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SOME BORDERENERGETIC AND EQUIENERGETIC GRAPHS

SAMIR K. VAIDYA[1] AND KALPESH M. POPAT[2]

ABSTRACT. The sum of absolute values of eigenvalues of a graph G is defined as energy of graph. If the energies of two non-isomorphic graphs are same then they are called equienergetic. The energy of complete graph with n vertices is $2(n-1)$ and the graphs whose energy is equal to $2(n-1)$ are called borderenergetic graphs. It has been revealed that the graphs upto 12 vertices are borderenergetic. It is very challenging and interesting as well to search for borderenergetic graphs with more than 14 vertices. The present work is leap ahead in this direction as we have found a family of borderenergetic graphs of arbitrarily large order. We have also obtained three pairs of equienergetic graphs.

1. INTRODUCTION

For standard terminology and notations in graph theory we follow West [19] while the terms related to algebra are used in sense of Lang [11].

Let G be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* denoted by $A(G)$ of G is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if v_i is adjacent with v_j, and 0 otherwise.

The eigenvalues of A are called the eigenvalues of G. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G then

$$\operatorname{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

The energy $E(G)$ of graph G is the sum of all absolute values of eigenvalues of G. The concept of energy of graph was introduced by Gutman [7] in 1978. A brief account on energy of graph can be found in Cvetković [2] and Li [12].

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The graphs of order n, whose energy exceeds than the energy of the complete graph K_n are called hyperenergetic graphs otherwise graphs of order n with $E(G) \leq E(K_n)$, are called non-hyperenergetic. As mentioned in Gutman [7] $E(K_n) = 2(n-1)$. Are there any graphs other than K_n with such behaviour?

This question motivated Gong et al. [6] to introduce a new concept. According to them, the graph G of order n satisfying $E(G) = 2(n-1)$ are called *borderenergetic*. Obviously, the complete graph K_n is borderenergetic. Gong et al. [6] have proved that such graphs exist for $n = 7, 8, 9$. Li et al. [13] and Shao et al. [15] have obtained the graphs with $n = 10$ and $n = 11$ respectively while Furtula and Gutman [4] have obtained the graphs with $n = 12$. A family of non-regular and non-integral borderenergetic graphs with particular behaviour were investigated by Hou and Tao [16]. Some new families of borderenergetic graphs were obtained by Jahfar et al. [10]. Recently, a survey on borderenergetic graphs was published by Ghorbani et al. [5].

We will introduce some concepts and also state some existing results for our ready reference.

Definition 1.1. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Proposition 1.1 ([17]). *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G, then $2n$ eigenvalues of $D_2(G)$ are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n, 0$ (n times).*

Proposition 1.2 ([3]). *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric block matrix. Then the spectrum of A is the union of spectra of $A_0 + A_1$ and $A_0 - A_1$.

Definition 1.2. The *extended shadow graph* $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G''.

A curious question: **How the energy of a given graph G can be correlated with the larger graph obtained by means of graph operations on G?** To quench this thirst we have considered shadow graph and extended shadow graph as these graphs are of same order but they are non isomorphic. Due to this specific characteristic, the said graphs are used to construct non-co spectral equienergetic graphs by constructing shadow graph of extended shadow graph as well as extended shadow graph of shadow graph.

2. Energy of Extended Shadow Graph

Theorem 2.1. *Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $|\lambda_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then $E(D_2^*(G)) = 2E(G) + n + \theta$, where θ is the difference between the number of positive and negative eigenvalues of G.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of graph G. Then the $A(G)$ is given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}.$$

Consider a second copy of graph G with vertices $u_1, u_2, u_3, \ldots, u_n$ and join u_i with neighbors of v_i and with v_i, $1 \le i \le n$, to obtain $D_2^*(G)$. Then the $A(D_2^*(G))$ can be written as a block matrix as follows

$$A(D_2^*(G)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdot & \boldsymbol{u_n} \end{array} \\ \left[\begin{array}{ccccc|ccccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} & 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 1 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \\ \hline 1 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 1 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 1 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(D_2^*(G)) = \begin{bmatrix} A(G) & A(G) + I \\ A(G) + I & A(G) \end{bmatrix}.$$

Hence, by Proposition 1.2 spectrum of $D_2^*(G)$ is union of spectra of $2A(G) + I$ and $-I$. Therefore, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G, then

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_i + 1 & -1 \\ n & n \end{pmatrix}.$$

Suppose that $|\lambda_i| \ge \frac{1}{2}$ for all $1 \le i \le n$, then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned}
E(D_2^*(G)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\
&= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\
&= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\
&= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\
&= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n \\
&= 2E(G) + n + \theta. \qquad \square
\end{aligned}$$

The following corollary proves the existence of borderenergetic graph of arbitrarily large order.

Corollary 2.1. $E(D_2^*(K_{n,n})) = E(K_{4n})$. *That is,* $D_2^*(K_{n,n})$ *is non complete borderenergetic graph.*

Proof. Consider complete bipartite graph $K_{n,n}$ of $2n$ vertices then

$$\operatorname{spec}(K_{n,n}) = \begin{pmatrix} n & -n & 0 \\ 1 & 1 & 2n-2 \end{pmatrix}.$$

Now, $D_2^*(K_{n,n})$ is a graph with $4n$ vertices and by Theorem 2.1 its spectrum is

$$\operatorname{spec}(D_2^*(K_{n,n})) = \begin{pmatrix} 2n+1 & -2n+1 & 1 & -1 \\ 1 & 1 & 2n-2 & 2n \end{pmatrix}. \tag{2.1}$$

Also,

$$\operatorname{spec}(K_{4n}) = \begin{pmatrix} 4n-1 & -1 \\ 1 & 4n-1 \end{pmatrix}. \tag{2.2}$$

Clearly from (2.1) and (2.2) $D_2^*(K_{n,n})$ and K_{4n} are non co-spectral and

$$\begin{aligned}
E(D_2^*(K_{n,n})) &= \sum_{i=1}^{4n} |\lambda_i| \\
&= (2n+1) + (2n-1) + (2n-2) + 2n \\
&= 8n - 2 = 2(4n-1) = E(K_{4n}).
\end{aligned}$$

Thus, $E(D_2^*(K_{n,n})) = E(K_{4n})$. Hence, $D_2^*(K_{n,n})$ is non complete borderenergetic graph. $\square$

3. Equienergetic Graphs

Definition 3.1. Two non-isomorphic graphs G_1 and G_2 of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$.

In 2007 Ramane et al. have proved that there exists a pair of connected non-cospectral, equienergetic graphs with n vertices for all $n \geq 9$.

Definition 3.2. The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in G.

Harary [8] defined the concept of iterated line graphs. According to him if G is graph and $L^1(G) = L(G)$ be its line graph, then $L^2(G) = L(L(G))$, $L^3(G) = L(L^2(G)), \dots, L^k(G) = L(L^{k-1}(G)), \dots$

Ramane et al. [14] have proved that if G_1 and G_2 are regular graphs of same order, then for $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.

Definition 3.3. The *cartesian product* of graphs G and H is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The following result gives the spectra of the Cartesian product of graphs.

Proposition 3.1 ([1])**.** *Let G_1 and G_2 are two graphs having spectra as $\mu_1, \mu_2, \dots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2}$, respectively. Then spectra of $G = G_1 \times G_2$ is $\mu_i + \sigma_j$, where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

Theorem 3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of graph G. Then $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of graph G. By Proposition 3.1

$$\operatorname{spec}(G \times K_2) = \begin{pmatrix} \lambda_1 + 1 & \lambda_2 + 1 & \cdots & \lambda_n + 1 & \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_n - 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

According to Theorem 2.1,

$$\operatorname{spec}(D_2^*(G \times K_2)) = \begin{pmatrix} 2\lambda_1 + 3 & \cdots & 2\lambda_n + 3 & 2\lambda_1 - 1 & \cdots & 2\lambda_n - 1 & -1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 2n \end{pmatrix}. \tag{3.1}$$

Moreover, by Theorem 2.1,

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1 + 1 & 2\lambda_2 + 1 & \cdots & 2\lambda_n + 1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Proposition 1.1,

$$\operatorname{spec}(D_2(D_2^*(G))) = \begin{pmatrix} 4\lambda_1 + 2 & 4\lambda_2 + 2 & \cdots & 4\lambda_n + 2 & -2 & 0 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.2}$$

If for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2}$, then

$$\left|\lambda_i + \frac{3}{2}\right| = \begin{cases} |\lambda_i| + \frac{3}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Also for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2} > \frac{1}{2}$,

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i < 0, \end{cases}$$

$$\left|\lambda_i - \frac{1}{2}\right| = \begin{cases} |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i < 0. \end{cases}$$

From (3.1) and (3.2)

$$\begin{aligned} E(D_2^*(G \times K_2)) &= \sum_{i=1}^{n} |2\lambda_i + 3| + \sum_{i=1}^{n} |2\lambda_i - 1| + 2n \\ &= 2\sum_{i=1}^{n} |\lambda_i + \frac{3}{2}| + 2\sum_{i=1}^{n} |\lambda_i - \frac{1}{2}| + 2n \\ &= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i>0} \left|\lambda_i - \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i - \frac{1}{2}\right|\right) + 2n \\ &= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{3}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{3}{2}\right)\right) \\ &\quad + 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| - \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| + \frac{1}{2}\right)\right) + 2n \\ &= 2\left(2\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i>0} 1\right)\right) + 2n \\ &= 4E(G) + 2\theta + 2n \qquad (3.3) \end{aligned}$$

and

$$\begin{aligned} E(D_2(D_2^*(G))) &= \sum_{i=1}^{n} |4\lambda_i + 2| + 2n \\ &= 4\sum_{i=1}^{n} |4\lambda_i + \frac{1}{2}| + 2n \\ &= 4\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + 2n \\ &= 4\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + 2n \end{aligned}$$

$$= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 2n$$

$$= 4E(G) + 2\theta + 2n. \tag{3.4}$$

Hence, from (5) and (6), $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$ □

Let $D_2^{**}(G)$ be extended shadow graph of $D_2^*(G)$, i.e., $D_2^{**}(G) = D_2^*(D_2^*(G))$ and if G be a bipartite graph, then it is well-known that the spectra of G is symmetric about the origin, so half of the non-zero eigenvalues of G lies to the left and half lies to the right of the origin. Therefore if G is a bipartite graph having all its eigenvalues nonzero, the number of positive and negative eigenvalues of G are same. Keeping this into mind we have the following result.

Theorem 3.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of a bipartite graph G. Then $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic if and only if $|\lambda_i| \geq \frac{3}{4}$ for $1 \leq i \leq n$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of bipartite graph G. By Theorem 2.1

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1+1 & 2\lambda_2+1 & \cdots & 2\lambda_n+1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

and

$$\operatorname{spec}(D_2^{**}(G)) = \begin{pmatrix} 4\lambda_1+3 & 4\lambda_2+3 & \cdots & 4\lambda_n+3 & -1 \\ 1 & 1 & \cdots & 1 & 3n \end{pmatrix}. \tag{3.5}$$

Moreover, by Proposition 1.1,

$$\operatorname{spec}(D_2(G)) = \begin{pmatrix} 2\lambda_1 & 2\lambda_2 & \cdots & 2\lambda_n & 0 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Theorem 2.1,

$$\operatorname{spec}(D_2^*(D_2(G))) = \begin{pmatrix} 4\lambda_1+1 & 4\lambda_2+1 & \cdots & 4\lambda_n+1 & 1 & -1 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.6}$$

Clearly, from (3.5) and (3.6), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are non-co spectral graphs. As G is bipartite graph we have,

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Assume that for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{4}$. Hence,

$$\left|\lambda_i + \frac{3}{4}\right| = \begin{cases} |\lambda_i| + \frac{3}{4}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{4}, & \text{if } \lambda_i < 0. \end{cases}$$

From (3.5)

$$E(D_2^{**}(G)) = \sum_{i=1}^{n}|4\lambda_i + 3| + \sum_{i=1}^{3n}|-1|$$

$$
\begin{aligned}
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{3}{4}\right| + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{3}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{3}{4}\right|\right) + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{3}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n\\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{3}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n\\
&= 4E(G) + 3n. \qquad (3.7)
\end{aligned}
$$

Also if for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{4} \geq \frac{1}{4}$, then

$$
\left|\lambda_i + \frac{1}{4}\right| = \begin{cases} |\lambda_i| + \frac{1}{4}, & \text{if } \lambda_i > 0,\\ |\lambda_i| - \frac{1}{4}, & \text{if } \lambda_i < 0. \end{cases}
$$

From (3.6)

$$
\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{n}|4\lambda_i + 1| + n + 2n\\
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{1}{4}\right| + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{1}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n\\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n\\
&= 4E(G) + 3n. \qquad (3.8)
\end{aligned}
$$

Thus, by (3.7) and (3.8), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are equienergetic graphs.

Conversely, suppose that the graphs $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic. We will show that $|\lambda_i| \geq \frac{3}{4}$ for $1 \leq i \leq n$.

Assume to the contrary that let $|\lambda_i| < \frac{3}{4}$ for some i. Then for the same i, $\left|\lambda_i + \frac{3}{4}\right| = \lambda_i + \frac{3}{4}$. Without loss of generality, suppose that the eigenvalues of G satisfy $|\lambda_i| \geq \frac{3}{4}$, for $i = 1, 2, \ldots, k$ and $|\lambda_i| < \frac{3}{4}$, for $i = k+1, k+2, \ldots, n$, since the eigenvalues are real and reordering does not affect the argument. We have the following cases to be considered.

Case I If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 6n. \qquad (3.9)
\end{aligned}$$

Case II If $\lambda_i > 0$ for $i = 1, 2, \cdots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \cdots, n$. If θ_0 is the number of zero eigenvalues of G, we have

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{3}{4}\right)\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}\theta_0\right) + 3n. \qquad (3.10)
\end{aligned}$$

Case III If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\theta_0\right) + 3n. \qquad (3.11)
\end{aligned}$$

Case IV If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}n\right) + 3n. \qquad (3.12)
\end{aligned}$$

While

$$E(D_2^*(D_2(G))) = 4\sum_{i=1}^{n} |\lambda_i| + 3n,$$

which remain same in each of the above cases only if $|\lambda_i| \geq \frac{1}{4}$ for $i = k+1, k+2, \ldots, n$.

If $|\lambda_i| < \frac{1}{4}$ for $i = k+1, k+2, \ldots, n$, then we have the following.

Case I If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{1}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 4n. \qquad (3.13)
\end{aligned}$$

Case II If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$, and if θ_0 is the number of zero eigenvalues of G, then we have

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{1}{4}\right)\right) + 3n
\end{aligned}$$

$$> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n$$

$$= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}\theta_0\right) + 3n. \tag{3.14}$$

Case III If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n}|\lambda_i| + \frac{1}{4}\theta_0\right) + 3n.
\end{aligned}$$

Case IV If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}n\right) + 3n. \qquad (3.15)
\end{aligned}$$

Clearly, in all the cases discussed above, we have $E(D_2^{**}(G)) \neq E(D_2^*(D_2(G)))$, a contradiction. Hence, the result follows. ☐

Corollary 3.1. *If G_1 and G_2 are two equienergetic bipartite graphs with $|\lambda_i| \geq \frac{1}{2}$ and $|\mu_i| \geq \frac{1}{2}$, where λ_i and μ_i are the eigenvalues of G_1 and G_2, respectively, for all $1 \leq i \leq n$, then $D_2^*(G_1)$ and $D_2^*(G_2)$ are non cospectral equienergetic.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ be eigenvalues of G_1 and G_2, respectively. Then by Theorem 2.1 spectrum of G_1 and G_2 are given by

$$\operatorname{spec}(D_2^*(G_1)) = \begin{pmatrix} 2\lambda_i + 1 & -1 \\ n & n \end{pmatrix}, \quad \operatorname{spec}(D_2^*(G_2)) = \begin{pmatrix} 2\mu_i + 1 & -1 \\ n & n \end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$ for $i = 1, 2, \ldots, n$. Then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned} E(D_2^*(G_1)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\ &= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\ &= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\ &= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\ &= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n. \end{aligned}$$

As G_1 is bipartite graph

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Hence, $E(D_2^*(G_1)) = 2E(G_1) + n$.

Similarly, if $|\mu_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then

$$E(D_2^*(G_2)) = 2E(G_2) + n.$$

Since, G_1 and G_2 are equienergetic graphs $D_2^*(G_1)$ and $D_2^*(G_1)$ are equienergetic. □

4. Extended m-Shadow Graph and Graph Energy

Definition 4.1. The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$. Vaidya and Popat [18] have proved that $E(D_m(G)) = mE(G)$.

Definition 4.2. The extended *m-shadow graph* $D_m^*(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v and with v in G_j, $1 \leq i, j \leq m$.

Definition 4.3. Let $A \in R^{m\times n}$, $B \in R^{p\times q}$. Then the *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Proposition 4.1 ([9]). *Let $A \in M^m$ and $B \in M^n$. Furthermore, let λ is an eigenvalues of matrix A with corresponding eigenvector x and μ is an eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Theorem 4.1. *Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with $|\lambda_i| \geq \frac{m-1}{m}$, for all $1 \leq i \leq n$. Then $E(D_m^*(G)) = mE(G) + (m-1)n + (m-1)\theta$, where θ is the difference between the number of positive and negative eigenvalues of G.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the graph G. Then its adjacency matrix of G is same as in the proof of Theorem 2.2. Consider m copies of graph G say $G_1, G_2, \dots, G_k$ with vertices $v_i^1, v_i^2, \dots, v_i^m$, $1 \leq i \leq n$, to obtain $D_m^*(G)$ such that each vertex u in G_j is joined to the neighbors of the corresponding vertex v as well as with v in G_k, $1 \leq j, k \leq m$. Then the $A(D_m^*(G))$ can be written as a block matrix as follow

$$A(D_m^*(G)) = \begin{bmatrix} A(G) & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G) & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G) \end{bmatrix}_m,$$

$$\begin{aligned} A(D_m^*(G)) + I_{mn} &= \begin{bmatrix} A(G)+I & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G)+I & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G)+I \end{bmatrix}_m \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_m \otimes (A(G)+I) \\ &= J_m \otimes (A(G)+I). \end{aligned}$$

Hence, by Proposition 4.1, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of G, then

$$\begin{aligned} \operatorname{spec}(D_m^*(G)+I) &= \begin{pmatrix} m(\lambda_i+1) & 0(\lambda_i+1) \\ n & mn-n \end{pmatrix} \\ &= \begin{pmatrix} m(\lambda_i+1) & 0 \\ n & mn-n \end{pmatrix}, \end{aligned}$$

$$\operatorname{spec}(D_m^*(G)) = \begin{pmatrix} m\lambda_i + (m-1) & -1 \\ n & mn-n \end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{m-1}{m}$ for all $1 \leq i \leq n$. Then

$$\left|\lambda_i + \frac{m-1}{m}\right| = \begin{cases} |\lambda_i| + \frac{m-1}{m}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{m-1}{m}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned} E(D_m^*(G)) &= \sum_{i=1}^{n} |m\lambda_i + (m-1)| + \sum_{i=1}^{(m-1)n} |-1| \\ &= m\sum_{i=1}^{n} \left|\lambda_i + \frac{m-1}{m}\right| + (m-1)n \\ &= m\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{m-1}{m}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{m-1}{m}\right|\right) + (m-1)n \\ &= m\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{m-1}{m}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{m-1}{m}\right)\right) + (m-1)n \\ &= m\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{m-1}{m}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + (m-1)n \\ &= mE(G) + (m-1)n + (m-1)\theta. \end{aligned}$$

□

5. Concluding Remarks

The energy of extended shadow graph has been obtained and using it a new family of non complete borderenergetic graphs and new pairs of non cospectral equienergetic graphs have been investigated.

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[1]Department of Mathematics,
Saurashtra University,
Rajkot(Gujarat), India
Email address: samirkvaidya@yahoo.co.in

[2]Department of Computer Applications-MCA,
Atmiya University,
Rajkot(Gujarat), India
Email address: kalpeshmpopat@gmail.com