Kragujevac Journal of Mathematics
Volume 48(6) (2024), Pages 827–844.

ENERGY LANDSCAPES AND NON-ARCHIMEDEAN PSEUDO-DIFFERENTIAL OPERATORS AS TOOLS FOR STUDYING THE SPREADING OF INFECTIOUS DISEASES IN A SITUATION OF EXTREME SOCIAL ISOLATION

VICTOR A. AGUILAR-ARTEAGA[1], I. S. GUTIÉRREZ[2],
AND ANSELMO TORRESBLANCA-BADILLO

Abstract. In this article, we introduce a new type of pseudo-differential equations naturally connected with non-archimedean pseudo-differential operators and whose symbols are new classes of negative definite functions in the p-adic context and in arbitrary dimension. These equations are proposed as a mathematical models to study the spreading of infectious diseases (say COVID-19) through a random walk on a complex energy landscape and taking into account social clusters in a situation of extreme social isolation.

1. Introduction

In the archimedean setting the nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)dy - u(x,t) \tag{1.1}$$

have been widely used to model diffusion processes. Here, $J : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, radial, continuous function with

$$\int_{\mathbb{R}^n} J(z)dz = 1.$$

Key words and phrases. Energy landscapes, pseudo-differential operators, negative definite functions, evolution equations, strong Markov processes, p-adic analysis.
2020 *Mathematics Subject Classification.* Primary: 11E95, 11F85. Secondary: 12H25.
DOI 10.46793/KgJMat2406.827AA
Received: February 27, 2021.
Accepted: September 04, 2021.

The model (1.1) can be interpreted as follows: if $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then

$$\int_{\mathbb{R}^n} J(y-x)u(y,t)dy = (J*u)(x,t)$$

is the rate at which individuals are arriving at position x from all other places. In the same way, $-u(x,t) = -\int_{\mathbb{R}^n} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1). For further details the reader may consult [12, 16], and the references therein. In [6, 7] Avetisov et al. developed a class of p-adic pseudo-differential equations in dimension one with the aim of studying the dynamics of a large class of complex systems such as macromolecules, glasses and proteins. In these models, the time-evolution of the system is controlled by a master equation of the form

$$\frac{\partial u\left(x,t\right)}{\partial t} = \int_{\mathbb{Q}_p} j\left(|x-y|_p\right)\left\{u\left(y,t\right)-u\left(x,t\right)\right\}dy, \quad t \geq 0,$$

where $j:\mathbb{Q}_p\times\mathbb{Q}_p\to\mathbb{R}_+$ is the probability of transition from state y to the state x per unit time, and the function $u(x,t):\mathbb{Q}_p\times\mathbb{R}_+\to\mathbb{R}_+$ is a probability density distribution.

In the latest years, p-adic nonlocal evolution equations and variations of it have been studied extensively. For example, modeling of geological processes (such as petroleum reservoir dynamics, fluid flows in porous media such as rock); the dynamics of myoglobin (myoglobin is a monomeric protein that gives muscle its red color); the relaxation of spin glasses, etc., see e.g. [1, 2, 5, 11, 21, 22, 24–29, 37] and the references therein. In this models the dynamics of complex systems is described by a random walk on a complex energy landscape. An energy landscape (or simply a landscape) is a continuous function $\mathbb{U}:X\to\mathbb{R}$ that assigns to each physical state of a system its energy. The term complex landscape means that function $\mathbb{U}$ has many local minima. In this case the method of *interbasin kinetics* is applied, in this approach, the study of a random walk on a complex landscape is based on a description of the kinetics generated by transitions between groups of states (basins). This concept can be outlined as follows. A complex system is assumed to have a large number of metastable configurations which realize local minima on the potential energy surface. The local minima are clustered in hierarchically nested basins of minima, namely, each large basin consists of smaller basins, each of these consisting of even smaller ones, and so on. Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. The transition rate between basins depends on the energy barrier between these basins. By using these methods, the configuration space of the system is approximated by an ultrametric space (a rooted tree) and by a function on the tree which describes by stochastic motions the distribution of the activation energies. Procedures for constructing hierarchies of basins kinetics from any energy landscapes have been studied extensively, see e.g. [6–8, 22, 27, 31, 32, 37] and the references therein.

In [7] an Arrhenius type relation was used, that is,

$$j\,(x \mid y) \sim A(T) \exp\left\{-\frac{\mathbb{U}\,(x \mid y)}{kT}\right\},$$

where $\mathbb{U}\,(x \mid y)$ is the height of the activation barrier for the transition from the state y to state x, k is the Boltzmann constant and T is the temperature. This formula establishes a relation between the structure of the energy landscape $\mathbb{U}\,(x \mid y)$ and the transition function $j\,(x \mid y)$.

In this paper, we introduce new classes of p-adic pseudo-differential equations naturally connected with certain types of non-archimedean pseudo-differential operators whose symbols are associated with new classes of negative definite functions on the p-adic numbers. This type of pseudo-differential equations may be seen as a generalization of the equations studied in [6, 7, 30, 36–38] and the references therein.

We establish rigorously that such equations are ultradiffusion equations, i.e., we show that the fundamental solutions of the Cauchy problems naturally associated to these equations are transition density functions of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$, see Theorem 3.2, Theorem 4.2 and Theorem 5.1.

Given the non-archimedean topology of $\mathbb{Q}_p^n$ we have that two balls in $\mathbb{Q}_p^n$ have nonempty intersection if and only if one is contained in the other. Moreover, any ball can be represented as disjoint union of balls of smaller radius, each of the latter can be represented in the same way with even smaller radius and so on, see e.g. [4, 38]. The above implies that every ball in $\mathbb{Q}_p^n$ can be identified with a rooted tree. For this reason, a particular population group in a human society can be represented as a system of hierarchically coupled disjoint clusters. Any cluster is slit into disjoint sub-cluster, each of the latter is split into its own disjoint) sub-clusters and so on. Therefore, the ultrametric spaces (in particular the p-adic numbers) are proposed as a natural, necessary and essential structure to study the spreading of infectious diseases (say COVID-19) through a random walk on a complex energy landscape and taking into account social clusters in a situation of extreme social isolation. For more details, the reader can consult [3, 19, 20, 23, 34].

From a mathematical, physical and computational point of view, we consider that the spread of infectious diseases (such as the COVID-19 epidemic and new variants with very high rates of contagion) on social clusters in a situation of extreme social isolation, can be modeled as a random walk in an complex energy landscape. Therefore, an interesting open problem consists in determine if our p-adic pseudo-differential equations (in combination with the method of interbasin kinetics) can be applied, among other things, to study the dynamics of the spread infectious diseases through in a random walk in a complex energy landscape.

It should also be noted that, recently in [36] and [37], the authors Torresblanca-Badillo and Zúñiga-Galindo introduce a large class of non-archimedean pseudo-differential operators whose symbols are negative definite functions. Since then, in the last four years, the first author and his collaborators have been studied new classes of non-archimedean pseudo-differential operators whose symbols are associated with negative definite functions on the p-adic numbers, see [9, 13–15, 34, 35].

This article is organized as follows. In Section 2, we will collect some basic results on the p-adic analysis and fix the notation that we will use through the article. In Section 3, we introduce a large class of negative definite functions of the semi-smooth and elliptic types, see Theorem 3.1 and Corollary 3.1, respectively. These functions are the symbols of a large class of non-archimedean pseudo-differential operators (denoted by $\mathcal{A}$) which determine certain ultradiffusion equations on $\mathbb{Q}_p^n$, see Theorem 3.2. In Section 4 we also introduced new classes of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions (in the p-adic context) associated with logarithmic functions, see Theorem 4.1 and Corollary 4.1. This operators determine certain Lévy process $\mathfrak{X}(t,\omega)$ with state space $\mathbb{Q}_p^n$, see Theorem 4.2. In Section 5 we will study a new class of non-archimedean operators (denoted by $\mathcal{A}_{\boldsymbol{\psi}}$) associated with a non-archimedean negative definite function $\boldsymbol{\psi}$. Imposing certain conditions to the function $\boldsymbol{\psi}$ we obtain that $\mathcal{A}_{\boldsymbol{\psi}}$ is a pseudo-differential operator which also determine ultradiffusion equations, see Theorem 5.1.

2. Fourier Analysis on $\mathbb{Q}_p^n$: Essential Ideas

2.1. **The field of p-adic numbers.** Along this article p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if } x = 0, \\ p^{-\gamma}, & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of x*.

Any p-adic number $x \neq 0$ has a unique expansion of the form $x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if } x = 0 \text{ or } ord(x) \geq 0, \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j, & \text{if } ord(x) < 0. \end{cases}$$

We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n \mid ||x - a||_p \leq p^r\}$ *the ball of radius p^r with center at* $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$ and take $B_r^n(0) =: B_r^n$.

Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p \mid |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of p-adic integers of* $\mathbb{Q}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n \mid ||x - a||_p = p^r\}$ *the sphere of radius p^r with center at* $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) =: S_r^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. The group of invertible elements in $\mathbb{Z}_p$ constitutes the set $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid |x|_p = 1\}$. As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [38, Section 1.3], or [4, Section 1.8]. The balls and spheres are compact subsets. Thus, $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

We will use $\Omega\left(p^{-r}||x - a||_p\right)$ to denote the characteristic function of the ball $B_r^n(a)$. We will use the notation 1_A for the characteristic function of a set A. Along the article $d^n x$ will denote a Haar measure on $\mathbb{Q}_p^n$ normalized such that $\int_{\mathbb{Z}_p^n} d^n x = 1$.

2.2. **Some function spaces.** A complex-valued function φ defined on $\mathbb{Q}_p^n$ is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x), \quad \text{for } x' \in B_{l(x)}^n.$$

Denote by $\mathcal{E}(\mathbb{Q}_p^n)$ the linear space of locally constant $\mathbb{C}$-value functions on $\mathbb{Q}_p^n$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function* (*or a test function*) if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n) =: \mathcal{D}$. Let $\mathcal{D}'(\mathbb{Q}_p^n) =: \mathcal{D}'$ denote the set of all continuous functional (distributions) on $\mathcal{D}$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^n) \times \mathcal{D}(\mathbb{Q}_p^n) \to \mathbb{C}$ is denoted as $\langle T, \varphi \rangle$ for $T \in \mathcal{D}'(\mathbb{Q}_p^n)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [4, Section 4.4].

Denote by $L_{loc}^1(\mathbb{Q}_p^n) := L_{loc}^1$ the set of functions $f : \mathbb{Q}_p^n \to \mathbb{C}$ such that $f \in L^1(K)$ for any compact $K \subset \mathbb{Q}_p^n$. Every $f \in L_{loc}^1$ defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ by the formula

$$\langle f, \varphi \rangle = \int_{\mathbb{Q}_p^n} f(x)\, \varphi(x)\, d^n x.$$

Such distributions are called *regular distributions.*

Given $\rho \in [0, \infty)$, we denote by $L^\rho\left(\mathbb{Q}_p^n, d^n x\right) = L^\rho\left(\mathbb{Q}_p^n\right) := L^\rho$, the $\mathbb{C}-$vector space of all the complex valued functions g satisfying $\int_{\mathbb{Q}_p^n} |g(x)|^\rho d^n x < \infty$, $L^\infty := L^\infty\left(\mathbb{Q}_p^n\right) = L^\infty\left(\mathbb{Q}_p^n, d^n x\right)$ denotes the $\mathbb{C}-$vector space of all the complex valued functions g such that the essential supremum of $|g|$ is bounded.

Let denote by $C(\mathbb{Q}_p^n, \mathbb{C}) =: C_\mathbb{C}$ the $\mathbb{C}-$vector space of all the complex valued functions which are continuous, by $C(\mathbb{Q}_p^n, \mathbb{R}) =: C_\mathbb{R}$ the $\mathbb{R}-$vector space of continuous functions. Set

$$C_0(\mathbb{Q}_p^n, \mathbb{C}) := C_0(\mathbb{Q}_p^n) = \left\{ f : \mathbb{Q}_p^n \to \mathbb{C} \mid \ f \text{ is continuous and } \lim_{||x||_p \to \infty} f(x) = 0 \right\},$$

where $\lim_{||x||_p\to\infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in \mathbb{Q}_p^n \backslash B(\epsilon)$. We recall that $(C_0(\mathbb{Q}_p^n, \mathbb{C}), ||\cdot||_{L^\infty})$ is a Banach space.

2.3. **Fourier transform.** Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0+x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \mapsto \chi_p(\xi x)$, see e.g. [4, Section 2.3].

Given $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, we set $x \cdot \xi := \sum_{j=1}^n x_j\xi_j$. If $f \in L^1(\mathbb{Q}_p^n)$, its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \mathcal{F}_{x\to\xi}(f) = \widehat{f}(\xi) := \int_{\mathbb{Q}_p^n} \chi_p(\xi\cdot x) f(x) d^n x, \quad \text{for } \xi \in \mathbb{Q}_p^n.$$

The inverse Fourier transform of a function $f \in L^1(\mathbb{Q}_p^n)$ is

$$(\mathcal{F}^{-1}f)(x) = \mathcal{F}^{-1}_{\xi\to x}(f) = \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi) f(\xi) d^n\xi, \quad \text{for } x \in \mathbb{Q}_p^n.$$

The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}f))(\xi) = f(-\xi),$$

for every $f \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [4, Section 4.8].

The set $L^2(\mathbb{Q}_p^n)$ is the Hilbert space with the scalar product

$$(f, g) = \int_{\mathbb{Q}_p^n} f(x)\overline{g}(x) d^n x, \quad f, g \in L^2(\mathbb{Q}_p^n),$$

so that $||f||_{L^2} = \sqrt{(f, f)}$.

If $f \in L^2(\mathbb{Q}_p^n)$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||\le p^k} \chi_p(\xi\cdot x) f(x) d^n x, \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in $L^2(\mathbb{Q}_p^n)$. We recall that the Fourier transform is unitary on $L^2(\mathbb{Q}_p^n)$, i.e., $||f||_{L^2} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2(\mathbb{Q}_p^n)$ and that (2.3) is also valid in $L^2(\mathbb{Q}_p^n)$, see e.g. [33, Chapter III, Section 2].

3. Non-Archimedean Pseudo-Differential Operators with Semi-Smooth and Elliptic Symbols

In this section we introduce a large class of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions on p-adic numbers. Moreover, we introduce a new class of non-archimedean ultradiffusion equations. From now on denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of (positive) natural numbers and by $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \ge 0\}$ the set of non-negative real numbers.

Definition 3.1. A function $\psi : \mathbb{Q}_p^n \to \mathbb{C}$ is called negative definite, if

$$\sum_{i,j=1}^{m} \left(\psi(\xi_i) + \overline{\psi(\xi_j)} - \psi(\xi_i - \xi_j)\right) \lambda_i \overline{\lambda_j} \geq 0,$$

for all $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_m \in \mathbb{Q}_p^n$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Remark 3.1. We denote by $\mathcal{N}(\mathbb{Q}_p^n)$ the set of negative definite functions on $\mathbb{Q}_p^n$ and by $\mathcal{CN}(\mathbb{Q}_p^n)$ the set of continuous negative definite functions on $\mathbb{Q}_p^n$. The following assertions hold:

(i) $\mathcal{N}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$;
(ii) the non-negative constant functions belong to $\mathcal{N}(\mathbb{Q}_p^n)$;
(iii) $\mathcal{CN}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of compact convergence on $\mathbb{Q}_p^n$;
(iv) if $\psi : \mathbb{Q}_p^n \to \mathbb{R}$ is negative definite function, then $\psi(-x) = \psi(x)$ and $\psi(x) \geq \psi(0) \geq 0$ for all $x \in \mathbb{Q}_p^n$.

For the basic results on negative definite functions the reader may consult [10].

Remark 3.2. (i) It is relevant to mention that for any locally bounded negative definite function $\psi \in \mathcal{N}(\mathbb{R}^n)$ there exists a constant $C_\psi > 0$ such that $|\psi(\xi)|_{\mathbb{R}^n} \leq C_\psi(1+|\xi|_{\mathbb{R}^n}^2)$, for all $\xi \in \mathbb{R}^n$, see e.g. [17, Lemma 3.6.22]. However, in the p-adic context this is not always the case, see e.g. [36].

(ii) Another aspect to be highlighted is the fact that the function $y \mapsto ||y||^\alpha$ is continuous and negative definite on $\mathbb{R}^n$ for all $\alpha \in (0, 2]$, see [10, 10.5, page 74]. However, in the p-adic context for all fixed $\alpha > 0$ and $\beta > 0$, the function $y \mapsto \alpha||y||_p^\beta$ is continuous and negative definite on $\mathbb{Q}_p^n$, see [36, Example 3.5].

Definition 3.2. We say that a function $\mathfrak{a} : \mathbb{Q}_p^n \to \mathbb{R}_+$ is a semi-smooth symbol, if it satisfies the following properties.

(i) $\mathfrak{a}$ is a continuous function.
(ii) $\mathfrak{a}$ is a increasing function with respect to $|| \cdot ||_p$.
(iii) $\mathfrak{a}$ is a *radial function* on $\mathbb{Q}_p^n$, i.e. $\mathfrak{a}(x) = g(||x||_p)$ for some $g : \mathbb{R}_+ \to \mathbb{R}_+$. To make the radial dependence clear we use the notation $\mathfrak{a}(x) = \mathfrak{a}(||x||_p)$ for all $x \in \mathbb{Q}_p^n$.
(iv) $\mathfrak{a}(||x||_p) = 0 \Leftrightarrow x = 0$.
(v) There exist positive constants $C_0 := C_0(\mathfrak{a})$ and $d := d(\mathfrak{a})$ such that

$$C_0||x||_p^d \leq \mathfrak{a}(||x||_p), \quad \text{for every } x \in \mathbb{Q}_p^n.$$

Example 3.1. (i) The simplest example of semi-smooth symbols is the elliptic polynomial of degree d (in particular the p-adic norm $|| \cdot ||_p$). For more details the reader may consult [27].

(ii) Taking $C_0 = d = 1$ in the above definition, we have that the function $\mathfrak{a}(x) := e^{||x||_p} - 1$, $x \in \mathbb{Q}_p^n$, is a semi-smooth symbol.

Remark 3.3. (i) For $t > 0$ note that

$$\begin{aligned}\int_{\mathbb{Q}_p^n} e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi &= \sum_{j=0}^{\infty} e^{-t\mathfrak{a}(p^{-j})} \int_{||\xi||_p = p^{-j}} d^n\xi + \sum_{j=1}^{\infty} e^{-t\mathfrak{a}(p^j)} \int_{||\xi||_p = p^j} d^n\xi \\ &= (1-p^{-n}) \left(\sum_{j=0}^{\infty} e^{-t\mathfrak{a}(p^{-j})} p^{-nj} + \sum_{j=1}^{\infty} e^{-t\mathfrak{a}(p^j)} p^{nj} \right) \\ &\leq (1-p^{-n}) \left(\sum_{j=0}^{\infty} p^{-nj} + \sum_{j=1}^{\infty} e^{-tC_0 p^{jd}} p^{nj} \right) < \infty,\end{aligned}$$

i.e., $e^{-t\mathfrak{a}} \in L^1(\mathbb{Q}_p^n)$.

Consider the operator non-archimedean pseudo-differential operator $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A}(\varphi)(x) &:= \mathcal{F}^{-1}_{\xi \to x} \left(\mathfrak{a}(||\xi||_p) \widehat{\varphi}(\xi) \right) \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \mathfrak{a}(||\xi||_p) \widehat{\varphi}(\xi) d^n\xi,\end{aligned}$$

where $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and $\mathfrak{a}$ is a semi-smooth symbol, and the Cauchy problem (or p-adic heat equation)

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \mathcal{A}u(x,t), \quad t \in [0,\infty),\ x \in \mathbb{Q}_p^n, \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n). \end{cases} \tag{3.1}$$

Then, the fundamental solution (or heat Kernel) of the Cauchy problem (3.1) is defined as

$$Z_t(x) = Z(x,t) := \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi, \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t > 0. \tag{3.2}$$

Therefore, by [33, (1.6), page 118] we have that $Z_t(x) \in C_0(\mathbb{Q}_p^n)$.

(ii) For all $t > 0$ we have that $Z_t(\cdot) \in L^1_{loc}$, i.e., $Z_t(\cdot)$ is a regular distribution on $\mathbb{Q}_p^n$. Therefore, for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and [4, Section 4.9] we have that

$$\langle \mathcal{F}(Z(x,t)), \varphi \rangle = \langle Z(x,t), \widehat{\varphi} \rangle = \left\langle e^{-t\mathfrak{a}(||x||_p)}, \varphi \right\rangle.$$

The above implies that $\mathcal{F}(Z(x,t)) = e^{-t\mathfrak{a}(||x||_p)}$.

Lemma 3.1. *The fundamental solution $Z_t(x)$ has the following properties:*

(i) $Z_t(x) \geq 0$ *for any* $t > 0$;
(ii) $\int_{\mathbb{Q}_p^n} Z_t(x) d^n x = 1$ *for any* $t > 0$;
(iii) $Z_{t+s}(x) = \int_{\mathbb{Q}_p^n} Z_t(x-y) Z_s(y) d^n y$ *for all* $t, s > 0$;
(iv) $Z_t(x) \leq t||x||_p^{-n}$ *for all* $t > 0$ *and* $x \in \mathbb{Q}_p^n \setminus \{0\}$.

Proof. (i) If $x = 0$, the assertion is clear. Then, for $x \in \mathbb{Q}_p^n \setminus \{0\}$ with $||x||_p = p^{-\gamma}$, $\gamma \in \mathbb{Z}$, $t > 0$ and making the change of variable $w = p^j\xi$, we have that

$$Z_t(x) = \sum_{-\infty < j < \infty} e^{-t\mathfrak{a}(p^j)} \int_{||\xi||_p = p^j} \chi_p(-x \cdot \xi) d^n\xi$$

$$
\begin{aligned}
&= \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(p^j)} \int_{||p^j\xi||_p=1} \chi_p\left(-x\cdot\xi\right) d^n\xi \\
&= \sum_{-\infty<j<\infty} p^{nj} e^{-t\mathfrak{a}(p^j)} \int_{||w||_p=1} \chi_p\left(-p^{-j}x\cdot w\right) d^n w.
\end{aligned}
$$

By using the formula

$$
\int_{||w||_p=1} \chi_p\left(-p^{-j}x\cdot w\right) d^n w = \begin{cases} 1-p^{-n}, & \text{if } j\le\gamma, \\ -p^{-n}, & \text{if } j=\gamma+1, \\ 0, & \text{if } j\ge\gamma+2, \end{cases}
$$

we have that

$$
\begin{aligned}
Z_t(x) &= (1-p^{-n}) \sum_{j=-\gamma}^{\infty} p^{-nj} e^{-t\mathfrak{a}(p^{-j})} - p^{n\gamma} e^{-t\mathfrak{a}(p^{\gamma+1})} \\
&\ge e^{-t\mathfrak{a}(p^{\gamma})} \sum_{j=-\gamma}^{\infty} (1-p^{-n}) p^{-nj} - p^{n\gamma} e^{-t\mathfrak{a}(p^{\gamma+1})} \\
&= p^{n\gamma}\left(e^{-t\mathfrak{a}(p^{\gamma})} - e^{-t\mathfrak{a}(p^{\gamma+1})}\right) \\
&\ge 0.
\end{aligned}
$$

(ii) As a direct consequence of Remark 3.3 (ii) we have that $\mathcal{F}(Z(0,t)) = 1$. On the other hand, $\mathcal{F}(Z(x,t)) = \int_{\mathbb{Q}_p^n} \chi_p(\xi\cdot x) Z(x,t) d^n x$ and $\mathcal{F}(Z(0,t)) = \int_{\mathbb{Q}_p^n} Z(x,t) d^n x$. Therefore, $\int_{\mathbb{Q}_p^n} Z(x,t) d^n x = 1$ for all $t>0$.

(iii) For $t,s>0$, we have by (3.2) that

$$
\begin{aligned}
Z_{t+s}(x) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-x\cdot\xi\right) e^{-t\mathfrak{a}(||\xi||_p)} e^{-s\mathfrak{a}(||\xi||_p)} d^n\xi \\
&= Z_t(x) * Z_s(x) \\
&= \int_{\mathbb{Q}_p^n} Z_t(x-y) Z_s(y) d^n y.
\end{aligned}
$$

(iv) For $t>0$, $x = p^{\gamma}x_0 \neq 0$ such that $\gamma\in\mathbb{Z}$ and $||x_0||_p = 1$, and making the change of variable $z = p^{\gamma}\xi$, we have that

$$
\begin{aligned}
Z(x,t) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-p^{\gamma}\xi\cdot x_0\right) e^{-t\mathfrak{a}(||\xi||_p)} d^n\xi \\
&= ||x||_p^{-n} \int_{\mathbb{Q}_p^n} \chi_p\left(-x_0\cdot z\right) e^{-t\mathfrak{a}(p^{\gamma}||z||_p)} d^n z \\
&= ||x||_p^{-n} \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(||x||_p^{-1}p^j)} \int_{||p^j z||_p=1} \chi_p(-x_0\cdot z) d^n z \\
&= ||x||_p^{-n} \sum_{-\infty<j<\infty} e^{-t\mathfrak{a}(||x||_p^{-1}p^j)} p^{nj} \int_{||z||_p=1} \chi_p(-x_0 p^{-j}\cdot z) d^n z.
\end{aligned}
$$

By using the formula

$$\int_{||z||_p=1} \chi_p\left(-x_0 p^{-j} \cdot z\right) d^n z = \begin{cases} 1-p^{-n}, & \text{if } j \leq 0, \\ -p^{-n}, & \text{if } j = 1, \\ 0, & \text{if } j \geq 2, \end{cases}$$

we get

$$\begin{aligned} Z(x,t) &= ||x||_p^{-n} \left\{ (1-p^{-n}) \sum_{j=0}^{\infty} p^{-nj} e^{-t\mathfrak{a}(||x||_p^{-1} p^{-j})} - e^{-t\mathfrak{a}(||x||_p^{-1} p)} \right\} \\ &\leq ||x||_p^{-n} \left\{ 1 - e^{-t\mathfrak{a}(||x||_p^{-1} p)} \right\}. \end{aligned}$$

By applying the Mean value theorem to the real function $g(v) = e^{-v\mathfrak{a}(||x||_p^{-1} p)}$ on $[0,t]$, $t > 0$, we have that

$$1 - e^{-t\mathfrak{a}(||x||_p^{-1} p)} = t e^{-\tau\mathfrak{a}(||x||_p^{-1} p)},$$

for some $\tau \in (0,t)$. So that,

$$Z(x,t) \leq t||x||_p^{-n}. \qquad \square$$

Theorem 3.1. *If $\mathfrak{a}$ is a semi-smooth symbol, then $\mathfrak{a}$ is a negative definite function.*

Proof. Due to Lemma 3.1 the proof of this theorem is completely similar to the proof given in [14, Theorem 3]. $\square$

The converse of the previous theorem generally does not hold. For example, the non-negative constant functions are negative definite functions but they are not semi-smooth symbol.

Definition 3.3. A function $f : \mathbb{Q}_p^n \to \mathbb{R}_+$ is called an elliptic symbol, if it satisfies the following properties:

(i) f is a continuous and radial function on $\mathbb{Q}_p^n$;
(ii) $f(||x||_p) = 0 \Leftrightarrow x = 0$;
(iii) f is a increasing functions with respect to $||\cdot||_p$ and there exist positive constants $C_0 := C_0(f)$, $C_1 := C_1(f)$ and $d := d(f)$ such that

$$C_0||x||_p^d \leq f(||x||_p) \leq C_1||x||_p^d,$$

for every $x \in \mathbb{Q}_p^n$.

Example 3.2. (i) For any $d > 0$ and $\beta > 0$, the function $f(x) = \beta||x||_p^d$, $x \in \mathbb{Q}_p^n$, is an elliptic symbol.

(ii) Let $h(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$ with $h(0) = 0$ be a non constant homogeneous polynomial of degree d with coefficients in $\mathbb{Z}_p^{\times}$ such that $h(x)$ is strongly elliptic modulo p, see [30, Definition 3]. Defining $f(x) = |h(x)|_p$ with $x \in \mathbb{Q}_p^n$, by [30, Lemma 15] we have that f is a elliptic symbol.

(iii) [27] For any $n \in \mathbb{N}$ and $p \neq 2$, there exists an elliptic polynomial $h(\xi_1, \ldots, \xi_n)$ with coefficients in $\mathbb{Z}_p^\times$ and degree $2d(n) := 2d$ such that

$$|h(\xi_1, \ldots, \xi_n)|_p = ||(\xi_1, \ldots, \xi_n)||_p^{2d}.$$

Therefore, proceeding analogously to the previous case, we can obtain infinitely many elliptic symbols.

Since every elliptic symbol is a semi-smooth symbol, then as a direct consequence of Theorem 3.1 we obtain the following result.

Corollary 3.1. *If f is an elliptic symbol, then f is a negative definite function.*

Next we will show that the heat Kernel Z_t associated with the non-archimedean pseudo-differential operator $\mathcal{A}$ determine a transition function of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$.
Let $\mathcal{B}(\mathbb{Q}_p^n)$ denote the σ-algebra of the Borel sets of $(\mathbb{Q}_p^n)$. For the basic results on positive bounded measure and Markov processes the reader may consult, respectively, [10] and [18].

Definition 3.4. A function $p_t(x, E)$, defined for all $t \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$, is called a Markov transition function on $\mathbb{Q}_p^n$ if it satisfies the following four conditions:

(i) $p_t(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$ and $p_t(x, \mathbb{Q}_p^n) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{Q}_p^n$;
(ii) $p_t(\cdot, E)$ is a Borel measurable function for all $t \geq 0$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$;
(iii) $p_0(x, \{x\}) = 1$ for all $x \in \mathbb{Q}_p^n$;
(iv) **(The Chapman-Kolmogorov equation)** for all $t, s \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we have the equations

$$p_{t+s}(x, E) = \int_{\mathbb{Q}_p^n} p_t(x, d^n y) p_s(y, E).$$

Definition 3.5. For $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we define

$$p_t(x, E) = \begin{cases} Z_t(x) * 1_E(x), & \text{for } t > 0, x \in \mathbb{Q}_p^n, \\ 1_E(x), & \text{for } t = 0, x \in \mathbb{Q}_p^n, \end{cases}$$

where $Z_t(x)$ is the fundamental solution defined in (3.2).

Theorem 3.2. *$p_t(x, \cdot)$ is a transition function of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$ whose paths are right continuous and have no discontinuities other than jumps.*

Proof. The result follows from Lemma 3.1 by using the argument given in the proof of [14, Theorem 2]. □

4. Non-Archimedean Pseudo-Differential Operators with Negative Definite Logarithmic Symbols

In this section we introduce a large class of non-archimedean pseudo-differential operators whose symbols are new classes of negative definite functions (in the p-adic context) associated with logarithmic functions. Moreover, we introduce a new class of non-archimedean ultradiffusion equations.

Definition 4.1. (i) A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called positive definite, if

$$\sum\nolimits_{i,j=1}^{m} \varphi(x_i - x_j)\lambda_i\overline{\lambda_j} \geq 0,$$

for all $m \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_m \in \mathbb{Q}_p^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

(ii) A C^∞-function $f : (0, \infty) \to \mathbb{R}$ is said to be a Bernstein function, if

$$f \geq 0 \quad \text{and} \quad (-1)^m D^m f \leq 0, \quad \text{for all integers } m \geq 1.$$

The set of positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{P}(\mathbb{Q}_p^n)$ and the subset of $\mathcal{P}(\mathbb{Q}_p^n)$ consisting of the continuous positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{CP}(\mathbb{Q}_p^n)$. For a more detailed discussion of positive definite functions and its properties the reader may consult [10].

Remark 4.1. The following assertions hold:

(i) $\mathcal{P}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$;

(ii) if φ_1, $\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$, then $\varphi_1\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$; the non-negative constant functions belong to $\mathcal{P}(\mathbb{Q}_p^n)$;

(iii) $\mathcal{CP}(\mathbb{Q}_p^n)$ is a convex cone which is a closed subset of the set of continuous complex-valued functions in the topology of compact convergence.

Theorem 4.1. *Let $\psi : \mathbb{Q}_p^n \to [1, \infty)$ be a continuous negative definite function. Then, the function $\ln(\psi) : \mathbb{Q}_p^n \to \mathbb{R}_+$ is negative definite.*

Proof. First, for fixed $t \in [0, 1]$ we consider the function $f_t : (0, \infty) \to \mathbb{R}_+$ given by $f_t(x) = x^t$. Clearly, $f \geq 0$ and by a direct calculation one verifies that

$$(-1)^m D^m(f) = (-1)^m \prod_{i=0}^{m-1}(t - i)x^{t-m} \leq 0,$$

i.e., f_t is a Bernstein function. Then, for fixed $t \in [0, 1]$ and by [10, 9.20, page 69] we have that

$$(f_t \circ \psi)(\xi) = \psi^t(\xi) : \mathbb{Q}_p^n \to [1, \infty) \tag{4.1}$$

is a continuous negative definite function. Moreover, by [10, Corollary 7.9] we have that $\frac{1}{\psi^t}$ is a positive definite function for fixed $t \in [0, 1]$.

On the other hand, for fixed $t_1, t_2 \in [0, 1]$ we have that the product $\frac{1}{\psi^{t_1}} \cdot \frac{1}{\psi^{t_2}}$ is a continuous positive definite function on $\mathbb{Q}_p^n$, see [10, Proposition 3.6]. Therefore,

$\frac{1}{\psi^t} = e^{-t\ln(\psi)}$ is a continuous positive definite function on $\mathbb{Q}_p^n$ for all $t > 0$, and by [10, Theorem 7.8] we have that the function $\ln(\psi)$ is negative definite. $\square$

As an immediate consequence of the theorem above and Remark 3.1, the following corollary is obtained.

Corollary 4.1. *Let $\psi_j : \mathbb{Q}_p^n \to \mathbb{R}_+$, $j = 1, \ldots, m$, be radial, continuous, negative definite functions such that at least one function ψ_j satisfies $\psi_j : \mathbb{Q}_p^n \to [1, \infty)$. Then the function $\ln\left(\sum_{j=1}^m \psi_j(||\xi||_p)\right)$ is negative definite.*

Example 4.1. For every fixed $k > 1$, $\alpha, \beta > 0$, the function $\psi : \mathbb{Q}_p^n \to \mathbb{R}_+$ given by $\psi(\xi) = \ln\left(k + \alpha||\xi||_p^\beta\right)$ is negative definite. By Remark 3.2-(*ii*), we have that in the real context the function $\psi' : \mathbb{R}^n \to \mathbb{R}_+$ given by $\psi'(\xi) = \ln\left(k + ||\xi||_{\mathbb{R}^n}^\beta\right)$ is not a negative definite function for $\beta > 2$.

Corollary 4.2. *Let $\psi : \mathbb{Q}_p^n \to [1, \infty)$ be a continuous negative definite function. Then the function $\ln^\alpha(\psi) : \mathbb{Q}_p^n \to [1, \infty)$, $\alpha > 1$, is negative definite.*

Proof. The result follows from Theorem 4.1, (4.1) and Remark 4.1 (ii). $\square$

Example 4.2. By Remark 3.1 and Remark 3.2 we have that the function $f(x) = 1 + ||x||_p$, $x \in \mathbb{Q}_p^n$, is negative definite and $f(x) \geq 1$ for all $x \in \mathbb{Q}_p^n$. Then, by above corollary we have that $\ln^\alpha(1 + ||x||_p) : \mathbb{Q}_p^n \to [1, \infty)$, $\alpha > 1$, is a negative definite function. Moreover, by [10, Corollary 7.9] we have that $\frac{1}{\ln^\alpha(1+||x||_p)}$, $\alpha > 1$, is a positive definite function.

Consider the operator non-archimedean pseudo-differential operator $\widetilde{\mathcal{A}}$ given by

$$\begin{aligned}\widetilde{\mathcal{A}}(\varphi)(x) &:= \mathcal{F}^{-1}_{\xi \to x}\left\{\ln(\psi(\xi))\widehat{\varphi}(\xi)\right\} \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \ln(\psi(\xi))\widehat{\varphi}(\xi) d^n\xi, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n),\end{aligned}$$

and the Cauchy problem (or p-adic heat equation)

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \widetilde{\mathcal{A}} u(x, t), & t \in [0, \infty),\ x \in \mathbb{Q}_p^n, \\ u(x, 0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n), \end{cases} \tag{4.2}$$

where ψ is a continuous negative definite function satisfies hypothesis of Theorem 4.1. Then the fundamental solution (or heat Kernel) of the Cauchy problem (4.2) is defined as

$$\widetilde{Z}_t(x) = \widetilde{Z}(x, t) := \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi)\, e^{-t\ln(\psi(\xi))} d^n\xi = \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi)\, \frac{1}{\psi^t(\xi)} d^n\xi,$$

for $x \in \mathbb{Q}_p^n$ and $t > 0$.

Lemma 4.1. *The family $\left(\widetilde{Z}_t\right)_{t>0}$ determine a convolution semigroup on $\mathbb{Q}_p^n$, i.e., $\left(\widetilde{Z}_t\right)_{t>0}$ satisfies the following properties:*

(i) *for all* $t>0$*,* $\widetilde{Z}_t$ *is a positive bounded measure on* $\mathbb{Q}_p^n$*;*
(ii) *for all* $t>0$*,* $\widetilde{Z}_t(\mathbb{Q}_p^n)\leq 1$*;*
(iii) *for all* $t,s>0$*, we have that* $\widetilde{Z}_t * \widetilde{Z}_s = \widetilde{Z}_{t+s}$*;*
(iv) $\lim_{t\to 0}\widetilde{Z}_t=\delta$*, where* δ *is the Dirac delta function.*

Proof. Following the proof of Theorem 4.1, we have that $\frac{1}{\psi^t}$, $t>0$, is a continuous positive definite function on $\mathbb{Q}_p^n$. Moreover, by [10, Theorem 3.12] we have that $\widetilde{Z}_t$, $t>0$, is a positive bounded measure on $\mathbb{Q}_p^n$. The desired result follows by application of [10, Theorem 8.3]. ☐

Theorem 4.2. *There exists a Lévy process* $\mathfrak{X}(t,\omega)$ *with state space* $\mathbb{Q}_p^n$ *and transition function* $\widetilde{p}_t(x,\cdot)$ *given by*

$$\widetilde{p}_t(x,E)=\begin{cases}\widetilde{Z}_t(x)*1_E(x), & \text{for } t>0 x\in\mathbb{Q}_p^n,\\ 1_E(x), & \text{for } t=0, x\in\mathbb{Q}_p^n,\end{cases}$$

for $E\in\mathcal{B}(\mathbb{Q}_p^n)$*,*

Proof. Due to Lemma 4.1 the proof of this Theorem is completely similar to the proof given in [37, Theorem 2]. ☐

5. Other Classes of Non-Archimedean Pseudo-Differential Operators Associated with Certain Types of Negative Definite Functions

In this section we will study a new class of non-archimedean operators (denoted by $\mathcal{A}_\psi$) associated with a non-archimedean negative definite function $\boldsymbol{\psi}$. Imposing certain conditions to the function $\boldsymbol{\psi}$ we obtain that $\mathcal{A}_\psi$ is a pseudo-differential operator which also determine ultradiffusion equations.

Along this section $\boldsymbol{\psi}:\mathbb{Q}_p^n\to\mathbb{R}_+\setminus\{0\}$ will denote a radial, continuous and negative definite function such that there exist positive real constants C_2 and β, $\beta>n$, such that $\boldsymbol{\psi}(||x||_p)\geq C_2||x||_p^\beta$ for all $x\in\mathbb{Q}_p^n$.

By Remark 3.1 (iv) note that $\boldsymbol{\psi}(||x||_p)\geq\boldsymbol{\psi}(0)>0$ for all $x\in\mathbb{Q}_p^n$. For examples of this type of functions, the reader can consult [36].

We now note that

$$\int_{\mathbb{Q}_p^n}\frac{d^nx}{\boldsymbol{\psi}(||x||_p)}=\int_{\mathbb{Z}_p^n}\frac{d^nx}{\boldsymbol{\psi}(||x||_p)}+\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}\frac{d^nx}{\boldsymbol{\psi}(||x||_p)}=I_1+I_2.$$

Now, since $\boldsymbol{\psi}$ is a continuous function on $\mathbb{Z}_p^n$ and given the normalization of the norm $||\cdot||_p$, we have that $I_1<\infty$.

On the other hand, note that

$$I_2\leq\frac{1}{C_2}\sum_{j=1}^{\infty}\frac{1}{p^{j\beta}}\int_{||x||_p=p^j}d^nx=\frac{1-p^{-n}}{C_2}\sum_{j=1}^{\infty}p^{j(n-\beta)}<\infty.$$

Therefore, $\frac{1}{\boldsymbol{\psi}} \in L^1(\mathbb{Q}_p^n)$ and consequently there is a positive real constant C such that

$$C \int_{\mathbb{Q}_p^n} \frac{d^n x}{\boldsymbol{\psi}(||x||_p)} = 1. \tag{5.1}$$

We define the operator

$$\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) = C \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{\boldsymbol{\psi}(||y||_p)} d^n y, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

where C is the constant given by (5.1).

Lemma 5.1. *The application*

$$\mathcal{D}\left(\mathbb{Q}_p^n\right) \to \mathcal{D}\left(\mathbb{Q}_p^n\right), \quad \varphi \to \mathcal{A}_{\boldsymbol{\psi}}(\varphi),$$

is a well-defined non-archimedean pseudo-differential operator.

Proof. The condition (5.1) implies that

$$\begin{aligned}
\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) &= \left(\frac{C}{\boldsymbol{\psi}} * \varphi\right)(x) - \varphi(x) \\
&= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\widehat{\varphi}(\xi) d^n\xi - \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi)\, \widehat{\varphi}(\xi) d^n\xi \\
&= -\int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) \left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi) d^n\xi \\
&= -\mathcal{F}^{-1}_{\xi \to x}\left(\left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi)\right),
\end{aligned}$$

i.e., $\mathcal{A}_{\boldsymbol{\psi}}$ is a pseudo-differential operator with symbol $1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}$.

On the other hand, since $\frac{C}{\boldsymbol{\psi}}$ is a radial function, then by [34, Lemma 1] and the n-dimensional version of [38, Example 8, page 43] we have that

$$\left(1 - \widehat{\left(\frac{C}{\boldsymbol{\psi}}\right)}(||\xi||_p)\right) \widehat{\varphi}(\xi) \in \mathcal{D}\left(\mathbb{Q}_p^n\right).$$

Therefore, by [38, VII, Section 2] we have that $\mathcal{A}_{\boldsymbol{\psi}}(\varphi)(x) \in \mathcal{D}\left(\mathbb{Q}_p^n\right)$. $\square$

Due to the condition (5.1) we have that $\frac{C}{\boldsymbol{\psi}}$ codify the structure of the function J given in [9] and [37]. Therefore, by Lemma 5.1 and proceeding analogous to these references, we can prove the following theorem.

Theorem 5.1. *There exists a Lévy process $\mathfrak{X}(t,\omega)$ with state space $\mathbb{Q}_p^n$ and transition function $q_t(x,\cdot)$ given by*

$$q_t(x,E)=\begin{cases} Z_{\psi}(t,x)*1_E(x), & \text{for } t>0, x\in\mathbb{Q}_p^n,\\ 1_E(x), & \text{for } t=0, x\in\mathbb{Q}_p^n,\end{cases}$$

where $Z_{\psi}(t,x)$ is the fundamental solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)=\mathcal{A}_{\psi}u(x,t), \quad t\in[0,\infty), x\in\mathbb{Q}_p^n,\\ u(x,0)=u_0(x)\in\mathcal{D}(\mathbb{Q}_p^n).\end{cases}$$

Remark 5.1. Note that the above Cauchy problem corresponds to the p-adic non local evolution equation

$$\frac{\partial u(x,t)}{\partial t}=\int_{\mathbb{Q}_p^n}\left(\frac{C}{\boldsymbol{\psi}}\right)(x-y)u(y,t)dy-u(x,t)=\left(\left(\frac{C}{\boldsymbol{\psi}}\right)*u-u\right)(x,t).$$

For further details the reader may consult [7] and [6].

References

[1] V. A. Aguilar-Arteaga, M. Cruz-López and S. Estala-Arias, *Non-archimedean analysis and a wave-type pseudodifferential equation on finite adéles.*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 1139–1181. `https://doi.org/10.1007/s11868-020-00343-1`

[2] V. A. Aguilar-Arteaga and S. Estala-Arias, *Pseudodifferential operators and Markov processes on adéles*, p-Adic Numbers Ultrametric Analysis and Applications **11** (2019), 89–113. `https://doi.org/10.1134/S2070046619020018`

[3] S. Albeverio, A. Khrennikov and P. Kloeden, *Memory retrieval as a p-adic dynamical system*, Biosystems **49** (1999), 105–115. `https://doi.org/10.1016/s0303-2647(98)00035-5`

[4] S. Albeverio, A. Y. Khrennikov and V. M. Shelkovich, *Theory of p-Adic Distributions: Linear and Nonlinear Models*, London Mathematical Society Lecture Note Series **370**, Cambridge University Press, Cambridge, 2010. `https://doi.org/10.1017/CBO9781139107167`

[5] A. V. Antoniouk, K. Oleschko, A. N. Kochubei and A. Y. Khrennikov, *A stochastic p-adic model of the capillary flow in porous random medium*, Physica A **505** (2018), 763–777. `https://doi.org/10.1016/j.physa.2018.03.049`

[6] V. A. Avetisov, A. H. Bikulov, S. V. Kozyrev and V. A. Osipov, *p-Adic models of ultrametric diffusion constrained by hierarchical energy landscapes*, J. Phys. A **35** (2002), 177–189. `https://doi.org/10.1088/0305-4470/35/2/301`

[7] V. A. Avetisov, A. K. Bikulov and V. A. Osipov, *p-Adic description of characteristic relaxation in complex systems*, J. Phys. A **36** (2003), 4239–4246. `https://doi.org/10.1088/0305-4470/36/15/301`

[8] O. M. Becker and M. Karplus, *The topology of multidimensional protein energy surfaces: theory and application to peptide structure and kinetics*, J. Chem. Phys. **106** (1997), 1495–1517. `https://doi.org/10.1063/1.473299`

[9] L. F. Chacón-Cortes, I. G. García, A. Torresblanca-Badillo and A. Vargas, *Finite time blow-up for a p-adic nonlocal semilinear ultradiffusion equation*, J. Math. Anal. Appl. **494**(2) (2021), Article ID 124599. `https://doi.org/10.1016/j.jmaa.2020.124599`

[10] B. Christian and F. Gunnar, *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag, New York, Heidelberg, 1975. `https://doi.org/10.1007/978-3-642-66128-0`

[11] H. Frauenfelder, B. H. McMahon and P. W. Fenimore, *Myoglobin: the hydrogen atom of biology and paradigm of complexity*, PNAS **100** (2003), 8615–8617. https://doi.org/10.1073/pnas.1633688100

[12] A. V. Fuensanta, J. M. Mazón, J. D. Rossi and T. M. J. Julián, *Nonlocal Diffusion Problems*, Mathematical Surveys and Monographs **165**, Amer. Math. Soc., Real Sociedad Matemática Española, Providence, Madrid, 2010. http://dx.doi.org/10.1090/surv/165

[13] I. G. García and A. Torresblanca-Badillo, *Some classes of non-archimedean pseudo-differential operators related to Bessel potentials*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 1111–1137. https://doi.org/10.1007/s11868-020-00333-3

[14] I. G. García and A. Torresblanca-Badillo, *Strong Markov processes and negative definite functions associated with non-archimedean elliptic pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 345–362. https://doi.org/10.1007/s11868-019-00293-3

[15] I. G. García and A. Torresblanca-Badillo, *Probability density functions and the dynamics of complex systems associated to some classes of non-archimedean pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article ID 12. https://doi.org/10.1007/s11868-021-00381-3

[16] C. A. Gómez and J. D. Rossi, *A nonlocal diffusion problem that approximates the heat equation with neumann boundary conditions*, Journal of King Saud University **32** (2020), 17–20. https://doi.org/10.1016/j.jksus.2017.08.008

[17] N. Jacob, *Pseudo Differential Operators and Markov Processes. Vol. I. Fourier Analysis and Semigroups*, Imperial College Press, London, 2001. https://doi.org/10.1142/p245

[18] T. Kazuaki, *Boundary Value Problems and Markov Processes*, Second Edition, Lecture Notes in Mathematics, Springer-Verlag, 2009. https://doi.org/10.1007/978-3-642-01677-6

[19] A. Y. Khrennikov, *Human subconscious as a p-adic dynamical system*, Journal of Theoretical Biology **193** (1998), 179–196. https://doi.org/10.1006/jtbi.1997.0604

[20] A. Y. Khrennikov, *Toward an adequate mathematical model of mental space: Conscious/unconscious dynamics on m-adic trees*, Biosystems **90** (2007), 656–675. https://doi.org/10.1016/j.biosystems.2007.02.004

[21] A. Y. Khrennikov and A. N. Kochubei, *p-Adic analogue of the porous medium equation*, J. Fourier Anal. Appl. **24** (2018), 1401–1424. https://doi.org/10.1007/s00041-017-9556-4.

[22] A. Y. Khrennikov, S. V. Kozyrev and W. A. Zúñiga-Galindo, *Ultrametric pseudodifferential equations and applications*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2018. https://doi.org/10.1017/9781316986707

[23] A. Y. Khrennikov and K. Oleschko, *An ultrametric random walk model for disease spread taking into account social clustering of the population*, Entropy **22**(9) (202), Article ID 931. https://doi.org/10.3390/e22090931

[24] A. Y. Khrennikov, K. Oleschko and M. C. López, *Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks.*, Entropy **18**(7) (2016), Article ID 249. https://doi.org/10.3390/e18070249

[25] S. V. Kozyrev, *p-Adic pseudodifferential operators and p-adic wavelets*, Theoret. and Math. Phys. **138** (2004), 322–332. https://doi.org/10.1023/B:TAMP.0000018449.72502.6f

[26] S. V. Kozyrev, *Dynamics on rugged landscapes of energy and ultrametric diffusion*, p-Adic Numbers, Ultrametric Analysis, and Applications **2** (2010), 122–132. https://doi.org/10.1134/S2070046610020044

[27] W. A. Zúñiga-Galindo, *Pseudodifferential Equations over Non-Archimedean Spaces*, Lecture Notes in Mathematics **2174**, Springer International Publishing, 2016. https://doi.org/10.1007/978-3-319-46738-2

[28] K. Oleschko and A. Y. Khrennikov, *Applications of p-adics to geophysics: Linear and quasilinear diffusion of water-in-oil and oil-in-water emulsions*, Theoret. and Math. Phys. **19** (2017), 154–163. https://doi.org/10.4213/tmf9142

[29] E. Pourhadi, A. Y. Khrennikov, R. Saadati, K. Oleschko and M. C. López, *Solvability of the p-adic analogue of navier-stokes equation via the wavelet theory*, Entropy **21**(11) (2019), Article ID 1129. https://doi.org/10.3390/e21111129

[30] J. J. Rodríguez-Vega and W. A. Zúñiga-Galindo, *Taibleson operators, p-adic parabolic equations and ultrametric diffusion*, Pacific J. Math. **237** (2008), 327–347. https://doi.org/10.2140/PJM.2008.237.327

[31] F. H. Stillinger and T. A. Weber, *Hidden structure in liquids*, Phys. Rev. A **25** (1982), 978–989. https://doi.org/10.1103/PhysRevA.25.978

[32] F. H. Stillinger and T. A. Weber, *Packing structures and transitions in liquids and solids*, Science **225** (1984), 983–989. https://doi.org/10.1126/science.225.4666.983

[33] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, New Jersey, 1975. http://www.jstor.org/stable/j.ctt130hkh3

[34] A. Torresblanca-Badillo, *Non-archimedean generalized bessel potentials and their applications*, J. Math. Anal. Appl. https://doi.org/10.1016/j.jmaa.2020.124874

[35] A. Torresblanca-Badillo, *Non-archimedean pseudo-differential operators on Sobolev spaces related to negative definite functions*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article ID 7. https://doi.org/10.1007/s11868-021-00385-z

[36] A. Torresblanca-Badillo and W. A. Zúñiga-Galindo, *Non-archimedean pseudodifferential operators and feller semigroups*, p-Adic Numbers, Ultrametric Analysis and Applications **10** (2018), 57–73. https://doi.org/10.1134/S2070046618010041

[37] A. Torresblanca-Badillo and W. A. Zúñiga-Galindo, *Ultrametric diffusion, exponential landscapes, and the first passage time problem*, Acta Appl. Math. **157** (2018), 93–116. https://doi.org/10.1007/s10440-018-0165-2

[38] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, Series on Soviet and East European Mathematics, World Scientific, 1994. https://doi.org/10.1142/1581

[1]Facultad de Ingeniería,
Universidad Autónoma de Querétaro,
Cerro de las Campanas S/N, C. P. 76010 Querétaro, Qro., México
Email address: victor.aguilara@uaq.mx

[2]Departamento de Matemáticas y Estadística,
Universidad del Norte,
Km. 5 Vía Puerto Colombia, Barranquilla, Colombia
Email address: isgutier@uninorte.edu.co

[3]Universidad del Norte, Departamento de Matemáticas y Estadística,
Km 5 Vía Pto. Colombia,
Barranquilla, Colombia
Email address: atorresblanca@uninorte.edu.co