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EXISTENCE RESULTS FOR KIRCHHOFF NONLOCAL FRACTIONAL EQUATIONS

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Abstract. Fractional and nonlocal operators of elliptic type arise in a quite natural way in many different contexts. In this paper, we study the existence of solutions for a class of fractional equations, while the nonlinear part of the problem admits some perturbation property. We obtain some new criteria for existence of two and infinitely many solutions, using critical point theory. Some recent results are extended and improved. Several examples are presented to demonstrate the applications of our main results.

1. Introduction

In this paper we investigate the existence of multiple nontrivial weak solutions for Kirchhoff fractional problem

$$(\mathcal{L}_f^\lambda) \qquad \begin{cases} -\mathcal{L}_K u = \lambda f(u), & \text{in}\,\Omega,\\ u = 0, & \text{in}\,\mathbb{R}^n\backslash\Omega, \end{cases}$$

where Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with $n > 2s$, $s \in (0,1)$ and $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$, with smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $\lambda > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, $\mathcal{L}_K$ is the nonlocal operator defined as follows:

$$\begin{aligned}\mathcal{L}_K u(x) =& M\left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\\ &\times \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big)K(y)\mathrm{d}y,\end{aligned}$$

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where $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $Q := (\mathbb{R}^n \times \mathbb{R}^n)\backslash \mathcal{O}$ with $\mathcal{O} := (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{C}\Omega := \mathbb{R}^n\backslash\{0\}$, $K : \mathbb{R}^n\backslash\{0\} \to (0, +\infty)$ is a function with the properties that:

(κ_1) $\gamma K \in \mathrm{L}^1(\mathbb{R}^n)$ where $\gamma(x) = \min\{|x|^2, 1\}$;
(κ_2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n\backslash\{0\}$;
(κ_3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n\backslash\{0\}$.

A special case of $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

which corresponds to the case $M \equiv 1$ and $K(x) = |x|^{-(n+2s)}$. One typical feature of problem ($\mathcal{L}_f^\lambda$) is the nonlocality, in the sense that the value of $(-\Delta)^s u$ at any point $x \in \Omega$ depends not only on Ω, but actually on the entire space $\mathbb{R}^n$. In the special case, fractional Laplacian operator $-(-\Delta)^s$ (up to normalization constants) may be defined as

$$-(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where $P.V.$ is a particular value. It may be seen as the infinitesimal small generators of a Lévy motion stable diffusion operations [1]. This operator has been used in modelling various applied phenomena, like phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics, etc. On the other hand, and more importantly, fractional and non-fractional operators find many specific applications also in bio-mathematics and physics, which nowadays is a rather fashionable field of research; we, for instance, refer to [15, 20, 21]. To see more features, you can see [30, 34] and references therein. Recently, a lot of research work has been done to the study of semiclassical standing waves for the non-linear fractional Schrödinger equation of the form

$$i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s} (-\Delta)^s \psi + P(x)\psi - f(x, |\psi|), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where ε is a small positive constant, which corresponds to the Planck constant, $(-\Delta)^s$, $0 < s < 1$, is the fractional Laplacian, $P(x)$ is a potential function. Problem (1.1) models naturally many physical problems, such as phase transition, conservation laws, especially in fractional quantum mechanics, etc. (see [16]). It was introduced by Laskin [19] as a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy process. We refer to [12] for more physical background. To obtain standing waves of the fractional non-linear Schrödinger equation (1.1), we set $\psi(x, t) = e^{\frac{-iEt}{\varepsilon}} u(x)$ for some function $u \in H^s(\mathbb{R}^n)$, and let $V(x) = P(x) - E$. Then problem (1.1) is reduced to the following equation:

$$\varepsilon^{2s} (-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^n. \tag{1.2}$$

In quantum mechanics, when ε tends to 0, the existence and multiplicity of solutions to (1.2) is of particular importance.

In the nonlocal case, that is, when $s \in (0,1)$, the nonlocal model has attracted much attentions recently. For the case of a bounded domain, Ricceri [33] established a theorem tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. In [8], Molica Bisci and Repovš studied a class of nonlocal fractional Laplacian equations depending on two real parameters. More precisely, by using an appropriate analytical context on fractional Sobolev spaces due to Servadei and Valdinoci, they established the existence of three weak solutions for nonlocal fractional problems exploiting an abstract critical point result for smooth functionals. They emphasized that the dependence of the underlying equation from one of the real parameters is not necessarily of affine type. For more related results, we refer the reader to [24–26] and the references therein.

The interest in studying problems like problem $(\mathcal{L}_f^\lambda)$ relies not only on mathematical purposes but also on their significance in real models. For example, in the Appendix of paper [17], the authors constructed a stationary Kirchhoff variational problem, which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string.

Kirchhoff models take into consideration the length changes of the string produced by transverse vibrations (see [18]). Fractional and nonlocal operators of elliptic type which is modeled by the singularity at infinity is an emerging research field. From the physical viewpoint, nonlocal operators play a considerable role in characterizing a set of phenomena. A general reference for this issue is [39], where the author explained two models of flow in porous media, including nonlocal diffusion effects, providing a long list of references related to physical phenomena and nonlocal operators. The first model is based on Darcy's law, and the pressure is associated with the density by an inverse fractional Laplacian operator. The second model mostly follows fractional Laplacian flows but it is nonlinear. In contrast to the usual porous medium flows, it has infinite speed of propagation. On the other hand, fractional nonlocal operators arise in a quite natural way in many different contexts. See for instance the references [5–7] and [2,4,8,13,25,28,38]. For example, Molica Bisci in [25] studied the existence of infinitely many weak solutions to the problem $(\mathcal{L}_f^\lambda)$ where $f(x,u)$ replaced by $f(u)$ with $x \in \Omega$ in the case $\lambda = 1$ and $M \equiv 1$. We have shown in Remark 4.1 that our results in Theorem 1.2 are different from [25, Theorem 1.1].

Recently, some researchers have studied the existence and multiplicity of solutions for fractional equations of Kirchhoff type; we refer the reader to [3,10,11,14,23,29,40,42] and the references therein. For example Chen and Deng in [10] based on Ekeland's variational principle investigated the existence of solutions to a Kirchhoff type problem involving the fractional p-Laplacian operator. It established in [23] the multiplicity of weak solutions for a Kirchhoff-type problem driven by a fractional

p-Laplacian operator with homogeneous Dirichlet boundary conditions:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)_p^s u(x) = f(x,u), & \text{in}\, \Omega, \\ u = 0, & \text{in}\, \mathbb{R}^n \backslash \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with Lipshcitz boundary $\partial\Omega$, $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0 < s < 1 < p < N$ such that $sp < N$, M is a continuous function and f is a Carathéodory function satisfying the Ambrosetti-Rabinowitz condition. When f satisfies the suplinear growth condition, they obtained the existence of a sequence of nontrivial solutions by using the symmetric mountain pass theorem, and when f satisfies the sub-linear growth condition, they obtained infinitely many pairs of nontrivial solutions by applying the Krasnoselskii genus theory. By using an appropriate analytical context on fractional Sobolev spaces, Molica Bisci and Tulone in [29] obtained the existence of one non-trivial weak solution for nonlocal fractional problem $(\mathcal{L}_f^\lambda)$ in the case $M(x) = a + bx$ where a, b are positive numbers. Xiang et al. in [40] studied the problem

$$\begin{cases} M\left(x, [u]_{s,p}^p\right)(-\Delta)_p^s u(x) = f(x, u, [u]_{s,p}^p), & \text{in}\, \Omega, \\ u = 0, & \text{in}\, \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y$, $(-\Delta)_p^s$ is a fractional p-Laplace operator, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $M : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function satisfying the Ambrosetti-Rabinowitz condition. They obtained the existence of nonnegative solutions by using the Mountain Pass Theorem and an iterative scheme.

The present paper focuses on this issue since it is clear that in problem $(\mathcal{L}_f^\lambda)$ there is a singularity in the term $\mathcal{L}_k(u)$, which causes difficulties in the proof. In this paper, we are concerned with the existence results for the problem $(\mathcal{L}_f^\lambda)$, and prove at least two weak solutions and infinitely many weak solutions for the problem $(\mathcal{L}_f^\lambda)$. Several special cases of the main results and two illustrating examples are also presented. We use the following assumptions throughout this paper:

- $(\mathcal{M})$ $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that satisfies $m_0 t^{\alpha-1} \leq M(t) \leq m_1 t^{\alpha-1}$ for all $t \in \mathbb{R}^+$, where $m_1 > m_0 > 0$ and $1 < \alpha < \frac{2n}{n-2s}$;
- $(\mathcal{F}_1)$ there exists a constant $\beta > \frac{2m_1\alpha}{m_0}$ with $0 < \beta F(t) \leq \xi f(t)$ for all $t \in \mathbb{R}\backslash\{0\}$;
- $(\mathcal{F}_2)$ $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{\alpha-1}} = 0$, i.e., f is $(\alpha - 1)$-sublinear at infinity.

The main results of this paper are presented as follows.

Theorem 1.1. *Assume that the assumptions* $(\mathcal{M})$, $(\mathcal{F}_1)$ *and* $(\mathcal{F}_2)$ *hold. Then, if* $f(t) \geq 0$ *for all* $t \in \mathbb{R}$, *the problem* $(\mathcal{L}_f^\lambda)$ *has at least two weak solutions.*

Theorem 1.2. *Assume that the assumptions* $(\mathcal{M})$, $(\mathcal{F}_1)$ *and* $(\mathcal{F}_2)$ *hold. Then, if* $f(t)$ *is odd, the problem* $(\mathcal{L}_f^\lambda)$ *has infinitely many weak solutions.*

2. Preliminaries

In this part, we discuss some preliminary results which can be found in [34]. The functional space E denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in E belongs to $\mathrm{L}^2(\Omega)$ and

$$\Big((x,y)\mapsto (u(x)-u(y))\sqrt{K(x-y)}\Big)\in \mathrm{L}^2\Big((\mathbb{R}^n\times\mathbb{R}^n)\backslash(\complement\Omega\times\complement\Omega),\mathrm{d}x\mathrm{d}y\Big).$$

We denote by E_0 the following linear subspace of E

$$\mathrm{E}_0 := \{\mathrm{u}\in \mathrm{E}:\ \mathrm{u}=0 \text{ a.e. in } \mathbb{R}^{\mathrm{n}}\backslash\Omega\}.$$

We remark that E and E_0 are nonempty, since $\mathrm{C}^2_0(\Omega)\subseteq \mathrm{E}_0$ by [34, Lemma 11]. Moreover, the space E is endowed with the norm defined as

$$\|u\|_{\mathrm{E}} := \|u\|_{\mathrm{L}^2(\Omega)} + \left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

It is easily seen that $\|\cdot\|_{\mathrm{E}}$ is a norm on E (see [35]). By [35, Lemmas 6 and 7] in the sequel we can take the function

$$\mathrm{E}_0 \ni u \mapsto \|u\|_{\mathrm{E}_0} := \left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/2} \tag{2.1}$$

as norm on E_0. Also $(\mathrm{E}_0, \|\cdot\|_{\mathrm{E}_0})$ is a Hilbert space with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y.$$

See [35, Lemma 7]. Note that in (2.1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n\times\mathbb{R}^n$, since $v\in \mathrm{E}_0$ (and so $v=0$ a.e. in $\mathbb{R}^n\backslash\Omega$). While for a general kernel K satisfying conditions from (κ_1)-(κ_3) we have that $\mathrm{E}_0\subset \mathrm{H}^s(\mathbb{R}^n)$, in the model case $K(x) := |x|^{-(n+2s)}$ the space E_0 consists of all the functions of the usual fractional Sobolev space $\mathrm{H}^s(\mathbb{R}^n)$ which vanish a.e. outside Ω (see [37, Lemma 7]). Here $\mathrm{H}^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|u\|_{\mathrm{E}} := \|u\|_{\mathrm{H}^s(\mathbb{R}^n)} = \|u\|_{\mathrm{L}^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

Remark 2.1. By [34, Lemma 8], the embedding $j:\mathrm{E}_0\hookrightarrow \mathrm{L}^\nu(\mathbb{R}^n)$ is continuous for any $\nu\in[1,2^*]$, while it is compact whenever $\nu\in[1,2^*)$, where $2^* := \frac{2n}{n-2s}$ denotes the fractional critical Sobolev exponent. For further details on the fractional Sobolev spaces we refer to [12] and to the references therein, while for other details on E and E_0 we refer to [12], where these functional spaces were introduced, and also to [35–37], where various properties of these spaces were proved.

Definition 2.1 ([24])**.** We say that $u \in \mathrm{E}_0$ is a weak solution of $(\mathcal{L}_f^\lambda)$ if for all $v \in \mathrm{E}_0$

$$M\left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y$$
$$-\lambda\int_\Omega f(u(x))v(x)\mathrm{d}x = 0.$$

We refer the reader to [22, 32] for the following notations and results.

Theorem 2.1 ([22, Theorem 4.4])**.** *Let X be a Banach space, $\phi : X \to \mathbb{R}$ a function bounded from below and differentiable on X. If ϕ satisfies the* $(\mathrm{PS})_{\mathrm{c}}$*-condition with $c = \inf_X \phi$, then ϕ has a minimum on X.*

It is clear that the (PS)-condition implies the $(\mathrm{PS})_{\mathrm{c}}$-condition for each $c \in \mathbb{R}$.

Theorem 2.2 ([22, Theorem 4.10])**.** *Let $\varphi \in \mathrm{C}^1(X,\mathbb{R})$, and φ satisfy the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and $\inf_{v\in\partial\Omega}\varphi(v) > \max\{\varphi(u_0),\varphi(u_1)\}$, then there exists a critical point u of φ, i.e., $\varphi'(u) = 0$, with $\varphi(u) > \max\{\varphi(u_0),\varphi(u_1)\}$.*

Theorem 2.3 ([32, Theorem 9.12])**.** *Let X be an infinite dimensional real Banach space. Let $\varphi \in \mathrm{C}^1(X,\mathbb{R})$ be an even functional which satisfies the* (PS)*-condition and $\varphi(0) = 0$. Suppose that $X = V \bigoplus \mathrm{E}$, where V is infinite dimensional, and φ satisfies that*

(*i*) *there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*
(*ii*) *for any finite dimensional subspace $W \subset X$, there is $R = R(W)$ such that $\varphi(u) \geq 0$ on $W \setminus B_{R(W)}$.*

Then φ possesses an unbounded sequence of critical values.

We refer the reader to the paper [9, 41] in which Theorems 2.2 and 2.3 were successfully employed to ensure the multiple solutions of degenerate nonlocal problems and nonlinear impulsive differential equations with Dirichlet boundary conditions, respectively.

Corresponding to the functions f and M we introduce the functions $F : \mathbb{R} \to \mathbb{R}$ and $\widehat{M} : [0,+\infty) \to \mathbb{R}$, respectively, as $F(t) := \int_0^t f(\xi)\mathrm{d}\xi$ for all $t \in \mathbb{R}$ and $\widehat{M}(t) := \int_0^t M(\xi)\mathrm{d}\xi$ for all $t \in [0,+\infty)$, and consider the functionals $\Phi, \Psi : \mathrm{E}_0 \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\widehat{M}\left(\|u\|_{\mathrm{E}_0}^2\right) \quad \text{and} \quad \Psi(u) = \int_\Omega F(u(x))\mathrm{d}x, \tag{2.2}$$

for all $u \in \mathrm{E}_0$. Thus, by the assumption $(\mathcal{M})$ we have

$$\frac{m_0}{2\alpha}\|u\|_{\mathrm{E}_0}^{2\alpha} \leq \Phi(u) \leq \frac{m_1}{2\alpha}\|u\|_{\mathrm{E}_0}^{2\alpha},$$

which means that the functional $\Phi : \mathrm{E}_0 \to \mathbb{R}$ is coercive. On the other hand, Φ and Ψ are continuously Gâteaux differentiable. More precisely, we have

$$\begin{aligned}\Phi'(u)(v) =& M\left(\int_Q |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\\ &\times \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y\end{aligned}$$

and

$$\Psi'(u)(v) = \int_\Omega f(u(x))v(x)\mathrm{d}x,$$

for every $u, v \in \mathrm{E}_0$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $u \in \mathrm{E}_0$ such that $\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for every $v \in \mathrm{E}_0$. Hence, the critical points of the functional J_λ are weak solutions of problem $(\mathcal{L}_f^\lambda)$.

3. Proofs of Main Results

We prove Theorems 1.1 and 1.2 in this section. For this we need the following remark and lemma.

Remark 3.1. If the assumption $(\mathcal{F}_1)$ holds and $m = \min_{|t|=1} F(t)$, then by the same argument as in [9, Remark 3.1], there exists a constant C_2 such that $F(t) \geq m|t|^\beta - C_2$ for all $t \in \mathbb{R}$.

Lemma 3.1. *Assume that* $(\mathcal{F}_1)$ *holds and* $\lambda > 0$. *Then* $J_\lambda(u)$ *satisfies the* (PS)-*condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ such that $\{J_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|J_\lambda(u_n)| \leq c_0$, $|J'_\lambda(u_n)| \leq c_0$ for all $n \in \mathbb{N}$. Therefore, we infer to deduce from the definition of J'_λ and the assumption $(\mathcal{F}_1)$ that

$$\begin{aligned}c_0 + c_1\|u_n\|_{\mathrm{E}_0} \geq& \beta J_\lambda(u_n) - J'_\lambda(u_n)(u_n)\\ \geq& \left(\frac{2\beta}{\alpha}m_0 - m_1\right)\|u_n\|^\alpha_{\mathrm{E}_0} - \lambda\int_\Omega \left(\beta F(u_n(t)) - f(u_n(t))(u_n(t))\right)\mathrm{d}t\\ \geq& \left(\frac{2\beta}{\alpha}m_0 - m_1\right)\|u_n\|^\alpha_{\mathrm{E}_0},\end{aligned}$$

for some $c_1 > 0$. Since $\beta > \frac{2m_1\alpha}{m_0}$, this implies that (u_n) is bounded. Now, as the same argument in [10, Lemma 2.2 (i)], we can prove that $\{u_n\}$ converges strongly to u in E_0. Consequently, J_λ satisfies (PS)-condition. □

3.1. Proof of Theorem 1.1.

Proof. In our case it is clear that $J_\lambda(0) = 0$. Lemma 3.1 has shown that J_λ satisfies the (PS)-condition.

Step 1. Since $1 \le \alpha < \frac{2n}{n-2s}$, by Remark 2.1 the embedding $\mathrm{E}_0 \hookrightarrow \mathrm{L}^\alpha(\mathbb{R}^n)$ is compact and there exists $C_1 > 0$ such that for all $u \in \mathrm{E}_0$, $C_1\|u\|_{\mathrm{L}^\alpha(\mathbb{R}^n)} \le \|u\|_{\mathrm{E}_0}$ or

$$C_1^{2\alpha}\int_\Omega |u(x)|^\alpha \mathrm{d}x \le \Big(\int_Q |u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\Big)^\alpha,$$

which implies that

$$\lambda_\alpha := \inf_{u\in \mathrm{E}_0\setminus\{0\}} \frac{\int_Q |u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega |u(x)|^{2\alpha}\mathrm{d}x} > 0.$$

By the assumptions $(\mathcal{M})$ and $(\mathcal{F}_2)$, and since $f(t) \ge 0$ for all $t \in \mathbb{R}$, we can take $\varepsilon < 2\alpha$ sufficiently small such that for sufficiently great $\sigma > 0$, $|f(t)| \le \frac{\varepsilon m_0}{\alpha}|t|^{2\alpha-1}$ for all $|t| \ge \sigma$ and $|F(t)| \le \frac{\varepsilon m_0}{2\alpha^2}|t|^{2\alpha} + (\max_{|t|\le\sigma} f(t))|t|$. Thus, for every $u \in \mathrm{E}_0$

$$\Psi(u) \le \frac{\varepsilon m_0}{2\alpha^2}\int_\Omega |u(x)|^\alpha \mathrm{d}x + \max_{|t|\le\sigma} f(t)\int_\Omega |u(x)|\mathrm{d}x. \tag{3.1}$$

By Hölder inequality, we have

$$\int_\Omega |u(x)|\mathrm{d}x \le \sqrt{|\Omega|}\Big(\int_\Omega |u(x)|^2\mathrm{d}x\Big)^{\frac12}.$$

Then, by (3.1)

$$\begin{aligned}\Psi(u) &\le \frac{\varepsilon m_0}{2\alpha^2}\|u\|^{2\alpha}_{\mathrm{L}^\alpha(\Omega)} + \sqrt{|\Omega|}\max_{|t|\le\sigma} f(t)\Big(\int_\Omega |u(x)|^2\mathrm{d}x\Big)^{\frac12}\\ &\le \frac{\varepsilon m_0}{2\alpha^2}\|u\|^{2\alpha}_{\mathrm{L}^\alpha(\Omega)} + \sqrt{|\Omega|}\max_{|t|\le\sigma} f(t)\,\lambda_1^{-\frac12}\|u\|_{\mathrm{E}_0}\\ &\le \frac{\varepsilon m_0}{2\alpha^2}\|u\|^{2\alpha}_{\mathrm{E}_0} + \sqrt{|\Omega|}\max_{|t|\le\sigma} f(t)\,\lambda_1^{-\frac12}\|u\|_{\mathrm{E}_0}.\end{aligned}$$

Then, for any $u \in X$ by (2.2)

$$J_\lambda(u) \ge \frac{m_0}{2\alpha}\Big(1-\frac{\varepsilon}{\alpha}\Big)\|u\|^{2\alpha}_{\mathrm{E}_0} - C_2\max_{|t|\le\sigma} f(t)\,\|u\|_{\mathrm{E}_0}, \tag{3.2}$$

where $C_2 = \sqrt{\frac{|\Omega|}{\lambda_1}}$. Now, by means of $\alpha > \frac{\varepsilon}{2}$, $p > 1$ and (3.2), it follows that J_λ is a coercive functional and is bounded from below. Since J_λ satisfies (PS)-condition by Lemma 3.1, Theorem 2.1 follows that there exists a minimum point u_0 of J_λ on E_0 and $0 = J_\lambda(0) \ge J_\lambda(u_0)$ and $J'_\lambda(u_0) = 0$.

Step 2. Since u_0 is a minimum point of J_λ on E_0 we can consider $L > 0$ sufficiently large such that $J_\lambda(u_0) \le 0 < \inf_{u\in\partial B_L} J_\lambda(u)$ where $B_L = \{u \in \mathrm{E}_0 : \|u\|_{\mathrm{E}_0} < L\}$. Now we will show that there exists u_1 with $\|u_1\|_{\mathrm{E}_0} > L$ such that $J_\lambda(u_1) < \inf_{\partial B_L} J_\lambda(u)$. For this, let $\ell_1(t) \in \mathrm{E}_0$ and $u_1 = r\ell_1$, $r > 0$ where ℓ_1 corresponding to λ_1 is the first eigenfunction of $(\mathcal{L}^\lambda_f)$ and $\|\ell_1\|_{\mathrm{E}_0} = 1$. By Remark 3.1, there exist constants $a_1, a_2 > 0$

such that $F(t) \geq a_1|t|^\beta - a_2$ for all $t \in \mathbb{R}$. Thus,

$$J_\lambda(u_1) = (\Phi - \lambda\Psi)(r\ell_1) \leq \frac{m_1}{2\alpha}\|r\ell_1\|_{\mathrm{E}_0}^{2\alpha} - \lambda \int_\Omega F(r\ell_1(x))\mathrm{d}x$$
$$\leq \frac{m_1 r^{2\alpha}}{2\alpha} - \lambda r^\beta a_1 \int_\Omega |\ell_1(x)|^\beta \mathrm{d}x + \lambda a_2|\Omega|.$$

So by $\beta \geq \frac{2m_1\alpha}{m_0}$, there exists sufficiently large $r > L > 0$ such that $J_\lambda(r\ell_1) < 0$. Therefore, $\max\{J_\lambda(u_0), J_\lambda(u_1)\} < \inf_{u\in\partial B_L} J_\lambda(u)$. Then, Theorem 2.2 by $X := \mathrm{E}_0$ and $\varphi := J_\lambda$ gives the critical point u^*. Therefore, u_0 and u^* are two critical points of J_λ, which are two solutions of $(\mathcal{L}_f^\lambda)$. □

3.2. **Proof of Theorem** 1.2.

Proof. Put $X := \mathrm{E}_0$. It is clear that, J_λ is continuously Gâteaux differentiable. In view of (2.2) it is obvious that $J_\lambda(u)$ is even and $J_\lambda(0) = 0$.

Step 1. We will show that J_λ satisfies condition (i) in Theorem 2.3. The inequality (3.2) shows the coercivity of J_λ and together with (PS)-condition, by minimization theorem [22, Theorem 4.4] the functional J_λ has a minimum critical point u with $J_\lambda(u) \geq \alpha > 0$ and $\|u\|_{\mathrm{E}_0} = \rho$ for $\rho > 0$ small enough.

Step 2. We will show that J_λ satisfies condition (ii) in Theorem 2.3. Let $W \subset \mathrm{E}_0$ be a finite dimensional subspace. By Remark 3.1, there exist constants $a_1, a_2 > 0$ such that $F(t) \geq a_1|t|^\beta - a_2$ for all $t \in \mathbb{R}$. Now, For every $r > 0$ and $u \in W \setminus \{0\}$ with $\|u\|_{\mathrm{E}_0} = 1$, one has

$$J_\lambda(ru) = (\Phi - \lambda\Psi)(ru) \leq \frac{m_1}{2\alpha}\|ru\|_{\mathrm{E}_0}^{2\alpha} - \lambda \int_\Omega F(ru(x))\mathrm{d}x$$
$$\leq \frac{m_1 r^{2\alpha}}{2\alpha}\|u\|_{\mathrm{E}_0}^{2\alpha} - \lambda r^\beta a_1 \int_\Omega |u(x)|^\beta \mathrm{d}x + \lambda a_2|\Omega| \to -\infty, \quad r \to +\infty.$$

The above inequality implies that there exists r_0 such that $\|ru\|_{\mathrm{E}_0} > \rho$ and $J_\lambda(ru) < 0$ for every $r \geq r_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that $J_\lambda(u) \leq 0$ on $W \setminus B_{R(W)}$. According to Theorem 2.3, the functional $J_\lambda(u)$ possesses infinitely many critical points, i.e., the problem $(\mathcal{L}_f^\lambda)$ has infinitely many weak solutions. □

4. Examples and Remarks

In this section we present two examples and some remarks of our main results.

Example 4.1. Let $n = 2$, $s = \frac{1}{2}$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\} \subset \mathbb{R}^2$, $M(t) = tK(t)$ for all $t \in \mathbb{R}^+$ where $K(t)$ is 2-periodic extension of the function $k(t) = 2 - |t-1|$, $0 \leq t \leq 2$, $f(t) = 1 + t^8$ for all $t \in \mathbb{R}$. We observe that $\frac{2n}{n-2s} = 4$, thus M satisfies the condition $(\mathcal{M})$ by $m_0 = 1$, $m_1 = 2$ and $\alpha = 2$. Also, M and f are two continuous functions, $f(t) \geq 0$ for all $t \in \mathbb{R}$, $\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{\alpha-1}} = \lim_{\xi\to 0^+} \frac{1+\xi^8}{\xi} = +\infty$, thus the assumption $(\mathcal{F}_2)$ is satisfied. Moreover, taking into account that $\lim_{|\xi|\to+\infty} \frac{\xi f(\xi)}{F(\xi)} =$

$\lim_{|\xi|\to+\infty}\frac{\xi+\xi^9}{\xi+\frac{1}{9}\xi^9}=9>8=\frac{2m_1\alpha}{m_0}$, by choosing $\beta=9>8=\frac{2m_1\alpha}{m_0}$, there exists $\varrho>1$ such that the assumption $(\mathcal{F}_1)$ is fulfilled for all $|\xi|>\varrho$. Hence, by applying Theorem 1.1, for every $\lambda>0$, the problem

$$\begin{cases} -M\left(\displaystyle\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus(\Omega\times\Omega)}\frac{|u(x)-u(y)|^2}{|x-y|^3}\mathrm{d}x\mathrm{d}y\right) & \\ \times\displaystyle\int_{\mathbb{R}^2}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^3}\mathrm{d}y=\lambda(1+u^8), & \text{in}\,\Omega,\\ u=0, & \text{on}\,\partial\Omega,\end{cases}$$

possesses at least two nontrivial weak solutions in the space

$$\mathrm{H}_0^{1/2}:=\left\{u\in\mathrm{H}^{1/2}(\mathbb{R}^2):\,u=0\text{ a.e. in }\mathbb{R}^2\setminus\Omega\right\}.$$

Example 4.2. Let $n=2$, $s=\frac{1}{2}$, $\Omega=\{(x_1,x_2)\in\mathbb{R}^2:\,x_1^2+x_2^2\leq 4\}\subset\mathbb{R}^2$, $M(t)=(\frac{3}{2}+\frac{1}{2}\sin t)t$ for all $t\in\mathbb{R}^+$, $f(t)=1+t^9$ for all $t\in\mathbb{R}$. We observe that $\frac{2n}{n-2s}=4$, thus M satisfies the condition $(\mathcal{M})$ by $m_0=1$, $m_1=2$ and $\alpha=2$. Also, M and f are two continuous functions, f is odd and $\lim_{\xi\to0^+}\frac{f(\xi)}{\xi^{\alpha-1}}=\lim_{\xi\to0^+}\frac{1+\xi^9}{\xi}=+\infty$, thus the assumption $(\mathcal{F}_2)$ is satisfied. Moreover, taking into account that $\lim_{|\xi|\to+\infty}\frac{\xi f(\xi)}{F(\xi)}=\lim_{|\xi|\to+\infty}\frac{\xi+\xi^{10}}{\frac{1}{2}\xi+\frac{1}{10}\xi^{10}}=10>8=\frac{2m_1\alpha}{m_0}$, by choosing $\beta=10>8=\frac{2m_1\alpha}{m_0}$, the assumption $(\mathcal{F}_1)$ is fulfilled. Hence, by choosing $\sigma=\frac{1}{2}$ and applying Theorem 1.2, for every $\lambda>0$, the problem

$$\begin{cases} -M\left(\displaystyle\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus(\Omega\times\Omega)}\frac{|u(x)-u(y)|^2}{|x-y|^3}\mathrm{d}x\mathrm{d}y\right) & \\ \times\displaystyle\int_{\mathbb{R}^2}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^3}\mathrm{d}y=\lambda(1+u^9), & \text{in}\,\Omega,\\ u=0, & \text{on}\,\partial\Omega,\end{cases}$$

has infinitely many weak solutions in the space $\mathrm{H}_0^{1/2}$.

Remark 4.1. Example 4.2 shows that our existence results to establish infinitely many solutions for the problem $(\mathcal{L}_f^\lambda)$ in Theorem 1.2 is different from the existence results of Molica Bicsi in [25, Theorem 1.1]. Because, firstly in Example 4.2 we have $M\neq 1$, while in [25, Theorem 1.1], $M\equiv 1$, and the second the function f in [25, Theorem 1.1] should satisfy in

$$|f(t)|\leq a_1+a_2|t|^{q-1},\qquad a_1,a_2>0,\,q\in\left(2,\frac{2n}{n-2s}\right),\,t\in\mathbb{R},\tag{4.1}$$

while in Example 4.2, $\frac{2n}{n-2s}=4$ and $f(t)=1+t^9$, and so f does not apply to (4.1).

Remark 4.2. By [28, Subsection 1.1], if $f(0)\neq 0$, then Theorem 1.1 ensures the existence of two nontrivial weak solutions for the problem $(\mathcal{L}_f^\lambda)$. If the condition $f(0)\neq 0$ does not hold, the second solution u_2 of the problem $(\mathcal{L}_f^\lambda)$ may be trivial, but the problem has at least a nontrivial solution. Moreover, by the same argument as [28, Corollary 3] we can prove that, under the condition that $f(0)=0$, the solutions given by Theorem 1.1 has constant sign, i.e., Theorem 1.1 provides non-negative (non-positive) solutions.

Remark 4.3. By the similar arguments as given in the proof of [28, Subsection 4.1] the non-triviality of the second weak solution ensured by Theorem 1.1 can be achieved also in the case $f(0) = 0$ requiring the extra condition at zero in the form of

$$\limsup_{\xi \to 0^+} \frac{f(\xi)}{|\xi|} = \infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{f(\xi)}{|\xi|} > -\infty. \tag{4.2}$$

Indeed, let $\lambda > 0$ and let Φ and Ψ be as given in Section 3. Due to Theorem 2.1 and Lemma 3.1, $J_\lambda = \Phi - \lambda\Psi$ has a critical point u_λ that is a global minimum of J_λ. We will prove that the function u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{4.3}$$

Owing to the assumptions (4.2), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $0 < \sigma < 1$) such that $\lim_{n\to+\infty} \frac{f(\xi_n)}{|\xi_n|} = +\infty$ and $F(\xi) \geq \kappa|\xi|^2$ for every $\xi \in [0, \sigma]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in X$ such that $v(t) \in [0, 1]$ for every $t \in \Omega$, $v(t) = 1$ for every $t \in \mathcal{G}$ and $v(t) = 0$ for every $x \in \Omega \setminus D$. Hence, fix $N > 0$ and consider a real positive number η with

$$N < \frac{2\alpha\eta|\mathcal{G}| + 2\alpha\kappa \int_{D\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_1\|v\|^{2\alpha}_{\mathrm{E}_0}}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and $F(\xi_n) \geq \eta|\xi_n|^2$ for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n large enough), one has

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} \geq \frac{F(\xi_n)|\mathcal{G}| + \int_{D\setminus\mathcal{G}} F(\xi_n v(t))\mathrm{d}t}{\Phi(\xi_n v)} > \frac{2\alpha\eta|\mathcal{G}| + 2\alpha\kappa \int_{D\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_1\|v\|^{2\alpha}_{\mathrm{E}_0}} > N.$$

Since N could be arbitrarily large, we get $\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty$, from which (4.3) clearly follows. So, there exists a sequence $\{\zeta_n\} \subset X$ strongly converging to zero such that, for n large enough, $J_\lambda(\zeta_n) = \Phi(\zeta_n) - \lambda\Psi(\zeta_n) < 0$. Since u_λ is a global minimum of J_λ, we obtain $J_\lambda(u_\lambda) < 0$, so that u_λ is not trivial.

Remark 4.4. We observe that if f is non-negative, Theorem 1.1 is a bifurcation result in the sense that the pair $(0, 0) \in \mathrm{E}^\lambda_f \subset \mathrm{E}_0 \times \mathbb{R}$ with

$$\mathrm{E}^\lambda_f := \Big\{(u_\lambda, \lambda) \in \mathrm{E}_0 \times (0, \infty) : u_\lambda \text{ is a non-trivial weak solution of } (\mathcal{L}^\lambda_f)\Big\}.$$

Practically, by the proof of Theorem 1.1, $\|u_\lambda\|_{\mathrm{E}_0} \to 0$ as $\lambda \to 0$. Hence, there exist two sequences $\{u_j\}$ in E_0 and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that $\lambda_j \to 0^+$ and $\|u_j\| \to 0$, as $j \to \infty$. Moreover, since f is nonnegative, $\Psi(u) < 0$ for all $u \in \mathbb{R}$ and thus the mapping $(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing. Hence, for every

$\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, the weak solutions u_{λ_1} and u_{λ_2} ensured by Theorem 1.1 are different.

Remark 4.5. If $f(u)$ is an odd function we can give the same result as Theorem 1.2 by setting the following assumptions on nonlinear term:

($\mathcal{F}_3$) there exist constants $R > 0$ and $0 < \lambda L_1 < \frac{1}{2}\min\{1, m_0\}$ such that $F(u) \leq L_1|u|^2$ for all $u \in \mathbb{R}$ with $|u| \leq R$;

($\mathcal{F}_4$) there exist constants $R_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \beta$ such that $F(u) \geq \delta_1|u|^{\alpha_1}$, for all $u \in \mathbb{R}$ with $|u| \geq R$;

($\mathcal{F}_5$) there exist constants $\beta > \frac{m_1\alpha}{m_0}$, $\delta_1 \geq 0$ and $0 < \alpha_2 < 2$ such that $\nu F(\xi) - \xi f(\xi) \leq \delta_2|u|^{\alpha_2}$.

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