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AN APPROACH TO LAGRANGE'S THEOREM IN PYTHAGOREAN FUZZY SUBGROUPS

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Abstract. The Pythagorean fuzzy environment is a modern way of depicting uncertainty. The concept of Pythagorean fuzzy semi-level subgroups of any group is described in this paper. The Pythagorean fuzzy order of an element in a Pythagorean fuzzy subgroup is introduced and established various algebraic attributes. The relation between the Pythagorean fuzzy order of an element of a group and the order of that group is established. The Pythagorean fuzzy normalizer and Pythagorean fuzzy centralizer of Pythagorean fuzzy subgroups are discussed. Further, the concept of Pythagorean fuzzy quotient group and the index of a Pythagorean fuzzy subgroup are defined. Finally, a framework is developed for proving Lagrange's theorem in Pythagorean fuzzy subgroups.

1. Introduction

One of the most important theorems in Abstract algebra is Lagrange's theorem. This theorem is very crucial in case of finite groups because it provides an overview of subgroup size. Lagrange's theorem has various applications in number theory. For further details, we refer to [16].

Uncertainty is an unavoidable element of our lives. This universe isn't built on assumptions or precise measures. It is not always feasible to make straightforward decisions. We face a significant problem in dealing with errors in decision-making situations. In 1965, Zadeh [19] proposed the fuzzy set as a way to deal with ambiguity in real-world problems. Following that, fuzzy sets become a worldwide study trend. Rosenfeld [15] was the first to examine the concept of fuzzy subgroup and its features

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in 1971. The concepts of fuzzy coset and fuzzy normal subgroup were introduced by Ajmal and Prajapati [2]. Dixit et al. [10] addressed fuzzy level subgroups and the union of fuzzy subgroups in 1990. Biswas[8] was the first to suggest the concept of an anti-fuzzy subgroup. The concepts of fuzzy normal subgroup, fuzzy coset, and fuzzy quotient subgroup were presented by Ajmal and Prajapati [2] in 1992. Chakraborty and Khare [9] investigated a variety of fuzzy homomorphism features. Ajmal [3] also looked into homomorphisms of fuzzy subgroups. Kim [11] established the order of fuzzy subgroups and fuzzy p-subgroups in 1994. In 1999, Ray [14] proposed the product of fuzzy subgroups. Many researchers have been studying the features of fuzzy groups in recent years. In 2015, Tarnauceanu [17] developed fuzzy normal subgroups of finite groups. Addis [1] proposed fuzzy homomorphism theorems for groups in 2018. In 2021, Bhunia [5] and Ghorai [7] presented the concept of (α, β)-Pythagorean fuzzy sets and characterized (α, β)-Pythagorean fuzzy subgroups.

When it comes to decision-making, assigning membership values isn't always adequate. Atanassov [4] established the intuitionistic fuzzy set in 1986 by attributing non-membership degrees to membership degrees. Yager [18] defined Pythagorean fuzzy set (PFS) in 2013 using this approach. In comparison to intuitionistic fuzzy sets, this set provides a modern technique to model vagueness and uncertainty with high precision and accuracy. Peng [13] and Yang presented some results relating to it. Bhunia et al. [6] started exploring Pythagorean fuzzy subgroups (PFSG) in 2021. Pythagorean fuzzy subgroup was shown to be a larger class of Intuitionistic fuzzy subgroup. The major goal of this study is to prove Lagrange's theorem in Pythagorean fuzzy subgroups. This article is designed in such a way that we can approach Lagrange's theorem.

This paper's outline is as follows: in Section 2, we review several key definitions and ideas. In Section 3, we define Pythagorean fuzzy order of elements of groups and go over some of its features. Section 4 discusses the algebraic properties of the Pythagorean fuzzy subgroup. We introduce the concept of a Pythagorean fuzzy quotient group and prove Lagrange's theorem in Section 5. In Section 6, we come to a conclusion.

2. Preliminaries

This section covers some definitions and concepts that are crucial for the development of subsequent sections.

Definition 2.1 ([18])**.** A PFS ψ on a set C is defined by $\psi = \{(m, \mu(m), \nu(m)) \mid m \in C\}$ where $\mu(m) \in [0,1]$ and $\nu(m) \in [0,1]$ are the degree of membership and non membership of $m \in C$, respectively, which fulfill the condition $0 \leq \mu^2(m) + \nu^2(m) \leq 1$ for all $m \in C$.

PFS will be denoted as $\psi = (\mu, \nu)$ rather than $\psi = \{(m, \mu(m), \nu(m)) \mid m \in C\}$.

Definition 2.2 ([6])**.** Let $\psi = (\mu, \nu)$ be a PFS on a group $(C, \circ)$. Then ψ is a PFSG of C if:

(i) $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$;
(ii) $\mu^2(m^{-1}) \geq \mu^2(m)$ and $\nu^2(m^{-1}) \leq \nu^2(m)$ for all $m \in C$.

Here, $\mu^2(m) = \{\mu(m)\}^2$ and $\nu^2(m) = \{\nu(m)\}^2$ for all $m \in C$.

Proposition 2.1 ([6]). *Let $\psi = (\mu, \nu)$ be a PFS on a group $(C, \circ)$. Then ψ is a PFSG of $(C, \circ)$ if and only if $\mu^2(m \circ n^{-1}) \geq \mu^2(m) \wedge \mu^2(n)$ and $\nu^2(m \circ n^{-1}) \leq \nu^2(m) \vee \nu^2(n)$ for all $m, n \in C$.*

Definition 2.3 ([6]). Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then for $m \in C$, the PFLC of ψ is the PFS $m\psi = (m\mu, m\nu)$, defined by $(m\mu)^2(u) = \mu^2(m^{-1} \circ u)$, $(m\nu)^2(u) = \nu^2(m^{-1} \circ u)$ and the PFRC of ψ is the PFS $\psi m = (\mu m, \nu m)$, defined by $(\mu m)^2(u) = \mu^2(u \circ m^{-1})$, $(\nu m)^2(u) = \nu^2(u \circ m^{-1})$ for all $u \in C$.

Definition 2.4 ([6]). Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG on the group $(C, \circ)$ if every PFLC of ψ is a PFRC of ψ on C.

Equivalently, $m\psi = \psi m$ for all $m \in C$.

Proposition 2.2 ([6]). *Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG on C if and only if $\mu^2(m \circ n) = \mu^2(n \circ m)$ and $\nu^2(m \circ n) = \nu^2(n \circ m)$ for all $m, n \in C$.*

Proposition 2.3 ([6]). *Let $\psi = (\mu, \nu)$ be a PFSG on a group $(C, \circ)$. Then ψ is a PFNSG of C if and only if $\mu^2(k \circ u \circ k^{-1}) = \mu^2(u)$ and $\nu^2(k \circ u \circ k^{-1}) = \nu^2(u)$ for all $u, k \in C$.*

3. Pythagorean Fuzzy Order of Elements in PFSG

This section establishes the Pythagorean fuzzy order of elements in PFSGs and introduce the concept of Pythagorean fuzzy semi-level subgroups of any group. We also compare the fuzzy order of elements in fuzzy subgroups with the Pythagorean fuzzy order of elements in PFSGs. We also go over some of the algebraic features of Pythagorean fuzzy order of elements in PFSGs.

Theorem 3.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$ is a subgroup of C.*

Proof. We have $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$, where $m \in C$. So, $\Gamma(m) \subset C$ as $m \in \Gamma(m)$. Also, $e \in \Gamma(m)$ as $\mu^2(e) \geq \mu^2(m)$ and $\nu^2(e) \leq \nu^2(m)$. Let $p, q \in \Gamma(m)$. Then

$$\mu^2(pq^{-1}) \geq \mu^2(p) \wedge \mu^2(q^{-1}) = \mu^2(p) \wedge \mu^2(q) \geq \mu^2(m).$$

In the same way, we can prove that $\nu^2(pq^{-1}) \leq \nu^2(m)$. Thus, $pq^{-1} \in \Gamma(m)$. Therefore, $\Gamma(m)$ is a subgroup of C. □

Definition 3.1. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then the subgroup $\Gamma(m)$ is a Pythagorean fuzzy semi-level subgroup of C corresponding to m.

Definition 3.2. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. Then the Pythagorean fuzzy order (PFO) of m in ψ is denoted by $PFO(m)_\psi$ and defined by the order of the Pythagorean fuzzy semi-level subgroup of m in C.

Therefore, $PFO(m)_\psi = O(\Gamma(m))$ for all $m \in C$.

Example 3.1. Consider the group $(\mathbb{Z}_4, +_4)$.

Assign membership and non-membership degree of the elements of $\mathbb{Z}_4$ by

$$\mu(0) = 0.95, \quad \mu(1) = 0.65, \quad \mu(2) = 0.65, \quad \mu(3) = 0.85,$$
$$\nu(0) = 0.25, \quad \nu(1) = 0.75, \quad \nu(2) = 0.75, \quad \nu(3) = 0.45.$$

Clearly, $\psi = (\mu, \nu)$ is a PFSG on $\mathbb{Z}_4$. Then PFO of the elements of $\mathbb{Z}_4$ in ψ is presented by

$$PFO(0)_\psi = O(\Gamma(0)) = 2, \quad PFO(1)_\psi = O(\Gamma(1)) = 4,$$
$$PFO(2)_\psi = O(\Gamma(2)) = 4, \quad PFO(3)_\psi = O(\Gamma(3)) = 2.$$

From above example, we see that $PFO(0)_\psi \neq O(0)$ and $PFO(0)_\psi = PFO(3)_\psi = 2$.

Remark 3.1. The PFO of an element in PFSG may not always be same to the element's order in the group.

Proposition 3.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $PFO(e)_\psi \leq PFO(m)_\psi$ for all $m \in C$, where e is group's identity.*

Proof. Let $PFO(e)_\psi = s$, where $s \in \mathbb{Z}^+$. Assume that $\Gamma(e) = \{m_1, m_2, \ldots, m_s\}$, where $m_i \neq m_j$ for all i, j.

Then $\mu^2(m_1) = \mu^2(m_2) = \cdots = \mu^2(m_s) = \mu^2(e)$ and $\nu^2(m_1) = \nu^2(m_2) = \cdots = \nu^2(m_s) = \nu^2(e)$.

As $\psi = (\mu, \nu)$ is a PFSG on C, $\mu^2(e) \geq \mu^2(m)$ and $\nu^2(e) \leq \nu^2(m)$ for all $m \in C$. So, $m_1, m_2, \ldots, m_s \in \Gamma(u)$. Then $\Gamma(e) \subseteq \Gamma(u)$. Thus, $O(\Gamma(e)) \leq O(\Gamma(m))$ for all $m \in C$. Therefore, $PFO(e)_\psi \leq PFO(m)_\psi$ for all $m \in C$. □

The next result represents a relation between the order and PFO of an element in a group.

Theorem 3.2. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $O(m)$ divides $PFO(m)_\psi$ for all $m \in C$.*

Proof. Let $m \in C$ and $O(m) = k$, where $k \in \mathbb{Z}^+$. Then $m^k = e$. Consider $D = \langle m \rangle$ as a subgroup of C.

Now, $\mu^2(m^2) \geq \mu^2(m) \wedge \mu^2(m) = \mu^2(m)$ and $\nu^2(m^2) \leq \nu^2(m) \vee \nu^2(m) = \nu^2(m)$. Therefore, by induction, $\mu^2(m^p) \geq \mu^2(m)$ and $\nu^2(m^p) \leq \nu^2(m)$ for all $p \in \mathbb{Z}^+$.

So, $m, m^2, \ldots, m^k \in \Gamma(m)$. Consequently, $D \subseteq \Gamma(m)$. Therefore, D is a subgroup of $\Gamma(m)$.

Thus, by Lagrange's theorem, $O(D)|O(\Gamma(m))$. Therefore, $O(m)|PFO(m)_\psi$. Since m is a random element of C, $O(m)|PFO(m)_\psi$ for all $m \in C$. □

We will now construct a relationship between the PFO of an element of a group in PFSG and the group's order.

Theorem 3.3. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFO of each element of C in ψ divides the order of C.*

Proof. According to the definition, $PFO(m)_\psi = O(\Gamma(m))$ for all $m \in C$.

From Theorem 3.1, $\Gamma(m)$ is a subgroup of C. Therefore, by Lagrange's theorem, the order of $\Gamma(m)$ divides the order of C. That is $O(\Gamma(m))|O(C)$.

This represent that $PFO(m)_\psi|O(C)$ for all $m \in C$. Hence, the PFO of each element of C in ψ divides the order of C. □

Theorem 3.4. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $PFO(m)_\psi = PFO(m^{-1})_\psi$ for all $m \in C$.*

Proof. Let $m \in C$. Then $PFO(m)_\psi = O(\Gamma(m))$.

As $\psi = (\mu, \nu)$ is a PFSG on C, then $\mu^2(m) = \mu^2(m^{-1})$ and $\nu^2(m) = \nu^\beta(m^{-1})$. Therefore, $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m^{-1}),\ \nu^2(n) \leq \nu^2(m^{-1})\} = \Gamma(m^{-1})$.

This proves that, $O(\Gamma(m)) = O(\Gamma(m^{-1}))$. That is $PFO(m)_\psi = PFO(m^{-1})_\psi$. Therefore, $PFO(m)_\psi = PFO(m^{-1})_\psi$ for all $m \in C$. □

Now, we will introduce the PFO of a PFSG on a group.

Definition 3.3. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then the PFO of the PFSG ψ is denoted by $PFO(\psi)$ and is defined by $PFO(\psi) = \bigvee\{PFO(m)_\psi \mid m \in C\}$.

Example 3.2. Consider the PFSG ψ on $\mathbb{Z}_4$ in Example 3.1.

The PFO of the elements of $\mathbb{Z}_4$ in ψ is presented by $PFO(0)_\psi = 2$, $PFO(1)_\psi = 4$, $PFO(2)_\psi = 4$ and $PFO(3)_\psi = 2$. Therefore, $PFO(\psi) = \bigvee\{PFO(m)_\psi \mid m \in \mathbb{Z}_4\}$=4.

Theorem 3.5. *The PFO of each PFSG on a group is the same as the group's order.*

Proof. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$.

Without sacrificing generality, we assume that $\mu^2(n) \geq \mu^2(m)$ and $\nu^2(n) \leq \nu^2(m)$ for all $n \in C$. Since $\Gamma(m) = \{n \in C \mid \mu^2(n) \geq \mu^2(m), \nu^2(n) \leq \nu^2(m)\}$, then $\Gamma(m) = C$. Also, $|\Gamma(m)| \geq |\Gamma(n)|$ for all $n \in C$. Consequently, $PFO(\psi) = PFO(m)_\psi$.

Again $PFO(m)_\psi = O(\Gamma(m))$. Therefore, $PFO(\psi) = O(C)$.

Hence, the PFO of an PFSG on a group is the same as group's order. □

Remark 3.2. For a PFSG on a group C, the PFO of an element of C divides the PFO of that PFSG.

Theorem 3.6. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$ such that $PFO(m)_\psi = s$. If $\gcd(s, t) = 1$, then $\mu^2(m^t) = \mu^2(m)$ and $\nu^2(m^t) = \nu^2(m)$.*

Proof. Since $PFO(m)_\psi = s$, then $m^s = e$. Also $\psi = (\mu, \nu)$ is a PFSG on C, then $\mu^2(m^t) \geq \mu^2(m)$ and $\nu^2(m^t) \leq \nu^2(m)$.

As $\gcd(s,t)=1$, then there exist a and b such that $as+bt=1$. Now

$$\mu^2(m) = \mu^2(m^{as+bt}) \geq \mu^2(m^{as}) \wedge \mu^2(m^{bt}) \geq \mu^2(e) \wedge \mu^2(m^t) = \mu^2(m^t).$$

Therefore, $\mu^2(m) \geq \mu^2(m^t)$. Same way we can prove that $\nu^2(m) \leq \nu^2(m^t)$. Hence, $\mu^2(m^t) = \mu^2(m)$ and $\nu^2(m^t) = \nu^2(m)$. □

Theorem 3.7. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $\mu^2(m^t) = \mu^2(e)$ and $\nu^2(m^t) = \nu^2(e)$ then $t|PFO(m)_\psi$, where $t \in \mathbb{Z}$.*

Proof. Let $PFO(m)_\psi = s$. We can suppose that q is the smallest integer for which $\mu^2(m^q) = \mu^2(e)$ and $\nu^2(m^q) = \nu^2(e)$ holds.

By division algorithm, there exist $a, b \in \mathbb{Z}$ such that $s = at + b$ where $0 \leq b < t$. Now

$$\begin{aligned}\mu^2(m^b) =& \mu^2(m^{s-at})\\ \geq& \mu^2(m^s) \wedge \mu^2((m^{-1})^{at})\\ =& \mu^2(m^s) \wedge \mu^2(m^{at}) = \mu^2(e) \wedge \mu^2((m^t)^a)\\ \geq& \mu^2(e) \wedge \mu^2(m^t)\\ =& \mu^2(e).\end{aligned}$$

Similarly, $\nu^2(m^b) \leq \nu^2(e)$. Thus, $\mu^2(m^b) = \mu^2(e)$ and $\nu^2(m^b) = \nu^2(e)$. This contradicts q's minimality as $0 \leq b < t$.

Therefore, $b = 0$, so $s = at$. Hence, $t|PFO(m)_\psi$. □

Theorem 3.8. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $PFO(m)_\psi = s$, then $PFO(m^v)_\psi = \frac{s}{\gcd(s,v)}$, where $v \in \mathbb{Z}$.*

Proof. Let $PFO(m^v)_\psi = a$ and $\gcd(s,v) = g$.

As $PFO(m)_\psi = s$ then by Theorem 3.7, $m^s = e$. Now

$$\mu^2((m^v)^{\frac{s}{g}}) = \mu^2((m^s)^{\frac{v}{g}}) = \mu^2(e^{\frac{v}{g}}) = \mu^2(e).$$

Similarly, $\nu^2((m^v)^{\frac{s}{g}}) = \nu^2(e)$. As a result of the Theorem 3.7, $\frac{s}{g}$ divides a.

Also, $\gcd(s,v) = g$, then there exist $p, q \in \mathbb{Z}$ such that $sp + vq = g$. Therefore,

$$\begin{aligned}\mu^2(m^{ga}) =& \mu^2(m^{(ps+vq)a})\\ =& \mu^2(m^{psa}m^{vqa})\\ \geq& \mu^2((m^s)^{pa}) \wedge \mu^2((m^{va})^q)\\ \geq& \mu^2(m^s) \wedge \mu^2((m^v)^a)\\ =& \mu^2(e) \wedge \mu^2(e)\\ =& \mu^2(e).\end{aligned}$$

As a result, the only option is $\mu^2(m^{ga}) = \mu^2(e)$. Similarly, $\nu^2(m^{ga}) = \nu^2(e)$.

Thus, by Theorem 3.7, $ga|s$, that is $a|\frac{s}{g}$. Therefore, $a = \frac{s}{g}$. Hence, $PFO(m^v)_\psi = \frac{s}{\gcd(s,v)}$. □

Theorem 3.9. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C and $m \in C$. If $PFO(m)_\psi = z$ and $g \cong h \pmod z$, then $PFO(m^g)_\psi = PFO(m^h)_\psi$, where $g, h, z \in \mathbb{Z}$.*

Proof. Let $PFO(m^g)_\psi = l_1$ and $PFO(m^h)_\psi = l_2$.
As $g \cong h \pmod z$, then $g = wz + h$, where $w \in \mathbb{Z}$. Then

$$\begin{aligned}\mu^2((m^g)^{l_2}) =&\mu^2((m^{zw+h})^{l_2}) = \mu^2(m^{wzl_2}m^{hl_2})\\ \geq&\mu^2((m^z)^{wl_2}) \wedge \mu^2((m^h)^{l_2})\\ =&\mu^2(e) \wedge \mu^2(e)\\ =&\mu^2(e).\end{aligned}$$

As a result, the only option is $\mu^2((m^g)^{l_2}) = \mu^2(e)$. Similarly, $\nu^2((m^g)^{l_2}) = \nu^2(e)$. Thus by Theorem 3.7, $l_2 | l_1$. In the same manner, we can prove that $l_1 | l_2$. Thus $l_1 = l_2$.
Hence $PFO(m^g)_\psi = PFO(m^h)_\psi$, where $g, h \in \mathbb{Z}$. □

Theorem 3.10. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a group C and $m \in C$. Then $PFO(m)_\psi = PFO(nmn^{-1})_\psi$ for all $n \in C$.*

Proof. Let n be any element of C.
As ψ is a PFNSG on the group C, then $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$. Therefore the Pythagorean fuzzy semi-level subgroup corresponding to m is equal to nmn^{-1}.
This implies that $\Gamma(m) = \Gamma(nmn^{-1})$. Consequently, $O(\Gamma(m)) = O(\Gamma(nmn^{-1}))$. Since n is a random element of C, hence $PFO(m)_\psi = PFO(nmn^{-1})_\psi$ for all $n \in C$. □

Theorem 3.11. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a group C. Then $PFO(mn)_\psi = PFO(nm)_\psi$ for all $m, n \in C$.*

Proof. Assume m and n are elements of C.
Then we have $\mu^2(mn) = \mu^2((n^{-1}n)(mn)) = \mu^2(n^{-1}(nm)n)$. Similarly, $\nu^2(mn) = \nu^2(n^{-1}(nm)n)$. Therefore, $\Gamma(mn) = \Gamma(n^{-1}(mn)(n^{-1})^{-1})$. Consequently, $PFO(mn)_\psi = PFO(n^{-1}(nm)(n^{-1})^{-1})_\psi$.
Using Theorem 3.10, we get $PFO(n(nm)n^{-1})_\psi = PFO(nm)_\psi$. As m and n are random elements of C, hence $PFO(mn)_\psi = PFO(nm)_\psi$ for all $m, n \in G$. □

Theorem 3.12. *Assume $\psi = (\mu, \nu)$ is a PFSG on a commutative group C and m, n are two elements of C such that $\gcd(PFO(m)_\psi, PFO(n)_\psi) = 1$. If $\mu^2(mn) = \mu^2(e)$ and $\nu^2(mn) = \nu^2(e)$, then $PFO(m)_\psi = PFO(n)_\psi = 1$.*

Proof. Assume $PFO(m)_\psi = p$ and $PFO(n)_\psi = q$. So, we get $\gcd(p, q) = 1$. Now

$$\mu^2(m^q n^q) = \mu^2((mn)^q) \geq \mu^2(mn) = \mu^2(e).$$

As a result, the only option is $\mu^2(m^q n^q) = \mu^2(e)$. Also,

$$\begin{aligned}\mu^2(m^q) =&\mu^2(m^q n^q v^{-q})\\ \geq&\mu^2(m^q n^q) \wedge \mu^2((n^{-1})^q)\\ =&\mu^2(e) \wedge \mu^2(e)\\ =&\mu^2(e).\end{aligned}$$

So, we get $\mu^2(m^q) = \mu^2(e)$. Similarly, anyone can verify that $\nu^2(m^q) = \nu^2(e)$.

Using Theorem 3.15, we get $q|p$. Again $\gcd(p, q) = 1$, thus $q = 1$. Similarly, we can present that $p = 1$.

Hence, $PFO(m)_\psi = PFO(n)_\psi = 1$. □

Theorem 3.13. *Generators of a cyclic group have same PFO in a PFSG.*

Proof. Assume C is a cyclic group of order k.

Let m, n are any two generators of C. Then $m^k = e = n^k$.

As m is a generator, then $n = m^p$ for some $p \in \mathbb{Z}^+$. Therefore, k and p are co-prime, so $\gcd(k, p) = 1$. Thus, by Theorem 3.6, we get $PFO(m)_\psi = PFO(m^p)_\psi = PFO(n)_\psi$. For an infinite cyclic group it has only two generator. If m is a generator of C, then m^{-1} is the only other generator. Thus, by Theorem 3.4, we get $PFO(m)_\psi = PFO(m^{-1})_\psi$.

Hence, any generators of a cyclic group have same PFO in a PFSG. □

4. Some Algebraic Attributes of PFSG

The concepts of Pythagorean fuzzy normalizer (PFNL) and Pythagorean fuzzy centralizer (PFCL) are developed in this section. We also look into a number of algebraic properties of it.

Definition 4.1. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFNL of ψ is denoted by $\delta(\psi)$ and defined by $\delta(\psi) = \{m \mid m \in C, \mu^2(x) = \mu^2(mxm^{-1})$ and $\nu^2(m) = \nu^2(mxm^{-1})\}$ for all $x \in G$.

Example 4.1. Consider the group $C = (\mathbb{Z}, +)$.

Assume $\psi = (\mu, \nu)$ is a PFS on $\mathbb{Z}$, which is presented by

$$\mu(m) = \begin{cases} 0.87, & \text{where } m \in 2\mathbb{Z},\\ 0.62, & \text{elsewhere,}\end{cases}$$

$$\nu(m) = \begin{cases} 0.31, & \text{where } m \in 2\mathbb{Z},\\ 0.68, & \text{elsewhere.}\end{cases}$$

We can clearly verify that $\psi = (\mu, \nu)$ is a PFSG on $\mathbb{Z}$. Then the PFNL of ψ is $\delta(\psi) = \mathbb{Z}$.

Theorem 4.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a finite group C. Then the PFNL $\delta(\psi)$ forms a subgroup of C.*

Proof. Let $m, n \in \delta(\psi)$. Then

$$\mu^2(p) = \mu^2(mpm^{-1}), \quad \nu^2(p) = \nu^2(mpm^{-1}), \quad \text{for all } p \in C, \tag{4.1}$$

and

$$\mu^2(q) = \mu^2(nqn^{-1}), \quad \nu^2(q) = \nu^2(nqn^{-1}), \quad \text{for all } q \in C. \tag{4.2}$$

Clearly, $e \in \delta(\psi)$, so $\delta(\psi)$ is a non-empty finite subset of C.

To show $\delta(\psi)$ is a subgroup of C, we need to show $mn \in \delta(\psi)$. Put $p = nqn^{-1}$ in (4.1), we get

$$\mu^2(nqn^{-1}) = \mu^2(mnqn^{-1}m^{-1}) \quad \text{and} \quad \nu^2(nqn^{-1}) = \nu^2(mnqn^{-1}m^{-1}). \tag{4.3}$$

Then applying (4.2) in (4.3), we have $\mu^2(q) = \mu^2(mnqn^{-1}m^{-1})$ and $\nu^2(q) = \nu^2(mnqn^{-1}m^{-1})$.

This shows that $\mu^2(q) = \mu^2((mn)q(mn)^{-1})$ and $\nu^\beta(q) = \nu^\beta((mn)q(mn)^{-1})$. Therefore, $mn \in \delta(\psi)$. Hence, $\delta(\psi)$ forms a subgroup of C. □

Proposition 4.1. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then $\psi = (\mu, \nu)$ is a PFNSG of C if and only if $\delta(\psi) = C$.*

Proof. We have $\delta(\psi) = \{m \mid m \in C, \mu^2(p) = \mu^2(mpm^{-1})$ and $\nu^2(m) = \nu^2(mpm^{-1})$ for all $p \in C$. Therefore, $\delta(\psi) \subseteq C$.

Assume $\psi = (\mu, \nu)$ is a PFNSG on C. Then we get $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$ for all $m, n \in C$.

This presents that $C \subseteq \delta(\psi)$. Hence, $\delta(\psi) = C$.

Conversely, let $\delta(\psi) = C$. Then $\mu^2(m) = \mu^2(nmn^{-1})$ and $\nu^2(m) = \nu^2(nmn^{-1})$ for all $m, n \in C$. Hence, $\psi = (\mu, \nu)$ forms a PFNSG on C. □

Theorem 4.2. *Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then ψ forms a PFNSG on the group $\delta(\psi)$.*

Proof. Let $m, n \in \delta(\psi)$. Then $\mu^2(w) = \mu^2(mwm^{-1})$ and $\nu^2(w) = \nu^2(mwm^{-1})$ for all $w \in C$. As $\delta(\psi)$ forms a subgroup of C, then $nm \in \delta(\psi)$. Putting $w = nm$ in above relation we have $\mu^2(nm) = \mu^2(mnmm^{-1})$ and $\nu^2(nm) = \nu^2(mnmm^{-1})$. This presents that $\mu^2(nm) = \mu^2(mn)$ and $\nu^2(nm) = \nu^2(mn)$. Hence, ψ forms a PFNSG on the group $\delta(\psi)$. □

Definition 4.2. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then PFCL of ψ is denoted by $\omega(\psi)$ and defined by $\omega(\psi) = \{m \mid m \in C, \mu^2(mn) = \mu^2(nm)$ and $\nu^2(mn) = \nu^2(nm)\}$ for all $n \in C$.

Example 4.2. From Example 3.1, consider the PFSG ψ on the group $\mathbb{Z}_4$. Then the PFCL of ψ is $\omega(\psi) = \mathbb{Z}_4$.

Theorem 4.3. *The PFCL of a PFSG on a group forms a subgroup of the group.*

Proof. Assume $\psi = (\mu, \nu)$ is a PFSG on a group C. Then the PFCL of ψ is presented by $\omega(\psi) = \{m \mid m \in C,\ \mu^2(mn) = \mu^2(nm)$ and $\nu^2(mn) = \nu^2(nm)\}$ for all $n \in C$.

Let $s, t \in \omega(\psi)$. Then for all $r \in C$, we get

$$\mu^2((st)r) = \mu^2(s(tr)) = \mu^2((tr)s) = \mu^2(t(rs)) = \mu^2((rs)t) = \mu^2(r(st)).$$

Thus, $\mu^2((st)r) = \mu^2(r(st))$ for all $r \in C$.

Similarly, we get $\nu^2((st)r) = \nu^2(r(st))$ for all $r \in C$. This presents that $st \in \omega(\psi)$. Also, for all $g \in C$, we get

$$\mu^2(s^{-1}g) = \mu^2((g^{-1}s)^{-1}) = \mu^2(g^{-1}s) = \mu^2(sg^{-1}) = \mu^2((gs^{-1})^{-1}) = \mu^2(gs^{-1}).$$

Thus, $\mu^2(s^{-1}g) = \mu^2(gs^{-1})$ for all $g \in C$.

Similarly, we get $\nu^2(s^{-1}g) = \nu^2(gs^{-1})$ for all $g \in C$. This presents that for $s \in \omega(\psi)$, we have $s^{-1} \in \omega(\psi)$. Hence, $\omega(\psi)$ forms a subgroup of C. □

5. Lagrange's Theorem in PFSG

This section revolves around the development of theories for Lagrange's theorem fuzzification in PFSG.

Theorem 5.1. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C and Λ is the set of all PFCs of ψ on C. Then Λ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$.*

Proof. To prove $(\Lambda, \circ)$ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$, we need to verify that $\circ$ is well defined.

Let $m, n, p, q \in C$ such that $m\psi = p\psi$ and $n\psi = q\psi$.

Therefore, $m\mu(x) = p\mu(x)$, $m\nu(x) = p\nu(x)$ and $n\mu(x) = q\mu(x)$, $n\nu(x) = q\nu(x)$ for all $x \in C$. This presents that for all $x \in C$

$$\mu^2(m^{-1}x) = \mu^2(p^{-1}x), \quad \nu^2(m^{-1}x) = \nu^2(p^{-1}x) \tag{5.1}$$

and

$$\mu^2(n^{-1}x) = \mu^2(q^{-1}x), \quad \nu^2(n^{-1}x) = \nu^2(q^{-1}x). \tag{5.2}$$

We need to verify that $m\psi \circ n\psi = p\psi \circ q\psi$. So, $(mn)\psi = (pq)\psi$. We get $(mn)\mu(x) = \mu^2(n^{-1}m^{-1}x)$ and $(pq)\mu(x) = \mu^2(q^{-1}p^{-1}x)$ for all $x \in C$. Then

$$\begin{aligned}\mu^2(n^{-1}m^{-1}x) =& \mu^2(n^{-1}m^{-1}pp^{-1}x)\\ =&\mu^2(n^{-1}m^{-1}pqq^{-1}p^{-1}x)\\ \geq&\mu^2(n^{-1}m^{-1}pq) \wedge \mu^2(q^{-1}p^{-1}x).\end{aligned}$$

So,

$$\mu^2(n^{-1}m^{-1}x) \geq \mu^2(n^{-1}m^{-1}pq) \wedge \mu^2(q^{-1}p^{-1}x). \tag{5.3}$$

Replace x with $m^{-1}pq$ in (5.2), then

$$\mu^2(n^{-1}m^{-1}pq) = \mu^2(q^{-1}m^{-1}pq).$$

As $\psi = (\mu, \nu)$ is a PFNSG on C, then $\mu^2(q^{-1}m^{-1}pq) = \mu^2(m^{-1}p)$. Replace x with p in (5.1), we get

$$\mu^2(m^{-1}p) = \mu^2(p^{-1}p) = \mu^2(e).$$

Consequently, $\mu^2(n^{-1}m^{-1}pq) = \mu^2(e)$.

From (5.3), we get $\mu^2(n^{-1}m^{-1}x) \geq \mu^2(q^{-1}p^{-1}x)$.

Similarly, $\mu^2(q^{-1}p^{-1}x) \geq \mu^2(n^{-1}m^{-1}x)$. Therefore, $\mu^2(n^{-1}m^{-1}x) = \mu^2(q^{-1}p^{-1}x)$, for all $x \in C$. Also, we can verify that $\nu^2(n^{-1}m^{-1}x) = \nu^2(q^{-1}p^{-1}x)$ for all $x \in C$. This presents that $(mn)\mu(x) = (pq)\mu(x)$ and $(mn)\nu(x) = (pq)\nu(x)$ for all $x \in C$. Consequently, $(mn)\psi = (pq)\psi$. Hence, $\circ$ is well defined on Λ. Clearly, Λ's identity element is $e\psi$. Also, $m^{-1}\psi \in \Lambda$ is the inverse of $m\psi$ in Λ. That is $(m\psi) \circ (m^{-1}\psi) = e\psi$. Therefore, $(\Lambda, \circ)$ constructs a group with the composition $m\psi \circ n\psi = (mn)\psi$ for all $m, n \in C$. □

Definition 5.1. The index of ψ is denoted by $[C : \psi]$ and defined by $[C : \psi] = O(\Lambda)$.

Example 5.1. Consider the group $C = (\mathbb{Z}_4, +_4)$. From Example 3.1, take the PFSG ψ on $\mathbb{Z}_4$. We can clearly show that ψ is a PFNSG on the group $C = (\mathbb{Z}_4, +_4)$. Then the set of all PFCs of ψ is $\Lambda = \{0\psi, 1\psi, 2\psi, 3\psi\}$.

Now $(1\mu)^2(1) = \mu^2(1^{-1} +_4 1) = \mu^2(3 +_4 1) = \mu^2(0) = 0.9025$, $(2\mu)^2(1) = \mu^2(2^{-1} +_4 1) = \mu^2(2 +_4 1) = \mu^2(3) = 0.7225$ and $(3\mu)^2(1) = \mu^2(3^{-1} +_4 1) = \mu^\alpha(1 +_4 1) = \mu^2(2) = 0.4225$.

Thus, $(1\mu)^2(1) \neq (2\mu)^2(1) \neq (3\mu)^2(1)$. This presents that $1\psi \neq 2\psi \neq 3\psi$. Therefore, the index of ψ is $[C : \psi] = O(\Lambda) = 4$.

Theorem 5.2. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C. Then a PFS $\Psi = (\mu_*, \nu_*)$ on Λ defined by $\mu_*(m\mu) = \mu(m)$ and $\nu_*(m\nu) = \nu(m)$ constructs a PFSG on $(\Lambda, \circ)$ for all $m \in C$.*

Proof. Let $m\psi, n\psi \in \Lambda$, where $m, n \in C$. Then

$$\begin{aligned} \mu_*^2((m\mu) \circ (n\mu)) =& \mu_*^2((mn)\mu) = \mu^2(mn) \\ \geq& \mu^2(m) \wedge \mu^2(n) \\ =& \mu_*^2(m\mu) \wedge \mu_*^2(n\mu). \end{aligned}$$

Therefore, $\mu_*^2((m\mu) \circ (n\mu)) \geq \mu_*^2(m\mu) \wedge \mu_*^2(n\mu)$.

Similarly, we get $\nu_*^2((m\nu) \circ (n\nu)) \leq \nu_*^2(m\nu) \vee \nu_*^2(n\nu)$. Also, $\mu_*^2(m^{-1}\mu) = \mu^2(m^{-1}) = \mu^2(m) = \mu_*^2(m\mu)$. Similarly, $\nu_*^2(m^{-1}\nu) = \nu_*^2(m\nu)$. Therefore, $\Psi = (\mu_*, \nu_*)$ constructs a PFSG on $(\Lambda, \circ)$. □

Definition 5.2. The PFSG $\Psi = (\mu_*, \nu_*)$ on the group $(\Lambda, \circ)$ is referred to as Pythagorean fuzzy quotient group (PFQG) on ψ.

Example 5.2. From Example 5.1, consider the PFNSG ψ on the group $(\mathbb{Z}_4, +_4)$. Then the set of all PFCs of ψ is $\Lambda = \{0\psi, 1\psi, 2\psi, 3\psi\}$. We create a PFS $\Psi = (\mu_*, \nu_*)$ on Λ by $\mu_*(m\mu) = \mu(m)$ and $\nu_((m\nu) = \nu(m)$.

Then $\mu_*(0\mu) = \mu(0) = 0.95$, $\mu_*(1\mu) = \mu(1) = 0.65$, $\mu_*(2\mu) = \mu(2) = 0.65$, $\mu_*(3\mu) = \mu(3) = 0.85$ and $\nu_*(0\nu) = \nu(0) = 0.25$, $\nu_*(1\nu) = \nu(1) = 0.75$, $\nu_*(2\nu) = \nu(2) = 0.75$, $\nu_*(3\nu) = \nu(3) = 0.45$.

We can clearly verify that $\Psi = (\mu_*, \nu_*)$ is a PFSG on Λ. Hence, $\Psi = (\mu_*, \nu_*)$ is the PFQG on ψ.

Theorem 5.3. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C and constructs a function $\kappa : C \to \Lambda$ by $\kappa(m) = m\psi$ for all $m \in C$. Then κ forms a group homomorphism with kernel* $\ker(\kappa) = \{m \in C \mid \mu^2(m) = \mu^2(e),\ \nu^2(m) = \nu^2(e)\}$.

Proof. Let $m, n \in C$.

Then $\kappa(mn) = (mn)\psi = (m\psi) \circ (m\psi) = \kappa(m) \circ \kappa(n)$. This presents that $\kappa : C \to \Lambda$ forms a group homomorphism. The kernel of κ is presented by

$$\begin{aligned}
\ker(\kappa) =& \{m \in C \mid \kappa(m) = e\psi\} \\
=& \{m \in C \mid m\psi = e\psi\} \\
=& \{m \in C \mid m\psi(n) = e\psi(n) \text{ for all } n \in C\} \\
=& \{m \in C \mid m\mu(n) = e\mu(n), m\nu(n) = e\nu(n) \text{ for all } n \in C\} \\
=& \{m \in C \mid \mu^2(m^{-1}n) = \mu^2(n), \nu^2(m^{-1}n) = \nu^2(n) \text{ for all } n \in C\} \\
=& \{m \in C \mid \mu^2(m) = \mu^2(e), \nu^2(m) = \nu^2(e)\}.
\end{aligned}$$

Hence, $\ker(\kappa) = \{m \in C \mid \mu^2(m) = \mu^2(e), \nu^2(m) = \nu^2(e)\}$. □

Remark 5.1. $\ker(\kappa)$ forms a subgroup of C.

Theorem 5.4. *Assume $\psi = (\mu, \nu)$ is a PFNSG on a finite group C. Then $[C : \psi]$ divides $O(C)$.*

Proof. $\Lambda = \{m\psi \mid m \in C\}$, the set of all PFC of ψ on C is finite as C is finite.

Theorem 5.3 proves that $\kappa : C \to \Lambda$ defined by $\kappa(m) = m\psi$ for all $m \in C$ is a group homomorphism.

We define $H = \{m \in C \mid m\psi = e\psi\} = \ker(\kappa)$, which is a subgroup of C. C is now decomposed as union of left cosets modulo p by

$$C = m_1H \cup m_2H \cup m_3H \cup \cdots \cup m_pH,$$

where $m_pH = H$.

We need to verify that there exists a one-one relation between Λ's elements and cosets m_iH of C.

We consider an element $p \in H$ and coset m_iH of C. Then we get $\kappa(m_ip) = m_ip\psi = (m_i\psi) \circ (p\psi) = (m_i\psi) \circ (e\psi) = (m_i\psi)$. This represents that κ maps elements of m_iH to $m_i\psi$.

We now create a mapping $\underline{\kappa}$ between $\{m_iH \mid 1 \le i \le p\}$ and Λ by $\underline{\kappa}(m_iH) = m_i\psi$. Let $m_a\psi = m_b\psi$. Then $m_b^{-1}m_a\psi = e\psi$. Therefore, $m_b^{-1}m_a \in H$. This presents that $m_aH = m_bH$.

Hence, $\underline{\kappa}(m_iH) = m_i\psi$ is a one-one map. As a result, we can establish that the number of distinct cosets is same as Λ's cardinality. That is $[C : H] = [C : \psi]$.

Since $[C : H]$ divides $O(C)$, then $[C : \psi]$ must divide $O(C)$. □

6. Conclusion

Various fuzzy algebraic structures have significant importance in decision making problems. This paper explores the study of PFSGs. We have arranged the sections of this paper in such a way that we can approach Lagrange's theorem at the end. In this paper, we have defined Pythagorean fuzzy semi-level subgroups. We have introduced the notion of PFO of an element in PFSG and discussed various algebraic properties of it. Further, We have developed the concept of PFNL and PFCL of a PFSG. Moreover, we have introduced PFQG and the index of a PFSG. Finally, we have presented Lagrange's theorem in the form of PFS. In future work, we will implement direct product of groups and group actions in PFSG.

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