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PSEUDO GE-ALGEBRAS AS THE EXTENSION OF GE-ALGEBRAS

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ABSTRACT. In this paper, the notion of a pseudo GE-algebra as an extension of a GE-algebra is introduced. Basic properties of pseudo GE-algebras are described. The concepts of strong pseudo BE-algebra, good pseudo BE-algebra, good pseudo GE-algebra, and the relationship between them are established. We provide a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

1. INTRODUCTION

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1]). In 1966, Y. Imai and K. Iseki [10, 12] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Pseudo-valuations were introduced and studied by Y. B. Jun [13]. Georgescu and Iorgulescu [9] introduced an extension of BCK-algebra called pseudo BCK-algebra. Di Nola et al. presented pseudo BL-algebras, which are non-commutative BL-algebras [5, 6]. Moreover, they gave the connection of pseudo BCK-algebra with pseudo MV-algebra and with pseudo BL-algebra. Pseudo BCI-algebras were introduced and studied by W. A. Dudek and Y.

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B Jun (see [7]), as generalizations of pseudo BCK-algebras and BCI-algebras, and they form an important tool for an algebraic axiomatization of implicational fragment of non-classical logic (see [8]). A. Walendziak [16] gave a system of axioms defining pseudo BCK-algebras. Pseudo BCK-algebras were intensively studied in [3, 11, 14]. R. A. Borzooei et al. [2] applied pseudo structure to BE-algebras and investigated its properties. They studied the concepts of pseudo-subalgebra, pseudo-filter and pseudo-upper-set and proved that every pseudo-filter is a union of pseudo-upper-sets. Later on, in 2019, Rezaei et al. defined pseudo CI-algebras, which are a generalization of the pseudo BE-algebras, pseudo BCK-algebras and pseudo MV-algebras [15].

In this paper, we introduce the notion of pseudo GE-algebra as a non-commutative generalization of GE-algebra and study its properties. We define the notion of $(\circledast, \boxplus)$-pseudo GE-algebra, $(\boxplus, \circledast)$-pseudo GE-algebra and investigate its properties. We define the concept of strong pseudo BE-algebra, good pseudo GE-algebra and study relation between them. Finally, we give a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

2. Preliminaries

Definition 2.1 ([1])**.** By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $u * u = 1$;

(GE2) $1 * u = u$;

(GE3) $u * (v * w) = u * (v * (u * w))$,

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \Leftrightarrow x * y = 1)\,.$$

Definition 2.2 ([1])**.** A GE-algebra X is said to be *transitive* if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.1}$$

Proposition 2.1 ([1])**.** *Every GE-algebra X satisfies the following items*

$$\begin{aligned}
&(\forall u \in X)\,(u * 1 = 1)\,,\\
&(\forall u, v \in X)\,(u * (u * v) = u * v)\,,\\
&(\forall u, v \in X)\,(u \leq v * u)\,,\\
&(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,,\\
&(\forall u \in X)\,(1 \leq u \Rightarrow u = 1)\,,\\
&(\forall u, v \in X)\,(u \leq (v * u) * u)\,,\\
&(\forall u, v \in X)\,(u \leq (u * v) * v)\,,\\
&(\forall u, v, w \in X)\,(u \leq v * w \Leftrightarrow v \leq u * w)\,.
\end{aligned}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w)\,,$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,.$$

Lemma 2.1 ([1]). *In a GE-algebra X, the following facts are equivalent to each other*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,,$$
$$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,.$$

3. Pseudo GE-Algebras

We consider the notion of a pseudo GE-algebra as a generalization of a GE-algebra. Let X be a set with two binary operations "$\circledast$" and "$\boxplus$". Then we can consider the following two cases:

$$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = x \boxplus (y \circledast (x \boxplus z)) \text{ and } x \boxplus (y \circledast z) = x \circledast (y \boxplus (x \circledast z))), \tag{3.1}$$

$$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \circledast z)) \text{ and } x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \boxplus z))). \tag{3.2}$$

Hence we can think of two types of pseudo GE-algebra so called type A and type B.

Definition 3.1. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo GE-algebra of type* A if it satisfies (3.1) and the following conditions:

$$(\forall x \in X)\,(x \circledast x = 1 \text{ and } x \boxplus x = 1)\,, \tag{3.3}$$

$$(\forall x \in X)\,(1 \circledast x = x \text{ and } 1 \boxplus x = x)\,. \tag{3.4}$$

Definition 3.2. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo GE-algebra of type* B if it satisfies (3.2), (3.3) and (3.4).

The pseudo GE-algebra $(X, \circledast, \boxplus, 1)$ of type A or type B is sometimes only shown as X. It is clear that if a pseudo GE-algebra X of type A or type B satisfies $x \circledast y = x \boxplus y$ for all $x, y \in X$, then X is a GE-algebra.

As you can see above, we have defined two types of pseudo GE-algebra. In considering the pseudo theory as a generalization for a given algebraic system, it is not desirable to have multiple types in the development of theory.

The following theorem shows that one of the two types has no meaning.

Theorem 3.1. *If X is a pseudo GE-algebra of type A, then it is just a GE-algebra.*

Proof. Let X be a pseudo GE-algebra of type A and let $x, y \in X$. Then $1 \circledast (x \boxplus y) = 1 \boxplus (x \circledast (1 \boxplus y))$ by (3.1), and so $x \boxplus y = x \circledast y$ by (3.4). Therefore, X is a GE-algebra. $\square$

So in thinking about pseudo theory, which is the generalization of GE-algebra, we can see that Definition 3.2 is the only definition expressed. Based on these discussions, we can call pseudo GE-algebra of type B just pseudo GE-algebra.

Now, we give examples of a pseudo GE-algebra.

Example 3.1. Let $X = \{1, a, b, c, d, e\}$ and define binary operations $\circledast$ and $\boxplus$ as follows:

$\circledast$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	d	d	d
b	1	a	1	1	d	d
c	1	a	1	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

,

$\boxplus$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	1	1	1
b	1	a	1	c	e	e
c	1	a	1	1	1	1
d	1	a	1	1	1	1
e	1	a	1	1	1	1

.

It is routine to verify that $(X, \circledast, \boxplus, 1)$ is a pseudo-GE-algebra.

Definition 3.3. A $(\circledast, \boxplus)$-*pseudo GE-algebra* is a structure $(X, \circledast, \boxplus, 1)$ in which X is set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$" satisfying the conditions (3.3), (3.4) and

$$(\forall x, y, z \in X)\,(x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \circledast z)))\,. \tag{3.5}$$

Example 3.2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	c	1
b	1	a	1	a
c	a	1	a	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra.

Definition 3.4. A $(\boxplus, \circledast)$-*pseudo GE-algebra* is a structure $(X, \circledast, \boxplus, 1)$ in which X is set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$" satisfying the conditions (3.3), (3.4) and

$$(\forall x, y, z \in X)\,(x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \boxplus z)))\,. \tag{3.6}$$

Example 3.3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	1
b	a	a	1	a
c	a	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra.

It is clear that if a structure $(X, \circledast, \boxplus, 1)$ is both a $(\circledast, \boxplus)$-pseudo GE-algebra and a $(\boxplus, \circledast)$-pseudo GE-algebra, then it is a pseudo GE-algebra.

Every $(\circledast, \boxplus)$-pseudo GE-algebra need not be a $(\boxplus, \circledast)$-pseudo GE-algebra. In Example 3.2, X is $(\circledast, \boxplus)$-pseudo GE-algebra. But X is not a $(\boxplus, \circledast)$-pseudo GE-algebra, since

$$a \boxplus (a \circledast b) = a \boxplus b = c \neq a = a \boxplus 1 = a \boxplus (a \circledast c) = a \boxplus (a \circledast (a \boxplus b)).$$

Every $(\boxplus, \circledast)$-pseudo GE-algebra need not be a $(\circledast, \boxplus)$-pseudo GE-algebra. In Example 3.3, X is $(\boxplus, \circledast)$-pseudo GE-algebra. But X is not a $(\circledast, \boxplus)$-pseudo GE-algebra, since

$$a \circledast (a \boxplus b) = a \circledast c = 1 \neq a = a \circledast 1 = a \circledast (a \boxplus a) = a \circledast (a \boxplus (a \circledast b)).$$

In a $(\circledast, \boxplus)$-pseudo GE-algebra or a $(\boxplus, \circledast)$-pseudo GE-algebra $(X, \circledast, \boxplus, 1)$, we define two binary operations "$\ll_\circledast$" and "$\ll_\boxplus$" as follows:

$$(\forall x, y \in X)(x \ll_\circledast y \Leftrightarrow x \circledast y = 1),$$
$$(\forall x, y \in X)(x \ll_\boxplus y \Leftrightarrow x \boxplus y = 1),$$

respectively. For every elements x and y of a pseudo GE-algebra X, if $x \ll_\circledast y$ and $x \ll_\boxplus y$ are formed at the same time, it is represented as $x \ll y$.

Proposition 3.1. *Every $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x \in X)(x \circledast 1 = 1), \tag{3.7}$$
$$(\forall x, y \in X)(x \circledast (x \circledast y) = x \circledast y), \tag{3.8}$$
$$(\forall x, y \in X)(x \ll_\circledast (x \circledast y) \boxplus y). \tag{3.9}$$

Proof. Let $x, y, z \in X$. Then

$$1 = x \circledast x = x \circledast (1 \boxplus x) = x \circledast ((x \circledast x) \boxplus x)) = x \circledast ((x \circledast x) \boxplus (x \circledast x)) = x \circledast 1$$

which proves (3.7). Using (3.4) and (3.5), we have $x \circledast y = x \circledast (1 \boxplus y) = x \circledast (1 \boxplus (x \circledast y)) = x \circledast (x \circledast y)$ which shows (3.8). Using (3.3), (3.5) and (3.7), we obtain

$$x \circledast ((x \circledast y) \boxplus y) = x \circledast ((x \circledast y) \boxplus (x \circledast y)) = x \circledast 1 = 1. \qquad \square$$

Proposition 3.2. *Every $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies:*

$$(\forall x \in X)(x \ll_\boxplus 1), \tag{3.10}$$
$$(\forall x, y \in X)(x \boxplus (x \boxplus y) = x \boxplus y), \tag{3.11}$$
$$(\forall x, y \in X)(x \ll_\boxplus (x \boxplus y) \circledast y). \tag{3.12}$$

Proof. Let $x, y, z \in X$. Then

$$1 = x \boxplus x = x \boxplus (1 \circledast x) = x \boxplus (1 \circledast (x \boxplus x)) = x \boxplus (1 \circledast 1) = x \boxplus 1,$$

by (3.3), (3.4) and (3.6), i.e., $x \ll_\boxplus 1$. Using (3.4) and (3.6) induces

$$x \boxplus (x \boxplus y) = x \boxplus (1 \circledast (x \boxplus y)) = x \boxplus (1 \circledast y) = x \boxplus y.$$

Using (3.3), (3.6) and (3.10), we obtain

$$x \boxplus ((x \boxplus y) \circledast y) = x \boxplus ((x \boxplus y) \circledast (x \boxplus y)) = x \boxplus 1 = 1,$$

and so (3.12) is valid. □

Lemma 3.1. *Every* $(\circledast, \boxplus)$*-pseudo GE-algebra or* $(\boxplus, \circledast)$*-pseudo GE-algebra* X *satisfies:*

$$(\forall x \in X)(1 \ll_{\circledast} x \Rightarrow x = 1),$$
$$(\forall x \in X)(1 \ll_{\boxplus} x \Rightarrow x = 1).$$

Proof. Straightforward. □

Proposition 3.3. *Every pseudo GE-algebra* X *satisfies:*

$$(\forall x \in X)(1 \ll x \Rightarrow x = 1). \tag{3.13}$$

Proof. Lemma 3.1 induces (3.13). □

The combination of Propositions 3.1 and 3.2 induces the next proposition.

Proposition 3.4. *Every pseudo GE-algebra* X *satisfies for all* $x, y, z \in X$

(1) $x \circledast 1 = 1$ *and* $x \boxplus 1 = 1$;
(2) $x \circledast (x \circledast y) = x \circledast y$ *and* $x \boxplus (x \boxplus y) = x \boxplus y$;
(3) $x \ll_{\circledast} (x \circledast y) \boxplus y$ *and* $x \ll_{\boxplus} (x \boxplus y) \circledast y$;
(4) $x \ll_{\boxplus} y \circledast x$ *and* $x \ll_{\circledast} y \boxplus x$;
(5) $x \ll_{\circledast} (y \circledast x) \boxplus x$ *and* $x \ll_{\boxplus} (y \boxplus x) \circledast x$;
(6) $x \ll_{\circledast} (x \circledast y) \boxplus x$ *and* $x \ll_{\boxplus} (x \boxplus y) \circledast x$;
(7) $y \ll_{\circledast} y \boxplus x \Rightarrow x \ll_{\boxplus} y \circledast (y \boxplus x)$;
(8) $y \ll_{\boxplus} y \circledast x \Rightarrow x \ll_{\circledast} y \boxplus (y \circledast x)$;
(9) $x \circledast (y \boxplus z) \ll_{\circledast} y \boxplus (x \circledast z)$ *and* $x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$;
(10) $y \ll_{\boxplus} x \circledast (y \boxplus z) \Rightarrow y \ll_{\boxplus} (x \circledast z)$;
(11) $y \ll_{\circledast} x \boxplus (y \circledast z) \Rightarrow y \ll_{\circledast} (x \boxplus z)$;
(12) $x \ll_{\boxplus} y \circledast z \Leftrightarrow y \ll_{\circledast} x \boxplus z$.

Proof. Propositions 3.1 and 3.2 prove (1), (2), (3). Now,

$$x \boxplus (y \circledast x) = x \boxplus (y \circledast (x \boxplus x)) = x \boxplus (y \circledast 1) = x \boxplus 1 = 1$$

and

$$x \circledast (y \boxplus x) = x \circledast (y \boxplus (x \circledast x)) = x \circledast (y \boxplus 1) = x \circledast 1 = 1.$$

Hence $x \ll_{\boxplus} y \circledast x$ and $x \ll_{\circledast} y \boxplus x$. Hence, (4) follows. (5) and (6) follow from (4). Assume that $y \ll_{\circledast} y \boxplus x$. Then $y \circledast (y \boxplus x) = 1$, which implies that

$$\begin{aligned} x \boxplus (y \circledast (y \boxplus x)) &= x \boxplus (y \circledast (x \boxplus (y \boxplus x))) \\ &= x \boxplus (y \circledast (x \boxplus (y \circledast (y \boxplus x)))) \\ &= x \boxplus (y \circledast (x \boxplus 1)) \\ &= x \boxplus (y \circledast 1) = x \boxplus 1 = 1. \end{aligned}$$

Thus $x \ll_{\boxplus} y \circledast (y \boxplus x)$. If $y \ll_{\boxplus} y \circledast x$, then $y \boxplus (y \circledast x) = 1$. Hence,

$$\begin{aligned} x \circledast (y \boxplus (y \circledast x)) &= x \circledast (y \boxplus (x \circledast (y \circledast x))) \\ &= x \circledast (y \boxplus (x \circledast (y \boxplus (y \circledast x)))) \\ &= x \circledast (y \boxplus (x \circledast 1)) \\ &= x \circledast (y \boxplus 1) = x \circledast 1 = 1, \end{aligned}$$

and so, $x \ll_{\circledast} y \boxplus (y \circledast x)$. The combination of (3.2), (3.3) and (1) induces

$$(x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast z)) = (x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast (y \boxplus z))) = 1, \quad \text{by (4)}$$

and

$$(x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus z)) = (x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus (y \circledast z))) = 1, \quad \text{by (4).}$$

Hence, $x \circledast (y \boxplus z) \ll_{\circledast} y \boxplus (x \circledast z)$ and $x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$, that is, (9) is true. If $y \ll_{\boxplus} x \circledast (y \boxplus z)$, then $y \boxplus (x \circledast (y \boxplus z)) = 1$ and hence $y \boxplus (x \circledast z) = 1$. Therefore $y \ll_{\boxplus} (x \circledast z)$. Thus (10) follows. (11) is similar to (10). If $x \ll_{\boxplus} y \circledast z$, then $1 = x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$ by (9) and so $y \circledast (x \boxplus z) = 1$ by Lemma 3.1, i.e., $y \ll_{\circledast} x \boxplus z$. Similarly, if $y \ll_{\circledast} x \boxplus z$, then $x \ll_{\boxplus} y \circledast z$. Therefore, (12) holds. $\square$

We consider the following four items

(3.14) $\quad (\forall x, y, z \in X)\,(x \circledast y \ll_{\circledast} (z \circledast x) \boxplus (z \circledast y))\,,$

(3.15) $\quad (\forall x, y, z \in X)\,(x \circledast y \ll_{\boxplus} (y \circledast z) \circledast (x \circledast z))\,,$

(3.16) $\quad (\forall x, y, z \in X)\,(x \boxplus y \ll_{\boxplus} (z \boxplus x) \circledast (z \boxplus y))\,,$

(3.17) $\quad (\forall x, y, z \in X)\,(x \boxplus y \ll_{\circledast} (y \boxplus z) \boxplus (x \boxplus z))\,.$

Example 3.4. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra. But it does not satisfy (3.14), since

$$(a \circledast 1) \circledast ((1 \circledast a) \boxplus (1 \circledast 1)) = a \circledast (a \boxplus 1) = a \circledast 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra. But it does not satisfy (3.15), since

$$(a \circledast 1) \boxplus ((1 \circledast b) \circledast (a \circledast b)) = 1 \boxplus (b \circledast c) = 1 \boxplus c = c \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	1	a	1	1
c	1	a	1	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra. But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \circledast (1 \boxplus 1)) = a \boxplus (a \circledast 1) = a \boxplus 1 = a \neq 1.$$

4. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	1
c	a	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	1	1	c
c	1	1	b	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra. But it does not satisfy (3.17), since

$$(a \boxplus 1) \circledast ((1 \boxplus b) \boxplus (a \boxplus b)) = 1 \circledast (b \boxplus c) = 1 \circledast c = c \neq 1.$$

Proposition 3.5. *Let X be a $(\boxplus, \circledast)$-pseudo GE-algebra. If X meets condition* (3.14)*, it also meets condition* (3.15)*.*

Proof. Assume that a $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies (3.14). Then $(x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1$ for all $x, y, z \in X$, and so

$$\begin{aligned}(x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) =&(x \circledast y) \boxplus ((y \circledast z) \circledast ((x \circledast y) \boxplus (x \circledast z)))\\ =&(x \circledast y) \boxplus 1 = 1,\end{aligned}$$

by (3.6) and (3.10). Therefore $x \circledast y \ll_{\boxplus} (y \circledast z) \circledast (x \circledast z)$ for all $x, y, z \in X$. □

Proposition 3.6. *Let X be a $(\circledast, \boxplus)$-pseudo GE-algebra. If X meets condition* (3.15)*, it also meets condition* (3.14)*.*

Proof. Assume that a $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies (3.15). Then $(x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1$ for all $x, y, z \in X$, which implies from (3.5) and (3.7) that

$$\begin{aligned}(x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) =&(x \circledast y) \circledast ((z \circledast x) \boxplus ((x \circledast y) \circledast (z \circledast y)))\\ =&(x \circledast y) \circledast 1 = 1.\end{aligned}$$

Thus, $x \circledast y \ll (z \circledast x) \boxplus (z \circledast y)$ for all $x, y, z \in X$. □

Corollary 3.1. *In a pseudo GE-algebra X,* (3.14) *and* (3.15) *are equivalent with each other.*

The following example shows that the converse of Propositions 3.5 and 3.6 is not true in general.

Example 3.5. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	a	a	1	a
c	a	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra satisfying (3.15). But it does not satisfy (3.14), since

$$(a \circledast 1) \circledast ((1 \circledast a) \boxplus (1 \circledast 1)) = a \circledast (a \boxplus 1) = a \circledast 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	a
c	1	1	1	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra satisfying (3.14). But it does not satisfy (3.15), since

$$(a \circledast b) \boxplus ((b \circledast c) \circledast (a \circledast c)) = b \boxplus (a \circledast c) = b \boxplus c = a \neq 1.$$

Proposition 3.7. *Let X be a $(\circledast, \boxplus)$-pseudo GE-algebra. If X meets condition* (3.16)*, it also meets condition* (3.17)*.*

Proof. If a $(\circledast, \boxplus)$-pseudo GE-algebra X meets condition (3.16), then $(x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1$ for all $x, y, z \in X$. Hence,

$$\begin{aligned}(x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) =& (x \boxplus y) \circledast ((y \boxplus z) \boxplus ((x \boxplus y) \circledast (x \boxplus z))) \\ =& (x \boxplus y) \circledast 1 = 1,\end{aligned}$$

by (3.5) and (3.7), that is, $x \boxplus y \ll_{\circledast} (y \boxplus z) \boxplus (x \boxplus z)$ for all $x, y, z \in X$. □

Proposition 3.8. *Let X be a $(\boxplus, \circledast)$-pseudo GE-algebra. If X meets condition* (3.17)*, it also meets condition* (3.16)*.*

Proof. Suppose that the condition (3.17) holds in a $(\boxplus, \circledast)$-pseudo GE-algebra X. Then

$$(x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

for all $x, y, z \in X$. Using (3.6) and (3.10) induces

$$\begin{aligned}(x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) =&(x \boxplus y) \boxplus ((z \boxplus x) \circledast ((x \boxplus y) \boxplus (z \boxplus y)))\\ =&(x \boxplus y) \boxplus 1 = 1,\end{aligned}$$

that is, $x \boxplus y \ll_{\boxplus} (z \boxplus x) \circledast (z \boxplus y)$ for all $x, y, z \in X$. □

Corollary 3.2. *In a pseudo GE-algebra X,* (3.16) *and* (3.17) *are equivalent with each other.*

The following example shows that the converse of Propositions 3.7 and 3.8 is not true in general.

Example 3.6. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	1	a	1	1
c	1	a	1	1

.

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra satisfying (3.17). But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \circledast (1 \boxplus 1)) = a \boxplus (a \circledast 1) = a \boxplus 1 = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	a
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra satisfying (3.16). But it does not satisfy (3.17), since

$$(a \boxplus b) \circledast ((b \boxplus c) \boxplus (a \boxplus c)) = b \circledast (a \boxplus c) = b \circledast c = a \neq 1.$$

Definition 3.5. A $(\circledast, \boxplus)$-pseudo GE-algebra X is said to be

- $\circledast$-*transitive* if it satisfies (3.15);
- $\boxplus$-*transitive* if it satisfies (3.16).

If a $(\circledast, \boxplus)$-pseudo GE-algebra X is both $\circledast$-transitive and $\boxplus$-transitive, we say X is a *transitive* $(\circledast, \boxplus)$*-pseudo GE-algebra.*

Example 3.7. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

Then X is a $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. But it is not $\boxplus$-transitive since

$$(a \boxplus b) \boxplus ((1 \boxplus a) \circledast (1 \boxplus b)) = a \boxplus (a \circledast b) = a \boxplus b = a \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	1
c	1	a	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	c
c	1	a	1	1

Then X is a $(\circledast, \boxplus)$-pseudo GE-algebra which is $\boxplus$-transitive. But it is not $\circledast$-transitive since

$$(a \circledast b) \boxplus ((b \circledast c) \circledast (a \circledast c)) = 1 \boxplus (1 \circledast c) = 1 \boxplus c = c \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	b
b	1	a	1	1
c	1	a	1	1

Then X is a transitive $(\circledast, \boxplus)$-pseudo GE-algebra.

Proposition 3.9. *Every $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies*

$$(\forall x, y, z \in X)(x \ll_{\circledast} y \Rightarrow y \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast y), \tag{3.18}$$

$$(\forall x, y, z \in X)(((x \circledast y) \boxplus y) \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast ((x \circledast y) \boxplus y)), \tag{3.19}$$

$$(\forall x, y, z \in X)(((y \circledast x) \boxplus x) \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast ((y \circledast x) \boxplus x)). \tag{3.20}$$

Proof. Let X be a $\circledast$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. If $x \ll_{\circledast} y$, then $x \circledast y = 1$ and thus

$$(y \circledast z) \circledast (x \circledast z) = 1 \boxplus ((y \circledast z) \circledast (x \circledast z)) = (x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1,$$

that is, $y \circledast z \ll_{\circledast} x \circledast z$. Since X satisfies (3.14) by Proposition 3.6, we have

$$(z \circledast x) \boxplus (z \circledast y) = 1 \circledast ((z \circledast x) \boxplus (z \circledast y)) = (x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1$$

and so, $z \circledast x \ll_{\boxplus} z \circledast y$. This proves (3.18). The combination of (3.9) and (3.18) induces (3.19). The combination of Proposition 3.4 (5) and (3.18) induces (3.20). $\square$

Proposition 3.10. *Every $\boxplus$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow z \boxplus x \ll_{\circledast} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z).$$

Proof. Let X be a $\boxplus$-transitive $(\circledast, \boxplus)$-pseudo GE-algebra. If $x \ll_{\boxplus} y$, then $x \boxplus y = 1$ and thus

$$(z \boxplus x) \circledast (z \boxplus y) = 1 \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1.$$

Thus, $z \boxplus x \ll_{\circledast} z \boxplus y$. By Proposition 3.7, we know that X satisfies (3.17). Hence,

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

and so $y \boxplus z \ll_{\boxplus} x \boxplus z$. $\square$

Corollary 3.3. *Every transitive $(\circledast, \boxplus)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)\left(x \ll y \Rightarrow \begin{cases} y \circledast z \ll_{\circledast} x \circledast z, z \circledast x \ll_{\boxplus} z \circledast y \\ z \boxplus x \ll_{\circledast} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z \end{cases} \right).$$

Definition 3.6. A $(\boxplus, \circledast)$-pseudo GE-algebra X is said to be

- $\circledast$-*transitive* if it satisfies (3.14);
- $\boxplus$-*transitive* if it satisfies (3.17).

If a $(\boxplus, \circledast)$-pseudo GE-algebra X is both $\circledast$-transitive and $\boxplus$-transitive, we say X is a *transitive* $(\boxplus, \circledast)$*-pseudo GE-algebra.*

Example 3.8. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	b
b	1	1	1	1
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	1
c	1	1	1	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra which is $\circledast$-transitive. But it is not $\boxplus$-transitive since

$$(a \boxplus b) \circledast ((b \boxplus c) \boxplus (a \boxplus c)) = 1 \circledast (1 \boxplus c) = 1 \circledast c = c \neq 1.$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

.

Then X is a $(\boxplus, \circledast)$-pseudo GE-algebra which is $\boxplus$-transitive. But it is not $\circledast$-transitive since

$$(a \circledast b) \circledast ((1 \circledast a) \boxplus (1 \circledast b)) = a \circledast (a \boxplus b) = a \circledast b = a \neq 1.$$

3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	1
c	1	a	1	1

Then X is a transitive $(\boxplus, \circledast)$-pseudo GE-algebra.

Proposition 3.11. *Every $\boxplus$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies*

(3.21) $\quad (\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus y),$

(3.22) $\quad (\forall x, y, z \in X)(((x \boxplus y) \circledast y) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus ((x \boxplus y) \circledast y)),$

(3.23) $\quad (\forall x, y, z \in X)(((y \boxplus x) \circledast x) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus ((y \boxplus x) \circledast x)).$

Proof. Let X be a $\boxplus$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra. Let $x, y \in X$ be such that $x \ll_{\boxplus} y$, Then $x \boxplus y = 1$ and so

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \circledast ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

by (3.4) and (3.17). Hence, $y \boxplus z \ll_{\boxplus} x \boxplus z$. We know that X satisfies (3.16) by Proposition 3.8. Thus,

$$(z \boxplus x) \circledast (z \boxplus y) = 1 \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \circledast (z \boxplus y)) = 1$$

and so $z \boxplus x \ll_{\circledast} z \boxplus y$, which proves (3.21). If we combine (3.21) and (3.12), then we have (3.22). The result (3.23) follows from the combination of (3.21) and Proposition 3.4 (5). $\square$

Proposition 3.12. *Every $\circledast$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra X satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\circledast} y \Rightarrow z \circledast x \ll_{\boxplus} z \circledast y, y \circledast z \ll_{\circledast} x \circledast z).$$

Proof. Let X be a $\circledast$-transitive $(\boxplus, \circledast)$-pseudo GE-algebra. If $x \ll_{\circledast} y$, then $x \circledast y = 1$ and so

$$(z \circledast x) \boxplus (z \circledast y) = 1 \circledast ((z \circledast x) \boxplus (z \circledast y)) = (x \circledast y) \circledast ((z \circledast x) \boxplus (z \circledast y)) = 1,$$

which shows that $z \circledast x \ll_{\boxplus} z \circledast y$. Using Proposition 3.5, we know that X satisfies condition (3.15). Thus,

$$(y \circledast z) \circledast (x \circledast z) = 1 \boxplus ((y \circledast z) \circledast (x \circledast z)) = (x \circledast y) \boxplus ((y \circledast z) \circledast (x \circledast z)) = 1,$$

and therefore $y \circledast z \ll_{\circledast} x \circledast z$. $\square$

Corollary 3.4. *Every transitive* $(\boxplus, \circledast)$*-pseudo GE-algebra* X *satisfies:*

$$(\forall x, y, z \in X)\left(x \ll y \Rightarrow \left\{ \begin{array}{l} y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\circledast} z \boxplus y \\ z \circledast x \ll_{\boxplus} z \circledast y, y \circledast z \ll_{\circledast} x \circledast z \end{array} \right. \right).$$

4. Relations Between Pseudo BE-Algebras and Pseudo GE-Algebras

As an extension of BE-algebras, Borzooei et al. introduced the notion of pseudo BE-algebras, and investigated its properties.

Definition 4.1 ([2])**.** Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *pseudo BE-algebra* if it satisfies (3.3), (3.4) and

$$(\forall x \in X)(x \circledast 1 = 1, x \boxplus 1 = 1), \tag{4.1}$$

$$(\forall x, y, z \in X)(x \circledast (y \boxplus z) = y \boxplus (x \circledast z)), \tag{4.2}$$

$$(\forall x, y \in X)(x \circledast y = 1 \Leftrightarrow x \boxplus y = 1). \tag{4.3}$$

Pseudo GE-algebra and pseudo BE-algebra basically form no relationship. In other words, the pseudo GE-algebra may not be the pseudo BE-algebra, and vice versa as seen in the following example.

Example 4.1. 1. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	1
c	1	a	1	1

.

Then X is a pseudo GE-algebra. But X is not a pseudo BE-algebra since

$$a \circledast (a \boxplus b) = a \circledast c = b \neq c = a \boxplus b = a \boxplus (a \circledast b).$$

2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	c	b
b	1	a	1	1
c	1	a	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	1
c	1	a	1	1

.

Then X is a pseudo BE-algebra. But X is not a pseudo GE-algebra since

$$a \circledast (a \boxplus b) = a \circledast b = c \neq b = a \circledast c = a \circledast (a \boxplus c) = a \circledast (a \boxplus (a \circledast b)).$$

The following example shows that when $(X, \circledast, \boxplus, 1)$ is a pseudo BE-algebra, $(X, \circledast, 1)$ or $(X, \boxplus, 1)$ does not need to be BE-algebra.

Example 4.2. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$$\begin{array}{c|cccc} \circledast & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & 1 & 1 & 1 & a \\ b & 1 & a & 1 & a \\ c & 1 & 1 & 1 & 1 \end{array}, \qquad \begin{array}{c|cccc} \boxplus & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & 1 & 1 & 1 & b \\ b & 1 & a & 1 & c \\ c & 1 & 1 & 1 & 1 \end{array}.$$

Then X is a pseudo BE-algebra. But $(X, \circledast, 1)$ is not a BE-algebra since

$$a \circledast (b \circledast c) = a \circledast a = 1 \neq a = b \circledast a = b \circledast (a \circledast c).$$

Also, $(X, \boxplus, 1)$ is not a BE-algebra since

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

Definition 4.2. A pseudo BE-algebra $(X, \circledast, \boxplus, 1)$ is said to be *strong* if $(X, \circledast, 1)$ and $(X, \boxplus, 1)$ are BE-algebras.

Example 4.3. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$$\begin{array}{c|cccc} \circledast & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & 1 & 1 & a & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 1 & 1 & a & 1 \end{array}, \qquad \begin{array}{c|cccc} \boxplus & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & 1 & 1 & c & 1 \\ b & 1 & 1 & 1 & 1 \\ c & 1 & 1 & a & 1 \end{array}.$$

Then X is a strong pseudo BE-algebra.

We have the following question.

Question 4.1. Does pseudo GE-algebra X satisfy the following conditions

$$(\forall x, y, z \in X) \begin{pmatrix} x \circledast (y \boxplus z) = x \circledast (y \boxplus (x \boxplus z)) \\ x \boxplus (y \circledast z) = x \boxplus (y \circledast (x \circledast z)) \end{pmatrix}? \tag{4.4}$$

The following example shows that the answer to Question 4.1 is negative.

Example 4.4. Let $X = \{1, a, b, c, d\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following table:

$$\begin{array}{c|ccccc} \circledast & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & c & c & 1 \\ b & 1 & 1 & 1 & 1 & 1 \\ c & 1 & 1 & 1 & 1 & 1 \\ d & 1 & 1 & 1 & 1 & 1 \end{array}, \qquad \begin{array}{c|ccccc} \boxplus & 1 & a & b & c & d \\ \hline 1 & 1 & a & b & c & d \\ a & 1 & 1 & d & 1 & d \\ b & 1 & a & 1 & 1 & a \\ c & 1 & a & a & 1 & a \\ d & 1 & a & a & 1 & 1 \end{array}.$$

Then X is a pseudo GE-algebra. But it does not satisfy (4.4) since

$$a \circledast (1 \boxplus b) = a \circledast b = c \neq 1 = a \circledast d = a \circledast (1 \boxplus d) = a \circledast (1 \boxplus (a \boxplus b)),$$

$$a \boxplus (1 \circledast b) = a \boxplus b = d \neq 1 = a \boxplus c = a \boxplus (1 \circledast c) = a \boxplus (1 \circledast (a \circledast b)).$$

Definition 4.3. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *good pseudo GE-algebra* if it satisfies (3.3), (3.4) and (4.4).

Example 4.5. Let $X = \{1, a, b, c, d\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following table:

$\circledast$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	c	c
c	1	a	1	1	1
d	1	a	1	1	1

,

$\boxplus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

.

Then X is a good pseudo GE-algebra.

Theorem 4.1. *Every good pseudo GE-algebra is a pseudo GE-algebra. But the converse is not true.*

Proof. Example 4.4 shows that any pseudo GE-algebra may not be a good pseudo GE-algebra. Let X be a good pseudo GE-algebra and let $x, y, z \in X$. Then

$$x \circledast y = x \circledast (1 \boxplus y) = x \circledast (1 \boxplus (x \boxplus y)) = x \circledast (x \boxplus y)$$

and

$$x \boxplus y = x \boxplus (1 \circledast y) = x \boxplus (1 \circledast (x \circledast y)) = x \boxplus (x \circledast y).$$

It follows from (4.4) that

$$x \circledast (y \boxplus (x \circledast z)) = x \circledast (y \boxplus (x \boxplus (x \circledast z))) = x \circledast (y \boxplus (x \boxplus z)) = x \circledast (y \boxplus z)$$

and

$$x \boxplus (y \circledast (x \boxplus z)) = x \boxplus (y \circledast (x \circledast (x \boxplus z))) = x \boxplus (y \circledast (x \circledast z)) = x \boxplus (y \circledast z).$$

Therefore, X is a pseudo GE-algebra. □

The following example shows that any pseudo BE-algebra X does not satisfy the condition

$$(\forall x, y, z \in X) \left(\begin{array}{c} x \circledast (y \boxplus z) = (x \circledast y) \boxplus (x \circledast z) \\ x \boxplus (y \circledast z) = (x \boxplus y) \circledast (x \boxplus z) \end{array} \right). \tag{4.5}$$

Example 4.6. Let $X = \{1, a, b, c, d\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	1
b	1	1	1	1	1
c	1	a	a	1	1
d	1	a	a	c	1

,

$\boxplus$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	d	c	1
b	1	1	1	1	1
c	1	a	b	1	1
d	1	a	b	c	1

.

Then X is a pseudo BE-algebra. But X does not satisfy (4.5) since

$$a \circledast (b \boxplus c) = a \circledast 1 = 1 \neq c = a \boxplus c = (a \circledast b) \boxplus (a \circledast c)$$

and

$$a \boxplus (b \circledast c) = a \boxplus 1 = 1 \neq c = d \circledast c = (a \boxplus b) \circledast (a \boxplus c).$$

Question 4.2. Does pseudo BE-algebra X satisfy the following conditions

$$(\forall x, y, z \in X) \left(\begin{array}{c} x \circledast (y \circledast z) = y \circledast (x \circledast z)) \\ x \boxplus (y \boxplus z) = y \boxplus (x \boxplus z)) \end{array} \right)? \tag{4.6}$$

The answer to Question 4.2 is negative as seen in the following example.

Example 4.7. Let $X = \{1, a, b, c\}$ be a set with binary operations "$\circledast$" and "$\boxplus$" given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

.

Then X is a pseudo BE-algebra. But it does not satisfy (4.6) since

$$a \circledast (b \circledast c) = a \circledast a = 1 \neq a = b \circledast a = b \circledast (a \circledast c)$$

and

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

Definition 4.4. Let X be a set with a constant 1 and two binary operations "$\circledast$" and "$\boxplus$". A structure $(X, \circledast, \boxplus, 1)$ is called a *good pseudo BE-algebra* if it satisfies (3.3), (3.4), (4.1), (4.3) and (4.6).

Example 4.8. Let $X = \{1, a, b, c\}$ be a set with binary operations $\circledast$, $\boxplus$ given in the following tables:

$\circledast$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	b	1

,

$\boxplus$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	1	1	1
c	1	a	b	1

.

Then X is a good pseudo BE-algebra. But X is not pseudo BE-algebra since

$$a \circledast (a \boxplus b) = a \circledast b = a \neq 1 = a \boxplus a = a \boxplus (a \circledast b).$$

We now consider conditions for a pseudo BE-algebra to be a pseudo GE-algebra.

Theorem 4.2. *If a good pseudo BE-algebra X satisfies the condition* (4.5)*, then it is a pseudo GE-algebra.*

Proof. Let X be a good pseudo BE-algebra that satisfies the conditions (4.5). It is sufficient to show that X satisfies the condition (4.4). Let $x, y, z \in X$. Then $x \circledast (x \boxplus y) = (x \circledast x) \boxplus (x \circledast y) = 1 \boxplus (x \circledast y) = x \circledast y$ and $x \boxplus (x \circledast y) = (x \boxplus x) \circledast (x \boxplus y) = 1 \circledast (x \boxplus y) = x \boxplus y$ by (3.3), (3.4) and (4.5). It follows that $x \circledast (y \boxplus z) = x \circledast (x \boxplus (y \boxplus z)) = x \circledast (y \boxplus (x \boxplus z))$ and $x \boxplus (y \circledast z) = x \boxplus (x \circledast (y \circledast z)) = x \boxplus (y \circledast (x \circledast z))$. Hence X is a good pseudo GE-algebra, and therefore it is a pseudo GE-algebra by Theorem 4.1. □

Corollary 4.1. *Every strong pseudo BE-algebra X satisfying the condition* (4.5) *is a* (*good*) *pseudo GE-algebra.*

We finally pose the following question.

Question 4.3. What conditions will be required to make pseudo GE-algebra into pseudo BE-algebra?

5. Concluding Remarks

In this paper, we generalized GE-algebras to the case of pseudo GE-algebras and studied some basic of those properties. In the last section we investigated among relation between pseudo BE-algebras and pseudo GE-algebras. Starting from these notions, one can define and investigate commutative pseudo GE-algebras, involutive pseudo-GE algebras and Smarandache pseudo GE-algebras. Another topic of research could be to define and investigate state and monadic on pseudo GE-algebras.

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