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SPECTRAL EXPANSION FOR CONFORMABLE FRACTIONAL STURM-LIOUVILLE PROBLEM ON THE WHOLE LINE

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Abstract. In this article, we discuss a conformable fractional Sturm-Liouville boundary-value problem on the whole line. We prove the existence of a spectral function for the singular conformable fractional Sturm-Lioville problem. Further, we establish a Parseval equality and spectral expansion formula by terms of the spectral function for conformable fractional Sturm-Liouville problem on the whole line.

1. Introduction

Fractional order differential equations first appeared towards the end of the 17th century with a letter of L'Hospital to Leibnitz in which he asked the meaning of "fractional order derivative". Up to the present time, many mathematicians such as Liouville, Riemann, Weyl, Fourier, Lagrange, Grönwald, Letnikov, Abel, and Caputo have made research in this field [2]. Fractional differential equations are used today in many fields such as transmission line theory, signal processing, chemical analysis, heat transfer, hydraulics of dams, material science, temperature field problems oil strata, diffusion problems, waves in liquids and gases, Schrödinger equation, and fractal equation [2–7]. Recently, based on the definition of the classical derivative, a new fractional derivative is put forward by Khalil et al. and named as conformable fractional derivative [1]. In this study Khalil et al. provided the linearity property for his new definition of the fractional derivative, they proved the product rule, the quotient rule, the fractional Rolle theorem, and the fractional mean value theorem. Later in [8],

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Abdeljawad defined the right and left conformable fractional derivatives, the fractional chain rule, and fractional integrals of higher orders. Conformable fractional derivative aims to expand the definition of the classical derivative by providing the natural characteristics of the classical derivative and gain new perspectives for the theory of differential equations [9]. Examples of these perspectives are [10–16, 31, 34]. In [10], the form of the Wronskian for conformable fractional linear differential equations with variable coefficients is discussed and Abel's formula for fractional differential equations with variable coefficients is proven. In [11], the exact solution of the heat conformable fractional differential equation is given. In [12], being existent and uniqueness theories of consecutive linear conformable fractional differential equations are demonstrated. In [13], some of the conformable fractional partial equations including the wave equation are solved. In [14], linear second-order conformable differential equations using a proportional derivative are shown to be formally self-adjoint equations with respect to a certain inner product and the associated self-adjoint boundary conditions. Defining a Wronskian, a Lagrange identity, and Abel's formula are established. Several reduction-of-order theorems are given. Solutions of the conformable second-order self-adjoint equation are then shown to be related to corresponding solutions of a first-order Riccati equation and a related quadratic functional and a conformable Picone identity. A Lyapunov inequality, factorizations of the second-order equation are established. Boundary value problems and Green's functions are studied. In [31], a regular fractional generalization of the Sturm-Liouville eigenvalue problems is suggested and some fundamental results of this suggested model are established. In [34], a general notion of fractional derivative for functions defined on arbitrary time scales is introduced. The basic tools for the time-scale fractional calculus are then developed. In [15], a conformable fractional Dirac system with separated boundary conditions on an arbitrary time scale is studied, some basic spectral properties of the classical Dirac system are extended to the conformable fractional case. In [16], a conformable fractional Sturm-Liouville equation with boundary conditions on an arbitrary time scale is analyzed, basic spectral properties of the classical Sturm-Liouville equation are extended to the conformable fractional case, some sufficient conditions are established to guarantee the existence of a solution for this conformable fractional Sturm-Liouville problem on $\mathbb{T}$ by using certain fixed point theorems.

Today, it is widely accepted that spectral expansion theorems are beneficial in science and engineering. If, for example, a partial differential equation is solved by the method of separation of variables (i.e., the Fourier method) then the problems of expanding an arbitrary function to a series of eigenfunctions and showing that the eigenfunctions form a complete system occur. The first study for the spectral expansion problem is constructed by Weyl [17] (see [18–29, 32, 33, 35, 36]).

The primary aim of this study is to prove the existence of a spectral function for singular conformable fractional (CF) Sturm-Liouville equation of the form

$$-T_{\alpha}^{2}y(t)+v(t)y(t)=\lambda y(t), \quad -\infty<t<\infty,$$

where λ is a complex parameter, $v(\cdot)$ is a real-valued conformable fractional locally integrable function on $(-\infty,\infty)$. The article is structured as follows. In Section 2, necessary concepts and properties are reviewed. In Section 3, we construct resolvent in view of Green's function. We show that the regular CF-Sturm-Liouville operator has a compact resolvent, so it has a purely discrete spectrum. Finally, in Section 4, we establish a Parseval equality and spectral expansion formula by terms of the spectral function for the CF-Sturm-Liouville problem on the whole line.

2. Preliminaries

In this section, our goal is to present some basic definitions and properties of conformable fractional calculus and operator theory. For more details, the reader may want to consult [1–8, 30, 37]. Throughout this paper, we will fix $\alpha \in (0,1)$.

Definition 2.1. Assume α be a positive number with $0<\alpha<1$. A function $f:\mathbb{R}\to\mathbb{R}=(-\infty,\infty)$ the conformable fractional derivative of order α of f at $t>0$ is defined by

$$T_\alpha f(t)=\lim_{\varepsilon\to\infty}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f\left(t\right)}{\varepsilon} \tag{2.1}$$

and the fractional derivative at 0 is defined by

$$\left(T_\alpha f\right)(0)=\lim_{t\to 0^+}T_\alpha f(t).$$

Definition 2.2. The left conformable fractional derivative starting from a of a function $f:[a,\infty)\to\mathbb{R}$ of order α is defined by

$$(T_\alpha^a\ f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon\left(t-a\right)^{1-\alpha})-f\left(t\right)}{\varepsilon},\quad 0<\alpha\le 1.$$

Definition 2.3. The right conformable fractional derivative of order $0<\alpha\le 1$ of f is defined by

$$({}_{\alpha}^{b}Tf)(t)=-\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(b-t)^{1-\alpha})-f(t)}{\varepsilon},$$

where f is terminating at b and $({}^{b}T_\alpha f)(t)=\lim_{t\to b^-}({}^{b}T_\alpha f)(t)$.

In the next lemma, we consider some properties of conformable derivatives.

Lemma 2.1. *Let f,g be conformable differentiable of order α, $0<\alpha\le 1$, at a point t. Then*

(i) $T_\alpha\left(\lambda f+\delta g\right)=\lambda T_\alpha\left(f\right)+\delta T_\alpha\left(g\right)$, $\lambda,\delta\in\mathbb{R}$;
(ii) $T_\alpha\left(fg\right)=fT_\alpha\left(g\right)+gT_\alpha\left(f\right)$;
(iii) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$;
(iv) *f is differentiable then* $T_\alpha^a\left(f\right)(t)=\left(t-a\right)^{1-\alpha}f'\left(t\right)$.
(v) $T_\alpha\left(t^n\right)=nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$.

Next, we present the conformable fractional integral and some of its properties.

Definition 2.4. The conformable fractional integral starting from a of a function f of order $0 < \alpha \leq 1$ is defined by

$$\left(I_{\alpha}^{a} f\right)(t)=\int_{a}^{t} f(x) d\alpha(x, a)=\int_{a}^{t}(x-a)^{\alpha-1} f(x) dx.$$

Similarly, in the right case, we have

$$\left({}^{b} I_{\alpha} f\right)(t)=\int_{t}^{b} f(x) d\alpha(b, x)=\int_{t}^{b}(b-x)^{\alpha-1} f(x) dx.$$

Lemma 2.2. *Assume that f is a continuous function on (a,∞) and $0<\alpha<1$. Then we have*

$$T_{\alpha}^{a} I_{\alpha}^{a} f(t)=f(t),$$

for all $t > a$.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that f and g are conformable fractional differentiable. So, we have*

$$\int_{a}^{b} f(t) T_{\alpha}^{a}(g)(t)\, d\alpha(t, a)+\int_{a}^{b} g(t)\, T_{\alpha}^{a}(f)(t)\, d\alpha(t, a)=f(b) g(b)-f(a) g(a).$$

Let $L_{\alpha}^{2}(-b,b)$, $-\infty \leq -b < b \leq \infty$, is the space of all complex-valued functions defined on $(-b,b)$ as

$$\|f\|:=\left(\int_{-b}^{b}|f(t)|^{2} d_{\alpha}(t)\right)^{1/2}=\left(\int_{-b}^{b}|f(t)|^{2} t^{\alpha-1} dt\right)^{1/2}<\infty.$$

The space $L_{\alpha}^{2}(-b,b)$ is a Hilbert space with the inner product

$$(f, g):=\int_{-b}^{b} f(t)\, \overline{g(t)} d_{\alpha}(t), \quad f, g \in L_{\alpha}^{2}(-b, b).$$

Let us define the conformable α-Wronskian of x and y by

$$W_{\alpha}(x, y)(t)=x(t) T_{\alpha} y(t)-y(t) T_{\alpha} x(t), \quad t \in(-b, b).$$

Definition 2.5. A function $M(t,x)$ in $\mathbb{C}^2$ of two variables with $-b < t$, $x < b$ is called the α-Hilbert-Schmidt kernel if

$$\int_{-b}^{b} \int_{-b}^{b}|M(t, x)|^{2} d_{\alpha}(t) d_{\alpha}(x)<\infty.$$

Theorem 2.2. *If*

$$\sum_{i, k=1}^{\infty}\left|a_{i k}\right|^{2}<\infty, \tag{2.2}$$

then the operator A defined by the formula

$$A\left\{x_{i}\right\}=\left\{y_{i}\right\}, \quad i \in \mathbb{N}:=\{1,2,3, \ldots\},$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \quad i \in \mathbb{N}, \tag{2.3}$$

is compact in the sequence space ℓ^2 [30].

Theorem 2.3 ([37]). *Let* $(w_n)_{n\in\mathbb{N}}$ *be a uniformly bounded sequence of real non-decreasing functions on a finite interval* $[c,d]$. *Then there exists a subsequence* $(w_{n_k})_{k\in\mathbb{N}}$ *and a non-decreasing function* w *such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),$$

where $\lambda \in [c,d]$.

Theorem 2.4 ([37]). *Assume* $(w_n)_{n\in\mathbb{N}}$ *is a real, uniformly bounded, sequence of non-decreasing functions on a finite interval* $[c,d]$, *and suppose*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),$$

where $\lambda \in [c,d]$. *If* f *is any continuous function on* $[c,d]$, *then*

$$\lim_{n\to\infty} \int_c^d f(\lambda)\, dw_n(\lambda) = \int_c^d f(\lambda)\, dw(\lambda).$$

3. Regular CF-Sturm-Liouville Problem

In this part, we construct Green's function and prove that the regular CF-Sturm-Liouville operator has a compact resolvent, so it has a purely discrete spectrum.

We consider the regular CF-Sturm-Liouville equation defined by

$$-T_\alpha^2 y(t) + v(t)y(t) = \lambda y(t), \quad -\infty < -b < x < b < \infty. \tag{3.1}$$

Let γ and β be arbitrary real numbers and let $y(t,\lambda)$ satisfies the boundary conditions

$$y(-b,\lambda)\cos\beta + T_\alpha y(-b,\lambda)\sin\beta = 0, \tag{3.2}$$

$$y(b,\lambda)\cos\gamma + T_\alpha y(b,\lambda)\sin\gamma = 0, \tag{3.3}$$

in which λ is a complex eigenvalue parameter, $v(t)$ is a real-valued continuous function defined on $\mathbb{R}$ and $v \in L^1_{\alpha,loc}(\mathbb{R})$, where

$$L^1_{\alpha,loc}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{-b}^{b} |f(t)|\, d_\alpha(t) < \infty \text{ for all } b \in \mathbb{R} \right\}.$$

Denote by $\theta_1(t,\lambda)$, and $\theta_2(t,\lambda)$ the linearly independent solutions of the (3.1) subject to the initial conditions

$$\theta_1(-b,\lambda) = \sin\beta, \quad T_\alpha\theta_1(-b,\lambda) = -\cos\beta, \tag{3.4}$$

$$\theta_2(b,\lambda) = \sin\gamma, \quad T_\alpha\theta_2(b,\lambda) = -\cos\gamma. \tag{3.5}$$

In this way, the *Green's function* of the problem is defined by (3.1)–(3.4) (see [23])

$$G(t,x,\lambda)=\begin{cases}\frac{\theta_1(t,\lambda)\theta_2(x,\lambda)}{W_\alpha(\theta_1,\theta_2)}, & -b\le x<t,\\ \frac{\theta_2(t,\lambda)\theta_1(x,\lambda)}{W_\alpha(\theta_1,\theta_2)}, & t<x\le b.\end{cases} \tag{3.6}$$

In the next results, without restriction of generality, we assume that $\lambda=0$ is not an eigenvalue of the problem (3.1)–(3.3).

Theorem 3.1. *$G(t,x)$ defined by* (3.6) *is a α-Hilbert-Schmidt kernel.*

Proof. By the upper half of the formula (3.6), we obtain

$$\int_{-b}^{b}d_\alpha(t)\int_{-b}^{t}|G(t,x)|^2 d_\alpha(x)<\infty,$$

and by the lower half of (3.6), we have

$$\int_{-b}^{b}d_\alpha(t)\int_{t}^{b}|G(t,x)|^2 d_\alpha(x)<\infty,$$

because the inner integral exists and is products $\theta_1(x)\,\theta_2(t)$, and these products belong to $L^2_\alpha(-b,b)\times L^2_\alpha(-b,b)$ because each of the factors belongs to $L^2_\alpha(-b,b)$. Then, we obtain

$$\int_{-b}^{b}\int_{-b}^{b}|G(t,x)|^2 d_\alpha(t)d_\alpha(x)<\infty. \tag{3.7}$$

□

Theorem 3.2. *The operator* $\mathbf{S}$ *defined by the formula*

$$(\mathbf{S}f)(t)=\int_{-b}^{b}G(t,x)f(x)d_\alpha(x)$$

is compact and self-adjoint on $L^2_\alpha(-b,b)$.

Proof. Let $\phi_i=\phi_i(x)$, $i\in\mathbb{N}$, is an orthonormal basis of $L^2_\alpha(-b,b)$. Because $G(t,x)$ is a $\alpha-$Hilbert-Schmidt kernel, it can be defined as

$$\begin{aligned} t_i&=(f,\phi_i)=\int_{-b}^{b}f(x)\overline{\phi_i(x)}d_\alpha(x),\\ y_i&=(g,\phi_i)=\int_{-b}^{b}g(x)\overline{\phi_i(x)}d_\alpha(x),\\ a_{ik}&=\int_{-b}^{b}\int_{-b}^{b}G(t,x)\phi_i(t)\overline{\phi_k(x)}d_\alpha(t)d_\alpha(x),\quad i,k\in\mathbb{N}.\end{aligned}$$

Then, $L^2_\alpha(-b,b)$ is mapped isometrically ℓ^2. As a consequence, the integral operator turns into the operator which is defined by the formula (2.3) in the space ℓ^2 by this mapping, and the condition (3.7) is translated into the condition (2.2). So, the original operator is compact.

Let $f,g\in L^2_\alpha(-b,b)$. As $G(t,x)=G(x,t)$ and we have

$$(\mathbf{S}f,g)=\int_{-b}^{b}(\mathbf{S}f)(t)\overline{g(t)}d_\alpha(t)$$

$$= \int_{-b}^{b}\int_{-b}^{b} G(t,x)f(x)d_\alpha(x)\overline{g(t)}d_\alpha(t)$$
$$= \int_{-b}^{b} f(x)\Big(\overline{\int_0^b G(x,t)g(t)d_\alpha(t)}\Big)d_\alpha(x) = (f, \mathbf{S}g).$$

Thus, we have proved that the operator $\mathbf{S}$ is self-adjoint. □

4. Parseval Equality and Spectral Expansion In The Case of The Whole Line

In this part, the existence of a spectral function for singular Sturm-Liouville problem (3.1)–(3.2) will be proven. A Parseval equality and spectral expansion formula by terms of the spectral function is set up.

Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues and $y_1, y_1, \ldots$ the corresponding eigenfunctions of the problem (3.1)–(3.3). Let $\theta_1(t,\lambda)$ and $\theta_2(t,\lambda)$ be solutions of the problem (3.1)–(3.2) satisfying the initial conditions

$$\theta_1(0,\lambda) = 0, \quad T_\alpha\theta_1(0,\lambda) = 1, \quad \theta_2(0,\lambda) = 1, \quad T_\alpha\theta_2(0,\lambda) = 0,$$

and let

$$y_n(t) = c_n\theta_1(t,\lambda_n) + d_n\theta_2(0,\lambda_n).$$

Let f be a real-valued function and $f \in L^2_\alpha(-b,b)$. Then it follows from Theorem 3.2 and the Hilbert-Schmidt theorem that

$$\begin{aligned}
\int_{-b}^{b} f^2(t)d_\alpha(t) &= \sum_{n=1}^{\infty}\left\{\int_{-b}^{b} f(t)y_n(t)d_\alpha(t)\right\}^2 \\
&= \sum_{n=1}^{\infty}\left\{\int_{-b}^{b} f(t)\{c_n\theta_1(t,\lambda_n) + d_n\theta_2(t,\lambda_n)\}d_\alpha(t)\right\}^2 \\
&= \sum_{n=1}^{\infty} c_n^2\left\{\int_{-b}^{b} f(t)\theta_1(t,\lambda_n)d_\alpha(t)\right\}^2 \\
&\quad + 2\sum_{n=1}^{\infty} c_nd_n\left\{\int_{-b}^{b} f(t)\theta_1(t,\lambda_n)d_\alpha(t)\int_{-b}^{b} f(t)\theta_2(t,\lambda_n)d_\alpha(t)\right\} \\
&\quad + \sum_{n=1}^{\infty} d_n^2\left\{\int_{-b}^{b} f(t)\theta_2(t,\lambda_n)d_\alpha(t)\right\}^2. \qquad (4.1)
\end{aligned}$$

Now, we will define the step functions by

$$\xi_{-b,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} c_n^2, & \text{for } \lambda \le 0, \\ \sum\limits_{0\le\lambda_n<\lambda} c_n^2 & \text{for } \lambda > 0, \end{cases}$$

$$\zeta_{-b,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} c_nd_n, & \text{for } \lambda \le 0, \\ \sum\limits_{0\le\lambda_n<\lambda} c_nd_n & \text{for } \lambda > 0, \end{cases}$$

$$\varsigma_{-b,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} d_n^2, & \text{for } \lambda \leq 0, \\ \sum\limits_{0\leq\lambda_n<\lambda} d_n^2 & \text{for } \lambda > 0. \end{cases}$$

Then the Parseval equality (4.1) can be stated as

$$\begin{aligned} \int_{-b}^{b} f^2(t)d_\alpha(t) &= \int_{-\infty}^{\infty} \left\{ \int_{-b}^{b} f(t)c_n\theta_1(t,\lambda_n)d_\alpha(t) \right\}^2 d\xi_{-b,b}(\lambda) \\ &+ 2\int_{-\infty}^{\infty} \left\{ \int_{-b}^{b} f(t)c_n\theta_1(t,\lambda_n)d_\alpha(t) \right\} \left\{ \int_{-b}^{b} f(t)d_n\theta_2(t,\lambda_n)d_\alpha(t) \right\} d\zeta_{-b,b}(\lambda) \\ &+ \int_{-\infty}^{\infty} \left\{ \int_{-b}^{b} f(t)d_n\theta_2(t,\lambda_n)d_\alpha(t) \right\}^2 d\varsigma_{-b,b}(\lambda). \end{aligned} \tag{4.2}$$

In the sequel, we shall present a lemma.

Lemma 4.1. *For any $s > 0$, there exists a positive constant $M = M(S)$ not depending on b such that*

$$\overset{S}{\underset{-S}{V}} \{\varrho_{ij,b}(\lambda)\} < M, \quad i,j = 1,2, \tag{4.3}$$

where

$$\varrho_{11,b}(\lambda) = \xi_{-b,b}(\lambda), \quad \varrho_{12,b}(\lambda) = \varrho_{21,b}(\lambda) = \zeta_{-b,b}(\lambda), \quad \varrho_{22,b}(\lambda) = \varsigma_{-b,b}(\lambda).$$

Proof. To see the validity of (4.3), it suffices to put $i = j$, because

$$\overset{S}{\underset{-S}{V}} \{\varrho_{12,b}(\lambda)\} \leq \frac{1}{2}\{\varrho_{11,b}(S) - \varrho_{11,b}(-S) + \varrho_{22,b}(S) - \varrho_{22,b}(S)\}.$$

The Parseval equality (4.2) then takes the form

$$\int_{-b}^{b} f^2(t)d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)F_j(\lambda)d\varrho_{ij,b}(\lambda), \tag{4.4}$$

where

$$F_i(\lambda) = \int_{-b}^{b} f(t)\theta_i(t,\lambda)d_\alpha(t), \quad i = 1,2.$$

If follows from (4.4) that

$$T_\alpha^{(j-1)}\theta_i(0,\lambda) = \delta_{i,j} \quad i,j = 1,2,$$

where $\delta_{i,j}$ is the Kronecker delta. Thus, for any $\varepsilon > 0$ and given $S > 0$, there exists a $r > 0$ such that

$$|T_\alpha^{(j-1)}\theta_i(t,\lambda) - \delta_{i,j}| < \varepsilon, \tag{4.5}$$

where $|\lambda| \leq S,\ t \in [0,r]$. □

Let $f_r(t)$ be a non-negative twice continuously differentiable function such that $f_r(t)$ vanishes outside the interval $[0,r]$, with

$$\int_0^r f_r(t)d_\alpha(t) = 1. \tag{4.6}$$

Now, if we apply the Parseval equality (4.4) to the functions $T_\alpha^{(j-1)} f_r(t)$, $j = 1, 2$, then we get

$$\int_0^r \left|T_\alpha^{(j-1)} f_r(t)\right|^2 d_\alpha(t) \geq \int_{-S}^S \sum_{i,k=1}^2 F_{i,j}(\lambda)F_{kj}(\lambda)d\varrho_{ik,b}(\lambda), \tag{4.7}$$

where

$$F_{ij}(\lambda) = \int_0^r T_\alpha^{(j-1)} f_r(t)\theta_i(t,\lambda)d_\alpha(t) = \pm\int_0^r f_r(t)T_\alpha^{(j-1)}\theta_i(t,\lambda)d_\alpha(t).$$

Using (4.5) and (4.6), we obtain

$$|F_{ij}(\lambda) - \delta_{i,j}| < \varepsilon, \quad i,j = 1,2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\int_0^r \left|T_\alpha^{(j-1)} f_r\left(t\right)\right|^2 d_\alpha(t) \geq \int_{-S}^S \sum_{i,k=1}^2 \left(\delta_{ij} - \varepsilon\right)\left(\delta_{kj} - \varepsilon\right)\left|d\varrho_{ik,b}\left(\lambda\right)\right|. \tag{4.9}$$

If we take $j = 1$ in (4.8), we have

$$\begin{aligned}\int_0^r f_r^2\left(t\right)d_\alpha(t) &> \left(1-\varepsilon\right)^2\int_{-S}^S \left|d\varrho_{11,b}\left(\lambda\right)\right| - \varepsilon\left(1+\varepsilon\right)\int_{-S}^S\left|d\varrho_{12,b}\left(\lambda\right)\right| \\ &\quad - \varepsilon\left(1+\varepsilon\right)\int_{-S}^S\left|d\varrho_{21,b}\left(\lambda\right)\right| + \varepsilon^2\int_{-S}^S\left|d\varrho_{22,b}\left(\lambda\right)\right| \\ &> \left(1-\varepsilon\right)^2\left(\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right)\right) - 2\varepsilon\left(1+\varepsilon\right)\bigvee_{-S}^{S}\left\{\varrho_{12,b}\left(\lambda\right)\right\}.\end{aligned}$$

Since

$$\bigvee_{-S}^{S}\left\{\varrho_{12,b}\left(\lambda\right)\right\} \leq \frac{1}{2}\left[\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right) + \varrho_{22,b}\left(S\right) - \varrho_{22,b}\left(-S\right)\right], \tag{4.10}$$

we obtain

$$\begin{aligned}&\int_0^r f_r^2\left(t\right)d_\alpha(t) \\ &> \left(\varepsilon^2 - 2\varepsilon + 1\right)\left\{\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right)\right\} \\ &\quad - \varepsilon\left(1+\varepsilon\right)\left\{\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right) + \varrho_{22,b}\left(S\right) - \varrho_{22,b}\left(-S\right)\right\} \\ &= \left(1-3\varepsilon\right)\left\{\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right)\right\} - \varepsilon\left(1+\varepsilon\right)\left\{\varrho_{22,b}\left(S\right) - \varrho_{22,b}\left(-S\right)\right\}.\end{aligned} \tag{4.11}$$

Putting $j = 2$ in (4.9), we see that

$$\begin{aligned}\int_0^r \left|T_\alpha f_r\left(t\right)\right|^2 d_\alpha(t) \geq &\left(1-3\varepsilon\right)\left\{\varrho_{22,b}\left(S\right) - \varrho_{22,b}\left(-S\right)\right\} \\ &- \varepsilon\left(1+\varepsilon\right)\left\{\varrho_{11,b}\left(S\right) - \varrho_{11,b}\left(-S\right)\right\}.\end{aligned} \tag{4.12}$$

If we add the inequalities (4.11) and (4.12), then we deduce that

$$\int_0^r f_r^2(t)\, d_\alpha(t) + \int_0^r |T_\alpha f_r(t)|^2\, d_\alpha(t)$$
$$\geq (2\varepsilon - 1)^2 \left(\varrho_{11,b}(S) - \varrho_{11,b}(-S) + \varrho_{22,b}(S) - \varrho_{22,b}(-S)\right).$$

If we choose $\varepsilon > 0$ such that $1 - 4\varepsilon - \varepsilon^2 > 0$, then we have the assertion of the lemma for the functions $\varrho_{11,b}(\lambda)$ and $\varrho_{22,b}(\lambda)$ relying on their monotonicity. From (4.10), we have the assertion of the lemma for the function $\varrho_{12,b}(\lambda)$.

Let ϱ be any non-decreasing function on $-\infty < \lambda < \infty$. Denote by $L_\varrho^2(\mathbb{R})$ the Hilbert space of all functions $f : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^2(\lambda)\, d\varrho(\lambda) < \infty,$$

with the inner product

$$(f, g)_\varrho := \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, d\varrho(\lambda).$$

The main results of this paper are the following three theorems.

Theorem 4.1. *Let f is a real-valued function and $f \in L_\alpha^2(\mathbb{R})$. Then there exist monotonic functions $\varrho_{11}(\lambda)$ and $\varrho_{22}(\lambda)$ which are bounded over every finite interval, and a function $\varrho_{12}(\lambda)$ which is of bounded variation over every finite interval with the property*

$$\int_{-\infty}^{\infty} f^2(t)\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, F_j(\lambda)\, d\varrho_{ij}(\lambda),$$

where

$$F_i(\lambda) = \lim_{b\to\infty} \int_{-b}^{b} f(t)\, \theta_i(t, \lambda)\, d_\alpha(t).$$

We note that the matrix-valued function $\varrho = (\varrho_{ij})_{i,j=1}^2$ $(\varrho_{12} = \varrho_{21})$ is called a *spectral function* for (3.1).

Proof. Assume that the real-valued function $f_n(t)$ satisfies the following conditions.

1) $f_n(t)$ vanishes outside the interval $[-n, n]$, where $n < b$.

2) The functions $f_n(t)$ and $T_\alpha f_n(t)$ are continuous.

If we apply the Parseval equality to $f_n(t)$, we get

$$\int_{-n}^{n} f_n^2(t)\, d_\alpha(t) = \sum_{k=1}^{\infty} \left\{ \int_{-b}^{b} f_n(t)\, y_k(t)\, d_\alpha(t) \right\}^2. \tag{4.13}$$

Then, by integrating by parts, we obtain

$$\int_{-n}^{n} f_n(t) y_k(t) d_\alpha(t) = \frac{1}{\lambda_k} \int_{-b}^{b} f_n(t) \left[-T_\alpha^2 y_k(t) + v(t) y_k(t)\right] d_\alpha(t),$$
$$= \frac{1}{\lambda_k} \int_{-b}^{b} \left[-T_\alpha^2 f_n(t) + v(t) f_n(t)\right] y_k(t) d_\alpha(t).$$

Thus we have

$$\begin{aligned}&\sum_{|\lambda_k|\geq\mu}\left\{\int_{-b}^{b} f_n\left(t\right) y_k\left(t\right) d_\alpha(t)\right\}^2\\
\leq&\frac{1}{\mu^2}\sum_{|\lambda_k|\geq\mu}\left\{\int_{-b}^{b}\left[-T_\alpha^2 f_\xi\left(t\right)+v\left(t\right) f_n\left(t\right)\right] y_k\left(t\right) d_\alpha(t)\right\}^2\\
\leq&\frac{1}{\mu^2}\sum_{k=1}^{\infty}\left\{\int_{-b}^{b}\left[-T_\alpha^2 f_n\left(t\right)+v\left(t\right) f_n\left(t\right)\right] y_k\left(t\right) d_\alpha(t)\right\}^2\\
=&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n\left(t\right)+v\left(t\right) f_n\left(t\right)\right]^2 d_\alpha(t).\end{aligned}$$

Using (4.13), we conclude that

$$\begin{aligned}&\left|\int_{-n}^{n} f_n\left(t\right) y_k\left(t\right) d_\alpha(t)-\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n\left(t\right) y_k\left(t\right) d_\alpha(t)\right\}^2\right|\\
\leq&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n\left(t\right)+v\left(t\right) f_n\left(t\right)\right]^2 d_\alpha(t).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n\left(t\right) y_k\left(t\right) d_\alpha(t)\right\}^2\\
=&\sum_{-\mu\leq\lambda_k\leq\mu}\left\{\int_{-b}^{b} f_n\left(t\right)\left\{c_n\theta_1\left(t,\lambda_k\right)+d_n\theta_2\left(t,\lambda_k\right)\right\} d_\alpha(t)\right\}^2\\
=&\int_{-\mu}^{\mu}\sum_{i,j=1}^{2} F_{in}\left(\lambda\right) F_{jn}\left(\lambda\right) d\varrho_{ij,b}\left(\lambda\right),\end{aligned}$$

where

$$F_{in}\left(\lambda\right)=\int_{-b}^{b} f_n\left(t\right)\theta_i\left(t,\lambda\right) d_\alpha(t),\quad i=1,2.$$

Consequently, we get

$$\begin{aligned}&\left|\int_{-n}^{n} f_n^2\left(t\right) d_\alpha(t)-\int_{-\mu}^{\mu}\sum_{i,j=1}^{2} F_{in}\left(\lambda\right) F_{jn}\left(\lambda\right) d\varrho_{ij,b}\left(\lambda\right)\right|\\
\leq&\frac{1}{\mu^2}\int_{-n}^{n}\left[-T_\alpha^2 f_n\left(t\right)+v\left(t\right) f_n\left(t\right)\right]^2 d_\alpha(t).\end{aligned}\tag{4.14}$$

By Lemma 4.1 and Theorems 2.3 and 2.4, we can find sequences $\{-b_k\}$ and $\{b_k\}$ $(b_k\to\infty)$ such that the function $\varrho_{ij,b_k}\left(\lambda\right)$ converges to a monotone function $\varrho_{ij}\left(\lambda\right)$. Passing to the limit with respect to $\{-b_k\}$ and $\{b_k\}$ in (4.14), we have

$$\left|\int_{-n}^{n} f_n^2\left(t\right) d_\alpha(t)-\int_{-\mu}^{\mu}\sum_{i,j=1}^{2} F_{in}\left(\lambda\right) F_{jn}\left(\lambda\right) d\varrho_{ij}\left(\lambda\right)\right|$$

$$\leq \frac{1}{\mu^2} \int_{-n}^{n} \left[-T_\alpha^2 f_n(t) + v(t) f_n(t) \right]^2 d_\alpha(t).$$

As $\mu \to \infty$, we get

$$\int_{-n}^{n} f_n^2(t) \, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \, d\varrho_{ij}(\lambda).$$

Let $f(\cdot) \in L_\alpha^2(\mathbb{R})$. Choose functions $\{f_\xi(t)\}$ satisfying the conditions 1)–2) and such that

$$\lim_{\xi \to \infty} \int_{-\infty}^{\infty} (f(t) - f_\xi(t))^2 \, d_\alpha(t) = 0.$$

Let

$$F_{i\xi}(\lambda) = \int_{-\infty}^{\infty} f_\xi(t) \, \theta_i(t, \lambda) \, d_\alpha(t), \quad i = 1, 2.$$

Then we have

$$\int_{-\infty}^{\infty} f_\xi^2(t) \, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{i\xi}(\lambda) F_{j\xi}(\lambda) \, d\varrho_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{\infty} (f_{\xi_1}(t) - f_{\xi_2}(t))^2 \, d_\alpha(t) \to 0 \quad \text{as} \quad \xi_1, \xi_2 \to \infty,$$

we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \sum_{i=1}^{2} (F_{i\xi_1}(\lambda) F_{j\xi_1}(\lambda) - F_{i\xi_2}(\lambda) F_{j\xi_2}(\lambda)) \, d\varrho_{ij}(\lambda) \\ &= \int_{-\infty}^{\infty} (f_{\xi_1}(t) - f_{\xi_2}(t))^2 \, d_\alpha(t) \to 0, \end{aligned}$$

as $\xi_1, \xi_2 \to \infty$. Therefore, there is a limit function F_i, $i = 1, 2$, that satisfies

$$\int_{-\infty}^{\infty} f^2(t) \, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, d\varrho_{ij}(\lambda),$$

by the completeness of the space $L_\varrho^2(\mathbb{R})$.

Now we will show that the sequence $(K_{i\xi})$ defined by

$$K_{i\xi}(\lambda) = \int_{-\xi}^{\xi} f(t) \, \theta_i(t, \lambda) \, d_\alpha(t), \quad i = 1, 2,$$

converges as $\xi \to \infty$ to $F_i(\lambda)$, $i = 1, 2$, in the metric of space $L_\varrho^2(\mathbb{R})$. Let g be another function in $L_\alpha^2(\mathbb{R})$. By a similar argument, $G_i(\lambda)$, $i = 1, 2$, be defined by g. It is obvious that

$$\int_{0}^{\infty} (f(t) - g(t))^2 \, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \{(F_i(\lambda) - G_i(\lambda))(F_j(\lambda) - G_j(\lambda))\} \, d\varrho_{ij}(\lambda).$$

Let

$$g(t) = \begin{cases} f(t), & t \in [-\xi, \xi], \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left\{ \left(F_i(\lambda) - K_{i\xi}(\lambda)\right)\left(F_j(\lambda) - K_{i\xi}(\lambda)\right) \right\} d\varrho_{ij}(\lambda)$$

$$= \int_{-\infty}^{-\xi} f^2(t)\, d_\alpha(t) + \int_{\xi}^{\infty} f^2(t)\, d_\alpha(t) \to 0, \quad \xi \to \infty,$$

which proves that (K_ξ) converges to F in $L^2_\varrho(\mathbb{R})$ as $\xi \to \infty$. □

Theorem 4.2. *Suppose that the real-valued functions $f(\cdot)$ and $g(\cdot)$ are in $L^2_\alpha(\mathbb{R})$, and $F_i(\lambda)$ and $G_i(\lambda)$, $i = 1, 2$, are their Fourier transforms. Then we have*

$$\int_{-\infty}^{\infty} f(t)\, g(t)\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\varrho_{ij}(\lambda),$$

which is called the generalized Parseval equality.

Proof. It is clear that $F_i \mp G_i$, $i = 1, 2$, are transforms of $f \mp g$. Therefore, we have

$$\int_{-\infty}^{\infty} (f(t) + g(t))^2\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) + G_i(\lambda)\right)\left(F_j(\lambda) + G_j(\lambda)\right) d\varrho_{ij}(\lambda),$$

$$\int_{-\infty}^{\infty} (f(t) - g(t))^2\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) - G_i(\lambda)\right)\left(F_j(\lambda) - G_j(\lambda)\right) d\varrho_{ij}(\lambda).$$

Subtracting one of these equalities from the other one, we get the desired result. □

Theorem 4.3. *Let f be a real-valued function and $f \in L^2_\alpha(\mathbb{R})$. Then, the integrals*

$$\int_{-\infty}^{\infty} F_i(\lambda)\, \theta_j(t, \lambda)\, d\varrho_{ij}(\lambda), \quad i, j = 1, 2,$$

converge in $L^2_\alpha(\mathbb{R})$. Consequently, we have

$$f(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, \theta_j(t, \lambda)\, d\varrho_{ij}(\lambda),$$

which is called the spectral expansion formula.

Proof. Take any function $f_\xi \in L^2_\alpha(\mathbb{R})$ and any positive number ξ, and set

$$f_\xi(t) = \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, \theta_j(t, \lambda)\, d\varrho_{ij}(\lambda).$$

Let $g(\cdot) \in L^2_\alpha(\mathbb{R})$ be a real-valued function which equals zero outside the finite interval $[-\tau, \tau]$, where $\tau > 0$. Thus, we obtain

$$\int_{-\tau}^{\tau} f_\xi(t)\, g(t)\, d_\alpha(t)$$

$$= \int_{-\tau}^{\tau} \left(\int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, \theta_j(t, \lambda)\, d\varrho_{ij}(\lambda) \right) g(t)\, d_\alpha(t)$$

$$= \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda) \left\{ \int_{-\tau}^{-\tau} g(t)\, \theta_j(t,\lambda)\, d_\alpha(t) \right\} d\varrho_{ij}(\lambda)$$

$$= \int_{-\xi}^{\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\varrho_{ij}(\lambda). \tag{4.15}$$

From Theorem 4.2, we get

$$\int_{-\infty}^{\infty} f(t)\, g(t)\, d_\alpha(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\varrho_{ij}(\lambda). \tag{4.16}$$

By (4.15) and, (4.16) we have

$$\int_{-\infty}^{\infty} (f(t) - f_\xi(t))\, g(t)\, d_\alpha(t) = \int_{|\lambda|>\xi} \sum_{i,j=1}^{2} F_i(\lambda)\, G_j(\lambda)\, d\alpha_{ij}(\lambda).$$

If we apply this equality to the function

$$g(t) = \begin{cases} f(t) - f_\xi(t), & t \in [-\xi, \xi], \\ 0, & \text{otherwise,} \end{cases}$$

then we get

$$\int_{-\infty}^{\infty} (f(t) - f_\xi(t))^2\, d_\alpha(t) \leq \sum_{i,j=1}^{2} \int_{|\lambda|>\xi} F_i(\lambda)\, F_j(\lambda)\, d\varrho_{ij}(\lambda).$$

Letting $\xi \to \infty$ yields the expansion result. $\square$

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