartheta \le 1$, and $\delta < 1$, if it satisfies the inequality

$$\left| \frac{(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1}{2\tau(1-\delta) + (1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1} \right| < 1, \quad z \in \mathbb{D}.$$

Lemma 1.3 ([24])**.** *If $f \in \mathcal{R}^{\tau}(\vartheta,\delta)$ is of the form* (1.1)*, then*

$$|a_n| \le \frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N}\setminus\{1\}. \tag{1.3}$$

The bounds given in (1.3) *is sharp for*

$$f(z) = \frac{1}{\vartheta z^{1-\frac{1}{\vartheta}}} \int_0^z t^{1-\frac{1}{\vartheta}} \left[1 + \frac{2(1-\delta)\tau\ t^{n-1}}{1-2^{n-1}} \right] dt.$$

Now we define the following new linear operator based on convolution (Hadamard) product.

For real parameters α, β, with $\alpha, \beta, \notin \{0, -1, -2, \dots\}$ and $E_{\alpha,\beta}$ be given by (1.2), we define the linear operator $\Lambda^\alpha_\beta : \mathcal{A} \to \mathcal{A}$ with the aid of the convolution product

$$\Lambda^\alpha_\beta f(z) := f(z) * E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} a_n z^n, \quad z \in \mathbb{D}.$$

Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example [5,10,13,21,22, 24]) and by the recent investigations related with distribution series (see for example [4,7,17–19], we obtain sufficient condition for the function $E_{\alpha,\beta}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions belonging in $\mathcal{R}^\tau(\vartheta,\delta)$ by using the convolution operator Λ^α_β. Finally, we determined conditions for the integral operator $\Psi^\alpha_\beta(z) = \int_0^z \frac{E_{\alpha,\beta}(t)}{t} dt$ to belong to the above classes.

2. Inclusion Results

In order to prove our main results, unless otherwise stated throughout this paper, we will use the notation (1.2), therefore

$$E_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{2.1}$$

$$E'_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{2.2}$$

$$E''_{\alpha,\beta}(1) = \sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \tag{2.3}$$

Theorem 2.1. *If*

$$[(1-\rho)\sec\xi + \rho(1-\gamma)]E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1) \le 2(1-\gamma), \tag{2.4}$$

then $E_{\alpha,\beta} \in \mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since $E_{\alpha,\beta}$ is defined by (1.2), according to Lemma 1.1 it is sufficient to show that

$$\sum_{n=2}^{\infty} \Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma. \tag{2.5}$$

Since the left-hand side of the inequality (2.5) could be written as

$$\begin{aligned} Q_1(\xi,\gamma,\rho) :=& \sum_{n=2}^{\infty} [(1-\rho)\sec\xi(n-1) + (1-\gamma)(1+n\rho-\rho)] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ =& [(1-\rho)\sec\xi + \rho(1-\gamma)] \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &+ (1-\rho)(1-\gamma-\sec\xi) \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \end{aligned}$$

therefore, by using (2.1) and (2.2), we get

$$\begin{aligned}
Q_1(\xi,\gamma,\rho) =&\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\Big[E'_{\alpha,\beta}(1)-1\Big]\\
&+(1-\rho)(1-\gamma-\sec\xi)\left[E_{\alpha,\beta}(1)-1\right]\\
=&\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E'_{\alpha,\beta}(1)+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1)\\
&-(1-\gamma).
\end{aligned}$$

Thus, from the assumption (2.4) it follows that $Q_1(\xi,\gamma,\rho)\leq 1-\gamma$, that is (2.5) holds, therefore $E_{\alpha,\beta}\in\mathcal{S}(\xi,\gamma,\rho)$. □

Theorem 2.2. *If*

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E''_{\alpha,\beta}(1)+(1-\gamma)E'_{\alpha,\beta}(1)\leq 2(1-\gamma), \tag{2.6}$$

then $E_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$.

Proof. Using the definition (1.2) of $E_{\alpha,\beta}$, in view of Lemma 1.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\leq 1-\gamma. \tag{2.7}$$

The left-hand side of the inequality (2.7) could be written as

$$\begin{aligned}
Q_2(\xi,\gamma,\rho) :=& \sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\
=&\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\
&+(1-\gamma)\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)},
\end{aligned}$$

and from (2.2) and (2.3) we get

$$Q_2(\xi,\gamma,\rho)=\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E''_{\alpha,\beta}(1)+(1-\gamma)[E'_{\alpha,\beta}(1)-1].$$

Hence, the assumption (2.6) implies that $Q_2(\xi,\gamma,\rho)\leq 1-\gamma$ that is (2.7) holds, and consequently $E_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$. □

3. Image Properties of Λ_β^α Operator

Making use of the Lemma 1.1 and Lemma 1.3 we will focus the influence of the Λ_β^α operator for the functions of the class $\mathcal{R}^\tau(\vartheta,\delta)$, and we will give sufficient conditions such that these images are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

Theorem 3.1. *If*

$$\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\left[E_{\alpha,\beta}(1)-1\right]$$

$$+\,(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt\le 1-\gamma, \tag{3.1}$$

then

$$\Lambda_\beta^\alpha\left(\mathcal{R}^\tau(\vartheta,\delta)\right)\subset\mathcal{S}(\xi,\gamma,\rho).$$

Proof. Let $f\in\mathcal{R}^\tau(\vartheta,\delta)$ be of the form (1.1). To prove that $\Lambda_\beta^\alpha(f)\in\mathcal{S}(\xi,\gamma,\rho)$, in view of Lemma 1.1 it is required to show that

$$\sum_{n=2}^{\infty}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\le 1-\gamma.$$

Let we denote the left-hand side of the above inequality by

$$Q_3(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|.$$

Since $f\in\mathcal{R}^\tau(\vartheta,\delta)$, by Lemma 1.3 we have

$$|a_n|\le\frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)},\quad n\in\mathbb{N}\setminus\{1\},$$

and using the inequality $1+\vartheta(n-1)\ge\vartheta n$ we obtain that

$$\begin{aligned}Q_3(\xi,\gamma,\rho)\le&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}\frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\right.\\&\left.\times\ \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}\\=&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right.\\&\left.+\ (1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\,\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}.\end{aligned}$$

From the above inequality, using (2.1), we get

$$\begin{aligned}Q_3(\xi,\gamma,\rho)\le&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\left[E_{\alpha,\beta}-1\right]\\&+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt,\end{aligned}$$

hence, the assumption (3.1) implies then $Q_3(\xi,\gamma,\rho)\le 1-\gamma$, that is $\Lambda_\beta^\alpha(f)\in\mathcal{S}(\xi,\gamma,\rho)$. □

Using Lemma 1.2 and following the same procedure as in the proof of Theorem 2.2, we have the subsequent result.

Theorem 3.2. *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}\left\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1)-(1-\gamma)\right\}\leq 1-\gamma, \tag{3.2}$$

then

$$\Lambda_\beta^\alpha\left(\mathcal{R}^\tau(\vartheta,\delta)\right)\subset\mathcal{K}(\xi,\gamma,\rho).$$

Proof. Let $f\in\mathcal{R}^\tau(\vartheta,\delta)$ be of the form (1.1). In view of Lemma 1.2, to prove that $\Lambda_\beta^\alpha(f)\in\mathcal{K}(\xi,\gamma,\rho)$ we have to show that

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\leq 1-\gamma. \tag{3.3}$$

Since $f\in\mathcal{R}^\tau(\vartheta,\delta)$, then by Lemma 1.3 we have

$$|a_n|\leq\frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)},\quad n\in\mathbb{N}\setminus\{1\},$$

and $1+\vartheta(n-1)\geq\vartheta n$. Denoting the left-hand side of the inequality (3.3) by

$$Q_4(\xi,\gamma,\rho):=\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|,$$

we deduce that

$$\begin{aligned}Q_4(\xi,\gamma,\rho)\leq&\frac{2|\tau|(1-\delta)}{\vartheta}\sum_{n=2}^{\infty}\Big[(1-\rho)\sec\xi(n-1)+(1-\gamma)(1+n\rho-\rho)\Big]\\&\times\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\=&\frac{2|\tau|(1-\delta)}{\vartheta}\left\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right.\\&\left.+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}.\end{aligned}$$

Now, by using (2.1) and (2.2), the above inequality yields to

$$\begin{aligned}Q_4(\xi,\gamma,\rho)\leq&\frac{2|\tau|(1-\delta)}{\vartheta}\left\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big][E'_{\alpha,\beta}(1)-1]\right.\\&\left.+(1-\rho)(1-\gamma-\sec\xi)[E_{\alpha,\beta}(1)-1]\right\}\\=&\frac{2|\tau|(1-\delta)}{\vartheta}\left\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E'_{\alpha,\beta}(1)\right.\\&\left.+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1)-(1-\gamma)\right\}.\end{aligned}$$

Therefore, the assumption (3.2) yields to $Q_4(\xi,\gamma,\rho) \le 1-\gamma$, which implies the inequality (3.3), that is $\Lambda^\alpha_\beta(f) \in \mathcal{K}(\xi,\gamma,\rho)$. □

4. The Alexander Integral Operator for $E_{\alpha,\beta}$

Theorem 4.1. *Let the function Ψ^α_β be given by*

$$\Psi^\alpha_\beta(z) = \int_0^z \frac{E_{\alpha,\beta}(t)}{t}dt, \quad z \in \mathbb{D}. \tag{4.1}$$

If

$$\Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big]E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1) \le 2(1-\gamma),$$

then $\Psi^\alpha_\beta \in \mathcal{K}(\xi,\gamma,\rho)$.

Proof. Since

$$\Psi^\alpha_\beta(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \cdot \frac{z^n}{n}, \quad z \in \mathbb{D}, \tag{4.2}$$

according to Lemma 1.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{1}{n}\cdot\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma,$$

or, equivalently

$$\sum_{n=2}^{\infty} \Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma.$$

Now, the proof of Theorem 4.1 is parallel to that of Theorem 2.1, and so it will be omitted. □

Theorem 4.2. *Let the function Ψ^α_β be given by* (4.1)*. If*

$$\begin{aligned}&\Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big](E_{\alpha,\beta}(1)-1)\\ &+ (1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt \le 1-\gamma,\end{aligned} \tag{4.3}$$

then $\Psi^\alpha_\beta \in \mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since Ψ^α_β has the power series expansion (4.2), then by Lemma 1.1 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma.$$

The left-hand side of the above inequality could be rewritten as

$$Q_5(\xi,\gamma,\rho) = \sum_{n=2}^{\infty} \frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big] \times \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}$$
$$= \sum_{n=2}^{\infty}\Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} + (1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)},$$

and using (2.1) we get

$$Q_5(\xi,\gamma,\rho) \le \Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big]\left[E_{\alpha,\beta}(1)-1\right] + (1-\rho)(1-\gamma-\sec\xi)\int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt.$$

Therefore, if the assumption (4.3) holds, then $Q_5(\xi,\gamma,\rho) \le 1-\gamma$. Hence, $\Psi_\beta^\alpha \in \mathcal{S}(\xi,\gamma,\rho)$. □

Remark 4.1. By taking $\rho = 0$ in Theorems 2.1–4.2, we can easily attain the sufficient condition for $E_{\alpha,\beta} \in \mathcal{S}(\xi,\gamma)$ and $E_{\alpha,\beta} \in \mathcal{K}(\xi,\gamma)$. The function $E_{\alpha,\beta}$ is associated with Mittag-Leffler functions and has not been studied sofar. We left this as an exercise to interested readers.

For the special case $\alpha = 1/2$ and $\beta = 1$, that is connected with the error function can derive some results based on the error function. Thus, a simple computation shows that if

$$\mathcal{E}(z) := E_{\frac{1}{2},1}(z) = \sum_{n=1}^{\infty}\frac{z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

then

$$\mathcal{E}(1) = \sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)},\quad \mathcal{E}'(1) = \sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)},\quad \mathcal{E}''(1) = \sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)},$$
$$\int_0^1\left(\frac{\mathcal{E}(t)}{t}-1\right)dt = \sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)},$$

(4.4) $$\mathcal{L} := \Lambda_1^{1/2}f(z) = f(z)*\mathcal{E}(z) = z + \sum_{n=2}^{\infty}\frac{a_n z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

(4.5) $$\mathcal{P} := \Psi_1^{1/2}(z) = \int_0^z \frac{\mathcal{E}(t)}{t}dt = \sum_{n=1}^{\infty}\frac{z^n}{n\Gamma\left(\frac{n+1}{2}\right)}.$$

Using the above relations, from Theorems 2.1 and 2.2 we get, respectively.

Example 4.1. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 2(1-\gamma),$$

then $\mathcal{E}\in\mathcal{S}(\xi,\gamma,\rho)$.

Example 4.2. If

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\gamma)\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\leq 2(1-\gamma),$$

then $\mathcal{E}\in\mathcal{K}(\xi,\gamma,\rho)$.

Similarly, Theorems 4.1 and 4.2 give us the next examples.

Example 4.3. If

$$\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}\leq 1-\gamma,$$

then

$$\mathcal{L}\left(\mathcal{R}^{\tau}(\vartheta,\delta)\right)\subset\mathcal{S}(\xi,\gamma,\rho),$$

where $\mathcal{L}$ is defined by (4.4).

Example 4.4. If

$$\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\Bigg\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}-(1-\gamma)\Bigg\}\leq 1-\gamma,$$

then

$$\mathcal{L}\left(\mathcal{R}^{\tau}(\vartheta,\delta)\right)\subset\mathcal{K}(\xi,\gamma,\rho),$$

where $\mathcal{L}$ is defined by (4.4).

Finally, from Theorems 4.1 and 4.2 we have the following.

Example 4.5. If

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 2(1-\gamma),$$

then $\mathcal{P}\in\mathcal{K}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5).

Example 4.6. If

$$\left[(1-\rho)\sec\xi+\rho(1-\gamma)\right]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 1-\gamma,$$

then $\mathcal{P}\in\mathcal{S}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5).

5. Conclusions

In this investigation we obtained sufficient conditions and inclusion results for functions $f\in\mathcal{A}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions by applying convolution operator with Mittag-Leffler functions.

The investigation methods are based on some recent results and techniques found in [15] and [16], and we determined sufficient conditions for the functions $E_{\alpha,\beta}$ to belongs to the new defined classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

Moreover, we found sufficient conditions such that the images of the functions belonging to the class $\mathcal{R}^{\tau}(\vartheta,\delta)$ by the new defined convolution operator Λ_{β}^{α} are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

Finally, we determined sufficient conditions such that the functions Ψ_{β}^{α} obtained as images of $E_{\alpha,\beta}$ via the Alexander integral operator belong to the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

We emphasize that till now such kind of results doesn't appeared in any previous articles: the general classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$ are completely new and introduced in [15, 16], while any type of such results were not studied previously.

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[1]School of Advanced Sciences,
Vellore Institute of Technology, deemed to be university,
Vellore-632014, Tamilnadu, India
Email address: gmsmoorthy@yahoo.com
ORCID iD: http://orcid.org/0000-0001-8285-6619

[2]Faculty of Mathematics and Computer Science,
Babeş-Bolyai University,
400084 Cluj-Napoca, Romania
Email address: bulboaca@math.ubbcluj.ro

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SOME BORDERENERGETIC AND EQUIENERGETIC GRAPHS

SAMIR K. VAIDYA[1] AND KALPESH M. POPAT[2]

Abstract. The sum of absolute values of eigenvalues of a graph G is defined as energy of graph. If the energies of two non-isomorphic graphs are same then they are called equienergetic. The energy of complete graph with n vertices is $2(n-1)$ and the graphs whose energy is equal to $2(n-1)$ are called borderenergetic graphs. It has been revealed that the graphs upto 12 vertices are borderenergetic. It is very challenging and interesting as well to search for borderenergetic graphs with more than 14 vertices. The present work is leap ahead in this direction as we have found a family of borderenergetic graphs of arbitrarily large order. We have also obtained three pairs of equienergetic graphs.

1. Introduction

For standard terminology and notations in graph theory we follow West [19] while the terms related to algebra are used in sense of Lang [11].

Let G be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* denoted by $A(G)$ of G is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if v_i is adjacent with v_j, and 0 otherwise.

The eigenvalues of A are called the eigenvalues of G. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G then

$$\operatorname{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

The energy $E(G)$ of graph G is the sum of all absolute values of eigenvalues of G. The concept of energy of graph was introduced by Gutman [7] in 1978. A brief account on energy of graph can be found in Cvetković [2] and Li [12].

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The graphs of order n, whose energy exceeds than the energy of the complete graph K_n are called hyperenergetic graphs otherwise graphs of order n with $E(G) \leq E(K_n)$, are called non-hyperenergetic. As mentioned in Gutman [7] $E(K_n) = 2(n-1)$. Are there any graphs other than K_n with such behaviour?

This question motivated Gong et al. [6] to introduce a new concept. According to them, the graph G of order n satisfying $E(G) = 2(n-1)$ are called *borderenergetic*. Obviously, the complete graph K_n is borderenergetic. Gong et al. [6] have proved that such graphs exist for $n = 7, 8, 9$. Li et al. [13] and Shao et al. [15] have obtained the graphs with $n = 10$ and $n = 11$ respectively while Furtula and Gutman [4] have obtained the graphs with $n = 12$. A family of non-regular and non-integral borderenergetic graphs with particular behaviour were investigated by Hou and Tao [16]. Some new families of borderenergetic graphs were obtained by Jahfar et al. [10]. Recently, a survey on borderenergetic graphs was published by Ghorbani et al. [5].

We will introduce some concepts and also state some existing results for our ready reference.

Definition 1.1. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Proposition 1.1 ([17]). *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G, then $2n$ eigenvalues of $D_2(G)$ are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n, 0$ (n times).*

Proposition 1.2 ([3]). *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric block matrix. Then the spectrum of A is the union of spectra of $A_0 + A_1$ and $A_0 - A_1$.

Definition 1.2. The *extended shadow graph* $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G''.

A curious question: **How the energy of a given graph G can be correlated with the larger graph obtained by means of graph operations on G?** To quench this thirst we have considered shadow graph and extended shadow graph as these graphs are of same order but they are non isomorphic. Due to this specific characteristic, the said graphs are used to construct non-co spectral equienergetic graphs by constructing shadow graph of extended shadow graph as well as extended shadow graph of shadow graph.

2. Energy of Extended Shadow Graph

Theorem 2.1. *Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $|\lambda_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then $E(D_2^*(G)) = 2E(G) + n + \theta$, where θ is the difference between the number of positive and negative eigenvalues of G.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of graph G. Then the $A(G)$ is given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}.$$

Consider a second copy of graph G with vertices $u_1, u_2, u_3, \ldots, u_n$ and join u_i with neighbors of v_i and with v_i, $1 \leq i \leq n$, to obtain $D_2^*(G)$. Then the $A(D_2^*(G))$ can be written as a block matrix as follows

$$A(D_2^*(G)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdot & \boldsymbol{u_n} \end{array} \\ \left[\begin{array}{ccccc|ccccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} & 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 1 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \\ \hline 1 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 1 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 1 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(D_2^*(G)) = \begin{bmatrix} A(G) & A(G)+I \\ A(G)+I & A(G) \end{bmatrix}.$$

Hence, by Proposition 1.2 spectrum of $D_2^*(G)$ is union of spectra of $2A(G)+I$ and $-I$. Therefore, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G, then

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_i + 1 & -1 \\ n & n \end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned}
E(D_2^*(G)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\
&= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\
&= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\
&= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\
&= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n \\
&= 2E(G) + n + \theta.
\end{aligned}$$

□

The following corollary proves the existence of borderenergetic graph of arbitrarily large order.

Corollary 2.1. $E(D_2^*(K_{n,n)}) = E(K_{4n})$. *That is,* $D_2^*(K_{n,n})$ *is non complete borderenergetic graph.*

Proof. Consider complete bipartite graph $K_{n,n}$ of $2n$ vertices then

$$\operatorname{spec}(K_{n,n}) = \begin{pmatrix} n & -n & 0 \\ 1 & 1 & 2n-2 \end{pmatrix}.$$

Now, $D_2^*(K_{n,n})$ is a graph with $4n$ vertices and by Theorem 2.1 its spectrum is

$$\operatorname{spec}(D_2^*(K_{n,n})) = \begin{pmatrix} 2n+1 & -2n+1 & 1 & -1 \\ 1 & 1 & 2n-2 & 2n \end{pmatrix}. \tag{2.1}$$

Also,

$$\operatorname{spec}(K_{4n}) = \begin{pmatrix} 4n-1 & -1 \\ 1 & 4n-1 \end{pmatrix}. \tag{2.2}$$

Clearly from (2.1) and (2.2) $D_2^*(K_{n,n})$ and K_{4n} are non co-spectral and

$$\begin{aligned}
E(D_2^*(K_{n,n})) &= \sum_{i=1}^{4n} |\lambda_i| \\
&= (2n+1) + (2n-1) + (2n-2) + 2n \\
&= 8n - 2 = 2(4n-1) = E(K_{4n}).
\end{aligned}$$

Thus, $E(D_2^*(K_{n,n})) = E(K_{4n})$. Hence, $D_2^*(K_{n,n})$ is non complete borderenergetic graph. □

3. Equienergetic Graphs

Definition 3.1. Two non-isomorphic graphs G_1 and G_2 of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$.

In 2007 Ramane et al. have proved that there exists a pair of connected non-cospectral, equienergetic graphs with n vertices for all $n \geq 9$.

Definition 3.2. The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in G.

Harary [8] defined the concept of iterated line graphs. According to him if G is graph and $L^1(G) = L(G)$ be its line graph, then $L^2(G) = L(L(G))$, $L^3(G) = L(L^2(G)), \dots, L^k(G) = L(L^{k-1}(G)), \dots$

Ramane et al. [14] have proved that if G_1 and G_2 are regular graphs of same order, then for $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.

Definition 3.3. The *cartesian product* of graphs G and H is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The following result gives the spectra of the Cartesian product of graphs.

Proposition 3.1 ([1]). *Let G_1 and G_2 are two graphs having spectra as $\mu_1, \mu_2, \dots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2}$, respectively. Then spectra of $G = G_1 \times G_2$ is $\mu_i + \sigma_j$, where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

Theorem 3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of graph G. Then $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of graph G. By Proposition 3.1

$$\operatorname{spec}(G \times K_2) = \begin{pmatrix} \lambda_1 + 1 & \lambda_2 + 1 & \cdots & \lambda_n + 1 & \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_n - 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

According to Theorem 2.1,

$$\operatorname{spec}(D_2^*(G \times K_2)) = \begin{pmatrix} 2\lambda_1 + 3 & \cdots & 2\lambda_n + 3 & 2\lambda_1 - 1 & \cdots & 2\lambda_n - 1 & -1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 2n \end{pmatrix}. \tag{3.1}$$

Moreover, by Theorem 2.1,

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1 + 1 & 2\lambda_2 + 1 & \cdots & 2\lambda_n + 1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Proposition 1.1,

$$\operatorname{spec}(D_2(D_2^*(G))) = \begin{pmatrix} 4\lambda_1 + 2 & 4\lambda_2 + 2 & \cdots & 4\lambda_n + 2 & -2 & 0 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.2}$$

If for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2}$, then

$$\left|\lambda_i + \frac{3}{2}\right| = \begin{cases} |\lambda_i| + \frac{3}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Also for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2} > \frac{1}{2}$,

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i < 0, \end{cases}$$

$$\left|\lambda_i - \frac{1}{2}\right| = \begin{cases} |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i < 0. \end{cases}$$

From (3.1) and (3.2)

$$\begin{aligned} E(D_2^*(G \times K_2)) &= \sum_{i=1}^{n} |2\lambda_i + 3| + \sum_{i=1}^{n} |2\lambda_i - 1| + 2n \\ &= 2\sum_{i=1}^{n} |\lambda_i + \frac{3}{2}| + 2\sum_{i=1}^{n} |\lambda_i - \frac{1}{2}| + 2n \\ &= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i>0} \left|\lambda_i - \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i - \frac{1}{2}\right|\right) + 2n \\ &= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{3}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{3}{2}\right)\right) \\ &\quad + 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| - \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| + \frac{1}{2}\right)\right) + 2n \\ &= 2\left(2\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i>0} 1\right)\right) + 2n \\ &= 4E(G) + 2\theta + 2n \qquad (3.3) \end{aligned}$$

and

$$\begin{aligned} E(D_2(D_2^*(G))) &= \sum_{i=1}^{n} |4\lambda_i + 2| + 2n \\ &= 4\sum_{i=1}^{n} |4\lambda_i + \frac{1}{2}| + 2n \\ &= 4\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + 2n \\ &= 4\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + 2n \end{aligned}$$

$$= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 2n$$

$$= 4E(G) + 2\theta + 2n. \tag{3.4}$$

Hence, from (5) and (6), $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$ □

Let $D_2^{**}(G)$ be extended shadow graph of $D_2^*(G)$, i.e., $D_2^{**}(G) = D_2^*(D_2^*(G))$ and if G be a bipartite graph, then it is well-known that the spectra of G is symmetric about the origin, so half of the non-zero eigenvalues of G lies to the left and half lies to the right of the origin. Therefore if G is a bipartite graph having all its eigenvalues nonzero, the number of positive and negative eigenvalues of G are same. Keeping this into mind we have the following result.

Theorem 3.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of a bipartite graph G. Then $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic if and only if $|\lambda_i| \geq \frac{3}{4}$ for $1 \leq i \leq n$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of bipartite graph G. By Theorem 2.1

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1+1 & 2\lambda_2+1 & \cdots & 2\lambda_n+1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

and

$$\operatorname{spec}(D_2^{**}(G)) = \begin{pmatrix} 4\lambda_1+3 & 4\lambda_2+3 & \cdots & 4\lambda_n+3 & -1 \\ 1 & 1 & \cdots & 1 & 3n \end{pmatrix}. \tag{3.5}$$

Moreover, by Proposition 1.1,

$$\operatorname{spec}(D_2(G)) = \begin{pmatrix} 2\lambda_1 & 2\lambda_2 & \cdots & 2\lambda_n & 0 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Theorem 2.1,

$$\operatorname{spec}(D_2^*(D_2(G))) = \begin{pmatrix} 4\lambda_1+1 & 4\lambda_2+1 & \cdots & 4\lambda_n+1 & 1 & -1 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.6}$$

Clearly, from (3.5) and (3.6), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are non-co spectral graphs. As G is bipartite graph we have,

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Assume that for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{4}$. Hence,

$$\left|\lambda_i + \frac{3}{4}\right| = \begin{cases} |\lambda_i| + \frac{3}{4}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{4}, & \text{if } \lambda_i < 0. \end{cases}$$

From (3.5)

$$E(D_2^{**}(G)) = \sum_{i=1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1|$$

$$
\begin{aligned}
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{3}{4}\right| + 3n \\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{3}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{3}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{3}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n \\
&= 4E(G) + 3n. \qquad (3.7)
\end{aligned}
$$

Also if for all $1 \le i \le n$, $|\lambda_i| \ge \frac{3}{4} \ge \frac{1}{4}$, then

$$
\left|\lambda_i + \frac{1}{4}\right| = \begin{cases} |\lambda_i| + \frac{1}{4}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{4}, & \text{if } \lambda_i < 0. \end{cases}
$$

From (3.6)

$$
\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{n}|4\lambda_i + 1| + n + 2n \\
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{1}{4}\right| + 3n \\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{1}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n \\
&= 4E(G) + 3n. \qquad (3.8)
\end{aligned}
$$

Thus, by (3.7) and (3.8), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are equienergetic graphs.

Conversely, suppose that the graphs $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic. We will show that $|\lambda_i| \ge \frac{3}{4}$ for $1 \le i \le n$.

Assume to the contrary that let $|\lambda_i| < \frac{3}{4}$ for some i. Then for the same i, $\left|\lambda_i + \frac{3}{4}\right| = \lambda_i + \frac{3}{4}$. Without loss of generality, suppose that the eigenvalues of G satisfy $|\lambda_i| \ge \frac{3}{4}$, for $i = 1, 2, \ldots, k$ and $|\lambda_i| < \frac{3}{4}$, for $i = k+1, k+2, \ldots, n$, since the eigenvalues are real and reordering does not affect the argument. We have the following cases to be considered.

Case I If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 6n. \qquad (3.9)
\end{aligned}$$

Case II If $\lambda_i > 0$ for $i = 1, 2, \cdots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \cdots, n$. If θ_0 is the number of zero eigenvalues of G, we have

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{3}{4}\right)\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}\theta_0\right) + 3n. \qquad (3.10)
\end{aligned}$$

Case III If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\theta_0\right) + 3n. \qquad (3.11)
\end{aligned}$$

Case IV If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$,

$$
\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}n\right) + 3n. \qquad (3.12)
\end{aligned}
$$

While

$$
E(D_2^*(D_2(G))) = 4\sum_{i=1}^{n} |\lambda_i| + 3n,
$$

which remain same in each of the above cases only if $|\lambda_i| \geq \frac{1}{4}$ for $i = k+1, k+2, \ldots, n$.

If $|\lambda_i| < \frac{1}{4}$ for $i = k+1, k+2, \ldots, n$, then we have the following.

Case I If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$
\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{1}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 4n. \qquad (3.13)
\end{aligned}
$$

Case II If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$, and if θ_0 is the number of zero eigenvalues of G, then we have

$$
\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{1}{4}\right)\right) + 3n
\end{aligned}
$$

$$> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n$$

$$= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}\theta_0\right) + 3n. \tag{3.14}$$

Case III If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n\\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n\\
&= 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n\\
&= 4\left(\sum_{i=1}^{n}|\lambda_i| + \frac{1}{4}\theta_0\right) + 3n.
\end{aligned}$$

Case IV If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n\\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n\\
&> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n
\end{aligned}$$

$$= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}n\right) + 3n. \tag{3.15}$$

Clearly, in all the cases discussed above, we have $E(D_2^{**}(G)) \neq E(D_2^*(D_2(G)))$, a contradiction. Hence, the result follows. □

Corollary 3.1. *If G_1 and G_2 are two equienergetic bipartite graphs with $|\lambda_i| \geq \frac{1}{2}$ and $|\mu_i| \geq \frac{1}{2}$, where λ_i and μ_i are the eigenvalues of G_1 and G_2, respectively, for all $1 \leq i \leq n$, then $D_2^*(G_1)$ and $D_2^*(G_2)$ are non cospectral equienergetic.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ be eigenvalues of G_1 and G_2, respectively. Then by Theorem 2.1 spectrum of G_1 and G_2 are given by

$$\operatorname{spec}(D_2^*(G_1)) = \begin{pmatrix} 2\lambda_i + 1 & -1\\ n & n\end{pmatrix}, \quad \operatorname{spec}(D_2^*(G_2)) = \begin{pmatrix} 2\mu_i + 1 & -1\\ n & n\end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$ for $i = 1, 2, \ldots, n$. Then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned} E(D_2^*(G_1)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\ &= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\ &= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\ &= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\ &= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n. \end{aligned}$$

As G_1 is bipartite graph

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Hence, $E(D_2^*(G_1)) = 2E(G_1) + n$.

Similarly, if $|\mu_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then

$$E(D_2^*(G_2)) = 2E(G_2) + n.$$

Since, G_1 and G_2 are equienergetic graphs $D_2^*(G_1)$ and $D_2^*(G_1)$ are equienergetic. □

4. Extended m-Shadow Graph and Graph Energy

Definition 4.1. The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$. Vaidya and Popat [18] have proved that $E(D_m(G)) = mE(G)$.

Definition 4.2. The extended *m-shadow graph* $D_m^*(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v and with v in G_j, $1 \leq i, j \leq m$.

Definition 4.3. Let $A \in R^{m\times n}$, $B \in R^{p\times q}$. Then the *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Proposition 4.1 ([9]). *Let $A \in M^m$ and $B \in M^n$. Furthermore, let λ is an eigenvalues of matrix A with corresponding eigenvector x and μ is an eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Theorem 4.1. *Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with $|\lambda_i| \geq \frac{m-1}{m}$, for all $1 \leq i \leq n$. Then $E(D_m^*(G)) = mE(G) + (m-1)n + (m-1)\theta$, where θ is the difference between the number of positive and negative eigenvalues of G.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph G. Then its adjacency matrix of G is same as in the proof of Theorem 2.2. Consider m copies of graph G say $G_1, G_2, \dots, G_k$ with vertices $v_i^1, v_i^2, \dots, v_i^m$, $1 \leq i \leq n$, to obtain $D_m^*(G)$ such that each vertex u in G_j is joined to the neighbors of the corresponding vertex v as well as with v in G_k, $1 \leq j, k \leq m$. Then the $A(D_m^*(G))$ can be written as a block matrix as follow

$$A(D_m^*(G)) = \begin{bmatrix} A(G) & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G) & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G) \end{bmatrix}_m,$$

$$\begin{aligned} A(D_m^*(G)) + I_{mn} &= \begin{bmatrix} A(G)+I & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G)+I & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G)+I \end{bmatrix}_m \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_m \otimes (A(G)+I) \\ &= J_m \otimes (A(G)+I). \end{aligned}$$

Hence, by Proposition 4.1, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G, then

$$\begin{aligned} \operatorname{spec}(D_m^*(G) + I) &= \begin{pmatrix} m(\lambda_i + 1) & 0(\lambda_i + 1) \\ n & mn - n \end{pmatrix} \\ &= \begin{pmatrix} m(\lambda_i + 1) & 0 \\ n & mn - n \end{pmatrix}, \\ \operatorname{spec}(D_m^*(G)) &= \begin{pmatrix} m\lambda_i + (m-1) & -1 \\ n & mn - n \end{pmatrix}. \end{aligned}$$

Suppose that $|\lambda_i| \geq \frac{m-1}{m}$ for all $1 \leq i \leq n$. Then

$$\left|\lambda_i + \frac{m-1}{m}\right| = \begin{cases} |\lambda_i| + \frac{m-1}{m}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{m-1}{m}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned} E(D_m^*(G)) &= \sum_{i=1}^{n} |m\lambda_i + (m-1)| + \sum_{i=1}^{(m-1)n} |-1| \\ &= m\sum_{i=1}^{n}\left|\lambda_i + \frac{m-1}{m}\right| + (m-1)n \\ &= m\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{m-1}{m}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{m-1}{m}\right|\right) + (m-1)n \\ &= m\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{m-1}{m}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{m-1}{m}\right)\right) + (m-1)n \\ &= m\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{m-1}{m}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + (m-1)n \\ &= mE(G) + (m-1)n + (m-1)\theta. \end{aligned}$$ □

5. Concluding Remarks

The energy of extended shadow graph has been obtained and using it a new family of non complete borderenergetic graphs and new pairs of non cospectral equienergetic graphs have been investigated.

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[1]Department of Mathematics,
Saurashtra University,
Rajkot(Gujarat), India
Email address: samirkvaidya@yahoo.co.in
ORCID iD: https://orcid.org/0000-0002-5394-1424

[2]Department of Computer Applications-MCA,
Atmiya University,
Rajkot(Gujarat), India
Email address: kalpeshmpopat@gmail.com
ORCID iD: https://orcid.org/0000-0002-8504-2466

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COMMON FIXED POINTS OF GENERALIZED α-NONEXPANSIVE MULTIVALUED MAPPINGS VIA MODIFIED S-TYPE ITERATION

R. SADHU[1,2] AND C. NAHAK[1]

Abstract. In this paper, we introduce a class of generalized α-nonexpansive multivalued mapping and study some of its important properties. In particular, this class is a multivalued version of the single-valued nonexpansive mapping, called generalized α-nonexpansive mapping proposed by Pant and Shukla [11]. A modified S-type iteration scheme is proposed to approximate the common fixed point of two multivalued mappings. Our algorithm provides a multivalued extension of the method given by Khan et al. [6]. Strong and weak convergence of the iterative process are also proved under suitable assumptions.

1. Introduction and Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and C be a nonempty subset of E. In this article $CB(C)$ denotes the collection of all nonempty closed and bounded subsets of C, $KC(C)$ denotes the collection of all nonempty compact convex subsets of C and $P(C)$ the collection of nonempty proximinal bounded subsets of C. A set C is said to be proximinal if for any $x \in E$, there exists an element $y \in C$ such that $\|x-y\| = d(x, C)$, where $d(x, C) = \inf\{\|x - y\| : y \in C\}$. Let A and B be two nonempty closed and bounded subsets of C. Then the Hausdorff distance H between A and B is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\right\}. \tag{1.1}$$

A multivalued mapping $T : C \to CB(C)$ is said to have a fixed point in C, if there exists a point $p \in C$ such that $p \in T(p)$. The set of all fixed points of T is denoted

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by $F(T)$, i.e., $F(T) = \{x \in C : x \in Tx\}$. We define $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$.

A mapping $T : C \to CB(C)$ is said to be

(a) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad \text{for all } x, y \in C;$$

(b) quasi nonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \quad \text{for all } x \in C \text{ and } p \in F(T);$$

(c) satisfying condition (C) if for $x, y \in C$

$$\frac{1}{2}d(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|;$$

(d) α-nonexpansive if for any $\alpha \in [0, 1)$

$$H(Tx, Ty)^2 \leq \alpha d(x, Ty)^2 + \alpha d(y, Tx)^2 + (1 - 2\alpha)\|x - y\|^2, \quad \text{for all } x, y \in C.$$

The following definitions and lemmas are useful in the subsequent sections.

Definition 1.1 (Opial's property [10])**.** A Banach space E is said to satisfy the Opials condition if for any sequence $\{x_n\}$ converging to z weakly and $z \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 1.1 ([12])**.** *Let E be a uniformly convex Banach space, with $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \in (0, 1)$ for all $n \geq 1$. $\{x_n\}$ and $\{y_n\}$ be sequences of E such that* $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ *and* $\lim_{n\to\infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = a$ *for some $a \geq 0$. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Fixed point theories for multivalued contractive and nonexpansive mappings, using the Hausdorff metric was first coined by Markin [8] and Nadler [9]. Since then several new classes of nonexpansive multivalued mapping appeared in the literature. In 2008, Suzuki [13] introduced the class of nonexpansive mapping, termed as generalized nonexpansive mapping (i.e., mapping satisfying condition (C)). A multivalued analog of [13] was proposed by Eslamian and Abkar [1] in a uniformly convex Banach space. Several other interesting generalizations for multivalued mappings are available in literature see for example [3–5].

In 2017, another new class of nonexpansive single-valued mapping, called generalized α-nonexpansive mapping, which properly contains the class of Suzuki-type mapping, was proposed in [11]. Motivated by the work of [11], in this paper we introduce the class of generalized α-nonexpansive multivalued mapping and prove several of its important properties in a uniformly convex Banach space.

Different iterative processes have been instrumented to approximate the fixed points of single-valued and multi-valued nonexpansive mappings. To describe some relevant iterative processes, let C be a nonempty convex subset of a Banach space E and $\mathcal{T} : C \to C$, then

(a) Mann iterates

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{T} x_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is in $(0,1)$;

(b) Ishikawa iterates

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{T} y_n, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$;

(c) Agarwal et al. [2] introduced the S-iterates

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)\mathfrak{T} x_n + \alpha_n \mathfrak{T} y_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They proved that this scheme converges at a rate faster than both Picard iteration scheme and Mann iteration scheme for contractions. Following their method, it was observed that S-iteration scheme also converges faster than Ishikawa iteration scheme.

The multivalued extension of [2] was proposed by Khan and Yildirim [7]. Let $T : C \to CB(C)$ be a multivalued mapping then, their scheme runs as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{1.5}$$

where $v_n \in P_T x_n$, $u_n \in P_T y_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

A modification of [2] captures the common fixed point of two single valued mapping $\mathcal{S}$ and $\mathfrak{T}$ was proposed by Khan et al. [6], the modified S-iteration is as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)\mathfrak{T} x_n + \alpha_n \mathcal{S} y_n, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

We modify the iteration process which is given by (1.6) in [7] to the case of two multivalued mapping T and S as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{1.7}$$

where $v_n \in P_T x_n$, $u_n \in P_S y_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

In this present paper we study strong and weak convergence of the iteration process (1.7), for finding a common fixed point of a pair of generalized α-nonexpansive multivalued mapping in a uniformly convex Banach space. Our work generalizes several convergence results in the existing literature.

2. Generalised α-Nonexpansive Mapping

In this section, we define a new class of multivalued mapping and study some of its properties.

Definition 2.1. Let C be a nonempty subset of a Banach space E. A multivalued mapping $T : C \to CB(C)$ is said to satisfy condition $(C - \alpha)$ if there exists $\alpha \in [0, 1)$ such that for any $x, y \in C$

$$\begin{aligned} &\frac{1}{2}d(x, Tx) \leq \|x - y\| \\ \Rightarrow &H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1 - 2\alpha)\|x - y\|. \end{aligned}$$

A multivalued mapping satisfying condition $(C - \alpha)$ is termed as generalized α-nonexpansive multivalued mapping.

Some important properties of the mapping are discussed below.

Proposition 2.1. *Let $T : E \to CB(E)$ be a multivalued mapping. Then the followings hold.*

(i) *If T satisfies condition (C), then T satisfies condition $(C - \alpha)$ for some $\alpha \in [0, 1)$.*
(ii) *If T satisfies condition $(C - \alpha)$ and $F(T)$ is nonempty, then T is quasi-nonexpansive.*

Proof. (i) If T satisfies condition (C), then it is trivially seen that T satisfies condition $(C - \alpha)$ for $\alpha = 0$.

(ii) Let $p \in F(T)$ (as $F(T)$ is nonempty). So, $p \in Tp$. Therefore, $\frac{1}{2}d(p, Tp) = 0 \leq \|x - p\|$ for all $x \in E$. Since T satisfies condition $(C - \alpha)$, there exists $\alpha \in [0, 1)$ such that

$$\begin{aligned} H(Tx, Tp) &\leq \alpha d(x, Tp) + \alpha d(p, Tx) + (1 - 2\alpha)\|x - p\| \\ &\leq \alpha\|x - p\| + \alpha H(Tp, Tx) + (1 - 2\alpha)\|x - p\|. \end{aligned}$$

That is, $(1 - \alpha)H(Tx, Tp) \leq (1 - \alpha)\|x - p\|$ for all $x \in E$. Since $1 - \alpha > 0$, it follows that

$$H(Tx, Tp) \leq \|x - p\|, \quad \text{for all } x \in E \text{ and } p \in F(T). \qquad \square$$

Remark 2.1. The converse of (i) in the above proposition is not true in general, i.e., if a multivalued mapping satisfies condition $(C - \alpha)$, it does not necessarily imply that the mapping satisfy condition (C). The following example gives a clear instance of the above situation.

Example 2.1. Let $C = [0,4]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $T : C \to CB(C)$ by

$$Tx = \begin{cases} [0, \frac{x}{4}], & \text{if } x \neq 4, \\ [0,2], & \text{if } x = 4. \end{cases}$$

Then, for $x = 4$ and $y \in (8/3, 3]$, we have $\frac{1}{2}d(x,Tx) \leq \|x-y\|$ but, $H(Tx,Ty) > \|x-y\|$. Hence, T does not satisfy condition (C). Also, for $x \in (8/3, 32/11]$ and $y = 4$, we have $\frac{1}{2}d(x,Tx) \leq \|x-y\|$, but $H(Tx,Ty) > \|x-y\|$. This again gives another instant which shows that the mapping T does not satisfies condition (C). But it is interesting to note that for any α with $\frac{1}{9} \leq \alpha \leq \frac{3}{11}$, T satisfies condition $(C-\alpha)$ and so, T is a generalized α-nonexpansive multivalued mapping. This example confirms that the new class of nonexpansive multivalued mappings proposed in this paper properly contains the class of Suzuki-type multivalued mappings.

We now prove some important inequalities related to the multivalued mapping satisfying condition $(C-\alpha)$.

Proposition 2.2. *Let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ a generalized α-nonexpansive multivalued mapping. Then for each $x, y \in C$*

(i) *$H(Tx,Tz) \leq \|x-z\|$ for all $z \in Tx$;*
(ii) *either $\frac{1}{2}d(x,Tx) \leq \|x-y\|$ or $\frac{1}{2}d(z,Tz) \leq \|z-y\|$ for all $z \in Tx$;*
(iii) *either $H(Tx,Ty) \leq \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\|$ or $H(Tz,Ty) \leq \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|z-y\|$ for all $z \in Tx$.*

Proof. Since, $\frac{1}{2}d(x,Tx) \leq d(x,Tx) \leq \|x-z\|$ for all $z \in Tx$, we have

$$\begin{aligned} H(Tx,Tz) &\leq \alpha d(z,Tx) + \alpha d(x,Tz) + (1-2\alpha)\|z-x\| \\ &= \alpha d(x,Tz) + (1-2\alpha)\|z-x\| \\ &\leq \alpha\|x-z\| + \alpha d(z,Tz) + (1-2\alpha)\|z-x\| \\ &\leq \alpha H(Tx,Tz) + (1-\alpha)\|z-x\|. \end{aligned}$$

Simplifying we get $H(Tx,Tz) \leq \|x-z\|$.

We prove (ii) by contradiction. Suppose that $\frac{1}{2}d(x,Tx) > \|x-y\|$ and $\frac{1}{2}d(z,Tz) > \|z-y\|$ for some $z \in Tx$.

Thus, from (i) we get

$$\begin{aligned} d(z,Tz) &\leq H(Tx,Tz) \leq \|x-z\| \\ &\leq \|x-y\| + \|y-z\| \\ &< \frac{1}{2}d(x,Tx) + \frac{1}{2}d(z,Tz) \\ &< d(x,Tx). \end{aligned}$$

Now,

$$d(x,Tx) \leq \|x-z\| \leq \|x-y\| + \|y-z\| < \frac{1}{2}d(x,Tx) + \frac{1}{2}d(z,Tz),$$

i.e., $d(x, Tx) < d(x, Tx)$, which is a contradiction. Thus, (ii) holds.

The condition (iii) directly follows from (ii). □

Proposition 2.3. *Let C be a nonempty subset of a Banach space E. If $T : C \to P(C)$ satisfies the condition $(C-\alpha)$, then*

$$H(Tx, Ty) \leq 2\frac{1+\alpha}{1-\alpha}d(x, Tx) + \|x-y\|,$$

for all $x, y \in C$.

Proof. Let $x \in C$. Since Tx is proximal, there exists $z \in Tx$ such that $\|z - x\| = d(x, Tx)$. Thus $\frac{1}{2}d(x, Tx) \leq \|z - x\|$. Since T satisfies condition $(C-\alpha)$, we have

$$\begin{aligned} H(Tx, Tz) &\leq \alpha d(x, Tz) + \alpha d(z, Tx) + (1-2\alpha)\|x-z\| \\ &= \alpha d(x, Tz) + (1-2\alpha)\|x-z\| \\ &\leq \alpha d(x, Tx) + \alpha H(Tx, Tz) + (1-2\alpha)\|x-z\| \\ &= \alpha H(Tx, Tz) + (1-\alpha)\|x-z\|. \end{aligned}$$

Simplifying we get

$$H(Tx, Tz) \leq \|x-z\|. \tag{2.1}$$

Now, by Proposition 2.2 we get for all $x, y \in C$, either

$$H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1-2\alpha)\|x-y\| \tag{2.2}$$

or

$$H(Tz, Ty) \leq \alpha d(z, Ty) + \alpha d(y, Tz) + (1-2\alpha)\|z-y\|. \tag{2.3}$$

If (2.2) holds, we have

$$\begin{aligned} & H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1-2\alpha)\|x-y\| \\ \Rightarrow\ & H(Tx, Ty) \leq \alpha d(x, Tx) + \alpha H(Tx, Ty) + \alpha d(x, Tx) + (1-\alpha)\|x-y\| \\ \Rightarrow\ & H(Tx, Ty) \leq \frac{2\alpha}{1-\alpha}d(x, Tx) + \|x-y\|, \end{aligned}$$

else (2.3) holds and by (2.1) we have

$$\begin{aligned} H(Tx, Ty) \leq& H(Tx, Tz) + H(Tz, Ty) \\ \leq& \|x-z\| + \alpha d(z, Ty) + \alpha d(y, Tz) + (1-2\alpha)\|z-y\| \\ \leq& 2(1-\alpha)\|x-z\| + \alpha d(z, Ty) + \alpha d(y, Tz) + (1-2\alpha)\|x-y\| \\ \leq& 2(1-\alpha)\|x-z\| + \alpha\|z-x\| + \alpha d(x, Tx) + \alpha H(Tx, Ty) \\ & + \alpha\|x-y\| + \alpha d(x, Tx) + \alpha H(Tx, Tz) + (1-2\alpha)\|x-y\| \\ =& (2+\alpha)\|x-z\| + \alpha H(Tx, Ty) + \alpha H(Tx, Tz) + (1-\alpha)\|x-y\| \\ \leq& 2(1+\alpha)\|x-z\| + \alpha H(Tx, Ty) + (1-\alpha)\|x-y\|. \end{aligned}$$

Thus, simplifying and dividing both side of the above relation by $(1-\alpha)$, we get

$$H(Tx,Ty) \leq 2\frac{1+\alpha}{1-\alpha}\|x-z\| + \|x-y\|$$

i.e.,

$$H(Tx,Ty) \leq 2\frac{1+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|.$$

Hence, our desired inequality is proved in either cases. □

The following lemma will be useful in our next section.

Lemma 2.1. *Let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ be a generalized α-nonexpansive mapping. Then for all $x, y \in C$*

$$d(x,Ty) \leq \frac{3+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \tag{2.4}$$

Proof. From the Proposition 2.2, we have for all $x, y \in C$ and $z \in Tx$, either

$$H(Tx,Ty) \leq \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\|$$

or

$$H(Tz,Ty) \leq \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|z-y\|.$$

For the first case, we have

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx) + H(Tx,Ty) \\ &\leq d(x,Tx) + \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\|. \end{aligned}$$

Hence,

$$\begin{aligned} (1-\alpha)d(x,Ty) &\leq (1+\alpha)d(x,Tx) + (1-\alpha)\|x-y\| \\ \Rightarrow d(x,Ty) &\leq \frac{1+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \end{aligned}$$

To prove the other case, let $z' \in Tx$ be such that $\|x-z'\| = d(x,Tx)$. So, by using (i) and (iii) of Proposition 2.2, we obtain

$$\begin{aligned} d(x,Ty) \leq& d(x,Tx) + H(Tx,Tz') + H(Tz',Ty) \\ \leq& d(x,Tx) + \|x-z'\| + H(Tz',Ty) \\ \leq& 2d(x,Tx) + \alpha d(z',Ty) + \alpha d(y,Tz') + (1-2\alpha)\|z'-y\| \\ \leq& 2d(x,Tx) + \alpha\|z'-x\| + \alpha d(x,Ty) + \alpha d(y,Tx) \\ &+ \alpha H(Tx,Tz') + (1-2\alpha)\|z'-y\|. \end{aligned}$$

This yields,

$$\begin{aligned} (1-\alpha)d(x,Ty) &\leq (3+\alpha)d(x,Tx) + (1-\alpha)\|x-y\|, \\ d(x,Ty) &\leq \frac{3+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \end{aligned}$$

Therefore, in both the cases, we get the desired results. □

We conclude this section with the property of demiclosedness.

Theorem 2.1 (Demiclosed principle)**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with Opial's property. $T : C \to CB(C)$ a multivalued mapping satisfying condition $(C - \alpha)$ and $\{x_n\}$ be a sequence in E. If $\{x_n\}$ converges weakly to some point $x \in C$ and $\limsup_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in Tx$, i.e., $(I - T)$ is demiclosed at zero.*

Proof. Since $x \in C$ and Tx is closed and bounded, for each $n \in \mathbb{N}$ there exist $z_n \in Tx$ such that $\|x_n - z_n\| = \mathrm{d}(x_n, Tx)$. Then by Proposition 2.3,

$$\begin{aligned}\|x_n - z_n\| &= \mathrm{d}(x_n, Tx) \le \mathrm{d}(x_n, Tx_n) + H(Tx_n, Tx)\\ &\le \mathrm{d}(x_n, Tx_n) + 2\frac{1+\alpha}{1-\alpha}\mathrm{d}(x_n, Tx_n) + \|x_n - x\|.\end{aligned}$$

Taking limsup on both side and using $\limsup_{n\to\infty} \mathrm{d}(x_n, Tx_n) = 0$, we obtain

$$\limsup_{n\to\infty} \|x_n - z_n\| \le \limsup_{n\to\infty} \|x_n - x\|, \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

As the sequence $\{x_n\}$ converges weakly to x and E possesses Opail's property, for any $n \in \mathbb{N}$ if $z_n \neq x$ then it follows that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z_n\|,$$

which contradicts (2.5), therefore we can infer $z_n = x$ for all $n \in \mathbb{N}$. As a consequence of $z_n \in Tx$ we have $x \in Tx$, i.e., $(I - T)$ is demiclosed at zero. □

3. Convergence Theorems

In this section we propose a modified S-type iterative process for finding a common fixed point of a pair of multivalued mapping satisfying condition $(C - \alpha)$. We prove strong and weak convergence of the iterated sequence under suitable assumptions. For ths purpose let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ be a multivalued mapping, our iterative scheme is defined as follows

$$\begin{cases} x_1 \in C,\\ y_n = (1 - \beta_n)x_n + \beta_n v_n,\\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{3.1}$$

where $v_n \in P_T x_n$, $u_n \in P_S y_n$ and $0 < a \le \alpha_n, \beta_n \le b < 1$.

It is interesting to note that the iteration process (3.1) reduces to

- (1.5) when $S = T$;
- (1.4) when $S = T$ and T is single-valued;
- (1.6) when S and T are single-valued;
- (1.2) when $T = I$ and S is single-valued.

Thus all the results which are proved in this section, also holds for the iteration processes (1.5), (1.4), (1.5) and (1.2). Before stating our main convergence results, let us first prove some important lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space E and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C - \alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be a sequence generated by the modified S-iteration scheme* (3.1), *then for any $w \in \mathbb{F}$, the following assertions hold:*

(a) $\max\{\|x_{n+1} - w\|, \|y_n - w\|\} \leq \|x_n - w\|$ *for all* $n \in \mathbb{N}$;
(b) $\lim_{n\to\infty} \|x_n - w\|$ *exists.*

Proof. By (3.1) and Proposition 2.1 we have

$$\begin{aligned}
\|y_n - w\| &= \|(1 - \beta_n)x_n + \beta_n v_n - w\| \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|v_n - w\| \\
&= (1 - \beta_n)\|x_n - w\| + \beta_n d(v_n, Tw) \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned}$$

Also by using (3.1) and Proposition 2.1 we obtain

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1 - \alpha_n)v_n + \alpha_n u_n - w\| \\
&\leq (1 - \alpha_n)\|v_n - w\| + \alpha_n\|u_n - w\| \\
&\leq (1 - \alpha_n)d(v_n, Tw) + \alpha_n d(u_n, Sw) \\
&\leq (1 - \alpha_n)H(Tx_n, Tw) + \alpha_n H(Sy_n, Sw) \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|y_n - w\| \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\
&\leq \|x_n - w\|.
\end{aligned}$$

This shows that the sequence $\{\|x_n - w\|\}$ is nonincreasing and bounded below. Thus, we can conclude $\lim_{n\to\infty} \|x_n - w\|$ exist. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C - \alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be a sequence generated by the modified S-iteration scheme* (3.1), *then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n).$$

Proof. Let $w \in \mathbb{F}$, then by Lemma 3.1 we have $\lim_{n\to\infty} \|x_n - w\|$ exists. Suppose

$$\lim_{n\to\infty} \|x_n - w\| = r. \tag{3.2}$$

Since $v_n \in P_T x_n$, by Proposition 2.1 we write

$$\|v_n - w\| = d(v_n, Tw) \leq H(Tx_n, T_w) \leq \|x_n - w\|,$$

taking lim sup and using (3.2),

$$\limsup_{n\to\infty} \|v_n - w\| \le r. \tag{3.3}$$

Again $u_n \in P_S y_n$, using Proposition 2.1 and Lemma 3.1 we write

$$\|u_n - w\| = d(u_n, Sw) \le H(Sy_n, S_w) \le \|y_n - w\| \le \|x_n - w\|,$$

taking lim sup and using (3.2),

$$\limsup_{n\to\infty} \|u_n - w\| \le r. \tag{3.4}$$

Moreover by (3.2) we can write

$$\begin{aligned} r &= \lim_{n\to\infty} \|x_{n+1} - w\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)v_n + \alpha_n u_n - w\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(v_n - w) + \alpha_n(u_n - w)\|. \end{aligned} \tag{3.5}$$

In view of Lemma 1.1, by (3.4), (3.3) and (3.5) we can confirm

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.6}$$

Now,

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)v_n + \alpha_n u_n - w\| \\ &= \|(v_n - w) + \alpha_n(u_n - v_n)\| \\ &\le \|(v_n - w)\| + \alpha_n\|(u_n - v_n)\|. \end{aligned}$$

Taking $\liminf$ we obtain

$$\liminf_{n\to\infty} \|v_n - w\| \ge r, \tag{3.7}$$

by (3.3) and (3.7) we conclude

$$\lim_{n\to\infty} \|v_n - w\| = r. \tag{3.8}$$

Again we can write

$$\begin{aligned} \|v_n - w\| &\le \|v_n - u_n\| + \|u_n - w\| \\ &= \|v_n - u_n\| + d(u_n, Sw) \\ &\le \|v_n - u_n\| + H(Sy_n, Sw) \\ &\le \|v_n - u_n\| + \|y_n - w\|. \end{aligned}$$

Taking limit on both side and using (3.8) we get

$$\lim_{n\to\infty} \|y_n - w\| \ge r, \tag{3.9}$$

also by Lemma 3.1

$$\lim_{n\to\infty} \|y_n - w\| \le \lim_{n\to\infty} \|x_n - w\| = r.$$

Therefore, $\lim_{n\to\infty} \|\|\|y_n - w\| = r$. Alternatively, we can write

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n v_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(v_n - w)\|. \end{aligned}$$

Hence, by Lemma 1.1 and (3.2), (3.7)

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Consequently, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n v_n - x_n\| = \lim_{n\to\infty} \beta_n \|v_n - x_n\| = 0.$$

Also,

$$\|u_n - x_n\| \le \|u_n - v_n\| + \|v_n - x_n\|.$$

Therefore, by (3.6), $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Now, using Lemma 2.1

$$\begin{aligned} d(x_n, Sx_n) &\le \|x_n - y_n\| + d(y_n, Sx_n) \\ &\le \|x_n - y_n\| + \frac{3+\alpha}{1-\alpha} d(y_n, Sy_n) + \|x_n - y_n\| \\ &\le 2\|x_n - y_n\| + \frac{3+\alpha}{1-\alpha}\|u_n - y_n\| \\ &\le \frac{5-\alpha}{1-\alpha}\|x_n - y_n\| + \frac{3+\alpha}{1-\alpha}\|u_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, with this we conclude this lemma. □

We first assert a weak convergence theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with opial's property and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C-\alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Then the sequence $\{x_n\}$ generated by the modified S-iteration scheme* (3.1) *converges weakly to a common fixed point of T and S.*

Proof. Let $w \in \mathbb{F}$, then by Lemma 3.1 $\lim\limits_{n\to\infty} \|x_n - w\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathbb{F}$. Let p and q be weak limits corresponding to the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of the sequence $\{x_n\}$, respectively. By lemma 3.2 and Lemma 2.1 we have $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ and $I - T$ is demiclosed with respect to zero, which together implies $p \in Tp$. Similarly we can conclude $p \in Sp$, therefore $p \in \mathbb{F}$. Again in the same manner, we can prove that $q \in \mathbb{F}$. Next, we prove the uniqueness. If possible let p and q be distinct, then by Opial's condition

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p\| &= \lim_{n\to\infty}\|x_{n_i}-p\| \\ &< \lim_{n\to\infty}\|x_{n_i}-q\| \\ &= \lim_{n\to\infty}\|x_n-q\| = \lim_{n\to\infty}\|x_{n_j}-q\| \\ &< \lim_{n\to\infty}\|x_{n_j}-p\| \\ &= \lim_{n\to\infty}\|x_n-p\|.\end{aligned}$$

This is a contradiction, therefore $p=q$. Hence, $\{x_n\}$ converges weakly to a unique point of $\mathbb{F}$ and this completes the proof. □

We have the following corollaries.

Corollary 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with opial's property and $T,S:C\to CB(C)$ be two multivalued mapping satisfying condition $(C-\alpha)$. Suppose $\mathbb{F}\neq\Phi$ and $Tw=\{w\}=Sw$ for all $w\in\mathbb{F}$. Let $\{x_n\}$ be generated by*

(i)

$$\begin{cases} x_1\in C, \\ y_n=(1-\beta_n)x_n+\beta_n v_n, \\ x_{n+1}=(1-\alpha_n)v_n+\alpha_n u_n, \end{cases} \tag{3.10}$$

where $v_n\in P_T x_n$, $u_n\in P_T y_n$;

(ii)

$$\begin{cases} x_1\in C, \\ x_{n+1}=(1-\alpha_n)x_n+\alpha_n u_n, \end{cases} \tag{3.11}$$

where $u_n\in P_S x_n$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0,1)$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T and S, respectively.

Proof. Letting $T=S$ in Theorem 3.1 we can prove that the iteration (3.10) converges weakly to a fixed point of T. On the other hand assuming $T=I$, the identity mapping we can conclude that the iteration (3.11) converges weakly to a fixed point of S. □

We now state our main result, ensuring strong convergence of the iteration process (3.1) to a common fixed point under suitable assumptions in a real Banach space.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space E, and let $T,S:C\to CB(C)$ satisfy condition $(C-\alpha)$. Also suppose that, $\mathbb{F}\neq\Phi$ and $Tw=\{w\}=Sw$ for all $w\in\mathbb{F}$. Let $\{x_n\}$ be the sequence generated by (3.1). Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\liminf_{n\to\infty} d(x_n,\mathbb{F})=0$.*

Proof. Let $\{x_n\}$ converge to a common fixed point x of T and S, i.e., $x \in \mathbb{F}$. Thus it obviously follows that $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathbb{F}) = 0$.

Conversely, let us suppose that $\liminf_{n\to\infty} d(x_n, \mathbb{F}) = 0$, then from Lemma 3.1 for each $w \in \mathbb{F}$ we have $\|x_{n+1} - w\| \leq \|x_n - w\|$, which implies

$$d(x_{n+1}, \mathbb{F}) \leq d(x_n, \mathbb{F}).$$

Hence, $\{d(x_n, \mathbb{F})\}$ is a decreasing sequence of real numbers which is bounded below and also $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathbb{F}) = 0$, implying $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$ We claim that, $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$, there exists $p \in \mathbb{N}$ such that for all $n \geq p$, we have

$$d(x_n, \mathbb{F}) < \frac{\epsilon}{2}.$$

In particular, $\inf\{\|x_p - w\| : w \in \mathbb{F}\} < \frac{\epsilon}{2}$, so there exists some $\bar{w} \in \mathbb{F}$ such that $\|x_p - \bar{w}\| < \frac{\epsilon}{2}$. Now, for $m, n \geq p$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - \bar{w}\| + \|x_n - \bar{w}\| < 2\|x_p - \bar{w}\| < 2\frac{\epsilon}{2} = \epsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Now C being a closed subset of E and $\{x_n\}$ is a Cauchy sequence in C, it must converge in C. Let $\lim_{n\to\infty} x_n = z$. Now

$$d(z, Tz) \leq \|z - x_n\| + d(x_n, Tx_n) + H(Tx_n, Tz).$$

Applying Proposition 2.3, we obtain

$$\begin{aligned} d(z, Tz) &\leq \|z - x_n\| + d(x_n, Tx_n) + 2\frac{1+\alpha}{1-\alpha} d(x_n, Tx_n) + \|x_n - z\| \\ &\leq 2\|z, x_n\| + \frac{3+\alpha}{1-\alpha} d(x_n, Tx_n). \end{aligned}$$

Since $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ (by Lemma 3.2) we conclude $d(z, Tz) = 0$ and hence we infer $z \in Tz$. In a similar fashion we can prove $z \in Sz$. Therefore, we conclude $z \in \mathbb{F}$. This completes the proof. □

Let C be a subset of a normed space E, two mappings $S, T : C \to CB(C)$ are said to satisfy the **Condition** (A), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that for all $x \in C$ either, $d(x, Tx) \geq f(d(x, \mathbb{F}))$ or $d(x, Sx) \geq f(d(x, \mathbb{F}))$.

Now we prove the strong convergence theorem by using Condition (A).

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to CB(C)$ and $S : C \to CB(C)$ be two multivalued mappings satisfying the condition $(C - \alpha)$ along with condition(A) Assume that $\mathbb{F} \neq \emptyset$ and $T(w) = \{w\} = Sw$ for each $w \in \mathbb{F}$. Let $\{x_n\}$ be the sequence defined as in* (3.1). *Then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. In view of Theorem 3.2 we have $\lim_{n\to\infty} \mathrm{d}(x_n, \mathbb{F})$ exists and by Condition (A), we have by Lemma 3.2, either

$$\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) \leq \lim_{n\to\infty} \mathrm{d}(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) \leq \lim_{n\to\infty} \mathrm{d}(x_n, Sx_n) = 0.$$

Thus, in any case $\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) = 0$. Since f is nondecreasing and $f(0) = 0$ implies $\lim_{n\to\infty} \mathrm{d}(x_n, \mathbb{F}) = 0$. Therefore our result follows as a direct consequence of Theorem 3.2. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to CB(C)$ be a multivalued mappings satisfying the condition $(C - \alpha)$ along with condition(A). Assume that $\mathbb{F} \neq \emptyset$ and $T(w) = \{w\}$ for each $w \in F(T)$. Let $\{x_n\}$ be the sequence defined as in* (3.10) *and* (3.11)*. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. The corollary can be proved as a direct consequence of Theorem 3.3, by letting $S = T$. □

4. Numerical Illustrations

In this section, the behavior of the modified S-type iteration is studied for a given set of problems. The iteration has been performed several times with different initial guesses. We also observe the effect on convergence speed, for different choices of the parameters α_n and β_n. It is worthy to mention that the iteration took more steps to converge when the parameters are chosen close to 0. In the table below we enlisted our observations on the iteration process to find the common fixed points of two multivalued mappings T and S. In this process, we choose the tolerance limit 10^{-4} as stopping criteria. The actual common fixed point of T and S is 0, while the approximate solutions are shown in the table.

$$Tx = \begin{cases} [0, \frac{x}{4}], & \text{if } x \neq 4, \\ [0, 2], & \text{if } x = 4, \end{cases} \qquad Sx = \begin{cases} [0, \frac{x}{5}], & \text{if } x \neq 5, \\ \{1\}, & \text{if } x = 5. \end{cases}$$

Table 1. Numerical illustrations with different initial guesses

Sl.	Initial guess	α_n	β_n	No. of iteration	Final solution
1	2	0.5	0.5	12	8.546 e-06
2	4	0.3	0.65	14	7.669 e-06
3	23.762	0.1	0.01	20	1.393 e-05
4	127.63	0.9	0.33	12	2.657 e-06
5	529.66	0.75	0.66	12	9.157 e-06
6	1000	0.8	0.8	12	3.214 e-06

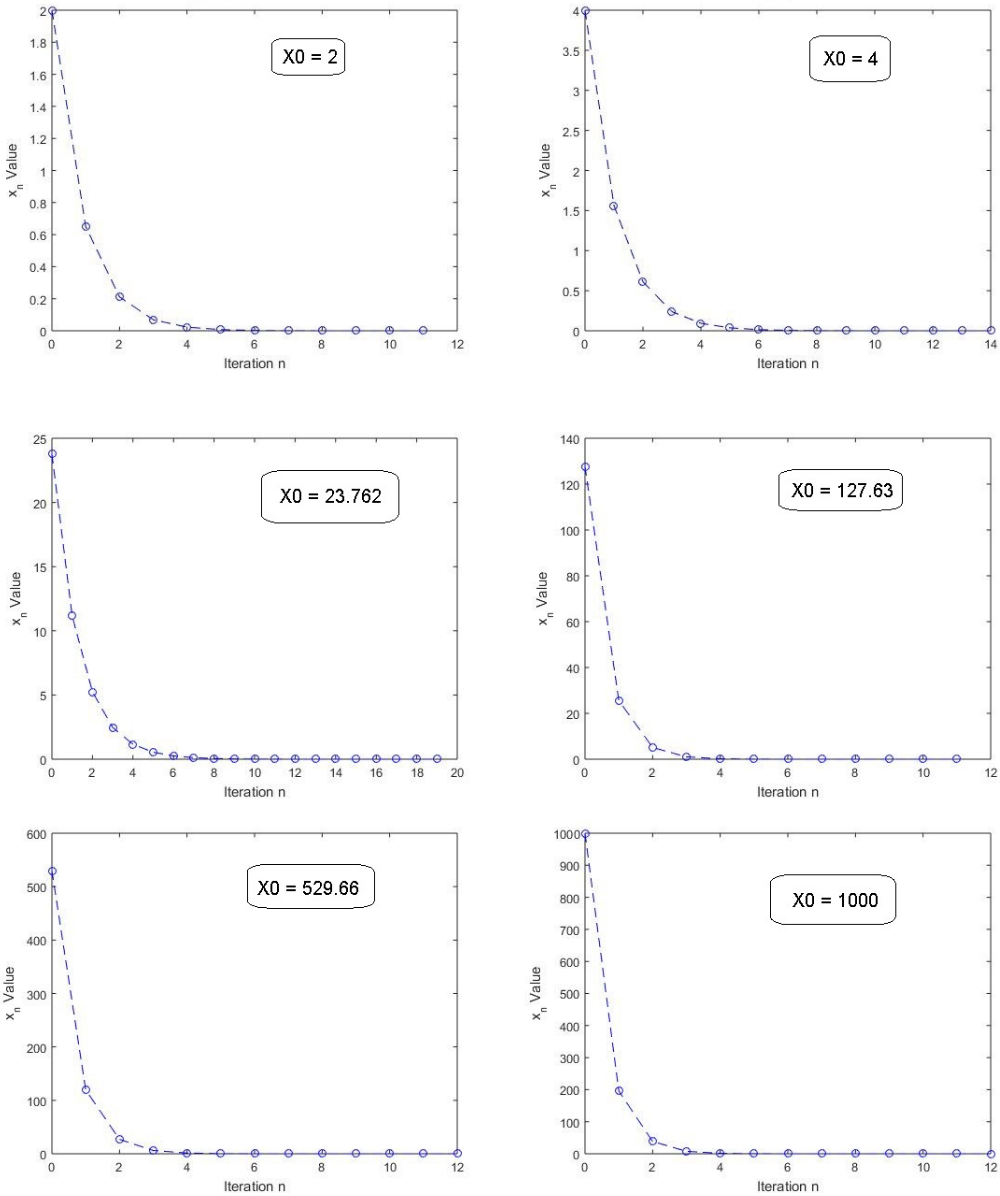

FIGURE 1. Plot of points against iteration for various initial guesses

5. CONCLUSION

The present article introduces a new class of nonexpansive multivalued mapping. A modified S-iteration process is employed to approximate a common fixed point of two multivalued mappings. Strong and weak convergence of the method are proved under suitable assumptions. Apart from Corollary 3.1 and Corollary 3.2 our results also generalize several theorems in the existing literature. In particular, if $T = S$ and T is chosen to be single-valued, then Theorems 5.8 (a), 5.9 and 5.10 of [11] follow from our results. Moreover, since the class of generalized α-nonexpansive mapping properly contains the class of nonexpansive and Suzuki-type mapping, our results also suit well in those settings.

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[1]Department of Mathematics,
Indian Institute of Technology Kharagpur,
Kharagpur, India

[2]Department of Mathematics,
Sreegopal Banerjee College, Magra,
West Bengal, India
Email address: rudrajitsadhu@gmail.com
ORCID iD: https://orcid.org/0000-0002-1487-1750
Email address: cnahak@maths.iitkgp.ac.in
ORCID iD: https://orcid.org/0000-0001-8208-5631

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