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HIGHER CODERIVATIONS ON COALGEBRAS AND CHARACTERIZATION

E. TAFAZOLI[1] AND M. MIRZAVAZIRI[2]

Abstract. In this paper we define higher coderivations on a coalgebra C and then we characterize them in terms of the coderivations on C. Indeed, we show that each higher coderivation is a combination of compositions of coderivations. Finally we prove a one to one correspondence between the set of all higher coderivations on C and all sequences of coderivations on C.

1. Introduction

A *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \kappa$, such that $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$, (coassociativity) and $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$, (counitary). The maps Δ and ε are called, respectively, *coproduct* and *counit* of the coalgebra C. Given an element c of the coalgebra (C, Δ, ε), we know that there exist elements $c_{1,i}$ and $c_{2,i}$ in C such that $\Delta(c) = \sum_i c_{1,i} \otimes c_{2,i}$. In *Sweedlers notation*, this is abbreviated to $\sum c_{(1)} \otimes c_{(2)}$. Here, the subscripts "(1)" and "(2)" indicate the order of the factors in the tensor product. For more about basic definitions in coalgebras notion, you can see [1] and [3].

A κ-linear map $f : C \to C$ on a κ-coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$. One can see examples and a general definition of coalgebras and coderivations in the sense of comodules in [2, 4, 6]. In this paper we define higher coderivations on a coalgebra C and then characterize them in terms of the coderivations on C. Indeed, we show that each higher coderivation is a combination of compositions of coderivations. As a corollary we characterize all higher coderivations which are ordinary. We have some nearly same properties for higher derivations, you

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can see in [5] and [7]. Throughout the paper, all coalgebras are assumed over a field of characteristic zero.

2. The Results

Throughout the paper, C denotes a coalgebra over a field of characteristic zero and I is the identity mapping on C. A *coalgebra* (C, Δ, ε) over a field κ is a κ-vector space C together with the κ-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \kappa$, such that $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$, (coassociativity), and $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$, (counitary). The maps Δ and ε are called, respectively, *coproduct* and *counit* of the coalgebra C. A κ-linear map $f : C \to C$ on a κ-coalgebra (C, Δ, ε) is called a *coderivation* if $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$.

Now we define a new concept, named higher coderivation and then characterize this, but at first we prove some properties, following.

Proposition 2.1. *If f is a coderivation on coalgebra (C, Δ, ε), then we have*

$$\Delta f^n = \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})\Delta, \tag{2.1}$$

for each nonnegative integer n.

Proof. We use induction on n. For $n = 1$ and $a \in C$ we have

$$\Delta f(a) = \sum a_{(1)} \otimes f(a_{(2)}) + f(a_{(1)}) \otimes a_{(2)},$$

and its true, since f is a coderivation on C. Now suppose that the equality is true for n, then for $n + 1$, in the left side of equality, we have

$$\Delta f^{n+1}(a) = \Delta f^n(f(a)) = \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})\Delta(f(a)),$$

because of f being a coderivation, we have

$$\begin{aligned}\Delta f^{n+1}(a) &= \sum_{k=0}^{n} \binom{n}{k} (f^k \otimes f^{n-k})(I \otimes f + f \otimes I)\Delta(a) \\ &= \sum_{k=0}^{n} \sum \binom{n}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) + f^{k+1}(a_{(1)}) \otimes f^{n-k}(a_{(2)}).\end{aligned}$$

On the other side we have

$$\begin{aligned}&\sum_{k=0}^{n+1} \binom{n+1}{k} (f^k \otimes f^{n+1-k})\Delta(a) \\ &= \sum_{k=0}^{n+1} \sum \binom{n+1}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) \\ &= \left[\sum_{k=0}^{n} \sum \left(\binom{n}{k} + \binom{n}{k-1}\right) \left(f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)})\right)\right] + f^{n+1}(a_{(1)}) \otimes a_{(2)}\end{aligned}$$

$$
\begin{aligned}
=&\Bigg[\sum_{k=0}^{n}\sum\binom{n}{k}f^{k}(a_{(1)})\otimes f^{n+1-k}(a_{(2)})\\
&+\sum_{k=-1}^{n-1}\sum\binom{n}{k}\left(f^{k+1}(a_{(1)})\otimes f^{n+1-(k+1)}(a_{(2)})\right)\Bigg]+f^{n+1}(a_{(1)})\otimes a_{(2)}\\
=&\sum_{k=0}^{n}\sum\binom{n}{k}f^{k}(a_{(1)})\otimes f^{n+1-k}(a_{(2)})+\sum_{k=-1}^{n}\sum\binom{n}{k}f^{k+1}(a_{(1)})\otimes f^{n-k}(a_{(2)}),
\end{aligned}
$$

and we have the result. □

We name the relation (2.1) *general coLiebnitz rule for coderivations.*
If we define a sequence $\{f_n\}$ of linear mappings on C by $f_0 = I$ and $f_n = \frac{\lambda^n}{n!}$, where I is the identity mapping on C, then general coLeibniz rule ensures us that f_n's satisfy the condition

$$
\Delta f_n = \sum_{k=0}^{n}(f_k \otimes f_{n-k})\Delta, \tag{2.2}
$$

for each nonnegative integer n. This motivates us to consider the sequences $\{f_n\}$ of linear mappings on a coalgebra C satisfying (2.2). We call such a sequence a higher coderivation.

Definition 2.1. Let C be a coalgebra. We define a sequence $\{f_n\}$ of linear mappings on C a *higher coderivation* if $\Delta f_n(a) = \sum_{k=0}^{n}(f_k \otimes f_{n-k})\Delta(a)$ for each $a \in C$ and each nonnegative integer n.

Though, if $\lambda : C \to C$ is a coderivation then $f_n = \frac{\lambda^n}{n!}$ is a higher coderivation. We name this kind of higher coderivation an *ordinary higher coderivation.*

Proposition 2.2. *Let $\{f_n\}$ be a higher coderivation on a coalgebra C with $f_0 = I$. Then there is a sequence $\{\lambda_n\}$ of coderivations on C such that*

$$
(n+1)f_{n+1} = \sum_{k=0}^{n} f_{n-k}\lambda_{k+1},
$$

for each nonnegative integer n.

Proof. We use induction on n. Because of $\{f_n\}$ being a higher coderivation, for $n = 0$ we have

$$
\begin{aligned}
\Delta f_1(a) &= [(f_0 \otimes f_1) + (f_1 \otimes f_0)]\Delta(a)\\
&= \sum f_0(a_{(1)}) \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes f_0(a_{(2)})\\
&= \sum a_{(1)} \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes a_{(2)}.
\end{aligned}
$$

Thus, if $\lambda_0 = I$ and $\lambda_1 = f_1$, then λ_1 is a coderivation on $\mathcal{A}$ and

$$
\Delta(f_0\lambda_1)(a) = \Delta(\lambda_1(a)) = \sum \lambda_0(a_{(1)}) \otimes \lambda_1(a_{(2)}) + \lambda_1(a_{(1)}) \otimes \lambda_0(a_{(2)}).
$$

Now suppose that λ_k it is defined and is a coderivation for $k \leq n$. Putting $\lambda_{n+1} = (n+1)f_{n+1} - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}$, we show that the well-defined mapping λ_{n+1} is a coderivation on C. For $a \in C$, since $\{f_n\}$ is a higher coderivation and $\lambda_1, \ldots, \lambda_n$ are coderivations, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) = (n+1)\Delta f_{n+1}(a) - \sum_{k=0}^{n-1} \Delta(f_{n-k}\lambda_{k+1})(a)\\
=&(n+1)\Delta f_{n+1}(a)\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum(f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right)\\
=&(n+1)\sum_{k=0}^{n+1}(f_k \otimes f_{n+1-k})\Delta(a)\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum(f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right)\\
=&(n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum(f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right)\\
=&(n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum f_l(a_{(1)}) \otimes f_{n-k-l}(\lambda_{k+1}(a_{(2)})) + f_l(\lambda_{k+1}(a_{(1)})) \otimes f_{n-k-l}(a_{(2)}).
\end{aligned}$$

Now, by properties of tensor product, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) =&\sum_{k=0}^{n+1}\sum(k+n+1-k)\left(f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\right)\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum(f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right)\\
=&\sum_{k=0}^{n+1}\sum kf_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) + f_k(a_{(1)}) \otimes (n+1-k)f_{n+1-k}(a_{(2)})\\
&- \sum_{k=0}^{n-1}\sum_{l=0}^{n-k}\sum(f_l \otimes f_{n-k-l})\left(a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right).
\end{aligned}$$

Writing

$$K = \sum_{k=0}^{n+1}\sum kf_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}\sum f_\ell\lambda_{k+1}(a_{(1)}) \otimes f_{n-k-\ell}(a_{(2)}),$$

$$\begin{aligned}L =&\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes (n+1-k)f_{n+1-k}(a_{(2)})\\&-\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}\sum f_\ell(a_{(1)}) \otimes f_{n-k-\ell}\lambda_{k+1}(a_{(2)}),\end{aligned}$$

we have $\Delta\lambda_{n+1}(a) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}$, we have $0 \le k + \ell \le n$ and $k \neq n$. Thus, if we put $r = k + \ell$ then we can write it as the form $\sum_{r=0}^{n}\sum_{k+\ell=r,k\neq n}$. Putting $\ell = r - k$ we indeed have

$$\begin{aligned}K =&\sum_{k=0}^{n+1}\sum kf_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\\&-\sum_{r=0}^{n}\sum_{0\le k\le r,k\neq n}\sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)})\\=&\sum_{k=0}^{n+1}\sum kf_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)})\\&-\sum\left(\sum_{r=0}^{n-1}\sum_{k=0}^{r} f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}\right) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)}.\end{aligned}$$

Putting $r+1$ instead of k in the first summation we have

$$\begin{aligned}&K + \sum_{k=0}^{n-1}\sum f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\\=&\sum_{r=0}^{n}\sum(r+1)f_{r+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) - \sum_{r=0}^{n-1}\sum_{k=0}^{r}\sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)})\\=&\sum\left(\sum_{r=0}^{n-1}\left[(r+1)f_{r+1}(a_{(1)}) - \sum_{k=0}^{r} f_{r-k}\lambda_{k+1}(a_{(1)})\right] \otimes f_{n-r}(a_{(2)}\right)\\&+ (n+1)f_{n+1}(a_{(1)}) \otimes a_{(2)}).\end{aligned}$$

By our assumption

$$(r+1)f_{r+1}(a) = \sum_{k=0}^{r}(f_{r-k}\lambda_{k+1})(a),$$

for $r = 0, \dots, n-1$. We can therefore deduce that

$$K = \sum\left[(n+1)f_{n+1}(a_{(1)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)})\right] \otimes a_{(2)} = \sum \lambda_{n+1}(a_{(1)}) \otimes a_{(2)}.$$

By a similar argument we have

$$L = \sum a_{(1)} \otimes \left[(n+1)f_{n+1}(a_{(2)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(2)})\right] = \sum a_{(1)} \otimes \lambda_{n+1}(a_{(2)}).$$

Thus,

$$\Delta\lambda_{n+1}(a) = K + L = (I \otimes \lambda_{n+1} + \lambda_{n+1} \otimes I)\Delta(a),$$

whence λ_{n+1} is a coderivation on C. □

To illustrate the recursive relation mentioned in Proposition 2.2, let us compute some terms of $\{d_n\}$.

Example 2.1. Using Proposition 2.2, the first five terms of $\{f_n\}$ are

$$\begin{aligned}
f_0 &= I,\\
f_1(a) &= f_0(\lambda_1(a)) = \lambda_1(a) \to f_1 = \lambda_1,\\
2f_2(a) &= f_1(\lambda_1(a)) + f_0(\lambda_2(a)) = {\lambda_1}^2(a) + \lambda_2(a) \to 2f_2 = \lambda_1^2 + \lambda_2,\\
f_2 &= \frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2,\\
3f_3 &= f_2\lambda_1 + f_1\lambda_2 + f_0\lambda_3 = \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_1 + \lambda_1\lambda_2 + \lambda_3,\\
f_3 &= \frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3,\\
4f_4 &= f_3\lambda_1 + f_2\lambda_2 + f_1\lambda_3 + f_0\lambda_4\\
&= \left(\frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3\right)\lambda_1 + \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_2 + \lambda_1\lambda_3 + \lambda_4,\\
f_4 &= \frac{1}{24}\lambda_1^4 + \frac{1}{24}\lambda_2\lambda_1^2 + \frac{1}{12}\lambda_1\lambda_2\lambda_1 + \frac{1}{12}\lambda_3\lambda_1 + \frac{1}{8}\lambda_1^2\lambda_2 + \frac{1}{8}\lambda_2^2 + \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{4}\lambda_4.
\end{aligned}$$

Theorem 2.1. *Let $\{f_n\}$ be a higher coderivation on a coalgebra C with $f_0 = I$. Then there is a sequence $\{\lambda_n\}$ of coderivations on C such that*

$$(n+1)f_{n+1} = \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_i + \cdots + r_j}\right)\lambda_{r_i}\cdots\lambda_{r_1}\right),$$

where the inner summation is taken over all positive integers r_j, with $\sum_{j=1}^{i} r_j = n$.

Proof. We show that if f_n is of the above form then it satisfies the recursive relation of Proposition 2.2. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation we put $a_{r_i,\ldots,r_1} = \prod_{j=1}^{i}\frac{1}{r_i+\cdots+r_j}$. Note that if $r_1 + \cdots + r_i = n+1$ then $(n+1)a_{r_i,\ldots,r_1} = a_{r_i,\ldots,r_2}$. Moreover, $a_{n+1} = \frac{1}{n+1}$. Now we have

$$\begin{aligned}
(n+1)f_{n+1} &= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1} a_{r_i,\ldots,r_1}(n+1)\lambda_{r_i}\cdots\lambda_{r_1}\right) + \lambda_{n+1}\\
&= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}\sum_{\sum_{j=2}^{i} r_j = n+1-r_1} a_{r_i,\ldots,r_2}\lambda_{r_i}\cdots\lambda_{r_2}\right)\lambda_{r_1} + \lambda_{n+1}\\
&= \sum_{r_1=1}^{n}\sum_{i=2}^{n-(r_1-1)}\left(\sum_{\sum_{j=2}^{i} r_j = n-(r_1-1)} a_{r_i,\ldots,r_2}\lambda_{r_i}\cdots\lambda_{r_2}\right)\lambda_{r_1} + \lambda_{n+1}
\end{aligned}$$

$$= \sum_{r_1=1}^{n} f_{n-(r_1-1)}\lambda_{r_1} + \lambda_{n+1}$$

$$= \sum_{k=0}^{n} f_{n-k}\lambda_{k+1}. \qquad \square$$

Example 2.2. We evaluate the coefficients $a_{r_i,\ldots,r_1}$ for the case $n = 4$.

For $n = 4$ we can write

$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1.$$

By the definition of $a_{r_i,\ldots,r_1}$ we have

$$\begin{aligned}
a_4 &= \frac{1}{4},\\
a_{1,3} &= \frac{1}{1+3}\cdot\frac{1}{3} = \frac{1}{12},\\
a_{3,1} &= \frac{1}{3+1}\cdot\frac{1}{1} = \frac{1}{4},\\
a_{2,2} &= \frac{1}{2+2}\cdot\frac{1}{2} = \frac{1}{8},\\
a_{1,1,2} &= \frac{1}{1+1+2}\cdot\frac{1}{1+2}\cdot\frac{1}{2} = \frac{1}{24},\\
a_{1,2,1} &= \frac{1}{1+2+1}\cdot\frac{1}{2+1}\cdot\frac{1}{1} = \frac{1}{12},\\
a_{2,1,1} &= \frac{1}{2+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{8},\\
a_{1,1,1,1} &= \frac{1}{1+1+1+1}\cdot\frac{1}{1+1+1}\cdot\frac{1}{1+1}\cdot\frac{1}{1} = \frac{1}{24}.
\end{aligned}$$

We can therefore deduce that

$$f_4 = \frac{1}{4}\lambda_4 + \frac{1}{12}\lambda_3\lambda_1 + \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{8}\lambda_2\lambda_2 + \frac{1}{24}\lambda_2\lambda_1\lambda_1 + \frac{1}{12}\lambda_1\lambda_2\lambda_1 + \frac{1}{8}\lambda_1\lambda_1\lambda_2 + \frac{1}{24}\lambda_1\lambda_1\lambda_1\lambda_1.$$

Theorem 2.2. *Let C be a coalgebra, F be the set of all higher coderivations $\{f_n\}_{n=0,1,\ldots}$ on C with $f_0 = I$ and Λ be the set of all sequences $\{\lambda_n\}_{n=0,1,\ldots}$ of coderivations on C with $\lambda_0 = 0$. Then there is a one to one correspondence between F and Λ.*

Proof. Let $\{\lambda_n\} \in \Lambda$. Define $f_n : C \to C$ by $f_0 = I$ and

$$f_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_i+\cdots+r_j}\right)\lambda_{r_i}\cdots\lambda_{r_1}\right).$$

We show that $\{f_n\} \in F$. By Theorem 2.1, $\{f_n\}$ satisfies the recursive relation

$$(n+1)f_{n+1} = \sum_{k=0}^{n} f_{n-k}\lambda_{k+1}.$$

To show that $\{f_n\}$ is a higher coderivation, we use induction on n. For $n=0$ we have

$$\Delta f_0(a) = \Delta(a) = \sum a_{(1)} \otimes a_{(2)} = \sum f_0(a_{(1)}) \otimes f_0(a_{(2)}) = \sum (f_0(a))_{(1)} \otimes (f_0(a))_{(2)}.$$

Let us assume that $\Delta f_k(a) = \sum_{i=0}^{k} (f_i \otimes f_{k-i})\Delta(a)$ for $k \leq n$. Thus, we have

$$\begin{aligned}
(n+1)\Delta f_{n+1}(a) =& \sum_{k=0}^{n} \Delta f_{n-k}\lambda_{k+1}(a) \\
=& \sum_{k=0}^{n}\sum_{i=0}^{n-k} (f_i \otimes f_{n-k-i})\Delta\lambda_{k+1}(a) \\
=& \sum_{k=0}^{n}\sum_{i=0}^{n-k} (f_i \otimes f_{n-k-i})(I \otimes \lambda_{k+1} + \lambda_{k+1} \otimes I)\Delta(a) \\
=& \sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum (f_i \otimes f_{n-k-i}) \left(\sum a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) \otimes \lambda_{k+1}(a_{(1)}) \otimes a_{(2)}\right) \\
=& \sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum f_i(a_{(1)}) \otimes f_{n-k-i}(\lambda_{k+1}(a_{(2)})) \\
&+ f_i(\lambda_{k+1}(a_{(1)}) \otimes f_{n-k-i}(a_{(2)}).
\end{aligned}$$

Using our assumption, we can write

$$\begin{aligned}
(n+1)\Delta f_{n+1}(a) =& \sum_{i=0}^{n}\sum f_i(a_{(1)}) \otimes (n-i+1) f_{n-i+1}(a_{(2)}) \\
&+ \sum_{i=0}^{n}\sum (n-i+1)\left(f_{n-i+1}(a_{(1)}) \otimes f_i(a_{(2)})\right) \\
=& \sum_{i=0}^{n}\sum (n+1-i) f_i(a_{(1)}) \otimes f_{n+1-i}(a_{(2)}) \\
&+ \sum_{i=1}^{n+1}\sum i(f_i(a_{(1)}) \otimes f_{n+1-i}(a_{(2)}) \\
=& (n+1)\sum_{k=0}^{n+1}\sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\
=& (n+1)\sum_{k=0}^{n+1}(f_k \otimes f_{n+1-k})\Delta(a).
\end{aligned}$$

Thus, $\{f_n\} \in F$.

Conversely, suppose that $\{f_n\} \in F$. Define $\lambda_n : C \to C$ by $\lambda_0 = 0$ and

$$\lambda_n = nf_n - \sum_{k=0}^{n-2} f_{n-1-k}\lambda_{k+1}.$$

Then Proposition 2.2 ensures us that $\{\lambda_n\} \in \Lambda$. Now define $\varphi : \Lambda \to F$ by $\varphi(\{\lambda_n\}) = \{f_n\}$, where

$$f_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_i + \cdots + r_j} \right) \lambda_{r_i} \cdots \lambda_{r_1} \right).$$

Now φ is clearly a one to one correspondence. □

Recall that a higher coderivation $\{f_n\}$ is called ordinary if there is a coderivation λ such that $f_n = \frac{\lambda^n}{n!}$ for all n.

Corollary 2.1. *A higher coderivation $\{f_n\} = \varphi(\{\lambda_n\})$ on a coalgebra C is ordinary if and only if $\lambda_n = 0$ for $n \geq 2$. In this case $f_n = \frac{f_1^n}{n!}$.*

3. Conclusion

In this paper proving an equality for a coderivation on a coalgebra C, named general coLiebnitz rule for coderivations, we defined higher coderivations on a coalgebra C and then we characterized them in terms of the coderivations on C. Indeed, we showed that each higher coderivation is a combination of compositions of coderivations. Finally we proved there is a one to one correspondence between the set of all higher coderivations on C and all sequences of coderivations on C. As a corollary we characterize all higher coderivations which are ordinary.

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EXISTENCE RESULTS FOR KIRCHHOFF NONLOCAL FRACTIONAL EQUATIONS

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ABSTRACT. Fractional and nonlocal operators of elliptic type arise in a quite natural way in many different contexts. In this paper, we study the existence of solutions for a class of fractional equations, while the nonlinear part of the problem admits some perturbation property. We obtain some new criteria for existence of two and infinitely many solutions, using critical point theory. Some recent results are extended and improved. Several examples are presented to demonstrate the applications of our main results.

1. INTRODUCTION

In this paper we investigate the existence of multiple nontrivial weak solutions for Kirchhoff fractional problem

$$(\mathcal{L}_f^\lambda) \qquad \begin{cases} -\mathcal{L}_K u = \lambda f(u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with $n > 2s$, $s \in (0,1)$ and $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$, with smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $\lambda > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, $\mathcal{L}_K$ is the nonlocal operator defined as follows:

$$\begin{aligned} \mathcal{L}_K u(x) =& M\left(\int_Q |u(x)-u(y)|^2 K(x-y) \mathrm{d}x\mathrm{d}y\right) \\ & \times \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big) K(y) \mathrm{d}y, \end{aligned}$$

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where $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $Q := (\mathbb{R}^n \times \mathbb{R}^n) \backslash \mathcal{O}$ with $\mathcal{O} := (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{C}\Omega := \mathbb{R}^n \backslash \{0\}$, $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ is a function with the properties that:

(κ_1) $\gamma K \in \mathrm{L}^1(\mathbb{R}^n)$ where $\gamma(x) = \min\{|x|^2, 1\}$;
(κ_2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \backslash \{0\}$;
(κ_3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \backslash \{0\}$.

A special case of $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

which corresponds to the case $M \equiv 1$ and $K(x) = |x|^{-(n+2s)}$. One typical feature of problem $(\mathcal{L}_f^\lambda)$ is the nonlocality, in the sense that the value of $(-\Delta)^s u$ at any point $x \in \Omega$ depends not only on Ω, but actually on the entire space $\mathbb{R}^n$. In the special case, fractional Laplacian operator $-(-\Delta)^s$ (up to normalization constants) may be defined as

$$-(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where $P.V.$ is a particular value. It may be seen as the infinitesimal small generators of a Lévy motion stable diffusion operations [1]. This operator has been used in modelling various applied phenomena, like phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics, etc. On the other hand, and more importantly, fractional and non-fractional operators find many specific applications also in bio-mathematics and physics, which nowadays is a rather fashionable field of research; we, for instance, refer to [15,20,21]. To see more features, you can see [30, 34] and references therein. Recently, a lot of research work has been done to the study of semiclassical standing waves for the non-linear fractional Schrödinger equation of the form

$$i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s} (-\Delta)^s \psi + P(x)\psi - f(x, |\psi|), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where ε is a small positive constant, which corresponds to the Planck constant, $(-\Delta)^s$, $0 < s < 1$, is the fractional Laplacian, $P(x)$ is a potential function. Problem (1.1) models naturally many physical problems, such as phase transition, conservation laws, especially in fractional quantum mechanics, etc. (see [16]). It was introduced by Laskin [19] as a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy process. We refer to [12] for more physical background. To obtain standing waves of the fractional non-linear Schrödinger equation (1.1), we set $\psi(x,t) = e^{\frac{-iEt}{\varepsilon}} u(x)$ for some function $u \in H^s(\mathbb{R}^n)$, and let $V(x) = P(x) - E$. Then problem (1.1) is reduced to the following equation:

$$\varepsilon^{2s} (-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^n. \tag{1.2}$$

In quantum mechanics, when ε tends to 0, the existence and multiplicity of solutions to (1.2) is of particular importance.

In the nonlocal case, that is, when $s \in (0,1)$, the nonlocal model has attracted much attentions recently. For the case of a bounded domain, Ricceri [33] established a theorem tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. In [8], Molica Bisci and Repovš studied a class of nonlocal fractional Laplacian equations depending on two real parameters. More precisely, by using an appropriate analytical context on fractional Sobolev spaces due to Servadei and Valdinoci, they established the existence of three weak solutions for nonlocal fractional problems exploiting an abstract critical point result for smooth functionals. They emphasized that the dependence of the underlying equation from one of the real parameters is not necessarily of affine type. For more related results, we refer the reader to [24–26] and the references therein.

The interest in studying problems like problem $(\mathcal{L}_f^\lambda)$ relies not only on mathematical purposes but also on their significance in real models. For example, in the Appendix of paper [17], the authors constructed a stationary Kirchhoff variational problem, which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string.

Kirchhoff models take into consideration the length changes of the string produced by transverse vibrations (see [18]). Fractional and nonlocal operators of elliptic type which is modeled by the singularity at infinity is an emerging research field. From the physical viewpoint, nonlocal operators play a considerable role in characterizing a set of phenomena. A general reference for this issue is [39], where the author explained two models of flow in porous media, including nonlocal diffusion effects, providing a long list of references related to physical phenomena and nonlocal operators. The first model is based on Darcy's law, and the pressure is associated with the density by an inverse fractional Laplacian operator. The second model mostly follows fractional Laplacian flows but it is nonlinear. In contrast to the usual porous medium flows, it has infinite speed of propagation. On the other hand, fractional nonlocal operators arise in a quite natural way in many different contexts. See for instance the references [5–7] and [2, 4, 8, 13, 25, 28, 38]. For example, Molica Bisci in [25] studied the existence of infinitely many weak solutions to the problem $(\mathcal{L}_f^\lambda)$ where $f(x,u)$ replaced by $f(u)$ with $x \in \Omega$ in the case $\lambda = 1$ and $M \equiv 1$. We have shown in Remark 4.1 that our results in Theorem 1.2 are different from [25, Theorem 1.1].

Recently, some researchers have studied the existence and multiplicity of solutions for fractional equations of Kirchhoff type; we refer the reader to [3, 10, 11, 14, 23, 29, 40, 42] and the references therein. For example Chen and Deng in [10] based on Ekeland's variational principle investigated the existence of solutions to a Kirchhoff type problem involving the fractional p-Laplacian operator. It established in [23] the multiplicity of weak solutions for a Kirchhoff-type problem driven by a fractional

p-Laplacian operator with homogeneous Dirichlet boundary conditions:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u(x) = f(x,u), & \text{in}\,\Omega, \\ u = 0, & \text{in}\,\mathbb{R}^n\backslash\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with Lipshcitz boundary $\partial\Omega$, $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0 < s < 1 < p < N$ such that $sp < N$, M is a continuous function and f is a Carathéodory function satisfying the Ambrosetti-Rabinowitz condition. When f satisfies the suplinear growth condition, they obtained the existence of a sequence of nontrivial solutions by using the symmetric mountain pass theorem, and when f satisfies the sub-linear growth condition, they obtained infinitely many pairs of nontrivial solutions by applying the Krasnoselskii genus theory. By using an appropriate analytical context on fractional Sobolev spaces, Molica Bisci and Tulone in [29] obtained the existence of one non-trivial weak solution for nonlocal fractional problem $(\mathcal{L}_f^\lambda)$ in the case $M(x) = a + bx$ where a, b are positive numbers. Xiang et al. in [40] studied the problem

$$\begin{cases} M\left(x, [u]_{s,p}^p\right)(-\Delta)_p^s u(x) = f(x,u,[u]_{s,p}^p), & \text{in}\,\Omega, \\ u = 0, & \text{in}\,\mathbb{R}^n\backslash\Omega, \end{cases}$$

where $[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y$, $(-\Delta)_p^s$ is a fractional p-Laplace operator, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $M : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function satisfying the Ambrosetti-Rabinowitz condition. They obtained the existence of nonnegative solutions by using the Mountain Pass Theorem and an iterative scheme.

The present paper focuses on this issue since it is clear that in problem $(\mathcal{L}_f^\lambda)$ there is a singularity in the term $\mathcal{L}_k(u)$, which causes difficulties in the proof. In this paper, we are concerned with the existence results for the problem $(\mathcal{L}_f^\lambda)$, and prove at least two weak solutions and infinitely many weak solutions for the problem $(\mathcal{L}_f^\lambda)$. Several special cases of the main results and two illustrating examples are also presented. We use the following assumptions throughout this paper:

- $(\mathcal{M})$ $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that satisfies $m_0 t^{\alpha-1} \le M(t) \le m_1 t^{\alpha-1}$ for all $t \in \mathbb{R}^+$, where $m_1 > m_0 > 0$ and $1 < \alpha < \frac{2n}{n-2s}$;
- $(\mathcal{F}_1)$ there exists a constant $\beta > \frac{2m_1\alpha}{m_0}$ with $0 < \beta F(t) \le \xi f(t)$ for all $t \in \mathbb{R}\backslash\{0\}$;
- $(\mathcal{F}_2)$ $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{\alpha-1}} = 0$, i.e., f is $(\alpha-1)$-sublinear at infinity.

The main results of this paper are presented as follows.

Theorem 1.1. *Assume that the assumptions* $(\mathcal{M})$, $(\mathcal{F}_1)$ *and* $(\mathcal{F}_2)$ *hold. Then, if* $f(t) \ge 0$ *for all* $t \in \mathbb{R}$, *the problem* $(\mathcal{L}_f^\lambda)$ *has at least two weak solutions.*

Theorem 1.2. *Assume that the assumptions* $(\mathcal{M})$, $(\mathcal{F}_1)$ *and* $(\mathcal{F}_2)$ *hold. Then, if* $f(t)$ *is odd, the problem* $(\mathcal{L}_f^\lambda)$ *has infinitely many weak solutions.*

2. Preliminaries

In this part, we discuss some preliminary results which can be found in [34]. The functional space E denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in E belongs to $\mathrm{L}^2(\Omega)$ and

$$\Big((x,y)\mapsto (u(x)-u(y))\sqrt{K(x-y)}\Big)\in \mathrm{L}^2\Big((\mathbb{R}^n\times\mathbb{R}^n)\backslash(\complement\Omega\times\complement\Omega),\mathrm{d}x\mathrm{d}y\Big).$$

We denote by E_0 the following linear subspace of E

$$\mathrm{E}_0 := \{\mathrm{u}\in\mathrm{E}:\ \mathrm{u}=0 \text{ a.e. in } \mathbb{R}^{\mathrm{n}}\backslash\Omega\}.$$

We remark that E and E_0 are nonempty, since $\mathrm{C}^2_0(\Omega)\subseteq \mathrm{E}_0$ by [34, Lemma 11]. Moreover, the space E is endowed with the norm defined as

$$\|u\|_{\mathrm{E}} := \|u\|_{\mathrm{L}^2(\Omega)} + \Big(\int_Q |u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\Big)^{1/2}.$$

It is easily seen that $\|\cdot\|_{\mathrm{E}}$ is a norm on E (see [35]). By [35, Lemmas 6 and 7] in the sequel we can take the function

$$\mathrm{E}_0\ni u\mapsto \|u\|_{\mathrm{E}_0} := \Big(\int_Q |u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\Big)^{1/2} \tag{2.1}$$

as norm on E_0. Also $(\mathrm{E}_0,\|\cdot\|_{\mathrm{E}_0})$ is a Hilbert space with scalar product

$$\langle u,v\rangle_{X_0} := \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y.$$

See [35, Lemma 7]. Note that in (2.1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n\times\mathbb{R}^n$, since $v\in\mathrm{E}_0$ (and so $v=0$ a.e. in $\mathbb{R}^n\backslash\Omega$). While for a general kernel K satisfying conditions from (κ_1)-(κ_3) we have that $\mathrm{E}_0\subset\mathrm{H}^s(\mathbb{R}^n)$, in the model case $K(x):=|x|^{-(n+2s)}$ the space E_0 consists of all the functions of the usual fractional Sobolev space $\mathrm{H}^s(\mathbb{R}^n)$ which vanish a.e. outside Ω (see [37, Lemma 7]). Here $\mathrm{H}^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|u\|_{\mathrm{E}} := \|u\|_{\mathrm{H}^s(\mathbb{R}^n)} = \|u\|_{\mathrm{L}^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\Big)^{1/2}.$$

Remark 2.1. By [34, Lemma 8], the embedding $j:\mathrm{E}_0\hookrightarrow\mathrm{L}^\nu(\mathbb{R}^n)$ is continuous for any $\nu\in[1,2^*]$, while it is compact whenever $\nu\in[1,2^*)$, where $2^*:=\frac{2n}{n-2s}$ denotes the fractional critical Sobolev exponent. For further details on the fractional Sobolev spaces we refer to [12] and to the references therein, while for other details on E and E_0 we refer to [12], where these functional spaces were introduced, and also to [35–37], where various properties of these spaces were proved.

Definition 2.1 ([24])**.** We say that $u \in \mathrm{E}_0$ is a weak solution of $(\mathcal{L}_f^\lambda)$ if for all $v \in \mathrm{E}_0$

$$M\left(\int_Q |u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\right)\int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y$$
$$-\lambda\int_\Omega f(u(x))v(x)\mathrm{d}x = 0.$$

We refer the reader to [22, 32] for the following notations and results.

Theorem 2.1 ([22, Theorem 4.4])**.** *Let X be a Banach space, $\phi : X \to \mathbb{R}$ a function bounded from below and differentiable on X. If ϕ satisfies the* $(\mathrm{PS})_\mathrm{c}$*-condition with $c = \inf_X \phi$, then ϕ has a minimum on X.*

It is clear that the (PS)-condition implies the $(\mathrm{PS})_\mathrm{c}$-condition for each $c \in \mathbb{R}$.

Theorem 2.2 ([22, Theorem 4.10])**.** *Let $\varphi \in \mathrm{C}^1(X,\mathbb{R})$, and φ satisfy the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and $\inf_{v\in\partial\Omega}\varphi(v) > \max\{\varphi(u_0),\varphi(u_1)\}$, then there exists a critical point u of φ, i.e., $\varphi'(u) = 0$, with $\varphi(u) > \max\{\varphi(u_0),\varphi(u_1)\}$.*

Theorem 2.3 ([32, Theorem 9.12])**.** *Let X be an infinite dimensional real Banach space. Let $\varphi \in \mathrm{C}^1(X,\mathbb{R})$ be an even functional which satisfies the* (PS)*-condition and $\varphi(0) = 0$. Suppose that $X = V \bigoplus \mathrm{E}$, where V is infinite dimensional, and φ satisfies that*

- *(i) there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*
- *(ii) for any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(u) \geq 0$ on $W \setminus B_{R(W)}$.*

Then φ possesses an unbounded sequence of critical values.

We refer the reader to the paper [9, 41] in which Theorems 2.2 and 2.3 were successfully employed to ensure the multiple solutions of degenerate nonlocal problems and nonlinear impulsive differential equations with Dirichlet boundary conditions, respectively.

Corresponding to the functions f and M we introduce the functions $F : \mathbb{R} \to \mathbb{R}$ and $\widehat{M} : [0,+\infty) \to \mathbb{R}$, respectively, as $F(t) := \int_0^t f(\xi)\mathrm{d}\xi$ for all $t \in \mathbb{R}$ and $\widehat{M}(t) := \int_0^t M(\xi)\mathrm{d}\xi$ for all $t \in [0,+\infty)$, and consider the functionals $\Phi, \Psi : \mathrm{E}_0 \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\widehat{M}\left(\|u\|^2_{\mathrm{E}_0}\right) \quad \text{and} \quad \Psi(u) = \int_\Omega F(u(x))\mathrm{d}x, \tag{2.2}$$

for all $u \in \mathrm{E}_0$. Thus, by the assumption $(\mathcal{M})$ we have

$$\frac{m_0}{2\alpha}\|u\|^{2\alpha}_{\mathrm{E}_0} \leq \Phi(u) \leq \frac{m_1}{2\alpha}\|u\|^{2\alpha}_{\mathrm{E}_0},$$

which means that the functional $\Phi : \mathrm{E}_0 \to \mathbb{R}$ is coercive. On the other hand, Φ and Ψ are continuously Gâteaux differentiable. More precisely, we have

$$\begin{aligned}\Phi'(u)(v) =& M\left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\\ &\times \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y\end{aligned}$$

and

$$\Psi'(u)(v) = \int_\Omega f(u(x))v(x)\mathrm{d}x,$$

for every $u, v \in \mathrm{E}_0$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $u \in \mathrm{E}_0$ such that $\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for every $v \in \mathrm{E}_0$. Hence, the critical points of the functional J_λ are weak solutions of problem $(\mathcal{L}_f^\lambda)$.

3. Proofs of Main Results

We prove Theorems 1.1 and 1.2 in this section. For this we need the following remark and lemma.

Remark 3.1. If the assumption $(\mathcal{F}_1)$ holds and $m = \min_{|t|=1} F(t)$, then by the same argument as in [9, Remark 3.1], there exists a constant C_2 such that $F(t) \geq m|t|^\beta - C_2$ for all $t \in \mathbb{R}$.

Lemma 3.1. *Assume that* $(\mathcal{F}_1)$ *holds and* $\lambda > 0$. *Then* $J_\lambda(u)$ *satisfies the* (PS)-*condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ such that $\{J_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|J_\lambda(u_n)| \leq c_0$, $|J'_\lambda(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Therefore, we infer to deduce from the definition of J'_λ and the assumption $(\mathcal{F}_1)$ that

$$\begin{aligned}c_0 + c_1\|u_n\|_{\mathrm{E}_0} \geq& \beta J_\lambda(u_n) - J'_\lambda(u_n)(u_n)\\ &\geq \left(\frac{2\beta}{\alpha}m_0 - m_1\right)\|u_n\|^\alpha_{\mathrm{E}_0} - \lambda\int_\Omega (\beta F(u_n(t)) - f(u_n(t))(u_n(t)))\,\mathrm{d}t\\ &\geq \left(\frac{2\beta}{\alpha}m_0 - m_1\right)\|u_n\|^\alpha_{\mathrm{E}_0},\end{aligned}$$

for some $c_1 > 0$. Since $\beta > \frac{2m_1\alpha}{m_0}$, this implies that (u_n) is bounded. Now, as the same argument in [10, Lemma 2.2 (i)], we can prove that $\{u_n\}$ converges strongly to u in E_0. Consequently, J_λ satisfies (PS)-condition. □

3.1. **Proof of Theorem** 1.1.

Proof. In our case it is clear that $J_\lambda(0) = 0$. Lemma 3.1 has shown that J_λ satisfies the (PS)-condition.

Step 1. Since $1 \leq \alpha < \frac{2n}{n-2s}$, by Remark 2.1 the embedding $\mathrm{E}_0 \hookrightarrow \mathrm{L}^{\alpha}(\mathbb{R}^n)$ is compact and there exists $C_1 > 0$ such that for all $u \in \mathrm{E}_0$, $C_1 \|u\|_{\mathrm{L}^{\alpha}(\mathbb{R}^n)} \leq \|u\|_{\mathrm{E}_0}$ or

$$C_1^{2\alpha} \int_{\Omega} |u(x)|^{\alpha} \mathrm{d}x \leq \Big(\int_Q |u(x) - u(y)|^2 K(x-y) \mathrm{d}x \mathrm{d}y \Big)^{\alpha},$$

which implies that

$$\lambda_{\alpha} := \inf_{u \in \mathrm{E}_0 \setminus \{0\}} \frac{\int_Q |u(x) - u(y)|^2 K(x-y) \mathrm{d}x \mathrm{d}y}{\int_{\Omega} |u(x)|^{2\alpha} \mathrm{d}x} > 0.$$

By the assumptions $(\mathcal{M})$ and $(\mathcal{F}_2)$, and since $f(t) \geq 0$ for all $t \in \mathbb{R}$, we can take $\varepsilon < 2\alpha$ sufficiently small such that for sufficiently great $\sigma > 0$, $|f(t)| \leq \frac{\varepsilon m_0}{\alpha} |t|^{2\alpha - 1}$ for all $|t| \geq \sigma$ and $|F(t)| \leq \frac{\varepsilon m_0}{2\alpha^2} |t|^{2\alpha} + (\max_{|t| \leq \sigma} f(t))|t|$. Thus, for every $u \in \mathrm{E}_0$

$$\Psi(u) \leq \frac{\varepsilon m_0}{2\alpha^2} \int_{\Omega} |u(x)|^{\alpha} \mathrm{d}x + \max_{|t| \leq \sigma} f(t) \int_{\Omega} |u(x)| \mathrm{d}x. \tag{3.1}$$

By Hölder inequality, we have

$$\int_{\Omega} |u(x)| \mathrm{d}x \leq \sqrt{|\Omega|} \Big(\int_{\Omega} |u(x)|^2 \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Then, by (3.1)

$$\begin{aligned} \Psi(u) &\leq \frac{\varepsilon m_0}{2\alpha^2} \|u\|_{\mathrm{L}^{\alpha}(\Omega)}^{2\alpha} + \sqrt{|\Omega|} \max_{|t| \leq \sigma} f(t) \Big(\int_{\Omega} |u(x)|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon m_0}{2\alpha^2} \|u\|_{\mathrm{L}^{\alpha}(\Omega)}^{2\alpha} + \sqrt{|\Omega|} \max_{|t| \leq \sigma} f(t)\, \lambda_1^{-\frac{1}{2}} \|u\|_{\mathrm{E}_0} \\ &\leq \frac{\varepsilon m_0}{2\alpha^2} \|u\|_{\mathrm{E}_0}^{2\alpha} + \sqrt{|\Omega|} \max_{|t| \leq \sigma} f(t)\, \lambda_1^{-\frac{1}{2}} \|u\|_{\mathrm{E}_0}. \end{aligned}$$

Then, for any $u \in X$ by (2.2)

$$J_{\lambda}(u) \geq \frac{m_0}{2\alpha} \Big(1 - \frac{\varepsilon}{\alpha} \Big) \|u\|_{\mathrm{E}_0}^{2\alpha} - C_2 \max_{|t| \leq \sigma} f(t) \, \|u\|_{\mathrm{E}_0}, \tag{3.2}$$

where $C_2 = \sqrt{\frac{|\Omega|}{\lambda_1}}$. Now, by means of $\alpha > \frac{\varepsilon}{2}$, $p > 1$ and (3.2), it follows that J_{λ} is a coercive functional and is bounded from below. Since J_{λ} satisfies (PS)-condition by Lemma 3.1, Theorem 2.1 follows that there exists a minimum point u_0 of J_{λ} on E_0 and $0 = J_{\lambda}(0) \geq J_{\lambda}(u_0)$ and $J'_{\lambda}(u_0) = 0$.

Step 2. Since u_0 is a minimum point of J_{λ} on E_0 we can consider $L > 0$ sufficiently large such that $J_{\lambda}(u_0) \leq 0 < \inf_{u \in \partial B_L} J_{\lambda}(u)$ where $B_L = \{u \in \mathrm{E}_0 : \|u\|_{\mathrm{E}_0} < L\}$. Now we will show that there exists u_1 with $\|u_1\|_{\mathrm{E}_0} > L$ such that $J_{\lambda}(u_1) < \inf_{\partial B_L} J_{\lambda}(u)$. For this, let $\ell_1(t) \in \mathrm{E}_0$ and $u_1 = r\ell_1$, $r > 0$ where ℓ_1 corresponding to λ_1 is the first eigenfunction of $(\mathcal{L}_f^{\lambda})$ and $\|\ell_1\|_{\mathrm{E}_0} = 1$. By Remark 3.1, there exist constants $a_1, a_2 > 0$

such that $F(t) \geq a_1|t|^\beta - a_2$ for all $t \in \mathbb{R}$. Thus,

$$J_\lambda(u_1) = (\Phi - \lambda\Psi)(r\ell_1) \leq \frac{m_1}{2\alpha}\|r\ell_1\|_{\mathrm{E}_0}^{2\alpha} - \lambda \int_\Omega F(r\ell_1(x))\mathrm{d}x$$
$$\leq \frac{m_1 r^{2\alpha}}{2\alpha} - \lambda r^\beta a_1 \int_\Omega |\ell_1(x)|^\beta \mathrm{d}x + \lambda a_2|\Omega|.$$

So by $\beta \geq \frac{2m_1\alpha}{m_0}$, there exists sufficiently large $r > L > 0$ such that $J_\lambda(r\ell_1) < 0$. Therefore, $\max\{J_\lambda(u_0), J_\lambda(u_1)\} < \inf_{u\in\partial B_L} J_\lambda(u)$. Then, Theorem 2.2 by $X := \mathrm{E}_0$ and $\varphi := J_\lambda$ gives the critical point u^*. Therefore, u_0 and u^* are two critical points of J_λ, which are two solutions of $(\mathcal{L}_f^\lambda)$. □

3.2. **Proof of Theorem** 1.2.

Proof. Put $X := \mathrm{E}_0$. It is clear that, J_λ is continuously Gâteaux differentiable. In view of (2.2) it is obvious that $J_\lambda(u)$ is even and $J_\lambda(0) = 0$.

Step 1. We will show that J_λ satisfies condition (i) in Theorem 2.3. The inequality (3.2) shows the coercivity of J_λ and together with (PS)-condition, by minimization theorem [22, Theorem 4.4] the functional J_λ has a minimum critical point u with $J_\lambda(u) \geq \alpha > 0$ and $\|u\|_{\mathrm{E}_0} = \rho$ for $\rho > 0$ small enough.

Step 2. We will show that J_λ satisfies condition (ii) in Theorem 2.3. Let $W \subset \mathrm{E}_0$ be a finite dimensional subspace. By Remark 3.1, there exist constants $a_1, a_2 > 0$ such that $F(t) \geq a_1|t|^\beta - a_2$ for all $t \in \mathbb{R}$. Now, For every $r > 0$ and $u \in W \setminus \{0\}$ with $\|u\|_{\mathrm{E}_0} = 1$, one has

$$J_\lambda(ru) = (\Phi - \lambda\Psi)(ru) \leq \frac{m_1}{2\alpha}\|ru\|_{\mathrm{E}_0}^{2\alpha} - \lambda \int_\Omega F(ru(x))\mathrm{d}x$$
$$\leq \frac{m_1 r^{2\alpha}}{2\alpha}\|u\|_{\mathrm{E}_0}^{2\alpha} - \lambda r^\beta a_1 \int_\Omega |u(x)|^\beta \mathrm{d}x + \lambda a_2|\Omega| \to -\infty, \quad r \to +\infty.$$

The above inequality implies that there exists r_0 such that $\|ru\|_{\mathrm{E}_0} > \rho$ and $J_\lambda(ru) < 0$ for every $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that $J_\lambda(u) \leq 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.3, the functional $J_\lambda(u)$ possesses infinitely many critical points, i.e., the problem $(\mathcal{L}_f^\lambda)$ has infinitely many weak solutions. □

4. Examples and Remarks

In this section we present two examples and some remarks of our main results.

Example 4.1. Let $n = 2$, $s = \frac{1}{2}$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\} \subset \mathbb{R}^2$, $M(t) = tK(t)$ for all $t \in \mathbb{R}^+$ where $K(t)$ is 2-periodic extension of the function $k(t) = 2 - |t - 1|$, $0 \leq t \leq 2$, $f(t) = 1 + t^8$ for all $t \in \mathbb{R}$. We observe that $\frac{2n}{n-2s} = 4$, thus M satisfies the condition $(\mathcal{M})$ by $m_0 = 1$, $m_1 = 2$ and $\alpha = 2$. Also, M and f are two continuous functions, $f(t) \geq 0$ for all $t \in \mathbb{R}$, $\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{\alpha-1}} = \lim_{\xi\to 0^+} \frac{1+\xi^8}{\xi} = +\infty$, thus the assumption $(\mathcal{F}_2)$ is satisfied. Moreover, taking into account that $\lim_{|\xi|\to+\infty} \frac{\xi f(\xi)}{F(\xi)} =$

$\lim_{|\xi|\to+\infty} \frac{\xi+\xi^9}{\xi+\frac{1}{9}\xi^9} = 9 > 8 = \frac{2m_1\alpha}{m_0}$, by choosing $\beta = 9 > 8 = \frac{2m_1\alpha}{m_0}$, there exists $\varrho > 1$ such that the assumption $(\mathcal{F}_1)$ is fulfilled for all $|\xi| > \varrho$. Hence, by applying Theorem 1.1, for every $\lambda > 0$, the problem

$$\begin{cases} -M\bigg(\displaystyle\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus(\Omega\times\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^3}\mathrm{d}x\mathrm{d}y\bigg) \\ \times \displaystyle\int_{\mathbb{R}^2} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^3}\mathrm{d}y = \lambda(1+u^8), & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega, \end{cases}$$

possesses at least two nontrivial weak solutions in the space

$$\mathrm{H}_0^{1/2} := \left\{u \in \mathrm{H}^{1/2}(\mathbb{R}^2) : u = 0 \text{ a.e. in } \mathbb{R}^2\setminus\Omega\right\}.$$

Example 4.2. Let $n = 2$, $s = \frac{1}{2}$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\} \subset \mathbb{R}^2$, $M(t) = (\frac{3}{2} + \frac{1}{2}\sin t)t$ for all $t \in \mathbb{R}^+$, $f(t) = 1 + t^9$ for all $t \in \mathbb{R}$. We observe that $\frac{2n}{n-2s} = 4$, thus M satisfies the condition $(\mathcal{M})$ by $m_0 = 1$, $m_1 = 2$ and $\alpha = 2$. Also, M and f are two continuous functions, f is odd and $\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{\alpha-1}} = \lim_{\xi\to 0^+} \frac{1+\xi^9}{\xi} = +\infty$, thus the assumption $(\mathcal{F}_2)$ is satisfied. Moreover, taking into account that $\lim_{|\xi|\to+\infty} \frac{\xi f(\xi)}{F(\xi)} = \lim_{|\xi|\to+\infty} \frac{\xi+\xi^{10}}{\frac{1}{2}\xi+\frac{1}{10}\xi^{10}} = 10 > 8 = \frac{2m_1\alpha}{m_0}$, by choosing $\beta = 10 > 8 = \frac{2m_1\alpha}{m_0}$, the assumption $(\mathcal{F}_1)$ is fulfilled. Hence, by choosing $\sigma = \frac{1}{2}$ and applying Theorem 1.2, for every $\lambda > 0$, the problem

$$\begin{cases} -M\bigg(\displaystyle\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus(\Omega\times\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^3}\mathrm{d}x\mathrm{d}y\bigg) \\ \times \displaystyle\int_{\mathbb{R}^2} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^3}\mathrm{d}y = \lambda(1+u^9), & \text{in}\,\Omega, \\ u = 0, & \text{on}\,\partial\Omega, \end{cases}$$

has infinitely many weak solutions in the space $\mathrm{H}_0^{1/2}$.

Remark 4.1. Example 4.2 shows that our existence results to establish infinitely many solutions for the problem $(\mathcal{L}_f^\lambda)$ in Theorem 1.2 is different from the existence results of Molica Bicsi in [25, Theorem 1.1]. Because, firstly in Example 4.2 we have $M \neq 1$, while in [25, Theorem 1.1], $M \equiv 1$, and the second the function f in [25, Theorem 1.1] should satisfy in

$$|f(t)| \leq a_1 + a_2|t|^{q-1}, \qquad a_1, a_2 > 0,\ q \in \left(2, \frac{2n}{n-2s}\right),\ t \in \mathbb{R}, \tag{4.1}$$

while in Example 4.2, $\frac{2n}{n-2s} = 4$ and $f(t) = 1 + t^9$, and so f does not apply to (4.1).

Remark 4.2. By [28, Subsection 1.1], if $f(0) \neq 0$, then Theorem 1.1 ensures the existence of two nontrivial weak solutions for the problem $(\mathcal{L}_f^\lambda)$. If the condition $f(0) \neq 0$ does not hold, the second solution u_2 of the problem $(\mathcal{L}_f^\lambda)$ may be trivial, but the problem has at least a nontrivial solution. Moreover, by the same argument as [28, Corollary 3] we can prove that, under the condition that $f(0) = 0$, the solutions given by Theorem 1.1 has constant sign, i.e., Theorem 1.1 provides non-negative (non-positive) solutions.

Remark 4.3. By the similar arguments as given in the proof of [28, Subsection 4.1] the non-triviality of the second weak solution ensured by Theorem 1.1 can be achieved also in the case $f(0) = 0$ requiring the extra condition at zero in the form of

$$\limsup_{\xi\to 0^+} \frac{f(\xi)}{|\xi|} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{f(\xi)}{|\xi|} > -\infty. \tag{4.2}$$

Indeed, let $\lambda > 0$ and let Φ and Ψ be as given in Section 3. Due to Theorem 2.1 and Lemma 3.1, $J_\lambda = \Phi - \lambda\Psi$ has a critical point u_λ that is a global minimum of J_λ. We will prove that the function u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{4.3}$$

Owing to the assumptions (4.2), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $0 < \sigma < 1$) such that $\lim_{n\to+\infty} \frac{f(\xi_n)}{|\xi_n|} = +\infty$ and $F(\xi) \geq \kappa|\xi|^2$ for every $\xi \in [0, \sigma]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in X$ such that $v(t) \in [0, 1]$ for every $t \in \Omega$, $v(t) = 1$ for every $t \in \mathcal{G}$ and $v(t) = 0$ for every $x \in \Omega \setminus D$. Hence, fix $N > 0$ and consider a real positive number η with

$$N < \frac{2\alpha\eta|\mathcal{G}| + 2\alpha\kappa \int_{D\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_1\|v\|_{\mathrm{E}_0}^{2\alpha}}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and $F(\xi_n) \geq \eta|\xi_n|^2$ for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n large enough), one has

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} \geq \frac{F(\xi_n)|\mathcal{G}| + \int_{D\setminus\mathcal{G}} F(\xi_n v(t))\mathrm{d}t}{\Phi(\xi_n v)} > \frac{2\alpha\eta|\mathcal{G}| + 2\alpha\kappa \int_{D\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_1\|v\|_{\mathrm{E}_0}^{2\alpha}} > N.$$

Since N could be arbitrarily large, we get $\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty$, from which (4.3) clearly follows. So, there exists a sequence $\{\zeta_n\} \subset X$ strongly converging to zero such that, for n large enough, $J_\lambda(\zeta_n) = \Phi(\zeta_n) - \lambda\Psi(\zeta_n) < 0$. Since u_λ is a global minimum of J_λ, we obtain $J_\lambda(u_\lambda) < 0$, so that u_λ is not trivial.

Remark 4.4. We observe that if f is non-negative, Theorem 1.1 is a bifurcation result in the sense that the pair $(0, 0) \in \mathrm{E}_f^\lambda \subset \mathrm{E}_0 \times \mathbb{R}$ with

$$\mathrm{E}_f^\lambda := \Big\{(u_\lambda, \lambda) \in \mathrm{E}_0 \times (0, \infty) : u_\lambda \text{ is a non-trivial weak solution of } (\mathcal{L}_f^\lambda)\Big\}.$$

Practically, by the proof of Theorem 1.1, $\|u_\lambda\|_{\mathrm{E}_0} \to 0$ as $\lambda \to 0$. Hence, there exist two sequences $\{u_j\}$ in E_0 and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\lambda_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to \infty$. Moreover, since f is nonnegative, $\Psi(u) < 0$ for all $u \in \mathbb{R}$ and thus the mapping $(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing. Hence, for every

$\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the weak solutions u_{λ_1} and u_{λ_2} ensured by Theorem 1.1 are different.

Remark 4.5. If $f(u)$ is an odd function we can give the same result as Theorem 1.2 by setting the following assumptions on nonlinear term:

($\mathcal{F}_3$) there exist constants $R > 0$ and $0 < \lambda L_1 < \frac{1}{2}\min\{1, m_0\}$ such that $F(u) \leq L_1|u|^2$ for all $u \in \mathbb{R}$ with $|u| \leq R$;

($\mathcal{F}_4$) there exist constants $R_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \beta$ such that $F(u) \geq \delta_1|u|^{\alpha_1}$, for all $u \in \mathbb{R}$ with $|u| \geq R$;

($\mathcal{F}_5$) there exist constants $\beta > \frac{m_1\alpha}{m_0}$, $\delta_1 \geq 0$ and $0 < \alpha_2 < 2$ such that $\nu F(\xi) - \xi f(\xi) \leq \delta_2|u|^{\alpha_2}$.

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GENERALIZATIONS OF SOME BERNSTEIN-TYPE INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. In this paper, we establish some new Bernstein-type bounds for the polar derivative of constrained polynomials on the unit circle in the plane. The obtained results sharpen some known estimates for the ordinary derivative of polynomials as special cases.

1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{v=0}^{n} c_v z^v$ of degree n. The extremal problems of functions of complex variables and the results where some approches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. A classical result due to Bernstein [2], that relates an estimate of the size of the derivative and the polynomial for the sup-norm on the unit circle states that: if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The above inequality (1.1) was proved by Bernstein in 1912. Later in 1985, Frappier, Rahman and Ruscheweyh [3] strengthened (1.1), by proving that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{1\leq l\leq 2n}\left|P(e^{\frac{il\pi}{n}})\right|. \tag{1.2}$$

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Clearly (1.2) represents a refinement of (1.1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than the maximum of $|P(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z) = z^n + ia, \ a > 0$. Following the approach of Frappier, Rahman and Ruscheweyh [3], Aziz [1] showed that the bound in (1.2) can be considerably improved. In fact, Aziz proved that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\Big|P'(z)\Big| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{1.3}$$

where

$$M_\alpha = \max_{1\leq l\leq n}\Big|P(e^{i(\alpha+2l\pi)/n})\Big|, \tag{1.4}$$

for all real α.

In the same paper, Aziz obtained a lower bound for the maximum of $|P'(z)|$ on $|z| = 1$, by proving that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\Big|P'(z)\Big| \geq \frac{n}{2}\Big\{2\max_{|z|=1}\Big|P(z)\Big| - \Big(M_0 + M_\pi\Big)\Big\}. \tag{1.5}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1}\Big|P'(z)\Big| \leq \frac{n}{2}\max_{|z|=1}\Big|P(z)\Big|, \tag{1.6}$$

whereas, if $P(z)$ has no zeros in $|z| > 1$, then

$$\max_{|z|=1}\Big|P'(z)\Big| \geq \frac{n}{2}\max_{|z|=1}\Big|P(z)\Big|. \tag{1.7}$$

Inequality (1.6) was conjectured by Erdős and later proved by Lax [6], whereas inequality (1.7) is due to Turán [18]. Ideally, it is natural to look for improving results in (1.3) when $P(z)$ does not vanish in the unit disk, and accordingly Aziz [1] proved that if $P \in \mathbb{P}_n$, and $P(z) \neq 0$ in $|z| < 1$, then for every real number α,

$$\max_{|z|=1}\Big|P'(z)\Big| \leq \frac{n}{2}\Big\{M_\alpha^2 + M_{\alpha+\pi}^2\Big\}^{\frac{1}{2}}, \tag{1.8}$$

where M_α is defined by (1.4).

It is important to mention that different versions of the Bernstein and Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. These inequalities have their own significance and importance in Approximation theory. One of such generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative. Before proceeding to our main results, let us remind that the polar derivative $D_\beta P(z)$ of $P(z)$ where $P \in \mathbb{P}_n$, with respect to the point β is defined as

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z).$$

Note that $D_\beta P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to β (see [7]). It generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\} := P'(z),$$

uniformly with respect to z for $|z|\leq R,\ R>0$.

More information on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [9] and Rahman and Schmeisser [17].

Over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z=0$, the modulus of largest root of $P(z)$, restrictions on coefficients, using higher order derivatives, etc. Many of these generalizations involve the comparison of polar derivative $D_\beta P(z)$ with various choices of $P(z)$, β and other parameters. The latest research and development on this topic can be found in the papers ([5, 8, 10, 11, 13–16, 20]).

The main purpose of this paper is to obtain some upper bound estimates for the maximal modulus of polar derivative of a polynomial on a disk under the assumption that the polynomial has no zeros either in the disk $|z|<k$ or in $|z|>k,\ k>0$. The obtained results sharpen as well generalize some already known estimates for the ordinary derivative of polynomials as special cases.

2. Main Results

Theorem 2.1. *If $P\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<k,\ k\geq 1$, then for every complex number β with $|\beta|\geq 1$*

$$\begin{aligned}\max_{|z|=1}\left|D_\beta P(z)\right| \leq & \frac{n}{2}\Bigg[2\max_{|z|=1}\left|P(z)\right| + \Big(|\beta|-1\Big)\Bigg\{M_\alpha^2+M_{\alpha+\pi}^2 \\ & -\frac{2}{(1+k)}\left((k-1)+\frac{2}{n}\left(\frac{|c_0|-k^n|c_n|}{|c_0|+k^n|c_n|}\right)\right)\left|P(z)\right|^2\Bigg\}^{\frac{1}{2}}\Bigg], \end{aligned} \tag{2.1}$$

where M_α is defined by (1.4)*.*

The result is best possible for $k=1$ and equality in (2.1) *holds for $P(z)=z^n+1$, with real $\beta\geq 1$.*

By taking $\alpha=0$ in (2.1), we get the following result.

Corollary 2.1. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$. If $t_1, t_2, \dots, t_n$ are the zeros of $z^n + 1$ and $s_1, s_2, \dots, s_n$ are the zeros of $z^n - 1$, then for $|\beta| \geq 1$*

$$\max_{|z|=1}\left|D_\beta P(z)\right| \leq \frac{n}{2}\left[2\max_{|z|=1}\left|P(z)\right| + \left(|\beta|-1\right)\left\{\left(\max_{1\leq j\leq n}|P(t_j)|\right)^2 + \left(\max_{1\leq j\leq n}|P(s_j)|\right)^2\right.\right.$$

$$\left.\left. - \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}}\right]. \tag{2.2}$$

The result is best possible for $k = 1$ and equality in (2.2) holds for $P(z) = z^n + 1$, with real $\beta \geq 1$.

Dividing both sides of inequality (2.1) by $|\beta|$ and letting $|\beta| \to \infty$, we get the following result.

Corollary 2.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then we have for every real α*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}},$$

where M_α is defined by (1.4).

It is easy to verify that Corollary 2.2 generalizes as well as sharpens inequality (1.8). Taking $k = 1$ in Corollary 2.2, we get the following result.

Corollary 2.3. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then we have for every real α,*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{n}\left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}},$$

where M_α is defined by (1.4).

The bound obtained in Corollary 2.3 is always sharper than the bound obtained from inequality (1.8), for this it needs to show that

$$\frac{|c_0| - |c_n|}{|c_0| + |c_n|} \geq 0,$$

which is equivalent to

$$|c_0| \geq |c_n|,$$

which is true as $P(z) \neq 0$ in $|z| < 1$.

If we divide both sides of inequality (2.2) by $|\beta|$ and let $|\beta| \to \infty$, we get the following result.

Corollary 2.4. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$. If $t_1, t_2, \dots, t_n$ are the zeros of $z^n + 1$, and $s_1, s_2, \dots, s_n$ are the zeros of $z^n - 1$, then*

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{2}\Bigg\{\left(\max_{1\leq j\leq n}|P(t_j)|\right)^2 + \left(\max_{1\leq j\leq n}|P(s_j)|\right)^2$$

$$- \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)\left|P(z)\right|^2\Bigg\}^{\frac{1}{2}}. \tag{2.3}$$

The result is best possible for $k = 1$ and equality in (2.3) holds for $P(z) = z^n + 1$.

Remark 2.1. It is easy to see that Corollary 2.4 generalizes the following result due to Wali and Shah [19, Corollary 1].

Theorem 2.2. *Let $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$. If $t_1, t_2, \dots, t_n$ are the zeros of $z^n + 1$, and $s_1, s_2, \dots, s_n$ are the zeros of $z^n - 1$, then for $|z| = 1$, we have*

$$\left|P'(z)\right| \leq \frac{n}{2}\left\{\left(\max_{1\leq j\leq n}\left|P(t_j)\right|\right)^2 + \left(\max_{1\leq j\leq n}\left|P(s_j)\right|\right)^2 - \frac{2}{n}\left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\left|P(z)\right|^2\right\}^{\frac{1}{2}}. \tag{2.4}$$

Equality in (2.4) holds for $P(z) = z^n + 1$.

If $P(z)$ has all its zeros on $|z| = k$, $k > 1$, then from Theorem 2.1, we get the following result.

Corollary 2.5. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros on $|z| = k$, $k > 1$, then for every complex number β, with $|\beta| \geq 1$*

$$\max_{|z|=1}\left|D_\beta P(z)\right| \leq \frac{n}{2}\left[2\max_{|z|=1}\left|P(z)\right| + \left(|\beta| - 1\right)\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\left(\frac{k-1}{k+1}\right)|P(z)|^2\right\}^{\frac{1}{2}}\right],$$

where M_α is defined by (1.4).

Next as an application of Theorem 2.1, we prove the following result.

Theorem 2.3. *Let $P(z) = \sum_{v=0}^{n} c_v z^v \in \mathbb{P}_n$, $c_0 \neq 0$, with $P(z) \neq 0$ in $|z| > k$, $k \leq 1$, then for every complex number γ with $|\gamma| \leq 1$, we have for $|z| = 1$*

$$\left|D_\gamma P(z)\right| \leq \frac{n}{2}\Bigg[2|\gamma|\max_{|z|=1}\left|P(z)\right| + \left(1 - |\gamma|\right)$$

$$\times\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}}\Bigg], \tag{2.5}$$

where M_α is defined by (1.4).

The result is best possible for $k = 1$ and equality in (2.5) holds for $P(z) = z^n + 1$, with real $\gamma \leq 1$.

Remark 2.2. If we take $\gamma = 0$ in (2.5), we get for $|z| = 1$

$$\begin{aligned}&\Big|nP(z) - zP'(z)\Big|\\ &\leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right]\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{2.6}$$

If $\max\limits_{|z|=1}\Big|P(z)\Big| = \Big|P(e^{i\phi})\Big|$, we get from (2.6) that

$$\begin{aligned}\Big|P'(e^{i\phi})\Big| \geq &\frac{n}{2}\Bigg[2\max_{|z|=1}\Big|P(z)\Big|\\ &-\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left((1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right)\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}\Bigg].\end{aligned} \tag{2.7}$$

Since $\max\limits_{|z|=1}\Big|P'(z)\Big| \geq \Big|P'(e^{i\phi})\Big|$, we get from (2.7) that

$$\begin{aligned}\max_{|z|=1}\Big|P'(z)\Big| \geq &\frac{n}{2}\Bigg[2\max_{|z|=1}\Big|P(z)\Big|\\ &-\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left((1-k) + \frac{2k}{n}\left(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\right)\right)\Big|P(z)\Big|^2\right\}^{\frac{1}{2}}\Bigg].\end{aligned} \tag{2.8}$$

Taking $k = 1$ in (2.8), we immediately get a refinement of (1.5) when all the zeros of $P(z)$ lie in $|z| \leq 1$.

Remark 2.3. It may be remarked here that for $k = 1$, Theorems 2.1 and 2.3 were recently established by Mir [11].

3. Lemmas

We need the following lemmas to prove our theorems.

Lemma 3.1. *If* $x_v,\ v = 1, 2, \dots, n$ *is a sequence of real numbers such that* $x_v \geq 1$ *for all* $v \in \mathbb{N}$, *then*

$$\sum_{v=1}^{n} \frac{x_v - 1}{x_v + 1} \geq \frac{\prod_{v=1}^{n} x_v - 1}{\prod_{v=1}^{n} x_v + 1}, \quad \textit{for all } n \in \mathbb{N}.$$

Proof of Lemma 3.1. We prove this result with the help of mathematical induction and we use induction on n. The result is trivially true for $n = 1$.

For $n = 2$

$$\frac{x_1 - 1}{x_1 + 1} + \frac{x_2 - 1}{x_2 + 1} \geq \frac{x_1x_2 - 1}{x_1x_2 + 1},$$

if

$$\frac{2(x_1x_2-1)}{1+x_1+x_2+x_1x_2} \geq \frac{x_1x_2-1}{x_1x_2+1},$$

i.e., if $(x_1-1)(x_2-1) \geq 0$, which is true, since $x_1, x_2 \geq 1$. This shows that the result holds for $n=2$. Assume the result is true for $n=r \in \mathbb{N}$. Now since $\prod_{v=1}^{r} x_v \geq 1$, we have

$$\begin{aligned}
\sum_{v=1}^{r+1} \frac{x_v-1}{x_v+1} &= \sum_{v=1}^{r} \frac{x_v-1}{x_v+1} + \frac{x_{r+1}-1}{x_{r+1}+1} \\
&\geq \frac{\prod_{v=1}^{r} x_v - 1}{\prod_{v=1}^{r} x_v + 1} + \frac{x_{r+1}-1}{x_{r+1}+1} \quad \text{(by induction hypothesis)} \\
&\geq \frac{\prod_{v=1}^{r+1} x_v - 1}{\prod_{v=1}^{r+1} x_v + 1} \quad \text{(by the case } n=2\text{)}.
\end{aligned}$$

This shows that the result holds for $n=r+1$ as well. Therefore by the principle of mathematical induction, it follows that the result holds for all $n \in \mathbb{N}$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for each point z on $|z| = 1$ for which $P(z) \neq 0$, we have*

$$\text{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{1}{1+k}\left\{n - \left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}. \tag{3.1}$$

Proof of Lemma 3.2. Recall that $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. If $z_1, z_2, \ldots, z_n$ are the zeros of $P(z) = \sum_{v=0}^{n} c_v z^v$ of degree n, then $|z_v| \geq k$, $k \geq 1$, and we can write $P(z) = c_n \prod_{v=1}^{n}(z - z_v)$. This gives

$$\frac{zP'(z)}{P(z)} = \sum_{v=1}^{n} \frac{z}{z - z_v}.$$

Now for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with $P(e^{i\theta}) \neq 0$, we have

$$\begin{aligned}
\text{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) &= \sum_{v=1}^{n} \text{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - z_v}\right) \\
&\leq \sum_{v=1}^{n} \frac{1}{1+|z_v|} \\
&= \frac{n}{1+k} - \frac{1}{1+k} \sum_{v=1}^{n} \frac{|z_v| - k}{|z_v| + 1} \\
&\leq \frac{n}{1+k} - \frac{1}{1+k} \sum_{v=1}^{n} \frac{|z_v| - k}{|z_v| + k} \quad \text{(as } k \geq 1\text{)} \\
&= \frac{n}{1+k} - \frac{1}{1+k} \sum_{v=1}^{n} \frac{|z_v|/k - 1}{|z_v|/k + 1}.
\end{aligned}$$

Since $|z_v|/k \geq 1,\ v = 1, 2, \dots, n$, we get on using Lemma 3.1 for the points $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with $P(e^{i\theta}) \neq 0$,

$$\begin{aligned} \text{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) \leq & \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{\prod_{v=1}^{n}|z_v|/k - 1}{\prod_{v=1}^{n}|z_v|/k + 1}\right) \\ = & \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{|c_0|/k^n|c_n| - 1}{|c_0|/k^n|c_n| + 1}\right), \end{aligned}$$

which is equivalent to (3.1). This completes the proof of Lemma 3.2. □

Lemma 3.3. *If $P \in \mathbb{P}_n$, then for $|z| = 1$, and for any real number α,*

$$\left|P'(z)\right|^2 + \left|Q'(z)\right|^2 \leq \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right),$$

where M_α is defined by (1.4).

The above lemma is due to Aziz [1].

Lemma 3.4. *If $P \in \mathbb{P}_n$, then for $|z| = 1$,*

$$\left|P'(z)\right| + \left|Q'(z)\right| \leq n \max_{|z|=1}|P(z)|.$$

The above lemma is a special case of a result due to Govil and Rahman [4].

4. Proof of the Theorems

Proof of Theorem 2.1. Recall that $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. First, we suppose that $P(z)$ has no zeros on $|z| = k,\ k > 1$ and therefore, all the zeros of $P(z)$ lie in $|z| > k$, we have by Lemma 3.2 for $|z| = 1$

$$2\text{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{2}{1+k}\left\{n - \left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right\}. \tag{4.1}$$

Also it easily follows that

$$\left|Q'(z)\right| = \left|nP(z) - zP'(z)\right|, \quad \text{for } |z| = 1, \tag{4.2}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$. This implies

$$\left|\frac{zQ'(z)}{P(z)}\right|^2 = \left|n - \frac{zP'(z)}{P(z)}\right|^2 = n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2n\text{Re}\left(\frac{zP'(z)}{P(z)}\right),$$

which by using (4.1) yields for $|z| = 1$

$$2\left|P'(z)\right|^2 \leq \left|P'(z)\right|^2 + \left|Q'(z)\right|^2 + \left[\frac{2n^2}{1+k} - \frac{2n}{1+k}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right) - n^2\right]|P(z)|^2. \tag{4.3}$$

On combining Lemma 3.3 and inequality (4.3), we get for $|z| = 1$

$$\left|P'(z)\right| \leq \frac{n}{2}\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{(1+k)}\left[(k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right]\left|P(z)\right|^2\right\}^{\frac{1}{2}}. \tag{4.4}$$

The above inequality (4.4) trivially holds for $k = 1$ as well as for points z on $|z| = 1$ for which $P(z) = 0$ by (1.8). Using the definition of polar derivative of a polynomial $P \in \mathbb{P}_n$ with respect to the complex number β, we have

$$\begin{aligned}\left|D_\beta P(z)\right| =& |nP(z) + (\beta - z)P'(z)|\\ \leq& \left|nP(z) - zP'(z)\right| + |\beta|\left|P'(z)\right|\\ =& \left|Q'(z)\right| + |\beta|\left|P'(z)\right| \quad \text{(using (4.2))}\\ \leq& n\max_{|z|=1}|P(z)| + (|\beta| - 1)\left|P'(z)\right| \quad \text{(by Lemma 3.4).} \qquad (4.5)\end{aligned}$$

Inequality (4.5) in conjunction with inequality (4.4) gives,

$$\begin{aligned}\max_{|z|=1}\left|D_\beta P(z)\right| \leq& \frac{n}{2}\Bigg[2\max_{|z|=1}\left|P(z)\right| + \left(|\beta| - 1\right)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2\\ &- \frac{2}{(1+k)}\left((k-1) + \frac{2}{n}\left(\frac{|c_0| - k^n|c_n|}{|c_0| + k^n|c_n|}\right)\right)|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg].\end{aligned}$$

This completes the proof of Theorem 2.1. ☐

Proof of Theorem 2.3. By hypothesis, the polynomial $P(z) = \sum_{v=0}^{n} c_v z^v$, $c_0 \neq 0$ has all its zeros in $|z| \leq k$, $k \leq 1$, therefore, the polynomial $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ has no zeros in $|z| < 1/k$, $1/k \geq 1$. Applying Theorem 2.1 to the polynomial $Q(z)$, we get for $|\beta| \geq 1$ and $|z| = 1$

$$\begin{aligned}\left|D_\beta Q(z)\right| \leq& \frac{n}{2}\Bigg[2\max_{|z|=1}|Q(z)| + \left(|\beta| - 1\right)\Bigg\{Y_\alpha^2 + Y_{\alpha+\pi}^2\\ &- \frac{2}{(1+1/k)}\left[(1/k - 1) + \frac{2}{n}\left(\frac{|c_n| - 1/k^n|c_0|}{|c_n| + 1/k^n|c_0|}\right)\right]|Q(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg]. \qquad (4.6)\end{aligned}$$

Since $|P(z)| = |Q(z)|$ for $|z| = 1$, it follows that

$$Y_\alpha = \max_{1\leq l\leq n}\left|Q(e^{i(\alpha+2l\pi)/n})\right| = \max_{1\leq l\leq n}\left|P(e^{i(\alpha+2l\pi)/n})\right| = M_\alpha.$$

Using this in (4.6), we get for $|\beta| \geq 1$ and $|z| = 1$

$$\Big|D_\beta Q(z)\Big| \leq \frac{n}{2}\Bigg[2\max_{|z|=1}|P(z)| + \Big(|\beta| - 1\Big)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2$$

$$- \frac{2}{(1+k)}\bigg[(1-k) + \frac{2k}{n}\bigg(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\bigg)\bigg]|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg]. \tag{4.7}$$

For $|z| = 1$, we have $z\overline{z} = 1$, then it is easy to verify (for example, see [11]), that for $|\alpha| \neq 0$

$$\Big|D_\beta Q(z)\Big| = \Big|\bar{\beta}\Big|\Big|D_{1/\bar{\beta}}P(z)\Big|.$$

Replacing $1/\bar{\beta}$ by γ, so that $|\gamma| \leq 1$, we obtain from (4.7), that

$$\Big|D_\gamma P(z)\Big| \leq \frac{n}{2}\Bigg[2|\gamma|\max_{|z|=1}|P(z)| + \Big(1 - |\gamma|\Big)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2$$

$$- \frac{2}{(1+k)}\bigg[(1-k) + \frac{2k}{n}\bigg(\frac{k^n|c_n| - |c_0|}{k^n|c_n| + |c_0|}\bigg)\bigg]|P(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg],$$

for $|z| = 1$ and $|\gamma| \leq 1$.

This completes the proof of Theorem 2.3. □

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NOTE ON THE MULTIFRACTAL FORMALISM OF COVERING NUMBER ON THE GALTON-WATSON TREE

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Abstract. We consider, for t in the boundary of Galton-Watson tree ($\partial\mathsf{T}$), the covering number $\mathsf{N}_n(t)$ by cylinder of generation n. For a suitable set I and a sequence $(s_{n,\gamma})$, we establish almost surely, and uniformly on γ, the Hausdorff and packing dimensions of the set $\{t \in \partial\mathsf{T} : \mathsf{N}_n(t) - nb \sim s_{n,\gamma}\}$ for $b \in I$.

1. Introduction and main results

Let (N, X) be a random vector with independent components taking values in $\mathbb{N}^2$, where $\mathbb{N}$ denotes the set of non-negative integers. Then let $\{(N_u, X_u)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ be a family of independent copies of the vector (N, X) indexed by the set of finite words over the alphabet $\mathbb{N}_+$: the set of positive integers ($n = 0$ corresponds to the empty sequence denoted $\emptyset$). Let T be the Galton-Watson tree with defining elements $\{N_u\}$: we have $\emptyset \in \mathsf{T}$, if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$ then ui, the concatenation of u and i, belongs to T if and only if $1 \leq i \leq N_u$ and if $ui \in \mathsf{T}$, then $u \in \mathsf{T}$. Similarly, for each $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, denote by $\mathsf{T}(u)$ the Galton-Watson tree rooted at u and defined by the $\{N_{uv}\}$, $v \in \bigcup_{n\geq 0}\mathbb{N}_+^n$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton-Watson tree is supercritical. We also assume that the probability of extinction is equal to 0, so that $\mathbb{P}(N \geq 1) = 1$.

For each infinite word $t = t_1t_2\cdots \in \mathbb{N}_+^{\mathbb{N}_+}$ and $n \geq 0$, we set $t_{|n} = t_1\cdots t_n \in \mathbb{N}_+^n$ ($t_{|0} = \emptyset$). If $u \in \mathbb{N}_+^n$ for some $n \geq 0$, then n is the length of u and it is denoted by $|u|$. We denote by $[u]$ the set of infinite words $t \in \mathbb{N}_+^{\mathbb{N}_+}$ such that $t_{||u|} = u$.

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The set $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d : (u, v) \mapsto e^{-\sup\{|w|: u\in[w], v\in[w]\}},$$

with the convention $\exp(-\infty) = 0$. The boundary of the Galton-Watson tree T is defined as the compact set

$$\partial\mathsf{T} = \bigcap_{n\geq 1} \bigcup_{u\in\mathsf{T}_n} [u],$$

where $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$.

We consider X_u as the covering number of the cylinder $[u]$, that is to say, the cylinder $[u]$ is cut off with probability $p_0 = \mathbb{P}(X = 0)$ and is covered m times with probability $p_m = \mathbb{P}(X = m)$, $m = 1, 2, \ldots$

For $t \in \partial\mathsf{T}$, set

$$\mathsf{N}_n(t) = \sum_{k=1}^{n} X_{t_1\cdots t_k}.$$

Since this quantity depends on $t_1 \cdots t_n$ only, we also denote by $\mathsf{N}_n(u)$ the constant value of $\mathsf{N}_n(\cdot)$ over $[u]$ whenever $u \in \mathsf{T}_n$. The quantity $\mathsf{N}_n(t)$ is called the covered number (or more precisely the n-covered number) of the point t by cylinder of generation k, $k = 1, 2, \ldots, n$.

Consider an individual infinite branch $t_1 \cdots t_n \cdots$ in $\partial\mathsf{T}$. When $\mathbb{E}(X)$ is defined, the strong law of large number yields $\lim_{n\to\infty} n^{-1}\mathsf{N}_n(t) = \mathbb{E}(X)$. It is also well known, in the theory of the birth process, (see [15]) that almost surely (a.s.) $\lim_{n\to\infty} \mathsf{N}_n(t) = +\infty$ for every $t \in \mathcal{D} = \{0, 1\}^{\mathbb{N}}$ if and only if

$$p_0 = \mathbb{P}(X = 0) < \frac{1}{2}.$$

If this condition is satisfied, then a.s. every point is infinitely covered.

We consider, for $b \in \mathbb{R}$, the set

$$E_b = \Big\{t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{\mathsf{N}_n(t)}{n} = b\Big\}.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [3,8,11,14,16,21] and [4,7] for a general case. All these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [1,2,19] for the study of Mandelbrot measures dimension).

We will assume that the free energy of X defined as

$$\tau(q) = \log \mathbb{E}\Big(\sum_{i=1}^{N} e^{qX_i}\Big)$$

is finite over $\mathbb{R}$. We will assume, without loss of generality, that X is not constant so that the function τ is strictly convex. Let τ^* stand for the Legendre transform of the function τ, defined as

$$\tau^*(b) := \inf_{q\in\mathbb{R}} \Big(\tau(q) - qb\Big), \quad b \in \mathbb{R}.$$

We say that the multifractal formalism holds at $b \in \mathbb{R}$ if

$$\dim E_b = \mathrm{Dim}\, E_b = \tau^*(b),$$

where dim E_b is the Hausdorff dimension of E_b and Dim E_b is the packing dimension of E_b (see Section A for the definition). In the following, we define the sets

$$J = \Big\{ q \in \mathbb{R}; \tau(q) - q\tau'(q) > 0 \Big\},$$

$$\Omega^1_\alpha = \mathrm{int}\Big\{ q : \mathbb{E}\Big[\Big| \sum_{i=1}^{N} e^{qX_i} \Big|^\alpha \Big] < \infty \Big\},$$

$$\Omega^1 = \bigcup_{\alpha \in (1,2]} \Omega^1_\alpha,$$

$$\mathcal{J} = J \cap \Omega^1 \quad \text{and} \quad I = \Big\{ \tau'(q); q \in \mathcal{J} \Big\}.$$

Remark 1.1. It is well known, see [6, Proposition 3.1], that $L = \{\alpha \in \mathbb{R},\ \tau^*(\alpha) \geq 0\}$, is a convex, compact and non-empty set. In addition, if we assume that $J = \mathcal{J}$ then $I = \mathrm{int}(L)$, where $\mathrm{int}(L)$ is the interior of L (see also [6, Proposition 3.1.]) In particular, I is an interval.

Next, we define for $b, \gamma \in \mathbb{R}$ and for any positive sequence $s^\gamma = \{s_{n,\gamma}\}_n$ such that $s_{n,\gamma} = \mathrm{o}(n)$ and $\gamma \mapsto s_{n,\gamma}$ is analytic function, the set

$$E_{b,s^\gamma} = \Big\{ t \in \partial\mathsf{T} : \mathsf{N}_n(t) - nb \sim s_{n,\gamma} \text{ as } n \to +\infty \Big\},$$

where $\mathsf{N}_n(t) - nb \sim s_{n,\gamma}$ means that $(\mathsf{N}_n(t) - nb)_n$ and $(s_{n,\gamma})_n$ are two equivalent sequences. It is clear that $E_{b,s^\gamma} \subset E_b$. So, we can get with a simple covering argument, with probability 1, for all $b \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$\dim E_{b,s^\gamma} \leq \dim E_b \leq \mathrm{Dim}\, E_b \leq \tau^*(b), \tag{1.1}$$

(see Proposition 1 in [5] and Proposition 2.7 in [4]). Let us mention that the methods used to compute Hausdorff dimension of the sets E_b in, for example, [4, 7, 17, 18]) do not give results on dim E_{b,s^γ}. These sets were considered by Kahane and Fan in [15]. The authors considered the space $\{0,1\}^{\mathbb{N}}$ and they compute, for each b, almost surely (a.s.), the Hausdorff dimension of E_{b,s^γ} under the hypothesis :

$$s_{n,\gamma} = \mathrm{o}(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = \mathrm{o}(1) \quad \text{and} \quad \sqrt{n \ln\ln n} = \mathrm{o}(s_{n,\gamma}).$$

A special case of a sequence satisfying the above hypothesis is $s_{n,\gamma} = n^\gamma$ with $\gamma \in (1/2, 1)$. Later, Attia in [5], gives a stronger result in the sense that, a.s. for all $b \in I$, he computed the Hausdorff dimensions of the sets E_{b,s^γ} under the hypothesis

$$s_{n,\gamma} = \mathrm{o}(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = \mathrm{o}(1) \tag{1.2}$$

and there exists $\epsilon_n \to 0$ such that

$$\sum_{n\geq 1} \exp\Big(-\epsilon \sum_{k=1}^{n} \epsilon_k\, \eta_k(\gamma)^2 \Big) < +\infty, \quad \text{for all } \epsilon > 0. \tag{1.3}$$

In particular, we can choose

$$s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma} \quad \text{with } \gamma \in (0, 1/2).$$

Theorem 1.1 ([5]). *Let s^γ be a positive sequence satisfying* (1.2) *and* (1.3). *Then, a.s. for all $b \in I$*

$$\dim E_{b,s^\gamma} = \dim E_b = \tau^*(b).$$

This requires, for a given sequence s^γ, a simultaneous building of an inhomogeneous Mandelbrot measure and a computing of their dimensions. In particular, for

$$s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma},$$

we have for all $\gamma \in (0, 1/2)$, a.s. $\dim E_{b,s^\gamma} = \tau^*(b)$. To state our main result, let $s^\gamma = (s_{n,\gamma})_n$ be a positive sequence and we define the set Λ_s to be any set of $\mathbb{R}$ such that

$$\Lambda_s \subseteq \Big\{ \gamma \in \mathbb{R}, \text{ such that } (s_{n,\gamma}) \text{ satisfies (1.2) and (1.3)} \Big\} \tag{1.4}$$

and, for $k \geq 1$

$$\widetilde{\eta}_k = \inf_{\gamma \in \Lambda_s} \eta_k(\gamma) > 0. \tag{1.5}$$

We suppose the following hypothesis.

Hypothesis 1.2. There exists a sequence $\epsilon_n \to 0$ such that

$$\sum_{n \geq 1} \exp\Big(-\epsilon \sum_{k=1}^{n} \epsilon_k \widetilde{\eta}_k^2 \Big) < +\infty, \quad \text{for all } \epsilon > 0.$$

Clearly this hypothesis is satisfied, for $s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma}$, with $\Lambda_s = [\epsilon, 1/2)$, $\epsilon > 0$. Applying the previous theorem we get the conclusion for each $\gamma \in \Lambda_s$ a.s. The goal of this note is to give a uniform result on γ. In addition, we determine the packing dimensions of the sets E_{b,s^γ}. More precisely we have the following result.

Theorem 1.3. *Let $s^\gamma = (s_{n,\gamma})_{n \geq 1}$ be a positive sequence and consider a set Λ_s satisfying* (1.4) *and* (1.5). *Under Hypothesis* 1.2, *we have, a.s.. for all $b \in I$ and for all $\gamma \in \Lambda_s$*

$$\dim E_{b,s^\gamma} = \dim E_b = \operatorname{Dim} E_b = \operatorname{Dim} E_{b,s^\gamma} = \tau^*(b).$$

2. Construction of Inhomogeneous Mandelbrot Measures

We define, for $(q, p) \in \mathcal{J} \times [1, \infty)$, the function

$$\varphi(p, q) = \exp\Big(\tau(pq) - p\tau(q)\Big).$$

From [5], for all nontrivial compact sets $K \subset \mathfrak{J}$ there exist $1 < p_K < 2$ and $\widetilde{p}_K > 1$ such that we have

$$\sup_{q \in K} \varphi(p_K, q) < 1, \quad \text{for all } 1 < p \leq p_K, \tag{2.1}$$

and

$$\sup_{q \in K} \mathbb{E}\Big(\Big(\sum_{i=1}^{N} e^{qX_i} \Big)^{\widetilde{p}_K} \Big) < \infty. \tag{2.2}$$

Now, we will construct the inhomogeneous Mandelbrot measure. For $q \in \mathfrak{J}$ and $k \geq 1$, we define $\psi_k(q, \gamma)$ as the unique t, such that

$$\tau'(t) = \tau'(q) + \eta_k(\gamma).$$

For $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ and $q \in \mathfrak{J}$ we define, for $1 \leq i \leq N_u$

$$V(ui, q) = \frac{\exp\Big(qX_{ui}\Big)}{\mathbb{E}\Big(\sum_{i=1}^{N} \exp\Big(qX_i\Big)\Big)} = \exp\Big(qX_{ui} - \tau(q)\Big)$$

and, for all $n \geq 0$

$$Y_n(q, \gamma, u) = \sum_{v_1 \cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^{n} V\Big(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(q, \gamma)\Big).$$

When $u = \emptyset$, this quantity will be denoted by $Y_n(q, \gamma)$ and when $n = 0$, their values equals 1.

The sequence $\Big(Y_n(q, \gamma, u)\Big)_{n \geq 1}$ is a positive martingale with expectation 1, which converges almost surely and in L^1 norm to a positive random variable $Y(q, \gamma, u)$ (see [9] or [10, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits.

Proposition 2.1. (a) *Let* $\mathsf{K} = K \times K_\gamma$ *be a compact subset of* $\mathfrak{J} \times \Lambda_s$. *There exists* $p_{\mathsf{K}} \in (1, 2]$ *such that for all* $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ *the continuous functions* $(q, \gamma) \in \mathsf{K} \mapsto Y_n(q, \gamma, u)$ *converge uniformly, almost surely and in* $L_{p_{\mathsf{K}}}$ *norm, to a limit* $(q, \gamma) \in \mathsf{K} \mapsto Y(q, \gamma, u)$. *In particular,* $\mathbb{E}(\sup_{(q,\gamma) \in \mathsf{K}} Y(q, \gamma, u)^{p_{\mathsf{K}}}) < \infty$. *Moreover,* $Y(\cdot, \cdot, u)$ *is positive almost surely.*

In addition, for all $n \geq 0$, $\sigma\Big(\{(X_{u1}, \ldots, X_{uN_u}), u \in \mathsf{T}_n\}\Big)$ *and* $\sigma\Big(\{Y(\cdot, \cdot, u), u \in \mathsf{T}_{n+1}\}\Big)$ *are independent, and the random functions* $Y(\cdot, \cdot, u), u \in \mathsf{T}_{n+1}$, *are independent copies of* $Y(\cdot, \cdot) := Y(\cdot, \cdot, \emptyset)$.

(b) *With probability* 1, *for all* $q \in \mathfrak{J}$ *and* $\gamma \in \Lambda_s$, *the weights*

$$\mu_q^\gamma\Big([u]\Big) = \Big[\prod_{k=1}^{n} \exp\Big(\psi_k(q, \gamma) X_{u_1 \ldots u_k} - \tau(\psi_k(q, \gamma))\Big)\Big] Y(q, \gamma, u)$$

define a measure on $\partial \mathsf{T}$, *where* $n = |u|$.

The measure μ_q^γ will be used to approximate from below the Hausdorff dimension of the set E_{b,s^γ}.

Proof. (a) Fix a compact $K \subset \mathfrak{J}$ and a compact $K_\delta \subset \Lambda_s$. Since $\eta_k(\gamma) = \circ(1)$, we can fix, without loss of generality, a compact neighborhood $K' \subset \mathfrak{J}$ of K and suppose that,

$$\forall (q,\gamma) \in \mathsf{K} = K \times K_\gamma, \quad \text{for all k} \geq 1, \psi_k(q,\gamma) \in K'.$$

Fix a compact neighborhood $\mathsf{K}'' = K'' \times K''_\gamma$ of $K' \times K_\gamma$. By (2.2), we can find $\widetilde{p}_{\mathsf{K}''} > 1$, such that

$$\sup_{q \in \mathsf{K}''} \mathbb{E}\Big(\Big(\sum_{i=1}^N e^{qX_i}\Big)^{\widetilde{p}_{\mathsf{K}''}}\Big) < \infty.$$

By (2.1), we can fix $1 < p_{\mathsf{K}} \leq \min(2, \widetilde{p}_{\mathsf{K}''})$ such that $\sup_{q \in K''} \varphi(p_{\mathsf{K}}, q) < 1$. Then for each $(q,\gamma) \in K' \times K$, there exists a neighborhood $V_q \times V_\gamma \subset \mathbb{C}^2$ of (q,γ), whose projection to $\mathbb{R}^2$ is contained in K'', and such that for all $u \in \mathsf{T}$, $(z,z') \in V_q \times V_\gamma$ and $k \geq 1$, the random variable

$$V(u,z) = \frac{\exp(zX_u)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i)\Big)}, \quad \Gamma(z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N X_i \exp(zX_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i)\Big)}$$

and the analytic extension of η_k, denoted also by η_k, are well defined. For $(z,z') \in V_q \times V_\gamma$ and $k \geq 1$, we define $\psi_k(z,z')$ as the unique t such that

$$\Gamma(t) = \Gamma(z) + |\eta_k(z')|.$$

Moreover, we have

$$\sup_{z \in V_q} \varphi(p_{\mathsf{K}}, z) < 1, \quad \text{where } \varphi(p_{\mathsf{K}}, z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N |e^{zX_i}|^{p_{\mathsf{K}}}\Big)}{\Big|\mathbb{E}\Big(\sum_{i=1}^N e^{zX_i}\Big)\Big|^{p_{\mathsf{K}}}}.$$

By extracting a finite covering of $K' \times K_\gamma$ from $\bigcup_{q,\gamma} V_q \times V_\gamma$, we find a neighborhood $\mathsf{V} = V_K \times V_{K\gamma} \subset \mathbb{C}^2$ of $K' \times K_\gamma$ such that

$$\sup_{z \in V_K} \varphi(p_{\mathsf{K}}, z) < 1$$

and for all $(z,z') \in \mathsf{V}$, $\psi_k(z,z')$ is defined and belongs to V_K. Since the projection of V_K to $\mathbb{R}$ is included in K'' and the mapping $z \mapsto \mathbb{E}\Big(\sum_{i=1}^N e^{zX_i}\Big)$ is continuous and does not vanish on V_K, by considering a smaller neighborhood of K' included in V_K if necessary, we can assume that

$$C_{V_K} = \sup_{z \in V_K} \mathbb{E}\Big(\Big|\sum_{i=1}^N e^{zX_i}\Big|^{p_{\mathsf{K}}}\Big)\Big|\mathbb{E}\Big(\sum_{i=1}^N e^{zX_i}\Big)\Big|^{-p_{\mathsf{K}}} < \infty.$$

Now, for $u \in \mathsf{T}$, we define the analytic extension to V of $Y_n(q,\gamma,u)$ given by

$$\begin{aligned} Y_n(z,z',u) &= \sum_{v\in \mathsf{T}_n(u)} \prod_{k=1}^{n} V(u\cdot v_1\cdots v_k, \psi_{|u|+k}(z,z')) \\ &= \Big[\prod_{k=1}^{n} \mathbb{E}\Big(\sum_{i=1}^{N} e^{\psi_k(z,z')X_i}\Big)\Big]^{-1} \sum_{v\in \mathsf{T}_n(u)} \prod_{k=1}^{n} e^{\psi_{|u|+k}(z,z')X(uv_{|k})}. \end{aligned}$$

We denote also $Y_n(z,z',\emptyset)$ by $Y_n(z,z')$. By Lemma 3 in [5], there exists a constant C_{p_K} such that for all $(z,z')\in \mathsf{V}$

$$\begin{aligned} &\mathbb{E}\Big(|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\Big) \\ \le & C_{p_\mathsf{K}}\mathbb{E}\left(\left|\sum_{i=1}^{N} V(i,\psi_n(z,z'))\right|^{p_\mathsf{K}}\right)\prod_{k=1}^{n-1}\mathbb{E}\left(\sum_{i=1}^{N}|V(i,\psi_k(z,z'))|^{p_\mathsf{K}}\right). \end{aligned}$$

Notice that $\mathbb{E}\Big(\sum_{i=1}^{N}|V(i,\psi_k(z,z'))|^{p_\mathsf{K}}\Big)=\varphi(p_\mathsf{K},\psi_k(z,z'))$. Then

$$\begin{aligned} \mathbb{E}\Big(|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\Big) &\le C_{p_\mathsf{K}}\mathbb{E}\left(\left|\sum_{i=1}^{N} V(i,\psi_n(z,z'))\right|^{p_\mathsf{K}}\right)\prod_{k=1}^{n-1}\varphi\Big(p_\mathsf{K},\psi_k(z,z')\Big). \\ &\le C_{p_\mathsf{K}} C_{V_K}\prod_{k=1}^{n-1}\sup_{z\in V_K}\varphi(p_\mathsf{K},z), \end{aligned}$$

where we have used the fact that $\psi_k(z,z')\in V_K$ for all $k\ge 1$. With probability 1, the functions $(z,z')\in \mathsf{V}\mapsto Y_n(z,z')$, $n\ge 0$, are analytic. Fix a closed polydisc $D(z_0,2\rho)\subset \mathsf{V}$ with $z_0=(z_1,z_1')$ and $\rho=(\rho_1,\rho_2)$. Theorem B.1 gives

$$\sup_{(z,z')\in D(z_0,\rho)}|Y_n(z,z')-Y_{n-1}(z,z')|\le 4\int_{[0,1]^2}|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|\,dt,$$

where, for $t=(t_1,t_2)\in[0,1]^2$

$$\zeta(t)=(\zeta_1(t_1),\zeta_2(t_2))=(z_1+\rho_1e^{i2\pi t_1}, z_1'+\rho_2e^{i2\pi t_2}).$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned} \mathbb{E}\left(\sup_{z\in D(z_0,\rho)}|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\right) &\le \mathbb{E}\left(\left(4\int_{[0,1]^2}|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|\,dt\right)^{p_\mathsf{K}}\right) \\ &\le 4^{p_\mathsf{K}}\mathbb{E}\left(\int_{[0,1]^2}|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|^{p_\mathsf{K}}\,dt\right) \\ &= 4^{p_\mathsf{K}}\int_{[0,1]^2}\mathbb{E}\,|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|^{p_\mathsf{K}}\,dt \\ &\le 4^{p_\mathsf{K}}C_{V_K}C_{p_K}\prod_{k=1}^{n-1}\sup_{z\in V_K}\varphi(p_\mathsf{K},z). \end{aligned}$$

Since $\sup_{z\in V_K} \varphi(p_{\mathsf{K}}, z) < 1$, it follows that

$$\sum_{n\geq 1} \left\| \sup_{(z,z')\in D(z_0,\rho)} |Y_n(z,z') - Y_{n-1}(z,z')| \right\|_{p_{\mathsf{K}}} < \infty.$$

This implies, $(z,z') \mapsto Y_n(z,z')$ converges uniformly, almost surely and in $L^{p_{\mathsf{K}}}$ norm over the compact $D(z_0,\rho)$ to a limit $(z,z') \mapsto Y(z,z')$. This also implies that

$$\left\| \sup_{z\in D(z_0,\rho)} Y(z,z') \right\|_{p_{\mathsf{K}}} < \infty.$$

Since K can be covered by finitely many such discs $D(z_0,\rho)$ we get the uniform convergence, almost surely and in $L^{p_{\mathsf{K}}}$ norm, of the sequence $((q,\gamma) \in \mathsf{K} \mapsto Y_n(q,\gamma))_{n\geq 1}$ to $(q,\gamma) \in \mathsf{K} \mapsto Y(q,\gamma)$. Moreover, since $\mathcal{J} \times \Lambda_s$ can be covered by a countable union of such compact K we get the simultaneous convergence for all $(q,\gamma) \in \mathcal{J} \times \Lambda_s$. The same holds simultaneously for all the functions $(q,\gamma) \in \mathcal{J} \times \Lambda_s \mapsto Y_n(q,\gamma,u)$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, because $\bigcup_{n\geq 0} \mathbb{N}_+^n$ is countable.

To finish the proof of Proposition 2.1 (1), we must show that with probability 1, $(q,\gamma) \in \mathsf{K} \mapsto Y(q,\gamma)$ does not vanish. Without loss of generality we can suppose that $\mathsf{K} = [0,1]^2$. If I is a dyadic closed subcube of $[0,1]^2$, we denote by E_I the event $\{\exists\ (q,\gamma) \in I : Y(q,\gamma) = 0\}$. Let $I_0, I_1,\ I_2, I_3$ stand for the 2^2 dyadic intervals of I in the next generation. The event E_I being a tail event of probability 0 or 1. If we suppose that $\mathbb{P}(E_I) = 1$, then there exists $j \in \{0,1,2,3\}$ such that $\mathbb{P}(E_{I_j}) = 1$. Suppose now that $\mathbb{P}(E_{\mathsf{K}}) = 1$. The previous remark allows to construct a decreasing sequence $(I(n))_{n\geq 0}$ of dyadic subcubes of K such that $\mathbb{P}(E_{I(n)}) = 1$. Let (q_0,γ_0) be the unique element of $\cap_{n\geq 0} I(n)$. Since $(q,\gamma) \mapsto Y(q,\gamma)$ is continuous we have $\mathbb{P}(Y(q_0,\gamma_0) = 0) = 1$, which contradicts the fact that $(Y_n(q_0,\gamma_0))_{n\geq 1}$ converges to $Y(q_0,\gamma_0)$ in L^1.

(b) It is a consequence of the branching property

$$Y_{n+1}(q,\gamma,u) = \sum_{i=1}^{N} \exp\Big(\psi_{n+1}(q,\gamma) X_{ui} - \tau(\psi_{n+1}(q,\gamma))\Big)\, Y_n(q,\gamma,ui). \qquad \square$$

3. Proof of Theorem 1.3

The proof of Theorem 1.3 can be deduced from the two following propositions. Their proof are developed in the next section.

Proposition 3.1. *Suppose Hypothesis* 1.2*, with probability* 1*, for all* $q \in \mathcal{J}$ *and* $\gamma \in \Lambda_s$,

$$\mathsf{N}_n(t) - nb \sim s_{n,\gamma}, \qquad \text{for } \mu_q^\gamma\text{-almost every } t \in \partial\mathsf{T},$$

where $b = \tau'(q)$.

Proposition 3.2. *With probability* 1*, for all* $(q,\gamma) \in \mathcal{J} \times \Lambda_s$*, for* μ_q^γ*-almost every* $t \in \partial\mathsf{T}$

$$\lim_{n\to\infty} \frac{\log Y(q,\gamma,t_{|n})}{n} = 0.$$

From Proposition 3.1, we have with probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$, that $\mu_q^\gamma\Big(E_{b,s^\gamma}\Big) = 1$, $(b = \tau'(q))$. In addition, with probability 1, for all $(q,\gamma) \in \mathcal{J} \times \Lambda_s$, for μ_q^γ-almost every $t \in E_{b,s^\gamma}$, from the same Proposition and proposition 3.2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\log(\mu_q^\gamma[t_{|n}])}{\log(\mathrm{diam}([t_{|n}]))} \\
&= \lim_{n\to\infty} -\frac{1}{n}\log\left(\prod_{k=1}^n \exp\Big(\psi_k(q,\gamma)X_{t_1\dots t_k} - \tau(\psi_k(q,\gamma))\Big)Y(q,\gamma,t_{|n})\right) \\
&= \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^n \psi_k(q,\gamma)X_{t_1\dots t_k} + \frac{1}{n}\sum_{k=1}^n \tau(\psi_k(q,\gamma)) - \frac{\log Y(q,\gamma,t_{|n})}{n} \\
&= \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^n \psi_k(q,\gamma)X_{t_1\dots t_k} + \frac{1}{n}\sum_{k=1}^n \tau(\psi_k(q,\gamma)).
\end{aligned}$$

Since $\eta_k(\gamma) = \circ(1)$ and then $\psi_k(q,\gamma) \to q$, we get

$$\lim_{n\to\infty} \frac{\log(\mu_q^\gamma[t_{|n}])}{\log(\mathrm{diam}([t_{|n}]))} = -q\tau'(q) + \tau(q) = \tau^*(\tau'(q)).$$

We deduce the result from the mass distribution principle (Theorem A.1) and (1.1).

4. Proof of Propositions 3.1 and 3.2

4.1. **Proof of Proposition** 3.1. Let $\mathsf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $b = \tau'(q)$, $q \in \mathcal{J}$, $\gamma \in \Lambda_s$, $n \geq 1$, $\epsilon > 0$ and $s^\gamma = (s_{n,\gamma})_{n\geq 1}$ we set

$$\begin{aligned}
E^1_{b,n,\gamma,\epsilon} &= \Big\{t \in \partial\mathsf{T} : \sum_{k=1}^n \Big(X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma)\Big) \geq \epsilon\sum_{k=1}^n \eta_k(\gamma)\Big\}, \\
E^{-1}_{b,n,\gamma,\epsilon} &= \Big\{t \in \partial\mathsf{T} : \sum_{k=1}^n \Big(X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma)\Big) \leq -\epsilon\sum_{k=1}^n \eta_k(\gamma)\Big\}.
\end{aligned}$$

Suppose that we have shown that for, $\lambda \in \{-1,1\}$, we have:

$$\mathbb{E}\Big(\sup_{(q,\gamma)\in\mathsf{K}} \sum_{n\geq 1} \mu_q^\gamma(E^\lambda_{b,n,\gamma,\epsilon})\Big) < \infty. \tag{4.1}$$

Then, with probability 1, for all $(q,\gamma) \in \mathcal{J} \times \Lambda_s$, $\lambda \in \{-1,1\}$, and $\epsilon \in \mathbb{Q}_+^*$,

$$\sum_{n\geq 1} \mu_q^\gamma(E^\lambda_{b,n,\gamma,\epsilon}) < \infty,$$

consequently, by the Borel-Cantelli lemma, for μ_q^γ-almost every t, we have

$$\sum_{k=1}^n X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma) = \circ\Big(\sum_{k=1}^n \eta_k(\gamma)\Big), \quad \text{so } \mathsf{N}_n(t) - nb \sim s_{n,\gamma},$$

which yields the desired result.

Let us prove (4.1) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\theta = (\theta_n)$ be a positive sequence and $(q, \gamma) \in \mathsf{K}$. One has

$$\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma\Big(E^1_{b,n,\gamma,\epsilon}\Big) \le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu_q^\gamma([u])\, \mathbf{1}_{\Big\{E^1_{b,n,\gamma,\epsilon}\Big\}}(t_u),$$

where t_u is any point in $[u]$. Denote t_u simply by t, then

$$\begin{aligned}
&\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma\Big(E^1_{b,n,\gamma,\epsilon}\Big)\\
&\le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu_q^\gamma[u] \prod_{k=1}^n \exp\Big(\theta_k X_{t_1\cdots t_k} - \theta_k b - \theta_k \eta_k(\gamma)(1+\epsilon)\Big)\\
&\le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n \exp\Big((\psi_k(q,\gamma)+\theta_k) X_{t_1\cdots t_k} - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1+\epsilon)\Big)\\
&\quad \times \quad Y(q,\gamma,u).
\end{aligned}$$

For $(q, \gamma) \in \mathsf{K}$, $\theta = (\theta_n)$ and $n \ge 1$, we set

$$\begin{aligned}
&H_n(q,\gamma,\theta)\\
&= \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n \exp\Big((\psi_k(q,\gamma)+\theta_k) X_{t_1\cdots t_k} - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1+\epsilon)\Big) M(u),
\end{aligned}$$

where

$$M(u) = \sup_{(q,\gamma)\in\mathsf{K}} Y(q,\gamma,u).$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathsf{V} = V_K \times V_{K_\gamma} \subset \mathbb{C}^2$ of $\mathsf{K} = K \times K_\gamma$ such that

$$\Gamma(z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N X_i \exp(zX_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i)\Big)}$$

is well defined for $z \in V_K$, for $k \ge 1$, $\eta_k(z')$ is defined for $z' \in V_{K_\gamma}$ and $\forall (z, z') \in \mathsf{V}$, $\psi_k(z, z')$ is defined and belongs to V_K.

For $\epsilon > 0$, $(z, z') \in \mathsf{V}$ and $n \ge 1$, we define

$$\begin{aligned}
H_n(z,z',\theta) = \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n &\exp\Big((\psi_k(z,z')+\theta_k) X_{u_{|k}} - \theta_k\Gamma(z) - \theta_k\eta_k(z')(1+\epsilon)\Big)\\
&\times \mathbb{E}\Big(\sum_{i=1}^N \exp\Big(\psi_k(z,z')X_i\Big)\Big)^{-1} M(u).
\end{aligned}$$

Proposition 4.1. *There exist a neighborhood* $\mathsf{V}' \subset \mathsf{V}$ *of* K*, a positive constant* $\mathcal{C}_{\mathsf{K}}$ *and a positive sequence* θ *such that for all* $(z, z') \in \mathsf{V}'$*, for all* $n \in \mathbb{N}^*$

$$\mathbb{E}(|H_n(z,z',\theta)|) \le \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big),$$

where the sequences $(\epsilon_n)_n$ *and* $(\widetilde{\eta}_n)_n$ *are the sequences used in Hypothesis* 1.2.

Lemma 4.1. *There exist a positive sequence* $\theta = (\theta_n)$ *and a positive constant* $\mathcal{C}_{\mathsf{K}}$ *such that for all* $(q,\gamma) \in \mathsf{K}$ *we have*

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \leq \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{2}\sum_{k=1}^{n} \epsilon_k \widetilde{\eta}_k^2\Big).$$

Proof of Lemma 4.1. Let $\theta = (\theta_n)$ be a positive sequence, clearly we have

$$\begin{aligned}\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) =& \prod_{k=1}^{n} \mathbb{E}\Big(\sum_{i=1}^{N} \exp\Big((\psi_k(q,\gamma)+\theta_k)X_i\Big) \\ &\times \exp\Big(-\tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1+\epsilon)\Big)\mathbb{E}(M(u)) \\ \leq& \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(\tau(\psi_k(q,\gamma)+\theta_k) - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big),\end{aligned}$$

where, by Proposition 2.1, $\mathcal{C}'_{\mathsf{K}} = \mathbb{E}\Big(M(u)\Big) = \mathbb{E}\Big(M(\emptyset)\Big) < \infty$ for all $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$.

Since $\eta_k(\gamma) = \mathrm{o}(1)$, we can fix a compact neighborhood K' of K and suppose that for all $k \geq 1$ and $(q,\gamma) \in \mathsf{K}$, we have $\psi_k(q,\gamma) \in K'$. For $(q,\gamma) \in \mathsf{K}$ and $k \geq 1$, writing the Taylor expansion with integral rest of order 2 of the function $g : \theta \mapsto \tau(\psi_k(q,\gamma)+\theta)$ at 0, we get

$$g(\theta) = g(0) + \theta g'(0) + \theta^2 \int_0^1 (1-t) g''(t\theta)dt,$$

with $g''(t\theta) \leq m_{\mathsf{K}} = \sup_{t\in[0,1]} \sup_{q\in K'} \sup_{\gamma\in K_\gamma} g''(t\theta)$. It follows that for all $k \geq 1$

$$\tau(\psi_k(q,\gamma)+\theta_k) - \tau((\psi_k(q,\gamma)) - \theta_k \tau'((\psi_k(q,\gamma)) \leq \theta_k^2 m_{\mathsf{K}}.$$

Recall that $\tau'(\psi_k(q,\gamma)) = \tau'(q) + \eta_k(\gamma)$. Then

$$\begin{aligned}\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) &\leq \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(\tau(\psi_k(q,\gamma)+\theta_k) - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big), \\ &\leq \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(-\theta_k\eta_k(\gamma)\epsilon + \theta_k^2 m_{\mathsf{K}}\Big).\end{aligned}$$

Choose the sequence θ such that $\theta_k = \epsilon_k \widetilde{\eta}_k$. Then

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \leq \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^{n} \exp\Big(-\epsilon_k \widetilde{\eta}_k^2(\epsilon - \epsilon_k m_{\mathsf{K}})\Big).$$

Since $\epsilon_k \to 0$ then for k large enough we have $\epsilon - \epsilon_k m_{\mathsf{K}} > \dfrac{\epsilon}{2}$. Then, there exists a constant $\mathcal{C}_{\mathsf{K}}$ such that

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \leq \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{2}\sum_{k=1}^{n} \epsilon_k \widetilde{\eta}_k^2\Big).$$

□

Proof of Proposition 4.1. Since $\mathbb{E}(|H_n(q,\gamma,\theta)|) \le \mathcal{C}_{\mathsf{K}} \exp\Big(- \frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2 \Big)$ for $q \in K$, there exists a neighborhood $\mathsf{V}_{q,\gamma} \subset \mathsf{V}$ of (q,γ) such that for all $(z,z') \in \mathsf{V}_{q,\gamma}$ we have

$$\mathbb{E}(|H_n(z,z',\theta)|) \le \mathcal{C}_{\mathsf{K}} \exp\Big(- \frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2 \Big).$$

By extracting a finite covering of K from $\bigcup_{(q,\gamma)\in\mathsf{K}} V_{q,\gamma}$, we find a neighborhood $\mathsf{V}' \subset \mathsf{V}$ of K such that

$$\mathbb{E}(|H_n(z,z',\theta)|) \le \mathcal{C}_{\mathsf{K}} \exp\Big(- \frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2 \Big). \qquad \square$$

With probability 1, the functions $(z,z') \in \mathsf{V}' \mapsto H_n(z,z',\theta)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$, with $z_0 = (z_1, z_1')$ and $\rho = (\rho_1, \rho_2)$. Theorem B.1 gives

$$\sup_{(z,z')\in D(z_0,\rho)} \Big| H_n(z,z',\theta) \Big| \le 2 \int_{[0,1]^2} \Big| H_n(\zeta(t),\theta) \Big| dt,$$

where for $t = (t_1, t_2) \in [0,1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z_1' + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned}
\mathbb{E}\left(\sup_{z\in D(z_0,\rho)} |H_n^s(z,z',\theta)| \right) &\le \mathbb{E}\Big(2 \int_{[0,1]^2} |H_n(\zeta(t),\theta)|\, dt \Big) \\
&\le 4 \int_{[0,1]^2} \mathbb{E}\, |H_n(\zeta(t),\theta)|\, dt \\
&\le 4 \exp\left(- \frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2 \right).
\end{aligned}$$

Finally, we get

$$\mathbb{E}\left(\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma \left(E^1_{b,n,\gamma,\epsilon} \right) \right) \le 4 \exp\Big(- \frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2 \Big)$$

and, then, under Hypothesis 1.2, we get (4.1), which finish the proof of Proposition 3.1.

4.2. **Proof of Propostion** 3.2. Let $\mathsf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $a > 1$, $(q,\gamma) \in \mathsf{K}$ and $n \ge 1$, we set

$$E^+_{n,a} = \Big\{ t \in \partial\mathsf{T} : Y(q,\gamma,t_{|n}) > a^n \Big\}$$

and

$$E^-_{n,a} = \Big\{ t \in \partial\mathsf{T} : Y(q,\gamma,t_{|n}) < a^{-n} \Big\}.$$

It is sufficient to show that for $E \in \{E^+_{n,a}, E^-_{n,a}\}$

$$\mathbb{E}\Big(\sup_{(q,\gamma)\in\mathsf{K}} \sum_{n\ge1} \mu_q^\gamma(E) \Big) < \infty. \tag{4.2}$$

Indeed, if this holds, then with probability 1, for each $(q,\gamma)\in \mathsf{K}$ and $E\in\{E^+_{n,a},E^-_{n,a}\}$, $\sum_{n\ge 1}\mu_q^\gamma(E)<\infty$, hence by the Borel-Cantelli lemma, for μ_q^γ-almost every $t\in\partial\mathsf{T}$, if n is big enough we have

$$-\log a\le \liminf_{n\to\infty}\frac{1}{n}\log Y(q,\gamma,t_{|n})\le \limsup_{n\to\infty}\frac{1}{n}\log Y(q,\gamma,t_{|n})\le \log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (4.2) for $E=E^+_{n,a}$ (the case $E=E^-_{n,a}$ is similar). At first we have,

$$\begin{aligned}\sup_{(q,\gamma)\in\mathsf{K}}\mu_q^\gamma(E^+_{n,a}) &= \sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}\mu_q^\gamma([u])\mathbf{1}_{\left\{Y(q,\gamma,u)>a^n\right\}}\\ &=\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}Y(q,\gamma,u)\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X(u)-\tau\big(\psi_k(q,\gamma)\big)\Big)\mathbf{1}_{\left\{Y(q,\gamma,u)>a^n\right\}}\\ &\le\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}(Y(q,\gamma,u))^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big((\psi_k(q,\gamma)\big)\Big)a^{-\nu},\\ &\le\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big(\psi_k(q,\gamma)\big)\Big)a^{-\nu},\end{aligned}$$

where $M(u)=\sup_{(q,\gamma)\in\mathsf{K}}Y(q,\gamma,u)$ and $\nu>0$ is an arbitrary parameter. For $q\in K$, $\gamma\in K_\gamma$ and $\nu>0$ we set

$$L_n(q,\gamma,\nu)=\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big(\psi_k(q,\gamma)\big)\Big)a^{-\nu}.$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathsf{V}\subset\mathbb{C}^2$ of K such that for all $(z,z')\in\mathsf{V}$ and $k\ge 1$ $\psi_k(z,z')$ is well defined and $\mathbb{E}\left(\sum_{i=1}^N e^{\psi_k(z,z')X_i}\right)\neq 0$.

Lemma 4.2. *Fix $a>1$. For $(z,z')\in\mathsf{V}$ and $\nu>0$, let*

$$\begin{aligned}L_n(z,z',\nu)&=\left[\prod_{k=1}^n\mathbb{E}\bigg(\sum_{i=1}^N\exp\Big(\psi_k(z,z')X_i\Big)\bigg)^{-1}\right]\\ &\quad\times\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(z,z')X_{u_{|k}}\Big)a^{-\nu}.\end{aligned}$$

There exist a neighborhood $\mathsf{V}'\subset\mathbb{C}^2$ of K and a positive constant C_K such that, for all $(z,z')\in\mathsf{V}'$, for all integer $n\ge 1$

$$\mathbb{E}\Big(\Big|L_n(z,z',p_\mathsf{K}-1)\Big|\Big)\le C_\mathsf{K}a^{-n(p_\mathsf{K}-1)/4},\tag{4.3}$$

where p_K provided by Proposition 2.1.

Proof. Write $V=V_K\times V_{K_\gamma}$. For $z\in V_K$ and $\nu>0$, let

$$\widetilde{L}_1(z,\nu)=\left|\mathbb{E}\bigg(\sum_{i=1}^N\exp\Big(zX_i\Big)\bigg)\right|^{-1}\mathbb{E}\bigg(\sum_{i=1}^N\Big|\exp\Big(zX_i\Big)\Big|\bigg)\ a^{-\nu}.$$

Let $q \in K$. Since $\mathbb{E}(\widetilde{L}_1(q,\nu)) = a^{-\nu}$, there exists a neighborhood $V_q \subset V_K$ of q such that for all $z \in V_q$ we have $\mathbb{E}\Big(\big|\widetilde{L}_1(z,\nu)\big|\Big) \le a^{-\nu/2}$. Let $\gamma \in K_\gamma$. Recall the proof of Proposition 2.1 and since $\eta_k(\gamma) = \circ(1)$, we can find a neighborhood $V_\gamma \subset V_{K_\gamma}$ of K_γ such that, for all $k \ge 1$, $(z,z') \in V_q \times V_\gamma$, we have

$$\mathbb{E}\Big(\big|\widetilde{L}_1(\psi_k(z,z'),\nu)\big|\Big) \le a^{-\nu/3}.$$

By extracting a finite covering of K from $\bigcup_{(q,\gamma)} V_q \times V_\gamma$, we find a neighborhood $\mathsf{V}' \subset \mathsf{V}$ of K such that for all $(z,z') \in \mathsf{V}'$ and $k \ge 1$

$$\mathbb{E}\Big(\big|\widetilde{L}_1(\psi_k(z,z'),\nu)\big|\Big) \le a^{-\nu/4}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big(\big|L_n(z,z',\nu)\big|\Big)\\
=&\left[\prod_{k=1}^{n}\left|\mathbb{E}\Big(\sum_{i=1}^{N}\exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right]\mathbb{E}\left(\left|\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^{n}\exp\Big(\psi_k(z,z')X_u\Big)\right|\right)a^{-n\nu}\\
\le&\left[\prod_{k=1}^{n}\left|\mathbb{E}\Big(\sum_{i=1}^{N}\exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right]\mathbb{E}\left(\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^{n}\left|\exp\Big(\psi_k(z,z')X_u\Big)\right|\right)a^{-n\nu}.
\end{aligned}$$

By Proposition 2.1, there exists $p_{\mathsf{K}} \in (1,2]$ such that for all $u \in \bigcup_{n\ge 0}\mathbb{N}_+^n$,

$$\mathbb{E}\Big(M(u)^{p_{\mathsf{K}}}\Big) = \mathbb{E}\Big(M(\emptyset)^{p_{\mathsf{K}}}\Big) = C_{\mathsf{K}} < \infty.$$

Now take $\nu = p_{\mathsf{K}} - 1$ in the last calculation, it follows, from the independence of $\sigma\Big(\{Y(\cdot,\cdot,u),\, u \in \mathsf{T}_n\}\Big)$ and $\sigma\Big(\{(X_{u1},\dots,X_{uN_u}),\, u \in \mathsf{T}_{n-1}\}\Big)$ for all $n \ge 1$, that

$$\begin{aligned}
&\mathbb{E}\Big(\big|L_n(z,z',p_{\mathsf{K}}-1)\big|\Big)\\
\le&\left[\prod_{k=1}^{n}\left|\mathbb{E}\Big(\sum_{i=1}^{N}\exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right]\prod_{k=1}^{n}\mathbb{E}\Big(\sum_{i=1}^{N}\left|\exp\Big(\psi_k(z,z')X_i\Big)\right|\Big)^{n}C_{\mathsf{K}}a^{-n(p_{\mathsf{K}}-1)}\\
=&C_{\mathsf{K}}\prod_{k=1}^{n}\mathbb{E}\Big(\big|\widetilde{L}_1(\psi_k(z,z'),p_{\mathsf{K}}-1)\big|\Big)\\
\le&C_{\mathsf{K}}a^{-n(p_K-1)/4},
\end{aligned}$$

then lemma is now proved. □

With probability 1, the functions $(z,z') \in \mathsf{V}' \mapsto L_n(z,z',\nu)$ are analytic. Fix a closed polydisc $D(z_0,2\rho) \subset \mathsf{V}'$, with $z_0 = (z_1,z_1')$ and $\rho = (\rho_1,\rho_2)$. Theorem B.1 gives

$$\sup_{z\in D(z_0,\rho)}\big|L_n(z,p_{\mathsf{K}}-1)\big| \le 4\int_{[0,1]^2}\big|L_n(\zeta(t),p_{\mathsf{K}}-1)\big|dt,$$

where, for $t = (t_1, t_2) \in [0,1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z_1' + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned}\mathbb{E}\left(\sup_{z\in D(z_0,\rho)} |L_n(z, p_{\mathsf{K}} - 1)|\right) &\le \mathbb{E}\left(4\int_{[0,1]^2} |L_n(\zeta(t), p_{\mathsf{K}} - 1)|\, dt\right)\\ &\le 4\int_{[0,1]^2} \mathbb{E}\,|L_n(\zeta(t), p_{\mathsf{K}} - 1)|\, dt\\ &\le 4C_{\mathsf{K}} a^{-n(p_{\mathsf{K}}-1)/4}.\end{aligned}$$

Since $a > 1$ and $p_{\mathsf{K}} - 1 > 0$, we get (4.2).

Appendix A. Hausdorff and Packing Dimensions

Given a subset K of $\mathbb{N}_+^{\mathbb{N}_+}$ endowed with a metric d making it σ-compact, $s > 0$ and E a subset of K, the s-dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \inf\left\{\sum_{i\in\mathbb{N}} (\mathrm{diam}(U_i)^s\right\},$$

the infimum being taken over all the countable coverings $(U_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ. Then, the Hausdorff dimension of E is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given $s > 0$ and $E \subset K$ as above, one first defines

$$\overline{P}^s(E) = \lim_{\delta\to 0^+} \sup\left\{\sum_{i\in\mathbb{N}} (\mathrm{diam}(B_i)^s\right\},$$

the supremum being taken over all the packings $\{B_i\}_{i\in\mathbb{N}}$ of E by balls centered on E and with diameter smaller than or equal to δ. Then, the s-dimensional packing measure of E is defined as

$$P^s(E) = \lim_{\delta\to 0^+} \inf\left\{\sum_{i\in\mathbb{N}} \overline{P}^s(E_i)\right\},$$

the infimum being taken over all the countable coverings $(E_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ. Then, the packing dimension of E is defined as

$$\mathrm{Dim}\, E = \sup\{s > 0 : P^s(E) = \infty\} = \inf\{s > 0 : P^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. For more details the reader is referred to [13, 20].

If μ is a positive and finite Borel measure supported on K, then its lower Hausdorff and packing dimensions is defined as

$$\underline{\dim}(\mu) = \inf\Big\{ \dim F : F \text{ Borel}, \ \mu(F) > 0 \Big\}$$
$$\underline{\operatorname{Dim}}\,(\mu) = \inf\Big\{ \operatorname{Dim} F : F \text{ Borel}, \ \mu(F) > 0 \Big\}$$

and its upper Hausdorff and packing dimensions are defined as

$$\overline{\dim}(\mu) = \inf\Big\{ \dim F : F \text{ Borel}, \ \mu(F) = \|\mu\| \Big\}$$
$$\overline{\operatorname{Dim}}\,(\mu) = \inf\Big\{ \operatorname{Dim} F : F \text{ Borel}, \ \mu(F) = \|\mu\| \Big\}.$$

We have (see [12])

$$\underline{\dim}(\mu) = \operatorname{ess\,inf}_\mu \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$
$$\underline{\operatorname{Dim}}\,(\mu) = \operatorname{ess\,inf}_\mu \limsup_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)}$$

and

$$\overline{\dim}(\mu) = \operatorname{ess\,sup}_\mu \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$
$$\overline{\operatorname{Dim}}\,(\mu) = \operatorname{ess\,sup}_\mu \limsup_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$

where $B(t, r)$ stands for the closed ball of radius r centered at t. If $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{\operatorname{Dim}}\,(\mu) = \overline{\operatorname{Dim}}\,(\mu)$), this common value is denoted $\dim \mu$ (resp. $\operatorname{Dim}(\mu)$), and if $\dim \mu = \operatorname{Dim} \mu$, one says that μ is exact dimensional.

Recall the mass distribution principle.

Theorem A.1. ([13, Theorem 4.2]). *Let ν be a positive and finite Borel probability measure on a compact metric space (X, d). Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and*

$$M \subseteq \left\{ t \in X : \liminf_{r\to 0^+} \frac{\log \nu(B(t,r))}{\log r} \ge \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ.

Appendix B. Cauchy Formula in Several Variables

Let us recall the Cauchy formula for holomorphic functions in several variables.

Definition B.1. Let $d \ge 1$, a subset D of $\mathbb{C}^d$ is an open polydisc if there exist open discs $D_1, \ldots, D_d$ of $\mathbb{C}$ such that $D = D_1 \times \cdots \times D_d$. If we denote by ζ_j the centre of D_j, then $\zeta = (\zeta_1, \ldots, \zeta_d)$ is the centre of D and if r_j is the radius of D_j then $r = (r_1, \ldots, r_d)$ is the multiradius of D. The set $\partial D = \partial D_1 \times \cdots \times \partial D_d$ is the distinguished boundary of D. We denote by $D(\zeta, r)$ the polydisc with center ζ and radius r.

Let $D = D(\zeta, r)$ be a polydisc of $\mathbb{C}^d$ and $g \in C(\partial D)$ a continuous function on ∂D. We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta_1 \cdots d\zeta_d = (2i\pi)^d r_1 \cdots r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1} \cdots e^{i2\pi\theta_d} d\theta_1 \cdots d\theta_d,$$

where $\zeta(\theta) = (\zeta_1(\theta), \ldots, \zeta_d(\theta))$ and $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$ for $j = 1, \ldots, d$.

Theorem B.1. *Let $D = D(a, r)$ be polydisc in $\mathbb{C}^d$ with a multiradius whose components are positive, and f be a holomorphic function in a neiborhood of D. Then, for all $z \in D$*

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_d}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| \, d\theta_1 \cdots d\theta_d.$$

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BERTRAND'S PARADOX: NEW PROBABILISTIC MODELS

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Abstract. In this paper two new generating procedure of a random chord are obtained and thereby new solutions of Bertrand's paradox are proposed.

1. Introduction

Paradox, on its own, is a puzzle that confronts some already established principles. Bertand's paradox was developed as a probability question that raised severe objections on the principle of indifference while dealing with geometrical probability. The question that defines this paradox: "What is the probability that a chord selected "at random" in a circle is larger than a side of the inscribed equilateral triangle?"

In [3], Bertrand obtained probabilities 1/3, 1/2 and 1/4 by different random chord generation procedures: by choosing a chord with one end at a vertex of the inscribed equilateral triangle in a circle; by choosing a chord perpendicular to the diameter which is the right bisector of the equilateral triangle; and selecting a point inside a circle and denoting it as a chord midpoint, respectively. This puzzle has fascinated many since its discovery and a series of papers with outstanding solutions of this problem have been published, see e.g. [1, 2, 4–9]. Here, we provide two new models of random chord construction in a circle and obtain associated probabilities of Bertrand's paradox.

The paper is organized as follows. In Section 2, we propose two new procedures for generating a random chord in a circle and obtain probabilities of Bertrand's paradox for each case. Section 3 concludes this paper.

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2. New Models and Solutions

In [7], an attempt was made to look at classical models of Bertrand's paradox as limits of a continuous family of planar probabilistic models. Such family is seen by fixing a point, say A, at a distance $h > 1$ from a unit circle and constructing lines that intersect the circle and point A. However, this family of chord constructing models undermines the randomness selection of distance h and, so, it yields inappropriate results with respect to Bertrand's paradox. Motivated by this issue, in [10] a chord generating procedure is presented that overcomes this obstacle. Here, we additionally provide two new methods of generating random chords in a circle with the same intention.

For both models, we will denote X as the distance from the center of the circle and the chord and L as the corresponding chord length.

2.1. **First model.** The first method is obtained as follows.

Step 1. Let a point A be such that its distance from the center of the circle OA is a random variable $Y \sim U(0,1)$ and is lying on the x axis.
Step 2. Using the circle invariance property we can obtain a point on a x axis, say P, so that the relation $OP \cdot OA = 1$ holds.
Step 3. Angle ϕ is determined by the circle tangent and the x axis, with P as its vertex;
Step 4. Select a line which is directed by an angle $\theta \in U(0,\phi)$, with P as its starting point. A chord is formed by its intersection with the circle (Figure 1.).

In this case, we have $X = \sqrt{1-\frac{L^2}{4}}$, $\phi = \arcsin(Y)$ and $\theta = \arcsin\left(Y\sqrt{1-\frac{L^2}{4}}\right)$.

Using transformation technique, the distribution function of L can be found as

$$F_I(l) = \int_0^1 \int_0^l \frac{xy}{4\arcsin(y)\sqrt{1-\frac{x^2}{4}}\sqrt{1-(1-\frac{x^2}{4})y^2}}\,dx\,dy$$

$$= \int_0^1 \frac{\arcsin(y) - \arcsin\left(\frac{y\sqrt{4-l^2}}{2}\right)}{\arcsin(y)}\,dy, \quad 0 < l < 2. \tag{2.1}$$

Integral (2.1) cannot be obtained explicitly, so we can only provide numerical solutions. For the Bertrand's case $l = \sqrt{3}$ we have

$$P\left\{L_I \geq \sqrt{3}\right\} = 1 - F_I(\sqrt{3}) = 0.4694. \tag{2.2}$$

2.2. **Second model.** The second method is obtained as follows.

Step 1. Let a point A be determined by a random angle $\phi \sim U(0,\pi/2)$ on a circumference of a circle.
Step 2. Let a tangent t of a circle be determined by point A.
Step 3. Angle δ is determined by the circle tangent and the x axis, with P as its vertex.
Step 4. Select a line which is directed by an angle $\theta \in U(0,\delta)$, with P as its starting point. A chord is formed by its intersection with the circle (Figure 2).

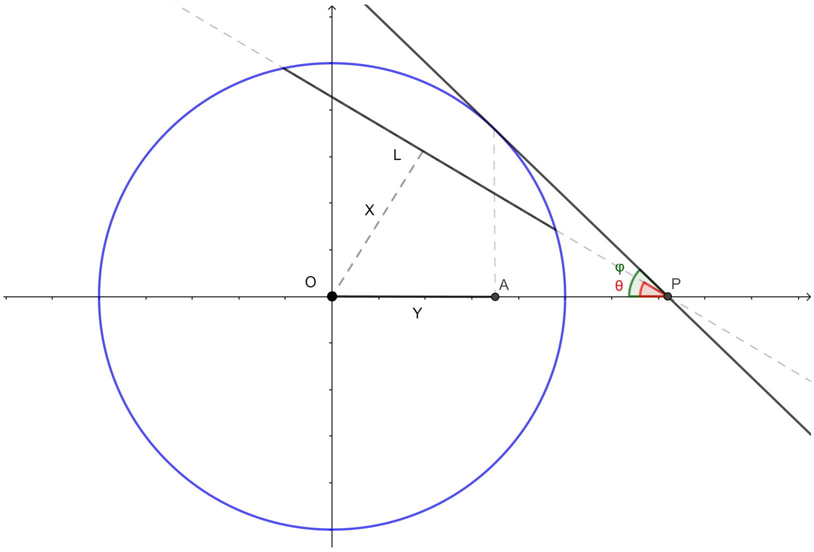

FIGURE 1. Solution I.

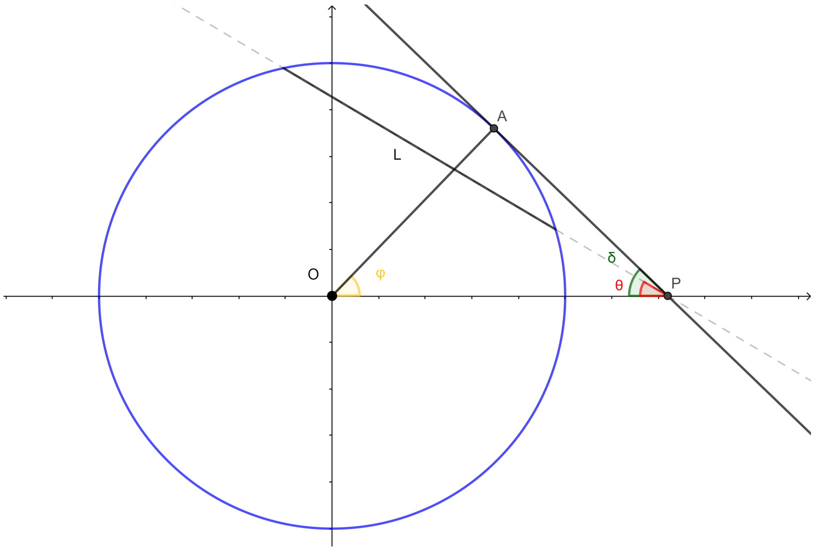

FIGURE 2. Solution II.

For this case, we have $X = \sqrt{1 - \frac{L^2}{4}}$ and $\sin\theta = \cos\phi\sqrt{1 - \frac{L^2}{4}}$. Further, the distribution function of L can be obtained as

$$F_{II}(l) = \frac{2}{\pi}\int_0^{\pi/2}\int_0^l \frac{x\cos y}{4(\frac{\pi}{2} - y)\sqrt{1 - \frac{x^2}{4}}\sqrt{1 - (1 - \frac{x^2}{4})\cos^2 y}}\,dx\,dy$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\operatorname{arcsec}\left(\frac{2\sec(y)}{\sqrt{4 - l^2}}\right) - y}{\pi - 2y}\,dy, \quad 0 < l < 2. \tag{2.3}$$

As above, integral (2.3) cannot be obtained explicitly, so we can obtain numerical solutions. For the Bertrand's case $l = \sqrt{3}$ we have

$$P\left\{L_{II} \geq \sqrt{3}\right\} = 1 - F_{II}(\sqrt{3}) = 0.4454. \tag{2.4}$$

3. Conclusion

Overall, in this paper we presented two new generating procedures of random chords in a circle. The distribution function (2.3) is also obtained in [10] using a different method of constructing random chords. The results presented in this paper extend those can be found in [4, 9, 10] on Bertrand's paradox.

In [6], procedures of chord construction were classified by disjoint procedures: (i) inside the circle, (ii) on the circle circumference and (iii) outside of the circle. Proposed generating models connect procedures (i), (ii) and (iii), and confronts such classification. This may be a motivation to overlook Bertrand's paradox in a quite different manner.

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ON Z-SYMMETRIC MANIFOLD WITH CONHARMONIC CURVATURE TENSOR IN SPECIAL CONDITIONS

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Abstract. The object of the present paper is to study the Z-symmetric manifold with conharmonic curvature tensor in special conditions. In this paper, we prove some theorems about these manifolds by using the properties of the Z-tensor.

1. Introduction

Conformal geometry has deep importance in pure mathematics, such as complex analysis, Riemann surface theory, differential geometry and algebraic topology, [2, 21, 22]. Computational conformal geometry is important in digital geometry processing. Discrete conformal geometry has been presented to compute conformal mapping which has been broadly applied in numerous practical fields, including computer vision and graphics, visualization, medical imaging, etc. In medical imaging, conformal geometry has been applied to surface parametrization and extract intrinsic features for natural objects like brain, colon, spleen and other human organs.

Historically, conformal mappings have been considered in many monographs, surveys and papers. Also, the theory of conformal mappings has very important applications in general relativity.

Let (M, g) and $(\bar{M}, \bar{g})$ be two n-dimensional Riemannian manifolds with metric tensors g_{ij} and $\bar{g}_{ij}$, respectively. Both metrics are defined in a common coordinate system (x^i). The correspondence between (M, g) and $(\bar{M}, \bar{g})$ is conformal, if the

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fundamental tensors g_{ij} and $\bar{g}_{ij}$ of two manifolds M and $\bar{M}$ are in the relation

$$\bar{g}_{ij}(x) = e^{2\sigma(X)} g_{ij}(x), \tag{1.1}$$

where $\sigma(x)$ is a scalar function of the x's.

By the transformation (1.1), it also follows that the relation between the Christoffel symbols Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ compatible with the metrics g_{ij} and $\bar{g}_{ij}$, respectively is given by

$$\bar{\Gamma}^h_{ij} = \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{1.2}$$

where $\sigma_i = \frac{\partial \sigma}{\partial x^i}$, $\sigma^h = g^{hi}\sigma_i$, g^{ij} are the components of the inverse matrix to g_{ij}, and δ^h_i is the Kronecker delta.

A conformal mapping is called homothetic if the function σ is a constant, that is, $\bar{g}_{ij}(x) = c g_{ij}(x)$. The condition is equivalent to $\sigma_i = 0$, hence, the mapping is also an affine one.

Denoting R^h_{ijk} and $\bar{R}^h_{ijk}$ are the Riemann tensors of the manifolds M and $\bar{M}$, respectively, then we have [11, 20]

$$\begin{aligned}\bar{R}^h_{ijk} =& R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{hl}(\sigma_{lk} g_{ij} - \sigma_{li} g_{jk}) + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1 \sigma,\\ \bar{S}_{ij} =& S_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij},\\ \bar{r} =& e^{-2\sigma}(r + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma),\end{aligned} \tag{1.3}$$

where $\sigma_i = \partial_i \sigma$, $\Delta_l \sigma = g^{ij}\sigma_i \sigma_j$, $\Delta_2 \sigma = g^{ij}\sigma_{i,j}$, $\sigma_{ij} = \sigma_{i,j} - \sigma_i \sigma_j$. We denote that $S_{ij} = R^h_{ijh}$ and $\bar{S}_{ij} = \bar{R}^h_{ijh}$ are their Ricci tensors and $r = S_{ij} g^{ij}$ and $\bar{r} = \bar{S}_{ij}\bar{g}^{ij}$ are their scalar curvatures.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In generally, the harmonic function is not invariant under the conformal transformation. In [14], Ishii obtained the conditions which a harmonic function remains invariant and he introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition [14]

$$\sigma^h_{,h} + \sigma^h_{,h} \sigma^h_{,} = 0, \tag{1.4}$$

where comma denotes the covariant differentiation with respect to the metric g.

Thus, we can say that the conharmonic transformation which is a special type of conformal transformations preserves the harmonicity of smooth functions. It is well known that such transformations have an invariant tensor, so-called the conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor, that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

A rank-four tensor L that remains invariant under conharmonic transformation of a Riemannian manifold (M, g) is given by

$$\begin{aligned}L(X,Y,Z,U) =& R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\\ &+ g(X,U)S(Y,Z) - g(Y,U)S(X,Z)],\end{aligned} \tag{1.5}$$

where R and S denote the Riemannian curvature tensor of type $(0,4)$ defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type $(0,2)$, respectively. The curvature tensor defined by (1.5) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness.

Q denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type $(0,2)$, that is

$$g(QX,Y) = S(X,Y). \tag{1.6}$$

Let $\Big\{e_i, i = 1,2,\dots,n\Big\}$ be an orthonormal basis of the tangent space at each point of the manifold. From (1.5), we have

$$\overline{L}(X,Y) = \sum_{i=1}^{n} L(X,e_i,e_i,Y) = \sum_{i=1}^{n} L(e_i,X,Y,e_i) = -\frac{r}{n-2}g(X,Y) \tag{1.7}$$

and

$$\sum_{i=1}^{n} L(e_i,e_i,X,Y) = \sum_{i=1}^{n} L(X,Y,e_i,e_i) = 0, \tag{1.8}$$

where r is the scalar curvature of the manifold. Also, from (1.5) it follows that [26]

$$\begin{aligned} &L(X,Y,Z,U) = -L(Y,X,Z,U),\\ &L(X,Y,Z,U) = -L(X,Y,U,Z),\\ &L(X,Y,Z,U) = L(Z,U,X,Y),\\ &L(X,Y,Z,U) + L(X,Z,U,Y) + L(X,U,Y,Z) = 0. \end{aligned} \tag{1.9}$$

In [26], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetries and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. This tensor has valuable applications in general relativity. In [1], Abdussatter investigated its physical significance in the theory of general relativity. The conharmonic transformation has also been studied by Siddique and Ahsan [27], Ghosh, De and Taleshian [12], and many others.

A non-flat Riemannian manifold which is called a recurrent manifold [25] if the curvature tensor of this manifold satisfies the relation

$$(\nabla_W R)(X,Y,Z,U) = A(W)R(X,Y,Z,U), \tag{1.10}$$

where A is a non-zero 1-form. A non-flat Riemannian manifold which is called a Ricci-recurrent manifold if the Ricci tensor of this manifold satisfies the relation [5,23,28]

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z), \tag{1.11}$$

where A is a non-zero 1-form.

A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition $\nabla_X \xi = \alpha X + \lambda(X)\xi$, where $X \in TM$, $\lambda(X)$ is a linear form and α is a function, [4, 19, 29].

In the local transcription, this reads

$$\xi^h_{,i} = \alpha \delta^h_i + \xi^h \lambda_i, \tag{1.12}$$

where ξ^h and λ_i are the components of ξ and λ respectively, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called, [19, 29],

i) recurrent if $\alpha = 0$, i.e.,

$$\xi^h_{,i} = \xi^h \lambda_i; \tag{1.13}$$

ii) concircular if the form λ_i is gradient covector (i.e., $\lambda_i = \lambda_{,i}$), i.e.,

$$\xi^h_{,i} = \alpha \delta^h_i; \tag{1.14}$$

iii) convergent if it is concircular and $\alpha = const.exp(\lambda)$.

A $\varphi(Ric)$-vector field is a vector field on an n dimensional Riemannian manifold (M, g) with metric g and Levi-Civita connection ∇, which satisfies the condition [13]

$$\nabla \varphi = \mu Ric, \tag{1.15}$$

where μ is some constant and Ric is the Ricci tensor. Obviously, when (M, g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu = 0$, the vector field φ is covariantly constant. In the following we suppose that $\mu \neq 0$ and (M, g) is neither an Einstein space nor a vacuum solution of the Einstein equations. In a locally coordinate neighbourhood $U(x)$, the equation (1.15) is written as

$$\varphi^h_{,i} = \mu S^h_i, \tag{1.16}$$

where φ^i and S^h_i are components of φ and Ric, respectively. After lowering indices, (1.16) has the form

$$\varphi_{i,j} = \mu S_{ij}, \tag{1.17}$$

where $\varphi_i = \varphi^\alpha g_{i\alpha}$ and $S_{ij} = g_{i\alpha} S^\alpha_j$.

2. Z-Tensor on a Riemannian Manifold

In 2012, Mantica and Molinari defined a generalized symmetric tensor of type $(0, 2)$ which is called Z-tensor and given by [15]

$$Z_{kl} = S_{kl} + \phi g_{kl}, \tag{2.1}$$

where ϕ is an arbitrary scalar function. The scalar $\bar{Z}$ is the trace of Z-tensor and from (2.1), it can be written as

$$\bar{Z} = g^{kl} Z_{kl} = r + n\phi. \tag{2.2}$$

The classical Z-tensor is obtained with the choice $\phi = -\frac{1}{n} r$. Shortly, the generalized Z-tensor is called as the Z-tensor. In some cases, the Z-tensor gives the several well known structures on Riemannian manifolds. For example, i) if $Z_{kl} = 0$ (i.e, Z-flat), then this manifold reduces to an Einstein manifold [3]; ii) if $\nabla_j Z_{kl} = \lambda_j Z_{kl}$ (Z-recurrent), then this manifold reduces to a generalized Ricci recurrent manifold [6]; iii) if $\nabla_j Z_{kl} = \nabla_k Z_{jl}$ (Codazzi tensor), then we find $\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)r$ [10]. This result gives us that this manifold is a nearly conformal symmetric manifold $(NCS)_n$ [24]. iv) The relation between the Z-tensor and the energy-stress tensor of Einstein' s equations [9], with cosmological constant Λ is $Z_{jl} = kT_{jl}$, where $\phi = -\frac{1}{2}r + \Lambda$ and k is the gravitational constant. In this case, the Z-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution ($Z = 0$) determines an Einstein space $\Lambda = \left(\frac{n-2}{2n}\right)r$; the conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla_j Z_{kl} = 0$ then this spacetime gives the conserved enery-momentum density.

This manifold has received a great deal of attention and is studied in considerable detail by many authors [7, 8, 15–18, 30, 31]), etc. Motivated by the above studies, in the present, we examine the properties of a Z-symmetric manifold with conharmonic curvature tensor.

The present paper is organized as follows. In Section 1 and Section 2, after reviewing the basics about symmetric spaces and Z-tensor, respectively. In Section 3 we will discuss Z-symmetric manifolds with conharmonic curvature tensor and mention some properties of these manifolds. We will concentrate on this paper that will be of relevance in our forthcoming paper.

3. Z-Symmetric Manifold with Conharmonic Curvature Tensor

In this section, we consider a Z-symmetric manifold with conharmonic curvature tensor. In the local coordinates, consider the equations (1.5) and (2.1), the relation between the Z-tensor and the conharmonic curvature tensor is found as

$$L_{hijk} = R_{hijk} - \frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{3.1}$$

By taking the covariant derivative of (3.1), we can find

$$\begin{aligned} L_{hijk,l} = & R_{hijk,l} - \frac{1}{n-2}[g_{ij}Z_{hk,l} - g_{ik}Z_{hj,l} + g_{hk}Z_{ij,l} - g_{hj}Z_{ik,l}] \\ & + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \end{aligned} \tag{3.2}$$

Suppose now that our manifold is Z-recurrent. Considering the equation (1.11) for Z-tensor, we can write $Z_{ij,l} = \lambda_l Z_{ij}$. Hence, we see from (3.2) that

$$L_{hijk,l} = R_{hijk,l} - \frac{\lambda_l}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{3.3}$$

It is obtained by (3.1)

(3.4)

$$\frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] = R_{hijk} - L_{hijk} + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}].$$

By the aid of (3.4), the expression (3.3) can be written as

$$L_{hijk,l} - \lambda_l L_{hijk} = R_{hijk,l} - \lambda_l R_{hijk} + \frac{2}{n-2}(g_{ij}g_{hk} - g_{ik}g_{hj})(\phi_l - \lambda_l\phi). \tag{3.5}$$

In the following theorems, a Riemannian manifold admitting covariantly constant conharmonic curvature tensor and recurrent Z-tensor with the recurrence vector field λ_l will be shown by (M, g).

Theorem 3.1. *The vector field ϕ_l and the recurrence vector field λ_l of (M, g) must be parallel and they satisfy the relation*

$$\phi_l = \left(\frac{r}{n} + \phi\right)\lambda_l.$$

Proof. Differentiating covariantly of (1.5) and assuming that the conharmonic curvature tensor is covariantly constant, it is not hard to see that the scalar curvature must be constant. If the Z-tensor is recurrent tensor admitting λ_l recurrence vector field then we have from (1.11) and (2.1)

$$\lambda_l Z_{ij} = S_{ij,l} + \phi_l g_{ij}. \tag{3.6}$$

Multiplying (3.6) by g^{ij}, we get

$$\lambda_l \bar{Z} = r_{,l} + n\phi_l. \tag{3.7}$$

Since r must be constant, from (2.2), the equation (3.7) takes the following form

$$\lambda_l(r + n\phi) = n\phi_l. \tag{3.8}$$

Arraying the equation (3.8), finally we obtain

$$\phi_l = \left(\frac{r}{n} + \phi\right)\lambda_l. \tag{3.9}$$

Hence, the proof is completed. □

Theorem 3.2. *On (M, g), $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector δ defined by $\lambda(X) = g(X, \delta)$.*

Proof. Suppose that the Z-tensor is recurrent tensor. As we already know from the equation (1.11), we have

$$Z_{ij,l} = \lambda_l Z_{ij}. \tag{3.10}$$

Multiplying (3.10) by g^{il}, we get

$$Z^l_{j,l} = \lambda^l Z_{jl}. \tag{3.11}$$

We remark that in a Riemannian manifold with covariantly constant conharmonic curvature tensor, the scalar curvature is constant. From this result and the Ricci Identity, we have $S^l_{j,l} = 0$. Thus, we see that

$$Z^l_{j,l} = \phi_j. \tag{3.12}$$

From (2.1), (3.11) and (3.12), one can show that

$$\phi_j = \lambda^l (S_{jl} + \phi g_{jl}). \tag{3.13}$$

On the other hand, if we use the equation (3.9), (3.13) takes the form

$$\left(\frac{r}{n} + \phi\right) \lambda_j = \lambda^l S_{jl} + \phi \lambda_j. \tag{3.14}$$

Finally, the equation (3.14) shows that

$$\lambda^l S_{jl} = \frac{r}{n} \lambda_j. \tag{3.15}$$

Hence, the proof is completed. □

Theorem 3.3. *A necessary and sufficient condition for the vector field ϕ^l generated by the scalar function ϕ of (M, g) to be divergence-free is that the divergence of the vector field λ^l be of negative value in the form*

$$\lambda^l_{,l} = -||\lambda||^2.$$

Proof. From Theorem 3.1, we know that the relation between ϕ_l and λ_l vector fields is in the form

$$\phi_l = \left(\frac{r}{n} + \phi\right) \lambda_l. \tag{3.16}$$

Taking the covariant derivative of (3.16), we get

$$\phi_{l,m} = \phi_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.17}$$

Substituting the equation (3.16) in (3.17), one can prove the relation

$$\phi_{l,m} = \left(\frac{r}{n} + \phi\right) \lambda_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.18}$$

Multiplying (3.18) by g^{lm}, we find

$$\phi^l_{,l} = \left(\frac{r}{n} + \phi\right) \left(||\lambda||^2 + \lambda^l_{,l}\right). \tag{3.19}$$

Now, suppose that the vector field ϕ_l is divergence-free. Of course, $r \neq -n\phi$ from (3.16), then by using (3.19), we obtain

$$\lambda^l_{,l} = -||\lambda||^2. \tag{3.20}$$

Conversely, if the equation (3.20) is satisfied then from (3.19), we can find $\phi^l_{,l} = 0$. Hence, the proof is completed. □

Theorem 3.4. *If the vector field λ_l on (M, g) is divergence-free then the divergence of the vector field ϕ_l is in the form*

$$\phi^l_{,l} = \frac{n}{r + n\phi}||\phi||^2.$$

Proof. From Theorem 3.3, we know that the relation (3.17) holds. In this case, if we use (3.16) and (3.17) then we get

$$\phi_{l,m} = \left(\frac{n}{r + n\phi}\right)\phi_m\phi_l + \left(\frac{r + n\phi}{n}\right)\lambda_{l,m}. \tag{3.21}$$

Multiplying (3.21) by g^{lm}, we find

$$\phi^l_{,l} = \left(\frac{n}{r + n\phi}\right)||\phi||^2 + \left(\frac{r + n\phi}{n}\right)\lambda^l_{,l}. \tag{3.22}$$

Now, suppose that the vector field λ_l is divergence-free. Finally, the divergence of the vector field ϕ_l is found in the following form

$$\phi^l_{,l} = \frac{n}{r + n\phi}||\phi||^2. \tag{3.23}$$

Thus, the proof is completed. □

Theorem 3.5. *If (M, g) admits a torse-forming vector field associated by the 1-form ϕ_l in the relation $\phi_{l,m} = \rho g_{lm} + \alpha_m\phi_l$, then the vector field λ_l is also torse-forming vector field satisfying the equation*

$$\lambda_{l,m} = \gamma g_{lm} + \beta_m\lambda_l,$$

where $\gamma = \frac{n\rho}{r+n\phi}$ and $\beta_m = \alpha_m - \lambda_m$.

Proof. Assume that the vector field ϕ_l is a torse-forming vector field with a scalar function ρ and a vector field α_m. As we know from (1.12) that

$$\phi_{l,m} = \rho g_{lm} + \alpha_m\phi_l. \tag{3.24}$$

Substituting the equation (3.24) in (3.17), thus we see that

$$\rho g_{lm} + \alpha_m\phi_l = \phi_m\lambda_l + \left(\frac{r}{n} + \phi\right)\lambda_{l,m}. \tag{3.25}$$

Also, we can use the equation (3.16) in (3.25). Then

$$\lambda_{l,m} = \frac{n\rho}{r + n\phi}g_{lm} + (\alpha_m - \lambda_m)\lambda_l. \tag{3.26}$$

Defining $\gamma = \frac{n\rho}{r+n\phi}$ and $\beta_m = \alpha_m - \lambda_m$, (3.26) takes the form

$$\lambda_{l,m} = \gamma g_{lm} + \beta_m\lambda_l. \tag{3.27}$$

Thus, the vector field λ_l is a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.6. *If (M,g) admits a torse-forming vector field associated by the 1-form ϕ_l in the relation $\phi_{l,m}=\rho g_{lm}+\lambda_m\phi_l$, then the vector field λ_l forms a concircular vector field in the form $\lambda_{l,m}=\gamma g_{lm}$, where $\gamma=\frac{n\rho}{r+n\phi}$.*

Proof. Assume that the vector field ϕ_l is a torse-forming vector field with a scalar function ρ and a vector field λ_l. If we take $\lambda_m=\alpha_m$ in (3.26), we get

$$\lambda_{l,m}=\frac{n\rho}{r+n\phi}g_{lm}. \tag{3.28}$$

Taking $\gamma=\frac{n\rho}{r+n\phi}$, we obtain

$$\lambda_{l,m}=\gamma g_{lm}. \tag{3.29}$$

Thus, the vector field λ_l forms a concircular vector field. Hence, the proof is completed. □

Theorem 3.7. *If the vector field ϕ_l of (M,g) is a concircular vector field, then the vector field λ_l forms a torse-forming vector field in the relation*

$$\lambda_{l,m}=\frac{n\rho}{r+n\phi}g_{lm}-\lambda_l\lambda_m.$$

Proof. Assume that the vector field ϕ_l is a concircular vector field with a scalar function ρ, i.e.,

$$\phi_{l,m}=\rho g_{lm}. \tag{3.30}$$

Using the equation (3.30) in (3.18), we get

$$\rho g_{lm}=\left(\frac{r}{n}+\phi\right)\lambda_l\lambda_m+\left(\frac{r}{n}+\phi\right)\lambda_{l,m}. \tag{3.31}$$

Finally, from (3.31), we obtain

$$\lambda_{l,m}=\frac{n\rho}{r+n\phi}g_{lm}-\lambda_l\lambda_m. \tag{3.32}$$

Thus, the vector field λ_l forms a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.8. *If the vector field λ_l of (M,g) has constant length and the vector field ϕ_l is a concircular vector field, then the equation $\rho=c^2(\frac{r}{n}+\phi)$ holds, where $||\lambda||=c$.*

Proof. Assume that the vector field λ_l is of constant length, i.e., $\lambda_l\lambda^l=c^2$. Multiplying (3.32) by λ^l, we find

$$\lambda^l\lambda_{l,m}=\left(\frac{n\rho}{r+n\phi}-\lambda_l\lambda^l\right)\lambda_m. \tag{3.33}$$

Since λ_l is of constant length, then we have $\lambda^l\lambda_{l,m}=0$. By substituting the last relation and $\lambda_l\lambda^l=c^2$ in (3.33), finally we obtain

$$\rho=c^2\left(\frac{r}{n}+\phi\right). \tag{3.34}$$

Thus, the proof is completed. □

Theorem 3.9. *If the vector field λ_l of (M, g) is a concircular vector field in the form $\lambda_{l,m} = \rho g_{lm}$, then the vector field ϕ_l is a torse-forming vector field satisfying the equation*

$$\phi_{l,m} = \frac{n}{r + n\phi}\phi_m\phi_l + \frac{\rho(r + n\phi)}{n}g_{lm}.$$

Proof. Suppose that the vector field λ_l of (M, g) is a concircular vector field, i.e., the equation

$$\lambda_{l,m} = \rho g_{lm} \tag{3.35}$$

holds. Using the equations (3.21) and (3.35), we see that

$$\phi_{l,m} = \left(\frac{n}{r + n\phi}\right)\phi_m\phi_l + \frac{\rho(r + n\phi)}{n}g_{lm}. \tag{3.36}$$

This result shows that ϕ_l is a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.10. *If the vector field λ_l of (M, g) is a concircular vector field in the form $\lambda_{l,m} = \rho g_{lm}$ and the vector field ϕ_l has constant length, then the scalar function ρ generating the vector field λ_l has negative value and it satisfies the equation $\rho = -(\frac{nc}{r+n\phi})^2$.*

Proof. Let the vector field λ_l be a concircular vector field and the vector field ϕ_l be of constant length. Multiplying (3.36) by ϕ^l, we get

$$\phi^l\phi_{l,m} = \left(\frac{n}{r + n\phi}\right)\phi_m\phi_l\phi^l + \frac{\rho(r + n\phi)}{n}\phi_m. \tag{3.37}$$

Since the vector field ϕ_l is of constant length, then we have $\phi^l\phi_{l,m} = 0$. If we take $||\phi|| = c$, the equation (3.37) reduces to

$$\left(\frac{nc^2}{r + n\phi}\right)\phi_m + \frac{\rho(r + n\phi)}{n}\phi_m = 0. \tag{3.38}$$

Finally, from (3.38), we obtain

$$\rho = -\left(\frac{nc}{r + n\phi}\right)^2.$$

Thus, the proof is completed. □

Theorem 3.11. *If the vector field ϕ_l of (M, g) is a recurrent vector field in the form $\phi_{l,m} = \alpha_m\phi_l$, then the vector field λ_l is also recurrent vector field in the form $\lambda_{l,m} = (\alpha_m - \lambda_m)\lambda_l$.*

Proof. Suppose that the vector field ϕ_l is recurrent vector field. Thus, we have

$$\phi_{l,m} = \alpha_m\phi_l. \tag{3.39}$$

Substituting the equation (3.39) in (3.17), we get

$$\alpha_m \phi_l = \phi_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.40}$$

Finally, from (3.40), it can be obtained that

$$\lambda_{l,m} = (\alpha_m - \lambda_m)\lambda_l. \tag{3.41}$$

Thus, the proof is completed. □

Theorem 3.12. *A recurrent vector field ϕ_m, with the recurrence vector field α_m of (M, g) admits the relation $\alpha_m = \lambda_m$ if and only if the vector field λ_m is covariantly constant or is of constant length.*

Proof. If we take $\alpha_m = \lambda_m$ in Theorem 3.11 then from (3.41), we get

$$\lambda_{l,m} = 0. \tag{3.42}$$

Thus, we can say that the vector field λ_l is covariantly constant. Conversely, if the relation (3.42) is satisfied, from (3.41) we have $\alpha_m = \lambda_m$. Similarly, suppose that the vector field λ_l has constant length. If we multiply the equation (3.41) by λ^l, then we have $\alpha_m = \lambda_m$. The converse is also true. Hence, the proof is completed. □

Theorem 3.13. *Let the vector field λ_l of (M, g) be a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$. A necessary and sufficient condition the vector field ϕ_l to be divergence-free is that the scalar function μ to be in the form*

$$\mu = -\left(\frac{n}{r + n\phi}\right)^2 \frac{||\phi||^2}{r}.$$

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field, from (1.17),

$$\lambda_{l,m} = \mu S_{lm}, \tag{3.43}$$

where μ is a scalar function. Putting the equation (3.43) in (3.21), one can easily obtain that

$$\phi_{l,m} = \left(\frac{n}{r + n\phi}\right) \phi_l \phi_m + \mu \left(\frac{r + n\phi}{n}\right) S_{lm}. \tag{3.44}$$

Multiplying the equation (3.44) by g^{lm}, it is found that

$$\phi^l_{,l} = \left(\frac{n}{r + n\phi}\right) ||\phi||^2 + \mu \left(\frac{r + n\phi}{n}\right) r. \tag{3.45}$$

Now, assume that the vector field ϕ_l is divergence-free. In this case, the equation (3.45) reduces to

$$\mu = -\left(\frac{n}{r + n\phi}\right)^2 \frac{||\phi||^2}{r}. \tag{3.46}$$

Conversely, if the scalar function μ satisfies the relation (3.46), from (3.45), it can be obtained that ϕ_l is divergence-free. Thus, the proof is completed. □

Theorem 3.14. *If the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) has constant length, then the value $-\frac{1}{\mu}(\frac{n||\phi||}{r+n\phi})^2$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector δ defined by $\phi(X) = g(X, \delta)$.*

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l is of constant length. Multiplying (3.44) by ϕ^l then we get

$$\phi^l \phi_{l,m} = \left(\frac{n}{r+n\phi}\right) ||\phi||^2 \phi_m + \mu \left(\frac{r+n\phi}{n}\right) \phi^l S_{lm}. \tag{3.47}$$

Because ϕ_l is of constant length, we have $\phi^l \phi_{l,m} = 0$. In this case, from (3.47), we obtain

$$\phi^l S_{lm} = -\frac{1}{\mu} \left(\frac{n||\phi||}{r+n\phi}\right)^2 \phi_m. \tag{3.48}$$

Thus, the proof is completed. □

Theorem 3.15. *If the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) is a concircular vector field, then the Ricci tensor is in the following form*

$$S_{lm} = a g_{lm} + b \phi_m \phi_l,$$

which is a quasi-Einstein manifold where $a = \frac{n\rho}{\mu(r+n\phi)}$, $b = -\frac{1}{\mu}(\frac{n}{r+n\phi})^2$.

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) is a concircular vector field. From the equation (3.44), one can obtain that

$$\rho g_{lm} = \left(\frac{n}{r+n\phi}\right) \phi_l \phi_m + \left(\frac{r+n\phi}{n}\right) \mu S_{lm}. \tag{3.49}$$

We easily find from (3.49) that

$$S_{lm} = \frac{n\rho}{\mu(r+n\phi)} g_{lm} - \frac{1}{\mu} \left(\frac{n}{r+n\phi}\right)^2 \phi_l \phi_m. \tag{3.50}$$

Finally, the Ricci tensor can be written in the form

$$S_{lm} = a g_{lm} + b \phi_m \phi_l, \tag{3.51}$$

where $a = \frac{n\rho}{\mu(r+n\phi)}$, $b = -\frac{1}{\mu}(\frac{n}{r+n\phi})^2$. Therefore, this manifold is a quasi-Einstein manifold. In this case, the proof is completed. □

Theorem 3.16. *If the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields in the forms $\lambda_{l,m} = \mu S_{lm}$ and $\phi_{l,m} = \alpha S_{lm}$, respectively, then the Ricci tensor is in the following form*

$$S_{lm} = \gamma \phi_l \phi_m,$$

where $\gamma = \frac{n^2}{(r+n\phi)(n\alpha-\mu(r+n\phi))}$, $r + n\phi \neq 0$, $\alpha \neq \mu(\frac{r+n\phi}{n})$ and α, μ, γ are scalar functions. Thus, this manifold reduces to a quasi-Einstein manifold.

Proof. Assume that the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields, respectively. In this case, we have from (1.17)

$$\lambda_{l,m} = \mu S_{lm} \quad \text{and} \quad \phi_{l,m} = \alpha S_{lm}. \tag{3.52}$$

Substituting the relations (3.52) in (3.21), we get

$$\alpha S_{lm} = \frac{n}{r+n\phi}\phi_l\phi_m + \mu\left(\frac{r+n\phi}{n}\right)S_{lm}. \tag{3.53}$$

Finally, Ricci tensor takes the form

$$S_{lm} = \gamma\phi_l\phi_m, \tag{3.54}$$

where $\gamma = \frac{n^2}{(r+n\phi)(n\alpha-\mu(r+n\phi))}$ and $r+n\phi \neq 0$, $\alpha \neq \mu(\frac{r+n\phi}{n})$. This means that this manifold reduces to a quasi-Einstein manifold. Hence, the proof is completed. □

Theorem 3.17. *The vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields in the forms $\lambda_{l,m} = \mu S_{lm}$ and $\phi_{l,m} = \alpha S_{lm}$, respectively. If the eigenvalue determined by the vector field α_k is r, then the eigenvalue determined by the vector field μ_k is also r.*

Proof. Assume that the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields, respectively. As we already know the relations in (3.52), if we use the equations (3.18) and (3.52), then we get

$$\alpha S_{lm} = \left(\frac{r}{n}+\phi\right)(\mu S_{lm} + \lambda_l\lambda_m). \tag{3.55}$$

Arraying the equation (3.55), we find

$$\left(\alpha - \left(\frac{r}{n}+\phi\right)\mu\right)S_{lm} = \left(\frac{r}{n}+\phi\right)\lambda_l\lambda_m. \tag{3.56}$$

Now, let's find the covariant derivative of (3.56) and use the equation (3.16), one can easily see that

$$\begin{aligned}&\left[\alpha_k - \left(\frac{r}{n}+\phi\right)(\lambda_k\mu + \mu_k)\right]S_{lm} + \left(\alpha - \left(\frac{r}{n}+\phi\right)\mu\right)S_{lm,k}\\ &= \left(\frac{r}{n}+\phi\right)[\mu(\lambda_l S_{mk} + \lambda_m S_{lk}) + \lambda_l\lambda_m\lambda_k].\end{aligned} \tag{3.57}$$

We arrive at the following relation multiplying (3.57) by g^{lm}

$$\left[\alpha_k - \left(\frac{r}{n}+\phi\right)\mu_k\right]r = \left(\frac{r}{n}+\phi\right)[2\mu\lambda^l S_{lk} + ||\lambda||^2\lambda_k + \lambda_k\mu r]. \tag{3.58}$$

Again, multiplying (3.57) by g^{lk}, we get

$$\alpha^l S_{lk} - \left(\frac{r}{n}+\phi\right)\mu^l S_{lk} = \left(\frac{r}{n}+\phi\right)[2\mu(\lambda^l S_{lk} + ||\lambda||^2\lambda_k + \lambda_k\mu r]. \tag{3.59}$$

At the end, substracting the equations (3.58) and (3.59), we obtain

$$\alpha^l S_{lk} - \alpha_k r = \left(\frac{r}{n}+\phi\right)(\mu^l S_{lk} - \mu_k r). \tag{3.60}$$

So, we can say that if the eigenvalue determined by the vector field α_k is r, then the eigenvalue determined by the vector field μ_k is also r. Thus, the proof is completed. □

Theorem 3.18. *If the vector field ϕ_l of (M,g) is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$, then the Laplacian of the trace function of the Z-tensor is*

$$\Delta \bar{Z} = n\alpha r.$$

Proof. As we know that in a Riemannian manifold with covariantly constant conharmonic curvature tensor, the scalar curvature must be constant. Thus, going back to the relation (2.1), we get

$$\bar{Z}_{,k} = n\phi_k. \tag{3.61}$$

By taking the covariant derivative of (3.61), it can be found

$$\bar{Z}_{,kl} = n\phi_{k,l}. \tag{3.62}$$

Now, let us asuume that the vector field ϕ_l is a $\phi(Ric)$ vector field. In this case, the equation (3.62) takes the form

$$\bar{Z}_{,kl} = n\alpha S_{kl}. \tag{3.63}$$

Multiplying the equation (3.63) by g^{kl}, we obtain

$$g^{kl}\bar{Z}_{,kl} = \Delta \bar{Z} = n\alpha r. \tag{3.64}$$

Hence, the proof is completed. □

Theorem 3.19. *If the vector field ϕ_l of (M,g) is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$, then the scalar curvature satisfies the relation*

$$r = \frac{n\phi\delta}{n\alpha - \delta},$$

where $\delta \neq n\alpha$.

Proof. Assume that the vector field ϕ_l is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$. Hence, from (3.18), one can easily find that

$$\alpha S_{lm} = \left(\frac{r + n\phi}{n}\right)(\lambda_l \lambda_m + \lambda_{l,m}). \tag{3.65}$$

Let's multiply (3.65) by g^{lm}. Thus, it takes the form

$$\alpha r = \left(\frac{r + n\phi}{n}\right)\left(||\lambda||^2 + \lambda^l_{,l}\right). \tag{3.66}$$

Now, let's take $\delta = ||\lambda||^2 + \lambda^l_{,l}$ and $\delta \neq n\alpha$. Finally, it is obtained that

$$r = \frac{n\phi\delta}{n\alpha - \delta}. \tag{3.67}$$

Hence, this completes the proof. □

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A STUDY ON THE BLOW-UP OF SOLUTIONS FOR A LAMÉ SYSTEM OF INVERSE PROBLEM

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Abstract. We consider the Lamé system of inverse problem in a bounded domain with nonlinear boundary condition. When $2 < m \leq \frac{p}{4}$, we obtain the blow-up result for the weak solution with positive initial energy and sufficiently large initial data.

1. Introduction

We study the following Lamé system of inverse problem of determining a pair of functions $\{u(x,t), f(t)\}$ that satisfy:

(1.1)
$$u_{tt} - \Delta_e u - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) + h(x,t,u,\nabla u) = |u|^{p-2}u + f(t)\omega(x), \quad x \in \Omega, t > 0,$$

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \mu \frac{\partial u}{\partial \nu}(x,t) + |\frac{\partial u}{\partial \nu}|^{m-2}\frac{\partial u}{\partial \nu} + (\lambda+\mu)\operatorname{div} u = 0, & x \in \Gamma_1, t > 0, \end{cases} \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$\int_\Omega u(x,t)\omega(x)dx = 1, \quad t > 0, \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and ν is the unit outward normal to $\partial\Omega$. Let $u = (u^1, \dots, u^n)$ be a vector function, $\operatorname{div} u = u^1_{x_1} + u^2_{x_2} + \cdots + u^n_{x_n}$ be the divergence of u, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. We write

$$\Delta u = \left(\sum_{i=1}^n u^1_{x_i x_i}, \sum_{i=1}^n u^2_{x_i x_i}, \dots, \sum_{i=1}^n u^n_{x_i x_i}\right)^T.$$

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Here Δ_e denotes the elasticity operator, which is the $n \times n$ matrix-valued differential operator defined by

$$\Delta_e u = \mu\Delta u + (\lambda + \mu)\nabla(\operatorname{div} u),$$

μ and λ are the Lamé constants which satisfy the following conditions

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

Also, m and p are constants such that $p, m > 2$. In addition, $h(x,t,u,\nabla u)$ and $\omega(x)$ are real functions that satisfy specific conditions that will be enunciated later (see **(A1)**-**(A3)**).

Elasticity systems with constants Lamé coefficients in direct problems ($\omega(x) \equiv 0$) has attracted considerable attention in recent years, where diverse type of dissipative mechanisms have been introduced and several results have been obtained. In [1] Bchatnia and Daolati studied behavior of the energy for solutions to a Lamé system on a bounded domain with localized nonlinear damping and external force. Later, Bchatnia and Guesmia [2] considered the Lamé system in 3-dimension bounded domain with infinite memories and proved that system is well-possed and stable. Moreover, they established solutions converge to zero at infinity in terms of the growth of the infinite memories. Li and Bao [19] investigated the following memory-type elasticity problem

$$\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega \times (0,\infty),\\
&u = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
&\mu\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + (\mu+\lambda)(\operatorname{div} u)\nu + h(u_t) = 0, \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \text{in } \Omega.
\end{aligned}$$

The authors obtained global existence and the general energy decay of solutions by using perturbed energy method.

Boulaaras [6] proved asymptotic stability result of global solution for a coupled Lamé system with a viscoelastic term and the logarithmic nonlinearity. He obtained this result taking into account that the kernel is not necessarily decreasing. Recently, Bocanegra-Rodríguez et al. [5] investigated the longtime dynamics of the following semilinear Lamé systems

$$\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\partial_t u + f(u) = b,$$

defined in bounded domains of $\mathbb{R}^3$ with Dirichlet boundary condition. They proved the existence of finite dimensional global attractors subjected to a critical forcing $f(u)$. Moreover, they showed the upper-semicontinuity of attractors with respect to the parameter when $(\lambda+\mu) \to 0$ (see also [3, 4, 9, 10]).

Inverse problems are the problems that consist of finding an unknown property of an object, or medium, to a probing signal (see [21]). In contrast with the extensive literature on global behaviour of solutions in direct problems, we know little about the inverse problems. For instance, Eden and Kalantarov in [8] studied the following inverse source problem:

$$\begin{aligned}
&u_t - \Delta u - |u|^p u + b(x,t,u,\nabla u) = F(t)\omega(x), \quad x \in \Omega, t > 0,\\
&u(x,t) = 0, \quad x \in \partial\Omega, t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega,\\
&\int_\Omega u(x,t)\omega(x)dx = 1, \quad t > 0,
\end{aligned}$$

and by using the modified concavity method established global nonexistence results as well as stability results depending on the sign of nonlinearity. For more information about the concavity argument, we refer the readers to [16–18]. In [26] Shahrouzi and Tahamtani by using the same method found conditions on data that guaranteeing the global nonexistence and asymptotic stability results for a class of Petrovsky inverse source problems (see also [22–24, 27]). Bukhgeǐm et al. [7] considered an inverse problem for the stationary elasticity system with constant Lamé coefficients and variable matrix coefficient depending on the spatial variables and frequency. They proved uniqueness theorem by reduction of the inverse problem to a family of equations with the M. Riesz potential. For more results on the Lamé system of inverse problems, we refer the reader to [11–15, 25] and references therein.

The paper is organized as follows. In Section 2, we present some notations, assumptions and known results needed for our work and state our main result: Theorem 2.1. Section 3 is devoted to the proof of the blow-up result.

2. Preliminaries and Main Result

We begin this section by introducing some hypotheses and our main result. We shall assume that the functions $\omega(x), h(x,t,u,\nabla u)$ and the functions appearing in the data satisfy the following conditions:

(A1) $u_0 \in H_0^1(\Omega) \cap L^{p+2}(\Omega)$, $u_1 \in L^2(\Omega)$, $\int_\Omega u_0(x)\omega(x)dx = 1$;

(A2) $\omega \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p+2}(\Omega)$, $\int_\Omega \omega^2(x)dx = 1$;

(A3) for some positive M_1, M_2 we have $|h(x,t,u,\nabla u)| \le M_1|u|^{\frac{p}{2}} + M_2|\nabla u|^{\frac{m}{2}}$.

Throughout this paper all the functions considered are real-valued. We denote by $\|\cdot\|_q$ the L^q-norm over Ω . In particular, the L^2-norm is denoted $\|\cdot\|$ in Ω and $\|\cdot\|_{\Gamma_i}$ in Γ_i. Also we use familiar function spaces H_0^1, H^2.

We recall the trace Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1), \quad \text{for } 2 \le q < \frac{2(n-1)}{n-2},$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \le B_q \|\nabla u\|_2,$$

where B_q is the optimal constant.

We sometimes use the Young's inequality

$$ab \leq \beta a^q + C(\beta, q) b^{q'}, \quad a, b \geq 0, \beta > 0, \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.1}$$

where $C(\beta, q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$ are constants.

The following lemma was introduced in [16]. It will be used in the next section in order to prove the blow-up result.

Lemma 2.1. *Let $\alpha > 0$, $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that*

$$\psi''\psi - (1+\alpha)\left[\psi'\right]^2 \geq -2c_1\psi\psi' - c_2[\psi]^2,$$

for all $t \geq 0$. If

$$\psi(0) > 0 \quad \text{and} \quad \psi'(0) + \gamma_2\alpha^{-1}\psi(0) > 0,$$

then

$$\psi(t) \to +\infty, \quad \text{as } t \to t_1 \leq t_2 = \frac{1}{2\sqrt{c_1^2 + \alpha c_2}} \log \frac{\gamma_1\psi(0) + \alpha\psi'(0)}{\gamma_2\psi(0) + \alpha\psi'(0)}.$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \alpha c_2} \quad \text{and} \quad \gamma_2 = -c_1 - \sqrt{c_1^2 + \alpha c_2}.$$

We consider the following problem that is obtained from (1.1)–(1.4) by substituting $u(x,t) = e^{\xi t} v(x,t)$:

$$\begin{aligned} &v_{tt} + 2\xi v_t + \xi^2 v - \Delta_e v - e^{\xi(m-2)t}\text{div}\left(|\nabla v|^{m-2}\nabla v\right) + e^{-\xi t}\hat{h}(t,v) \\ &= e^{\xi(p-2)t}|v|^{p-2}v + e^{-\xi t} f(t)\omega(t), \quad x \in \Omega, t > 0, \end{aligned} \tag{2.2}$$

$$\begin{cases} v(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \mu\frac{\partial v}{\partial \nu}(x,t) + e^{\xi(m-2)t}|\frac{\partial v}{\partial \nu}|^{m-2}\frac{\partial v}{\partial \nu} + (\lambda+\mu)\text{div}\, v = 0, & x \in \Gamma_1, t > 0, \end{cases} \tag{2.3}$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) - \xi u_0(x), \quad x \in \Omega, \tag{2.4}$$

$$\int_\Omega v(x,t)\omega(x)dx = e^{-\xi t}, \quad t > 0, \tag{2.5}$$

where

$$\hat{h}(t,v) := h(x,t,e^{\xi t}v, e^{\xi t}\nabla v),$$

and the value of the parameter ξ will be prescribed later.

By using the idea of Prilepko et al. [20] and **(A2)**, one can easily see that the problem (2.2)–(2.5) is equivalent to (2.2)–(2.4) in which the unknown function $f(t)$ is replaced by

$$
\begin{aligned}
e^{-\xi t}f(t) =&\mu\int_\Omega \nabla v\nabla\omega dx + (\lambda+\mu)\int_\Omega(\operatorname{div} v)(\operatorname{div}\omega(x))dx\\
&+ e^{\xi(m-2)t}\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega dx + e^{-\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx\\
&- e^{\xi(p-2)t}\int_\Omega |v|^{p-2}v\omega(x)dx. \qquad (2.6)
\end{aligned}
$$

Define the total energy functional associated with problem (2.2)–(2.4) as follows

$$
E_\xi(t) = \frac{1}{p}e^{\xi(p-2)t}\|v\|_p^p - \frac{1}{2}I(t), \qquad (2.7)
$$

where

$$
I(t) = \|v_t\|^2 + \xi^2\|v\|^2 + \mu\|\nabla v\|^2 + (\lambda+\mu)\int_\Omega(\operatorname{div} v)^2dx + \frac{2}{m}e^{\xi(m-2)t}\|\nabla v\|_m^m.
$$

Now, we are in a position to state blow-up result.

Theorem 2.1. *Let the conditions* **(A1)-(A3)** *be satisfied. Assume that* $2 < m \leq \frac{p}{4}$ *and for sufficiently large initial data and* $\xi > 0$

$$
\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}} \leq \xi < \frac{(m-1)\int_\Omega u_0u_1dx}{(m+1)\|u_0\|^2}, \qquad (2.8)
$$

$$
E_\xi(0) \geq \frac{2D_1}{\xi(p+2)} + \frac{D_2}{2m},
$$

where

$$
\begin{aligned}
D_1 =&\frac{\mu\xi}{p-2}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)\xi}{p-2}\int_\Omega(\operatorname{div}\omega(x))^2dx + \frac{\xi^2(pM_1^2+2mM_2^2)}{4p+8m}\|\omega\|^2\\
&+ \frac{\xi\|\nabla\omega\|_m^m}{m\left[\frac{p-2}{12(m-1)}\right]^{m-1}} + \frac{\xi\|\omega\|_p^p}{p\left[\frac{p-2}{6(p-1)}\right]^{p-1}}, \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
D_2 =&\frac{\mu}{2m}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)}{2m}\int_\Omega(\operatorname{div}\omega(x))^2dx + \frac{\xi^2(3pM_1^2+6mM_2^2)}{8m}\|\omega\|^2\\
&+ \frac{\xi^m\|\nabla\omega\|_m^m}{m^m\left[\frac{1}{3(m-1)}\right]^{m-1}} + \frac{\xi^p\|\omega\|_p^p}{p\left[\frac{2m}{3(p-1)}\right]^{p-1}}. \qquad (2.10)
\end{aligned}
$$

Then there exists a finite time t_1 *such that the solution of the problem* (1.1)–(1.4) *blows up in* t_1*, that is*

$$
\|u(t)\| \to +\infty, \quad \textit{as } t \to t_1.
$$

3. Blow-up

In this section we are going to prove that for sufficiently large initial data some of the solutions blow up in a finite time. To prove the blow-up result (Theorem 2.1) for certain solutions with positive initial energy, we need the following lemma for the problem (2.2)–(2.5).

Lemma 3.1. *Under the conditions of Theorem* 2.1*, the energy functional* $E_\xi(t)$*, defined by* (2.7)*, satisfies*

$$E_\xi(t) \geq E_\xi(0) - \frac{2D_1}{\xi(p+2)}.$$

Proof. A multiplication of equation (2.2) by v_t and integrating over Ω gives

$$\begin{aligned} E'_\xi(t) =& 2\xi\|v_t\|^2 - \frac{\xi(m-2)}{m} e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi(p-2)}{p} e^{\xi(p-2)t}\|v\|_p^p \\ &+ e^{-\xi t}\int_\Omega v_t \hat{h}(t,v)dx + \xi e^{-2\xi t} f(t). \end{aligned} \tag{3.1}$$

Plugging definition of $f(t)$, (2.6) into (3.1), we obtain

$$\begin{aligned} E'_\xi(t) =& 2\xi\|v_t\|^2 - \frac{\xi(m-2)}{m} e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi(p-2)}{p} e^{\xi(p-2)t}\|v\|_p^p + e^{-\xi t}\int_\Omega v_t \hat{h}(t,v)dx \\ &+ \xi\mu e^{-\xi t}\int_\Omega \nabla v \nabla \omega dx + \xi(\lambda+\mu)e^{-\xi t}\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))\, dx \\ &+ \xi e^{\xi(m-3)t}\int_\Omega |\nabla v|^{m-2}\nabla v \nabla\omega(x)dx + \xi e^{-2\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx \\ &- \xi e^{\xi(p-3)t}\int_\Omega |v|^p v\omega(x)dx. \end{aligned} \tag{3.2}$$

Next, we estimate the terms on the right-hand side of (3.2). Using **(A3)**, Cauchy-Schwartz and Young's inequality (2.1), we obtain

$$\begin{aligned} e^{-\xi t}\left|\int_\Omega v_t\hat{h}(t,v)dx\right| \leq& M_1\int_\Omega v_t e^{\xi(\frac{p}{2}-1)t}|v|^{\frac{p}{2}}dx + M_2\int_\Omega v_t e^{\xi(\frac{m}{2}-1)t}|\nabla v|^{\frac{m}{2}}dx \\ \leq& M_1\|v_t\| e^{\xi(\frac{p}{2}-1)t}\|v\|_p^{\frac{p}{2}} + M_2\|v_t\| e^{\xi(\frac{m}{2}-1)t}\|\nabla v\|_m^{\frac{m}{2}} \\ \leq& \beta_1 e^{\xi(p-2)t}\|v\|_p^p + \beta_2 e^{\xi(m-2)t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\|v_t\|^2, \end{aligned} \tag{3.3}$$

where β_1 and β_2 are arbitrary positive constants

$$\mu\xi e^{-\xi t}\left|\int_\Omega \nabla v\nabla\omega dx\right| \leq \frac{\mu\xi(p-2)}{4}\|\nabla v\|^2 + \frac{\mu\xi}{p-2}e^{-2\xi t}\|\nabla\omega\|^2, \tag{3.4}$$

$$\begin{aligned} &\xi(\lambda+\mu)e^{-\xi t}\left|\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))\, dx\right| \\ \leq& \frac{\xi(\lambda+\mu)(p-2)}{4}\int_\Omega (\operatorname{div} v)^2\, dx + \frac{\xi(\lambda+\mu)}{p-2}e^{-2\xi t}\int_\Omega (\operatorname{div}\omega(x))^2\, dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\xi e^{\xi(m-3)t}\left|\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega(x)dx\right| \leq \xi e^{\xi(m-2)t}\|\nabla v\|_m^{m-1} e^{-\xi t}\|\nabla\omega\|_m \\ \leq& \beta_3 e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi^m e^{-2\xi t}}{m\left[\frac{\beta_3 m}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m, \end{aligned} \tag{3.6}$$

where β_3 is an arbitrary positive constant,

$$\begin{aligned}
&\xi e^{-2\xi t}\left|\int_\Omega \hat{h}(t,v)\omega(x)dx\right| \\
\leq & M_1\int_\Omega e^{\xi(\frac{p}{2}-1)t}|v|^{\frac{p}{2}}\xi e^{-\xi t}\omega(x)dx + M_2\int_\Omega e^{\xi(\frac{m}{2}-1)t}|\nabla v|^{\frac{m}{2}}\xi e^{-\xi t}\omega(x)dx \\
\leq & e^{\xi(\frac{p}{2}-1)t}\|v\|_p^{\frac{p}{2}}.M_1\xi e^{-\xi t}\|\omega\| + e^{\xi(\frac{m}{2}-1)t}\|\nabla v\|_m^{\frac{m}{2}}.M_2\xi e^{-\xi t}\|\omega\| \\
\leq & \beta_4 e^{\xi(p-2)t}\|v\|_p^p + \beta_5 e^{\xi(m-2)t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\beta_4}+\frac{M_2^2}{4\beta_5}\right)\xi^2 e^{-2\xi t}\|\omega\|^2, \qquad (3.7)
\end{aligned}$$

where β_4 and β_5 are arbitrary positive constants.

Finally, we have for any positive β_6:

$$\begin{aligned}
\xi e^{\xi(p-3)t}\left|\int_\Omega |v|^p v\omega(x)dx\right| \leq & \xi E^{\xi(p-2)t}\|v\|_p^{p-1}e^{-\xi t}\|\omega\|_p \\
\leq & \beta_6 e^{\xi(p-2)t}\|v\|_p^p + \frac{\xi^p e^{-2\xi t}}{p\left[\frac{\beta_6 p}{p-1}\right]^{p-1}}\|\omega\|_p^p. \qquad (3.8)
\end{aligned}$$

Combining (3.3)–(3.8) with (3.2), we deduce

$$\begin{aligned}
E'_\xi(t) \geq & \left[2\xi - \left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\right]\|v_t\|^2 - \left(\frac{\xi(m-2)}{m}+\beta_2+\beta_3+\beta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\
&+\left(\frac{\xi(p-2)}{p}-\beta_1-\beta_4-\beta_6\right)e^{\xi(p-2)t}\|v\|_p^p - \frac{\mu\xi(p-2)}{4}\|\nabla v\|^2 \\
&-\frac{\xi(\lambda+\mu)(p-2)}{4}\int_\Omega (\operatorname{div} v)^2\,dx - e^{-2\xi t}D_1, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
D_1 = & \frac{\mu\xi}{p-2}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)\xi}{p-2}\int_\Omega (\operatorname{div}\omega(x))^2\,dx + \frac{\xi^2(\beta_5 M_1^2+\beta_4 M_2^2)}{4(\beta_4+\beta_5)}\|\omega\|^2 \\
&+\frac{\xi^m\|\nabla\omega\|_m^m}{m\left[\frac{\beta_3 m}{m-1}\right]^{m-1}} + \frac{\xi^p\|\omega\|_p^p}{p\left[\frac{\beta_6 p}{p-1}\right]^{p-1}}.
\end{aligned}$$

By virtue of (3.9), we obtain from (2.7) the following inequality

$$\begin{aligned}
E'_\xi(t) - \frac{\xi(p-2)}{2}E_\xi(t) \geq & \left(\frac{\xi(p-2m+2)}{2m}-\beta_2-\beta_3-\beta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\
&+\left(\frac{\xi(p-2)}{2p}-\beta_1-\beta_4-\beta_6\right)e^{\xi(p-2)t}\|v\|_p^p \\
&+\left[\frac{\xi(p+6)}{4}-\left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\right]\|v_t\|^2 - e^{-2\xi t}D_1.
\end{aligned}$$

At this point if we choose $\beta_1 = \beta_4 = \beta_6 = \frac{\xi(p-2)}{6p}$ and $\beta_2 = \beta_3 - \beta_5 = \frac{\xi(p-2)}{12m}$, then we gain

$$E'_\xi(t) - \frac{\xi(p-2)}{2}E_\xi(t) \geq \frac{\xi(p-4m+6)}{4m}\|\nabla v\|_m^m + \left[\frac{\xi(p+6)}{4} - \frac{3pM_1^2 + 6mM_2^2}{2\xi(p-2)}\right]\|v_t\|^2 - e^{-2\xi t}D_1.$$

Hence, by choosing $m \leq \frac{p+6}{4}$ and

$$\xi \geq \sqrt{\frac{6(pM_1^2 + 2mM_2^2)}{p^2 + 4p - 12}},$$

we get

$$E'_\xi(t) - \frac{\xi(p-2)}{2}E_\xi(t) \geq -e^{-2\xi t}D_1. \tag{3.10}$$

Integrating the differential inequality (3.10) between 0 and t gives that

$$E_\xi(t) \geq E_\xi(0) - \frac{D_1}{\xi(p+2)},$$

where D_1 satisfies (2.9) and proof of Lemma 3.1 is completed. □

Proof of Theorem 2.1**.** For obtain the blow-up result, the choice of the following functional is standard (see [17, 18])

$$\psi(t) = \|v(t)\|^2, \tag{3.11}$$

then

$$\psi'(t) = 2\int_\Omega v v_t dx, \quad \psi''(t) = 2\int_\Omega v v_{tt} dx + 2\|v_t\|^2. \tag{3.12}$$

A multiplication of equation (2.2) by v and integrating over Ω gives

$$\begin{aligned}\int_\Omega v v_{tt} dx =& -2\xi\int_\Omega v v_t dx - \xi^2\|v\|^2 - \mu\|\nabla v\|^2 - (\lambda+\mu)\int_\Omega (\operatorname{div} v)^2 dx\\ &- e^{\xi(m-2)t}\|\nabla v\|_m^m - e^{-\xi t}\int_\Omega v\hat{h}(t,v)dx + e^{\xi(p-2)t}\|v\|_p^p\\ &+ \mu e^{-\xi t}\int_\Omega \nabla v \nabla\omega dx + (\lambda+\mu)e^{-\xi t}\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))dx\\ &+ e^{\xi(m-3)t}\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega dx + e^{-2\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx\\ &- e^{\xi(p-3)t}\int_\Omega |v|^{p-2}v\omega(x)dx,\end{aligned} \tag{3.13}$$

where the definition of unknown function (2.6) has been used.

By combining (2.7) with (3.13), one can easily verify that

$$\begin{aligned}\int_\Omega vv_{tt}dx =&\eta E_\xi(t) - 2\xi\int_\Omega vv_t dx + \frac{\eta}{2}\|v_t\|^2 + \xi^2\left(\frac{\eta}{2}-1\right)\|v\|^2 + \mu\left(\frac{\eta}{2}-1\right)\|\nabla v\|^2\\&+(\lambda+\mu)\left(\frac{\eta}{2}-1\right)\int_\Omega(\operatorname{div} v)^2dx + \left(\frac{\eta}{m}-1\right)e^{\xi(m-2)t}\|\nabla v\|_m^m\\&+\left(1-\frac{\eta}{p}\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-\xi t}\int_\Omega v\hat{h}(t,v)dx + \mu e^{-\xi t}\int_\Omega \nabla v\nabla\omega dx\\&+(\lambda+\mu)e^{-\xi t}\int_\Omega(\operatorname{div} v)(\operatorname{div}\omega(x))dx + e^{\xi(m-3)t}\int_\Omega|\nabla v|^{m-2}\nabla v\nabla\omega dx\\&+e^{-2\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx - e^{\xi(p-3)t}\int_\Omega|v|^{p-2}v\omega(x)dx.\end{aligned}\tag{3.14}$$

Applying **(A3)**, Cauchy-Schwartz inequality and the Young's inequality (2.1) to estimate the terms on the right-hand side of (3.14)

$$\begin{aligned}e^{-\xi t}\left|\int_\Omega v\hat{h}(t,v)dx\right| \le& M_1\|v\|e^{(\frac{p}{2}-1)\xi t}\|v\|_p^{\frac{p}{2}} + M_2\|v\|e^{(\frac{m}{2}-1)\xi t}\|\nabla v\|_m^{\frac{m}{2}}\\ \le&\theta_1 e^{(p-2)\xi t}\|v\|_p^p + \theta_2 e^{(m-2)\xi t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\theta_1}+\frac{M_2^2}{4\theta_2}\right)\|v\|^2,\end{aligned}\tag{3.15}$$

where θ_1 and θ_2 are positive constants,

$$\mu e^{-\xi t}\left|\int_\Omega \nabla v\nabla\omega dx\right| \le \frac{\mu\eta}{4}\|\nabla v\|^2 + \frac{\mu e^{-2\xi t}}{\eta}\|\nabla\omega\|^2,\tag{3.16}$$

$$\begin{aligned}&(\lambda+\mu)e^{-\xi t}\left|\int_\Omega(\operatorname{div} v)(\operatorname{div}\omega(x))dx\right|\\ \le&\frac{\eta(\lambda+\mu)}{4}\int_\Omega(\operatorname{div} v)^2dx + \frac{(\lambda+\mu)e^{-2\xi t}}{\eta}\int_\Omega(\operatorname{div}\omega(x))^2dx,\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}e^{\xi(m-3)t}\left|\int_\Omega|\nabla v|^{m-2}\nabla v\nabla\omega dx\right| \le&\xi e^{\xi(m-2)t}\|\nabla v\|_m^{m-1}e^{-\xi t}\|\nabla\omega\|_m\\ \le&\theta_3 e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi^m e^{-2\xi t}}{m\left[\frac{m\theta_3}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m,\end{aligned}\tag{3.18}$$

where θ_3 is an arbitrary positive constant. Also, similar to (3.7), we have

$$\begin{aligned}e^{-2\xi t}\left|\int_\Omega\hat{h}(t,v)\omega(x)dx\right| \le&\theta_4 e^{\xi(p-2)t}\|v\|_p^p + \theta_5 e^{\xi(m-2)t}\|\nabla v\|_m^m\\&+\left(\frac{M_1^2}{4\theta_4}+\frac{M_2^2}{4\theta_5}\right)\xi^2 e^{-2\xi t}\|\omega\|^2,\end{aligned}\tag{3.19}$$

where θ_4 and θ_5 are positive constants. Furthermore, for $\theta_6 > 0$ we derive

$$e^{\xi(p-3)t}\int_\Omega|v|^{p-2}v\omega(x)dx \le \theta_6 e^{\xi(p-2)t}\|v\|_p^p + \frac{\xi^p e^{-2\xi t}}{p\left[\frac{p\theta_6}{p-1}\right]^{p-1}}\|\omega\|_p^p.\tag{3.20}$$

Utilizing (3.15)–(3.20) with (3.14), we get

$$\begin{aligned}\int_\Omega vv_{tt}dx \geq &\eta E_\xi(t) - 2\xi\int_\Omega vv_t dx + \left[\xi^2(\frac{\eta}{2}-1) - \left(\frac{M_1^2}{4\theta_1}+\frac{M_2^2}{4\theta_2}\right)\right]\|v\|^2 \\ &+ \mu\left(\frac{\eta}{4}-1\right)\|\nabla v\|^2 + \frac{\eta}{2}\|v_t\|^2 + (\lambda+\mu)\left(\frac{\eta}{4}-1\right)\int_\Omega (\operatorname{div} v)^2 dx \\ &+ \left(\frac{\eta}{m} - 1 - \theta_2 - \theta_3 - \theta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\ &+ \left(1 - \frac{\eta}{p} - \theta_1 - \theta_4 - \theta_6\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-2\xi t}\Bigg(\frac{\mu}{\eta}\|\nabla\omega\|^2 \\ &+ \frac{\lambda+\mu}{\eta}\int_\Omega(\operatorname{div}\omega(x))^2dx + \frac{\xi^m}{m\left[\frac{m\theta_3}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m \\ &+ \left(\frac{M_1^2}{4\theta_4}+\frac{M_2^2}{4\theta_5}\right)\xi^2\|\omega\|^2 + \frac{\xi^p}{p\left[\frac{p\theta_6}{p-1}\right]^{p-1}}\|\omega\|_p^p\Bigg).\end{aligned}$$

Now by choosing $\eta = 2m$, $\theta_2 = \theta_3 = \theta_5 = \frac{1}{3}$ and $\theta_1 = \theta_4 = \theta_6 = \frac{2m}{3p}$, we conclude that

$$\begin{aligned}\int_\Omega vv_{tt}dx \geq &2mE_\xi(t) - 2\xi\int_\Omega vv_t dx + \frac{\eta}{2}\|v_t\|^2 + \left[\xi^2(m-1) - \left(\frac{3pM_1^2}{8m}+\frac{3M_2^2}{4}\right)\right]\|v\|^2 \\ &+ \mu\left(\frac{m}{2}-1\right)\|\nabla v\|^2 + (\lambda+\mu)\left(\frac{m}{2}-1\right)\int_\Omega (\operatorname{div} v)^2 dx \\ &+ \left(1 - \frac{4m}{p}\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-2\xi t}D_2,\end{aligned}$$

where D_2 satisfies (2.10). Let $2 < m \leq \frac{p}{4}$ and

$$\xi \geq \sqrt{\frac{3(pM_1^2 + 2mM_2^2)}{8m(m-1)}},$$

it holds that

$$\int_\Omega vv_{tt}dx \geq 2mE_\xi(t) - 2\xi\int_\Omega vv_t dx + m\|v_t\|^2 - e^{-2\xi t}D_2.$$

According to Lemma 3.1 and hypothesis of Theorem 2.1, we obtain

$$\int_\Omega vv_{tt}dx \geq -2\xi\int_\Omega vv_t dx + m\|v_t\|^2. \tag{3.21}$$

To this end, by substituting (3.11) and (3.12) in (3.21), we arrive at

$$\psi''(t) \geq -2\xi\psi'(t) + 2(m+1)\|v_t\|^2,$$

finally we get

$$\psi(t)\psi''(t) \geq \frac{(m+1)}{2}[\psi'(t)]^2 - 2\xi\psi(t)\psi'(t).$$

Considering $2 < m \leq \frac{p}{4}$, it is obvious that

$$\max\left\{\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}},\sqrt{\frac{6(pM_1^2+2mM_2^2)}{p^2+4p-12}}\right\}=\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}}.$$

Hence by attention to (2.8) we see that the hypotheses of Lemma 2.1 are fulfilled with $\alpha = \frac{m-1}{2}$, $c_1 = \xi$, $c_2 = 0$ and

$$\psi'(0) - \frac{4\xi}{m-1}\psi(0) > 0,$$

thus conclusion of Lemma 2.1 gives us that some solutions of problem (2.2)–(2.5) blow up in a finite time and since this system is equivalent to (1.1)–(1.4), the proof is completed.

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UNIFORM ULTIMATE BOUNDEDNESS RESULTS FOR SOME SYSTEM OF THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. The paper is concerned with the study of the uniform ultimate boundedness of solutions of the third-order system of nonlinear delay differential equation

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t-r)) = P(t, X, \dot{X}, \ddot{X}),$$

where A, B are real $n \times n$ constant symmetric matrices, r is a positive real constant and $X \in \mathbb{R}^n$, using the Lyapunov-Krasovskii functional method and following the arguments used in [1] and [10], we obtained results which give an $n-$dimensional analogue of an earlier result of [13] and extend other earlier results for the case in which we do not necessarily require that $H(X(t-r))$ be differentiable.

1. Introduction

Let $\mathbb{R}$ denote the real line, $-\infty < t < \infty$ and $\mathbb{R}^n$ denote the real n-dimensional Euclidean space $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (in n places) with the usual norm which will be represented throughout by $\|\cdot\|$.

Consider the delay differential equation of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t-r)) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

where $X \in \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, A and B are real $n \times n$ constant symmetric matrices, r is a positive real constant and the dots indicate differentiation with respect to t. We shall assume that H and P are continuous in their respective arguments.

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Equation (1.1) is the vector version for systems of real third-order nonlinear delay differential equations of the form

$$\dddot{x}_i + \sum_{k=1}^{n} a_{ik}\ddot{x}_k + \sum_{k=1}^{n} b_{ik}\dot{x}_k + h_i(x_1(t-r), x_2(t-r), \ldots, x_n(t-r))$$
$$= p_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ddot{x}_1, \ldots, \ddot{x}_n),$$

$i = 1, 2, \ldots, n$, in which a_{ik} and b_{ik} are real constants, r is a positive real constant and h_i, p_i are continuous in their respective arguments. The case when $n = 1$ and $r = 0$ which give rise to the nonlinear differential equations of the form

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.2}$$

have been greatly studied by several authors for stability, boundedness, convergence and periodicity of solutions (see [5, 8, 14]). Similarly, equations of the form (1.2) for which a, b are not necessarily constants have been studied by several authors in the literature (see [14]). For the case $n = 1$ and $r > 0$, delay differential equations of the form

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x(t-r)) = p(t, x, \dot{x}, \ddot{x}) \tag{1.3}$$

have been studied for stability, boundedness and periodicity of solutions by several authors in the literature. In [18], sufficient conditions which ensure the stability (for $p(t, x, \dot{x}, \ddot{x}) = 0$) and boundedness (for $p(t, x, \dot{x}, \ddot{x}) \neq 0$) of solutions of equation (1.3) were obtained. In [13], equation (1.3) (in which h is not necessarily differentiable) was studied, and the author obtained conditions which ensure that solutions are bounded. Similarly, equations of the form (1.3) for which a, b are not necessarily constants have been studied by several authors in the literature. It is worth mentioning that equation (1.1), when $r = 0$, gives rise to the nonlinear vector differential equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.4}$$

where A, B, H and P are as defined above. Equations of the form (1.4) have been studied by several authors for boundedness and periodicity of solutions ([1,6,10]). In [6] the authors studied equation (1.4) when $H(X)$ is continuous and differentiable, while in [1] and [10] the authors studied (1.4) when $H(X)$ is not necessarily differentiable. Similarly, qualitative properties of solutions of (1.4) for which A, B are not necessarily constants have been investigated by several authors (see [2,7,9,15]). However, there are few papers in connection with the qualitative properties of solutions of systems of third order nonlinear delay differential equations in literature. Recently, in [12], equation (1.1) in which $r = r(t)$, $H \in \mathcal{C}'(\mathbb{R}^n)$ and $P(t, X, \dot{X}, \ddot{X}) = P(t)$ was investigated for the boundedness of solutions, while in [17], the author studied the stability and boundedness of solutions of the equation

$$\dddot{X} + H(\dot{X})\ddot{X} + G(\dot{X}(t-r)) + cX(t-r) = P(t, X, \dot{X}, \ddot{X}),$$

where H, G are continuous and differentiable in their arguments and P is continuous in its arguments. To the best of our knowledge the extension of the results in [1] and [13] to equation (1.1) does not exist in literature. Throughout all the foregoing papers, the Lyapunov's second (or direct) method has been used as a main tool to carry out the proofs of the main results for scalar and vector ordinary differential equations, while the Lyapunov-Krasovskii functional method has been used for scalar and vector delay differential equations ([1] - [19]). In the present paper, we shall use the Lyapunov-Krasovskii functional method as a basic tool in our proofs. In the present paper, we also used the same method as a basic tool in our proofs. The motivation for the present work is derived from the papers mentioned above, and the object of this paper is to prove the uniform boundedness results under specified conditions on H and P. Specifically, unlike in [12], we shall only assume that H is not necessarily differentiable, and that for any $X, Y \in \mathbb{R}^n$ (following [1] and [10]), there exists an $n \times n$ operator $C(X, Y)$ such that

$$H(X) = H(Y) + C(X,Y)(X - Y) \tag{1.5}$$

for which the eigenvalues $\lambda_i(C(X, Y))$, $i = 1, 2, \ldots, n$, are continuous and satisfy

$$0 < \delta_h \leq \lambda_i(C(X,Y)) \leq \Delta_h$$

for fixed constants δ_h and Δ_h. Moreover, we shall assume that

$$\Delta_h \leq k\delta_a\delta_b, \quad k < 1,$$

where

$$k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}, \frac{\alpha(1-\beta)\delta_a}{(\delta_a+2\alpha)^2}\right\} \tag{1.6}$$

and

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a, \tag{1.7}$$

$$0 < \delta_b \leq \lambda_i(B) \leq \Delta_b, \tag{1.8}$$

with $\lambda_i(A)$ and $\lambda_i(B)$ as the eigenvalues of A and B, respectively.

The result in this paper is the n-dimensional analog of a result in [13]. Moreover, we shall improve on the results in [12] when $H(X(t-r))$ is not necessarily differentiable and $r(t) = r > 0$.

1.1. **Notation and definitions.** Given any X, Y in $\mathbb{R}^n$ the symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$, that is $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$. Thus $\|X\|^2 = \langle X, X\rangle$. The matrix A is said to be positive definite when $\langle AX, X\rangle > 0$ for all nonzero X in $\mathbb{R}^n$.

The following notations (see [12, 13]) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0$, $t_1 \in \mathbb{R}$, $C(t_1) = \{\phi : [t_1 - r, t_1] \to \mathbb{R}^n/\phi$ is continuous$\}$. In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = \sup_{-r\leq\theta\leq 0} |\phi(0)|$. $C_{\mathbf{H}}$ will denote the set of ϕ such that $\|\phi\| \leq \mathbf{H}$. For any continuous function $x(u)$ defined on $-h \leq u < A$, $A > 0$, and any fixed t, $0 \leq t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-r,t]$, that is, x_t is an element of C defined by $x_t(\theta)=x(t+\theta)$, $-r\le\theta\le 0$.

2. Some Preliminary Results

In this section, we shall state the algebraic results required in the proofs of our main results. The proofs are not given since they are found in [1, 2, 6, 7, 9–11, 15, 16].

Lemma 2.1 ([1, 2, 6, 7, 9–11, 15, 16])**.** *Let D be a real symmetric positive definite $n\times n$ matrix, then for any X in $\mathbb{R}^n$, we have*

$$\delta_d\|X\|^2\le\langle DX,X\rangle\le\Delta_d\|X\|^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 ([1, 2, 6, 7, 9–11, 15, 16])**.** *Let Q, D be any two real $n\times n$ commuting symmetric matrices. Then*

(i) *the eigenvalues $\lambda_i(QD)$, $i=1,2,\dots,n$, of the product matrix QD are all real and satisfy*

$$\min_{1\le j,k\le n}\lambda_j(Q)\lambda_k(D)\le\lambda_i(QD)\le\max_{1\le j,k\le n}\lambda_j(Q)\lambda_k(D);$$

(ii) *the eigenvalues $\lambda_i(Q+D)$, $i=1,2,\dots,n$, of the sum of matrices Q and D are real and satisfy*

$$\left\{\min_{1\le j\le n}\lambda_j(Q)+\min_{1\le k\le n}\lambda_k(D)\right\}\le\lambda_i(Q+D)\le\left\{\max_{1\le j\le n}\lambda_j(Q)+\max_{1\le k\le n}\lambda_k(D)\right\}.$$

Lemma 2.3. *Let $H\in\mathcal{C}(\mathbb{R}^n)$ be a continuous vector function and that $H(0)=0$. Then*

$$H(U)=C(U,0)X(t)-C(U,0)\int_{t-r}^{t}Y(s)ds,$$

where $U=X(t-r)$.

Proof of Lemma 2.3. From (1.5), we have that

$$H(X(t-r))=H(Y(t-r))+C(X(t-r),Y(t-r))(X(t-r)-Y(t-r)). \tag{2.1}$$

If we set $Y(t-r)=0$ in (2.1), we obtain

$$H(X(t-r))=C(X(t-r),0)X(t-r). \tag{2.2}$$

Since

$$X(t-r)=X(t)-\int_{t-r}^{t}Y(s)ds,$$

where

$$\dot{X}(t)=\frac{dX(t)}{dt}=Y(t),$$

it follows from (2.2) that

$$H(X(t-r))=C(X(t-r),0)X(t)-C(X(t-r),0)\int_{t-r}^{t}Y(s)ds.$$

Let $U=X(t-r)$, hence the result follows. □

Corollary 2.1. *If $r = 0$, then* (2.2) *reduces to $H(X) = C(X, 0)X$.*

3. Boundedness

First, consider a system of delay differential equations

$$\dot{x} = F(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0, \tag{3.1}$$

where $F : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets. The following lemma is a well-known result obtained in [4].

Lemma 3.1 ([4]). *Let $V(t, \phi) : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) $W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\right)$, *and*
(ii) $\dot{V}_{(3.1)} \le -W_3(|x(s)|) + M$,

for some $M > 0$, where $W(r)$, W_i, $i = 1, 2, 3$, are wedges, then the solutions of (3.1) *are uniformly bounded and uniformly ultimately bounded for bound* $\mathbf{B}$.

To study the boundedness of solutions of (1.1) for which $P(t, X, \dot{X}, \ddot{X}) \neq 0$, we would need to write (1.1) in the form

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -AZ - BY - H(X(t-r)) + P(t, X, Y, Z). \end{aligned} \tag{3.2}$$

Our main theorem in this paper stated with respect to (3.2), which is an n-dimensional analogue of a result in [13] is the following.

Theorem 3.1. *Consider* (3.2), *let $H(0) = 0$ and suppose that*

(i) *there exists an $n \times n$ real continuous operator $C(X, Y)$ for any vectors $X, Y \in \mathbb{R}^n$ such that*

$$H(X) = H(Y) + C(X, Y)(X - Y)$$

whose eigenvalues $\lambda_i(C(X, Y))$, $i = 1, 2, \ldots, n$, satisfy

$$0 < \delta_h \le \lambda_i(C(X, Y)) \le \Delta_h; \tag{3.3}$$

(ii) *the constant symmetric matrices A and B have positive eigenvalues, commute with themselves as well with the operator $C(X, Y)$ for any $X, Y \in \mathbb{R}^n$ and that*

$$\Delta_h \le k\delta_a\delta_b,$$

where k (< 1) is the constant defined in (1.6);

(iii) *there exist finite constants $\Delta_0 \ge 0$, $\Delta_1 \ge 0$, such that the vector P satisfies*

$$\|P(t, X, Y, Z)\| \le \Delta_0 + \Delta_1(\|X\| + \|Y\| + \|Z\|) \tag{3.4}$$

uniformly in t, for all arbitrary $X, Y, Z \in \mathbb{R}^n$. Then, if Δ_1 is sufficiently small, the solutions to the system (3.2) *are uniformly bounded and uniformly ultimately bounded provided*

$$r < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h}, \frac{2\beta\delta_a\delta_b}{\Delta_h[1+(1-\beta)\Delta_b + 2(\Delta_a+\alpha+\alpha\delta_a^{-1})]}, \frac{\alpha}{\Delta_h(1+2\alpha\delta_a^{-1})}\right\}.$$

Proof. The main tool in the proof of Theorem 3.1 is the Lyapunov functional

$$\begin{aligned} 2V(X_t, Y_t, Z_t) =& \beta(1-\beta)\langle BX, BX\rangle + \beta\langle BY, Y\rangle + 2\alpha\langle BY, A^{-1}Y\rangle \\ &+ \alpha\langle A^{-1}Z, Z\rangle + \alpha\langle A^{-1}(AY+Z), AY+Z\rangle \\ &+ \langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle \\ &+ \lambda\int_{-r}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds, \end{aligned} \tag{3.5}$$

where $0 < \beta < 1$ and $\alpha, \lambda > 0$ are constants.

Obviously, the function $V(X_t, Y_t, Z_t)$ is positive definite since each term of (3.5) is positive. Hence the condition (i) of Lemma 3.1 is satisfied. Now, let us compute the time derivative of the functional $V(X_t, Y_t, Z_t)$ for the solution (X_t, Y_t, Z_t) of system (3.2). By $\dot{V}$, we denote the time derivative of the function $V = V(X_t, Y_t, Z_t)$ for the solution (X_t, Y_t, Z_t) of the system (3.2). Then

$$\begin{aligned} \frac{dV}{dt} =& -\langle(1-\beta)BX, H(X(t-r))\rangle - \langle\alpha BY, Y\rangle - \langle\beta AY, BY\rangle \\ &- \langle(I+2\alpha A^{-1})Z, H(X(t-r))\rangle - \langle(\alpha I + A)Y, H(X(t-r))\rangle \\ &- \langle\alpha Z, Z\rangle + \langle\lambda rY, Y\rangle - \lambda\int_{t-r}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta \\ &+ \langle(1-\beta)BX + (\alpha I + A)Y + (I+2\alpha A^{-1})Z, P(t, X, Y, Z)\rangle. \end{aligned}$$

Upon using (2.2), we obtain

$$\begin{aligned} \frac{dV}{dt} =& -\langle(1-\beta)BX, C(U,0))X\rangle - \langle\alpha BY, Y\rangle - \langle\beta AY, BY\rangle \\ &- \langle\alpha Z, Z\rangle - \langle(I+2\alpha A^{-1})Z, C(U,0)X\rangle - \langle(\alpha I + A)Y, C(U,0)X\rangle \\ &+ \int_{t-r}^{t}\langle(1-\beta)BX(s) + (\alpha I + A)Y(s) \\ &+ (I+2\alpha A^{-1})Z(s), C(U,0)Y(s)\rangle ds \\ &+ \langle\lambda rY, Y\rangle - \lambda\int_{t-r}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta \\ &+ \langle(1-\beta)BX + (\alpha I + A)Y + (I+2\alpha A^{-1})Z, P(t, X, Y, Z)\rangle \\ =& -U_1 - U_2 - U_3 + U_4 + U_5, \end{aligned}$$

where

$$U_1 = \frac{1}{2}\langle X, (1-\beta)BC(U,0)X\rangle + \langle Y, \beta ABY\rangle + \frac{1}{2}\langle \alpha Z, Z\rangle,$$
$$U_2 = \frac{1}{4}\langle X, (1-\beta)BC(U,0)X\rangle + \langle (\alpha I + A)Y, C(U,0)X\rangle + \langle \alpha BY, Y\rangle,$$
$$U_3 = \frac{1}{4}\langle X, (1-\beta)BC(U,0)X\rangle + \langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle + \frac{1}{2}\langle \alpha Z, Z\rangle,$$
$$U_4 = \int_{t-r}^{t}\langle (1-\beta)BX(s) + (\alpha I + A)Y(s) + (I+2\alpha A^{-1})Z(s), C(U,0)Y(s)\rangle ds + \langle \lambda rY, Y\rangle - \lambda\int_{t-r}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta$$

and

$$U_5 = \langle (1-\beta)BX + (\alpha I + A)Y + (I + 2\alpha A^{-1})Z, P(t,X,Y,Z)\rangle.$$

From (1.7), (1.8) and (3.3), we have

$$U_1 \geq \frac{1-\beta}{2}\delta_b\delta_h\|X\|^2 + \beta\delta_a\delta_b\|Y\|^2 + \frac{\alpha}{2}\|Z\|^2 \geq \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{3.6}$$

where $\delta_1 = \min\left\{\frac{(1-\beta)}{2}\delta_b\delta_h, \beta\delta_a\delta_b, \frac{\alpha}{2}\right\}$.

Next, we give estimates for $\langle (\alpha I + A)Y, C(U,0)X\rangle$ and $\langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle$. For some $k_1 > 0$, $k_2 > 0$, conveniently chosen later, we obtain

$$\langle (\alpha I + A)Y, C(U,0)X\rangle = \left\|k_1(\alpha I + A)^{\frac{1}{2}}Y + \frac{1}{2}k_1^{-1}(\alpha I + A)^{\frac{1}{2}}C(U,0)X\right\|^2 - \langle k_1^2(\alpha I + A)Y, Y\rangle - \frac{1}{4}k_1^{-2}\langle (\alpha I + A)C(U,0)X, C(U,0)X\rangle$$

and

$$\langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle = \left\|k_2(I + 2\alpha A^{-1})^{\frac{1}{2}}Z + \frac{1}{2}k_2^{-1}(I + 2\alpha A^{-1})^{\frac{1}{2}}C(U,0)X\right\|^2 - \langle k_2^2(I + 2\alpha A^{-1})Z, Z\rangle - \frac{1}{4}k_2^{-2}\langle (I + 2\alpha A^{-1})C(U,0)X, C(U,0)X\rangle,$$

thus

$$U_2 = \left\|k_1(\alpha I + A)^{\frac{1}{2}}Y + \frac{1}{2}k_1^{-1}(\alpha I + A)^{\frac{1}{2}}C(U,0)X\right\|^2 + \left\langle \left\{\alpha B - k_1^2(\alpha I + A)\right\}Y, Y\right\rangle + \left\langle \frac{1}{4}\left\{(1-\beta)B - k_1^{-2}(\alpha I + A)C(U,0)\right\}C(U,0)X, X\right\rangle$$

and

$$U_3 = \left\| k_2(I + 2\alpha A^{-1})^{\frac{1}{2}} Z + \frac{1}{2} k_2^{-1}(I + 2\alpha A^{-1})^{\frac{1}{2}} C(U,0)X \right\|^2 + \left\langle \left\{ \alpha I - k_2^2(I + 2\alpha A^{-1}) \right\} Z, Z \right\rangle + \left\langle \frac{1}{4} \left\{ (1-\beta)B - k_2^{-2}(I + 2\alpha A^{-1})C(U,0) \right\} C(U,0)X, X \right\rangle.$$

By Lemma 2.1 and Lemma 2.2, we have

$$U_2 \geq \left\{ \alpha\delta_b - k_1^2(\alpha + \Delta_a) \right\} \|Y\|^2 + \frac{1}{4}\delta_h \left\{ (1-\beta)\delta_b - \frac{1}{k_1^2}(\alpha + \Delta_a)\Delta_h \right\} \|X\|^2 \geq 0, \tag{3.7}$$

provided

$$\frac{(\alpha + \Delta_a)\Delta_h}{(1-\beta)\delta_b} \leq k_1^2 \leq \frac{\alpha\delta_b}{\alpha + \Delta_a}$$

and

$$\Delta_h \leq \frac{\alpha\delta_b^2(1-\beta)}{(\alpha + \Delta_a)^2}. \tag{3.8}$$

In a similar manner,

$$U_3 \geq 0, \tag{3.9}$$

provided

$$\frac{(2\alpha + \delta_a)\Delta_h}{(1-\beta)\delta_a\delta_b} \leq k_2^2 \leq \frac{\alpha\delta_a}{2\alpha + \delta_a}$$

and

$$\Delta_h \leq \frac{\alpha\delta_b\delta_a^2(1-\beta)}{(2\alpha + \delta_a)^2}. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\Delta_h \leq k\delta_a\delta_b,$$

where

$$k = \min\left\{ \frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha + \Delta_a)^2}, \frac{\alpha(1-\beta)\delta_a}{(\delta_a + 2\alpha)^2} \right\} < 1.$$

For U_4, using the identity $2|\langle u, v\rangle| \leq \|u\|^2 + \|v\|^2$, we obtain

$$\begin{aligned} |U_4| \leq & \frac{1}{2}(1-\beta)\Delta_b\Delta_h r\|X\|^2 + \frac{1}{2}(\alpha + \Delta_a)\Delta_h r\|Y\|^2 \\ & + \frac{1}{2}(1 + 2\alpha\delta_a^{-1})\Delta_h r\|Z\|^2 + \left\{ \frac{1}{2}(1-\beta)\Delta_b\Delta_h \right. \\ & \left. + \frac{1}{2}(\alpha + \Delta_a)\Delta_h + \frac{1}{2}(1 + 2\alpha\delta_a^{-1})\Delta_h \right\} \int_{t-r}^{t} \langle Y(s), Y(s)\rangle ds \\ & + \langle \lambda r Y, Y\rangle - \lambda \int_{t-r}^{t} \langle Y(\theta), Y(\theta)\rangle d\theta. \end{aligned} \tag{3.11}$$

If we choose

$$\lambda = \frac{1}{2}\Delta_h\left[(1-\beta)\Delta_b + (\alpha+\Delta_a) + (1+2\alpha\delta_a^{-1})\right]$$

in (3.11), we obtain

$$|U_4| \le \frac{1}{2}(1-\beta)\Delta_b\Delta_h r\|X\|^2 + \frac{1}{2}(1+2\alpha\delta_a^{-1})\Delta_h r\|Z\|^2 + \frac{1}{2}\Delta_h r\left[1+(1-\beta)\Delta_b + 2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]\|Y\|^2. \tag{3.12}$$

Finally, we are left with U_5. Since $P(t,X,Y,Z)$ satisfies (3.4), by Schwarz's inequality we obtain

$$\begin{aligned} |U_5| &\le \left[(1-\beta)\Delta_b\|X\| + (\alpha+\Delta_a)\|Y\| + (1+2\alpha\delta_a^{-1})\|Z\|\right]\|P(t,X,Y,Z)\| \\ &\le \delta_2(\|X\|+\|Y\|+\|Z\|)\left[\Delta_0 + \Delta_1(\|X\|+\|Y\|+\|Z\|)\right], \end{aligned} \tag{3.13}$$

where $\delta_2 = \max\left\{(1-\beta)\Delta_b, (\alpha+\Delta_a), (1+2\alpha\delta_a^{-1})\right\}$.

Combining inequalities (3.6), (3.7), (3.9), (3.12) and (3.13), we obtain

$$\begin{aligned} \frac{dV}{dt} \le &-\frac{1}{2}(1-\beta)[\delta_b\delta_h - r\Delta_b\Delta_h]\|X\|^2 \\ &-\left(\beta\delta_a\delta_b - \frac{1}{2}\Delta_h r\left[1+(1-\beta)\Delta_b+2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]\right)\|Y\|^2 \\ &-\frac{1}{2}\left[\alpha - \Delta_h r(1+2\alpha\delta_a^{-1})\right]\|Z\|^2 \\ &+\delta_2(\|X\|+\|Y\|+\|Z\|)\left[\Delta_0+\Delta_1(\|X\|+\|Y\|+\|Z\|)\right]. \end{aligned}$$

Now if we choose

$$r < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h}, \frac{2\beta\delta_a\delta_b}{\Delta_h\left[1+(1-\beta)\Delta_b+2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]}, \frac{\alpha}{\Delta_h(1+2\alpha\delta_a^{-1})}\right\},$$

we get

$$\begin{aligned} \frac{dV}{dt} \le &-\gamma(\|X\|^2+\|Y\|^2+\|Z\|^2) + 3\delta_2\Delta_1(\|X\|^2+\|Y\|^2+\|Z\|^2) \\ &+\delta_2\Delta_0(\|X\|+\|Y\|+\|Z\|) \\ = &-(\gamma-3\delta_2\Delta_1)(\|X\|^2+\|Y\|^2+\|Z\|^2) + \delta_2\Delta_0(\|X\|+\|Y\|+\|Z\|). \end{aligned}$$

If we choose $\Delta_1 < \frac{\gamma}{3\delta_2}$, then there is some $\theta > 0$, such that

$$\begin{aligned} \frac{d}{dt}V(X_t,Y_t,Z_t) \le &-\theta(\|X\|^2+\|Y\|^2+\|Z\|^2) + n\theta(\|X\|+\|Y\|+\|Z\|) \\ = &-\frac{\theta}{2}(\|X\|^2+\|Y\|^2+\|Z\|^2) \\ &-\frac{\theta}{2}\left\{(\|X\|-n)^2+(\|Y\|-n)^2+(\|Z\|-n)^2\right\} + \frac{3\theta}{2}n^2 \\ \le &-\frac{\theta}{2}\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right) + \frac{3\theta}{2}n^2, \end{aligned}$$

for some $n, \theta > 0$.

This completes the proof. □

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LOGARITHMICALLY COMPLETE MONOTONICITY OF RECIPROCAL ARCTAN FUNCTION

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Abstract. We prove the conjecture stated in F. Qi and R. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. (2019), that the function $1/\arctan$ is logarithmically completely monotonic on $(0,\infty)$, but not a Stieltjes transform.

1. Introduction

By a *completely monotonic function* (shortly CM) we mean here an infinitely differentiable function $f:(0,\infty)\to\mathbb{R}$, such that

$$(-1)^n f^{(n)} \geq 0, \quad n = 0, 1, 2, \ldots$$

If f' is completely monotonic and $f \geq 0$, then we call f a *Bernstein function*. Here we are mostly interested in *logarithmically completely monotonic functions*, that is, infinitely differentiable functions $f:(0,\infty)\to(0,\infty)$ with the property

$$(-1)^n (\log f)^{(n)} \geq 0, \quad n = 1, 2, 3, \ldots$$

A basic fact concerning CM - functions is the Bernstein theorem: a function f is CM if and only if there exists a non-decreasing function α on $(0,\infty)$ satisfying

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

for all $x > 0$ (see [9, p. 161]). In some occasions it has been proven a stronger property which leads to complete monotonicity of a function f, namely that there exist $a \geq 0$

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and a non-negative Borel measure μ on $[0,\infty)$ for which the equality

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t}$$

holds for $x > 0$, where the measure μ fulfills the condition

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

Such functions are called *Stieltjes transforms*. We recall that all Stieltjes transforms are logarithmically completely monotonic (see [2] and further generalizations [3]), and the latter are CM (see [5], but also [7] and [8]).

In [6] the authors set the conjecture that the function $f(x) = \frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0,\infty)$, but not a Stieltjes transform. The aim of this paper is to justify these assertions. We will do it in the next section.

2. Formulations and Proofs

Theorem 2.1. *The function $f(x) = \frac{1}{\arctan x}$ is logarithmically completely monotonic on $(0,\infty)$.*

The idea of the proof of Theorem 2.1 is based on the Remark 1 in [1], where the authors suggest employing the residue theorem in an attempt to obtain integral representations of functions under consideration.

Proof. It suffices to prove that

$$g(x) = -(\log f(x))' = \frac{1}{(x^2+1)\arctan x}$$

is CM on $(0,\infty)$. In what follows we always assume that log denotes the principle value of logarithm, i.e., $\log z = \ln|z| + i\arg z$, with $\arg z \in (-\pi,\pi]$.

Let us consider the integral $\int_{\Gamma_{R,r}} G(z)\,dz$, over the "keyhole" contour $\Gamma_{R,r}$ given in Figure 1, where

$$G(z) = \frac{z+1}{z(z-z_0)\log z}$$

and $z_0 = \frac{i-x}{i+x}$ for $x > 0$.

We assume $R > 1$ and $r < 1$. Note that $|z_0| = 1$ and that 1, z_0 are the only singularities of G lying inside $\Gamma_{R,r}$. From the residue theorem, we have

$$\int_{\Gamma_{R,r}} G(z)\,dz = 2\pi i(\mathrm{Res}(G(z);z_0) + \mathrm{Res}(G(z);1)).$$

Since z_0 is a first-order pole, it follows

$$\mathrm{Res}(G(z);z_0) = \frac{1+z_0}{z_0\log z_0} = \frac{1+\frac{i-x}{i+x}}{\frac{i-x}{i+x}\log\frac{i-x}{i+x}} = \frac{2i}{(i-x)2i\arctan x} = -\frac{(i+x)}{(x^2+1)\arctan x},$$

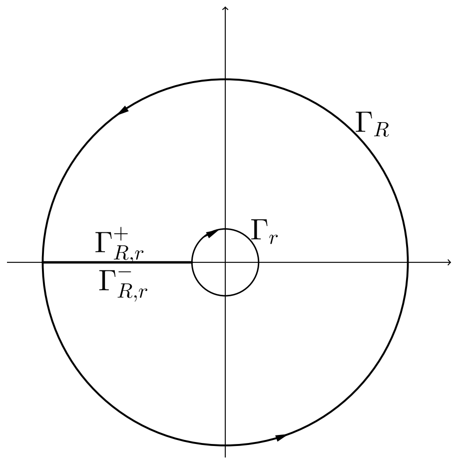

FIGURE 1. Keyhole contour $\Gamma_{R,r}$

where we used the fact that $\arctan x = \frac{1}{2i}\log\frac{1+ix}{1-ix}$, for $x>0$. Similarly,

$$\operatorname{Res}(G(z);1) = \lim_{z\to 1}(z-1)\frac{1+z}{z(z-z_0)\log z} = \frac{2}{1-z_0} = \frac{2}{1-\frac{i-x}{i+x}} = \frac{i+x}{x},$$

whence,

$$g(x) = \frac{1}{x} - \frac{1}{2\pi i(x+i)}\int_{\Gamma_{R,r}} G(z)\,dz. \tag{2.1}$$

Now, it remains to calculate the integral $\int_{\Gamma_{R,r}} G(z)\,dz$. In order to accomplish it, we start from the relation

$$\int_{\Gamma_{R,r}} G(z)\,dz = \int_{\Gamma_R} G(z)\,dz + \int_{\Gamma_r} G(z)\,dz + \int_{\Gamma^+_{R,r}} G(z)\,dz + \int_{\Gamma^-_{R,r}} G(z)\,dz. \tag{2.2}$$

The first two integrals vanish as $R\to\infty$ and $r\to 0+$. It follows from the estimates

$$\left|\int_{\Gamma_R} G(z)\,dz\right| \le 2R\pi \max_{|z|=R}\frac{|z+1|}{|z||\log z||z-z_0|} \le 2\pi\frac{R+1}{(\ln R-2\pi)(R-1)}$$

and

$$\left|\int_{\Gamma_r} G(z)\,dz\right| \le 2r\pi \max_{|z|=r}\frac{|z+1|}{|z||\log z||z-z_0|} \le 2\pi\frac{1+r}{(-\ln r-2\pi)(1-r)}.$$

We also have for $t<0$

$$\lim_{\substack{z\to t\\ \Im z>0}} G(z) = \frac{t+1}{t(\ln(-t)+\pi i)(t-z_0)} = G^+(t)$$

and

$$\lim_{\substack{z\to t\\ \Im z<0}} G(z) = \frac{t+1}{t(\ln(-t)-\pi i)(t-z_0)} = G^-(t).$$

Consequently,

$$\int_{\Gamma^+_{R,r}} G(z)\,dz + \int_{\Gamma^-_{R,r}} G(z)\,dz = \int_{-R}^{-r} [G^+(t)-G^-(t)]\,dt. \tag{2.3}$$

Let us denote $I = \lim\limits_{\substack{R\to\infty\\ r\to 0+}} \int_{\Gamma_{R,r}} G(z)\,dz$. From (2.2) and (2.3) we obtain

$$\begin{aligned} I &= \int_{-\infty}^{0} [G^+(t)-G^-(t)]\,dt \\ &= \int_{-\infty}^{0} \frac{2\pi i(t+1)\,dt}{t(\log^2(-t)+\pi^2)(t-z_0)} \\ &= 2\pi i \int_0^\infty \frac{(1-t)\,dt}{t(\log^2 t+\pi^2)(t+z_0)}. \end{aligned}$$

Using $z_0 = \frac{i-x}{i+x}$, we have

$$\begin{aligned} I &= \int_0^\infty \frac{2\pi i(1-t)\,dt}{t(\log^2 t+\pi^2)(t+\frac{i-x}{i+x})} \\ &= \int_0^\infty \frac{2\pi i(i+x)(1-t)\,dt}{t(\log^2 t+\pi^2)(x(t-1)+i(t+1))} \\ &= -2\pi i(i+x)\int_0^\infty \frac{((1-t)^2x+i(1-t^2))\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \end{aligned}$$

Note that (2.1) implies

$$g(x) = \frac{1}{x} - \frac{1}{2\pi i(x+i)}\, I \tag{2.4}$$

and since $\dfrac{1}{2\pi i(x+i)} I$ is real, we conclude that

$$\int_0^\infty \frac{(1-t^2)\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)} = 0.$$

Therefore, from (2.4), it follows

$$g(x) = \frac{1}{x} + \int_0^\infty \frac{(1-t)^2x\,dt}{t(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \tag{2.5}$$

Employing

$$\frac{1}{x} = \int_0^\infty \frac{dt}{xt(\log^2 t+\pi^2)},$$

we get

$$g(x) = \int_0^\infty \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}.$$

The substitution $t \mapsto \frac{1}{t}$ implies

$$\int_0^1 \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)} = \int_1^\infty \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}.$$

Hence,

$$g(x) = 2\int_0^1 \frac{(2(1-t)^2x+(1+t)^2)\,dt}{xt(x^2(1-t)^2+(1+t)^2)(\log^2 t+\pi^2)}. \tag{2.6}$$

For $a, b, x > 0$ it is

$$\frac{2a^2x^2+b^2}{x(a^2x^2+b^2)} = \frac{1}{x} + \frac{1}{2}\left(\frac{1}{x+\frac{bi}{a}} + \frac{1}{x-\frac{bi}{a}}\right)$$

and using

$$\frac{1}{x} = \int_0^\infty e^{-xs}\,ds, \qquad \frac{1}{x+\frac{bi}{a}} = \int_0^\infty e^{-xs}e^{-\frac{bi}{a}s}\,ds, \qquad \frac{1}{x-\frac{bi}{a}} = \int_0^\infty e^{-xs}e^{\frac{bi}{a}s}\,ds,$$

one obtains

$$\frac{2a^2x^2+b^2}{x(a^2x^2+b^2)} = \int_0^\infty e^{-xs}\left(1+\cos\frac{bs}{a}\right)ds.$$

Setting $a = 1-t$ and $b = 1+t$ yields

$$\frac{2(1-t)^2x+(1+t)^2}{x(x^2(1-t)^2+(1+t)^2)} = \int_0^\infty e^{-xs}\left(1+\cos\frac{1+t}{1-t}s\right)ds.$$

From (2.6), we have

$$g(x) = 2\int_0^1 \left(\int_0^\infty \frac{e^{-xs}(1+\cos\frac{1+t}{1-t}s)\,ds}{t(\ln^2 t+\pi^2)}\right)dt,$$

and, finally, after interchanging integration order, we obtain

$$g(x) = \int_0^\infty \left(\int_0^1 \frac{2(1+\cos\frac{1+t}{1-t}s)\,dt}{t(\ln^2 t+\pi^2)}\right)e^{-xs}\,ds. \tag{2.7}$$

Now, it is evident that (2.7) implies complete monotonicity of g. ☐

Theorem 2.2. *The function $f(x) = \frac{1}{\arctan x}$ is not a Stieltjes transform on $(0,\infty)$.*

For the proof of this theorem, we the use following result on Stieltjes transforms from [4].

Proposition 2.1. *If $f \neq 0$ is a Stieltjes transform, then $\frac{1}{f}$ is a Bernstein function.*

Proof of Theorem 2.2. The function $h(x) = \frac{1}{f(x)} = \arctan x$ is not a Bernstein function, since

$$h^{(3)}(x) = -2\,\frac{3x^2-1}{(1+x^2)^3}$$

changes its sign on $(0,\infty)$. Therefore, according to Proposition 2.1, f is not a Stieltjes transform. ☐

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APPROXIMATION BY MODIFIED SZÁSZ OPERATORS WITH A NEW MODIFICATION OF BRENKE TYPE POLYNOMIALS

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Abstract. In the present article we study the approximation properties of modified Szász operators with a new modification of Brenke type polynomials. First, we estimate the rate of convergence, for the newly defined operators, by means of modulus of smoothness, Peetre's K-functional and Lipschitz type functions. Furthermore, we also prove a Voronovskaja type asymptotic theorem.

1. Introduction and Preliminaries

In 1950, Szász [18] extended the theory of well known Bernstein operators for the finite interval $[0,1]$ to infinite interval $\mathbb{R}_0^+ := [0,\infty)$ and established the convergence properties in the infinite interval $\mathbb{R}_0^+$ by defining the operators for $f \in C(\mathbb{R}_0^+)$ as

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}. \tag{1.1}$$

A generalization of (1.1) was established by Jakimovski-Leviatan in [12] with the help of the Appell polynomials as

$$P_n(f;x) := \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}, \tag{1.2}$$

where $A(x) = \sum_{n=0}^{\infty} b_n x^n$, $b_n \in \mathbb{R}$, is an analytic function on the disk $|x| < R$, $R > 1$, with $A(1) \neq 0$. The polynomials $p_k(x) = \sum_{i=0}^{k} b_i \frac{x^{k-i}}{(k-i)!}$, $k \in \mathbb{N}$, are the Appell polynomials which are generated by $A(z)e^{zx} = \sum_{k=0}^{\infty} p_k(x) z^k$ under the assumption

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that $p_k(x) \geq 0$ for all $x \in [0,\infty)$. In particular, if $A(z) = 1$, then $p_k(x) = \frac{x^k}{k!}$, and the operators (1.2) reduce to the operators (1.1).

Ismail [11] defined another generalization of (1.1) and (1.2) with the help of Sheffer type polynomials $\{u_k(x)\}_{k\geq 1}$, which are generated by

$$A(s)e^{tB(s)} = \sum_{k=0}^{\infty} u_k(t)s^k, \quad |s| < R,$$

where $A(s) = \sum_{k=0}^{\infty} a_k s^k$, $a_0 \neq 0$ and $B(s) = \sum_{k=1}^{\infty} b_k s^k$, $b_1 \neq 0$, are analytic functions on the disc $|s| < R,\ R > 1$, and a_k and b_k are the real coefficients. Under the following assumptions:

(i) for $t \in \mathbb{R}_0^+$, $u_k(t) \geq 0$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
(ii) $A(1) \neq 0$ and $B^{(1)}(1) = 1$,

Ismail introduced and studied some important approximation properties of the following operators

$$Q_n(f;x) = \frac{e^{-nxB(1)}}{A(1)} \sum_{k=0}^{\infty} u_k(nx) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}. \tag{1.3}$$

In particular, when $A(t) = t$ and $B(t) = 1$, the operator (1.3) reduces to the Szász operator (1.1) and for the case $B(t) = t$, the operator $Q_n(f;x)$ yields the operator $P_n(f;x)$ defined in (1.2).

Let $v_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r$, $k \in \mathbb{N} \cup \{0\}$, be the Brenke type polynomials on the disk $|x| < R, (R > 1)$ which are generated by

$$A(s)B(xs) = \sum_{k=0}^{\infty} v_k(x)s^k, \tag{1.4}$$

where $A(s) = \sum_{k=0}^{\infty} a_k s^k, a_0 \neq 0$, and $B(s) = \sum_{k=0}^{\infty} b_k s^k$, $b_k \neq 0$, are analytic functions on the disk $|s| < R$, $R > 1$.

Under the following assumptions:

(i) $A(1) \neq 0$, $\frac{a_{r-k} b_r}{A(1)} \geq 0$, $0 \leq r \leq k$, $k \in \mathbb{N} \cup \{0\}$;
(ii) $B : \mathbb{R}_0^+ \to (0,\infty)$;
(iii) (1.4) and the power series $A(t)$ and $B(t)$ converge for $|t| < R$, $R > 1$.

Varma et al. [20] presented a generalization of Szász operators by means of the Brenke type polynomials as

$$R_n(f;x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} v_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0, n \in \mathbb{N}. \tag{1.5}$$

In particular, if $B(t) = e^t$, the operator (1.5) reduces to the operator (1.2) and if $B(t) = e^t$ and $A(t) = 1$ the operator (1.5) reproduces the Szász operator (1.1).

Cheikh and Romdhane [6] defined the d-symmetric d-orthogonal polynomials of Brenke type as

$$\mathcal{A}(t^{\rho+1})\mathcal{B}(xt)=\sum_{k=0}^{\infty}q_k(x)t^k, \tag{1.6}$$

where $\mathcal{A}(t)=\sum_{k=0}^{\infty}a_kt^k$, $\mathcal{B}(t)=\sum_{k=0}^{\infty}b_kt^k$ with $a_0b_k\neq 0$ for all $k\in\mathbb{N}$, are analytic functions on the disk $|t|<R$, $R>1$, and ρ is a positive integer. In particular case, $\mathcal{A}(t)=\exp(x)$ and $\mathcal{B}(t)=\exp(x)$, the polynomials (1.6) reduce to the Gould-Hopper polynomials [10] and also when $\rho=0$, (1.6) reduces to (1.4).

Motivated by the work above, we present a new modification of Szász operators with the generalized form of Brenke type polynomials $q_k(x)$ as

$$\mathcal{D}_n(f;x):=\frac{1}{\mathcal{A}(1)\mathcal{B}(nx)}\sum_{k=0}^{\infty}q_k(nx)f\left(\frac{k}{n}\right),\quad x\geq 0, n\in\mathbb{N}, \tag{1.7}$$

where $q_k(x)$ is defined in (1.6). The purpose of this article is to establish some approximation properties for the operator (1.7), under the following certain conditions

(i) $\mathcal{A}(1)\neq 0$, $\frac{a_{k-m}b_k}{\mathcal{A}(1)}\geq 0$, $0\leq k\leq m$, $m\in\mathbb{N}_0$;
(ii) $\mathcal{B}:\mathbb{R}_0^+\to(0,\infty)$;
(iii) (1.6) and the power series for $\mathcal{A}(t)$ and $\mathcal{B}(t)$ converge for $|t|<R$, $R>1$.

In particular, the operator $\mathcal{D}_n(f;x)$ have the following reductions

(i) if $\rho=0$, the operator (1.7) reduces to the operator (1.5);
(ii) if $\rho=0$, and $\mathcal{B}(t)=e^t$, the operator (1.7) reduces to the operator (1.2);
(iii) if $\rho=0$, $\mathcal{A}(t)=e^t$ and $\mathcal{B}(t)=1$, the operator (1.7) reproduces the Szász operator (1.1).

For some other recent papers on the topic dealing with the generalization of Szász type operators using different classes of polynomials, see [1–3, 5, 7, 8, 13–15, 17, 19, 21] and the references cited therein.

The rest of the paper is organised as follows. In Section 2, we present some auxiliary results. In Section 3, we estimate the rate of convergence with the help of classical and second-order modulus of smoothness and Peetre's K-functional and also give the order of approximation for the Lipschitz type space. Lastly, we discuss a quantitative Voronovskaja-type theorem.

2. Auxiliary Results

In this section, we present some important auxiliary results which will be used in this later work.

Lemma 2.1. *From the generating function* (1.6) *of the Brenke type polynomials, we have the following equalities:*

$$\sum_{k=0}^{\infty}q_k(nx)=\mathcal{A}(1)\mathcal{B}(nx),$$

$$\begin{aligned}
\sum_{k=0}^{\infty} kq_k(nx) =& (\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx) + nx\mathcal{B}^{(1)}(nx)\mathcal{A}(1),\\
\sum_{k=0}^{\infty} k^2q_k(nx) =& (\rho+1)^2(\mathcal{A}^{(2)}(1)+\mathcal{A}^{(1)}(1))\mathcal{B}(nx) + 2n(\rho+1)x\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx)\\
&+ n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx) + nx\mathcal{A}(1)\mathcal{B}^{(1)}(nx),\\
\sum_{k=0}^{\infty} k^3q_k(nx) =& (\rho+1)^3(\mathcal{A}^{(3)}(1)+3\mathcal{A}^{(2)}(1))\mathcal{B}(nx) + (\rho^2+1)(\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx)\\
&+ 3n(\rho+1)^2x\mathcal{A}^{(2)}(1)\mathcal{B}^{(1)}(nx) + 3n(\rho+1)(\rho+2)x\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx)\\
&+ 3(\rho+1)n^2x^2\mathcal{A}^{(1)}(1)\mathcal{B}^{(2)}(nx) + n^3x^3\mathcal{A}(1)\mathcal{B}^{(3)}(nx)\\
&+ 3n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx) + nx\mathcal{A}^{(1)}(1)\mathcal{B}(nx),\\
\sum_{k=0}^{\infty} k^4q_k(nx) =& (\rho+1)^4(\mathcal{A}^{(4)}(1)+6\mathcal{A}^{(3)}(1)+7\mathcal{A}^{(2)}(1))\mathcal{B}(nx)\\
&+ (\rho+1)(\rho^3+3\rho^2-9\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx)\\
&+ 4nx(\rho+1)^3\mathcal{A}^{(3)}(1)\mathcal{B}^{(1)}(nx)\\
&+ 9nx(\rho+2)(\rho+1)^2\mathcal{A}^{(2)}(1)\mathcal{B}^{(1)}(nx) + 6n^2x^2(\rho+1)^2\mathcal{A}^{(2)}(1)\mathcal{B}^{(2)}(nx)\\
&+ 3(\rho+7)(\rho+1)n^2x^2\mathcal{A}^{(1)}(1)\mathcal{B}^{(2)}(nx) + 4(\rho+1)n^3x^3\mathcal{A}^{(1)}(1)\mathcal{B}^{(3)}(nx)\\
&+ n^4x^4\mathcal{A}(1)\mathcal{B}^{(4)}(nx) + 6n^3x^3\mathcal{A}(1)\mathcal{B}^{(3)}(nx)\\
&+ 7n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx) + nx\mathcal{A}(1)\mathcal{B}^{(1)}(nx)\\
&+ (\rho+1)(\rho^2+20\rho+11)\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx),
\end{aligned}$$

where $\mathcal{A}^{(r)}(x) = \frac{d^r\mathcal{A}(x)}{dx^r}$ *and* $\mathcal{B}^{(r)}(x) = \frac{d^r\mathcal{B}(x)}{dx^r}$ *for all* $r \in \mathbb{N}$.

Proof. Differentiating (1.6) with respect to t, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} kq_k(x)t^{k-1} =& (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt),\\
\sum_{k=0}^{\infty} k^2q_k(x)t^{k-2} =& (\rho+1)^2t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt) + \rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})(2x\mathcal{B}^{(1)}(xt) + \mathcal{B}(xt)),\\
\sum_{k=0}^{\infty} k^3q_k(x)t^{k-3} =& (\rho+1)^3t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt) + (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})(3x^2\mathcal{B}^{(2)}(xt)\\
&+ \mathcal{B}(xt)) + 3\rho(\rho+1)^2t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ (\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})(3\rho\mathcal{B}(xt) + 3x(\rho+2)\mathcal{B}^{(1)}(xt))\\
&+ x^3\mathcal{A}(t^{\rho+1})\mathcal{B}^{(3)}(xt) + 3x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)
\end{aligned}$$

$$
\begin{aligned}
&+ (\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})(3x\mathcal{B}^{(1)}(xt)+3\mathcal{B}(xt))\\
&+ \rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt),\\
\sum_{k=0}^{\infty} k^4 q_k(x)t^{k-4} =& 4x^3(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(3)}(xt) + 6x^2(\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 6x^2(\rho+1)\rho t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 4x(\rho+1)^3 t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 6x(\rho+1)^2\rho t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 6x\rho(\rho+1)^2 t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 3x\rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^4 t^{4d}\mathcal{A}^{(4)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 6\rho(\rho+1)^3 t^{3d-1}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 3\rho(\rho+1)^2(2\rho-1)t^{2\rho-2}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x\rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ \rho(\rho^2-1)(\rho+1)t^{2\rho-2}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ \rho(\rho^2-1)(d-2)t^{d-3}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x^4\mathcal{A}(t^{\rho+1})\mathcal{B}^{(4)}(xt)\\
&+ 6\Bigg\{3x^2(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 3x(\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 3x\rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^3 t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 3\rho(\rho+1)^2 t^{2\rho-1}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt) + \rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x^3\mathcal{A}(t^{\rho+1})\mathcal{B}^{(3)}(xt) + 3\Bigg[2x(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ (\rho+1)^2 t^{2\rho}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt) + \rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt) + (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Bigg]\\
&- 2\Bigg[(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Bigg]\Bigg\}\\
&+ 6\Bigg\{(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Bigg\}\\
&- 11\Bigg\{2x(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^2 t^{2\rho}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ \rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt)
\end{aligned}
$$

$$+ (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Big\}.$$

The desired lemma is obtained by substituting $t = 1$ and $x = nx$, in the above computations. □

Lemma 2.2. *For $x \in \mathbb{R}_0^+$, the r^{th} order moments $\mathcal{D}_n(t^r;x)$, $r = 0,1,2,3,4$, of the operators $\mathcal{D}_n$ are defined as:*

$$\begin{aligned}
\mathcal{D}_n(1;x) &= 1,\\
\mathcal{D}_n(t;x) &= xb_1 + pa_1,\\
\mathcal{D}_n(t^2;x) &= x^2b_2 + \frac{x}{n}b_1\Big(1+2npa_1\Big) + p^2\Big(a_2+a_1\Big),\\
\mathcal{D}_n(t^3;x) &= 3px^2a_1b_2 + 3xp^2a_2b_1 + 3xp\Big(p+\frac{1}{n}\Big)a_1b_1 + p^3a_3 + 3p^3a_2\\
&\quad + p\Big(p^2 - \frac{2}{n} + \frac{2}{n^2}\Big)a_1 + x^3b_3 + \frac{3x^2}{n}b_2 + \frac{x}{n^2}a_1,\\
\mathcal{D}_n(t^4;x) &= x^4b_4 + \frac{x^3}{n}b_3\Big(6+4npa_1\Big) + \frac{x^2}{n^2}b_2\Big(7+6n^2p^2a_2+3np(np+6)a_1\Big)\\
&\quad + \frac{x}{n^3}b_1\Big(1+4n^3p^3a_3+9n^2p^2(np+1)a_2\Big) + \frac{1}{n^3}\Big(n^2p^3 - 18np^2 - 8p\Big)a_1b_1\\
&\quad + p^4a_4 + 6p^4a_3 + 7p^4a_2 + \Big(p^4 - \frac{12}{n^2}p^2 + \frac{12}{n^3}p\Big)a_1,
\end{aligned}$$

where $p = \frac{\rho+1}{n}$, $a_r = \frac{\mathcal{A}^{(r)}(1)}{\mathcal{A}(1)}$ and $b_r = \frac{\mathcal{B}^{(r)}(nx)}{\mathcal{B}(nx)}$, $r \in \mathbb{N}$. These notations will be used throughout the paper.

Proof. Using Lemma 2.1 and (1.7), the proof of this lemma can be easily obtained. Hence the details are omitted. □

As a consequence of Lemma 2.2, we have the following result.

Lemma 2.3. *For $x \in \mathbb{R}_0^+$, the central moments $\mathcal{D}_n((t-x)^m;x)$, $m = 1,2,4$, are defined by*

$$\begin{aligned}
\mathcal{D}_n(t-x;x) =& x(b_1-1) + pa_1,\\
\mathcal{D}_n((t-x)^2;x) =& x^2\Big(b_2-2b_1+1\Big) + 2xpa_1\Big(b_1-1\Big) + \frac{x}{n}b_1 + p^2(a_2+a_1),\\
\mathcal{D}_n((t-x)^4;x) =& x^4\Big(1-4b_1+6b_2-4b_3+b_4\Big) - x^3\Big(-4pa_1 - \frac{12}{n}b_2 + 4pa_1b_3 + \frac{6}{n}b_1\\
&+ 12pa_1b_1 + \frac{6}{n}b_3 - 12pa_1b_2\Big) + x^2\Big(6p^2a_1 - \frac{4}{n^2}a_1 + 6p^2a_2\\
&- 12p^2a_2b_1 - 12p\Big(p+\frac{1}{n}\Big)a_1b_1 + 6p^2a_2b_2 + 3p\Big(p+\frac{6}{n}\Big)a_1b_2 + \frac{7}{n^2}b_2\Big)\\
&- x\Big(4p^3a_3 - 12p^3a_2 - 4\Big(p^3 - \frac{2}{n}p^2 - \frac{2}{n^2}p\Big)a_1 + 4p^3a_3b_1
\end{aligned}$$

$$+ 9\Big(p^3 + \frac{p^2}{n}\Big)a_2 b_1 + \frac{1}{n^3}b_1\Big) + \Big(\frac{p^3}{n} + \frac{16}{n^2}p^2 - \frac{6}{n^3}p\Big)a_1 b_1 + p^4 a_4$$
$$+ 6p^4 a_3 + 7p^4 a_2 + \Big(p^4 - 12\frac{p^2}{n^2} + 12\frac{p}{n^3}\Big)a_1.$$

For the remainder of the work we denote $\xi_n^{\rho}(x) = \mathcal{D}_n((t-x)^2; x)$ and assume that

$$\lim_{s\to\infty} \frac{\frac{d^r \mathcal{B}(s)}{ds^r}}{\mathcal{B}(s)} = 1, \quad \text{for } 1 \le r \le k, k \in \mathbb{N}. \tag{2.1}$$

Also, let $C_E(\mathbb{R}_0^+)$ be the space of all continuous functions on the interval $\mathbb{R}_0^+$ with $|f(t)| \le \alpha e^{\beta x}$ for all $t \ge 0$ and positive finite numbers α and β.

Theorem 2.1. *Let $f \in C_E(\mathbb{R}_0^+)$. If $\rho \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \mathcal{D}_n(f; x) = f(x),$$

converges uniformly in each compact subset of $\mathbb{R}_0^+$.

Proof. With the help of Lemma 2.2 and condition (2.1), we have

$$\lim_{n\to\infty} \mathcal{D}_n(t^r; x) = x^r, \quad \text{for } r = 0, 1, 2.$$

The above convergence is satisfied uniformly in every compact subset of $\mathbb{R}_0^+$. Hence, by applying Korokin's type theorem (vi) of Theorem 4.1.4 in [4], we get the desired result. □

Next, we present some useful definitions which are needed in the sequel.

Definition 2.1. Let $\delta > 0$ and $f \in C^*(\mathbb{R}_0^+)$. Then the usual modulus of continuity $\omega(f; \delta)$ is defined as

$$\omega(f; \delta) := \sup_{|x-y|\le\delta} |f(x) - f(y)|, \quad \text{for all } x, y \in [0, \infty),$$

where $C^*(\mathbb{R}_0^+)$ be a space of uniformly continuous functions defined on $[0, \infty)$. It is also known that, for any $\delta > 0$,

$$|f(x) - f(y)| \le \omega(f; \delta)\Big(\frac{|x-y|}{\delta} + 1\Big), \quad \text{for all } x, y \in \mathbb{R}_0^+.$$

Definition 2.2. Let $f \in C_B(\mathbb{R}_0^+)$. Then the second order modulus of smoothness is defined by

$$\omega_2(f; \delta) := \sup_{0<t\le\delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B},$$

where $C_B(\mathbb{R}_0^+)$ is a class of bounded and uniformly continuous real-valued functions with the norm $\|f\|_{C_B} = \sup_{x\in\mathbb{R}_0^+} |f(x)|$.

Definition 2.3 ([9])**.** Let $f \in C_B(\mathbb{R}_0^+)$. The Peetre's K-functional is defined by

$$K(f;\delta) := \inf\left\{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}, \quad \text{for all } g \in C_B^2(\mathbb{R}_0^+), \tag{2.2}$$

where $C_B^2(\mathbb{R}_0^+) := \{g \in C_B(\mathbb{R}_0^+) : g' \in AC_{loc}(\mathbb{R}_0^+), g'' \in C_B(\mathbb{R}_0^+)\}$ endowed with the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$ and $g' \in AC_{loc}(\mathbb{R}_0^+)$ means that g' is locally absolutely continuous function. It is also known that from [9], there exists an absolute constant $C > 0$, such that

$$K(f;\delta) \le C\omega_2(f;\sqrt{\delta}). \tag{2.3}$$

It is clear that the following inequality

$$K(f,\delta) \le M\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\right\}, \tag{2.4}$$

is valid, for all $\delta > 0$. The constant $M > 0$ is independent of f and δ.

3. The Order of Approximation

In this section, we establish the rate of convergence for the operators $\mathcal{D}_n$ in terms of Peetre's K-functional, classical and second-order modulus of continuity.

Theorem 3.1. *Let $f \in C_E(\mathbb{R}_0^+)$ and $\rho \in \mathbb{N}$. Then the operators $\mathcal{D}_n$ satisfy the following inequality:*

$$|\mathcal{D}_n(f;x) - f(x)| \le 2\omega\left(f;\sqrt{\xi_n^\rho(x)}\right),$$

where $\xi := \xi_n^\rho(x) = \mathcal{D}_n((t-x)^2;x) = x^2\Big(b_2-2b_1+1\Big)+2xpa_1\Big(b_1-1\Big)+\frac{x}{n}b_1+p^2(a_2+a_1)$, see Lemma 2.3.

Proof. In view of the fact that $\mathcal{D}_n(1;x) = 1$ and (1.7), we have

$$\begin{aligned}
|\mathcal{D}_n(f;x) - f(x)| \le& \frac{1}{\mathcal{A}(1)\mathcal{B}(nx)}\sum_{k=0}^{\infty} q_k(nx)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\
\le& \frac{1}{\mathcal{A}(1)\mathcal{B}(nx)}\sum_{k=0}^{\infty} q_k(nx)\left(\frac{1}{\delta}\left|\frac{k}{n} - x\right| + 1\right)\omega(f;\delta) \\
\le& \left\{1 + \frac{1}{\delta\mathcal{A}(1)\mathcal{B}(nx)}\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f;\delta). \qquad (3.1)
\end{aligned}$$

In view of Lemma 2.3 and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right| \le& \left\{\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|^2\right\}^{1/2} \\
\le& \left(\sum_{k=0}^{\infty} q_k(nx)\right)^{1/2}\left(\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|^2\right)^{1/2} \\
=& \sqrt{\mathcal{A}(1)\mathcal{B}(nx)}\left(\mathcal{A}(1)\mathcal{B}(nx)\mathcal{D}_n((t-x)^2;x)\right)^{1/2}
\end{aligned}$$

$$=\mathcal{A}(1)\mathcal{B}(nx)\left(\mathcal{D}_n((t-x)^2;x)\right)^{1/2}$$

$$=\mathcal{A}(1)\mathcal{B}(nx)\sqrt{\xi_n^\rho(x)}. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$|\mathcal{D}_n(f;x)-f(x)|\leq\left\{1+\frac{\sqrt{\xi_n^\rho(x)}}{\delta}\right\}\omega(f;\delta).$$

Choosing $\delta=\sqrt{\xi_n^\rho(x)}$, we obtain the desired result. □

Remark 3.1. For $\rho=0$, Theorem 3.1 represents the Theorem 2 for the operators given by (1.5) (see [20]).

Theorem 3.2. *Let $f\in C_B^2(\mathbb{R}_0^+)$ and $\rho\in\mathbb{N}$. Then we have*

$$|\mathcal{D}_n(f;x)-f(x)|\leq\psi\|f\|_{C_B^2(\mathbb{R}_0^+)},$$

where $\psi:=\psi_n^\rho(x)=\left[\frac{1}{2}\Big(b_2-2b_1+1\Big)x^2+\Big\{n\Big(b_1-1\Big)\Big(pa_1+1\Big)+b_1\Big\}\frac{x}{n}+pa_1+p^2(a_2+a_1)\right]\|f\|_{C_B^2(\mathbb{R}_0^+)}$.

Proof. Let $x\in\mathbb{R}_0^+$. Applying Taylor's expansion to the function $f\in C_B^2(\mathbb{R}_0^+)$ and using the linearity of $\mathcal{D}_n$, we have

$$\mathcal{D}_n(f;x)-f(x)=f'(x)\mathcal{D}_n(t-x;x)+\frac{1}{2}f^{(2)}(\xi)\mathcal{D}_n((t-x)^2;x),\quad \xi\in(x,t).$$

Using Lemma 2.3, we have

$$\begin{aligned}|\mathcal{D}_n(f;x)-f(x)|\leq&\Big\{x(b_1-1)+pa_1\Big\}\|f'\|_{C_B(\mathbb{R}_0^+)}\\&+\frac{1}{2}\Big\{x^2\Big(b_2-2b_1+1\Big)+2xpa_1\Big(b_1-1\Big)\\&+\frac{x}{n}b_1+p^2(a_2+a_1)\Big\}\|f^{(2)}\|_{C_B(\mathbb{R}_0^+)}\\\leq&\left[\frac{1}{2}\Big(b_2-2b_1+1\Big)x^2+\Big\{n\Big(b_1-1\Big)\Big(pa_1+1\Big)+b_1\Big\}\frac{x}{n}\right.\\&\left.+pa_1+p^2(a_2+a_1)\right]\|f\|_{C_B^2(\mathbb{R}_0^+)}.\end{aligned}\tag{3.3}$$

This completes the proof of the theorem. □

Theorem 3.3. *Let $f\in C_B(\mathbb{R}_0^+)$. Then the following inequality satisfy:*

$$|\mathcal{D}_n(f;x)-f(x)|\leq 2M\left\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B(\mathbb{R}_0^+)}\right\},$$

where $\delta:=\delta_n^\rho(x)=\frac{1}{2}\psi_n^\rho(x)$ and M is a positive constant which is independent of the function f and δ. Also, $\psi_n^\rho(x)$ is defined in Theorem 3.2.

Proof. Let $h \in C_B^2(\mathbb{R}_0^+)$. In view of the Theorem 3.2, we have

$$\begin{aligned}|\mathcal{D}_n(f;x) - f(x)| =&|\mathcal{D}_n(f-h;x)| + |\mathcal{D}_n(h;x) - h(x)| + |f(x) - h(x)|\\ \leq& 2\|f-h\|_{C_B} + \psi\|h\|_{C_B^2(\mathbb{R}_0^+)}\\ \leq& 2\left[\|f-h\|_{C_B} + \delta\|h\|_{C_B^2(\mathbb{R}_0^+)}\right].\end{aligned}$$

Left-hand side of the above inequality is independent of $h \in C_B^2(\mathbb{R}_0^+)$, so

$$|\mathcal{D}_n(f;x) - f(x)| \leq 2K(f;\delta),$$

where $K(f;\delta)$ is defined in (2.2). Taking into account the relation (2.4) in the above inequality, we have

$$|\mathcal{D}_n(f;x) - f(x)| \leq 2M\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B(\mathbb{R}_0^+)}\right\}.$$

This is the required result. □

Theorem 3.4. *Let $x \in \mathbb{R}_0^+$ and $f \in C_B(\mathbb{R}_0^+)$. Then we have the following relation*

$$|\mathcal{D}_n(f;x) - f(x)| \leq 4\omega_2\Big(f;\sqrt{\lambda_n^\rho}\Big) + \omega(f;\gamma_n^\rho),$$

where

$$\lambda_n^\rho := \lambda_n^\rho(x) = \frac{1}{8}\left\{\xi_n^\rho(x) + \Big(x(b_1-1)+pa_1-x\Big)^2\right\} \tag{3.4}$$

and

$$\gamma_n^\rho := \gamma_n^\rho(x) = |x(b_1-1)+pa_1-x| = |\mathcal{D}_n((t-x);x) - x|. \tag{3.5}$$

Proof. Let us consider a new auxiliary operators $\tilde{\mathcal{D}}_n(f;x)$ on $C_B(\mathbb{R}_0^+)$ defined by

$$\tilde{\mathcal{D}}_n(f;x) := \mathcal{D}_n(f;x) - f\Big(x(b_1-1)+pa_1\Big) + f(x). \tag{3.6}$$

From the above auxiliary operators, it is observe that $\tilde{\mathcal{D}}_n(1;x) = 1$ and $\tilde{\mathcal{D}}_n(t;x) = x$. Let $h \in C_B^2(\mathbb{R}_0^+)$, $C_B^2(\mathbb{R}_0^+) = \{h \in C_B(\mathbb{R}_0^+) : h', h^{(2)} \in C_B(\mathbb{R}_0^+)\}$, then by Taylor series theorem, we have

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-\nu)h^{(2)}(\nu)d\nu.$$

Using Lemma 2.3 and (3.6) and applying the operators $\tilde{\mathcal{D}}_n$ on both sides of the above equation, we have

$$\tilde{\mathcal{D}}_n(h;x) - h(x) = \tilde{\mathcal{D}}_n\bigg(\int_x^t (t-\nu)h^{(2)}(\nu)d\nu; x\bigg).$$

It follows from (3.6) that

$$\begin{aligned}\tilde{\mathcal{D}}_n(h;x) - h(x) =&\mathcal{D}_n\bigg(\int_x^t (t-\nu)h^{(2)}(\nu)d\nu; x\bigg)\\ &+ \int_x^{x(b_1-1)+pa_1} \Big(x(b_1-1)+pa_1-\nu\Big)h^{(2)}(\nu)d\nu\end{aligned}$$

$$\begin{aligned}&\leq \frac{\|h^{(2)}\|}{2}\mathcal{D}_n((t-x)^2;x)+\frac{\|h^{(2)}\|}{2}\Big(x(b_1-1)+pa_1-x\Big)^2\\&=\frac{\|h^{(2)}\|}{2}\Big\{\xi_n^\rho(x)+\Big(x(b_1-1)+pa_1-x\Big)^2\Big\},\end{aligned}$$

considering (3.4), we obtain

$$|\tilde{\mathcal{D}}_n(h;x)-h(x)|\leq 4\lambda_n^\rho\|h^{(2)}\|, \tag{3.7}$$

where λ_n^ρ is given in (3.4).

In view of Lemma 2.3 and (3.6), we have

$$|\tilde{\mathcal{D}}_n(f;x)|\leq \|\mathcal{D}_n(f;x)\|+2\|f\|\leq 3\|f\|,\quad \text{for all } f\in C_B(\mathbb{R}_0^+). \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}|\mathcal{D}_n(f;x)-f(x)|\leq&|\tilde{\mathcal{D}}_n(f-h;x)-(f-h)(x)|+|\tilde{\mathcal{D}}_n(h;x)-h(x)|\\&+\Big|f\Big(x(b_1-1)+pa_1\Big)-f(x)\Big|\\\leq&4\Big(\|f-h\|+\lambda_n^\rho\|h^{(2)}\|\Big)+\omega\Big(f;\Big|x(b_1-2)+pa_1\Big|\Big),\end{aligned}$$

taking the infimum on the first term of the above inequality for $h\in C_B^2(\mathbb{R}_0^+)$ and using the inequalities (3.5) and (2.2), we have

$$|\mathcal{D}_n(f;x)-f(x)|\leq 4K(f;\lambda_n^\rho)+\omega(f;\gamma_n^\rho),$$

where γ_n^ρ is given in (3.5) and in view of the relation (2.3), we get our desired result. □

Remark 3.2. It is note that from Theorem 3.1- Theorem 3.4, the operators $\mathcal{D}_n(f;x)\to f(x)$, when λ_n^ρ, γ_n^ρ, ψ_n^α and ξ_n^α tend to zero as $n\to\infty$ with the assumption (2.1).

Now, we estimate the following local approximation result for the function belonging to Lipschitz-type space.

For $\mu\geq 0$, $\nu>0$ to be fixed, the class of two parameteric Lipschitz type functions [16] is defined as

$$Lip_M^{\zeta,\nu}(\alpha)=\left\{f\in C_B(\mathbb{R}_0^+):|f(t)-f(x)|\leq\frac{M|t-x|^\alpha}{(t+\zeta x^2+\nu x)^{\frac{\alpha}{2}}},t,x\in(0,\infty)\right\},$$

where M is positive constant and $0<\alpha\leq 1$. In particular, at $\zeta=0$ and $\nu=1$, the space $Lip_M^{0,1}(\alpha)$ reduced to the space $L_M^*(\alpha)$ defined in [18].

Theorem 3.5. *Let $f\in L_M^{\zeta,\nu}(\alpha)$ and $\rho\in\mathbb{N}$. Then, for all $x>0$, we have*

$$|\mathcal{D}_n(f,x)-f(x)|\leq M\left(\frac{\xi_n^\rho(x)}{\zeta x^2+\nu x}\right)^{\frac{\alpha}{2}},$$

where $\xi_n^\rho(x)$ is defined in Lemma 2.3.

Proof. Let $x \in (0,\infty)$ and $f \in L_M^{\zeta,\nu}(\alpha)$. We have

$$\begin{aligned}
|\mathcal{D}_n(f,x) - f(x)| &\le \mathcal{D}_n(|f(t) - f(x)|, x) \\
&\le M\mathcal{D}_n\left(\frac{|t-x|^\alpha}{(\zeta x^2 + \nu x + t)^{\frac{\alpha}{2}}}, x\right) \\
&\le \frac{M}{(\zeta x^2 + \nu x)^{\frac{\alpha}{2}}}\mathcal{D}_n\Big(|t-x|^\alpha, x\Big). \qquad (3.9)
\end{aligned}$$

First, we consider the case $\alpha = 1$. Applying Cauchy-Schwarz inequality in (3.9) at $\alpha = 1$, we obtain

$$|\mathcal{D}_n(f,x) - f(x)| \le \frac{M}{(\zeta x^2 + \nu x)^{\frac{1}{2}}}\Big(\mathcal{D}_n((t-x)^2, x)\Big)^{\frac{1}{2}} \le M\left(\frac{\xi_n^\rho(x)}{\zeta x^2 + \nu x}\right)^{\frac{1}{2}}.$$

Thus, the result holds for $\alpha = 1$.

Now, we prove the result is true for $0 < \alpha < 1$. Then, for $x \in (0,\infty)$, $f \in L_M^{\zeta,\nu}(\alpha)$ and applying Hölder's inequality in (3.9) by taking $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we obtain

$$|\mathcal{D}_n(f,x) - f(x)| \le \frac{M}{(\zeta x^2 + \nu x)^{\frac{\alpha}{2}}}\mathcal{D}_n\Big(|t-x|, x\Big)^\alpha.$$

Finally, applying the Cauchy-Schwartz inequality, we obtain

$$|\mathcal{D}_n(f;x) - f(x)| \le \frac{M}{(\zeta x^2 + \nu x)^{\frac{\alpha}{2}}}\Big\{\mathcal{D}_n((t-x)^2; x)\Big\}^{\frac{\alpha}{2}} = M\left(\frac{\xi_n^\alpha(x)}{\zeta x^2 + \nu x}\right)^{\frac{\alpha}{2}}.$$

This completes the proof of theorem. □

4. Voronovskaja-Type Result

The following assumptions are required to discuss a quantitative Voronovskaja-type result for the operators (1.7).

Assumptions:

(i) $\lim_{n\to\infty} n(b_1 - 1) = \alpha(x)$;
(ii) $\lim_{n\to\infty} n(b_2 - 2b_1 + 1) = \beta(x)$;
(iii) $\lim_{n\to\infty} n(b_3 - 2b_2 + b_1) = \lambda(x)$;
(iv) $\lim_{n\to\infty} n(b_3 - 3b_2 + 3b_1 - 1) = \delta(x)$;
(v) $\lim_{n\to\infty} n^2(b_4 - 4b_3 + 6b_2 - 4b_1 + 1) = \gamma(x)$;

where $\alpha(x)$, $\beta(x)$, $\lambda(x)$, $\delta(x)$ and $\gamma(x)$ are continuous and bounded functions on $\mathbb{R}_0^+$.

Taking into account (2.1), Lemma 2.3 and the above assumptions, we have the following.

Lemma 4.1. *The operators* (1.7) *verify:*

(i) $\lim_{n\to\infty} n\mathcal{D}_n((t-x); x) = x\alpha(x) + pa_1$;
(ii) $\lim_{n\to\infty} n\mathcal{D}_n((t-x)^2; x) = x^2\beta(x) + x$;
(iii) $\lim_{n\to\infty} n^2\mathcal{D}_n((t-x)^4; x) = x^4\gamma(x) - x^3\{4npa_1\delta(x) + 6\lambda(x)\} - (3n^2p^2 + 10np - 12)a_1 + 7$.

Theorem 4.1. *Let $f \in C_B^2(\mathbb{R}_0^+)$. Then we have*

$$\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} = \Big\{x\alpha(x) + pa_1\Big\}f'(x) + \Big\{x^2\beta(x) + x\Big\}\frac{f^{(2)}(x)}{2}.$$

Proof. Let $x \in \mathbb{R}_0^+$ be an arbitrary but fixed number. Applying the Taylor series theorem to the function $f \in C_B^2(\mathbb{R}_0^+)$, we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f^{(2)}(x) + \kappa(t,x)(t-x)^2, \tag{4.1}$$

where $\kappa(t,x) \in C_E(\mathbb{R}_0^+)$ and satisfies $\lim_{t\to x}\kappa(t,x) = 0$. Now, applying the operators $\mathcal{D}_n$ both sides on the equation (4.1), we get

$$\begin{aligned}\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} &= \lim_{n\to\infty} nf'(x)\mathcal{D}_n(t-x;x) + \lim_{n\to\infty} n\frac{1}{2}\mathcal{D}_n((t-x)^2;x)f^{(2)}(x)\\ &\quad + \lim_{n\to\infty} n\mathcal{D}_n(\kappa(t,x)(t-x)^2;x).\end{aligned} \tag{4.2}$$

In the last term of (4.2), we apply the Cauchy-Schwartz inequality

$$n\mathcal{D}_n(\kappa(t,x)(t-x)^2;x) \le \sqrt{n^2\mathcal{D}_n((t-x)^4;x)\mathcal{D}_n(\kappa^2(t,x);x)}. \tag{4.3}$$

Since $\kappa(t,x) \to 0$ as $t \to x$, it follows from Theorem 2.1 that

$$\lim_{n\to\infty} \mathcal{D}_n(\kappa^2(t,x);x) = \kappa^2(x,x) = 0, \tag{4.4}$$

uniformly for $x \in [0,b], b > 0$.

Combining the equations from (4.2)–(4.4) and taking into account the Lemma 4.1, we conclude that

$$\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} = \Big\{x\alpha(x) + pa_1\Big\}f'(x) + \Big\{x^2\beta(x) + x\Big\}\frac{f^{(2)}(x)}{2}.$$

This completes the proof of the theorem. ☐

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EXISTENCE OF CLASSICAL SOLUTIONS FOR BROER-KAUP EQUATIONS

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Abstract. In this paper we investigate the Cauchy problem for one dimensional Broer-Kaup equations for existence of global classical solutions. We give conditions under which the considered equations have at least one and at least two classical solutions. To prove our main results we propose a new approach based upon recent theoretical results.

1. Introduction

Study of existence of global classical solutions of nonlinear models is one of the important works in nonlinear science. In this paper, we investigate the Cauchy problem for a model describing the bi-directional propagation of long waves in shallow water which was proposed by Broer and Kaup [2,9] and called Broer-Kaup (BK) equations. Namely, we are concerned with the following system:

$$
\begin{aligned}
u_t + uu_x + v_x &= 0, \quad t \in (0,\infty), x \in \mathbb{R},\\
v_t + u_x + 2(uv)_x + u_{xxx} &= 0, \quad t \in (0,\infty), x \in \mathbb{R},\\
u(0,x) &= u_0(x), \quad x \in \mathbb{R},\\
v(0,x) &= v_0(x), \quad x \in \mathbb{R},
\end{aligned} \tag{1.1}
$$

where

(H1): $u_0, v_0 \in \mathcal{C}^1(\mathbb{R})$, $0 \leq u_0, v_0 \leq B$ on $\mathbb{R}$ for some positive constant B.

Here the unknowns $u = u(t,x)$ and $v = v(t,x)$ denote respectively, the horizontal velocity and the elevation of the water wave. The Broer-Kaup equations of system

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(1.1) can be obtained from the symmetry constraints of the Kadomtsev-Petviashvili (KP) equation and are a mathematical model of many nonlinear waves, see [12]. More precisely, they describe the evolution of the horizontal velocity component $u(t,x)$ of water waves of height $v(t,x)$ propagating in both directions in an infinite narrow channel of finite constant depth. Several methods have been used to capture different nature of solutions contained in Broer-Kaup equations like traveling wave solutions, periodic wave solutions, dromion solutions, solitary wave solutions and soliton-like solutions, see [21] and [29]. By qualitative analysis method, a sufficient condition for the existence of peaked periodic wave solutions to the Broer-Kaup equations was given in [8] and some exact explicit expressions of peaked periodic wave solutions were also presented. In [16], fission and fusion phenomena were revealed and soliton solutions were obtained. A family of traveling wave solutions is given in [18,19] and [7]. Solitary wave solutions to the Broer-Kaup equations are considered in [14] by using the first integral method. By application of the sub-ode method [25], new and more general form solutions are obtained for the Broer-Kaup equations. Using a consistent tanh expansion method, Chen et al. [1] gave the interaction solutions between the solitons and other different types of nonlinear waves. In [13], some smooth and peaked solitary wave solutions have been constructed by the bifurcation method of dynamical system. By using a Darboux transformation, Zhou et al. [27] obtained new exact solutions for Broer-Kaup system. In [6], new type of solitary wave solutions for the Broer-Kaup equations were presented by using the He's variational principle.

Various algebraic aspects of BK equations solutions have been studied. Kupershmidt [11] showed that BK equations are integrable and possess infinite number of conservation laws and tri-Hamiltonian structure. In [4], The geometric properties of non-Noether symmetries as well as their applications were discussed.

The analysis by many methods of the (2+1)-dimensional BK system can be found in [26] and [28] and the references therein. The (1+1)-dimensional and the $(2+1)$-dimensional higher order Broer-Kaup equation was considered for example in [23] and [20], respectively. Concerning generalized and variable coefficient Broer-Kaup equations, see for example [22] and [10]. Recently, fractional and stochastic Broer-Kaup system, were studied in [3] and [24].

The aim of this paper is to investigate the initial value problem (1.1) for existence and nonuniqueness of global classical solutions. For goal, a new topological approach which uses the abstract theory of the sum of two operators is used for investigations of existence of at least one and at least two classical solutions. This basic and new idea can be used for investigations for existence of global classical solutions for many of the interesting equations of mathematical physics. Here, by a classical solution to the Broer-Kaup equations we mean a solution at least three times continuously differentiable in x and once in t for any $t \geq 0$. In other words, (u,v) belongs to the space $\mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))\times\mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))$ of continuously differentiable functions on $[0,\infty)$ with values in the Banach space $\mathcal{C}^3(\mathbb{R})$.

The paper is organized as follows. In the next section, we give some properties of solutions of problem (1.1). First, we give an integral representation of these solutions, then we prove some *a priori* estimates in a sense that will be defined later on. In Section 3 we prove our main results about existence and multiplicity of solutions for the Broer-Kaup system (1.1). Finally, in Section 4 we give an example to illustrate our main results.

2. Some Properties of Solutions of Problem (1.1)

Let $X = X^1 \times X^1$, where $X^1 = \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$. For $(u,v) \in X$, define the operators S_1^1, S_1^2 and S_1 as follows.

$$S_1^1(u,v)(t,x) = u(t,x) - u_0(x) + \int_0^t (u(t_1,x)u_x(t_1,x) + v_x(t_1,x))dt_1,$$

$$S_1^2(u,v)(t,x) = v(t,x) - v(0,x) + \int_0^t \Big(u_x(t_1,x) + 2u_x(t_1,x)v(t_1,x) + 2u(t_1,x)v_x(t_1,x) + u_{xxx}(t_1,x)\Big)dt_1,$$

$$S_1(u,v)(t,x) = \left(S_1^1(u,v)(t,x), S_1^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

2.1. **Integral representation of the solutions.**

Lemma 2.1. *Suppose that* **(H1)** *is satisfied. If* $(u,v) \in X$ *satisfies the equation*

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2.1}$$

then (u,v) *is a solution of the IVP* (1.1).

Proof. Let $(u,v) \in X$ be a solution of the equation (2.1). Then

$$S_1^1(u,v)(t,x) = 0, \quad S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.2}$$

We differentiate both equations of (2.2) with respect to t and x and we find

$$u_t(t,x) + u(t,x)u_x(t,x) + v_x(t,x) = 0,$$
$$v_t(t,x) + u_x(t,x) + 2u_x(t,x)v(t,x) + 2u(t,x)v_x(t,x) + u_{xxx}(t,x) = 0,$$

$(t,x) \in [0,\infty) \times \mathbb{R}$. We put $t = 0$ in both equations of (2.2) and we arrive at

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}.$$

This completes the proof. □

Lemma 2.2. *Suppose* **(H1)** *and let* $h \in \mathcal{C}([0,\infty) \times \mathbb{R})$ *be a positive function almost everywhere on* $[0,\infty) \times \mathbb{R}$. *If* $(u,v) \in X$ *satisfies the following integral equations:*

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^3 h(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^3 h(t_1,x_1) S_1^2(u,v)(t_1,x_1)dx_1dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then (u, v) *is a solution to the IVP* (1.1).

Proof. We differentiate three times with respect to t and three times with respect to x the integral equations of Lemma 2.2 and we find

$$h(t,x)S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence and Lemma 2.1, we conclude that (u, v) is a solution to the IVP (1.1). This completes the proof. □

2.2. **A priori estimates.** In the sequel, $X = X^1 \times X^1$ where $X^1 = \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$ will be endowed with the norm

$$\|(u,v)\| = \max\left\{\|u\|_{X^1}, \|v\|_{X^1}\right\}, \quad (u,v) \in X,$$

with

$$\|u\|_{X^1} = \max\Bigg\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xx}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xxx}(t,x)| \Bigg\},$$

provided it exists. Let

$$B_1 = 4(B + B^2).$$

Lemma 2.3. *Under hypothesis* **(H1)** *and for* $(u, v) \in X$ *with* $\|(u, v)\| \le B$*, the following estimates hold:*

$$|S_1^1(u,v)(t,x)| \le B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$|S_1^2(u,v)(t,x)| \le B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. Suppose that **(H1)** is satisfied and let $(u, v) \in X$ with $\|(u, v)\| \le B$.

(i) Estimation of $|S_1^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} |S_1^1(u,v)(t,x)| &= \left| u(t,x) - u_0(x) + \int_0^t (u(t_1,x)u_x(t_1,x) + v_x(t_1,x))\, dt_1 \right| \\ &\le |u(t,x)| + |u_0(x)| + \int_0^t (|u(t_1,x)||u_x(t_1,x)| + |v_x(t_1,x)|)\, dt_1 \\ &\le 2B + (B + B^2)t \\ &\le (3B + B^2)(1+t) \\ &\le B_1(1+t). \end{aligned}$$

(ii) Estimation of $|S_1^2(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|S_1^2(u,v)(t,x)\right| &= \left|v(t,x)-v(0,x)+\int_0^x\Big(u_x(t_1,x)+2u_x(t_1,x)v(t_1,x)\right.\\ &\quad\left.+2u(t_1,x)v_x(t_1,x)+u_{xxx}(t_1,x)\Big)dx_1dt_1\right|\\ &\le |v(t,x)|+|v(0,x)|+\int_0^x\Big(|u_x(t_1,x)|+2|u_x(t_1,x)||v(t_1,x)|\\ &\quad+2|u(t_1,x)||v_x(t_1,x)|+|u_{xxx}(t_1,x)|\Big)dx_1dt_1\\ &\le 2B+(2B+4B^2)t\\ &\le 4(B+B^2)(1+t)\\ &= B_1(1+t).\end{aligned}$$

This completes the proof. □

Suppose

(H2): $g\in\mathcal{C}([0,\infty)\times\mathbb{R})$ is a positive function almost everywhere on $[0,\infty)\times\mathbb{R}$ such that

$$16(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\le A,$$

$(t,x)\in[0,\infty)\times\mathbb{R}$, for some constant $A>0$.

In the last section, we will give an example for a function g that satisfies **(H2)**. For $(u,v)\in X$, define the operators

$$\begin{aligned}S_2^1(u,v)(t,x) &= \int_0^t\int_0^x(t-t_1)(x-x_1)^3g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1,\\ S_2^2(u,v)(t,x) &= \int_0^t\int_0^x(t-t_1)(x-x_1)^3g(t_1,x_1)S_1^2(u,v)(t_1,x_1)dx_1dt_1\end{aligned}$$

and

$$S_2(u,v)(t,x)=\left(S_2^1(u,v)(t,x),S_2^2(u,v)(t,x)\right),\quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{2.3}$$

Lemma 2.4. *Under hypothesis* **(H1)** *and* **(H2)** *and for* $(u,v)\in X$*, with* $\|(u,v)\|\le B$*, the following estimate holds:*

$$\|S_2(u,v)\|\le AB_1.$$

Proof. Suppose **(H1)** and **(H2)** and let $(u,v)\in X$, with $\|(u,v)\|\le B$.

(i) Estimation of $|S_2^1(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}|S_2^1(u,v)(t,x)| &= \left|\int_0^t\int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\le \int_0^t\left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\le B_1(1+t)\int_0^t\left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)dx_1\right|dt_1\\ &\le 8B_1(1+t)^2|x|^3\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le 8B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le AB_1.\end{aligned}$$

(ii) Estimation of $\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right| &= \left|\int_0^t\int_0^x (x-x_1)^3 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\le \int_0^t\left|\int_0^x |x-x_1|^3 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\le B_1(1+t)\int_0^t\left|\int_0^x |x-x_1|^3 g(t_1,x_1)dx_1\right|dt_1\\ &\le 8B_1(1+t)^2|x|^3\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le 8B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le AB_1.\end{aligned}$$

(iii) Estimation of $\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right| &= 3\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\le 3B_1(1+t)\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2x^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le AB_1.\end{aligned}$$

(iv) Estimation of $\left|\frac{\partial^2}{\partial x^2}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2}S_2^1(u,v)(t,x)\right| &= 6\left|\int_0^t\int_0^x(t-t_1)(x-x_1)g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\leq 6\int_0^t\left|\int_0^x(t-t_1)|x-x_1|g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\leq 6B_1(1+t)\int_0^t\left|\int_0^x(t-t_1)|x-x_1|g(t_1,x_1)dx_1\right|dt_1\\ &\leq 12B_1(1+t)^2|x|\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\leq 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\leq AB_1.\end{aligned}$$

(v) Estimation of $\left|\frac{\partial^3}{\partial x^3}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial^3}{\partial x^3}S_2^1(u,v)(t,x)\right| &= 6\left|\int_0^t\int_0^x(t-t_1)g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\leq 6\int_0^t\left|\int_0^x(t-t_1)g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\leq 6B_1(1+t)\int_0^t\left|\int_0^x(t-t_1)g(t_1,x_1)dx_1\right|dt_1\\ &\leq 12B_1(1+t)^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\leq 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\leq AB_1.\end{aligned}$$

Similarly, the same estimates (i)-(v) can be proved for the operator S_2^2. Finally,

$$\|S_2(u,v)\|\leq AB_1.$$

This completes the proof. □

3. Main Results

3.1. **Existence of nonnegative solutions.** The following theorem (see its proof in [17]) will be used to prove Theorem 3.2.

Theorem 3.1. *Let E be a Banach space and*

$$E_1=\{x\in E:\|x\|\leq R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = -\epsilon x, \quad x \in E_1,$$

with $\epsilon > 0$ and $S : E_1 \to E$ be continuous and such that

(i) $(I - S)(E_1)$ *resides in a compact subset of E and*

(ii) $\{x \in E : x = \lambda(I - S)x, \|x\| = R\} = \emptyset$ *for any* $\lambda \in \left(0, \frac{1}{\epsilon}\right)$.

Then there exists $x^* \in E_1$ such that

$$Tx^* + Sx^* = x^*.$$

In the sequel, suppose that the constants B and A which appear in the conditions **(H1)** and **(H2)**, respectively, satisfy the following inequality:

(H3): $AB_1 < B$, where $B_1 = 4(B + B^2)$.

Our first main result for existence of classical solutions of the IVP (1.1) is as follows.

Theorem 3.2. *Assume that the hypotheses* **(H1)**, **(H2)** *and* **(H3)** *are satisfied. Then the IVP* (1.1) *has at least one nonnegative solution* $(u, v) \in \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$.

Proof. Choose $\epsilon \in (0, 1)$, such that $\epsilon B_1(1 + A) < B$.

For $(u, v) \in X = \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$, we will write

$$(u, v) \geq 0 \text{ if } u(t, x) \geq 0 \text{ and } v(t, x) \geq 0, \quad \text{for any } (t, x) \in [0, \infty) \times \mathbb{R}.$$

Let $\tilde{\tilde{\tilde{Y}}}$ denotes the set of all equi-continuous families in X with respect to the norm $\| \cdot \|$, $\tilde{\tilde{Y}} = \overline{\tilde{\tilde{\tilde{Y}}}}$ be the closure of $\tilde{\tilde{\tilde{Y}}}$, $\tilde{Y} = \tilde{\tilde{Y}} \cup \{(u_0, v_0)\}$ and

$$Y = \{(u, v) \in \tilde{Y} : (u, v) \geq 0, \|(u, v)\| \leq B\}.$$

Note that Y is a compact set in X. For $(u, v) \in X$, define the operators

$$T(u, v)(t, x) = -\epsilon(u, v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$
$$S(u, v)(t, x) = (u, v)(t, x) + \epsilon(u, v)(t, x) + \epsilon S_2(u, v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

For $(u, v) \in Y$ and by using Lemma 2.4, it follows that

$$\begin{aligned}
\|(I - S)(u, v)\| &= \|\epsilon(u, v) - \epsilon S_2(u, v)\| \\
&\leq \epsilon\|(u, v)\| + \epsilon\|S_2(u, v)\| \\
&\leq \epsilon B_1 + \epsilon A B_1 \\
&= \epsilon B_1(1 + A) \\
&< B.
\end{aligned}$$

Thus, $S : Y \to X$ is continuous and $(I - S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $(u, v) \in X$ so that $\|(u, v)\| = B$ and

$$(u, v) = \lambda(I - S)(u, v)$$

or

$$\frac{1}{\lambda}(u,v) = (I-S)(u,v) = -\epsilon(u,v) - \epsilon S_2(u,v)$$

or

$$\left(\frac{1}{\lambda}+\epsilon\right)(u,v) = -\epsilon S_2(u,v),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(u,v)\| \leq AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda}+\epsilon\right) B = \left(\frac{1}{\lambda}+\epsilon\right)\|(u,v)\| = \epsilon\|S_2(u,v)\| < \epsilon B,$$

which is a contradiction. In virtue of Theorem 3.1, the operator $T+S$ has a fixed point $(u^*,v^*) \in Y$. Therefore,

$$\begin{aligned}(u^*,v^*)(t,x) =& T(u^*,v^*)(t,x) + S(u^*,v^*)(t,x)\\ =& -\epsilon(u^*,v^*)(t,x) + (u^*,v^*)(t,x) + \epsilon(u^*,v^*)(t,x) + \epsilon S_2(u^*,v^*)(t,x),\end{aligned}$$

$(t,x) \in [0,\infty)\times\mathbb{R}$, whereupon

$$0 = S_2(u^*,v^*)(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Lemma 2.2 yields that (u^*,v^*) is a solution to the IVP (1.1). This completes the proof. □

3.2. **Multiplicity of nonnegative solutions.** Let E be a real Banach space.

Definition 3.1. A closed, convex set $\mathcal{P}$ in E is said to be cone if

(a) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$;
(b) $x, -x \in \mathcal{P}$ implies $x = 0$.

Definition 3.2. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 3.3. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x-y\|_X,$$

for any $x, y \in X$.

The following result (see details of its proof in [5] and [17]) will be used to prove Theorem 3.4.

Theorem 3.3. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $w_0 \in \mathcal{P}\backslash\{0\}$ such that the following conditions hold:*

(i) *$Sx \neq (I-T)(x - \lambda w_0)$ for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda w_0)$;*
(ii) *there exists $\varepsilon > 0$ such that $Sx \neq (I-T)(\lambda x)$ for all $\lambda \geq 1+\varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$;*

(iii) $Sx \neq (I - T)(x - \lambda w_0)$ *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda w_0)$.

Then $T + S$ *has at least two non-zero fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

In the sequel, suppose that the constants B and A which appear in the conditions **(H1)** and **(H2)**, respectively, satisfy the following inequality:

(H4): $AB_1 < \frac{L}{5}$, where $B_1 = 4(B+B^2)$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

with r and R_1 are positive constants and $m > 0$ is large enough.

Our second main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

Theorem 3.4. *Assume that the hypotheses* **(H1)**, **(H2)** *and* **(H4)** *are satisfied. Then the IVP* (1.1) *has at least two nonnegative solutions*

$$(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})).$$

Proof. Set $X = \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$ and let

$$\widetilde{P} = \{(u, v) \in X : (u, v) \geq 0 \text{ on } [0, \infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u, v) \in X$, define the operators

$$T_1(u, v)(t, x) = (1 + m\epsilon)(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

$$S_3(u, v)(t, x) = -\epsilon S_2(u, v)(t, x) - m\epsilon(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (2.3). Note that any fixed point $(u, v) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Now, let us define

$$U_1 = \mathcal{P}_r = \{(u, v) \in \mathcal{P} : \|(u, v)\| < r\},$$

$$U_2 = \mathcal{P}_L = \{(u, v) \in \mathcal{P} : \|(u, v)\| < L\},$$

$$U_3 = \mathcal{P}_{R_1} = \{(u, v) \in \mathcal{P} : \|(u, v)\| < R_1\},$$

$$\Omega = \overline{\mathcal{P}_{R_2}} = \{(u, v) \in \mathcal{P} : \|(u, v)\| \leq R_2\}, \quad \text{with } R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}.$$

(a) Let $(u_1, v_1), (u_2, v_2) \in \Omega$, then

$$\|T_1(u_1, v_1) - T_1(u_2, v_2)\| = (1 + m\epsilon)\|(u_1, v_1) - (u_2, v_2)\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(b) Let $(u,v) \in \overline{\mathcal{P}_{R_1}}$, then Lemma 2.4 yields

$$\|S_3(u,v)\| \le \epsilon\|S_2(u,v)\| + m\epsilon\|(u,v)\| + \epsilon\frac{L}{10} \le \epsilon\bigg(AB_1 + mR_1 + \frac{L}{10}\bigg).$$

Therefore, $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently, $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

(c) Let $(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$ and set

$$(u_2,v_2) = (u_1,v_1) + \frac{1}{m}S_2(u_1,v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2^1(u_1,v_1) + \frac{L}{5} \ge 0$, $S_2^2(u_1,v_1) + \frac{L}{5} \ge 0$ on $[0,\infty) \times \mathbb{R}$. We have $u_2, v_2 \ge 0$ on $[0,\infty) \times \mathbb{R}$ and

$$\|(u_2,v_2)\| \le \|(u_1,v_1)\| + \frac{1}{m}\|S_2(u_1,v_1)\| + \frac{L}{5m} \le R_1 + \frac{A}{m}B_1 + \frac{L}{5m} = R_2.$$

Therefore, $(u_2,v_2) \in \Omega$ and

$$-\epsilon m(u_2,v_2) = -\epsilon m(u_1,v_1) - \epsilon S_2(u_1,v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$(I - T_1)(u_2,v_2) = -\epsilon m(u_2,v_2) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = S_3(u_1,v_1).$$

Consequently, $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(d) Assume that for any $(w_0,z_0) \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$ there exist $\lambda \ge 0$ and $(u,v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(w_0,z_0))$ or $(u,v) \in \overline{\mathcal{P}_{R_1}} \cap (\Omega + \lambda(w_0,z_0))$ such that

$$S_3(u,v) = (I - T_1)((u,v) - \lambda(w_0,z_0)).$$

Then

$$-\epsilon S_2(u,v) - m\epsilon(u,v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u,v) - \lambda(w_0,z_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(u,v) = \lambda m(w_0,z_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(w_0,z_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This is a contradiction.

(e) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $(u_1,v_1) \in \partial\mathcal{P}_L$ and $\lambda_1 \ge 1 + \varepsilon_1$ such that $\lambda_1(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$ and

$$S_3(u_1,v_1) = (I - T_1)(\lambda_1(u_1,v_1)). \tag{3.1}$$

Since $(u_1,v_1) \in \partial\mathcal{P}_L$ and $\lambda_1(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$, it follows that

$$\left(\frac{2}{5m} + 1\right)L < \lambda_1 L = \lambda_1\|(u_1,v_1)\| \le R_1.$$

Moreover,

$$-\epsilon S_2(u_1,v_1) - m\epsilon(u_1,v_1) - \epsilon\left(\frac{L}{10},\frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1,v_1) + \epsilon\left(\frac{L}{10},\frac{L}{10}\right)$$

or

$$S_2(u_1,v_1) + \left(\frac{L}{5},\frac{L}{5}\right) = (\lambda_1 - 1)m(u_1,v_1).$$

From here,

$$2\frac{L}{5} \geq \left\| S_2(u_1,v_1) + \left(\frac{L}{5},\frac{L}{5}\right)\right\| = (\lambda_1-1)m\|(u_1,v_1)\| = (\lambda_1-1)mL$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore, all conditions of Theorem 3.4 hold. Hence, the IVP (1.1) has at least two solutions (u_1,v_1) and (u_2,v_2) so that

$$\|(u_1,v_1)\| = L < \|(u_2,v_2)\| \leq R_1$$

or

$$r \leq \|(u_1,v_1)\| < L < \|(u_2,v_2)\| \leq R_1.$$

□

4. An Example

Below, we will illustrate our main results. Let

$$h(s) = \log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s\in\mathbb{R}, \quad s\neq\pm1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})},$$
$$l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{20})}{1+s^{40}}, \quad s\in\mathbb{R}, \quad s\neq\pm1.$$

Therefore,

$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)h(s) < \infty,$$
$$-\infty < \lim_{s\to\pm\infty}(1+s+s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2+s^3+s^4+s^5+s^6)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$

$$(1+s+s^2+s^3+s^4+s^5+s^6)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim\limits_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [15, pp. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ such that

$$12(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t \left|\int_0^x g_1(t_1,x_1)dx_1\right| dt_1 \le C_2, \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C_2} g_1(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Then

$$12(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t \left|\int_0^x g(t_1,x_1)dx_1\right| dt_1 \le A, \quad (t,x) \in [0,\infty)\times\mathbb{R},$$

i.e., **(H2)** holds. Now, consider the initial value problem

$$\begin{aligned} u_t + uu_x + v_x &= 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ v_t + u_x + 2(uv)_x + u_{xxx} &= 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) &= \frac{1}{1+x^2+x^4}, \quad x \in \mathbb{R}, \\ v(0,x) &= \frac{1}{1+3x^2+x^8}, \quad x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

so that **(H1)** holds, with $B = 10$, for example. Take

$$B = 10 \quad \text{and} \quad A = \frac{1}{10^4}.$$

Then

$$AB_1 = A.4(B+B^2) = \frac{1}{10^4}\cdot 4(10+10^2) < B.$$

So, condition **(H3)** is fulfilled. Thus, the conditions **(H1)**, **(H2)** and **(H3)** are satisfied. Hence, by Theorem 3.2, it follows that problem (4.1) has at least one solution $(u,v) \in \mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))$.

In the sequel, take

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Clearly,

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5},$$

i.e., **(H4)** holds. Hence, by Theorem 3.4, it follows that the initial value problem (4.1) has at least two nonnegative solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$.

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SEMI-SLANT LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. The aim of our paper is to introduce the notion of semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. We give non-trivial examples of semi-slant lightlike submanifolds and provide a characterization theorem of such submanifolds. Further, we obtain necessary and sufficient conditions for integrability of the distributions and investigate the geometry of the leaves of the foliation determined by the distributions. We also obtain a necessary and sufficient condition for the induced connection to be a metric connection. Finally, we obtain necessary and sufficient condition for mixed-geodesic semi-slant lightlike submanifold of golden semi-Riemannian manifold.

1. Introduction

A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric on it is degenerate, i.e., there exists a non zero $X \in \Gamma(TM)$ such that $g(X, Z) = 0$ for all $Z \in \Gamma(TM)$. In [4], Duggal and Bejancu introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle and they studied the geometry of arbitrary lightlike submanifold of a semi-Riemannian manifold. Lightlike geometry has its applications in general relativity, particularly in black hole theory. Many authors have studied lightlike submanifolds in various spaces ([5, 17]). In [15], authors introduced a new class of lightlike submanifolds namely, semi-slant lightlike submanifolds of indefinite Kaehler manifolds. In [15], authors investigated the integrability of various distributions, obtained a characterization theorem of such lightlike submanifolds and

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established equivalent conditions for totally geodesic foliation of distributions. In [16], authors introduced a general notion of lightlike submanifolds namely, semi-slant lightlike submanifolds of indefinite Sasakian manifolds. In [16], authors found some equivalent conditions for integrability and totally geodesic foliation of distributions. Golden proportion ψ is the real positive root of the equation $x^2 - x - 1 = 0$ (thus $\psi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$). Inspired by the Golden proportion, Crasmareanu and Hretcanu defined golden structure $\tilde{P}$ which is a tensor field satisfying $\tilde{P}^2 - \tilde{P} - I = 0$ on $\overline{M}$ [3]. Golden structure was inspired by the Golden proportion, which was described by Kepler (1571–1630).

A Riemannian manifold $\overline{M}$ with a golden structure $\tilde{P}$ is called a golden Riemannian manifold and was studied in ([3,9]). In [9], authors studied invariant submanifolds of a golden Riemannian manifold. Submanifolds of golden manifolds in semi-Riemannian geometry were studied by Poyraz and Yasar [12]. In [12], they proved that there is no radical anti-invariant lightlike hypersurface of a golden semi-Riemannian manifold and also studied screen semi-invariant and screen conformal screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold. Transversal and Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds were studied in ([6,8]). In [13], authors proved that there is no radical anti-invariant lightlike submanifold of a golden semi-Riemannian manifolds. In [7], author studies the geometry of screen transversal lightlike submanifolds and radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and investigate the geometry of distributions. Screen pseudo-slant and golden GCR-lightlike submanifolds of a golden semi-Riemannian manifold were studied in ([1, 11]). In [10], N. Onen Poyraz introduced screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifolds and found the conditions of integrability of distributions. In [10], they proved some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.

The purpose of this paper is to study semi-slant lightlike submanifold of golden semi-Riemannian manifolds. The paper is arranged as follows. In Section 2, some definitions and basic results about lightlike submanifolds and golden semi-Riemannian manifold are given. In Section 3, we study semi-slant lightlike submanifolds of a golden semi-Riemannian manifold giving examples, provide a characterization theorem and investigate the integrability of distributions. We also obtain necessary and sufficient conditions for semi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be metric connection. In Section 4, we find necessary and sufficient conditions for totally geodesic foliation determined by distributions on a semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. We also obtain necessary and sufficient conditions for semi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be mixed geodesic.

2. Preliminaries

Let $\overline{M}$ be a C^∞-differentiable manifold. If a $(1,1)$ type tensor field $\tilde{P}$ on $\overline{M}$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is called a golden structure on $\overline{M}$, where I is the identity transformation. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\overline{M}$. If $\tilde{P}$ satisfies the following equation

$$\overline{g}(\tilde{P}U, W) = \overline{g}(U, \tilde{P}W), \tag{2.2}$$

then $(\overline{M}, \overline{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [14], also, if $\tilde{P}$ is integrable, then we have [3]

$$\overline{\nabla}_U \tilde{P}W = \tilde{P}\overline{\nabla}_U W. \tag{2.3}$$

Now, from (2.2) we get

$$\overline{g}(\tilde{P}U, \tilde{P}W) = \overline{g}(\tilde{P}U, W) + \overline{g}(U, W), \tag{2.4}$$

for all $U, W \in \Gamma(T\overline{M})$.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\overline{M}$, where g is the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold [4] of $\overline{M}$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is

$$TM = \mathrm{Rad}(TM) \oplus_{orth} S(TM). \tag{2.5}$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^\perp$. Let $\mathrm{tr}(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and $\mathrm{Rad}(TM)$ in $S(TM^\perp)^\perp$, respectively. Then

$$\mathrm{tr}(TM) = ltr(TM) \oplus_{orth} S(TM^\perp), \tag{2.6}$$

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM), \tag{2.7}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [\mathrm{Rad}(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^\perp). \tag{2.8}$$

Theorem 2.1 ([4])**.** *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1, 2, \dots, r\}$, is a basis of $\Gamma(\mathrm{Rad}(TM|_U))$. Then there exist a complementary vector subbundle $ltr(TM)$ of $\mathrm{Rad}(TM)$ in $S(TM^\perp)^\perp$ and a basis $\{N_i\}$, $i \in \{1, 2, \dots, r\}$, of $\Gamma(ltr(TM|_U))$ such that $\overline{g}(N_i, \xi_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$ for any $i, j \in \{1, 2, \dots, r\}$.*

Following are four cases of a lightlike submanifold $\left(M, g, S(TM), S(TM^{\perp})\right)$.
Case 1. r-lightlike if $r < \min(m, n)$.
Case 2. Co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$.
Case 3. Isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$.
Case 4. Totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.
The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(\mathrm{tr}(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $\mathrm{tr}(TM)$, respectively. From (2.9) and (2.10), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.11}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.12}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.13}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $\mathrm{tr}(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M, respectively.
Also by using (2.9), (2.11)–(2.13) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.14}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{2.15}$$

Now, denote the projection of TM on $S(TM)$ by S. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, we have

$$\nabla_X SY = \nabla^*_X SY + h^*(X, SY), \tag{2.16}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.17}$$

By using above equations, we obtain

$$\overline{g}(h^l(X, SY), \xi) = g(A^*_\xi X, SY). \tag{2.18}$$

It is important to note that in general ∇ is not a metric connection on M. Since $\overline{\nabla}$ is metric connection, by using (2.11), we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y), \tag{2.19}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

Definition 2.1 ([2])**.** An equivalence relation on an n-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ in which the equivalence classes are connected, immersed submanifolds (called the leaves of the foliation) of a common dimension k, $0 < k \leq n$, is called a foliation on $\overline{M}$. If each leaf of a foliation F on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally geodesic submanifold of $\overline{M}$, we say that F is a totally geodesic foliation.

3. Semi Slant Lightlike Submanifolds

In this section, we study semi-slant lightlike submanifolds of golden semi Riemannian manifolds. We now give the following lemmas which will be useful to define slant notion on the screen distribution.

Lemma 3.1. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Suppose that $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = \{0\}$. Then $\tilde{P}ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$.*

Proof. Since by hypothesis $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\tilde{P}\,\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = 0$, we have $\tilde{P}\,\mathrm{Rad}(TM) \subset S(TM)$. Now we claim that $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Let us suppose that $ltr(TM)$ is invariant with respect to $\tilde{P}$. Choose $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma ltr(TM)$ such that $\overline{g}(N, \xi) = 1$. Then from (2.4), we have $1 = \overline{g}(\xi, N) = \overline{g}(\tilde{P}\xi, \tilde{P}N) - \overline{g}(\tilde{P}\xi, N) = 0$, due to $\tilde{P}\xi \in \Gamma S(TM)$ and $\tilde{P}N \in \Gamma ltr(TM)$. This is a contradiction, so $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Also $\tilde{P}N$ does not belong to $S(TM^{\perp})$, since $S(TM^{\perp})$ is ortogonal to $S(TM)$, $\overline{g}(\tilde{P}N, \tilde{P}\xi)$ must be zero, but from (2.4) we have $\overline{g}(\tilde{P}N, \tilde{P}\xi) = \overline{g}(\tilde{P}\xi, N) + \overline{g}(N, \xi) \neq 0$, for some $\xi \in \Gamma\,\mathrm{Rad}(TM)$, this is again a contradiction. Thus, we conclude that $\tilde{P}ltr(TM)$ is a distribution on M. Moreover, $\tilde{P}N$ does not belong to $\mathrm{Rad}(TM)$. Indeed, if $\tilde{P}N \in \Gamma\,\mathrm{Rad}(TM)$, we would have $\tilde{P}^2 N = \tilde{P}N + N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, but this is impossible. Finally, let $\tilde{P}N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, we obtain $\tilde{P}^2 N = \tilde{P}N + N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM) + \mathrm{Rad}(TM))$, this is not possible. Hence, $\tilde{P}N$ does not belong to $\tilde{P}\,\mathrm{Rad}(TM)$. Thus, we conclude that $\tilde{P}ltr(TM) \subset S(TM)$ and $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$. □

Lemma 3.2. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Suppose $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = \{0\}$. Then any complementary distribution to $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let M be an m-dimensional q-lightlike submanifold of an $(m+n)$-dimensional golden semi-Riemannian manifold $\overline{M}$ of index $2q$. From Lemma 3.1, we have $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$ and $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM) \subset S(TM)$. We denote the complementary distribution to $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)$ in $S(TM)$ by D. Then we have a local orthonormal frame of fields on $\overline{M}$ along M $\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i, X_\alpha, W_a\}$,

$i \in \{1, 2, \ldots, q\}, \alpha \in \{3q+1, \ldots, m\}, a \in \{q+1, \ldots, n\}$, where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $\text{Rad}(TM)$ and $ltrTM$, respectively and $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal bases of D and $S(TM^\perp)$, respectively.

Now, from the bases $\{\xi_1, \ldots, \xi_q, N_1, \ldots, N_q, \tilde{P}\xi_1, \ldots, \tilde{P}\xi_q, \tilde{P}N_1, \ldots, \tilde{P}N_q\}$ of $\text{Rad}(TM) \oplus ltrTM \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)$, we can construct an orthonormal bases $\{U_1, \ldots, U_{2q}, V_1, \ldots, V_{2q}\}$ as follows:

$$
\begin{aligned}
U_1 =& \frac{1}{\sqrt{2}}(\xi_1 + N_1), \quad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),\\
U_3 =& \frac{1}{\sqrt{2}}(\xi_2 + N_2), \quad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),\\
&\vdots\\
U_{2q-1} =& \frac{1}{\sqrt{2}}(\xi_q + N_q), \quad U_{2q} = \frac{1}{\sqrt{2}}(\xi_q - N_q),\\
V_1 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 + \tilde{P}N_1), \quad V_2 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 - \tilde{P}N_1),\\
V_3 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 + \tilde{P}N_2), \quad V_4 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 - \tilde{P}N_2),\\
&\vdots\\
V_{2q-1} =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_q + \tilde{P}N_q), \quad V_{2q} = \frac{1}{\sqrt{2}}(\tilde{P}\xi_q - \tilde{P}N_q).
\end{aligned}
$$

Hence, $\text{Span}\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i\}$ is a non-degenerate space of constant index $2q$. Thus we conclude that $\text{Rad}(TM) \oplus ltr(TM) \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)$ is non-degenerate and of constant index $2q$ on $\overline{M}$. Since $\text{index}(T\overline{M}) = \text{index}(\text{Rad}(TM) \oplus ltr(TM) \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)) + \text{index}\ (D \oplus_{orth} S(TM^\perp))$, we have $2q = 2q + \text{index}\ (D + S(TM^\perp))$. Thus, $D \oplus_{orth} S(TM^\perp)$ is Riemannian, i.e., $\text{index}\ (D \oplus_{orth} S(TM^\perp)) = 0$. Hence, D is Riemannian. □

Definition 3.1. Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then we say that M is a semi-slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

(i) $\tilde{P}\,\text{Rad}(TM)$ is a distribution on M such that $\text{Rad}(TM) \cap \tilde{P}\,\text{Rad}(TM) = \{0\}$;

(ii) there exist non-degenerate orthogonal complementary distributions D_1 and D_2 on M such that $S(TM) = (\tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2$;

(iii) the distribution D_1 is an invariant distribution, i.e., $\tilde{P}D_1 = D_1$;

(iv) the distribution D_2 is slant with angle $\theta (\neq 0)$, i.e., for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\tilde{P}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called slant angle of distribution D_2. A semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = \mathrm{Rad}(TM) \oplus_{orth} (\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2. \tag{3.1}$$

Now, for any vector field X tangent to M, we put

$$\tilde{P}X = PX + FX, \tag{3.2}$$

where PX and FX are tangential and transversal parts of $\tilde{P}X$, respectively. Also for any $V \in \Gamma(\mathrm{tr}(TM))$, we write

$$\tilde{P}V = BV + CV, \tag{3.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$, respectively.

We denote the projections on $\mathrm{Rad}(TM)$, $\tilde{P}\,\mathrm{Rad}(TM)$, $\tilde{P}ltr(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4 and P_5, respectively. Similarly, we denote the projections of $\mathrm{tr}(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ by Q_1 and Q_2, respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X. \tag{3.4}$$

Now applying $\tilde{P}$ to (3.4), we have

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + \tilde{P}P_4X + \tilde{P}P_5X, \tag{3.5}$$

which gives

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + \tilde{P}P_4X + PP_5X + FP_5X, \tag{3.6}$$

where $\tilde{P}P_2X = K_1\tilde{P}P_2X + K_2\tilde{P}P_2X$, $\tilde{P}P_3X = L_1\tilde{P}P_3X + L_2\tilde{P}P_3X$ and PP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of $\tilde{P}P_5X$. Thus, we get $\tilde{P}P_1X \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, $K_1\tilde{P}P_2X \in \Gamma(\mathrm{Rad}(TM))$, $K_2\tilde{P}P_2X \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, $L_1\tilde{P}P_3X \in \Gamma(ltr(TM))$, $L_2\tilde{P}P_3X \in \Gamma(\tilde{P}ltr(TM))$, $\tilde{P}P_4X \in \Gamma(\tilde{P}D_1)$, $PP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(S(TM^{\perp}))$. Also, for any $W \in \Gamma(\mathrm{tr}(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.7}$$

Applying $\tilde{P}$ to (3.7), we obtain

$$\tilde{P}W = \tilde{P}Q_1W + \tilde{P}Q_2W, \tag{3.8}$$

which gives

$$\tilde{P}W = \tilde{P}Q_1W + BQ_2W + CQ_2W, \tag{3.9}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\tilde{P}Q_2W$. Thus, we get $\tilde{P}Q_1W \in \Gamma(\tilde{P}ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^{\perp}))$.

Proposition 3.1. *There exist no isotropic or totally lightlike proper semi-slant lightlike submanifolds of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g}, \tilde{P})$.

Proof. We suppose that M is isotropic or totally lightlike, then $S(TM) = 0$, hence $D_1 = 0$ and $D_2 = 0$. ☐

Lemma 3.3. *Let (M, g) be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then we have*

$$(\nabla_X P)Y = A_{FY}X + Bh(X,Y), \tag{3.10}$$

$$(\nabla^t_X F)Y = Ch(X,Y) - h(X,PY), \tag{3.11}$$

$$P^2X = PX + X - BFX, \tag{3.12}$$

$$FX = FPX + CFX, \tag{3.13}$$

$$PBV = BV - BCV, \tag{3.14}$$

$$C^2V = CV + V - FBV, \tag{3.15}$$

$$g(PX,Y) - g(X,PY) = g(X,FY) - g(FX,Y), \tag{3.16}$$

$$\begin{aligned} g(PX,PY) =& g(PX,Y) + g(X,Y) + g(FX,Y) - g(PX,FY) \\ & - g(FX,PY) - g(FX,FY), \end{aligned} \tag{3.17}$$

where $(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y$ *and* $(\nabla^t_X F)Y = \nabla^t_X FY - F\nabla_X Y$ *for all* $X, Y \in \Gamma(TM)$ *and* $V \in \Gamma(\mathrm{tr}(TM))$.

Proof. Using (2.3), (2.9), (2.10), (3.2) and (3.3), on comparing tangential and transversal parts of the resulting equation, we obtain (3.10) and (3.11). Applying $\tilde{P}$ to (3.2), using (2.1) and (3.2), taking tangential and transversal parts of the resulting equation, we get (3.12) and (3.13). Applying $\tilde{P}$ to (3.3), using (2.1) and (3.3), taking tangential and transversal parts of the resulting equation, we get (3.14) and (3.15). Finally, using (2.2), (2.4) and (3.2), we obtain (3.16) and (3.17). □

Proposition 3.2. *Let (M, g) be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then P is a golden structure on M if and only if $FX = 0$.*

Proof. Let P is a golden structure on M then, from (3.12), $FX = 0$. Conversely, let $FX = 0$. Then our result follows from (3.12). □

Example 3.1. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.

Taking, $\tilde{P}(\partial x^1, \ldots, \partial x^{12}) = ((1-\psi)\partial x^1, \psi\partial x^2, \psi\partial x^3, (1-\psi)\partial x^4, (1-\psi)\partial x^5, \psi\partial x^6, (1-\psi)\partial x^7, \psi\partial x^8, \psi\partial x^9, \psi\partial x^{10}, (1-\psi)\partial x^{11}, (1-\psi)\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus, $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = \psi u^1 + u^2 - u^3$, $x^2 = u^1 - \psi u^2 + \psi u^3$, $x^3 = u^1 + \psi u^2 + \psi u^3$, $x^4 = \psi u^1 - u^2 - u^3$, $x^5 = \psi u^4$, $x^6 = \psi u^5$, $x^7 = (1-\psi)u^4$, $x^8 = (1-\psi)u^5$, $x^9 = \psi u^6$, $x^{10} = \psi u^7$, $x^{11} = (1-\psi)u^6$, $x^{12} = (1-\psi)u^7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where $Z_1 = \psi\partial x^1 + \partial x^2 + \partial x^3 + \psi\partial x^4$, $Z_2 = \partial x^1 - \psi\partial x^2 + \psi\partial x^3 - \partial x^4$, $Z_3 = -\partial x^1 + \psi\partial x^2 + \psi\partial x^3 - \partial x^4$,

$Z_4 = \psi\partial x^5 + (1-\psi)\partial x^7$, $Z_5 = \psi\partial x^6 + (1-\psi)\partial x^8$, $Z_6 = \psi\partial x^9 + (1-\psi)\partial x^{11}$ and $Z_7 = \psi\partial x^{10} + (1-\psi)\partial x^{12}$.

Hence $\mathrm{Rad}(TM) = \mathrm{Span}\,\{Z_1\}$ and $S(TM) = \mathrm{Span}\,\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $ltr(TM)$ is spanned by $N_1 = \frac{1}{2(2+\psi)}(-\psi\partial x^1 - \partial x^2 + \partial x^3 + \psi\partial x^4)$ and $S(TM^\perp)$ is spanned by $W_1 = (1-\psi)\partial x^5 - \psi\partial x^7$, $W_2 = (1-\psi)\partial x^6 - \psi\partial x^8$, $W_3 = (1-\psi)\partial x^9 - \psi\partial x^{11}$ and $W_4 = (1-\psi)\partial x^{10} - \psi\partial x^{12}$.

It follows that $\tilde{P}Z_1 = Z_3$, $\tilde{P}N_1 = Z_2$ and $\tilde{P}Z_4 = (1-\psi)Z_4$, $\tilde{P}Z_5 = \psi Z_5$, which implies D_1 is invariant, i.e., $\tilde{P}D_1 = D_1$ and $D_1 = \mathrm{Span}\{Z_4, Z_5\}$ and distribution $D_2 = \mathrm{Span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(\frac{4}{\sqrt{21}})$. Hence M is a semi-slant 1-lightlike submanifold of $\mathbb{R}^{12}_2$.

Example 3.2. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(+,-,+,-,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.

Taking, $\tilde{P}(\partial x^1, \dots, \partial x^{12}) = (\psi\partial x^1, \psi\partial x^2, (1-\psi)\partial x^3, (1-\psi)\partial x^4, \psi\partial x^5, \psi\partial x^6, (1-\psi)\partial x^7, (1-\psi)\partial x^8, (1-\psi)\partial x^9, \psi\partial x^{10}, (1-\psi)\partial x^{11}, \psi\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u^1 + \psi u^2 - \psi u^3$, $x^2 = u^1 + \psi u^2 + \psi u^3$, $x^3 = \psi u^1 - u^2 + u^3$, $x^4 = \psi u^1 - u^2 - u^3$, $x^5 = \psi u^4$, $x^6 = (1-\psi)u^4$, $x^7 = \psi u^5$, $x^8 = (1-\psi)u^5$, $x^9 = \psi u^6$, $x^{10} = (1-\psi)u^6$, $x^{11} = \psi u^7$, $x^{12} = (1-\psi)u^7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where $Z_1 = \partial x^1 + \partial x^2 + \psi\partial x^3 + \psi\partial x^4$, $Z_2 = \psi\partial x^1 + \psi\partial x^2 - \partial x^3 - \partial x^4$, $Z_3 = -\psi\partial x^1 + \psi\partial x^2 + \partial x^3 - \partial x^4$, $Z_4 = \psi\partial x^5 + (1-\psi)\partial x^6$, $Z_5 = \psi\partial x^7 + (1-\psi)\partial x^8$, $Z_6 = \psi\partial x^9 + (1-\psi)\partial x^{10}$, $Z_7 = \psi\partial x^{11} + (1-\psi)\partial x^{12}$.

Hence, $\mathrm{Rad}(TM) = \mathrm{Span}\,\{Z_1\}$ and $S(TM) = \mathrm{Span}\,\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $ltr(TM)$ is spanned by $N_1 = \frac{1}{2(2+\psi)}(-\partial x^1 + \partial x^2 - \psi\partial x^3 + \psi\partial x^4)$ and $S(TM^\perp)$ is spanned by $W_1 = (1-\psi)\partial x^5 - \psi\partial x^6$, $W_2 = (1-\psi)\partial x^7 - \psi\partial x^8$, $W_3 = (1-\psi)\partial x^9 - \psi\partial x^{10}$, $W_4 = (1-\psi)\partial x^{11} - \psi\partial x^{12}$.

It follows that $\tilde{P}Z_1 = Z_2$, $\tilde{P}N_1 = Z_3$ and $\tilde{P}Z_4 = \psi Z_4$, $\tilde{P}Z_5 = (1-\psi)Z_5$, which implies D_1 is invariant, i.e., $\tilde{P}D_1 = D_1$ and $D_1 = \mathrm{Span}\{Z_4, Z_5\}$ and distribution $D_2 = \mathrm{Span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{6})$. Hence M is a semi-slant 1-lightlike submanifold of $\mathbb{R}^{12}_2$.

Theorem 3.1. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Then M is a semi-slant lightlike submanifold of $\overline{M}$ if and only if*

(i) $\tilde{P}\,\mathrm{Rad}(TM)$ *is a distribution on M such that* $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = 0$*;*

(ii) *the screen distribution $S(TM)$ split as* $S(TM) = (\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2$*;*

(iii) *there exists a constant $\lambda \in [0,1)$ such that $P^2X = \lambda(\tilde{P}X + X)$;*

for all $X \in \Gamma(D_2)$. Moreover, in this case $\lambda = \cos^2\theta$ and θ is the slant angle of D_2.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the distribution $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM)\cap\tilde{P}\,\mathrm{Rad}(TM)=0$ and $S(TM)=(\tilde{P}\,\mathrm{Rad}(TM)\oplus\tilde{P}ltr(TM))\oplus_{orth}D_1\oplus_{orth}D_2$.

Now for any $X\in\Gamma(D_2)$, we have $|PX|=|\tilde{P}X|\cos\theta$, which implies

$$\cos\theta=\frac{|PX|}{|\tilde{P}X|}. \tag{3.18}$$

In view of (3.18), we get $\cos^2\theta=\frac{|PX|^2}{|\tilde{P}X|^2}=\frac{g(PX,PX)}{g(\tilde{P}X,\tilde{P}X)}=\frac{g(X,P^2X)}{g(X,\tilde{P}^2X)}$, which gives

$$g(X,P^2X)=\cos^2\theta\, g(X,\tilde{P}^2X). \tag{3.19}$$

Since M is a semi-slant lightlike submanifold, $\cos^2\theta=\lambda$ (constant) $\in[0,1)$ and therefore from (3.19), we get $g(X,P^2X)=\lambda g(X,\tilde{P}^2X)=g(X,\lambda\tilde{P}^2X)$, which implies

$$g(X,(P^2-\lambda\tilde{P}^2)X)=0. \tag{3.20}$$

Since $(P^2-\lambda\tilde{P}^2)X\in\Gamma(D_2)$ and the induced metric $g=g|_{D_2\times D_2}$ is non-degenerate (positive definite), from (3.20), we have $(P^2-\lambda\tilde{P}^2)X=0$, which implies

$$P^2X=\lambda\tilde{P}^2X=\lambda(\tilde{P}X+X), \tag{3.21}$$

for all $X\in\Gamma(D_2)$. This proves (iii).

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. From (iii), we have $P^2X=\lambda\tilde{P}^2X$, for all $X\in\Gamma(D_2)$, where λ (constant) $\in[0,1)$.

Now

$$\cos\theta=\frac{g(\tilde{P}X,PX)}{|\tilde{P}X||PX|}=\frac{g(X,\tilde{P}PX)}{|\tilde{P}X||PX|}=\frac{g(X,P^2X)}{|\tilde{P}X||PX|}=\lambda\frac{g(X,\tilde{P}^2X)}{|\tilde{P}X||PX|}=\lambda\frac{g(\tilde{P}X,\tilde{P}X)}{|\tilde{P}X||PX|}.$$

From above equation, we get

$$\cos\theta=\lambda\frac{|\tilde{P}X|}{|PX|}. \tag{3.22}$$

Therefore, (3.18) and (3.22) give $\cos^2\theta=\lambda$ (constant). Hence, M is a semi-slant lightlike submanifold. □

Corollary 3.1. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$ with slant angle θ, then for any $X,Y\in\Gamma(D_2)$, we have*

(i) $g(PX,PY)=\cos^2\theta(g(X,Y)+g(X,PY))$;

(ii) $g(FX,FY)=\sin^2\theta(g(X,Y)+g(PX,Y))$.

Proof. From (2.2), (3.2) and (3.21), we obtain

$$g(PX,PY)=g(X,\lambda(\tilde{P}Y+Y))=\cos^2\theta(g(X,Y)+g(X,PY)).$$

Moreover, from (2.2), (3.2) and (i) part of Corollary 3.1, we get

$$g(FX,FY)=g(X,Y)+g(PX,Y)-g(PX,PY)=\sin^2\theta(g(X,Y)+g(PX,Y)).$$

Hence, the proof is complete. □

Theorem 3.2. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then* $\mathrm{Rad}(TM)$ *is integrable if and only if the following conditions hold:*

(i) $\overline{g}(h^l(X,\tilde{P}Y),\xi)=\overline{g}(h^l(Y,\tilde{P}X),\xi)$;

(ii) $\overline{g}(h^*(X,\tilde{P}Y),N)=\overline{g}(h^*(Y,\tilde{P}X),N)$;

(iii) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}Z_1)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z_1)$;

(iv) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z)$,

for any $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, $\mathrm{Rad}(TM)$ is integrable if and only if $\overline{g}([X,Y],\tilde{P}\xi)=\overline{g}([X,Y],\tilde{P}N)=\overline{g}([X,Y],Z_1)=\overline{g}([X,Y],Z)=0$, for all $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$. Then from (2.4), (2.11), (2.16) and (3.6), we obtain

$$\begin{aligned} g([X,Y],\tilde{P}\xi)&=\overline{g}(\overline{\nabla}_XY-\overline{\nabla}_YX,\tilde{P}\xi)=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\xi)\\ &=\overline{g}(h^l(X,\tilde{P}Y)-h^l(Y,\tilde{P}X),\xi), \end{aligned} \tag{3.23}$$

$$\begin{aligned} g([X,Y],\tilde{P}N)&=\overline{g}(\overline{\nabla}_XY-\overline{\nabla}_YX,\tilde{P}N)=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\ &=\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N), \end{aligned} \tag{3.24}$$

$$\begin{aligned} g([X,Y],Z_1)&=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z_1)-\overline{g}(\tilde{P}[X,Y],Z_1)\\ &=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}Z_1)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z_1)\\ &=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}Z_1)\\ &\quad-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z_1), \end{aligned} \tag{3.25}$$

$$\begin{aligned} g([X,Y],Z)&=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)\\ &=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,PZ+FZ)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z)\\ &=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)\\ &\quad-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z). \end{aligned} \tag{3.26}$$

From (3.23), (3.24), (3.25) and (3.26), we derive our theorem. □

Theorem 3.3. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then $\tilde{P}\,\mathrm{Rad}(TM)$ is integrable if and only if the following conditions hold:*

(i) $\overline{g}(h^l(\tilde{P}X,Y),\xi)=\overline{g}(h^l(\tilde{P}Y,X),\xi)$;

(ii) $\overline{g}(A^*_X\tilde{P}Y,\tilde{P}Z_1)=\overline{g}(A^*_Y\tilde{P}X,\tilde{P}Z_1)$;

(iii) $\overline{g}(A^*_X\tilde{P}Y-A^*_Y\tilde{P}X,PZ)=\overline{g}(h^s(\tilde{P}Y,X)-h^s(\tilde{P}X,Y),FZ)$;

(iv) $\overline{g}(A_N\tilde{P}X,\tilde{P}Y)=\overline{g}(A_N\tilde{P}Y,\tilde{P}X)$,

for any $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, $\tilde{P}\,\mathrm{Rad}(TM)$ is integrable if and only if $\overline{g}([\tilde{P}X,\tilde{P}Y],\tilde{P}\xi)=\overline{g}([\tilde{P}X,\tilde{P}Y],Z_1)=\overline{g}([\tilde{P}X,\tilde{P}Y],Z)=$

$\overline{g}([\tilde{P}X, \tilde{P}Y],$
$N) = 0$, for all $X, Y, \xi \in \Gamma(\mathrm{Rad}(TM))$, $Z_1 \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$. Since $\overline{\nabla}$ is metric connection and using (2.4), (2.11), (2.12), (2.17) and (3.6), we obtain

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], \tilde{P}\xi) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}\tilde{P}Y - \overline{\nabla}_{\tilde{P}Y}\tilde{P}X, \tilde{P}\xi) \\ =& \overline{g}(h^l(\tilde{P}X, Y) - h^l(\tilde{P}Y, X), \xi), \end{aligned} \tag{3.27}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z_1) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}Y, \tilde{P}Z_1) - \overline{g}(\overline{\nabla}_{\tilde{P}Y}X, \tilde{P}Z_1) \\ =& \overline{g}(A^*_X\tilde{P}Y, \tilde{P}Z_1) - \overline{g}(A^*_Y\tilde{P}X, \tilde{P}Z_1), \end{aligned} \tag{3.28}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}Y, PZ + FZ) - \overline{g}(\overline{\nabla}_{\tilde{P}Y}X, PZ + FZ) \\ =& \overline{g}(A^*_X\tilde{P}Y - A^*_Y\tilde{P}X, PZ) - \overline{g}(h^s(\tilde{P}Y, X) - h^s(\tilde{P}X, Y), FZ), \end{aligned} \tag{3.29}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z) =& -\overline{g}(\tilde{P}Y, \overline{\nabla}_{\tilde{P}X}N) + \overline{g}(\tilde{P}X, \overline{\nabla}_{\tilde{P}Y}N) \\ =& \overline{g}(A_N\tilde{P}X, \tilde{P}Y) - \overline{g}(A_N\tilde{P}Y, \tilde{P}X). \end{aligned} \tag{3.30}$$

From (3.27), (3.28), (3.29) and (3.30), proof is completed. □

Theorem 3.4. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then $\tilde{P}ltr(TM)$ is integrable if and only if the following conditions hold:*

(i) $\overline{g}(A_{N_1}\tilde{P}N_2, N) = \overline{g}(A_{N_2}\tilde{P}N_1, N)$*;*

(ii) $\overline{g}(A_{N_1}\tilde{P}N_2, \tilde{P}Z_1) = \overline{g}(A_{N_2}\tilde{P}N_1, \tilde{P}Z_1)$*;*

(iii) $\overline{g}(A_{N_1}\tilde{P}N_2 - A_{N_2}\tilde{P}N_1, PZ) = \overline{g}(D^s(\tilde{P}N_2, N_1) - D^s(\tilde{P}N_1, N_2), FZ)$*;*

(iv) $\overline{g}(A_N\tilde{P}N_1, \tilde{P}N_2) = \overline{g}(A_N\tilde{P}N_2, \tilde{P}N_1)$,

for any $N_1, N_2, N \in \Gamma(ltr(TM))$, $Z_1 \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. From the definition of semi-slant lightlike submanifolds, $\tilde{P}ltr(TM)$ is integrable if and only if $\overline{g}([\tilde{P}N_1, \tilde{P}N_2], \tilde{P}N) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], Z_1) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], Z) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], N) = 0$, for any $N_1, N_2, N \in \Gamma(ltr(TM))$, $Z_1 \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Taking $\overline{\nabla}$ is metric connection and from (2.4), (2.11), (2.12), (2.16) and (3.6), we obtain

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], \tilde{P}N) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}\tilde{P}N_2 - \overline{\nabla}_{\tilde{P}N_2}\tilde{P}N_1, \tilde{P}N) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2, N) - \overline{g}(A_{N_2}\tilde{P}N_1, N), \end{aligned} \tag{3.31}$$

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], Z_1) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}N_2, \tilde{P}Z_1) - \overline{g}(\overline{\nabla}_{\tilde{P}N_2}N_1, \tilde{P}Z_1) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2, \tilde{P}Z_1) - \overline{g}(A_{N_2}\tilde{P}N_1, \tilde{P}Z_1), \end{aligned} \tag{3.32}$$

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], Z) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}N_2, PZ + FZ) - \overline{g}(\overline{\nabla}_{\tilde{P}N_2}N_1, PZ + FZ) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2 - A_{N_2}\tilde{P}N_1, PZ) \\ & - \overline{g}(D^s(\tilde{P}N_2, N_1) - D^s(\tilde{P}N_1, N_2), FZ), \end{aligned} \tag{3.33}$$

$$g([\tilde{P}N_1, \tilde{P}N_2], N) = -\overline{g}(\tilde{P}N_2, \overline{\nabla}_{\tilde{P}N_1}N) + \overline{g}(\tilde{P}N_1, \overline{\nabla}_{\tilde{P}N_2}N)$$

$$=\overline{g}(A_N\tilde{P}N_1,\tilde{P}N_2)-\overline{g}(A_N\tilde{P}N_2,\tilde{P}N_1). \tag{3.34}$$

From (3.31), (3.32), (3.33) and (3.34), we derive our theorem. □

Theorem 3.5. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_1 is integrable if and only if the following conditions hold:*
(i) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z)$;
(ii) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}N)=\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N)$;
(iii) $\overline{g}(A_NX,\tilde{P}Y)=\overline{g}(A_NY,\tilde{P}X)$,
for any $X,Y\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, D_1 is integrable if and only if $\overline{g}([X,Y],Z)=\overline{g}([X,Y],N)=\overline{g}([X,Y],\tilde{P}N)=0$ for all $X,Y\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$. Since $\overline{\nabla}$ is metric connection and from (2.4), (2.11), (2.12), (2.16) and (3.6), we obtain

$$\begin{aligned}g([X,Y],Z)=&\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)\\=&\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,PZ+FZ)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z)\\=&\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)\\&-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z),\end{aligned} \tag{3.35}$$

$$\begin{aligned}g([X,Y],N)=&\overline{g}(\tilde{P}[X,Y],\tilde{P}N)-\overline{g}(\tilde{P}[X,Y],N)\\=&\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}N)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\=&\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}N)-\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N),\end{aligned} \tag{3.36}$$

$$\begin{aligned}g([X,Y],\tilde{P}N)=&-\overline{g}(\overline{\nabla}_XN,\tilde{P}Y)+\overline{g}(\overline{\nabla}_YN,\tilde{P}X)\\=&\overline{g}(A_NX,\tilde{P}Y)-\overline{g}(A_NY,\tilde{P}X).\end{aligned} \tag{3.37}$$

From (3.35), (3.36) and (3.37), proof is completed. □

Theorem 3.6. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_2 is integrable if and only if the following conditions hold:*
(i) $\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}Z)+\overline{g}(\nabla_YPX-A_{FX}Y,Z)=\overline{g}(\nabla_XPY-A_{FY}X,Z)+\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}Z)$;
(ii) $\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}N)+\overline{g}(\nabla_YPX-A_{FX}Y,N)=\overline{g}(\nabla_XPY-A_{FY}X,N)+\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}N)$;
(iii) $\overline{g}(\nabla_XPY-A_{FY}X,N)=\overline{g}(\nabla_YPX-A_{FX}Y,N)$,
for any $X,Y\in\Gamma(D_2)$, $Z\in\Gamma(D_1)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, D_2 is integrable if and only if $\overline{g}([X,Y],Z)=\overline{g}([X,Y],N)=\overline{g}([X,Y],\tilde{P}N)=0$ for all $X,Y\in\Gamma(D_2)$, $Z\in\Gamma(D_1)$ and $N\in\Gamma(ltr(TM))$. Then from (2.4), (2.11), (2.13) and (3.6), we obtain

$$g([X,Y],Z)=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)$$

$$
\begin{aligned}
&=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}Z)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z)\\
&=\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}Z)+\overline{g}(\nabla_YPX-A_{FX}Y,Z)\\
&\quad-\overline{g}(\nabla_XPY-A_{FY}X,Z)-\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}Z), \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
g([X,Y],N)&=\overline{g}(\tilde{P}[X,Y],\tilde{P}N)-\overline{g}(\tilde{P}[X,Y],N)\\
&=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}N)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\
&=\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}N)+\overline{g}(\nabla_YPX-A_{FX}Y,N)\\
&\quad-\overline{g}(\nabla_XPY-A_{FY}X,N)-\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}N), \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
g([X,Y],\tilde{P}N)&=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\
&=\overline{g}(\nabla_XPY-A_{FY}X,N)-\overline{g}(\nabla_YPX-A_{FX}Y,N). \qquad (3.40)
\end{aligned}
$$

From (3.38), (3.39) and (3.40), we derive our theorem. □

Theorem 3.7. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if*

(i) $P\nabla_X\tilde{P}Y\in\Gamma(\mathrm{Rad}(TM))$;
(ii) $Bh^l(X,\tilde{P}Y)=P_3\nabla_X\tilde{P}Y$ *and* $P_2\nabla_X\tilde{P}Y=0$;
(iii) $Bh^s(X,\tilde{P}Y)=P_5\nabla_X\tilde{P}Y$ *and* $P_4\nabla_X\tilde{P}Y=0$,

for all $X\in\Gamma(TM)$ and $Y\in\Gamma(\mathrm{Rad}(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $\mathrm{Rad}(TM)$ is parallel distribution with respect to ∇ [4]. For any $X\in\Gamma(TM)$ and $Y\in\Gamma(\mathrm{Rad}(TM))$, we have $\overline{\nabla}_XY=\tilde{P}\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_X\tilde{P}Y$, using (2.11), (3.2) and (3.9), we get $\overline{\nabla}_XY=P\nabla_X\tilde{P}Y+F\nabla_X\tilde{P}Y+Bh^l(X,\tilde{P}Y)+Bh^s(X,\tilde{P}Y)+Ch^s(X,\tilde{P}Y)-\nabla_X\tilde{P}Y-h^l(X,\tilde{P}Y)-h^s(X,\tilde{P}Y)$. By comparing tangential components of both sides of above equation, we obtain $\nabla_XY=P\nabla_X\tilde{P}Y+Bh^l(X,\tilde{P}Y)+Bh^s(X,\tilde{P}Y)-P_1\nabla_X\tilde{P}Y-P_2\nabla_X\tilde{P}Y-P_3\nabla_X\tilde{P}Y-P_4\nabla_X\tilde{P}Y-P_5\nabla_X\tilde{P}Y$, which completes the proof. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a semi-slant lightlike submanifold of a golden semi-Riemannian manifold to be totally geodesic.

Definition 4.1 ([5])**.** A semi-slant lightlike submanifold M of a golden semi-Riemannian manifold $\overline{M}$ is said to be a mixed geodesic if its second fundamental form h satisfies $h(X,Y)=0$, for all $X\in\Gamma(D_1)$ and $Y\in\Gamma(D_2)$. Thus M is mixed geodesic semi-slant lightlike submanifold if $h^l(X,Y)=0$ and $h^s(X,Y)=0$, for all $X\in\Gamma(D_1)$ and $Y\in\Gamma(D_2)$.

Theorem 4.1. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then* $\mathrm{Rad}(TM)$ *defines a totally geodesic foliation if and only* *if* $\overline{g}(h^l(X, K_1\tilde{P}P_2Z)+h^l(X, K_2\tilde{P}P_2Z)+\nabla^l_X L_1\tilde{P}P_3Z+h^l(X, L_2\tilde{P}P_3Z)+h^l(X, \tilde{P}P_4Z)+h^l(X, PP_5Z)+D^l(X, FP_5Z), Y) = \overline{g}(\nabla_X K_1\tilde{P}P_2Z+\nabla_X K_2\tilde{P}P_2Z-A_{L_1\tilde{P}P_3Z}X+\nabla_X L_2\tilde{P}P_3Z + \nabla_X\tilde{P}P_4Z + \nabla_X PP_5Z - A_{FP_5Z}X, \tilde{P}Y)$ *for all* $X, Y \in \Gamma(\mathrm{Rad}(TM))$ *and* $Z \in \Gamma(S(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution $\mathrm{Rad}(TM)$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(\mathrm{Rad}(TM))$, for all $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Since $\overline{\nabla}$ is a metric connection, using (2.4), (2.11), (2.12), (2.13) and (3.6), for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = \overline{g}(Y, \overline{\nabla}_X\tilde{P}P_2Z + \overline{\nabla}_X\tilde{P}P_3Z + \overline{\nabla}_X\tilde{P}P_4Z + \overline{\nabla}_X PP_5Z + \overline{\nabla}_X FP_5Z) - \overline{g}(\tilde{P}Y, \overline{\nabla}_X\tilde{P}P_2Z+\overline{\nabla}_X\tilde{P}P_3Z+\overline{\nabla}_X\tilde{P}P_4Z+\overline{\nabla}_X PP_5Z+\overline{\nabla}_X FP_5Z)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(h^l(X, K_1\tilde{P}P_2Z)+h^l(X, K_2\tilde{P}P_2Z)+\nabla^l_X L_1\tilde{P}P_3Z+h^l(X, L_2\tilde{P}P_3Z) + h^l(X, \tilde{P}P_4Z) + h^l(X, PP_5Z) + D^l(X, FP_5Z), Y) - \overline{g}(\nabla_X K_1\tilde{P}P_2Z + \nabla_X K_2\tilde{P}P_2Z - A_{L_1\tilde{P}P_3Z}X + \nabla_X L_2\tilde{P}P_3Z + \nabla_X\tilde{P}P_4Z + \nabla_X PP_5Z - A_{FP_5Z}X, \tilde{P}Y)$. Thus, the theorem is completed. □

Theorem 4.2. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X PZ - A_{FZ}X, \tilde{P}Y) = \overline{g}(\nabla_X PZ - A_{FZ}X, Y)$;

(ii) $\overline{g}(\nabla^*_X\tilde{P}Y, \tilde{P}N) = \overline{g}(h^*(X, \tilde{P}Y), N)$;

(iii) $h^*(X, \tilde{P}Y)$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ *and* $N \in \Gamma ltr(TM)$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, from (2.4), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ) + \overline{g}(Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \nabla_X PZ - A_{FZ}X) + \overline{g}(Y, \nabla_X PZ - A_{FZ}X)$. From (2.4), (2.11) and (2.16), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X\tilde{P}Y, \tilde{P}N) - \overline{g}(\overline{\nabla}_X\tilde{P}Y, N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla^*_X\tilde{P}Y, \tilde{P}N) - \overline{g}(h^*(X, \tilde{P}Y), N)$. Now, from (2.2), (2.11) and (2.16), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X\tilde{P}Y, N)$, which implies $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(h^*(X, \tilde{P}Y), N)$. This proves the theorem. □

Theorem 4.3. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X\tilde{P}Z, Y) - \overline{g}(PY, \nabla_X\tilde{P}Z) = \overline{g}(FY, h^s(X, \tilde{P}Z))$;

(ii) $\overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$;

(iii) $\nabla_X PY - A_{FY}X$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in$

$\Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z))$. In view of (2.4), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, \tilde{P}N) - \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}N) - \overline{g}(\nabla_X PY - A_{FY}X, N)$. Now, from (2.2), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$. Hence, the proof is completed. ☐

Theorem 4.4. *Let M be a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(PY, \nabla_X \tilde{P}Z) = \overline{g}(Y, \nabla_X \tilde{P}Z)$;

(ii) $\overline{g}(\nabla_X \tilde{P}N, Y) - \overline{g}(\nabla_X \tilde{P}N, PY) = \overline{g}(h^s(X, \tilde{P}N), FY)$;

(iii) $\nabla_X PY - A_{FY}X$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifolds $\overline{M}$, we have $h^s(X, \tilde{P}Z) = 0$, for all $X \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z)$. From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}N) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}N) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X \tilde{P}N, Y) - \overline{g}(\nabla_X \tilde{P}N, PY) - \overline{g}(h^s(X, \tilde{P}N), FY)$. Now, from (2.2), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$. Hence, the proof is completed. ☐

Theorem 4.5. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then M is mixed geodesic if and only if the following hold:*

(i) $F(\nabla_X PZ - A_{FZ}X) = -C(h^s(X, PZ) + \nabla^s_X FZ)$;

(ii) $h^l(X, PZ) + D^l(X, FZ) = h^s(X, PZ) + \nabla^s_X FZ$,

for any $X \in \Gamma(D_1)$ *and* $Z \in \Gamma(D_2)$.

Proof. From (2.9), (2.11), (2.13), (3.2), (3.6) and (3.9), we obtain

$$\begin{aligned} h(X,Z) =& \tilde{P}(\nabla_X PZ + h^l(X,PZ) + h^s(X,PZ) - A_{FZ}X + \nabla^s_X FZ + D^l(X,FZ)) \\ & - (\nabla_X PZ + h^l(X,PZ) + h^s(X,PZ) - A_{FZ}X + \nabla^s_X FZ + D^l(X,FZ)) \\ & - \nabla_X Z. \end{aligned}$$

Taking transversal part of this equation, we get

$$\begin{aligned} h(X,Z) =& F(\nabla_X PZ - A_{FZ}X) + C(h^s(X,PZ) + \nabla^s_X FZ) - h^l(X,PZ) - h^s(X,PZ) \\ & - \nabla^s_X FZ - D^l(X,FZ). \end{aligned}$$

Hence, $h(X,Z) = 0$ if and only if (i) and (ii) hold. Hence, the proof is completed. □

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