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SEMI-SLANT LIGHTLIKE SUBMANIFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. The aim of our paper is to introduce the notion of semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. We give non-trivial examples of semi-slant lightlike submanifolds and provide a characterization theorem of such submanifolds. Further, we obtain necessary and sufficient conditions for integrability of the distributions and investigate the geometry of the leaves of the foliation determined by the distributions. We also obtain a necessary and sufficient condition for the induced connection to be a metric connection. Finally, we obtain necessary and sufficient condition for mixed-geodesic semi-slant lightlike submanifold of golden semi-Riemannian manifold.

1. Introduction

A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric on it is degenerate, i.e., there exists a non zero $X \in \Gamma(TM)$ such that $g(X, Z) = 0$ for all $Z \in \Gamma(TM)$. In [4], Duggal and Bejancu introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle and they studied the geometry of arbitrary lightlike submanifold of a semi-Riemannian manifold. Lightlike geometry has its applications in general relativity, particularly in black hole theory. Many authors have studied lightlike submanifolds in various spaces ([5, 17]). In [15], authors introduced a new class of lightlike submanifolds namely, semi-slant lightlike submanifolds of indefinite Kaehler manifolds. In [15], authors investigated the integrability of various distributions, obtained a characterization theorem of such lightlike submanifolds and

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established equivalent conditions for totally geodesic foliation of distributions. In [16], authors introduced a general notion of lightlike submanifolds namely, semi-slant lightlike submanifolds of indefinite Sasakian manifolds. In [16], authors found some equivalent conditions for integrability and totally geodesic foliation of distributions. Golden proportion ψ is the real positive root of the equation $x^2 - x - 1 = 0$ (thus $\psi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$). Inspired by the Golden proportion, Crasmareanu and Hretcanu defined golden structure $\tilde{P}$ which is a tensor field satisfying $\tilde{P}^2 - \tilde{P} - I = 0$ on $\overline{M}$ [3]. Golden structure was inspired by the Golden proportion, which was described by Kepler (1571–1630).

A Riemannian manifold $\overline{M}$ with a golden structure $\tilde{P}$ is called a golden Riemannian manifold and was studied in ([3,9]). In [9], authors studied invariant submanifolds of a golden Riemannian manifold. Submanifolds of golden manifolds in semi-Riemannian geometry were studied by Poyraz and Yasar [12]. In [12], they proved that there is no radical anti-invariant lightlike hypersurface of a golden semi-Riemannian manifold and also studied screen semi-invariant and screen conformal screen semi-invariant lightlike hypersurfaces of a golden semi-Riemannian manifold. Transversal and Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds were studied in ([6,8]). In [13], authors proved that there is no radical anti-invariant lightlike submanifold of a golden semi-Riemannian manifolds. In [7], author studies the geometry of screen transversal lightlike submanifolds and radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and investigate the geometry of distributions. Screen pseudo-slant and golden GCR-lightlike submanifolds of a golden semi-Riemannian manifold were studied in ([1, 11]). In [10], N. Onen Poyraz introduced screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifolds and found the conditions of integrability of distributions. In [10], they proved some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.

The purpose of this paper is to study semi-slant lightlike submanifold of golden semi-Riemannian manifolds. The paper is arranged as follows. In Section 2, some definitions and basic results about lightlike submanifolds and golden semi-Riemannian manifold are given. In Section 3, we study semi-slant lightlike submanifolds of a golden semi-Riemannian manifold giving examples, provide a characterization theorem and investigate the integrability of distributions. We also obtain necessary and sufficient conditions for semi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be metric connection. In Section 4, we find necessary and sufficient conditions for totally geodesic foliation determined by distributions on a semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. We also obtain necessary and sufficient conditions for semi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be mixed geodesic.

2. Preliminaries

Let $\overline{M}$ be a C^∞-differentiable manifold. If a $(1,1)$ type tensor field $\tilde{P}$ on $\overline{M}$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I, \tag{2.1}$$

then $\tilde{P}$ is called a golden structure on $\overline{M}$, where I is the identity transformation. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\overline{M}$. If $\tilde{P}$ satisfies the following equation

$$\overline{g}(\tilde{P}U, W) = \overline{g}(U, \tilde{P}W), \tag{2.2}$$

then $(\overline{M}, \overline{g}, \tilde{P})$ is called a golden semi-Riemannian manifold [14], also, if $\tilde{P}$ is integrable, then we have [3]

$$\overline{\nabla}_U \tilde{P}W = \tilde{P}\overline{\nabla}_U W. \tag{2.3}$$

Now, from (2.2) we get

$$\overline{g}(\tilde{P}U, \tilde{P}W) = \overline{g}(\tilde{P}U, W) + \overline{g}(U, W), \tag{2.4}$$

for all $U, W \in \Gamma(T\overline{M})$.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\overline{M}$, where g is the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold [4] of $\overline{M}$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is

$$TM = \mathrm{Rad}(TM) \oplus_{orth} S(TM). \tag{2.5}$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^\perp$. Let $\mathrm{tr}(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and $\mathrm{Rad}(TM)$ in $S(TM^\perp)^\perp$, respectively. Then

$$\mathrm{tr}(TM) = ltr(TM) \oplus_{orth} S(TM^\perp), \tag{2.6}$$

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM), \tag{2.7}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [\mathrm{Rad}(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^\perp). \tag{2.8}$$

Theorem 2.1 ([4]). *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1, 2, \dots, r\}$, is a basis of $\Gamma(\mathrm{Rad}(TM|_U))$. Then there exist a complementary vector subbundle $ltr(TM)$ of $\mathrm{Rad}(TM)$ in $S(TM^\perp)^\perp$ and a basis $\{N_i\}$, $i \in \{1, 2, \dots, r\}$, of $\Gamma(ltr(TM|_U))$ such that $\overline{g}(N_i, \xi_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$ for any $i, j \in \{1, 2, \dots, r\}$.*

Following are four cases of a lightlike submanifold $\left(M, g, S(TM), S(TM^{\perp})\right)$.

Case 1. r-lightlike if $r < \min(m, n)$.

Case 2. Co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$.

Case 3. Isotropic if $r = m < n$, $S(TM) = \{0\}$.

Case 4. Totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(\mathrm{tr}(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $\mathrm{tr}(TM)$, respectively. From (2.9) and (2.10), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.11}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.12}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.13}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $\mathrm{tr}(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M, respectively.

Also by using (2.9), (2.11)–(2.13) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.14}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{2.15}$$

Now, denote the projection of TM on $S(TM)$ by S. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, we have

$$\nabla_X SY = \nabla^*_X SY + h^*(X, SY), \tag{2.16}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.17}$$

By using above equations, we obtain

$$\overline{g}(h^l(X, SY), \xi) = g(A^*_\xi X, SY). \tag{2.18}$$

It is important to note that in general ∇ is not a metric connection on M. Since $\overline{\nabla}$ is metric connection, by using (2.11), we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y), \tag{2.19}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$.

Definition 2.1 ([2])**.** An equivalence relation on an n-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ in which the equivalence classes are connected, immersed submanifolds (called the leaves of the foliation) of a common dimension k, $0 < k \leq n$, is called a foliation on $\overline{M}$. If each leaf of a foliation F on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally geodesic submanifold of $\overline{M}$, we say that F is a totally geodesic foliation.

3. Semi Slant Lightlike Submanifolds

In this section, we study semi-slant lightlike submanifolds of golden semi Riemannian manifolds. We now give the following lemmas which will be useful to define slant notion on the screen distribution.

Lemma 3.1. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Suppose that $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = \{0\}$. Then $\tilde{P}ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$.*

Proof. Since by hypothesis $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\tilde{P}\,\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = 0$, we have $\tilde{P}\,\mathrm{Rad}(TM) \subset S(TM)$. Now we claim that $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Let us suppose that $ltr(TM)$ is invariant with respect to $\tilde{P}$. Choose $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma ltr(TM)$ such that $\overline{g}(N, \xi) = 1$. Then from (2.4), we have $1 = \overline{g}(\xi, N) = \overline{g}(\tilde{P}\xi, \tilde{P}N) - \overline{g}(\tilde{P}\xi, N) = 0$, due to $\tilde{P}\xi \in \Gamma S(TM)$ and $\tilde{P}N \in \Gamma ltr(TM)$. This is a contradiction, so $ltr(TM)$ is not invariant with respect to $\tilde{P}$. Also $\tilde{P}N$ does not belong to $S(TM^{\perp})$, since $S(TM^{\perp})$ is ortogonal to $S(TM)$, $\overline{g}(\tilde{P}N, \tilde{P}\xi)$ must be zero, but from (2.4) we have $\overline{g}(\tilde{P}N, \tilde{P}\xi) = \overline{g}(\tilde{P}\xi, N) + \overline{g}(N, \xi) \neq 0$, for some $\xi \in \Gamma\,\mathrm{Rad}(TM)$, this is again a contradiction. Thus, we conclude that $\tilde{P}ltr(TM)$ is a distribution on M. Moreover, $\tilde{P}N$ does not belong to $\mathrm{Rad}(TM)$. Indeed, if $\tilde{P}N \in \Gamma\,\mathrm{Rad}(TM)$, we would have $\tilde{P}^2N = \tilde{P}N + N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, but this is impossible. Finally, let $\tilde{P}N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, we obtain $\tilde{P}^2N = \tilde{P}N + N \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM) + \mathrm{Rad}(TM))$, this is not possible. Hence, $\tilde{P}N$ does not belong to $\tilde{P}\,\mathrm{Rad}(TM)$. Thus, we conclude that $\tilde{P}ltr(TM) \subset S(TM)$ and $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$. □

Lemma 3.2. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Suppose $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = \{0\}$. Then any complementary distribution to $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let M be an m-dimensional q-lightlike submanifold of an $(m+n)$-dimensional golden semi-Riemannian manifold $\overline{M}$ of index $2q$. From Lemma 3.1, we have $\tilde{P}\,\mathrm{Rad}(TM) \cap \tilde{P}ltr(TM) = \{0\}$ and $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM) \subset S(TM)$. We denote the complementary distribution to $\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)$ in $S(TM)$ by D. Then we have a local orthonormal frame of fields on $\overline{M}$ along M $\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i, X_\alpha, W_a\}$,

$i \in \{1, 2, \ldots, q\}, \alpha \in \{3q+1, \ldots, m\}, a \in \{q+1, \ldots, n\}$, where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $\text{Rad}(TM)$ and $ltrTM$, respectively and $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal bases of D and $S(TM^\perp)$, respectively.

Now, from the bases $\{\xi_1, \ldots, \xi_q, N_1, \ldots, N_q, \tilde{P}\xi_1, \ldots, \tilde{P}\xi_q, \tilde{P}N_1, \ldots, \tilde{P}N_q\}$ of $\text{Rad}(TM) \oplus ltrTM \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)$, we can construct an orthonormal bases $\{U_1, \ldots, U_{2q}, V_1, \ldots, V_{2q}\}$ as follows:

$$
\begin{aligned}
U_1 =& \frac{1}{\sqrt{2}}(\xi_1 + N_1), \quad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),\\
U_3 =& \frac{1}{\sqrt{2}}(\xi_2 + N_2), \quad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),\\
&\vdots\\
U_{2q-1} =& \frac{1}{\sqrt{2}}(\xi_q + N_q), \quad U_{2q} = \frac{1}{\sqrt{2}}(\xi_q - N_q),\\
V_1 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 + \tilde{P}N_1), \quad V_2 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_1 - \tilde{P}N_1),\\
V_3 =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 + \tilde{P}N_2), \quad V_4 = \frac{1}{\sqrt{2}}(\tilde{P}\xi_2 - \tilde{P}N_2),\\
&\vdots\\
V_{2q-1} =& \frac{1}{\sqrt{2}}(\tilde{P}\xi_q + \tilde{P}N_q), \quad V_{2q} = \frac{1}{\sqrt{2}}(\tilde{P}\xi_q - \tilde{P}N_q).
\end{aligned}
$$

Hence, $\text{Span}\{\xi_i, N_i, \tilde{P}\xi_i, \tilde{P}N_i\}$ is a non-degenerate space of constant index $2q$. Thus we conclude that $\text{Rad}(TM) \oplus ltr(TM) \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)$ is non-degenerate and of constant index $2q$ on $\overline{M}$. Since $\text{index}(T\overline{M}) = \text{index}(\text{Rad}(TM) \oplus ltr(TM) \oplus \tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)) + \text{index}\ (D \oplus_{orth} S(TM^\perp))$, we have $2q = 2q + \text{index}\ (D + S(TM^\perp))$. Thus, $D \oplus_{orth} S(TM^\perp)$ is Riemannian, i.e., $\text{index}\ (D \oplus_{orth} S(TM^\perp)) = 0$. Hence, D is Riemannian. □

Definition 3.1. Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$ such that $2q < \dim(M)$. Then we say that M is a semi-slant lightlike submanifold of $\overline{M}$ if following conditions are satisfied:

(i) $\tilde{P}\,\text{Rad}(TM)$ is a distribution on M such that $\text{Rad}(TM) \cap \tilde{P}\,\text{Rad}(TM) = \{0\}$;

(ii) there exist non-degenerate orthogonal complementary distributions D_1 and D_2 on M such that $S(TM) = (\tilde{P}\,\text{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2$;

(iii) the distribution D_1 is an invariant distribution, i.e., $\tilde{P}D_1 = D_1$;

(iv) the distribution D_2 is slant with angle $\theta (\neq 0)$, i.e., for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\tilde{P}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called slant angle of distribution D_2. A semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = \mathrm{Rad}(TM) \oplus_{orth} (\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2. \tag{3.1}$$

Now, for any vector field X tangent to M, we put

$$\tilde{P}X = PX + FX, \tag{3.2}$$

where PX and FX are tangential and transversal parts of $\tilde{P}X$, respectively. Also for any $V \in \Gamma(\mathrm{tr}(TM))$, we write

$$\tilde{P}V = BV + CV, \tag{3.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$, respectively.

We denote the projections on $\mathrm{Rad}(TM)$, $\tilde{P}\,\mathrm{Rad}(TM)$, $\tilde{P}ltr(TM)$, D_1 and D_2 in TM by P_1, P_2, P_3, P_4 and P_5, respectively. Similarly, we denote the projections of $\mathrm{tr}(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2, respectively. Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X. \tag{3.4}$$

Now applying $\tilde{P}$ to (3.4), we have

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + \tilde{P}P_4X + \tilde{P}P_5X, \tag{3.5}$$

which gives

$$\tilde{P}X = \tilde{P}P_1X + \tilde{P}P_2X + \tilde{P}P_3X + \tilde{P}P_4X + PP_5X + FP_5X, \tag{3.6}$$

where $\tilde{P}P_2X = K_1\tilde{P}P_2X + K_2\tilde{P}P_2X$, $\tilde{P}P_3X = L_1\tilde{P}P_3X + L_2\tilde{P}P_3X$ and PP_5X (resp. FP_5X) denotes the tangential (resp. transversal) component of $\tilde{P}P_5X$. Thus, we get $\tilde{P}P_1X \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, $K_1\tilde{P}P_2X \in \Gamma(\mathrm{Rad}(TM))$, $K_2\tilde{P}P_2X \in \Gamma(\tilde{P}\,\mathrm{Rad}(TM))$, $L_1\tilde{P}P_3X \in \Gamma(ltr(TM))$, $L_2\tilde{P}P_3X \in \Gamma(\tilde{P}ltr(TM))$, $\tilde{P}P_4X \in \Gamma(\tilde{P}D_1)$, $PP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(\mathrm{tr}(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.7}$$

Applying $\tilde{P}$ to (3.7), we obtain

$$\tilde{P}W = \tilde{P}Q_1W + \tilde{P}Q_2W, \tag{3.8}$$

which gives

$$\tilde{P}W = \tilde{P}Q_1W + BQ_2W + CQ_2W, \tag{3.9}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\tilde{P}Q_2W$. Thus, we get $\tilde{P}Q_1W \in \Gamma(\tilde{P}ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Proposition 3.1. *There exist no isotropic or totally lightlike proper semi-slant lightlike submanifolds of a golden semi-Riemannian manifold* $(\overline{M}, \overline{g}, \tilde{P})$.

Proof. We suppose that M is isotropic or totally lightlike, then $S(TM) = 0$, hence $D_1 = 0$ and $D_2 = 0$. □

Lemma 3.3. *Let (M, g) be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then we have*

$$(\nabla_X P)Y = A_{FY}X + Bh(X,Y), \tag{3.10}$$

$$(\nabla^t_X F)Y = Ch(X,Y) - h(X,PY), \tag{3.11}$$

$$P^2X = PX + X - BFX, \tag{3.12}$$

$$FX = FPX + CFX, \tag{3.13}$$

$$PBV = BV - BCV, \tag{3.14}$$

$$C^2V = CV + V - FBV, \tag{3.15}$$

$$g(PX,Y) - g(X,PY) = g(X,FY) - g(FX,Y), \tag{3.16}$$

$$\begin{aligned} g(PX,PY) =& g(PX,Y) + g(X,Y) + g(FX,Y) - g(PX,FY) \\ & - g(FX,PY) - g(FX,FY), \end{aligned} \tag{3.17}$$

where $(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y$ and $(\nabla^t_X F)Y = \nabla^t_X FY - F\nabla_X Y$ for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$.

Proof. Using (2.3), (2.9), (2.10), (3.2) and (3.3), on comparing tangential and transversal parts of the resulting equation, we obtain (3.10) and (3.11). Applying $\tilde{P}$ to (3.2), using (2.1) and (3.2), taking tangential and transversal parts of the resulting equation, we get (3.12) and (3.13). Applying $\tilde{P}$ to (3.3), using (2.1) and (3.3), taking tangential and transversal parts of the resulting equation, we get (3.14) and (3.15). Finally, using (2.2), (2.4) and (3.2), we obtain (3.16) and (3.17). □

Proposition 3.2. *Let (M, g) be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then P is a golden structure on M if and only if $FX = 0$.*

Proof. Let P is a golden structure on M then, from (3.12), $FX = 0$. Conversely, let $FX = 0$. Then our result follows from (3.12). □

Example 3.1. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.

Taking, $\tilde{P}(\partial x^1, \dots, \partial x^{12}) = ((1-\psi)\partial x^1, \psi\partial x^2, \psi\partial x^3, (1-\psi)\partial x^4, (1-\psi)\partial x^5, \psi\partial x^6, (1-\psi)\partial x^7, \psi\partial x^8, \psi\partial x^9, \psi\partial x^{10}, (1-\psi)\partial x^{11}, (1-\psi)\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus, $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = \psi u^1 + u^2 - u^3$, $x^2 = u^1 - \psi u^2 + \psi u^3$, $x^3 = u^1 + \psi u^2 + \psi u^3$, $x^4 = \psi u^1 - u^2 - u^3$, $x^5 = \psi u^4$, $x^6 = \psi u^5$, $x^7 = (1-\psi)u^4$, $x^8 = (1-\psi)u^5$, $x^9 = \psi u^6$, $x^{10} = \psi u^7$, $x^{11} = (1-\psi)u^6$, $x^{12} = (1-\psi)u^7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where $Z_1 = \psi\partial x^1 + \partial x^2 + \partial x^3 + \psi\partial x^4$, $Z_2 = \partial x^1 - \psi\partial x^2 + \psi\partial x^3 - \partial x^4$, $Z_3 = -\partial x^1 + \psi\partial x^2 + \psi\partial x^3 - \partial x^4$,

$Z_4 = \psi\partial x^5 + (1-\psi)\partial x^7$, $Z_5 = \psi\partial x^6 + (1-\psi)\partial x^8$, $Z_6 = \psi\partial x^9 + (1-\psi)\partial x^{11}$ and $Z_7 = \psi\partial x^{10} + (1-\psi)\partial x^{12}$.

Hence $\mathrm{Rad}(TM) = \mathrm{Span}\{Z_1\}$ and $S(TM) = \mathrm{Span}\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $ltr(TM)$ is spanned by $N_1 = \frac{1}{2(2+\psi)}(-\psi\partial x^1 - \partial x^2 + \partial x^3 + \psi\partial x^4)$ and $S(TM^{\perp})$ is spanned by $W_1 = (1-\psi)\partial x^5 - \psi\partial x^7$, $W_2 = (1-\psi)\partial x^6 - \psi\partial x^8$, $W_3 = (1-\psi)\partial x^9 - \psi\partial x^{11}$ and $W_4 = (1-\psi)\partial x^{10} - \psi\partial x^{12}$.

It follows that $\tilde{P}Z_1 = Z_3$, $\tilde{P}N_1 = Z_2$ and $\tilde{P}Z_4 = (1-\psi)Z_4$, $\tilde{P}Z_5 = \psi Z_5$, which implies D_1 is invariant, i.e., $\tilde{P}D_1 = D_1$ and $D_1 = \mathrm{Span}\{Z_4, Z_5\}$ and distribution $D_2 = \mathrm{Span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(\frac{4}{\sqrt{21}})$. Hence M is a semi-slant 1-lightlike submanifold of $\mathbb{R}^{12}_2$.

Example 3.2. Let $(\mathbb{R}^{12}_2, \overline{g}, \tilde{P})$ be a golden semi-Riemannian manifold, where metric $\overline{g}$ is of signature $(+,-,+,-,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial x^5, \partial x^6, \partial x^7, \partial x^8, \partial x^9, \partial x^{10}, \partial x^{11}, \partial x^{12}\}$ and $(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12})$ be standard coordinate system of $\mathbb{R}^{12}_2$.

Taking, $\tilde{P}(\partial x^1, \ldots, \partial x^{12}) = (\psi\partial x^1, \psi\partial x^2, (1-\psi)\partial x^3, (1-\psi)\partial x^4, \psi\partial x^5, \psi\partial x^6, (1-\psi)\partial x^7, (1-\psi)\partial x^8, (1-\psi)\partial x^9, \psi\partial x^{10}, (1-\psi)\partial x^{11}, \psi\partial x^{12})$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $(1-\psi) = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Thus $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{12}_2$. Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u^1 + \psi u^2 - \psi u^3$, $x^2 = u^1 + \psi u^2 + \psi u^3$, $x^3 = \psi u^1 - u^2 + u^3$, $x^4 = \psi u^1 - u^2 - u^3$, $x^5 = \psi u^4$, $x^6 = (1-\psi)u^4$, $x^7 = \psi u^5$, $x^8 = (1-\psi)u^5$, $x^9 = \psi u^6$, $x^{10} = (1-\psi)u^6$, $x^{11} = \psi u^7$, $x^{12} = (1-\psi)u^7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where $Z_1 = \partial x^1 + \partial x^2 + \psi\partial x^3 + \psi\partial x^4$, $Z_2 = \psi\partial x^1 + \psi\partial x^2 - \partial x^3 - \partial x^4$, $Z_3 = -\psi\partial x^1 + \psi\partial x^2 + \partial x^3 - \partial x^4$, $Z_4 = \psi\partial x^5 + (1-\psi)\partial x^6$, $Z_5 = \psi\partial x^7 + (1-\psi)\partial x^8$, $Z_6 = \psi\partial x^9 + (1-\psi)\partial x^{10}$, $Z_7 = \psi\partial x^{11} + (1-\psi)\partial x^{12}$.

Hence, $\mathrm{Rad}(TM) = \mathrm{Span}\{Z_1\}$ and $S(TM) = \mathrm{Span}\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$.

Now $ltr(TM)$ is spanned by $N_1 = \frac{1}{2(2+\psi)}(-\partial x^1 + \partial x^2 - \psi\partial x^3 + \psi\partial x^4)$ and $S(TM^{\perp})$ is spanned by $W_1 = (1-\psi)\partial x^5 - \psi\partial x^6$, $W_2 = (1-\psi)\partial x^7 - \psi\partial x^8$, $W_3 = (1-\psi)\partial x^9 - \psi\partial x^{10}$, $W_4 = (1-\psi)\partial x^{11} - \psi\partial x^{12}$.

It follows that $\tilde{P}Z_1 = Z_2$, $\tilde{P}N_1 = Z_3$ and $\tilde{P}Z_4 = \psi Z_4$, $\tilde{P}Z_5 = (1-\psi)Z_5$, which implies D_1 is invariant, i.e., $\tilde{P}D_1 = D_1$ and $D_1 = \mathrm{Span}\{Z_4, Z_5\}$ and distribution $D_2 = \mathrm{Span}\{Z_6, Z_7\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{6})$. Hence M is a semi-slant 1-lightlike submanifold of $\mathbb{R}^{12}_2$.

Theorem 3.1. *Let M be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$ of index $2q$. Then M is a semi-slant lightlike submanifold of $\overline{M}$ if and only if*

(i) $\tilde{P}\,\mathrm{Rad}(TM)$ *is a distribution on M such that* $\mathrm{Rad}(TM) \cap \tilde{P}\,\mathrm{Rad}(TM) = 0$;

(ii) *the screen distribution $S(TM)$ split as* $S(TM) = (\tilde{P}\,\mathrm{Rad}(TM) \oplus \tilde{P}ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2$;

(iii) *there exists a constant $\lambda \in [0, 1)$ such that* $P^2X = \lambda(\tilde{P}X + X)$;

for all $X \in \Gamma(D_2)$. Moreover, in this case $\lambda = \cos^2\theta$ and θ is the slant angle of D_2.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the distribution $\tilde{P}\,\mathrm{Rad}(TM)$ is a distribution on M such that $\mathrm{Rad}(TM)\cap\tilde{P}\,\mathrm{Rad}(TM)=0$ and $S(TM)=(\tilde{P}\,\mathrm{Rad}(TM)\oplus\tilde{P}ltr(TM))\oplus_{orth}D_1\oplus_{orth}D_2$.

Now for any $X\in\Gamma(D_2)$, we have $|PX|=|\tilde{P}X|\cos\theta$, which implies

$$\cos\theta=\frac{|PX|}{|\tilde{P}X|}. \tag{3.18}$$

In view of (3.18), we get $\cos^2\theta=\frac{|PX|^2}{|\tilde{P}X|^2}=\frac{g(PX,PX)}{g(\tilde{P}X,\tilde{P}X)}=\frac{g(X,P^2X)}{g(X,\tilde{P}^2X)}$, which gives

$$g(X,P^2X)=\cos^2\theta\,g(X,\tilde{P}^2X). \tag{3.19}$$

Since M is a semi-slant lightlike submanifold, $\cos^2\theta=\lambda$ (constant) $\in[0,1)$ and therefore from (3.19), we get $g(X,P^2X)=\lambda g(X,\tilde{P}^2X)=g(X,\lambda\tilde{P}^2X)$, which implies

$$g(X,(P^2-\lambda\tilde{P}^2)X)=0. \tag{3.20}$$

Since $(P^2-\lambda\tilde{P}^2)X\in\Gamma(D_2)$ and the induced metric $g=g|_{D_2\times D_2}$ is non-degenerate (positive definite), from (3.20), we have $(P^2-\lambda\tilde{P}^2)X=0$, which implies

$$P^2X=\lambda\tilde{P}^2X=\lambda(\tilde{P}X+X), \tag{3.21}$$

for all $X\in\Gamma(D_2)$. This proves (iii).

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. From (iii), we have $P^2X=\lambda\tilde{P}^2X$, for all $X\in\Gamma(D_2)$, where λ (constant) $\in[0,1)$.

Now

$$\cos\theta=\frac{g(\tilde{P}X,PX)}{|\tilde{P}X||PX|}=\frac{g(X,\tilde{P}PX)}{|\tilde{P}X||PX|}=\frac{g(X,P^2X)}{|\tilde{P}X||PX|}=\lambda\frac{g(X,\tilde{P}^2X)}{|\tilde{P}X||PX|}=\lambda\frac{g(\tilde{P}X,\tilde{P}X)}{|\tilde{P}X||PX|}.$$

From above equation, we get

$$\cos\theta=\lambda\frac{|\tilde{P}X|}{|PX|}. \tag{3.22}$$

Therefore, (3.18) and (3.22) give $\cos^2\theta=\lambda$ (constant). Hence, M is a semi-slant lightlike submanifold. ☐

Corollary 3.1. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$ with slant angle θ, then for any $X,Y\in\Gamma(D_2)$, we have*
(i) $g(PX,PY)=\cos^2\theta(g(X,Y)+g(X,PY))$;
(ii) $g(FX,FY)=\sin^2\theta(g(X,Y)+g(PX,Y))$.

Proof. From (2.2), (3.2) and (3.21), we obtain

$$g(PX,PY)=g(X,\lambda(\tilde{P}Y+Y))=\cos^2\theta(g(X,Y)+g(X,PY)).$$

Moreover, from (2.2), (3.2) and (i) part of Corollary 3.1, we get

$$g(FX,FY)=g(X,Y)+g(PX,Y)-g(PX,PY)=\sin^2\theta(g(X,Y)+g(PX,Y)).$$

Hence, the proof is complete. ☐

Theorem 3.2. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then $\mathrm{Rad}(TM)$ is integrable if and only if the following conditions hold:*
(i) $\overline{g}(h^l(X,\tilde{P}Y),\xi)=\overline{g}(h^l(Y,\tilde{P}X),\xi)$;
(ii) $\overline{g}(h^*(X,\tilde{P}Y),N)=\overline{g}(h^*(Y,\tilde{P}X),N)$;
(iii) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}Z_1)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z_1)$;
(iv) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z)$,
for any $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, $\mathrm{Rad}(TM)$ is integrable if and only if $\overline{g}([X,Y],\tilde{P}\xi)=\overline{g}([X,Y],\tilde{P}N)=\overline{g}([X,Y],Z_1)=\overline{g}([X,Y],Z)=0$, for all $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$. Then from (2.4), (2.11), (2.16) and (3.6), we obtain

$$\begin{aligned} g([X,Y],\tilde{P}\xi) &=\overline{g}(\overline{\nabla}_XY-\overline{\nabla}_YX,\tilde{P}\xi)=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\xi)\\ &=\overline{g}(h^l(X,\tilde{P}Y)-h^l(Y,\tilde{P}X),\xi), \end{aligned}\tag{3.23}$$

$$\begin{aligned} g([X,Y],\tilde{P}N) &=\overline{g}(\overline{\nabla}_XY-\overline{\nabla}_YX,\tilde{P}N)=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\ &=\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N), \end{aligned}\tag{3.24}$$

$$\begin{aligned} g([X,Y],Z_1) &=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z_1)-\overline{g}(\tilde{P}[X,Y],Z_1)\\ &=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}Z_1)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z_1)\\ &=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}Z_1)\\ &\quad-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z_1), \end{aligned}\tag{3.25}$$

$$\begin{aligned} g([X,Y],Z) &=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)\\ &=\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,PZ+FZ)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z)\\ &=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)\\ &\quad-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z). \end{aligned}\tag{3.26}$$

From (3.23), (3.24), (3.25) and (3.26), we derive our theorem. ☐

Theorem 3.3. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then $\tilde{P}\,\mathrm{Rad}(TM)$ is integrable if and only if the following conditions hold:*
(i) $\overline{g}(h^l(\tilde{P}X,Y),\xi)=\overline{g}(h^l(\tilde{P}Y,X),\xi)$;
(ii) $\overline{g}(A^*_X\tilde{P}Y,\tilde{P}Z_1)=\overline{g}(A^*_Y\tilde{P}X,\tilde{P}Z_1)$;
(iii) $\overline{g}(A^*_X\tilde{P}Y-A^*_Y\tilde{P}X,PZ)=\overline{g}(h^s(\tilde{P}Y,X)-h^s(\tilde{P}X,Y),FZ)$;
(iv) $\overline{g}(A_N\tilde{P}X,\tilde{P}Y)=\overline{g}(A_N\tilde{P}Y,\tilde{P}X)$,
for any $X,Y,\xi\in\Gamma(\mathrm{Rad}(TM))$, $Z_1\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, $\tilde{P}\,\mathrm{Rad}(TM)$ is integrable if and only if $\overline{g}([\tilde{P}X,\tilde{P}Y],\tilde{P}\xi)=\overline{g}([\tilde{P}X,\tilde{P}Y],Z_1)=\overline{g}([\tilde{P}X,\tilde{P}Y],Z)=$

$\overline{g}([\tilde{P}X, \tilde{P}Y],$
$N) = 0$, for all $X, Y, \xi \in \Gamma(\mathrm{Rad}(TM))$, $Z_1 \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$. Since $\overline{\nabla}$ is metric connection and using (2.4), (2.11), (2.12), (2.17) and (3.6), we obtain

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], \tilde{P}\xi) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}\tilde{P}Y - \overline{\nabla}_{\tilde{P}Y}\tilde{P}X, \tilde{P}\xi) \\ =& \overline{g}(h^l(\tilde{P}X, Y) - h^l(\tilde{P}Y, X), \xi), \end{aligned} \tag{3.27}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z_1) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}Y, \tilde{P}Z_1) - \overline{g}(\overline{\nabla}_{\tilde{P}Y}X, \tilde{P}Z_1) \\ =& \overline{g}(A^*_X\tilde{P}Y, \tilde{P}Z_1) - \overline{g}(A^*_Y\tilde{P}X, \tilde{P}Z_1), \end{aligned} \tag{3.28}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z) =& \overline{g}(\overline{\nabla}_{\tilde{P}X}Y, PZ + FZ) - \overline{g}(\overline{\nabla}_{\tilde{P}Y}X, PZ + FZ) \\ =& \overline{g}(A^*_X\tilde{P}Y - A^*_Y\tilde{P}X, PZ) - \overline{g}(h^s(\tilde{P}Y, X) - h^s(\tilde{P}X, Y), FZ), \end{aligned} \tag{3.29}$$

$$\begin{aligned} g([\tilde{P}X, \tilde{P}Y], Z) =& -\overline{g}(\tilde{P}Y, \overline{\nabla}_{\tilde{P}X}N) + \overline{g}(\tilde{P}X, \overline{\nabla}_{\tilde{P}Y}N) \\ =& \overline{g}(A_N\tilde{P}X, \tilde{P}Y) - \overline{g}(A_N\tilde{P}Y, \tilde{P}X). \end{aligned} \tag{3.30}$$

From (3.27), (3.28), (3.29) and (3.30), proof is completed. □

Theorem 3.4. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then $\tilde{P}ltr(TM)$ is integrable if and only if the following conditions hold:*

(i) $\overline{g}(A_{N_1}\tilde{P}N_2, N) = \overline{g}(A_{N_2}\tilde{P}N_1, N)$*;*
(ii) $\overline{g}(A_{N_1}\tilde{P}N_2, \tilde{P}Z_1) = \overline{g}(A_{N_2}\tilde{P}N_1, \tilde{P}Z_1)$*;*
(iii) $\overline{g}(A_{N_1}\tilde{P}N_2 - A_{N_2}\tilde{P}N_1, PZ) = \overline{g}(D^s(\tilde{P}N_2, N_1) - D^s(\tilde{P}N_1, N_2), FZ)$*;*
(iv) $\overline{g}(A_N\tilde{P}N_1, \tilde{P}N_2) = \overline{g}(A_N\tilde{P}N_2, \tilde{P}N_1)$*,*

for any $N_1, N_2, N \in \Gamma(ltr(TM))$, $Z_1 \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. From the definition of semi-slant lightlike submanifolds, $\tilde{P}ltr(TM)$ is integrable if and only if $\overline{g}([\tilde{P}N_1, \tilde{P}N_2], \tilde{P}N) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], Z_1) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], Z) = \overline{g}([\tilde{P}N_1, \tilde{P}N_2], N) = 0$, for any $N_1, N_2, N \in \Gamma(ltr(TM))$, $Z_1 \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Taking $\overline{\nabla}$ is metric connection and from (2.4), (2.11), (2.12), (2.16) and (3.6), we obtain

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], \tilde{P}N) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}\tilde{P}N_2 - \overline{\nabla}_{\tilde{P}N_2}\tilde{P}N_1, \tilde{P}N) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2, N) - \overline{g}(A_{N_2}\tilde{P}N_1, N), \end{aligned} \tag{3.31}$$

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], Z_1) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}N_2, \tilde{P}Z_1) - \overline{g}(\overline{\nabla}_{\tilde{P}N_2}N_1, \tilde{P}Z_1) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2, \tilde{P}Z_1) - \overline{g}(A_{N_2}\tilde{P}N_1, \tilde{P}Z_1), \end{aligned} \tag{3.32}$$

$$\begin{aligned} g([\tilde{P}N_1, \tilde{P}N_2], Z) =& \overline{g}(\overline{\nabla}_{\tilde{P}N_1}N_2, PZ + FZ) - \overline{g}(\overline{\nabla}_{\tilde{P}N_2}N_1, PZ + FZ) \\ =& \overline{g}(A_{N_1}\tilde{P}N_2 - A_{N_2}\tilde{P}N_1, PZ) \\ & - \overline{g}(D^s(\tilde{P}N_2, N_1) - D^s(\tilde{P}N_1, N_2), FZ), \end{aligned} \tag{3.33}$$

$$g([\tilde{P}N_1, \tilde{P}N_2], N) = -\overline{g}(\tilde{P}N_2, \overline{\nabla}_{\tilde{P}N_1}N) + \overline{g}(\tilde{P}N_1, \overline{\nabla}_{\tilde{P}N_2}N)$$

$$=\overline{g}(A_N\tilde{P}N_1,\tilde{P}N_2)-\overline{g}(A_N\tilde{P}N_2,\tilde{P}N_1). \tag{3.34}$$

From (3.31), (3.32), (3.33) and (3.34), we derive our theorem. □

Theorem 3.5. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_1 is integrable if and only if the following conditions hold:*
(i) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)=\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z)$;
(ii) $\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}N)=\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N)$;
(iii) $\overline{g}(A_NX,\tilde{P}Y)=\overline{g}(A_NY,\tilde{P}X)$,
for any $X,Y\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, D_1 is integrable if and only if $\overline{g}([X,Y],Z)=\overline{g}([X,Y],N)=\overline{g}([X,Y],\tilde{P}N)=0$ for all $X,Y\in\Gamma(D_1)$, $Z\in\Gamma(D_2)$ and $N\in\Gamma(ltr(TM))$. Since $\overline{\nabla}$ is metric connection and from (2.4), (2.11), (2.12), (2.16) and (3.6), we obtain

$$\begin{aligned}g([X,Y],Z)=&\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)\\=&\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,PZ+FZ)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,Z)\\=&\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,PZ)+\overline{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),FZ)\\&-\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,Z),\end{aligned} \tag{3.35}$$

$$\begin{aligned}g([X,Y],N)=&\overline{g}(\tilde{P}[X,Y],\tilde{P}N)-\overline{g}(\tilde{P}[X,Y],N)\\=&\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,\tilde{P}N)-\overline{g}(\overline{\nabla}_X\tilde{P}Y-\overline{\nabla}_Y\tilde{P}X,N)\\=&\overline{g}(\nabla^*_X\tilde{P}Y-\nabla^*_Y\tilde{P}X,\tilde{P}N)-\overline{g}(h^*(X,\tilde{P}Y)-h^*(Y,\tilde{P}X),N),\end{aligned} \tag{3.36}$$

$$\begin{aligned}g([X,Y],\tilde{P}N)=&-\overline{g}(\overline{\nabla}_XN,\tilde{P}Y)+\overline{g}(\overline{\nabla}_YN,\tilde{P}X)\\=&\overline{g}(A_NX,\tilde{P}Y)-\overline{g}(A_NY,\tilde{P}X).\end{aligned} \tag{3.37}$$

From (3.35), (3.36) and (3.37), proof is completed. □

Theorem 3.6. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_2 is integrable if and only if the following conditions hold:*
(i) $\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}Z)+\overline{g}(\nabla_YPX-A_{FX}Y,Z)=\overline{g}(\nabla_XPY-A_{FY}X,Z)+\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}Z)$;
(ii) $\overline{g}(\nabla_XPY-A_{FY}X,\tilde{P}N)+\overline{g}(\nabla_YPX-A_{FX}Y,N)=\overline{g}(\nabla_XPY-A_{FY}X,N)+\overline{g}(\nabla_YPX-A_{FX}Y,\tilde{P}N)$;
(iii) $\overline{g}(\nabla_XPY-A_{FY}X,N)=\overline{g}(\nabla_YPX-A_{FX}Y,N)$,
for any $X,Y\in\Gamma(D_2)$, $Z\in\Gamma(D_1)$ and $N\in\Gamma(ltr(TM))$.

Proof. From the definition of semi-slant lightlike submanifolds, D_2 is integrable if and only if $\overline{g}([X,Y],Z)=\overline{g}([X,Y],N)=\overline{g}([X,Y],\tilde{P}N)=0$ for all $X,Y\in\Gamma(D_2)$, $Z\in\Gamma(D_1)$ and $N\in\Gamma(ltr(TM))$. Then from (2.4), (2.11), (2.13) and (3.6), we obtain

$$g([X,Y],Z)=\overline{g}(\tilde{P}[X,Y],\tilde{P}Z)-\overline{g}(\tilde{P}[X,Y],Z)$$

$$
\begin{aligned}
&= \overline{g}(\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_Y \tilde{P}X, \tilde{P}Z) - \overline{g}(\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_Y \tilde{P}X, Z)\\
&= \overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}Z) + \overline{g}(\nabla_Y PX - A_{FX}Y, Z)\\
&\quad - \overline{g}(\nabla_X PY - A_{FY}X, Z) - \overline{g}(\nabla_Y PX - A_{FX}Y, \tilde{P}Z), \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
g([X,Y],N) &= \overline{g}(\tilde{P}[X,Y], \tilde{P}N) - \overline{g}(\tilde{P}[X,Y], N)\\
&= \overline{g}(\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_Y \tilde{P}X, \tilde{P}N) - \overline{g}(\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_Y \tilde{P}X, N)\\
&= \overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}N) + \overline{g}(\nabla_Y PX - A_{FX}Y, N)\\
&\quad - \overline{g}(\nabla_X PY - A_{FY}X, N) - \overline{g}(\nabla_Y PX - A_{FX}Y, \tilde{P}N), \qquad (3.39)
\end{aligned}
$$

$$
\begin{aligned}
g([X,Y],\tilde{P}N) &= \overline{g}(\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_Y \tilde{P}X, N)\\
&= \overline{g}(\nabla_X PY - A_{FY}X, N) - \overline{g}(\nabla_Y PX - A_{FX}Y, N). \qquad (3.40)
\end{aligned}
$$

From (3.38), (3.39) and (3.40), we derive our theorem. □

Theorem 3.7. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if*

(i) $P\nabla_X \tilde{P}Y \in \Gamma(\mathrm{Rad}(TM))$;

(ii) $Bh^l(X, \tilde{P}Y) = P_3\nabla_X \tilde{P}Y$ *and* $P_2\nabla_X \tilde{P}Y = 0$;

(iii) $Bh^s(X, \tilde{P}Y) = P_5\nabla_X \tilde{P}Y$ *and* $P_4\nabla_X \tilde{P}Y = 0$,

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathrm{Rad}(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $\mathrm{Rad}(TM)$ is parallel distribution with respect to ∇ [4]. For any $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathrm{Rad}(TM))$, we have $\overline{\nabla}_X Y = \tilde{P}\overline{\nabla}_X \tilde{P}Y - \overline{\nabla}_X \tilde{P}Y$, using (2.11), (3.2) and (3.9), we get $\overline{\nabla}_X Y = P\nabla_X \tilde{P}Y + F\nabla_X \tilde{P}Y + Bh^l(X, \tilde{P}Y) + Bh^s(X, \tilde{P}Y) + Ch^s(X, \tilde{P}Y) - \nabla_X \tilde{P}Y - h^l(X, \tilde{P}Y) - h^s(X, \tilde{P}Y)$. By comparing tangential components of both sides of above equation, we obtain $\nabla_X Y = P\nabla_X \tilde{P}Y + Bh^l(X, \tilde{P}Y) + Bh^s(X, \tilde{P}Y) - P_1\nabla_X \tilde{P}Y - P_2\nabla_X \tilde{P}Y - P_3\nabla_X \tilde{P}Y - P_4\nabla_X \tilde{P}Y - P_5\nabla_X \tilde{P}Y$, which completes the proof. □

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a semi-slant lightlike submanifold of a golden semi-Riemannian manifold to be totally geodesic.

Definition 4.1 ([5])**.** A semi-slant lightlike submanifold M of a golden semi-Riemannian manifold $\overline{M}$ is said to be a mixed geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic semi-slant lightlike submanifold if $h^l(X, Y) = 0$ and $h^s(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then* $\mathrm{Rad}(TM)$ *defines a totally geodesic foliation if and only* *if* $\overline{g}(h^l(X, K_1\tilde{P}P_2Z)+h^l(X, K_2\tilde{P}P_2Z)+\nabla^l_X L_1\tilde{P}P_3Z+h^l(X, L_2\tilde{P}P_3Z)+h^l(X, \tilde{P}P_4Z)+h^l(X, PP_5Z)+D^l(X, FP_5Z), Y) = \overline{g}(\nabla_X K_1\tilde{P}P_2Z+\nabla_X K_2\tilde{P}P_2Z-A_{L_1\tilde{P}P_3Z}X+\nabla_X L_2\tilde{P}P_3Z + \nabla_X \tilde{P}P_4Z + \nabla_X PP_5Z - A_{FP_5Z}X, \tilde{P}Y)$ *for all* $X, Y \in \Gamma(\mathrm{Rad}(TM))$ *and* $Z \in \Gamma(S(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution $\mathrm{Rad}(TM)$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(\mathrm{Rad}(TM))$, for all $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Since $\overline{\nabla}$ is a metric connection, using (2.4), (2.11), (2.12), (2.13) and (3.6), for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y, Z) = \overline{g}(Y, \overline{\nabla}_X \tilde{P}P_2Z + \overline{\nabla}_X \tilde{P}P_3Z + \overline{\nabla}_X \tilde{P}P_4Z + \overline{\nabla}_X PP_5Z + \overline{\nabla}_X FP_5Z) - \overline{g}(\tilde{P}Y, \overline{\nabla}_X \tilde{P}P_2Z + \overline{\nabla}_X \tilde{P}P_3Z + \overline{\nabla}_X \tilde{P}P_4Z + \overline{\nabla}_X PP_5Z + \overline{\nabla}_X FP_5Z)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(h^l(X, K_1\tilde{P}P_2Z)+h^l(X, K_2\tilde{P}P_2Z)+\nabla^l_X L_1\tilde{P}P_3Z+h^l(X, L_2\tilde{P}P_3Z) + h^l(X, \tilde{P}P_4Z) + h^l(X, PP_5Z) + D^l(X, FP_5Z), Y) - \overline{g}(\nabla_X K_1\tilde{P}P_2Z + \nabla_X K_2\tilde{P}P_2Z - A_{L_1\tilde{P}P_3Z}X + \nabla_X L_2\tilde{P}P_3Z + \nabla_X \tilde{P}P_4Z + \nabla_X PP_5Z - A_{FP_5Z}X, \tilde{P}Y)$. Thus, the theorem is completed. □

Theorem 4.2. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X PZ - A_{FZ}X, \tilde{P}Y) = \overline{g}(\nabla_X PZ - A_{FZ}X, Y)$;
(ii) $\overline{g}(\nabla^*_X \tilde{P}Y, \tilde{P}N) = \overline{g}(h^*(X, \tilde{P}Y), N)$;
(iii) $h^*(X, \tilde{P}Y)$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ *and* $N \in \Gamma ltr(TM)$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, from (2.4), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ) + \overline{g}(Y, \overline{\nabla}_X PZ + \overline{\nabla}_X FZ)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\tilde{P}Y, \nabla_X PZ - A_{FZ}X) + \overline{g}(Y, \nabla_X PZ - A_{FZ}X)$. From (2.4), (2.11) and (2.16), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \tilde{P}Y, \tilde{P}N) - \overline{g}(\overline{\nabla}_X \tilde{P}Y, N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla^*_X \tilde{P}Y, \tilde{P}N) - \overline{g}(h^*(X, \tilde{P}Y), N)$. Now, from (2.2), (2.11) and (2.16), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X \tilde{P}Y, N)$, which implies $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(h^*(X, \tilde{P}Y), N)$. This proves the theorem. □

Theorem 4.3. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X \tilde{P}Z, Y) - \overline{g}(PY, \nabla_X \tilde{P}Z) = \overline{g}(FY, h^s(X, \tilde{P}Z))$;
(ii) $\overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$;
(iii) $\nabla_X PY - A_{FY}X$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in$

$\Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z))$. In view of (2.4), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, \tilde{P}N) - \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X PY - A_{FY}X, \tilde{P}N) - \overline{g}(\nabla_X PY - A_{FY}X, N)$. Now, from (2.2), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$. Hence, the proof is completed. □

Theorem 4.4. *Let M be a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(PY, \nabla_X \tilde{P}Z) = \overline{g}(Y, \nabla_X \tilde{P}Z)$;

(ii) $\overline{g}(\nabla_X \tilde{P}N, Y) - \overline{g}(\nabla_X \tilde{P}N, PY) = \overline{g}(h^s(X, \tilde{P}N), FY)$;

(iii) $\nabla_X PY - A_{FY}X$ *has no components in* $\Gamma(\mathrm{Rad}(TM))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a mixed geodesic semi-slant lightlike submanifold of a golden semi-Riemannian manifolds $\overline{M}$, we have $h^s(X, \tilde{P}Z) = 0$, for all $X \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}Z) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}Z) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(PY, \nabla_X \tilde{P}Z) - \overline{g}(FY, h^s(X, \tilde{P}Z)) + \overline{g}(Y, \nabla_X \tilde{P}Z)$. From (2.4), (2.11) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = -\overline{g}(PY, \overline{\nabla}_X \tilde{P}N) - \overline{g}(FY, \overline{\nabla}_X \tilde{P}N) + \overline{g}(Y, \overline{\nabla}_X \tilde{P}N)$, which gives $\overline{g}(\nabla_X Y, N) = \overline{g}(\nabla_X \tilde{P}N, Y) - \overline{g}(\nabla_X \tilde{P}N, PY) - \overline{g}(h^s(X, \tilde{P}N), FY)$. Now, from (2.2), (2.11), (2.13) and (3.6), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\overline{\nabla}_X PY + \overline{\nabla}_X FY, N)$, which gives $\overline{g}(\nabla_X Y, \tilde{P}N) = \overline{g}(\nabla_X PY - A_{FY}X, N)$. Hence, the proof is completed. □

Theorem 4.5. *Let M be a semi-slant lightlike submanifold of a golden semi Riemannian manifold $(\overline{M}, \overline{g}, \tilde{P})$. Then M is mixed geodesic if and only if the following hold:*

(i) $F(\nabla_X PZ - A_{FZ}X) = -C(h^s(X, PZ) + \nabla^s_X FZ)$;

(ii) $h^l(X, PZ) + D^l(X, FZ) = h^s(X, PZ) + \nabla^s_X FZ$,

for any $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. From (2.9), (2.11), (2.13), (3.2), (3.6) and (3.9), we obtain

$$\begin{aligned} h(X, Z) =& \tilde{P}(\nabla_X PZ + h^l(X, PZ) + h^s(X, PZ) - A_{FZ}X + \nabla^s_X FZ + D^l(X, FZ)) \\ &- (\nabla_X PZ + h^l(X, PZ) + h^s(X, PZ) - A_{FZ}X + \nabla^s_X FZ + D^l(X, FZ)) \\ &- \nabla_X Z. \end{aligned}$$

Taking transversal part of this equation, we get

$$h(X,Z) = F(\nabla_X PZ - A_{FZ}X) + C(h^s(X,PZ) + \nabla^s_X FZ) - h^l(X,PZ) - h^s(X,PZ) - \nabla^s_X FZ - D^l(X,FZ).$$

Hence, $h(X,Z)=0$ if and only if (i) and (ii) hold. Hence, the proof is completed. □

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