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SUFFICIENT CONDITIONS OF SUBCLASSES OF SPIRAL-LIKE FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTIONS

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ABSTRACT. The purpose of the present paper is to find the sufficient conditions for some subclasses of analytic functions associated with Mittag-Leffler functions to be in subclasses of spiral-like univalent functions. Further, we discuss geometric properties of an integral operator related to Mittag-Leffler functions.

1. INTRODUCTION AND DEFINITIONS

Let $\mathbf{E}_\alpha$ be the function defined by

$$\mathbf{E}_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C},\ \alpha \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0,$$

that was introduced by Mittag-Leffler [14] and commonly known as the *Mittag-Leffler function*. Wiman [25] defined a more general function $\mathbf{E}_{\alpha,\beta}$ generalizing the $\mathbf{E}_\alpha$ Mittag-Leffler function, that is

$$\mathbf{E}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C},\ \alpha, \beta \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0.$$

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When $\beta = 1$, it is abbreviated as $\mathbf{E}_\alpha(z) = \mathbf{E}_{\alpha,1}(z)$. Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\mathbf{E}_{1,1}(z) = e^z, \quad \mathbf{E}_{1,2}(z) = \frac{e^z - 1}{z}, \quad \mathbf{E}_{2,1}\left(z^2\right) = \cosh z,$$
$$\mathbf{E}_{2,1}\left(-z^2\right) = \cos z, \quad \mathbf{E}_{2,2}\left(z^2\right) = \frac{\sinh z}{z}, \quad \mathbf{E}_{2,2}\left(-z^2\right) = \frac{\sin z}{z},$$
$$\mathbf{E}_4(z) = \frac{1}{2}\left(\cos z^{1/4} + \cosh z^{1/4}\right), \quad \mathbf{E}_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2} z^{1/3}\right)\right].$$

We recall the error function erf given by [1, p. 297]

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

the complement of the error function erfc defined by

$$\operatorname{erfc}(z) := 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1},$$

and the normalized form of the error function erf denoted by Erf (normalized with the condition $\operatorname{Erf}'(0) = 1$) is given by

$$\operatorname{Erf}(z) := \frac{\sqrt{\pi z}}{2} \operatorname{erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(2n-1)} z^n.$$

It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1}\right).$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found for example in [2, 3, 8, 9, 11, 12]. We note that the above generalized (Mittag-Leffler) function $\mathbf{E}_{\alpha,\beta}$ does not belongs to the family $\mathcal{A}$, where $\mathcal{A}$ represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$. Thus, it is expected to define the following normalization of Mittag-Leffler functions as below, due to Bansal and Prajapat [3]:

$$E_{\alpha,\beta}(z) := z\Gamma(\beta)\,\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \tag{1.2}$$

that holds for the parameters $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $z \in \mathbb{C}$. In this paper we shall confine our attention to the case of real-valued parameters α and β, and we will consider that $z \in \mathbb{D}$.

For functions $f \in \mathcal{A}$ be given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$, we define the *Hadamard product* (*or convolution*) of f and g by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

The two well known subclasses of $\mathcal{S}$ are namely the class of starlike and convex functions (for details see Robertson [20]). Thus, a function $f \in \mathcal{A}$ given by (1.1) is said to be *starlike of order* γ, $0 \le \gamma < 1$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{D},$$

and this function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be *convex of order* γ, $0 \le \gamma < 1$, if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{D},$$

and this class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} := \mathcal{K}(0)$ represents the well-known standard class of convex functions. By Alexander's duality relation (see [6]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

A function $f \in \mathcal{A}$ is said to be *spiral-like* if

$$\operatorname{Re}\left(e^{-i\xi}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some $\xi \in \mathbb{C}$ with $|\xi| < \frac{\pi}{2}$, and the class of spiral-like functions was introduced in [23]. Also, the function f is said to be convex spiral-like if $zf'(z)$ is spiral-like. Due to Murugusundramoorthy [15, 16], we consider the following subclasses of spiral-like functions as below.

Definition 1.1. For $0 \le \rho < 1$, $0 \le \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{S}(\xi, \gamma, \rho)$ by

$$\mathcal{S}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\xi}\frac{zf'(z)}{(1-\rho)f(z) + \rho zf'(z)}\right) > \gamma\cos\xi,\ z \in \mathbb{D} \right\}.$$

By virtue of Alexander's relation (see [6]) we define the following subclass $\mathcal{K}(\xi,\gamma,\rho)$.

Definition 1.2. For $0\le\rho<1$, $0\le\gamma<1$ and $|\xi|<\frac{\pi}{2}$, let define the class $\mathcal{K}(\xi,\gamma,\rho)$ by

$$\mathcal{K}(\xi,\gamma,\rho):=\left\{f\in\mathcal{A}:\operatorname{Re}\left(e^{i\xi}\frac{zf''(z)+f'(z)}{f'(z)+\rho zf''(z)}\right)>\gamma\cos\xi,\ z\in\mathbb{D}\right\}.$$

By specializing the parameter $\rho=0$ in the above two definitions we obtain the subclasses $\mathcal{S}(\xi,\gamma):=\mathcal{S}(\xi,\gamma,0)$ and $\mathcal{K}(\xi,\gamma):=\mathcal{K}(\xi,\gamma,0)$, respectively.

Now we state a sufficient conditions for the function f to be in the above classes.

Lemma 1.1 ([15,16]). *A function f given by* (1.1) *is a member of* $\mathcal{S}(\xi,\gamma,\rho)$ *if*

$$\sum_{n=2}^{\infty}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]|a_n|\le 1-\gamma,$$

where $|\xi|<\dfrac{\pi}{2}$, $0\le\rho<1$, $0\le\gamma<1$.

Since $f\in\mathcal{K}(\xi,\gamma,\rho)$ if and only if $zf'(z)\in\mathcal{S}(\xi,\gamma,\rho)$, and from Lemma 1.1 we get the next result.

Lemma 1.2. *A function f given by* (1.1) *is a member of* $\mathcal{K}(\xi,\gamma,\rho)$ *if*

$$\sum_{n=2}^{\infty}n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]|a_n|\le 1-\gamma,$$

where $|\xi|<\frac{\pi}{2}$, $0\le\rho<1$, $0\le\gamma<1$.

The next class $\mathcal{R}^{\tau}(\vartheta,\delta)$ was introduced earlier by Swaminathan [24], and for special cases see the references cited there in.

Definition 1.3. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(\vartheta,\delta)$, where $\tau\in\mathbb{C}\setminus\{0\}$, $0<\vartheta\le1$, and $\delta<1$, if it satisfies the inequality

$$\left|\frac{(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}{2\tau(1-\delta)+(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}\right|<1,\quad z\in\mathbb{D}.$$

Lemma 1.3 ([24]). *If $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$ is of the form* (1.1), *then*

$$|a_n|\le\frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)},\quad n\in\mathbb{N}\setminus\{1\}.\tag{1.3}$$

The bounds given in (1.3) *is sharp for*

$$f(z)=\frac{1}{\vartheta z^{1-\frac{1}{\vartheta}}}\int_0^z t^{1-\frac{1}{\vartheta}}\left[1+\frac{2(1-\delta)\tau\ t^{n-1}}{1-2^{n-1}}\right]dt.$$

Now we define the following new linear operator based on convolution (Hadamard) product.

For real parameters α, β, with $\alpha, \beta, \notin \{0, -1, -2, \dots\}$ and $E_{\alpha,\beta}$ be given by (1.2), we define the linear operator $\Lambda_\beta^\alpha : \mathcal{A} \to \mathcal{A}$ with the aid of the convolution product

$$\Lambda_\beta^\alpha f(z) := f(z) * E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} a_n z^n, \quad z \in \mathbb{D}.$$

Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example [5,10,13,21,22, 24]) and by the recent investigations related with distribution series (see for example [4,7,17–19], we obtain sufficient condition for the function $E_{\alpha,\beta}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions belonging in $\mathcal{R}^\tau(\vartheta,\delta)$ by using the convolution operator Λ_β^α. Finally, we determined conditions for the integral operator $\Psi_\beta^\alpha(z) = \int_0^z \frac{E_{\alpha,\beta}(t)}{t} dt$ to belong to the above classes.

2. Inclusion Results

In order to prove our main results, unless otherwise stated throughout this paper, we will use the notation (1.2), therefore

$$E_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{2.1}$$

$$E'_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{2.2}$$

$$E''_{\alpha,\beta}(1) = \sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \tag{2.3}$$

Theorem 2.1. *If*

$$[(1-\rho)\sec\xi + \rho(1-\gamma)]E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1) \le 2(1-\gamma), \tag{2.4}$$

then $E_{\alpha,\beta} \in \mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since $E_{\alpha,\beta}$ is defined by (1.2), according to Lemma 1.1 it is sufficient to show that

$$\sum_{n=2}^{\infty} \left[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\right] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma. \tag{2.5}$$

Since the left-hand side of the inequality (2.5) could be written as

$$\begin{aligned} Q_1(\xi,\gamma,\rho) :=& \sum_{n=2}^{\infty} [(1-\rho)\sec\xi(n-1) + (1-\gamma)(1+n\rho-\rho)] \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ =& [(1-\rho)\sec\xi + \rho(1-\gamma)] \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &+ (1-\rho)(1-\gamma-\sec\xi) \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \end{aligned}$$

therefore, by using (2.1) and (2.2), we get

$$\begin{aligned} Q_1(\xi,\gamma,\rho) =& \Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\Big[E'_{\alpha,\beta}(1)-1\Big] \\ &+(1-\rho)(1-\gamma-\sec\xi)\left[E_{\alpha,\beta}(1)-1\right] \\ =& \Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E'_{\alpha,\beta}(1)+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1) \\ &-(1-\gamma). \end{aligned}$$

Thus, from the assumption (2.4) it follows that $Q_1(\xi,\gamma,\rho)\le 1-\gamma$, that is (2.5) holds, therefore $E_{\alpha,\beta}\in \mathcal{S}(\xi,\gamma,\rho)$. □

Theorem 2.2. *If*

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E''_{\alpha,\beta}(1)+(1-\gamma)E'_{\alpha,\beta}(1)\le 2(1-\gamma), \tag{2.6}$$

then $E_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$.

Proof. Using the definition (1.2) of $E_{\alpha,\beta}$, in view of Lemma 1.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma. \tag{2.7}$$

The left-hand side of the inequality (2.7) could be written as

$$\begin{aligned} Q_2(\xi,\gamma,\rho) :=& \sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ =& \Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &+(1-\gamma)\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \end{aligned}$$

and from (2.2) and (2.3) we get

$$Q_2(\xi,\gamma,\rho)=\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]E''_{\alpha,\beta}(1)+(1-\gamma)[E'_{\alpha,\beta}(1)-1].$$

Hence, the assumption (2.6) implies that $Q_2(\xi,\gamma,\rho)\le 1-\gamma$ that is (2.7) holds, and consequently $E_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$. □

3. Image Properties of Λ_β^α Operator

Making use of the Lemma 1.1 and Lemma 1.3 we will focus the influence of the Λ_β^α operator for the functions of the class $\mathcal{R}^\tau(\vartheta,\delta)$, and we will give sufficient conditions such that these images are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

Theorem 3.1. *If*

$$\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big][E_{\alpha,\beta}(1)-1] + (1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt \le 1-\gamma, \tag{3.1}$$

then

$$\Lambda_\beta^\alpha\left(\mathcal{R}^\tau(\vartheta,\delta)\right)\subset\mathcal{S}(\xi,\gamma,\rho).$$

Proof. Let $f\in\mathcal{R}^\tau(\vartheta,\delta)$ be of the form (1.1). To prove that $\Lambda_\beta^\alpha(f)\in\mathcal{S}(\xi,\gamma,\rho)$, in view of Lemma 1.1 it is required to show that

$$\sum_{n=2}^{\infty}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\le 1-\gamma.$$

Let we denote the left-hand side of the above inequality by

$$Q_3(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|.$$

Since $f\in\mathcal{R}^\tau(\vartheta,\delta)$, by Lemma 1.3 we have

$$|a_n|\le\frac{2\,|\tau|\,(1-\delta)}{1+\vartheta(n-1)},\quad n\in\mathbb{N}\setminus\{1\},$$

and using the inequality $1+\vartheta(n-1)\ge\vartheta n$ we obtain that

$$\begin{aligned}Q_3(\xi,\gamma,\rho)\le&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}\frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\right.\\&\left.\times\ \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}\\=&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right.\\&\left.+\ (1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\,\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}.\end{aligned}$$

From the above inequality, using (2.1), we get

$$\begin{aligned}Q_3(\xi,\gamma,\rho)\le&\frac{2\,|\tau|\,(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big][E_{\alpha,\beta}-1]\\&+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt,\end{aligned}$$

hence, the assumption (3.1) implies then $Q_3(\xi,\gamma,\rho)\le 1-\gamma$, that is $\Lambda_\beta^\alpha(f)\in\mathcal{S}(\xi,\gamma,\rho)$. □

Using Lemma 1.2 and following the same procedure as in the proof of Theorem 2.2, we have the subsequent result.

Theorem 3.2. *If*

$$\begin{aligned}&\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\left\{\left[(1-\rho)\sec\xi+\rho(1-\gamma)\right]E'_{\alpha,\beta}(1)\right.\\ &\left.+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1)-(1-\gamma)\right\}\leq 1-\gamma,\end{aligned}\tag{3.2}$$

then

$$\Lambda_{\beta}^{\alpha}\left(\mathcal{R}^{\tau}(\vartheta,\delta)\right)\subset\mathcal{K}(\xi,\gamma,\rho).$$

Proof. Let $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$ be of the form (1.1). In view of Lemma 1.2, to prove that $\Lambda_{\beta}^{\alpha}(f)\in\mathcal{K}(\xi,\gamma,\rho)$ we have to show that

$$\sum_{n=2}^{\infty}n\left[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\leq 1-\gamma.\tag{3.3}$$

Since $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$, then by Lemma 1.3 we have

$$|a_n|\leq\frac{2\left|\tau\right|(1-\delta)}{1+\vartheta(n-1)},\quad n\in\mathbb{N}\setminus\{1\},$$

and $1+\vartheta(n-1)\geq\vartheta n$. Denoting the left-hand side of the inequality (3.3) by

$$Q_4(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}n\left[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|,$$

we deduce that

$$\begin{aligned}Q_4(\xi,\gamma,\rho)\leq&\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\sum_{n=2}^{\infty}\left[(1-\rho)\sec\xi(n-1)+(1-\gamma)(1+n\rho-\rho)\right]\\ &\times\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\ =&\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\left\{\left[(1-\rho)\sec\xi+\rho(1-\gamma)\right]\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right.\\ &\left.+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right\}.\end{aligned}$$

Now, by using (2.1) and (2.2), the above inequality yields to

$$\begin{aligned}Q_4(\xi,\gamma,\rho)\leq&\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\left\{\left[(1-\rho)\sec\xi+\rho(1-\gamma)\right][E'_{\alpha,\beta}(1)-1]\right.\\ &\left.+(1-\rho)(1-\gamma-\sec\xi)[E_{\alpha,\beta}(1)-1]\right\}\\ =&\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\left\{\left[(1-\rho)\sec\xi+\rho(1-\gamma)\right]E'_{\alpha,\beta}(1)\right.\\ &\left.+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1)-(1-\gamma)\right\}.\end{aligned}$$

Therefore, the assumption (3.2) yields to $Q_4(\xi,\gamma,\rho) \le 1-\gamma$, which implies the inequality (3.3), that is $\Lambda^{\alpha}_{\beta}(f) \in \mathcal{K}(\xi,\gamma,\rho)$. $\square$

4. The Alexander Integral Operator for $E_{\alpha,\beta}$

Theorem 4.1. *Let the function* Ψ^{α}_{β} *be given by*

$$\Psi^{\alpha}_{\beta}(z) = \int_0^z \frac{E_{\alpha,\beta}(t)}{t}dt, \quad z \in \mathbb{D}. \tag{4.1}$$

If

$$\Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big]E'_{\alpha,\beta}(1) + (1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}(1) \le 2(1-\gamma),$$

then $\Psi^{\alpha}_{\beta} \in \mathcal{K}(\xi,\gamma,\rho)$.

Proof. Since

$$\Psi^{\alpha}_{\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \cdot \frac{z^n}{n}, \quad z \in \mathbb{D}, \tag{4.2}$$

according to Lemma 1.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{1}{n}\cdot\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma,$$

or, equivalently

$$\sum_{n=2}^{\infty} \Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma.$$

Now, the proof of Theorem 4.1 is parallel to that of Theorem 2.1, and so it will be omitted. $\square$

Theorem 4.2. *Let the function* Ψ^{α}_{β} *be given by* (4.1)*. If*

$$\begin{aligned} &\Big[(1-\rho)\sec\xi + \rho(1-\gamma)\Big](E_{\alpha,\beta}(1)-1) \\ &+ (1-\rho)(1-\gamma-\sec\xi)\int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t} - 1\right)dt \le 1-\gamma, \end{aligned} \tag{4.3}$$

then $\Psi^{\alpha}_{\beta} \in \mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since Ψ^{α}_{β} has the power series expansion (4.2), then by Lemma 1.1 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma.$$

The left-hand side of the above inequality could be rewritten as

$$\begin{aligned}Q_5(\xi,\gamma,\rho) =&\sum_{n=2}^{\infty}\frac{1}{n}\Big[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)\Big]\\&\times\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\=&\sum_{n=2}^{\infty}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\\&+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)},\end{aligned}$$

and using (2.1) we get

$$\begin{aligned}Q_5(\xi,\gamma,\rho)\le&\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big][E_{\alpha,\beta}(1)-1]\\&+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt.\end{aligned}$$

Therefore, if the assumption (4.3) holds, then $Q_5(\xi,\gamma,\rho)\le 1-\gamma$. Hence, $\Psi_\beta^\alpha\in\mathcal{S}(\xi,\gamma,\rho)$. ☐

Remark 4.1. By taking $\rho=0$ in Theorems 2.1–4.2, we can easily attain the sufficient condition for $E_{\alpha,\beta}\in\mathcal{S}(\xi,\gamma)$ and $E_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma)$. The function $E_{\alpha,\beta}$ is associated with Mittag-Leffler functions and has not been studied sofar. We left this as an exercise to interested readers.

For the special case $\alpha=1/2$ and $\beta=1$, that is connected with the error function can derive some results based on the error function. Thus, a simple computation shows that if

$$\mathcal{E}(z):=E_{\frac{1}{2},1}(z)=\sum_{n=1}^{\infty}\frac{z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

then

$$\mathcal{E}(1)=\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)},\quad \mathcal{E}'(1)=\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)},\quad \mathcal{E}''(1)=\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)},$$

$$\int_0^1\left(\frac{\mathcal{E}(t)}{t}-1\right)dt=\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)},$$

$$\mathcal{L}:=\Lambda_1^{1/2}f(z)=f(z)*\mathcal{E}(z)=z+\sum_{n=2}^{\infty}\frac{a_nz^n}{\Gamma\left(\frac{n+1}{2}\right)},\tag{4.4}$$

$$\mathcal{P}:=\Psi_1^{1/2}(z)=\int_0^z\frac{\mathcal{E}(t)}{t}dt=\sum_{n=1}^{\infty}\frac{z^n}{n\Gamma\left(\frac{n+1}{2}\right)}.\tag{4.5}$$

Using the above relations, from Theorems 2.1 and 2.2 we get, respectively.

Example 4.1. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 2(1-\gamma),$$

then $\mathcal{E}\in\mathcal{S}(\xi,\gamma,\rho)$.

Example 4.2. If

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\gamma)\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\leq 2(1-\gamma),$$

then $\mathcal{E}\in\mathcal{K}(\xi,\gamma,\rho)$.

Similarly, Theorems 4.1 and 4.2 give us the next examples.

Example 4.3. If

$$\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}\leq 1-\gamma,$$

then

$$\mathcal{L}\left(\mathcal{R}^{\tau}(\vartheta,\delta)\right)\subset\mathcal{S}(\xi,\gamma,\rho),$$

where $\mathcal{L}$ is defined by (4.4).

Example 4.4. If

$$\frac{2\left|\tau\right|(1-\delta)}{\vartheta}\Bigg\{\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}-(1-\gamma)\Bigg\}\leq 1-\gamma,$$

then

$$\mathcal{L}\left(\mathcal{R}^{\tau}(\vartheta,\delta)\right)\subset\mathcal{K}(\xi,\gamma,\rho),$$

where $\mathcal{L}$ is defined by (4.4).

Finally, from Theorems 4.1 and 4.2 we have the following.

Example 4.5. If

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 2(1-\gamma),$$

then $\mathcal{P}\in\mathcal{K}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5).

Example 4.6. If

$$\Big[(1-\rho)\sec\xi+\rho(1-\gamma)\Big]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 1-\gamma,$$

then $\mathcal{P}\in\mathcal{S}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5).

5. Conclusions

In this investigation we obtained sufficient conditions and inclusion results for functions $f\in\mathcal{A}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions by applying convolution operator with Mittag-Leffler functions.

The investigation methods are based on some recent results and techniques found in [15] and [16], and we determined sufficient conditions for the functions $E_{\alpha,\beta}$ to belongs to the new defined classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

Moreover, we found sufficient conditions such that the images of the functions belonging to the class $\mathcal{R}^{\tau}(\vartheta,\delta)$ by the new defined convolution operator Λ_{β}^{α} are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

Finally, we determined sufficient conditions such that the functions Ψ_{β}^{α} obtained as images of $E_{\alpha,\beta}$ via the Alexander integral operator belong to the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

We emphasize that till now such kind of results doesn't appeared in any previous articles: the general classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$ are completely new and introduced in [15, 16], while any type of such results were not studied previously.

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