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ON THE PROXIMAL POINT ALGORITHM OF HYBRID-TYPE IN FLAT HADAMARD SPACES WITH APPLICATIONS

LAWAL YUSUF HARUNA[1], GODWIN CHIDI UGWUNNADI[2,3], AND BASHIR ALI[4]

Abstract. In this paper, we introduce a hybrid-type proximal point algorithm for approximating zero of monotone operator in Hadamard-type spaces. We then prove that a sequence generated by the algorithm involving Mann-type iteration converges strongly to a zero of the said operator in the setting of flat Hadamard spaces. To the best of our knowledge, this result presents the first hybrid-type proximal point algorithm in the space. The result is applied to convex minimization and fixed point problems.

1. Introduction

Let (X, d) be a metric space, an isometry $c : [0, d(x, y)] \to X$ satisfying $c(0) = x$ and $c(d(x, y)) = y$ is called a geodesic path joining x to y for any $x, y \in X$. A geodesic segment between x and y is the image of a geodesic path joining x to y and is denoted by $[x, y]$ when it is unique. A geodesic space is a metric space (X, d) in which every two points of X are joined by a geodesic segment. It is said to be uniquely geodesic space if every two points of X are joined by only one geodesic segment. Let X be a uniquely geodesic space and $(1 - t)x \oplus ty$ denote the unique point z of the geodesic segment joining x to y for each $x, y \in X$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$. Set $[x, y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$, then a subset $C \subset X$ is said to be convex if $[x, y] \subset C$ for all $x, y \in C$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of points (the edges

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of Δ). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. A geodesic space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle Δ is said to satisfy the CAT(0) inequality if $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$ for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. A complete CAT(0) space is called a Hadamard space.

Definition 1.1. Let X be a Hadamard space and $g : X \to (-\infty, \infty)$ be a function with domain $\text{dom}(g) = \{x \in X : g(x) < +\infty\}$. Then g is said to be

(i) proper, if $\text{dom}(g) \neq \emptyset$;
(ii) convex, if $g(\alpha x \oplus (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$ for all $x, y \in X$ and $\alpha \in (0, 1)$;
(iii) lower semicontinuous at a point $x \in \text{dom}(g)$, if for each sequence $\{x_n\}$ in $\text{dom}(g)$ with $x_n \to x$ implies $g(x) \leq \liminf_{n\to\infty} g(x_n)$;
(iv) lower semicontinuous on $\text{dom}(g)$, if it is lower semicontinuouos at every point in $\text{dom}(g)$.

The concept of quasilinearisation was introduced by Berg and Nicolev [4] in a complete CAT(0) space. They denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. A quasilinearisation is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)),$$

for every $a, b, c, d \in X$. From the definition, it is easy to see that for all $a, b, c, d, e \in X$, $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$. The space X is said to satisfy Cauchy Schwartz inequality if for all $a, b, c, d \in X$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$. It is known from [4] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

Definition 1.2 ([16])**.** A Hadamard space is called flat if and only if for all $x, y, z \in X$ and $t \in [0, 1]$

$$d^2((1-t)x \oplus ty, z) = (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y).$$

It is worth mentioning that every Hilbert space is flat Hadamard space but the converse is not always true (see [16, Theorem 3.3]) for details. It is not hard to see that in a flat Hadamard space X, for each $x, y, z, w \in X$ and $t \in [0, 1]$

$$\langle \overrightarrow{xy}, \overrightarrow{x(tz \oplus (1-t)w)} \rangle = t\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle + (1-t)\langle \overrightarrow{xy}, \overrightarrow{xw} \rangle. \tag{1.1}$$

The necessary and sufficient conditions for nonemptyness of the subdifferential set (see Definition 1.3 below) with respect to convexity happened to be the basis for introducing the concept of flat Hadamard spaces. The authors [16] observed that various important results about subdifferentials which hold under topological vector spaces are not valid on Hadamard spaces in general. As such, they establish some basic properties of subdifferentials under the setting of flat Hadamard spaces.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and for $x \in X$ $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n)$, the asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. In a complete CAT(0) space, it is generally known that $A(\{x_n\})$ consists of exactly one point, see [6] for details. A sequence $\{x_n\}$ is said to be Δ-convergent to a point $x \in X$ if for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$. In this case x is called Δ-limit of $\{x_n\}$ and it is written as $\Delta - \lim_{n\to\infty} x_n = x$.

Kakavandi and Amini [9] introduced the concept of dual space in a complete CAT(0) space X as follow. Let $C(X, \mathbb{R})$ be the space of all continuous real-valued functions on X. Consider a map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax}\rangle, \quad t \in \mathbb{R}, a, b, x \in X.$$

The Cauchy-Schwartz inequality implies $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$, $t \in \mathbb{R}$, $a, b \in X$, where the Lipschitz semi-norm $L(\phi)$ of any function $\phi : X \to \mathbb{R}$ is given by $L(\phi) = \sup\left\{\frac{\phi(x)-\phi(y)}{d(x,y)} : x, y \in X, x \neq y\right\}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)),$$

for $t, s \in \mathbb{R}$ and $a, b, c, d \in X$. In a complete CAT(0) space, it is shown [9] that $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$. Thus D induces an equivalence relation on $\mathbb{R} \times X \times X$ with equivalence class defined by

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The pair (X^*, D) is called the dual space of the metric space (X, d), where $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ and the function D on X^* is a metric.

Definition 1.3 ([9])**.** Let X^* be a dual of a Hadamard space X and $g : X \to (-\infty, +\infty]$ be a proper function with effective domain $\mathrm{dom}(f) := \{x \in X : g(x) < +\infty\}$. A subdifferential of g is a multi-valued mapping $\delta g : X \to 2^{X^*}$ defined by

$$\delta g(x) = \{x^* \in X^* : g(y) - g(x) \geq \langle x^*, \overrightarrow{xy}\rangle \text{ for all } y \in X\},$$

for $x \in \mathrm{dom}(g)$ and $\delta g(x) = \emptyset$, otherwise.

Let X^* be a dual of the Hadamard space X and $A : X \to 2^{X^*}$ be a multivalued operator. Let the domain and range of A be respectively denoted by $D(A) := \{x \in X : Ax \neq \emptyset\}$ and $R(A) := \cup_{x\in X} Ax$, $A^{-1}x^* := \{x \in X : x^* \in Ax\}$. The multivalued operator $A : X \to 2^{X^*}$ is said to be monotone if and only if, for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$, $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$. The monotone inclusion problem (MIP) is to find a point

$$x \in D(A) \quad \text{such that} \quad 0 \in Ax, \tag{1.2}$$

where 0 is the zero element of the dual space X^*. We say that A satisfies the range condition if for every $z \in X$ and $\alpha > 0$, there exists an element $x \in X$ such $[\alpha\overrightarrow{xz}] \in Ax$. In a Hilbert space H, it is known that if A is a maximal monotone operator, then $R(I + \lambda A) = H$ for $\lambda > 0$. If A is monotone, then there exists a nonexpansive single-valued mapping $J_\lambda^A : R(I + \lambda A) \to \operatorname{dom}(A)$ defined by $J_\lambda^A = (I + \lambda A)^{-1}$, which is called the resolvent of A. A monotone operator A is said to satisfy the range condition in H if $\overline{\operatorname{dom}}(A) \subset R(I + \lambda A)$ for all $\lambda > 0$, where $Dom(A)$ denotes the closure of the domain of A. We know that in H, a monotone operator A which satisfy the range condition, $A^{-1}0 = F(J_\lambda^A)$ and every maximal monotone operator in H has range condition. Also the subdifferential function δg satisfies the range condition, whenever g is a proper, lower semicontinuous and convex function on a Hadamard space. However, it is not yet known whether every maximal monotone operator in Hadamard spaces satisfy the range condition. This could be seen as one of the significant issue of MIP in Hadamard spaces. But every maximal monotone operator has the range condition in a flat Hadamard space see [11, 18].

The said problem (MIP) is one of the most important problems in nonlinear and convex analysis due to its application in optimization and other related mathematical problems such as variational inequality problems (VIPs), and convex feasibility feasibility problems. Let the solution set of problem (1.2) be denoted by $A^{-1}(0)$. It is known (see [19]) that the set $A^{-1}(0)$ is closed and convex. The proximal point algorithm (PPA) which was introduced by Martinet [15] and further studied by Rockafellar [20] in Hilbert spaces, is a well-known method for approximating solutions of the MIPs. The said algorithm generates a sequence $\{x_n\}$ iteratively by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_{\lambda_n}^A x_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

where $J_{\lambda_n}^A = (I + \lambda_n A)^{-1}$ is the resolvent of the monotone operator A and (λ_n) is a sequence of positive real numbers. It is a fact proved by Rockafellar [20] that the sequence generated by the PPA converges weakly to a zero of the monotone operator A provided $\lambda_n \geq \lambda > 0$, for each $n \geq 1$. To get strong convergence, Solodov and Svaiter [21] modified the proximal point algorithm with resolvent $R_{\lambda_n} A := (I + \lambda_n A)^{-1}$ of A and generate a sequence $\{x_n\}$ iteratively by

$$\begin{cases} x_0 \in H, \\ y_n = R_{\lambda_n} A(x_n), \\ C_n = \{z \in H : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in H : \langle x_0 - x_n, z - x_n \rangle \leq 0\}, \\ x_{n+1} = \mathrm{P}_{C_n \cap Q_n}(x_0), \quad n \geq 0. \end{cases}$$

In 2013, Bacak [3] proved Δ-convergence of the PPA in Hadamard spaces by considering the operator A to be a subdifferential of a convex, proper and lower semicontinuous. Khatibzadeh and Ranjbar [11] studied PPA in Hadamard spaces when the operator A is monotone.

Remark 1.1. The well definedness of the PPA (1.3) (as a well known and most important method for solving the MIP) requires among others, the monotone operator A to satisfy the range condition and it is not known yet, whether every maximal monotone operators satisfy the range condition in Hadamard spaces as in the case of Hilbert spaces and Hadamard manifolds.

Ranjbar and Khatibzadeh [19] proved that the sequence $\{x_n\}$ defined by the following Mann-type PPA Δ-converges to zero of the monotone operator (see [19] for Halpern-type PPA that converges strongly to zero of the monotone operator A).

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J_{\lambda_n}^A x_n, \quad n \geq 0. \end{cases} \tag{1.4}$$

For more recent and related PPA results, the reader may consult [10, 12, 14, 22].

To the best of our knowledge, it appears in the literature that the only PPA that guaranteed the strong convergence in Hadamard spaces are the one generated by Halpern and viscosity-type algorithms (see for example,[2, 5, 7, 17, 23–25]) unlike in the setting of Hilbert spaces where the PPA of hybrid-type is proved to be among. As such, there arises a question: Can we establish a strong convergence of the Mann-type PPA (1.4) by hybrid method in Hadamard spaces?

In this paper, an affirmative answer is given to such question by introducing a hybrid-type PPA involving Mann-type iteration in the setting of flat Hadamard spaces. We also prove that the sequence generated by the said algorithm converges strongly to the zero of the monotone operator in the space.

Remark 1.2. The proposed method guaranteed the strong convergence of the Mann-type PPA rather than the Δ-convergence of the corresponding algorithm as in [19]. Also, the method does not require monotonicity assumption on the sequence as it can be proved. Hence, the two cases approach in proving the strong convergence is not required unlike in the existing methods. The result established generalized the corresponding ones in Hilbert spaces.

2. Preliminaries

Throughout this section, the symbols "$\to$" and "$\rightharpoonup$" represent the strong and Δ-convergence, respectively. The following results will play vital roles in establishing our main result.

Lemma 2.1 ([6])**.** *Let X be a $CAT(0)$ space and $x, y \in X$, $t \in [0,1]$. Then*

(i) $d(z, tx \oplus (1-t)y) \leq td(z,x) + (1-t)d(z,y)$;
(ii) $d^2(z, tx \oplus (1-t)y) \leq td^2(z,x) + (1-t)d^2(z,y) - t(1-t)d^2(x,y)$.

Lemma 2.2. *Let C be a nonempty convex subset of a $CAT(0)$ space X. For $x \in X$ and $u \in C$, then $u = P_C x$ if and only if*

$$\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \geq 0, \quad \textit{for all } y \in C.$$

Lemma 2.3 ([13]). *Let X be a complete CAT(0) space. Then every bounded sequence in X has a Δ-convergence subsequence.*

Lemma 2.4 ([8]). *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{yx}\rangle \leq 0$ for all $y \in X$.*

Lemma 2.5 ([9]). *Let X^* be a dual of the Hadamard space X and $g : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then*

(i) *g attains its minimum at $x \in X$ if and only if $0 \in \delta g(x)$;*
(ii) *$\delta g : X \to 2^{X^*}$ is a monotone operator;*
(iii) *for any $x \in X$ and $\alpha > 0$ there exists a unique point $y \in X$ such that $[\alpha\overrightarrow{xy}] \in \delta g(y)$, that is $\operatorname{dom}(J_\lambda^{\delta g}) = X$ for all $\lambda > 0$.*

Lemma 2.6 ([11]). *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function on a Hadamard space X with dual X^*. Then*

$$J_\lambda^{\delta g}(x) = \operatorname*{argmin}_{y\in X}\{g(y) + \frac{1}{2\lambda}d^2(y, x)\},$$

for all $\lambda > 0$ and $x \in X$.

Lemma 2.7 ([11]). *Let X be a CAT(0) space and J_λ^A be the resolvent of the monotone operator A with order λ. Then*

(i) *for any $\lambda > 0$, $R(J_\lambda^A) \subset \operatorname{dom}(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A,*
(ii) *if A is monotone, then J_λ^A is single-valued and firmly nonexpansive and hence nonexpansive,*
(iii) *if A is monotone and $\mu \geq \lambda > 0$, then $d(x, J_\lambda^A x) \leq d(x, J_\mu^A x)$.*

3. Main Results

In this section, C is considered to be nonempty closed convex subset of a flat Hadamard space X. We introduce a hybrid-type proximal point algorithm involving Mann-type iteration for approximating zero of monotone operator in flat Hadamard spaces.

Theorem 3.1. *Let X be a flat Hadamard space with its dual X^*. Let A be a multi-valued monotone operator of X into 2^{X^*} satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n)J_{\lambda_n}^A v_n, \\ D_n = \{v \in C : d(v, u_n) \leq d(v, v_n)\}, \\ E_n = \{v \in C : \langle \overrightarrow{v_0 v_n}, \overrightarrow{v v_n}\rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\lambda_n \in (0,\infty)$ with $\lambda_n \geq \lambda > 0$ and $\{\alpha_n\} \subset [0,1]$. Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}$ is the metric projection from X onto $A^{-1}(0)$.

Proof. We divide the proof into the following steps.

Step 1. We show that the set $D_n \cap E_n$ is closed and convex. From D_n and definition of quasilinearization, we see $d(v, u_n) \le d(v, v_n)$ if and only if $-d^2(v_n, u_n) + \langle \overrightarrow{vu_n}, \overrightarrow{v_nu_n} \rangle \le 0$. Thus, it is an evident from [1] that $D_n \cap E_n$ is closed and convex. For completeness sake, we give the proof here. Let $y_m \in D_n$ such that $\lim_{m\to\infty} y_m = y$ then we show that $y \in D_n$. But

$$\begin{aligned}-d^2(v_n, u_n) + \langle \overrightarrow{yu_n}, \overrightarrow{v_nu_n} \rangle &= -d^2(v_n, u_n) + \langle \overrightarrow{\lim_{m\to\infty} y_mu_n}, \overrightarrow{v_nu_n} \rangle \\ &= -d^2(v_n, u_n) + \lim_{m\to\infty} \langle \overrightarrow{y_mu_n}, \overrightarrow{v_nu_n} \rangle \\ &= \lim_{m\to\infty} (-d^2(v_n, u_n) + \langle \overrightarrow{y_mu_n}, \overrightarrow{v_nu_n} \rangle) \le 0.\end{aligned}$$

Thus, D_n is closed. For convexity, let $y_1, y_2 \in D_n$ then we show that $y = ry_1 \oplus (1 - r)y_2 \in D_n$ for $r \in [0, 1]$. Using equation (1.1), we get

$$\begin{aligned}-d^2(v_n, u_n) + \langle \overrightarrow{yu_n}, \overrightarrow{v_nu_n} \rangle =& - d^2(v_n, u_n) + \langle \overrightarrow{(ry_1 \oplus (1 - r)y_2)u_n}, \overrightarrow{v_nu_n} \rangle \\ =& - d^2(v_n, u_n) + r\langle \overrightarrow{y_1u_n}, \overrightarrow{v_nu_n} \rangle + (1 - r)\langle \overrightarrow{y_2u_n}, \overrightarrow{v_nu_n} \rangle \\ \le& 0.\end{aligned}$$

Thus, D_n is convex. Therefore, D_n is closed and convex. Similarly, for the set E_n, we take $y_m \in E_n$ with $\lim_{m\to\infty} y_m = y$ and by continuity of quasilinearization, we get

$$\langle \overrightarrow{v_0v_n}, \overrightarrow{yv_n} \rangle = \langle \overrightarrow{v_0v_n}, \overrightarrow{\lim_{m\to\infty} y_mv_n} \rangle = \lim_{m\to\infty} \langle \overrightarrow{v_0v_n}, \overrightarrow{y_mv_n} \rangle \le 0.$$

Thus, E_n is closed. Also, for $y = ry_1 \oplus (1 - r)y_2$ where $y_1, y_2 \in E_n$, we see that

$$\begin{aligned}\langle \overrightarrow{v_0v_n}, \overrightarrow{yv_n} \rangle &= \langle \overrightarrow{v_0v_n}, \overrightarrow{(ry_1 \oplus (1 - r)y_2)v_n} \rangle \\ &= r\langle \overrightarrow{v_0v_n}, \overrightarrow{y_1v_n} \rangle + (1 - r)\langle \overrightarrow{v_0v_n}, \overrightarrow{y_2v_n} \rangle \\ &\le 0.\end{aligned}$$

Thus, E_n is convex. Therefore, E_n is closed and convex. Hence $D_n \cap E_n$ is closed and convex.

Step 2. We show that the sequence $\{v_n\}$ is well-defined. The well-definedness of $\mathrm{P}_{A^{-1}(0)}$ follows from the fact that $A^{-1}(0)$ is closed and convex. Now let $w_n = J^A_{\lambda_n} v_n$ and $A^{-1}(0) \neq \emptyset$. Then, we can take $u = \mathrm{P}_{A^{-1}(0)} \subset A^{-1}(0)$ so that $J^A_{\lambda_n} u = u$. It follows from (3.1) and nonexpansivity of $J^A_{\lambda_n}$ that

$$d(u, w_n) = d(J^A_{\lambda_n} u, J^A_{\lambda_n} v_n) \le d(u, v_n). \tag{3.2}$$

Also, using (3.1) and (3.2) we get

$$\begin{aligned}d(u, u_n) &= d(u, \alpha_nv_n \oplus (1 - \alpha_n)w_n) \\ &\le \alpha_nd(u, v_n) + (1 - \alpha_n)d(u, w_n) \le \alpha_nd(u, v_n) + (1 - \alpha_n)d(u, v_n) \\ &= d(u, v_n).\end{aligned}$$

Thus, $u \in D_n$ and therefore $A^{-1}(0) \subset D_n$. Next we show that $A^{-1}(0) \subset D_n \cap E_n$, for all $n \in \mathbb{N}$. We do this by induction. Now let $n = 1$, we see that $\mathcal{F} \subset D_1 = E_1 = C$ and so $A^{-1}(0) \subset D_1 \cap E_1$. Suppose that $A^{-1}(0) \subset D_k \cap E_k$ for some $k > 1$. Since $v_{k+1} = P_{D_k \cap E_k}(v_0)$, then using Lemma 2.2, we get

$$\langle \overrightarrow{v_0 v_{k+1}}, \overrightarrow{p v_{k+1}} \rangle \leq 0,$$

for all $p \in D_k \cap E_k$. Also, since $A^{-1}(0) \subset D_k \cap E_k$, we have

$$\langle \overrightarrow{v_0 v_{k+1}}, \overrightarrow{u v_{k+1}} \rangle \leq 0,$$

for all $u \in A^{-1}(0)$. This implies $A^{-1}(0) \subset D_{k+1} \cap E_{k+1}$. Therefore, $A^{-1}(0) \subset D_n \cap E_n$ for all $n \in \mathbb{N}$. Hence, the sequence $\{v_n\}$ is well defined.

Step 3. The $\lim_{n\to\infty} d(v_n, v_0)$ exists. First we show that the sequence $\{v_n\}$ is bounded. Using the property of metric projection and the fact that $v_n = P_{E_n}(v_0)$, we get

$$d(v_n, v_0) = d(P_{E_n}(v_0), v_0) \leq d(u, v_0) - d(u, P_{E_n}(v_0)) = d(u, v_0).$$

This implies that the sequence $\{d(v_n, v_0)\}$ is bounded. Thus, the sequence $\{v_n\}$ is bounded too. Since $v_n = P_{E_n}(v_0)$ and $v_{n+1} \in E_n$, then using Lemma 2.2 and quasilinearization definition, we have

$$\begin{aligned} 0 &\leq \langle \overrightarrow{v_0 v_n}, \overrightarrow{v_n v_{n+1}} \rangle \\ &= d^2(v_0, v_{n+1}) + d^2(v_n, v_n) - d^2(v_0, v_n) - d^2(v_n, v_{n+1}) \\ &\leq d^2(v_0, v_{n+1}) - d^2(v_0, v_n). \end{aligned} \tag{3.3}$$

This implies $d(v_0, v_n) \leq d(v_0, v_{n+1})$. Thus, the sequence $\{d(v_0, v_n)\}$ is monotone increasing. Since it is bounded, then $\lim_{n\to\infty} d(v_n, v_0)$ exists.

Step 4. We show that $\lim_{n\to\infty} d(v_n, J_\lambda^A v_n) = 0$. From equation (3.3), we see that

$$d^2(v_n, v_{n+1}) \leq d^2(v_0, v_{n+1}) - d^2(v_0, v_n). \tag{3.4}$$

Using the fact that $\lim_{n\to\infty} d(v_n, v_0)$ exists, it follows from (3.4) that

$$\lim_{n\to\infty} d(v_n, v_{n+1}) = 0. \tag{3.5}$$

Since $v_{n+1} \in D_n$, then $d(v_{n+1}, u_n) \leq d(v_{n+1}, v_n)$. Thus, it follows from (3.5) that

$$\lim_{n\to\infty} d(v_{n+1}, u_n) = 0. \tag{3.6}$$

With the use of (3.5), (3.6) and the property of metric distance, we get

$$\lim_{n\to\infty} d(v_n, u_n) = 0. \tag{3.7}$$

On the other hand,

$$\begin{aligned} d^2(u, u_n) =& d^2(u, \alpha_n v_n \oplus (1-\alpha_n) w_n) \\ =& \alpha_n d^2(u, v_n) + (1-\alpha_n) d^2(u, w_n) - \alpha_n(1-\alpha_n) d^2(u_n, w_n) \\ \leq& d^2(u, v_n) - \alpha_n(1-\alpha_n) d^2(v_n, w_n). \end{aligned}$$

Thus, using quasilinearization definition, Cauchy-Schwartz inequality and (3.7) we get

$$\begin{aligned}\alpha_n(1-\alpha_n)d^2(v_n,w_n) &\leq d^2(u,v_n)-d^2(u,u_n)\\ &= d^2(v_n,v_n)-d^2(v_n,u_n)+2\langle\overrightarrow{uv_n},\overrightarrow{u_nv_n}\rangle\\ &\leq 2\langle\overrightarrow{uv_n},\overrightarrow{u_nv_n}\rangle \leq 2d(u,v_n)d(u_n,v_n)\\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Using the fact that $\alpha_n \in (0,1)$, we get

$$\lim_{n\to\infty} d(v_n, J^A_{\lambda_n}v_n) = \lim_{n\to\infty} d(v_n,w_n) = 0.$$

Since $\lambda_n \geq \lambda$, then by Lemma 2.7 (iii) we get

$$\lim_{n\to\infty} d(v_n, J^A_{\lambda}v_n) \leq 2\lim_{n\to\infty} d(v_n, J^A_{\lambda_n}v_n) = 0.$$

Since the sequence $\{v_n\}$ is bounded and the space X is Hadamard, then from Lemma 2.3 there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\Delta - \lim_{k\to\infty} v_{n_k} = w$. Since J_λ is nonexpansive then by demiclosedness of J_λ, we get $w \in F(J_\lambda) = A^{-1}(0)$.

Since $v_{n+1} = P_{D_n\cap E_n}(v_0)$, then by letting $q = P_{A^{-1}(0)}(v_0) \in D_n \cap E_n$, we get $d(v_{n+1},v_0) \leq d(q,v_0)$. Also, $v_{n_k} \rightharpoonup w$ and $d(\cdot,\cdot)$ is convex and lower semicontinuous hence Δ-lower semicontinuous (see [3]), we get

$$d(w,v_0) \leq \liminf_{k\to\infty} d(v_{n_k},v_0) \leq d(q,v_0).$$

From the definition of q, we can conclude that $w = q$ and so $v_n \rightharpoonup q$. It follows from Lemma 2.4 that the $\limsup_{n\to\infty}\langle\overrightarrow{qs},\overrightarrow{qv_n}\rangle \leq 0$ for all $s \in X$. Thus, it holds for $v_0 \in X$, i.e.,

$$\limsup_{n\to\infty}\langle\overrightarrow{qv_0},\overrightarrow{qv_n}\rangle \leq 0. \tag{3.8}$$

We now show that $v_n \to q$. Using quasilinearization definition, we see that

$$\begin{aligned}d^2(v_n,q) &= (d^2(q,v_0)+d^2(v_n,v_0)-2\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle)\\ &\leq (d^2(q,v_0)+d^2(q,v_0)-2\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle)\\ &= 2(d^2(q,v_0)-\langle\overrightarrow{qv_0},\overrightarrow{v_nv_0}\rangle) = 2(\langle\overrightarrow{qv_0},\overrightarrow{qv_0}\rangle+\langle\overrightarrow{qv_0},\overrightarrow{v_0v_n}\rangle)\\ &= 2\langle\overrightarrow{qv_0},\overrightarrow{qv_n}\rangle. \qquad (3.9)\end{aligned}$$

Taking lim sup of the inequality (3.9) as $n \to \infty$ together with the use of (3.8), we see that the $\limsup_{n\to\infty} d^2(v_n,q) = 0$. Thus, $\lim_{n\to\infty} d^2(v_n,q) = 0$ and hence the sequence $v_n \to q$. This completes the prove. □

In view of the fact that every closed convex subset of Hilbert spaces is flat Hadamard space, the following results can be obtained from Theorem 3.1 as corollaries.

Corollary 3.1. *Let X be a Hilbert space and A be a multi-valued monotone operator of X into 2^X satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n + (1-\alpha_n) J_{\lambda_n}^A v_n, \\ D_n = \{v \in C : \|v - u_n\| \leq \|v - v_n\|\}, \\ E_n = \{v \in C : \langle v_0 - v_n, v - v_n \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0, \infty)$ with $\lambda_n \geq \lambda > 0$ and $\{\alpha_n\} \subset [0,1]$. Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}$ is the metric projection from X onto $A^{-1}(0)$.

Proof. Since every closed convex subset of a Hilbert space is flat Hadamard space then by Theorem 3.1, we get the desired result. This completes the proof. ☐

Corollary 3.2 ([21])**.** *Let X be a Hilbert space and A be a multi-valued monotone operator of X into 2^X satisfying the range condition such that $A^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be iteratively defined by,*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = J_{\lambda_n}^A v_n, \\ D_n = \{v \in C : \|v - u_n\| \leq \|v - v_n\|\}, \\ E_n = \{v \in C : \langle v_0 - v_n, v - v_n \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0. \end{cases}$$

Then $\{v_n\}$ converges strongly to $u = P_{A^{-1}(0)}(v_0)$, where $P_{A^{-1}(0)}(v_0)$ is the metric projection from X onto $A^{-1}(0)$.

4. Application to Convex Minimization Problem

In this section, we consider an application to convex minimization problem. Recall that the minimization problem is a problem of finding a point

$$u \in X \quad \text{such that} \quad g(u) = \min_{v \in X} f(v).$$

In view of Theorem 2.5 (i) this problem can be formulated as follow: find $u \in X$ such that $0 \in \delta g(u)$. Thus, by setting $A = \delta g$ in Theorem 3.1 together with the use of Lemma 2.6, the following result can easily be obtained.

Theorem 4.1. *Let X be a flat Hadamard space with its dual X^* and $g : X \to (-\infty, +\infty]$ be proper, lower semicontinuous function such that $(\delta g)^{-1}(0) \neq \emptyset$. Let the sequence $\{v_n\} \subset C$ be defined by*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n) J_{\lambda_n}^{\delta g} v_n, \\ D_n = \{v \in C : d(v, u_n) \leq d(v, v_n)\}, \\ E_n = \{v \in C : \langle \overrightarrow{v_0 v_n}, \overrightarrow{v v_n} \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0,\infty)$ *with* $\lambda_n \geq \lambda > 0$ *and* $\{\alpha_n\} \subset [0,1]$. *Then* $\{v_n\}$ *converges strongly to* $u = P_{(\delta g)^{-1}(0)}(v_0)$, *where* $P_{(\delta g)^{-1}(0)}$ *is the metric projection from* X *onto* $(\delta g)^{-1}(0)$.

5. Application to Fixed Point Problem

In this section, we consider application to fixed point of nonexpansive mapping. Recall that a mapping T of a metric space X into itself is called nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$. If X is a Hilbert space, the operator $I-T$ is known to be maximal monotone and hence satisfies the range condition, where I is the identity mapping. For the operator $I-T$, the maximal monotonicity and the range condition are considered in Hadamard spaces by Khatibzadeh and Ranjbar [11] as can be seen from the following results.

Proposition 5.1 ([11]). *Let* X *be a Hadamard space and* $T : X \to X$ *be an arbitrary nonexpansive mapping. If the monotone operator* $Az = [\overrightarrow{Tzz}]$ *is maximal, then* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition.*

Proposition 5.2 ([11]). *Let* X *be a Hadamard space. For every nonexpansive mapping* $T : X \to X$, *the operator* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition if and only if for all* $x,y,z \in X$

$$d^2(\alpha x \oplus (1-\alpha)y, z) = \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y).$$

This is equivalent to saying that, for every nonexpansive mapping $T : X \to X$, *the operator* $Az = [\overrightarrow{Tzz}]$ *satisfies the range condition if and only if the Hadamard space* X *is flat.*

Thus, $F(T) = A^{-1}(0)$ (see Ranjbar and Khatibzadeh [19]), where $F(T) := \{x \in X : Tx = x\}$ and $Az = [\overrightarrow{Tzz}]$. Hence the following result follows from Theorem 3.1.

Theorem 5.1. *Let* X *be a flat Hadamard space with its dual* X^* *and* $T : X \to X$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$ *with the operator* $Az = [\overrightarrow{Tzz}]$. *Let the sequence* $\{v_n\} \subset C$ *be iteratively defined by*

$$\begin{cases} v_0 \in C = D_1 = E_1, \\ u_n = \alpha_n v_n \oplus (1-\alpha_n) J^A_{\lambda_n} v_n, \\ D_n = \{v \in C : d(v,u_n) \leq d(v,v_n)\}, \\ E_n = \{v \in C : \langle \overrightarrow{v_0 v_n}, \overrightarrow{v v_n} \rangle \leq 0\}, \\ v_{n+1} = P_{D_n \cap E_n}(v_0), \quad n \geq 0, \end{cases}$$

where $\lambda_n \in (0,\infty)$ *with* $\lambda_n \geq \lambda > 0$ *and* $\{\alpha_n\} \subset [0,1]$. *Then* $\{v_n\}$ *converges strongly to* $u = P_{T^{-1}(0)}(v_0)$, *where* $P_{T^{-1}(0)}$ *is the metric projection from* X *onto* $T^{-1}(0)$.

Proof. Since X is flat Hadamard space then by proposition 5.2, the operator $Az = [\overrightarrow{Tzz}]$ satisfies the range condition and $F(T) = A^{-1}(0)$. Thus, by Theorem 3.1, we get the desired result. □

6. Conclusion

In this article, a new Mann hybrid-type proximal point algorithm for solving monotone inclusion problem is presented in Hadamard-type spaces. It is shown that our algorithm converges strongly to a zero solution of the said operator in the setting of flat Hadamard spaces. To the best of our knowledge, this result presents the first hybrid-type proximal point algorithm in Hadamard-type spaces.

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[1]Department of Mathematical Sciences,
Kaduna State University,
Tafawa Balewa Way, Kaduna-Nigeria
Email address: yulah121@gmail.com

[2]Department of Mathematics,
University of Eswatini,
Kwaluseni, Eswatini

[3]Department of Mathematics and Applied Mathematics,
Sefako Makgato Health Science University,
P.O. Box 94, Pretoria 0204, South Africa
Email address: gc.ugwunnadi@mouau.edu.ng
Email address: ugwunnadi4u@yahoo.com

[4]Department of Mathematical Sciences,
Bayero University, Kano,
Kano-Nigeria
Email address: bashiralik@yahoo.com