

SUFFICIENT CONDITIONS OF SUBCLASSES OF SPIRAL-LIKE FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTIONS

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ABSTRACT. The purpose of the present paper is to find the sufficient conditions for some subclasses of analytic functions associated with Mittag-Leffler functions to be in subclasses of spiral-like univalent functions. Further, we discuss geometric properties of an integral operator related to Mittag-Leffler functions.

1. INTRODUCTION AND DEFINITIONS

Let \mathbf{E}_α be the function defined by

$$\mathbf{E}_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \alpha \in \mathbb{C}, \text{ with } \operatorname{Re} \alpha > 0,$$

that was introduced by Mittag-Leffler [14] and commonly known as the *Mittag-Leffler function*. Wiman [25] defined a more general function $\mathbf{E}_{\alpha,\beta}$ generalizing the \mathbf{E}_α Mittag-Leffler function, that is

$$\mathbf{E}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \alpha, \beta \in \mathbb{C}, \text{ with } \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0.$$

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When $\beta = 1$, it is abbreviated as $\mathbf{E}_\alpha(z) = \mathbf{E}_{\alpha,1}(z)$. Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\begin{aligned}\mathbf{E}_{1,1}(z) &= e^z, & \mathbf{E}_{1,2}(z) &= \frac{e^z - 1}{z}, & \mathbf{E}_{2,1}(z^2) &= \cosh z, \\ \mathbf{E}_{2,1}(-z^2) &= \cos z, & \mathbf{E}_{2,2}(z^2) &= \frac{\sinh z}{z}, & \mathbf{E}_{2,2}(-z^2) &= \frac{\sin z}{z}, \\ \mathbf{E}_4(z) &= \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4}), & \mathbf{E}_3(z) &= \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right].\end{aligned}$$

We recall the error function erf given by [1, p. 297]

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

the complement of the error function erfc defined by

$$\operatorname{erfc}(z) := 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

and the normalized form of the error function erf denoted by Erf (normalized with the condition $\operatorname{Erf}'(0) = 1$) is given by

$$\operatorname{Erf}(z) := \frac{\sqrt{\pi z}}{2} \operatorname{erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n.$$

It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1} \right).$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found for example in [2, 3, 8, 9, 11, 12]. We note that the above generalized (Mittag-Leffler) function $\mathbf{E}_{\alpha,\beta}$ does not belong to the family \mathcal{A} , where \mathcal{A} represents the class of functions whose members are of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let \mathcal{S} be the subclass of \mathcal{A} whose members

are univalent in \mathbb{D} . Thus, it is expected to define the following normalization of Mittag-Leffler functions as below, due to Bansal and Prajapat [3]:

$$(1.2) \quad E_{\alpha,\beta}(z) := z\Gamma(\beta) \mathbf{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} z^n,$$

that holds for the parameters $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0$ and $z \in \mathbb{C}$. In this paper we shall confine our attention to the case of real-valued parameters α and β , and we will consider that $z \in \mathbb{D}$.

For functions $f \in \mathcal{A}$ be given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$, we define the *Hadamard product (or convolution)* of f and g by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

The two well known subclasses of \mathcal{S} are namely the class of starlike and convex functions (for details see Robertson [20]). Thus, a function $f \in \mathcal{A}$ given by (1.1) is said to be *starlike of order γ* , $0 \leq \gamma < 1$, if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \gamma, \quad z \in \mathbb{D},$$

and this function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where \mathcal{S}^* denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be *convex of order γ* , $0 \leq \gamma < 1$, if and only if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \gamma, \quad z \in \mathbb{D},$$

and this class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} := \mathcal{K}(0)$ represents the well-known standard class of convex functions. By Alexander’s duality relation (see [6]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

A function $f \in \mathcal{A}$ is said to be *spiral-like* if

$$\operatorname{Re} \left(e^{-i\xi} \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D},$$

for some $\xi \in \mathbb{C}$ with $|\xi| < \frac{\pi}{2}$, and the class of spiral-like functions was introduced in [23]. Also, the function f is said to be *convex spiral-like* if $zf'(z)$ is spiral-like. Due to Murugusundramoorthy [15, 16], we consider the following subclasses of spiral-like functions as below.

Definition 1.1. For $0 \leq \rho < 1, 0 \leq \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{S}(\xi, \gamma, \rho)$ by

$$\mathcal{S}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(e^{i\xi} \frac{zf'(z)}{(1-\rho)f(z) + \rho zf'(z)} \right) > \gamma \cos \xi, \quad z \in \mathbb{D} \right\}.$$

By virtue of Alexander’s relation (see [6]) we define the following subclass $\mathcal{K}(\xi, \gamma, \rho)$.

Definition 1.2. For $0 \leq \rho < 1$, $0 \leq \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{K}(\xi, \gamma, \rho)$ by

$$\mathcal{K}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(e^{i\xi} \frac{zf''(z) + f'(z)}{f'(z) + \rho zf''(z)} \right) > \gamma \cos \xi, z \in \mathbb{D} \right\}.$$

By specializing the parameter $\rho = 0$ in the above two definitions we obtain the subclasses $\mathcal{S}(\xi, \gamma) := \mathcal{S}(\xi, \gamma, 0)$ and $\mathcal{K}(\xi, \gamma) := \mathcal{K}(\xi, \gamma, 0)$, respectively.

Now we state a sufficient conditions for the function f to be in the above classes.

Lemma 1.1 ([15, 16]). *A function f given by (1.1) is a member of $\mathcal{S}(\xi, \gamma, \rho)$ if*

$$\sum_{n=2}^{\infty} \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] |a_n| \leq 1 - \gamma,$$

where $|\xi| < \frac{\pi}{2}$, $0 \leq \rho < 1$, $0 \leq \gamma < 1$.

Since $f \in \mathcal{K}(\xi, \gamma, \rho)$ if and only if $zf'(z) \in \mathcal{S}(\xi, \gamma, \rho)$, and from Lemma 1.1 we get the next result.

Lemma 1.2. *A function f given by (1.1) is a member of $\mathcal{K}(\xi, \gamma, \rho)$ if*

$$\sum_{n=2}^{\infty} n \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] |a_n| \leq 1 - \gamma,$$

where $|\xi| < \frac{\pi}{2}$, $0 \leq \rho < 1$, $0 \leq \gamma < 1$.

The next class $\mathcal{R}^\tau(\vartheta, \delta)$ was introduced earlier by Swaminathan [24], and for special cases see the references cited there in.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(\vartheta, \delta)$, where $\tau \in \mathbb{C} \setminus \{0\}$, $0 < \vartheta \leq 1$, and $\delta < 1$, if it satisfies the inequality

$$\left| \frac{(1 - \vartheta) \frac{f(z)}{z} + \vartheta f'(z) - 1}{2\tau(1 - \delta) + (1 - \vartheta) \frac{f(z)}{z} + \vartheta f'(z) - 1} \right| < 1, \quad z \in \mathbb{D}.$$

Lemma 1.3 ([24]). *If $f \in \mathcal{R}^\tau(\vartheta, \delta)$ is of the form (1.1), then*

$$(1.3) \quad |a_n| \leq \frac{2|\tau|(1 - \delta)}{1 + \vartheta(n - 1)}, \quad n \in \mathbb{N} \setminus \{1\}.$$

The bounds given in (1.3) is sharp for

$$f(z) = \frac{1}{\vartheta z^{1-\frac{1}{\vartheta}}} \int_0^z t^{1-\frac{1}{\vartheta}} \left[1 + \frac{2(1 - \delta)\tau t^{n-1}}{1 - 2^{n-1}} \right] dt.$$

Now we define the following new linear operator based on convolution (Hadamard) product.

For real parameters α, β , with $\alpha, \beta, \notin \{0, -1, -2, \dots\}$ and $E_{\alpha, \beta}$ be given by (1.2), we define the linear operator $\Lambda_\beta^\alpha : \mathcal{A} \rightarrow \mathcal{A}$ with the aid of the convolution product

$$\Lambda_\beta^\alpha f(z) := f(z) * E_{\alpha, \beta}(z) = z + \sum_{n=2}^\infty \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} a_n z^n, \quad z \in \mathbb{D}.$$

Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example [5, 10, 13, 21, 22, 24]) and by the recent investigations related with distribution series (see for example [4, 7, 17–19]), we obtain sufficient condition for the function $E_{\alpha, \beta}$ to be in the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$, and information regarding the images of functions belonging in $\mathcal{R}^\tau(\vartheta, \delta)$ by using the convolution operator Λ_β^α . Finally, we determined conditions for the integral operator $\Psi_\beta^\alpha(z) = \int_0^z \frac{E_{\alpha, \beta}(t)}{t} dt$ to belong to the above classes.

2. INCLUSION RESULTS

In order to prove our main results, unless otherwise stated throughout this paper, we will use the notation (1.2), therefore

$$(2.1) \quad E_{\alpha, \beta}(1) - 1 = \sum_{n=2}^\infty \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)},$$

$$(2.2) \quad E'_{\alpha, \beta}(1) - 1 = \sum_{n=2}^\infty \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)},$$

$$(2.3) \quad E''_{\alpha, \beta}(1) = \sum_{n=2}^\infty \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)}.$$

Theorem 2.1. *If*

$$(2.4) \quad [(1 - \rho) \sec \xi + \rho(1 - \gamma)] E'_{\alpha, \beta}(1) + (1 - \rho)(1 - \gamma - \sec \xi) E_{\alpha, \beta}(1) \leq 2(1 - \gamma),$$

then $E_{\alpha, \beta} \in \mathcal{S}(\xi, \gamma, \rho)$.

Proof. Since $E_{\alpha, \beta}$ is defined by (1.2), according to Lemma 1.1 it is sufficient to show that

$$(2.5) \quad \sum_{n=2}^\infty \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \leq 1 - \gamma.$$

Since the left-hand side of the inequality (2.5) could be written as

$$\begin{aligned} Q_1(\xi, \gamma, \rho) &:= \sum_{n=2}^\infty [(1 - \rho) \sec \xi(n - 1) + (1 - \gamma)(1 + n\rho - \rho)] \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &= [(1 - \rho) \sec \xi + \rho(1 - \gamma)] \sum_{n=2}^\infty \frac{n\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &\quad + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=2}^\infty \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)}, \end{aligned}$$

therefore, by using (2.1) and (2.2), we get

$$\begin{aligned} Q_1(\xi, \gamma, \rho) &= \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \left[E'_{\alpha, \beta}(1) - 1 \right] \\ &\quad + (1 - \rho)(1 - \gamma - \sec \xi) \left[E_{\alpha, \beta}(1) - 1 \right] \\ &= \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] E'_{\alpha, \beta}(1) + (1 - \rho)(1 - \gamma - \sec \xi) E_{\alpha, \beta}(1) \\ &\quad - (1 - \gamma). \end{aligned}$$

Thus, from the assumption (2.4) it follows that $Q_1(\xi, \gamma, \rho) \leq 1 - \gamma$, that is (2.5) holds, therefore $E_{\alpha, \beta} \in \mathcal{S}(\xi, \gamma, \rho)$. □

Theorem 2.2. *If*

$$(2.6) \quad \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] E''_{\alpha, \beta}(1) + (1 - \gamma) E'_{\alpha, \beta}(1) \leq 2(1 - \gamma),$$

then $E_{\alpha, \beta} \in \mathcal{K}(\xi, \gamma, \rho)$.

Proof. Using the definition (1.2) of $E_{\alpha, \beta}$, in view of Lemma 1.2 it is sufficient to prove that

$$(2.7) \quad \sum_{n=2}^{\infty} n \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \leq 1 - \gamma.$$

The left-hand side of the inequality (2.7) could be written as

$$\begin{aligned} Q_2(\xi, \gamma, \rho) &:= \sum_{n=2}^{\infty} n \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &= \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=2}^{\infty} \frac{n(n - 1)\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &\quad + (1 - \gamma) \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)}, \end{aligned}$$

and from (2.2) and (2.3) we get

$$Q_2(\xi, \gamma, \rho) = \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] E''_{\alpha, \beta}(1) + (1 - \gamma) \left[E'_{\alpha, \beta}(1) - 1 \right].$$

Hence, the assumption (2.6) implies that $Q_2(\xi, \gamma, \rho) \leq 1 - \gamma$ that is (2.7) holds, and consequently $E_{\alpha, \beta} \in \mathcal{K}(\xi, \gamma, \rho)$. □

3. IMAGE PROPERTIES OF Λ_{β}^{α} OPERATOR

Making use of the Lemma 1.1 and Lemma 1.3 we will focus the influence of the Λ_{β}^{α} operator for the functions of the class $\mathcal{R}^{\tau}(\vartheta, \delta)$, and we will give sufficient conditions such that these images are in the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$, respectively.

Theorem 3.1. *If*

$$(3.1) \quad \frac{2|\tau|(1-\delta)}{\vartheta} \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] [E_{\alpha,\beta}(1) - 1] + (1-\rho)(1-\gamma - \sec\xi) \int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t} - 1 \right) dt \leq 1 - \gamma,$$

then

$$\Lambda_\beta^\alpha(\mathcal{R}^\tau(\vartheta, \delta)) \subset \mathcal{S}(\xi, \gamma, \rho).$$

Proof. Let $f \in \mathcal{R}^\tau(\vartheta, \delta)$ be of the form (1.1). To prove that $\Lambda_\beta^\alpha(f) \in \mathcal{S}(\xi, \gamma, \rho)$, in view of Lemma 1.1 it is required to show that

$$\sum_{n=2}^\infty \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} |a_n| \leq 1 - \gamma.$$

Let us denote the left-hand side of the above inequality by

$$Q_3(\xi, \gamma, \rho) := \sum_{n=2}^\infty \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} |a_n|.$$

Since $f \in \mathcal{R}^\tau(\vartheta, \delta)$, by Lemma 1.3 we have

$$|a_n| \leq \frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\},$$

and using the inequality $1 + \vartheta(n-1) \geq \vartheta n$ we obtain that

$$\begin{aligned} Q_3(\xi, \gamma, \rho) &\leq \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \sum_{n=2}^\infty \frac{1}{n} \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \right] \right. \\ &\quad \left. \times \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right\} \\ &= \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \sum_{n=2}^\infty \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right. \\ &\quad \left. + (1-\rho)(1-\gamma - \sec\xi) \sum_{n=2}^\infty \frac{1}{n} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right\}. \end{aligned}$$

From the above inequality, using (2.1), we get

$$\begin{aligned} Q_3(\xi, \gamma, \rho) &\leq \frac{2|\tau|(1-\delta)}{\vartheta} \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] [E_{\alpha,\beta} - 1] \\ &\quad + (1-\rho)(1-\gamma - \sec\xi) \int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t} - 1 \right) dt, \end{aligned}$$

hence, the assumption (3.1) implies then $Q_3(\xi, \gamma, \rho) \leq 1 - \gamma$, that is $\Lambda_\beta^\alpha(f) \in \mathcal{S}(\xi, \gamma, \rho)$. □

Using Lemma 1.2 and following the same procedure as in the proof of Theorem 2.2, we have the subsequent result.

Theorem 3.2. *If*

$$(3.2) \quad \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma - \sec\xi)E_{\alpha,\beta}(1) - (1-\gamma) \right\} \leq 1-\gamma,$$

then

$$\Lambda_{\beta}^{\alpha}(\mathcal{R}^{\tau}(\vartheta, \delta)) \subset \mathcal{K}(\xi, \gamma, \rho).$$

Proof. Let $f \in \mathcal{R}^{\tau}(\vartheta, \delta)$ be of the form (1.1). In view of Lemma 1.2, to prove that $\Lambda_{\beta}^{\alpha}(f) \in \mathcal{K}(\xi, \gamma, \rho)$ we have to show that

$$(3.3) \quad \sum_{n=2}^{\infty} n \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} |a_n| \leq 1-\gamma.$$

Since $f \in \mathcal{R}^{\tau}(\vartheta, \delta)$, then by Lemma 1.3 we have

$$|a_n| \leq \frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\},$$

and $1+\vartheta(n-1) \geq \vartheta n$. Denoting the left-hand side of the inequality (3.3) by

$$Q_4(\xi, \gamma, \rho) := \sum_{n=2}^{\infty} n \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} |a_n|,$$

we deduce that

$$\begin{aligned} Q_4(\xi, \gamma, \rho) &\leq \frac{2|\tau|(1-\delta)}{\vartheta} \sum_{n=2}^{\infty} \left[(1-\rho)\sec\xi(n-1) + (1-\gamma)(1+n\rho-\rho) \right] \\ &\quad \times \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &= \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right. \\ &\quad \left. + (1-\rho)(1-\gamma - \sec\xi) \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right\}. \end{aligned}$$

Now, by using (2.1) and (2.2), the above inequality yields to

$$\begin{aligned} Q_4(\xi, \gamma, \rho) &\leq \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] [E'_{\alpha,\beta}(1) - 1] \right. \\ &\quad \left. + (1-\rho)(1-\gamma - \sec\xi)[E_{\alpha,\beta}(1) - 1] \right\} \\ &= \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \left[(1-\rho)\sec\xi + \rho(1-\gamma) \right] E'_{\alpha,\beta}(1) \right. \\ &\quad \left. + (1-\rho)(1-\gamma - \sec\xi)E_{\alpha,\beta}(1) - (1-\gamma) \right\}. \end{aligned}$$

Therefore, the assumption (3.2) yields to $Q_4(\xi, \gamma, \rho) \leq 1 - \gamma$, which implies the inequality (3.3), that is $\Lambda_\beta^\alpha(f) \in \mathcal{K}(\xi, \gamma, \rho)$. \square

4. THE ALEXANDER INTEGRAL OPERATOR FOR $E_{\alpha,\beta}$

Theorem 4.1. *Let the function Ψ_β^α be given by*

$$(4.1) \quad \Psi_\beta^\alpha(z) = \int_0^z \frac{E_{\alpha,\beta}(t)}{t} dt, \quad z \in \mathbb{D}.$$

If

$$\left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] E'_{\alpha,\beta}(1) + (1 - \rho)(1 - \gamma - \sec \xi) E_{\alpha,\beta}(1) \leq 2(1 - \gamma),$$

then $\Psi_\beta^\alpha \in \mathcal{K}(\xi, \gamma, \rho)$.

Proof. Since

$$(4.2) \quad \Psi_\beta^\alpha(z) = z + \sum_{n=2}^\infty \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \cdot \frac{z^n}{n}, \quad z \in \mathbb{D},$$

according to Lemma 1.2, it is sufficient to prove that

$$\sum_{n=2}^\infty n \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{1}{n} \cdot \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \leq 1 - \gamma,$$

or, equivalently

$$\sum_{n=2}^\infty \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \leq 1 - \gamma.$$

Now, the proof of Theorem 4.1 is parallel to that of Theorem 2.1, and so it will be omitted. \square

Theorem 4.2. *Let the function Ψ_β^α be given by (4.1). If*

$$(4.3) \quad \begin{aligned} & \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] (E_{\alpha,\beta}(1) - 1) \\ & + (1 - \rho)(1 - \gamma - \sec \xi) \int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t} - 1 \right) dt \leq 1 - \gamma, \end{aligned}$$

then $\Psi_\beta^\alpha \in \mathcal{S}(\xi, \gamma, \rho)$.

Proof. Since Ψ_β^α has the power series expansion (4.2), then by Lemma 1.1 it is sufficient to prove that

$$\sum_{n=2}^\infty \frac{1}{n} \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \leq 1 - \gamma.$$

The left-hand side of the above inequality could be rewritten as

$$\begin{aligned} Q_5(\xi, \gamma, \rho) &= \sum_{n=2}^{\infty} \frac{1}{n} \left[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \\ &\quad \times \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &= \sum_{n=2}^{\infty} \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} \\ &\quad + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=2}^{\infty} \frac{1}{n} \cdot \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)}, \end{aligned}$$

and using (2.1) we get

$$\begin{aligned} Q_5(\xi, \gamma, \rho) &\leq \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] [E_{\alpha, \beta}(1) - 1] \\ &\quad + (1 - \rho)(1 - \gamma - \sec \xi) \int_0^1 \left(\frac{E_{\alpha, \beta}(t)}{t} - 1 \right) dt. \end{aligned}$$

Therefore, if the assumption (4.3) holds, then $Q_5(\xi, \gamma, \rho) \leq 1 - \gamma$. Hence, $\Psi_\beta^\alpha \in \mathcal{S}(\xi, \gamma, \rho)$. □

Remark 4.1. By taking $\rho = 0$ in Theorems 2.1–4.2, we can easily attain the sufficient condition for $E_{\alpha, \beta} \in \mathcal{S}(\xi, \gamma)$ and $E_{\alpha, \beta} \in \mathcal{K}(\xi, \gamma)$. The function $E_{\alpha, \beta}$ is associated with Mittag-Leffler functions and has not been studied sofar. We left this as an exercise to interested readers.

For the special case $\alpha = 1/2$ and $\beta = 1$, that is connected with the error function can derive some results based on the error function. Thus, a simple computation shows that if

$$\mathcal{E}(z) := E_{\frac{1}{2}, 1}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

then

$$\begin{aligned} \mathcal{E}(1) &= \sum_{n=1}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)}, \quad \mathcal{E}'(1) = \sum_{n=1}^{\infty} \frac{n}{\Gamma\left(\frac{n+1}{2}\right)}, \quad \mathcal{E}''(1) = \sum_{n=2}^{\infty} \frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}, \\ \int_0^1 \left(\frac{\mathcal{E}(t)}{t} - 1 \right) dt &= \sum_{n=2}^{\infty} \frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}, \end{aligned}$$

$$(4.4) \quad \mathcal{L} := \Lambda_1^{1/2} f(z) = f(z) * \mathcal{E}(z) = z + \sum_{n=2}^{\infty} \frac{a_n z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

$$(4.5) \quad \mathcal{P} := \Psi_1^{1/2}(z) = \int_0^z \frac{\mathcal{E}(t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n\Gamma\left(\frac{n+1}{2}\right)}.$$

Using the above relations, from Theorems 2.1 and 2.2 we get, respectively.

Example 4.1. If

$$\begin{aligned} & [(1 - \rho) \sec \xi + \rho(1 - \gamma)] \sum_{n=1}^{\infty} \frac{n}{\Gamma\left(\frac{n+1}{2}\right)} + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=1}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \\ & \leq 2(1 - \gamma), \end{aligned}$$

then $\mathcal{E} \in \mathcal{S}(\xi, \gamma, \rho)$.

Example 4.2. If

$$\left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=2}^{\infty} \frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)} + (1 - \gamma) \sum_{n=1}^{\infty} \frac{n}{\Gamma\left(\frac{n+1}{2}\right)} \leq 2(1 - \gamma),$$

then $\mathcal{E} \in \mathcal{K}(\xi, \gamma, \rho)$.

Similarly, Theorems 4.1 and 4.2 give us the next examples.

Example 4.3. If

$$\begin{aligned} & \frac{2|\tau|(1-\delta)}{\vartheta} \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=2}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \\ & + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=2}^{\infty} \frac{1}{n\Gamma\left(\frac{n+1}{2}\right)} \leq 1 - \gamma, \end{aligned}$$

then

$$\mathcal{L}(\mathcal{R}^\tau(\vartheta, \delta)) \subset \mathcal{S}(\xi, \gamma, \rho),$$

where \mathcal{L} is defined by (4.4).

Example 4.4. If

$$\begin{aligned} & \frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=1}^{\infty} \frac{n}{\Gamma\left(\frac{n+1}{2}\right)} \right. \\ & \left. + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=1}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} - (1 - \gamma) \right\} \leq 1 - \gamma, \end{aligned}$$

then

$$\mathcal{L}(\mathcal{R}^\tau(\vartheta, \delta)) \subset \mathcal{K}(\xi, \gamma, \rho),$$

where \mathcal{L} is defined by (4.4).

Finally, from Theorems 4.1 and 4.2 we have the following.

Example 4.5. If

$$\begin{aligned} & \left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=1}^{\infty} \frac{n}{\Gamma\left(\frac{n+1}{2}\right)} + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=1}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \\ & \leq 2(1 - \gamma), \end{aligned}$$

then $\mathcal{P} \in \mathcal{K}(\xi, \gamma, \rho)$, where \mathcal{L} is defined by (4.5).

Example 4.6. If

$$\left[(1 - \rho) \sec \xi + \rho(1 - \gamma) \right] \sum_{n=2}^{\infty} \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} + (1 - \rho)(1 - \gamma - \sec \xi) \sum_{n=2}^{\infty} \frac{1}{n\Gamma\left(\frac{n+1}{2}\right)} \leq 1 - \gamma,$$

then $\mathcal{P} \in \mathcal{S}(\xi, \gamma, \rho)$, where \mathcal{L} is defined by (4.5).

5. CONCLUSIONS

In this investigation we obtained sufficient conditions and inclusion results for functions $f \in \mathcal{A}$ to be in the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$, and information regarding the images of functions by applying convolution operator with Mittag-Leffler functions.

The investigation methods are based on some recent results and techniques found in [15] and [16], and we determined sufficient conditions for the functions $E_{\alpha, \beta}$ to belong to the new defined classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$.

Moreover, we found sufficient conditions such that the images of the functions belonging to the class $\mathcal{R}^{\tau}(\vartheta, \delta)$ by the new defined convolution operator Λ_{β}^{α} are in the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$, respectively.

Finally, we determined sufficient conditions such that the functions Ψ_{β}^{α} obtained as images of $E_{\alpha, \beta}$ via the Alexander integral operator belong to the classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$.

We emphasize that till now such kind of results doesn't appeared in any previous articles: the general classes $\mathcal{S}(\xi, \gamma, \rho)$ and $\mathcal{K}(\xi, \gamma, \rho)$ are completely new and introduced in [15, 16], while any type of such results were not studied previously.

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