Kragujevac Journal of Mathematics
Volume 48(6) (2024), Pages 951–966.

COMMON FIXED POINTS OF GENERALIZED α-NONEXPANSIVE MULTIVALUED MAPPINGS VIA MODIFIED S-TYPE ITERATION

R. SADHU[1,2] AND C. NAHAK[1]

Abstract. In this paper, we introduce a class of generalized α-nonexpansive multivalued mapping and study some of its important properties. In particular, this class is a multivalued version of the single-valued nonexpansive mapping, called generalized α-nonexpansive mapping proposed by Pant and Shukla [11]. A modified S-type iteration scheme is proposed to approximate the common fixed point of two multivalued mappings. Our algorithm provides a multivalued extension of the method given by Khan et al. [6]. Strong and weak convergence of the iterative process are also proved under suitable assumptions.

1. Introduction and Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and C be a nonempty subset of E. In this article $CB(C)$ denotes the collection of all nonempty closed and bounded subsets of C, $KC(C)$ denotes the collection of all nonempty compact convex subsets of C and $P(C)$ the collection of nonempty proximinal bounded subsets of C. A set C is said to be proximinal if for any $x \in E$, there exists an element $y \in C$ such that $\|x-y\| = d(x, C)$, where $d(x, C) = \inf\{\|x-y\| : y \in C\}$. Let A and B be two nonempty closed and bounded subsets of C. Then the Hausdorff distance H between A and B is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\right\}. \tag{1.1}$$

A multivalued mapping $T : C \to CB(C)$ is said to have a fixed point in C, if there exists a point $p \in C$ such that $p \in T(p)$. The set of all fixed points of T is denoted

Key words and phrases. Multivalued mapping, common fixed point, S-iteration.
2020 *Mathematics Subject Classification.* Primary: 47H10. Secondary: 54H25.
DOI 10.46793/KgJMat2406.951S
Received: July 06, 2019.
Accepted: October 03, 2021.

by $F(T)$, i.e., $F(T) = \{x \in C : x \in Tx\}$. We define $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$.

A mapping $T : C \to CB(C)$ is said to be

(a) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad \text{for all } x, y \in C;$$

(b) quasi nonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \quad \text{for all } x \in C \text{ and } p \in F(T);$$

(c) satisfying condition (C) if for $x, y \in C$

$$\frac{1}{2}d(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|;$$

(d) α-nonexpansive if for any $\alpha \in [0, 1)$

$$H(Tx, Ty)^2 \leq \alpha d(x, Ty)^2 + \alpha d(y, Tx)^2 + (1 - 2\alpha)\|x - y\|^2, \quad \text{for all } x, y \in C.$$

The following definitions and lemmas are useful in the subsequent sections.

Definition 1.1 (Opial's property [10])**.** A Banach space E is said to satisfy the Opials condition if for any sequence $\{x_n\}$ converging to z weakly and $z \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 1.1 ([12])**.** *Let E be a uniformly convex Banach space, with $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \in (0, 1)$ for all $n \geq 1$. $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = a$ for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Fixed point theories for multivalued contractive and nonexpansive mappings, using the Hausdorff metric was first coined by Markin [8] and Nadler [9]. Since then several new classes of nonexpansive multivalued mapping appeared in the literature. In 2008, Suzuki [13] introduced the class of nonexpansive mapping, termed as generalized nonexpansive mapping (i.e., mapping satisfying condition (C)). A multivalued analog of [13] was proposed by Eslamian and Abkar [1] in a uniformly convex Banach space. Several other interesting generalizations for multivalued mappings are available in literature see for example [3–5].

In 2017, another new class of nonexpansive single-valued mapping, called generalized α-nonexpansive mapping, which properly contains the class of Suzuki-type mapping, was proposed in [11]. Motivated by the work of [11], in this paper we introduce the class of generalized α-nonexpansive multivalued mapping and prove several of its important properties in a uniformly convex Banach space.

Different iterative processes have been instrumented to approximate the fixed points of single-valued and multi-valued nonexpansive mappings. To describe some relevant iterative processes, let C be a nonempty convex subset of a Banach space E and $\mathcal{T} : C \to C$, then

(a) Mann iterates

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{T} x_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is in $(0,1)$;

(b) Ishikawa iterates

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathfrak{T} y_n, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$;

(c) Agarwal et al. [2] introduced the S-iterates

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)\mathfrak{T} x_n + \alpha_n \mathfrak{T} y_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They proved that this scheme converges at a rate faster than both Picard iteration scheme and Mann iteration scheme for contractions. Following their method, it was observed that S-iteration scheme also converges faster than Ishikawa iteration scheme.

The multivalued extension of [2] was proposed by Khan and Yildirim [7]. Let $T : C \to CB(C)$ be a multivalued mapping then, their scheme runs as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{1.5}$$

where $v_n \in P_T x_n$, $u_n \in P_T y_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

A modification of [2] captures the common fixed point of two single valued mapping $\mathcal{S}$ and $\mathfrak{T}$ was proposed by Khan et al. [6], the modified S-iteration is as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n \mathfrak{T} x_n, \\ x_{n+1} = (1-\alpha_n)\mathfrak{T} x_n + \alpha_n \mathcal{S} y_n, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

We modify the iteration process which is given by (1.6) in [7] to the case of two multivalued mapping T and S as follows

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{1.7}$$

where $v_n \in P_T x_n$, $u_n \in P_S y_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

In this present paper we study strong and weak convergence of the iteration process (1.7), for finding a common fixed point of a pair of generalized α-nonexpansive multivalued mapping in a uniformly convex Banach space. Our work generalizes several convergence results in the existing literature.

2. Generalised α-Nonexpansive Mapping

In this section, we define a new class of multivalued mapping and study some of its properties.

Definition 2.1. Let C be a nonempty subset of a Banach space E. A multivalued mapping $T : C \to CB(C)$ is said to satisfy condition $(C - \alpha)$ if there exists $\alpha \in [0, 1)$ such that for any $x, y \in C$

$$\begin{aligned} &\frac{1}{2}d(x, Tx) \leq \|x - y\| \\ \Rightarrow &H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1 - 2\alpha)\|x - y\|. \end{aligned}$$

A multivalued mapping satisfying condition $(C - \alpha)$ is termed as generalized α-nonexpansive multivalued mapping.

Some important properties of the mapping are discussed below.

Proposition 2.1. *Let $T : E \to CB(E)$ be a multivalued mapping. Then the followings hold.*

(i) *If T satisfies condition (C), then T satisfies condition $(C - \alpha)$ for some $\alpha \in [0, 1)$.*
(ii) *If T satisfies condition $(C - \alpha)$ and $F(T)$ is nonempty, then T is quasi-nonexpansive.*

Proof. (i) If T satisfies condition (C), then it is trivially seen that T satisfies condition $(C - \alpha)$ for $\alpha = 0$.

(ii) Let $p \in F(T)$ (as $F(T)$ is nonempty). So, $p \in Tp$. Therefore, $\frac{1}{2}d(p, Tp) = 0 \leq \|x - p\|$ for all $x \in E$. Since T satisfies condition $(C - \alpha)$, there exists $\alpha \in [0, 1)$ such that

$$\begin{aligned} H(Tx, Tp) &\leq \alpha d(x, Tp) + \alpha d(p, Tx) + (1 - 2\alpha)\|x - p\| \\ &\leq \alpha\|x - p\| + \alpha H(Tp, Tx) + (1 - 2\alpha)\|x - p\|. \end{aligned}$$

That is, $(1 - \alpha)H(Tx, Tp) \leq (1 - \alpha)\|x - p\|$ for all $x \in E$. Since $1 - \alpha > 0$, it follows that

$$H(Tx, Tp) \leq \|x - p\|, \quad \text{for all } x \in E \text{ and } p \in F(T).$$ □

Remark 2.1. The converse of (i) in the above proposition is not true in general, i.e., if a multivalued mapping satisfies condition $(C - \alpha)$, it does not necessarily imply that the mapping satisfy condition (C). The following example gives a clear instance of the above situation.

Example 2.1. Let $C = [0, 4]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $T : C \to CB(C)$ by

$$Tx = \begin{cases} [0, \frac{x}{4}], & \text{if } x \neq 4, \\ [0, 2], & \text{if } x = 4. \end{cases}$$

Then, for $x = 4$ and $y \in (8/3, 3]$, we have $\frac{1}{2}d(x, Tx) \leq \|x - y\|$ but, $H(Tx, Ty) > \|x - y\|$. Hence, T does not satisfy condition (C). Also, for $x \in (8/3, 32/11]$ and $y = 4$, we have $\frac{1}{2}d(x, Tx) \leq \|x - y\|$, but $H(Tx, Ty) > \|x - y\|$. This again gives another instant which shows that the mapping T does not satisfies condition (C). But it is interesting to note that for any α with $\frac{1}{9} \leq \alpha \leq \frac{3}{11}$, T satisfies condition $(C - \alpha)$ and so, T is a generalized α-nonexpansive multivalued mapping. This example confirms that the new class of nonexpansive multivalued mappings proposed in this paper properly contains the class of Suzuki-type multivalued mappings.

We now prove some important inequalities related to the multivalued mapping satisfying condition $(C - \alpha)$.

Proposition 2.2. *Let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ a generalized α-nonexpansive multivalued mapping. Then for each $x, y \in C$*

(i) $H(Tx, Tz) \leq \|x - z\|$ *for all* $z \in Tx$*;*
(ii) *either* $\frac{1}{2}d(x, Tx) \leq \|x - y\|$ *or* $\frac{1}{2}d(z, Tz) \leq \|z - y\|$ *for all* $z \in Tx$*;*
(iii) *either* $H(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx) + (1 - 2\alpha)\|x - y\|$ *or* $H(Tz, Ty) \leq \alpha d(z, Ty) + \alpha d(y, Tz) + (1 - 2\alpha)\|z - y\|$ *for all* $z \in Tx$*.*

Proof. Since, $\frac{1}{2}d(x, Tx) \leq d(x, Tx) \leq \|x - z\|$ for all $z \in Tx$, we have

$$\begin{aligned} H(Tx, Tz) &\leq \alpha d(z, Tx) + \alpha d(x, Tz) + (1 - 2\alpha)\|z - x\| \\ &= \alpha d(x, Tz) + (1 - 2\alpha)\|z - x\| \\ &\leq \alpha\|x - z\| + \alpha d(z, Tz) + (1 - 2\alpha)\|z - x\| \\ &\leq \alpha H(Tx, Tz) + (1 - \alpha)\|z - x\|. \end{aligned}$$

Simplifying we get $H(Tx, Tz) \leq \|x - z\|$.

We prove (ii) by contradiction. Suppose that $\frac{1}{2}d(x, Tx) > \|x - y\|$ and $\frac{1}{2}d(z, Tz) > \|z - y\|$ for some $z \in Tx$.

Thus, from (i) we get

$$\begin{aligned} d(z, Tz) &\leq H(Tx, Tz) \leq \|x - z\| \\ &\leq \|x - y\| + \|y - z\| \\ &< \frac{1}{2}d(x, Tx) + \frac{1}{2}d(z, Tz) \\ &< d(x, Tx). \end{aligned}$$

Now,

$$d(x, Tx) \leq \|x - z\| \leq \|x - y\| + \|y - z\| < \frac{1}{2}d(x, Tx) + \frac{1}{2}d(z, Tz),$$

i.e., $d(x,Tx) < d(x,Tx)$, which is a contradiction. Thus, (ii) holds.

The condition (iii) directly follows from (ii). □

Proposition 2.3. *Let C be a nonempty subset of a Banach space E. If $T : C \to P(C)$ satisfies the condition $(C-\alpha)$, then*

$$H(Tx,Ty) \le 2\frac{1+\alpha}{1-\alpha} d(x,Tx) + \|x-y\|,$$

for all $x, y \in C$.

Proof. Let $x \in C$. Since Tx is proximal, there exists $z \in Tx$ such that $\|z-x\| = d(x,Tx)$. Thus $\frac{1}{2}d(x,Tx) \le \|z-x\|$. Since T satisfies condition $(C-\alpha)$, we have

$$\begin{aligned} H(Tx,Tz) &\le \alpha d(x,Tz) + \alpha d(z,Tx) + (1-2\alpha)\|x-z\| \\ &= \alpha d(x,Tz) + (1-2\alpha)\|x-z\| \\ &\le \alpha d(x,Tx) + \alpha H(Tx,Tz) + (1-2\alpha)\|x-z\| \\ &= \alpha H(Tx,Tz) + (1-\alpha)\|x-z\|. \end{aligned}$$

Simplifying we get

$$H(Tx,Tz) \le \|x-z\|. \tag{2.1}$$

Now, by Proposition 2.2 we get for all $x, y \in C$, either

$$H(Tx,Ty) \le \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\| \tag{2.2}$$

or

$$H(Tz,Ty) \le \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|z-y\|. \tag{2.3}$$

If (2.2) holds, we have

$$\begin{aligned} &H(Tx,Ty) \le \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\| \\ \Rightarrow\ &H(Tx,Ty) \le \alpha d(x,Tx) + \alpha H(Tx,Ty) + \alpha d(x,Tx) + (1-\alpha)\|x-y\| \\ \Rightarrow\ &H(Tx,Ty) \le \frac{2\alpha}{1-\alpha} d(x,Tx) + \|x-y\|, \end{aligned}$$

else (2.3) holds and by (2.1) we have

$$\begin{aligned} H(Tx,Ty) \le\ &H(Tx,Tz) + H(Tz,Ty) \\ \le\ &\|x-z\| + \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|z-y\| \\ \le\ &2(1-\alpha)\|x-z\| + \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|x-y\| \\ \le\ &2(1-\alpha)\|x-z\| + \alpha\|z-x\| + \alpha d(x,Tx) + \alpha H(Tx,Ty) \\ &+ \alpha\|x-y\| + \alpha d(x,Tx) + \alpha H(Tx,Tz) + (1-2\alpha)\|x-y\| \\ =\ &(2+\alpha)\|x-z\| + \alpha H(Tx,Ty) + \alpha H(Tx,Tz) + (1-\alpha)\|x-y\| \\ \le\ &2(1+\alpha)\|x-z\| + \alpha H(Tx,Ty) + (1-\alpha)\|x-y\|. \end{aligned}$$

Thus, simplifying and dividing both side of the above relation by $(1-\alpha)$, we get

$$H(Tx,Ty) \leq 2\frac{1+\alpha}{1-\alpha}\|x-z\| + \|x-y\|$$

i.e.,

$$H(Tx,Ty) \leq 2\frac{1+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|.$$

Hence, our desired inequality is proved in either cases. □

The following lemma will be useful in our next section.

Lemma 2.1. *Let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ be a generalized α-nonexpansive mapping. Then for all $x, y \in C$*

$$d(x,Ty) \leq \frac{3+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \tag{2.4}$$

Proof. From the Proposition 2.2, we have for all $x, y \in C$ and $z \in Tx$, either

$$H(Tx,Ty) \leq \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\|$$

or

$$H(Tz,Ty) \leq \alpha d(z,Ty) + \alpha d(y,Tz) + (1-2\alpha)\|z-y\|.$$

For the first case, we have

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx) + H(Tx,Ty) \\ &\leq d(x,Tx) + \alpha d(x,Ty) + \alpha d(y,Tx) + (1-2\alpha)\|x-y\|. \end{aligned}$$

Hence,

$$\begin{aligned} (1-\alpha)d(x,Ty) &\leq (1+\alpha)d(x,Tx) + (1-\alpha)\|x-y\| \\ \Rightarrow d(x,Ty) &\leq \frac{1+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \end{aligned}$$

To prove the other case, let $z' \in Tx$ be such that $\|x-z'\| = d(x,Tx)$. So, by using (i) and (iii) of Proposition 2.2, we obtain

$$\begin{aligned} d(x,Ty) \leq& d(x,Tx) + H(Tx,Tz') + H(Tz',Ty) \\ \leq& d(x,Tx) + \|x-z'\| + H(Tz',Ty) \\ \leq& 2d(x,Tx) + \alpha d(z',Ty) + \alpha d(y,Tz') + (1-2\alpha)\|z'-y\| \\ \leq& 2d(x,Tx) + \alpha\|z'-x\| + \alpha d(x,Ty) + \alpha d(y,Tx) \\ &+ \alpha H(Tx,Tz') + (1-2\alpha)\|z'-y\|. \end{aligned}$$

This yields,

$$\begin{aligned} (1-\alpha)d(x,Ty) &\leq (3+\alpha)d(x,Tx) + (1-\alpha)\|x-y\|, \\ d(x,Ty) &\leq \frac{3+\alpha}{1-\alpha}d(x,Tx) + \|x-y\|. \end{aligned}$$

Therefore, in both the cases, we get the desired results. □

We conclude this section with the property of demiclosedness.

Theorem 2.1 (Demiclosed principle)**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with Opial's property. $T : C \to CB(C)$ a multivalued mapping satisfying condition $(C-\alpha)$ and $\{x_n\}$ be a sequence in E. If $\{x_n\}$ converges weakly to some point $x \in C$ and $\limsup_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in Tx$, i.e., $(I - T)$ is demiclosed at zero.*

Proof. Since $x \in C$ and Tx is closed and bounded, for each $n \in \mathbb{N}$ there exist $z_n \in Tx$ such that $\|x_n - z_n\| = \mathrm{d}(x_n, Tx)$. Then by Proposition 2.3,

$$\begin{aligned}\|x_n - z_n\| &= \mathrm{d}(x_n, Tx) \leq \mathrm{d}(x_n, Tx_n) + H(Tx_n, Tx)\\ &\leq \mathrm{d}(x_n, Tx_n) + 2\frac{1+\alpha}{1-\alpha}\mathrm{d}(x_n, Tx_n) + \|x_n - x\|.\end{aligned}$$

Taking limsup on both side and using $\limsup_{n\to\infty} \mathrm{d}(x_n, Tx_n) = 0$, we obtain

$$\limsup_{n\to\infty} \|x_n - z_n\| \leq \limsup_{n\to\infty} \|x_n - x\|, \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

As the sequence $\{x_n\}$ converges weakly to x and E possesses Opail's property, for any $n \in \mathbb{N}$ if $z_n \neq x$ then it follows that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z_n\|,$$

which contradicts (2.5), therefore we can infer $z_n = x$ for all $n \in \mathbb{N}$. As a consequence of $z_n \in Tx$ we have $x \in Tx$, i.e., $(I - T)$ is demiclosed at zero. □

3. Convergence Theorems

In this section we propose a modified S-type iterative process for finding a common fixed point of a pair of multivalued mapping satisfying condition $(C-\alpha)$. We prove strong and weak convergence of the iterated sequence under suitable assumptions. For ths purpose let C be a nonempty subset of a Banach space E and $T : C \to CB(C)$ be a multivalued mapping, our iterative scheme is defined as follows

$$\begin{cases} x_1 \in C,\\ y_n = (1-\beta_n)x_n + \beta_n v_n,\\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n,\end{cases} \tag{3.1}$$

where $v_n \in P_T x_n$, $u_n \in P_S y_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

It is interesting to note that the iteration process (3.1) reduces to

- (1.5) when $S = T$;
- (1.4) when $S = T$ and T is single-valued;
- (1.6) when S and T are single-valued;
- (1.2) when $T = I$ and S is single-valued.

Thus all the results which are proved in this section, also holds for the iteration processes (1.5), (1.4), (1.5) and (1.2). Before stating our main convergence results, let us first prove some important lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space E and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C - \alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be a sequence generated by the modified S-iteration scheme* (3.1)*, then for any $w \in \mathbb{F}$, the following assertions hold:*

(a) $\max\{\|x_{n+1} - w\|, \|y_n - w\|\} \leq \|x_n - w\|$ *for all $n \in \mathbb{N}$;*
(b) $\lim_{n\to\infty} \|x_n - w\|$ *exists.*

Proof. By (3.1) and Proposition 2.1 we have

$$\begin{aligned}
\|y_n - w\| &= \|(1 - \beta_n)x_n + \beta_n v_n - w\| \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|v_n - w\| \\
&= (1 - \beta_n)\|x_n - w\| + \beta_n d(v_n, Tw) \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned}$$

Also by using (3.1) and Proposition 2.1 we obtain

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1 - \alpha_n)v_n + \alpha_n u_n - w\| \\
&\leq (1 - \alpha_n)\|v_n - w\| + \alpha_n\|u_n - w\| \\
&\leq (1 - \alpha_n)d(v_n, Tw) + \alpha_n d(u_n, Sw) \\
&\leq (1 - \alpha_n)H(Tx_n, Tw) + \alpha_n H(Sy_n, Sw) \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|y_n - w\| \\
&\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\
&\leq \|x_n - w\|.
\end{aligned}$$

This shows that the sequence $\{\|x_n - w\|\}$ is nonincreasing and bounded below. Thus, we can conclude $\lim_{n\to\infty} \|x_n - w\|$ exist. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C - \alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be a sequence generated by the modified S-iteration scheme* (3.1)*, then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n).$$

Proof. Let $w \in \mathbb{F}$, then by Lemma 3.1 we have $\lim_{n\to\infty} \|x_n - w\|$ exists. Suppose

$$\lim_{n\to\infty} \|x_n - w\| = r. \tag{3.2}$$

Since $v_n \in P_T x_n$, by Proposition 2.1 we write

$$\|v_n - w\| = d(v_n, Tw) \leq H(Tx_n, T_w) \leq \|x_n - w\|,$$

taking lim sup and using (3.2),

$$\limsup_{n\to\infty} \|v_n - w\| \leq r. \tag{3.3}$$

Again $u_n \in P_S y_n$, using Proposition 2.1 and Lemma 3.1 we write

$$\|u_n - w\| = d(u_n, Sw) \leq H(Sy_n, S_w) \leq \|y_n - w\| \leq \|x_n - w\|,$$

taking lim sup and using (3.2),

$$\limsup_{n\to\infty} \|u_n - w\| \leq r. \tag{3.4}$$

Moreover by (3.2) we can write

$$\begin{aligned} r &= \lim_{n\to\infty} \|x_{n+1} - w\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)v_n + \alpha_n u_n - w\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(v_n - w) + \alpha_n(u_n - w)\|. \end{aligned} \tag{3.5}$$

In view of Lemma 1.1, by (3.4), (3.3) and (3.5) we can confirm

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.6}$$

Now,

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)v_n + \alpha_n u_n - w\| \\ &= \|(v_n - w) + \alpha_n(u_n - v_n)\| \\ &\leq \|(v_n - w)\| + \alpha_n\|(u_n - v_n)\|. \end{aligned}$$

Taking lim inf we obtain

$$\liminf_{n\to\infty} \| \|v_n - w\| \geq r, \tag{3.7}$$

by (3.3) and (3.7) we conclude

$$\lim_{n\to\infty} \| \|v_n - w\| = r. \tag{3.8}$$

Again we can write

$$\begin{aligned} \|v_n - w\| &\leq \|v_n - u_n\| + \|u_n - w\| \\ &= \|v_n - u_n\| + d(u_n, Sw) \\ &\leq \|v_n - u_n\| + H(Sy_n, Sw) \\ &\leq \|v_n - u_n\| + \|y_n - w\|. \end{aligned}$$

Taking limit on both side and using (3.8) we get

$$\lim_{n\to\infty} \| \|y_n - w\| \geq r, \tag{3.9}$$

also by Lemma 3.1

$$\lim_{n\to\infty} \| \|y_n - w\| \leq \lim_{n\to\infty} \| \|x_n - w\| = r.$$

Therefore, $\lim_{n\to\infty} \|\|\|y_n - w\| = r$. Alternatively, we can write

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n v_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(v_n - w)\|. \end{aligned}$$

Hence, by Lemma 1.1 and (3.2), (3.7)

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Consequently, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n v_n - x_n\| = \lim_{n\to\infty} \beta_n \|v_n - x_n\| = 0.$$

Also,

$$\|u_n - x_n\| \leq \|u_n - v_n\| + \|v_n - x_n\|.$$

Therefore, by (3.6), $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Now, using Lemma 2.1

$$\begin{aligned} d(x_n, Sx_n) &\leq \|x_n - y_n\| + d(y_n, Sx_n) \\ &\leq \|x_n - y_n\| + \frac{3+\alpha}{1-\alpha} d(y_n, Sy_n) + \|x_n - y_n\| \\ &\leq 2\|x_n - y_n\| + \frac{3+\alpha}{1-\alpha}\|u_n - y_n\| \\ &\leq \frac{5-\alpha}{1-\alpha}\|x_n - y_n\| + \frac{3+\alpha}{1-\alpha}\|u_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, with this we conclude this lemma. □

We first assert a weak convergence theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with opial's property and $T, S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C-\alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Then the sequence $\{x_n\}$ generated by the modified S-iteration scheme* (3.1) *converges weakly to a common fixed point of T and S.*

Proof. Let $w \in \mathbb{F}$, then by Lemma 3.1 $\lim\limits_{n\to\infty} \|x_n - w\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathbb{F}$. Let p and q be weak limits corresponding to the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of the sequence $\{x_n\}$, respectively. By lemma 3.2 and Lemma 2.1 we have $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ and $I - T$ is demiclosed with respect to zero, which together implies $p \in Tp$. Similarly we can conclude $p \in Sp$, therefore $p \in \mathbb{F}$. Again in the same manner, we can prove that $q \in \mathbb{F}$. Next, we prove the uniqueness. If possible let p and q be distinct, then by Opial's condition

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p\| &= \lim_{n\to\infty}\|x_{n_i}-p\| \\ &< \lim_{n\to\infty}\|x_{n_i}-q\| \\ &= \lim_{n\to\infty}\|x_n-q\| = \lim_{n\to\infty}\|x_{n_j}-q\| \\ &< \lim_{n\to\infty}\|x_{n_j}-p\| \\ &= \lim_{n\to\infty}\|x_n-p\|.\end{aligned}$$

This is a contradiction, therefore $p=q$. Hence, $\{x_n\}$ converges weakly to a unique point of $\mathbb{F}$ and this completes the proof. □

We have the following corollaries.

Corollary 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with opial's property and $T,S : C \to CB(C)$ be two multivalued mapping satisfying condition $(C-\alpha)$. Suppose $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be generated by*

(i)

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n, \end{cases} \tag{3.10}$$

where $v_n \in P_T x_n$, $u_n \in P_T y_n$;

(ii)

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{3.11}$$

where $u_n \in P_S x_n$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0,1)$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T and S, respectively.

Proof. Letting $T=S$ in Theorem 3.1 we can prove that the iteration (3.10) converges weakly to a fixed point of T. On the other hand assuming $T=I$, the identity mapping we can conclude that the iteration (3.11) converges weakly to a fixed point of S. □

We now state our main result, ensuring strong convergence of the iteration process (3.1) to a common fixed point under suitable assumptions in a real Banach space.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Banach space E, and let $T,S : C \to CB(C)$ satisfy condition $(C-\alpha)$. Also suppose that, $\mathbb{F} \neq \Phi$ and $Tw = \{w\} = Sw$ for all $w \in \mathbb{F}$. Let $\{x_n\}$ be the sequence generated by (3.1). Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\liminf_{n\to\infty} d(x_n,\mathbb{F}) = 0$.*

Proof. Let $\{x_n\}$ converge to a common fixed point x of T and S, i.e., $x \in \mathbb{F}$. Thus it obviously follows that $\liminf\limits_{n\to\infty} \operatorname{dist}(x_n, \mathbb{F}) = 0$.

Conversely, let us suppose that $\liminf_{n\to\infty} d(x_n, \mathbb{F}) = 0$, then from Lemma 3.1 for each $w \in \mathbb{F}$ we have $\|x_{n+1} - w\| \leq \|x_n - w\|$, which implies

$$d(x_{n+1}, \mathbb{F}) \leq d(x_n, \mathbb{F}).$$

Hence, $\{d(x_n, \mathbb{F})\}$ is a decreasing sequence of real numbers which is bounded below and also $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathbb{F}) = 0$, implying $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$ We claim that, $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$, there exists $p \in \mathbb{N}$ such that for all $n \geq p$, we have

$$d(x_n, \mathbb{F}) < \frac{\epsilon}{2}.$$

In particular, $\inf\{\|x_p - w\| : w \in \mathbb{F}\} < \frac{\epsilon}{2}$, so there exists some $\bar{w} \in \mathbb{F}$ such that $\|x_p - \bar{w}\| < \frac{\epsilon}{2}$. Now, for $m, n \geq p$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - \bar{w}\| + \|x_n - \bar{w}\| < 2\|x_p - \bar{w}\| < 2\frac{\epsilon}{2} = \epsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Now C being a closed subset of E and $\{x_n\}$ is a Cauchy sequence in C, it must converge in C. Let $\lim_{n\to\infty} x_n = z$. Now

$$d(z, Tz) \leq \|z - x_n\| + d(x_n, Tx_n) + H(Tx_n, Tz).$$

Applying Proposition 2.3, we obtain

$$\begin{aligned} d(z, Tz) &\leq \|z - x_n\| + d(x_n, Tx_n) + 2\frac{1+\alpha}{1-\alpha} d(x_n, Tx_n) + \|x_n - z\| \\ &\leq 2\|z, x_n\| + \frac{3+\alpha}{1-\alpha} d(x_n, Tx_n). \end{aligned}$$

Since $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ (by Lemma 3.2) we conclude $d(z, Tz) = 0$ and hence we infer $z \in Tz$. In a similar fashion we can prove $z \in Sz$. Therefore, we conclude $z \in \mathbb{F}$. This completes the proof. ☐

Let C be a subset of a normed space E, two mappings $S, T : C \to CB(C)$ are said to satisfy the **Condition** (A), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that for all $x \in C$ either, $d(x, Tx) \geq f(d(x, \mathbb{F}))$ or $d(x, Sx) \geq f(d(x, \mathbb{F}))$.

Now we prove the strong convergence theorem by using Condition (A).

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to CB(C)$ and $S : C \to CB(C)$ be two multivalued mappings satisfying the condition $(C-\alpha)$ along with condition(A) Assume that $\mathbb{F} \neq \emptyset$ and $T(w) = \{w\} = Sw$ for each $w \in \mathbb{F}$. Let $\{x_n\}$ be the sequence defined as in* (3.1). *Then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. In view of Theorem 3.2 we have $\lim_{n\to\infty} \mathrm{d}(x_n, \mathbb{F})$ exists and by Condition (A), we have by Lemma 3.2, either

$$\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) \leq \lim_{n\to\infty} \mathrm{d}(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) \leq \lim_{n\to\infty} \mathrm{d}(x_n, Sx_n) = 0.$$

Thus, in any case $\lim_{n\to\infty} f(\mathrm{d}(x_n, \mathbb{F})) = 0$. Since f is nondecreasing and $f(0) = 0$ implies $\lim_{n\to\infty} \mathrm{d}(x_n, \mathbb{F}) = 0$. Therefore our result follows as a direct consequence of Theorem 3.2. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to CB(C)$ be a multivalued mappings satisfying the condition $(C - \alpha)$ along with condition(A). Assume that $\mathbb{F} \neq \emptyset$ and $T(w) = \{w\}$ for each $w \in F(T)$. Let $\{x_n\}$ be the sequence defined as in* (3.10) *and* (3.11)*. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. The corollary can be proved as a direct consequence of Theorem 3.3, by letting $S = T$. □

4. Numerical Illustrations

In this section, the behavior of the modified S-type iteration is studied for a given set of problems. The iteration has been performed several times with different initial guesses. We also observe the effect on convergence speed, for different choices of the parameters α_n and β_n. It is worthy to mention that the iteration took more steps to converge when the parameters are chosen close to 0. In the table below we enlisted our observations on the iteration process to find the common fixed points of two multivalued mappings T and S. In this process, we choose the tolerance limit 10^{-4} as stopping criteria. The actual common fixed point of T and S is 0, while the approximate solutions are shown in the table.

$$Tx = \begin{cases} [0, \frac{x}{4}], & \text{if } x \neq 4, \\ [0, 2], & \text{if } x = 4, \end{cases} \qquad Sx = \begin{cases} [0, \frac{x}{5}], & \text{if } x \neq 5, \\ \{1\}, & \text{if } x = 5. \end{cases}$$

Table 1. Numerical illustrations with different initial guesses

Sl.	Initial guess	α_n	β_n	No. of iteration	Final solution
1	2	0.5	0.5	12	8.546 e-06
2	4	0.3	0.65	14	7.669 e-06
3	23.762	0.1	0.01	20	1.393 e-05
4	127.63	0.9	0.33	12	2.657 e-06
5	529.66	0.75	0.66	12	9.157 e-06
6	1000	0.8	0.8	12	3.214 e-06

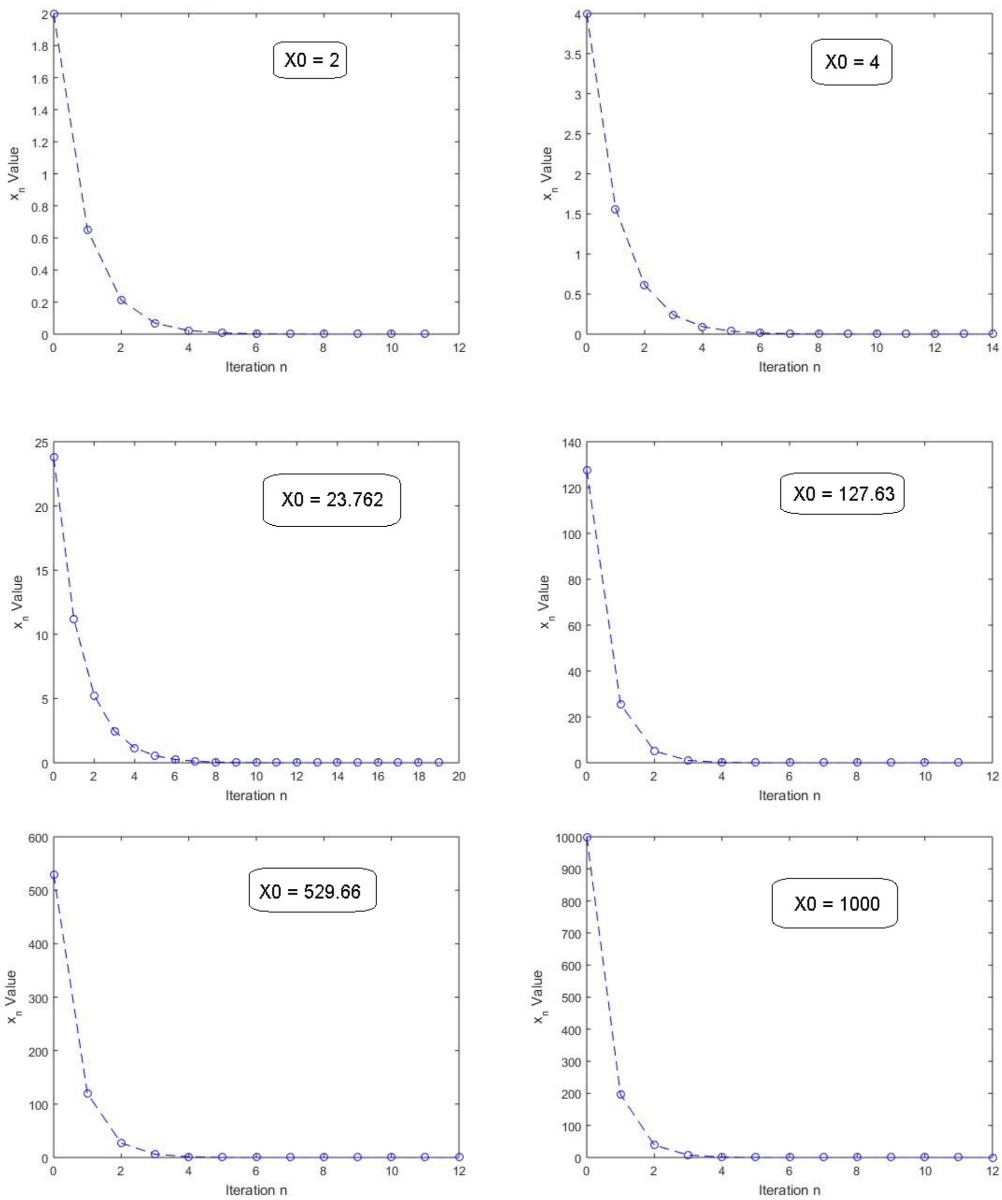

FIGURE 1. Plot of points against iteration for various initial guesses

5. CONCLUSION

The present article introduces a new class of nonexpansive multivalued mapping. A modified S-iteration process is employed to approximate a common fixed point of two multivalued mappings. Strong and weak convergence of the method are proved under suitable assumptions. Apart from Corollary 3.1 and Corollary 3.2 our results also generalize several theorems in the existing literature. In particular, if $T = S$ and T is chosen to be single-valued, then Theorems 5.8 (a), 5.9 and 5.10 of [11] follow from our results. Moreover, since the class of generalized α-nonexpansive mapping properly contains the class of nonexpansive and Suzuki-type mapping, our results also suit well in those settings.

References

[1] A. Abkar and M. Eslamian, *Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces*, Fixed Point Theory Appl. **2010** (2010), Article ID 457935, 10 pages. https://doi.org/10.1155/2010/457935

[2] R. Agarwal, D. O Regan and D. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal. **8**(1) (2007), 61–79.

[3] S. S. Chang, R. P. Agarwal and L. Wang, *Existence and convergence theorems of fixed points for multi-valued scc-, skc-, ksc-, scs- and c-type mappings in hyperbolic spaces*, Fixed Point Theory Appl. **2015**(83) (2015), 17 pages. https://doi.org/10.1186/s13663-015-0339-9

[4] S. S. Chang, Y. K. Tang, L. Wang, Y. G. Xu, Y. H. Zhao and G. Wang, *Convergence theorems for some multivalued generalized nonexpansive mappings*, Fixed Point Theory Appl. **2014**(1) (2014), Article ID 33, 11 pages. https://doi.org/10.1186/1687-1812-2014-33

[5] J. S. Jung, *Strong convergence theorems for multivalued nonexpansive nonself mappings in Banach spaces*, Nonlinear Anal. **66**(11) (2007), 2345–2354. https://doi.org/10.1016/j.na.2006.03.023

[6] S. H. Khan, Y. J. Cho and M. Abbas, *Convergence to common fixed points by a modified iteration process*, J. Appl. Math. Comput. **35**(1) (2011), 607–616. https://doi.org/10.1007/s12190-010-0381-z

[7] S. H. Khan and I. Yildirim, *Fixed points of multivalued nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2012**(73) (2012), Article ID 73, 9 pages. https://doi.org/10.1186/1687-1812-2012-73

[8] J. T. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc. **74**(4) (1968), 639–640.

[9] S. B. Nadler, JR, *Multivalued contraction mappings*, Pac. J. Appl. Math. **30**(2) (1969), 475–488.

[10] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73**(4) (1967), 591–597.

[11] R. Pant and R. Shukla, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim. **38**(2) (2017), 248–266. https://doi.org/10.1080/01630563.2016.1276075

[12] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc. **43**(1) (1991), 153–159.

[13] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340**(2) (2008), 1088–1095.

[1]Department of Mathematics,
Indian Institute of Technology Kharagpur,
Kharagpur, India

[2]Department of Mathematics,
Sreegopal Banerjee College, Magra,
West Bengal, India
Email address: rudrajitsadhu@gmail.com
Email address: cnahak@maths.iitkgp.ac.in