

## NOTE ON THE MULTIFRACTAL FORMALISM OF COVERING NUMBER ON THE GALTON-WATSON TREE

NAJMEDINE ATTIA<sup>1,2</sup> AND MERIEM BEN HADJ KHALIFA<sup>3</sup>

ABSTRACT. We consider, for  $t$  in the boundary of Galton-Watson tree ( $\partial\mathbb{T}$ ), the covering number  $\mathbb{N}_n(t)$  by cylinder of generation  $n$ . For a suitable set  $I$  and a sequence  $(s_{n,\gamma})$ , we establish almost surely, and uniformly on  $\gamma$ , the Hausdorff and packing dimensions of the set  $\{t \in \partial\mathbb{T} : \mathbb{N}_n(t) - nb \sim s_{n,\gamma}\}$  for  $b \in I$ .

### 1. INTRODUCTION AND MAIN RESULTS

Let  $(N, X)$  be a random vector with independent components taking values in  $\mathbb{N}^2$ , where  $\mathbb{N}$  denotes the set of non-negative integers. Then let  $\{(N_u, X_u)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$  be a family of independent copies of the vector  $(N, X)$  indexed by the set of finite words over the alphabet  $\mathbb{N}_+$ : the set of positive integers ( $n = 0$  corresponds to the empty sequence denoted  $\emptyset$ ). Let  $\mathbb{T}$  be the Galton-Watson tree with defining elements  $\{N_u\}$ : we have  $\emptyset \in \mathbb{T}$ , if  $u \in \mathbb{T}$  and  $i \in \mathbb{N}_+$  then  $ui$ , the concatenation of  $u$  and  $i$ , belongs to  $\mathbb{T}$  if and only if  $1 \leq i \leq N_u$  and if  $ui \in \mathbb{T}$ , then  $u \in \mathbb{T}$ . Similarly, for each  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ , denote by  $\mathbb{T}(u)$  the Galton-Watson tree rooted at  $u$  and defined by the  $\{N_{uv}\}$ ,  $v \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ .

We assume that  $\mathbb{E}(N) > 1$  so that the Galton-Watson tree is supercritical. We also assume that the probability of extinction is equal to 0, so that  $\mathbb{P}(N \geq 1) = 1$ .

For each infinite word  $t = t_1 t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}_+}$  and  $n \geq 0$ , we set  $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$  ( $t_{|0} = \emptyset$ ). If  $u \in \mathbb{N}_+^n$  for some  $n \geq 0$ , then  $n$  is the length of  $u$  and it is denoted by  $|u|$ . We denote by  $[u]$  the set of infinite words  $t \in \mathbb{N}_+^{\mathbb{N}_+}$  such that  $t_{|u} = u$ .

---

*Key words and phrases.* Random covering, Hausdorff dimension, indexed martingale, Galton-Watson tree.

2020 *Mathematics Subject Classification.* Primary: 60G50. Secondary: 28A78.

DOI 10.46793/KgJMat2501.043A

*Received:* April 05, 2021.

*Accepted:* October 28, 2021.

The set  $\mathbb{N}_+^{\mathbb{N}}$  is endowed with the standard ultrametric distance

$$d : (u, v) \mapsto e^{-\sup\{|w| : u \in [w], v \in [w]\}},$$

with the convention  $\exp(-\infty) = 0$ . The boundary of the Galton-Watson tree  $\mathbb{T}$  is defined as the compact set

$$\partial\mathbb{T} = \bigcap_{n \geq 1} \bigcup_{u \in \mathbb{T}_n} [u],$$

where  $\mathbb{T}_n = \mathbb{T} \cap \mathbb{N}_+^n$ .

We consider  $X_u$  as the covering number of the cylinder  $[u]$ , that is to say, the cylinder  $[u]$  is cut off with probability  $p_0 = \mathbb{P}(X = 0)$  and is covered  $m$  times with probability  $p_m = \mathbb{P}(X = m)$ ,  $m = 1, 2, \dots$

For  $t \in \partial\mathbb{T}$ , set

$$\mathbf{N}_n(t) = \sum_{k=1}^n X_{t_1 \dots t_k}.$$

Since this quantity depends on  $t_1 \dots t_n$  only, we also denote by  $\mathbf{N}_n(u)$  the constant value of  $\mathbf{N}_n(\cdot)$  over  $[u]$  whenever  $u \in \mathbb{T}_n$ . The quantity  $\mathbf{N}_n(t)$  is called the covered number (or more precisely the  $n$ -covered number) of the point  $t$  by cylinder of generation  $k$ ,  $k = 1, 2, \dots, n$ .

Consider an individual infinite branch  $t_1 \dots t_n \dots$  in  $\partial\mathbb{T}$ . When  $\mathbb{E}(X)$  is defined, the strong law of large number yields  $\lim_{n \rightarrow \infty} n^{-1} \mathbf{N}_n(t) = \mathbb{E}(X)$ . It is also well known, in the theory of the birth process, (see [15]) that almost surely (a.s.)  $\lim_{n \rightarrow \infty} \mathbf{N}_n(t) = +\infty$  for every  $t \in \mathcal{D} = \{0, 1\}^{\mathbb{N}}$  if and only if

$$p_0 = \mathbb{P}(X = 0) < \frac{1}{2}.$$

If this condition is satisfied, then a.s. every point is infinitely covered.

We consider, for  $b \in \mathbb{R}$ , the set

$$E_b = \left\{ t \in \partial\mathbb{T} : \lim_{n \rightarrow \infty} \frac{\mathbf{N}_n(t)}{n} = b \right\}.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [3, 8, 11, 14, 16, 21] and [4, 7] for a general case. All these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [1, 2, 19] for the study of Mandelbrot measures dimension).

We will assume that the free energy of  $X$  defined as

$$\tau(q) = \log \mathbb{E} \left( \sum_{i=1}^N e^{qX_i} \right)$$

is finite over  $\mathbb{R}$ . We will assume, without loss of generality, that  $X$  is not constant so that the function  $\tau$  is strictly convex. Let  $\tau^*$  stand for the Legendre transform of the function  $\tau$ , defined as

$$\tau^*(b) := \inf_{q \in \mathbb{R}} \left( \tau(q) - qb \right), \quad b \in \mathbb{R}.$$

We say that the multifractal formalism holds at  $b \in \mathbb{R}$  if

$$\dim E_b = \text{Dim } E_b = \tau^*(b),$$

where  $\dim E_b$  is the Hausdorff dimension of  $E_b$  and  $\text{Dim } E_b$  is the packing dimension of  $E_b$  (see Section A for the definition). In the following, we define the sets

$$\begin{aligned} J &= \left\{ q \in \mathbb{R}; \tau(q) - q\tau'(q) > 0 \right\}, \\ \Omega_\alpha^1 &= \text{int} \left\{ q : \mathbb{E} \left[ \left| \sum_{i=1}^N e^{qX_i} \right|^\alpha \right] < \infty \right\}, \\ \Omega^1 &= \bigcup_{\alpha \in (1,2]} \Omega_\alpha^1, \\ \mathcal{J} &= J \cap \Omega^1 \quad \text{and} \quad I = \left\{ \tau'(q); q \in \mathcal{J} \right\}. \end{aligned}$$

*Remark 1.1.* It is well known, see [6, Proposition 3.1], that  $L = \{\alpha \in \mathbb{R}, \tau^*(\alpha) \geq 0\}$ , is a convex, compact and non-empty set. In addition, if we assume that  $J = \mathcal{J}$  then  $I = \text{int}(L)$ , where  $\text{int}(L)$  is the interior of  $L$  (see also [6, Proposition 3.1.]) In particular,  $I$  is an interval.

Next, we define for  $b, \gamma \in \mathbb{R}$  and for any positive sequence  $s^\gamma = \{s_{n,\gamma}\}_n$  such that  $s_{n,\gamma} = o(n)$  and  $\gamma \mapsto s_{n,\gamma}$  is analytic function, the set

$$E_{b,s^\gamma} = \left\{ t \in \partial\mathbb{T} : \mathbf{N}_n(t) - nb \sim s_{n,\gamma} \text{ as } n \rightarrow +\infty \right\},$$

where  $\mathbf{N}_n(t) - nb \sim s_{n,\gamma}$  means that  $(\mathbf{N}_n(t) - nb)_n$  and  $(s_{n,\gamma})_n$  are two equivalent sequences. It is clear that  $E_{b,s^\gamma} \subset E_b$ . So, we can get with a simple covering argument, with probability 1, for all  $b \in \mathbb{R}$  and  $\gamma \in \mathbb{R}$ ,

$$(1.1) \quad \dim E_{b,s^\gamma} \leq \dim E_b \leq \text{Dim } E_b \leq \tau^*(b),$$

(see Proposition 1 in [5] and Proposition 2.7 in [4]). Let us mention that the methods used to compute Hausdorff dimension of the sets  $E_b$  in, for example, [4, 7, 17, 18]) do not give results on  $\dim E_{b,s^\gamma}$ . These sets were considered by Kahane and Fan in [15]. The authors considered the space  $\{0, 1\}^{\mathbb{N}}$  and they compute, for each  $b$ , almost surely (a.s.), the Hausdorff dimension of  $E_{b,s^\gamma}$  under the hypothesis :

$$s_{n,\gamma} = o(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = o(1) \quad \text{and} \quad \sqrt{n \ln \ln n} = o(s_{n,\gamma}).$$

A special case of a sequence satisfying the above hypothesis is  $s_{n,\gamma} = n^\gamma$  with  $\gamma \in (1/2, 1)$ . Later, Attia in [5], gives a stronger result in the sense that, a.s. for all  $b \in I$ , he computed the Hausdorff dimensions of the sets  $E_{b,s^\gamma}$  under the hypothesis

$$(1.2) \quad s_{n,\gamma} = o(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = o(1)$$

and there exists  $\epsilon_n \rightarrow 0$  such that

$$(1.3) \quad \sum_{n \geq 1} \exp \left( -\epsilon \sum_{k=1}^n \epsilon_k \eta_k(\gamma)^2 \right) < +\infty, \quad \text{for all } \epsilon > 0.$$

In particular, we can choose

$$s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma} \quad \text{with } \gamma \in (0, 1/2).$$

**Theorem 1.1** ([5]). *Let  $s^\gamma$  be a positive sequence satisfying (1.2) and (1.3). Then, a.s. for all  $b \in I$*

$$\dim E_{b,s^\gamma} = \dim E_b = \tau^*(b).$$

This requires, for a given sequence  $s^\gamma$ , a simultaneous building of an inhomogeneous Mandelbrot measure and a computing of their dimensions. In particular, for

$$s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma},$$

we have for all  $\gamma \in (0, 1/2)$ , a.s.  $\dim E_{b,s^\gamma} = \tau^*(b)$ . To state our main result, let  $s^\gamma = (s_{n,\gamma})_n$  be a positive sequence and we define the set  $\Lambda_s$  to be any set of  $\mathbb{R}$  such that

$$(1.4) \quad \Lambda_s \subseteq \left\{ \gamma \in \mathbb{R}, \text{ such that } (s_{n,\gamma}) \text{ satisfies (1.2) and (1.3)} \right\}$$

and, for  $k \geq 1$

$$(1.5) \quad \tilde{\eta}_k = \inf_{\gamma \in \Lambda_s} \eta_k(\gamma) > 0.$$

We suppose the following hypothesis.

*Hypothesis 1.2.* There exists a sequence  $\epsilon_n \rightarrow 0$  such that

$$\sum_{n \geq 1} \exp \left( - \epsilon \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right) < +\infty, \quad \text{for all } \epsilon > 0.$$

Clearly this hypothesis is satisfied, for  $s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma}$ , with  $\Lambda_s = [\epsilon, 1/2)$ ,  $\epsilon > 0$ . Applying the previous theorem we get the conclusion for each  $\gamma \in \Lambda_s$  a.s. The goal of this note is to give a uniform result on  $\gamma$ . In addition, we determine the packing dimensions of the sets  $E_{b,s^\gamma}$ . More precisely we have the following result.

**Theorem 1.3.** *Let  $s^\gamma = (s_{n,\gamma})_{n \geq 1}$  be a positive sequence and consider a set  $\Lambda_s$  satisfying (1.4) and (1.5). Under Hypothesis 1.2, we have, a.s.. for all  $b \in I$  and for all  $\gamma \in \Lambda_s$*

$$\dim E_{b,s^\gamma} = \dim E_b = \text{Dim } E_b = \text{Dim } E_{b,s^\gamma} = \tau^*(b).$$

## 2. CONSTRUCTION OF INHOMOGENEOUS MANDELBROT MEASURES

We define, for  $(q, p) \in \mathcal{J} \times [1, \infty)$ , the function

$$\varphi(p, q) = \exp \left( \tau(pq) - p\tau(q) \right).$$

From [5], for all nontrivial compact sets  $K \subset \mathcal{J}$  there exist  $1 < p_K < 2$  and  $\tilde{p}_K > 1$  such that we have

$$(2.1) \quad \sup_{q \in K} \varphi(p_K, q) < 1, \quad \text{for all } 1 < p \leq p_K,$$

and

$$(2.2) \quad \sup_{q \in K} \mathbb{E} \left( \left( \sum_{i=1}^N e^{qX_i} \right)^{\tilde{p}_K} \right) < \infty.$$

Now, we will construct the inhomogeneous Mandelbrot measure. For  $q \in \mathcal{J}$  and  $k \geq 1$ , we define  $\psi_k(q, \gamma)$  as the unique  $t$ , such that

$$\tau'(t) = \tau'(q) + \eta_k(\gamma).$$

For  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$  and  $q \in \mathcal{J}$  we define, for  $1 \leq i \leq N_u$

$$V(ui, q) = \frac{\exp(qX_{ui})}{\mathbb{E} \left( \sum_{i=1}^N \exp(qX_i) \right)} = \exp(qX_{ui} - \tau(q))$$

and, for all  $n \geq 0$

$$Y_n(q, \gamma, u) = \sum_{v_1 \cdots v_n \in \mathbb{T}_n(u)} \prod_{k=1}^n V(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(q, \gamma)).$$

When  $u = \emptyset$ , this quantity will be denoted by  $Y_n(q, \gamma)$  and when  $n = 0$ , their values equals 1.

The sequence  $(Y_n(q, \gamma, u))_{n \geq 1}$  is a positive martingale with expectation 1, which converges almost surely and in  $L^1$  norm to a positive random variable  $Y(q, \gamma, u)$  (see [9] or [10, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits.

**Proposition 2.1.** (a) *Let  $\mathbf{K} = K \times K_\gamma$  be a compact subset of  $\mathcal{J} \times \Lambda_s$ . There exists  $p_{\mathbf{K}} \in (1, 2]$  such that for all  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$  the continuous functions  $(q, \gamma) \in \mathbf{K} \mapsto Y_n(q, \gamma, u)$  converge uniformly, almost surely and in  $L_{p_{\mathbf{K}}}$  norm, to a limit  $(q, \gamma) \in \mathbf{K} \mapsto Y(q, \gamma, u)$ . In particular,  $\mathbb{E}(\sup_{(q, \gamma) \in \mathbf{K}} Y(q, \gamma, u)^{p_{\mathbf{K}}}) < \infty$ . Moreover,  $Y(\cdot, \cdot, u)$  is positive almost surely.*

*In addition, for all  $n \geq 0$ ,  $\sigma(\{(X_{u_1}, \dots, X_{u_{N_u}}), u \in \mathbb{T}_n\})$  and  $\sigma(\{Y(\cdot, \cdot, u), u \in \mathbb{T}_{n+1}\})$  are independent, and the random functions  $Y(\cdot, \cdot, u), u \in \mathbb{T}_{n+1}$ , are independent copies of  $Y(\cdot, \cdot) := Y(\cdot, \cdot, \emptyset)$ .*

(b) *With probability 1, for all  $q \in \mathcal{J}$  and  $\gamma \in \Lambda_s$ , the weights*

$$\mu_q^\gamma([u]) = \left[ \prod_{k=1}^n \exp(\psi_k(q, \gamma) X_{u_1 \dots u_k} - \tau(\psi_k(q, \gamma))) \right] Y(q, \gamma, u)$$

*define a measure on  $\partial \mathbb{T}$ , where  $n = |u|$ .*

The measure  $\mu_q^\gamma$  will be used to approximate from below the Hausdorff dimension of the set  $E_{b,s^\gamma}$ .

*Proof.* (a) Fix a compact  $K \subset \mathcal{J}$  and a compact  $K_\delta \subset \Lambda_s$ . Since  $\eta_k(\gamma) = o(1)$ , we can fix, without loss of generality, a compact neighborhood  $K' \subset \mathcal{J}$  of  $K$  and suppose that,

$$\forall (q, \gamma) \in \mathbf{K} = K \times K_\gamma, \quad \text{for all } k \geq 1, \psi_k(q, \gamma) \in K'.$$

Fix a compact neighborhood  $\mathbf{K}'' = K'' \times K''_\gamma$  of  $K' \times K_\gamma$ . By (2.2), we can find  $\tilde{p}_{\mathbf{K}''} > 1$ , such that

$$\sup_{q \in \mathbf{K}''} \mathbb{E} \left( \left( \sum_{i=1}^N e^{qX_i} \right)^{\tilde{p}_{\mathbf{K}''}} \right) < \infty.$$

By (2.1), we can fix  $1 < p_{\mathbf{K}} \leq \min(2, \tilde{p}_{\mathbf{K}''})$  such that  $\sup_{q \in \mathbf{K}''} \varphi(p_{\mathbf{K}}, q) < 1$ . Then for each  $(q, \gamma) \in K' \times K$ , there exists a neighborhood  $V_q \times V_\gamma \subset \mathbb{C}^2$  of  $(q, \gamma)$ , whose projection to  $\mathbb{R}^2$  is contained in  $\mathbf{K}''$ , and such that for all  $u \in \mathbb{T}$ ,  $(z, z') \in V_q \times V_\gamma$  and  $k \geq 1$ , the random variable

$$V(u, z) = \frac{\exp(zX_u)}{\mathbb{E} \left( \sum_{i=1}^N \exp(zX_i) \right)}, \quad \Gamma(z) = \frac{\mathbb{E} \left( \sum_{i=1}^N X_i \exp(zX_i) \right)}{\mathbb{E} \left( \sum_{i=1}^N \exp(zX_i) \right)}$$

and the analytic extension of  $\eta_k$ , denoted also by  $\eta_k$ , are well defined. For  $(z, z') \in V_q \times V_\gamma$  and  $k \geq 1$ , we define  $\psi_k(z, z')$  as the unique  $t$  such that

$$\Gamma(t) = \Gamma(z) + |\eta_k(z')|.$$

Moreover, we have

$$\sup_{z \in V_q} \varphi(p_{\mathbf{K}}, z) < 1, \quad \text{where } \varphi(p_{\mathbf{K}}, z) = \frac{\mathbb{E} \left( \sum_{i=1}^N |e^{zX_i}|^{p_{\mathbf{K}}} \right)}{\left| \mathbb{E} \left( \sum_{i=1}^N e^{zX_i} \right) \right|^{p_{\mathbf{K}}}}.$$

By extracting a finite covering of  $K' \times K_\gamma$  from  $\cup_{q, \gamma} V_q \times V_\gamma$ , we find a neighborhood  $\mathbf{V} = V_{\mathbf{K}} \times V_{\mathbf{K}\gamma} \subset \mathbb{C}^2$  of  $K' \times K_\gamma$  such that

$$\sup_{z \in V_{\mathbf{K}}} \varphi(p_{\mathbf{K}}, z) < 1$$

and for all  $(z, z') \in \mathbf{V}$ ,  $\psi_k(z, z')$  is defined and belongs to  $V_{\mathbf{K}}$ . Since the projection of  $V_{\mathbf{K}}$  to  $\mathbb{R}$  is included in  $\mathbf{K}''$  and the mapping  $z \mapsto \mathbb{E} \left( \sum_{i=1}^N e^{zX_i} \right)$  is continuous and does not vanish on  $V_{\mathbf{K}}$ , by considering a smaller neighborhood of  $K'$  included in  $V_{\mathbf{K}}$  if necessary, we can assume that

$$C_{V_{\mathbf{K}}} = \sup_{z \in V_{\mathbf{K}}} \mathbb{E} \left( \left| \sum_{i=1}^N e^{zX_i} \right|^{p_{\mathbf{K}}} \right) \left| \mathbb{E} \left( \sum_{i=1}^N e^{zX_i} \right) \right|^{-p_{\mathbf{K}}} < \infty.$$

Now, for  $u \in \mathbb{T}$ , we define the analytic extension to  $\mathbf{V}$  of  $Y_n(q, \gamma, u)$  given by

$$\begin{aligned} Y_n(z, z', u) &= \sum_{v \in \mathbb{T}_n(u)} \prod_{k=1}^n V(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(z, z')) \\ &= \left[ \prod_{k=1}^n \mathbb{E} \left( \sum_{i=1}^N e^{\psi_k(z, z') X_i} \right) \right]^{-1} \sum_{v \in \mathbb{T}_n(u)} \prod_{k=1}^n e^{\psi_{|u|+k}(z, z') X(uv_k)}. \end{aligned}$$

We denote also  $Y_n(z, z', \emptyset)$  by  $Y_n(z, z')$ . By Lemma 3 in [5], there exists a constant  $C_{p_K}$  such that for all  $(z, z') \in \mathbf{V}$

$$\begin{aligned} &\mathbb{E} \left( |Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) \\ &\leq C_{p_K} \mathbb{E} \left( \left| \sum_{i=1}^N V(i, \psi_n(z, z')) \right|^{p_K} \right) \prod_{k=1}^{n-1} \mathbb{E} \left( \sum_{i=1}^N |V(i, \psi_k(z, z'))|^{p_K} \right). \end{aligned}$$

Notice that  $\mathbb{E} \left( \sum_{i=1}^N |V(i, \psi_k(z, z'))|^{p_K} \right) = \varphi(p_K, \psi_k(z, z'))$ . Then

$$\begin{aligned} \mathbb{E} \left( |Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) &\leq C_{p_K} \mathbb{E} \left( \left| \sum_{i=1}^N V(i, \psi_n(z, z')) \right|^{p_K} \right) \prod_{k=1}^{n-1} \varphi(p_K, \psi_k(z, z')). \\ &\leq C_{p_K} C_{V_K} \prod_{k=1}^{n-1} \sup_{z \in V_K} \varphi(p_K, z), \end{aligned}$$

where we have used the fact that  $\psi_k(z, z') \in V_K$  for all  $k \geq 1$ . With probability 1, the functions  $(z, z') \in \mathbf{V} \mapsto Y_n(z, z')$ ,  $n \geq 0$ , are analytic. Fix a closed polydisc  $D(z_0, 2\rho) \subset \mathbf{V}$  with  $z_0 = (z_1, z'_1)$  and  $\rho = (\rho_1, \rho_2)$ . Theorem B.1 gives

$$\sup_{(z, z') \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')| \leq 4 \int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))| dt,$$

where, for  $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned} \mathbb{E} \left( \sup_{z \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) &\leq \mathbb{E} \left( \left( 4 \int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))| dt \right)^{p_K} \right) \\ &\leq 4^{p_K} \mathbb{E} \left( \int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))|^{p_K} dt \right) \\ &= 4^{p_K} \int_{[0,1]^2} \mathbb{E} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))|^{p_K} dt \\ &\leq 4^{p_K} C_{V_K} C_{p_K} \prod_{k=1}^{n-1} \sup_{z \in V_K} \varphi(p_K, z). \end{aligned}$$

Since  $\sup_{z \in V_K} \varphi(p_K, z) < 1$ , it follows that

$$\sum_{n \geq 1} \left\| \sup_{(z, z') \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')| \right\|_{p_K} < \infty.$$

This implies,  $(z, z') \mapsto Y_n(z, z')$  converges uniformly, almost surely and in  $L^{p_K}$  norm over the compact  $D(z_0, \rho)$  to a limit  $(z, z') \mapsto Y(z, z')$ . This also implies that

$$\left\| \sup_{z \in D(z_0, \rho)} Y(z, z') \right\|_{p_K} < \infty.$$

Since  $K$  can be covered by finitely many such discs  $D(z_0, \rho)$  we get the uniform convergence, almost surely and in  $L^{p_K}$  norm, of the sequence  $((q, \gamma) \in K \mapsto Y_n(q, \gamma))_{n \geq 1}$  to  $(q, \gamma) \in K \mapsto Y(q, \gamma)$ . Moreover, since  $\mathcal{J} \times \Lambda_s$  can be covered by a countable union of such compact  $K$  we get the simultaneous convergence for all  $(q, \gamma) \in \mathcal{J} \times \Lambda_s$ . The same holds simultaneously for all the functions  $(q, \gamma) \in \mathcal{J} \times \Lambda_s \mapsto Y_n(q, \gamma, u)$ ,  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ , because  $\bigcup_{n \geq 0} \mathbb{N}_+^n$  is countable.

To finish the proof of Proposition 2.1 (1), we must show that with probability 1,  $(q, \gamma) \in K \mapsto Y(q, \gamma)$  does not vanish. Without loss of generality we can suppose that  $K = [0, 1]^2$ . If  $I$  is a dyadic closed subcube of  $[0, 1]^2$ , we denote by  $E_I$  the event  $\{\exists (q, \gamma) \in I : Y(q, \gamma) = 0\}$ . Let  $I_0, I_1, I_2, I_3$  stand for the  $2^2$  dyadic intervals of  $I$  in the next generation. The event  $E_I$  being a tail event of probability 0 or 1. If we suppose that  $\mathbb{P}(E_I) = 1$ , then there exists  $j \in \{0, 1, 2, 3\}$  such that  $\mathbb{P}(E_{I_j}) = 1$ . Suppose now that  $\mathbb{P}(E_K) = 1$ . The previous remark allows to construct a decreasing sequence  $(I(n))_{n \geq 0}$  of dyadic subcubes of  $K$  such that  $\mathbb{P}(E_{I(n)}) = 1$ . Let  $(q_0, \gamma_0)$  be the unique element of  $\bigcap_{n \geq 0} I(n)$ . Since  $(q, \gamma) \mapsto Y(q, \gamma)$  is continuous we have  $\mathbb{P}(Y(q_0, \gamma_0) = 0) = 1$ , which contradicts the fact that  $(Y_n(q_0, \gamma_0))_{n \geq 1}$  converges to  $Y(q_0, \gamma_0)$  in  $L^1$ .

(b) It is a consequence of the branching property

$$Y_{n+1}(q, \gamma, u) = \sum_{i=1}^N \exp(\psi_{n+1}(q, \gamma) X_{ui} - \tau(\psi_{n+1}(q, \gamma))) Y_n(q, \gamma, ui). \quad \square$$

### 3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 can be deduced from the two following propositions. Their proof are developed in the next section.

**Proposition 3.1.** *Suppose Hypothesis 1.2, with probability 1, for all  $q \in \mathcal{J}$  and  $\gamma \in \Lambda_s$ ,*

$$N_n(t) - nb \sim s_{n, \gamma}, \quad \text{for } \mu_q^\gamma\text{-almost every } t \in \partial\mathbb{T},$$

where  $b = \tau'(q)$ .

**Proposition 3.2.** *With probability 1, for all  $(q, \gamma) \in \mathcal{J} \times \Lambda_s$ , for  $\mu_q^\gamma$ -almost every  $t \in \partial\mathbb{T}$*

$$\lim_{n \rightarrow \infty} \frac{\log Y(q, \gamma, t|_n)}{n} = 0.$$



From Proposition 3.1, we have with probability 1, for all  $q \in \mathcal{J}$  and  $\gamma \in \Lambda_s$ , that  $\mu_q^\gamma(E_{b,s^\gamma}) = 1$ , ( $b = \tau'(q)$ ). In addition, with probability 1, for all  $(q, \gamma) \in \mathcal{J} \times \Lambda_s$ , for  $\mu_q^\gamma$ -almost every  $t \in E_{b,s^\gamma}$ , from the same Proposition and proposition 3.2, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\log(\mu_q^\gamma[t_{|n}|])}{\log(\text{diam}([t_{|n}|]))} \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \log \left( \prod_{k=1}^n \exp(\psi_k(q, \gamma) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma))) Y(q, \gamma, t_{|n|}) \right) \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{k=1}^n \psi_k(q, \gamma) X_{t_1 \dots t_k} + \frac{1}{n} \sum_{k=1}^n \tau(\psi_k(q, \gamma)) - \frac{\log Y(q, \gamma, t_{|n|})}{n} \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{k=1}^n \psi_k(q, \gamma) X_{t_1 \dots t_k} + \frac{1}{n} \sum_{k=1}^n \tau(\psi_k(q, \gamma)). \end{aligned}$$

Since  $\eta_k(\gamma) = o(1)$  and then  $\psi_k(q, \gamma) \rightarrow q$ , we get

$$\lim_{n \rightarrow \infty} \frac{\log(\mu_q^\gamma[t_{|n}|])}{\log(\text{diam}([t_{|n}|]))} = -q\tau'(q) + \tau(q) = \tau^*(\tau'(q)).$$

We deduce the result from the mass distribution principle (Theorem A.1) and (1.1).

#### 4. PROOF OF PROPOSITIONS 3.1 AND 3.2

**4.1. Proof of Proposition 3.1.** Let  $K = K \times K_\gamma$  be a compact subset of  $\mathcal{J} \times \Lambda_s$ . For  $b = \tau'(q)$ ,  $q \in \mathcal{J}$ ,  $\gamma \in \Lambda_s$ ,  $n \geq 1$ ,  $\epsilon > 0$  and  $s^\gamma = (s_{n,\gamma})_{n \geq 1}$  we set

$$\begin{aligned} E_{b,n,\gamma,\epsilon}^1 &= \left\{ t \in \partial\mathbb{T} : \sum_{k=1}^n \left( X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) \right) \geq \epsilon \sum_{k=1}^n \eta_k(\gamma) \right\}, \\ E_{b,n,\gamma,\epsilon}^{-1} &= \left\{ t \in \partial\mathbb{T} : \sum_{k=1}^n \left( X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) \right) \leq -\epsilon \sum_{k=1}^n \eta_k(\gamma) \right\}. \end{aligned}$$

Suppose that we have shown that for,  $\lambda \in \{-1, 1\}$ , we have:

$$(4.1) \quad \mathbb{E} \left( \sup_{(q,\gamma) \in K} \sum_{n \geq 1} \mu_q^\gamma(E_{b,n,\gamma,\epsilon}^\lambda) \right) < \infty.$$

Then, with probability 1, for all  $(q, \gamma) \in \mathcal{J} \times \Lambda_s$ ,  $\lambda \in \{-1, 1\}$ , and  $\epsilon \in \mathbb{Q}_+^*$ ,

$$\sum_{n \geq 1} \mu_q^\gamma(E_{b,n,\gamma,\epsilon}^\lambda) < \infty,$$

consequently, by the Borel-Cantelli lemma, for  $\mu_q^\gamma$ -almost every  $t$ , we have

$$\sum_{k=1}^n X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) = o \left( \sum_{k=1}^n \eta_k(\gamma) \right), \quad \text{so } \mathbf{N}_n(t) - nb \sim s_{n,\gamma},$$

which yields the desired result.

Let us prove (4.1) when  $\lambda = 1$  (the case  $\lambda = -1$  is similar). Let  $\theta = (\theta_n)$  be a positive sequence and  $(q, \gamma) \in \mathbf{K}$ . One has

$$\sup_{(q, \gamma) \in \mathbf{K}} \mu_q^\gamma \left( E_{b, n, \gamma, \epsilon}^1 \right) \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \mu_q^\gamma([u]) \mathbf{1}_{\left\{ E_{b, n, \gamma, \epsilon}^1 \right\}}(t_u),$$

where  $t_u$  is any point in  $[u]$ . Denote  $t_u$  simply by  $t$ , then

$$\begin{aligned} & \sup_{(q, \gamma) \in \mathbf{K}} \mu_q^\gamma \left( E_{b, n, \gamma, \epsilon}^1 \right) \\ & \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \mu_q^\gamma[u] \prod_{k=1}^n \exp \left( \theta_k X_{t_1 \dots t_k} - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) \\ & \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left( (\psi_k(q, \gamma) + \theta_k) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) \\ & \quad \times Y(q, \gamma, u). \end{aligned}$$

For  $(q, \gamma) \in \mathbf{K}$ ,  $\theta = (\theta_n)$  and  $n \geq 1$ , we set

$$\begin{aligned} & H_n(q, \gamma, \theta) \\ & = \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left( (\psi_k(q, \gamma) + \theta_k) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) M(u), \end{aligned}$$

where

$$M(u) = \sup_{(q, \gamma) \in \mathbf{K}} Y(q, \gamma, u).$$

Recall the proof of Proposition 2.1, there exists a neighborhood  $\mathbf{V} = V_K \times V_{K_\gamma} \subset \mathbb{C}^2$  of  $\mathbf{K} = K \times K_\gamma$  such that

$$\Gamma(z) = \frac{\mathbb{E} \left( \sum_{i=1}^N X_i \exp(z X_i) \right)}{\mathbb{E} \left( \sum_{i=1}^N \exp(z X_i) \right)}$$

is well defined for  $z \in V_K$ , for  $k \geq 1$ ,  $\eta_k(z')$  is defined for  $z' \in V_{K_\gamma}$  and  $\forall (z, z') \in \mathbf{V}$ ,  $\psi_k(z, z')$  is defined and belongs to  $V_K$ .

For  $\epsilon > 0$ ,  $(z, z') \in \mathbf{V}$  and  $n \geq 1$ , we define

$$\begin{aligned} H_n(z, z', \theta) & = \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left( (\psi_k(z, z') + \theta_k) X_{u|_k} - \theta_k \Gamma(z) - \theta_k \eta_k(z')(1 + \epsilon) \right) \\ & \quad \times \mathbb{E} \left( \sum_{i=1}^N \exp(\psi_k(z, z') X_i) \right)^{-1} M(u). \end{aligned}$$

**Proposition 4.1.** *There exist a neighborhood  $\mathbf{V}' \subset \mathbf{V}$  of  $\mathbf{K}$ , a positive constant  $\mathcal{C}_\mathbf{K}$  and a positive sequence  $\theta$  such that for all  $(z, z') \in \mathbf{V}'$ , for all  $n \in \mathbb{N}^*$*

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathcal{C}_\mathbf{K} \exp \left( -\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right),$$

where the sequences  $(\epsilon_n)_n$  and  $(\tilde{\eta}_n)_n$  are the sequences used in Hypothesis 1.2.

**Lemma 4.1.** *There exist a positive sequence  $\theta = (\theta_n)$  and a positive constant  $\mathcal{C}_\mathbb{K}$  such that for all  $(q, \gamma) \in \mathbb{K}$  we have*

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}_\mathbb{K} \exp\left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right).$$

*Proof of Lemma 4.1.* Let  $\theta = (\theta_n)$  be a positive sequence, clearly we have

$$\begin{aligned} \mathbb{E}(H_n(q, \gamma, \theta)) &= \prod_{k=1}^n \mathbb{E}\left(\sum_{i=1}^N \exp\left((\psi_k(q, \gamma) + \theta_k)X_i\right)\right) \\ &\quad \times \exp\left(-\tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon)\right) \mathbb{E}(M(u)) \\ &\leq \mathcal{C}'_\mathbb{K} \prod_{k=1}^n \exp\left(\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon)\right), \end{aligned}$$

where, by Proposition 2.1,  $\mathcal{C}'_\mathbb{K} = \mathbb{E}(M(u)) = \mathbb{E}(M(\emptyset)) < \infty$  for all  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ .

Since  $\eta_k(\gamma) = o(1)$ , we can fix a compact neighborhood  $K'$  of  $K$  and suppose that for all  $k \geq 1$  and  $(q, \gamma) \in \mathbb{K}$ , we have  $\psi_k(q, \gamma) \in K'$ . For  $(q, \gamma) \in \mathbb{K}$  and  $k \geq 1$ , writing the Taylor expansion with integral rest of order 2 of the function  $g : \theta \mapsto \tau(\psi_k(q, \gamma) + \theta)$  at 0, we get

$$g(\theta) = g(0) + \theta g'(0) + \theta^2 \int_0^1 (1-t) g''(t\theta) dt,$$

with  $g''(t\theta) \leq m_\mathbb{K} = \sup_{t \in [0,1]} \sup_{q \in K'} \sup_{\gamma \in \tilde{K}_\gamma} g''(t\theta)$ . It follows that for all  $k \geq 1$

$$\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k \tau'(\psi_k(q, \gamma)) \leq \theta_k^2 m_\mathbb{K}.$$

Recall that  $\tau'(\psi_k(q, \gamma)) = \tau'(q) + \eta_k(\gamma)$ . Then

$$\begin{aligned} \mathbb{E}(H_n(q, \gamma, \theta)) &\leq \mathcal{C}'_\mathbb{K} \prod_{k=1}^n \exp\left(\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon)\right), \\ &\leq \mathcal{C}'_\mathbb{K} \prod_{k=1}^n \exp\left(-\theta_k \eta_k(\gamma) \epsilon + \theta_k^2 m_\mathbb{K}\right). \end{aligned}$$

Choose the sequence  $\theta$  such that  $\theta_k = \epsilon_k \tilde{\eta}_k$ . Then

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}'_\mathbb{K} \prod_{k=1}^n \exp\left(-\epsilon_k \tilde{\eta}_k^2 (\epsilon - \epsilon_k m_\mathbb{K})\right).$$

Since  $\epsilon_k \rightarrow 0$  then for  $k$  large enough we have  $\epsilon - \epsilon_k m_\mathbb{K} > \frac{\epsilon}{2}$ . Then, there exists a constant  $\mathcal{C}_\mathbb{K}$  such that

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}_\mathbb{K} \exp\left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right). \quad \square$$

*Proof of Proposition 4.1.* Since  $\mathbb{E}(|H_n(q, \gamma, \theta)|) \leq \mathfrak{C}_K \exp\left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right)$  for  $q \in K$ , there exists a neighborhood  $V_{q, \gamma} \subset V$  of  $(q, \gamma)$  such that for all  $(z, z') \in V_{q, \gamma}$  we have

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathfrak{C}_K \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right).$$

By extracting a finite covering of  $K$  from  $\cup_{(q, \gamma) \in K} V_{q, \gamma}$ , we find a neighborhood  $V' \subset V$  of  $K$  such that

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathfrak{C}_K \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right). \quad \square$$

With probability 1, the functions  $(z, z') \in V' \mapsto H_n(z, z', \theta)$  are analytic. Fix a closed polydisc  $D(z_0, 2\rho) \subset V$ , with  $z_0 = (z_1, z'_1)$  and  $\rho = (\rho_1, \rho_2)$ . Theorem B.1 gives

$$\sup_{(z, z') \in D(z_0, \rho)} |H_n(z, z', \theta)| \leq 2 \int_{[0, 1]^2} |H_n(\zeta(t), \theta)| dt,$$

where for  $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned} \mathbb{E}\left(\sup_{z \in D(z_0, \rho)} |H_n^s(z, z', \theta)|\right) &\leq \mathbb{E}\left(2 \int_{[0, 1]^2} |H_n(\zeta(t), \theta)| dt\right) \\ &\leq 4 \int_{[0, 1]^2} \mathbb{E} |H_n(\zeta(t), \theta)| dt \\ &\leq 4 \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right). \end{aligned}$$

Finally, we get

$$\mathbb{E}\left(\sup_{(q, \gamma) \in K} \mu_q^\gamma(E_{b, n, \gamma, \epsilon}^1)\right) \leq 4 \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right)$$

and, then, under Hypothesis 1.2, we get (4.1), which finish the proof of Proposition 3.1.

**4.2. Proof of Propostion 3.2.** Let  $K = K \times K_\gamma$  be a compact subset of  $\mathcal{J} \times \Lambda_s$ . For  $a > 1$ ,  $(q, \gamma) \in K$  and  $n \geq 1$ , we set

$$E_{n, a}^+ = \{t \in \partial\mathbb{T} : Y(q, \gamma, t|_n) > a^n\}$$

and

$$E_{n, a}^- = \{t \in \partial\mathbb{T} : Y(q, \gamma, t|_n) < a^{-n}\}.$$

It is sufficient to show that for  $E \in \{E_{n, a}^+, E_{n, a}^-\}$

$$(4.2) \quad \mathbb{E}\left(\sup_{(q, \gamma) \in K} \sum_{n \geq 1} \mu_q^\gamma(E)\right) < \infty.$$

Indeed, if this holds, then with probability 1, for each  $(q, \gamma) \in \mathbf{K}$  and  $E \in \{E_{n,a}^+, E_{n,a}^-\}$ ,  $\sum_{n \geq 1} \mu_q^\gamma(E) < \infty$ , hence by the Borel-Cantelli lemma, for  $\mu_q^\gamma$ -almost every  $t \in \partial\mathbb{T}$ , if  $n$  is big enough we have

$$-\log a \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log Y(q, \gamma, t|_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log Y(q, \gamma, t|_n) \leq \log a.$$

Letting  $a$  tend to 1 along a countable sequence yields the result.

Let us prove (4.2) for  $E = E_{n,a}^+$  (the case  $E = E_{n,a}^-$  is similar). At first we have,

$$\begin{aligned} \sup_{(q,\gamma) \in \mathbf{K}} \mu_q^\gamma(E_{n,a}^+) &= \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \mu_q^\gamma([u]) \mathbf{1}_{\{Y(q,\gamma,u) > a^n\}} \\ &= \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} Y(q, \gamma, u) \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X(u) - \tau(\psi_k(q, \gamma))\right) \mathbf{1}_{\{Y(q,\gamma,u) > a^n\}} \\ &\leq \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} (Y(q, \gamma, u))^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}, \\ &\leq \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}, \end{aligned}$$

where  $M(u) = \sup_{(q,\gamma) \in \mathbf{K}} Y(q, \gamma, u)$  and  $\nu > 0$  is an arbitrary parameter. For  $q \in K$ ,  $\gamma \in K_\gamma$  and  $\nu > 0$  we set

$$L_n(q, \gamma, \nu) = \sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}.$$

Recall the proof of Proposition 2.1, there exists a neighborhood  $\mathbf{V} \subset \mathbb{C}^2$  of  $\mathbf{K}$  such that for all  $(z, z') \in \mathbf{V}$  and  $k \geq 1$   $\psi_k(z, z')$  is well defined and  $\mathbb{E}\left(\sum_{i=1}^N e^{\psi_k(z, z') X_i}\right) \neq 0$ .

**Lemma 4.2.** *Fix  $a > 1$ . For  $(z, z') \in \mathbf{V}$  and  $\nu > 0$ , let*

$$\begin{aligned} L_n(z, z', \nu) &= \left[ \prod_{k=1}^n \mathbb{E}\left(\sum_{i=1}^N \exp(\psi_k(z, z') X_i)\right)^{-1} \right] \\ &\quad \times \sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(z, z') X_{u|_k}\right) a^{-\nu}. \end{aligned}$$

*There exist a neighborhood  $\mathbf{V}' \subset \mathbb{C}^2$  of  $\mathbf{K}$  and a positive constant  $C_{\mathbf{K}}$  such that, for all  $(z, z') \in \mathbf{V}'$ , for all integer  $n \geq 1$*

$$(4.3) \quad \mathbb{E}\left(\left|L_n(z, z', p_{\mathbf{K}} - 1)\right|\right) \leq C_{\mathbf{K}} a^{-n(p_{\mathbf{K}} - 1)/4},$$

*where  $p_{\mathbf{K}}$  provided by Proposition 2.1.*

*Proof.* Write  $V = V_K \times V_{K_\gamma}$ . For  $z \in V_K$  and  $\nu > 0$ , let

$$\tilde{L}_1(z, \nu) = \left| \mathbb{E}\left(\sum_{i=1}^N \exp(z X_i)\right) \right|^{-1} \mathbb{E}\left(\sum_{i=1}^N \left| \exp(z X_i) \right|\right) a^{-\nu}.$$

Let  $q \in K$ . Since  $\mathbb{E}(\tilde{L}_1(q, \nu)) = a^{-\nu}$ , there exists a neighborhood  $V_q \subset V_K$  of  $q$  such that for all  $z \in V_q$  we have  $\mathbb{E}\left(\left|\tilde{L}_1(z, \nu)\right|\right) \leq a^{-\nu/2}$ . Let  $\gamma \in K_\gamma$ . Recall the proof of Proposition 2.1 and since  $\eta_k(\gamma) = o(1)$ , we can find a neighborhood  $V_\gamma \subset V_{K_\gamma}$  of  $K_\gamma$  such that, for all  $k \geq 1$ ,  $(z, z') \in V_q \times V_\gamma$ , we have

$$\mathbb{E}\left(\left|\tilde{L}_1(\psi_k(z, z'), \nu)\right|\right) \leq a^{-\nu/3}.$$

By extracting a finite covering of  $K$  from  $\bigcup_{(q, \gamma)} V_q \times V_\gamma$ , we find a neighborhood  $V' \subset V$  of  $K$  such that for all  $(z, z') \in V'$  and  $k \geq 1$

$$\mathbb{E}\left(\left|\tilde{L}_1(\psi_k(z, z'), \nu)\right|\right) \leq a^{-\nu/4}.$$

Therefore,

$$\begin{aligned} & \mathbb{E}\left(\left|L_n(z, z', \nu)\right|\right) \\ &= \left[ \prod_{k=1}^n \left| \mathbb{E}\left(\sum_{i=1}^N \exp(\psi_k(z, z')X_i)\right) \right|^{-1} \right] \mathbb{E}\left(\left|\sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp(\psi_k(z, z')X_u)\right|\right) a^{-n\nu} \\ &\leq \left[ \prod_{k=1}^n \left| \mathbb{E}\left(\sum_{i=1}^N \exp(\psi_k(z, z')X_i)\right) \right|^{-1} \right] \mathbb{E}\left(\sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \left|\exp(\psi_k(z, z')X_u)\right|\right) a^{-n\nu}. \end{aligned}$$

By Proposition 2.1, there exists  $p_K \in (1, 2]$  such that for all  $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ ,

$$\mathbb{E}(M(u)^{p_K}) = \mathbb{E}(M(\emptyset)^{p_K}) = C_K < \infty.$$

Now take  $\nu = p_K - 1$  in the last calculation, it follows, from the independence of  $\sigma(\{Y(\cdot, \cdot, u), u \in \mathbb{T}_n\})$  and  $\sigma(\{(X_{u_1}, \dots, X_{u_{N_u}}), u \in \mathbb{T}_{n-1}\})$  for all  $n \geq 1$ , that

$$\begin{aligned} & \mathbb{E}\left(\left|L_n(z, z', p_K - 1)\right|\right) \\ &\leq \left[ \prod_{k=1}^n \left| \mathbb{E}\left(\sum_{i=1}^N \exp(\psi_k(z, z')X_i)\right) \right|^{-1} \right] \prod_{k=1}^n \mathbb{E}\left(\sum_{i=1}^N \left|\exp(\psi_k(z, z')X_i)\right|\right)^n C_K a^{-n(p_K - 1)} \\ &= C_K \prod_{k=1}^n \mathbb{E}\left(\left|\tilde{L}_1(\psi_k(z, z'), p_K - 1)\right|\right) \\ &\leq C_K a^{-n(p_K - 1)/4}, \end{aligned}$$

then lemma is now proved.  $\square$

With probability 1, the functions  $(z, z') \in V' \mapsto L_n(z, z', \nu)$  are analytic. Fix a closed polydisc  $D(z_0, 2\rho) \subset V'$ , with  $z_0 = (z_1, z'_1)$  and  $\rho = (\rho_1, \rho_2)$ . Theorem B.1 gives

$$\sup_{z \in D(z_0, \rho)} \left|L_n(z, p_K - 1)\right| \leq 4 \int_{[0,1]^2} \left|L_n(\zeta(t), p_K - 1)\right| dt,$$

where, for  $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned} \mathbb{E} \left( \sup_{z \in D(z_0, \rho)} |L_n(z, p_K - 1)| \right) &\leq \mathbb{E} \left( 4 \int_{[0,1]^2} |L_n(\zeta(t), p_K - 1)| dt \right) \\ &\leq 4 \int_{[0,1]^2} \mathbb{E} |L_n(\zeta(t), p_K - 1)| dt \\ &\leq 4C_K a^{-n(p_K - 1)/4}. \end{aligned}$$

Since  $a > 1$  and  $p_K - 1 > 0$ , we get (4.2).

#### APPENDIX A. HAUSDORFF AND PACKING DIMENSIONS

Given a subset  $K$  of  $\mathbb{N}_+^{\mathbb{N}}$  endowed with a metric  $d$  making it  $\sigma$ -compact,  $s > 0$  and  $E$  a subset of  $K$ , the  $s$ -dimensional Hausdorff measure of  $E$  is defined as

$$\mathcal{H}^s(E) = \liminf_{\delta \rightarrow 0^+} \left\{ \sum_{i \in \mathbb{N}} (\text{diam}(U_i))^s \right\},$$

the infimum being taken over all the countable coverings  $(U_i)_{i \in \mathbb{N}}$  of  $E$  by subsets of  $K$  of diameters less than or equal to  $\delta$ . Then, the Hausdorff dimension of  $E$  is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention  $\sup \emptyset = 0$  and  $\inf \emptyset = \infty$ .

Packing measures and dimensions are defined as follows. Given  $s > 0$  and  $E \subset K$  as above, one first defines

$$\overline{P}^s(E) = \limsup_{\delta \rightarrow 0^+} \left\{ \sum_{i \in \mathbb{N}} (\text{diam}(B_i))^s \right\},$$

the supremum being taken over all the packings  $\{B_i\}_{i \in \mathbb{N}}$  of  $E$  by balls centered on  $E$  and with diameter smaller than or equal to  $\delta$ . Then, the  $s$ -dimensional packing measure of  $E$  is defined as

$$P^s(E) = \liminf_{\delta \rightarrow 0^+} \left\{ \sum_{i \in \mathbb{N}} \overline{P}^s(E_i) \right\},$$

the infimum being taken over all the countable coverings  $(E_i)_{i \in \mathbb{N}}$  of  $E$  by subsets of  $K$  of diameters less than or equal to  $\delta$ . Then, the packing dimension of  $E$  is defined as

$$\text{Dim } E = \sup\{s > 0 : P^s(E) = \infty\} = \inf\{s > 0 : P^s(E) = 0\},$$

with the convention  $\sup \emptyset = 0$  and  $\inf \emptyset = \infty$ . For more details the reader is referred to [13, 20].

If  $\mu$  is a positive and finite Borel measure supported on  $K$ , then its lower Hausdorff and packing dimensions is defined as

$$\begin{aligned}\underline{\dim}(\mu) &= \inf \left\{ \dim F : F \text{ Borel}, \mu(F) > 0 \right\} \\ \underline{\text{Dim}}(\mu) &= \inf \left\{ \text{Dim } F : F \text{ Borel}, \mu(F) > 0 \right\}\end{aligned}$$

and its upper Hausdorff and packing dimensions are defined as

$$\begin{aligned}\overline{\dim}(\mu) &= \inf \left\{ \dim F : F \text{ Borel}, \mu(F) = \|\mu\| \right\} \\ \overline{\text{Dim}}(\mu) &= \inf \left\{ \text{Dim } F : F \text{ Borel}, \mu(F) = \|\mu\| \right\}.\end{aligned}$$

We have (see [12])

$$\begin{aligned}\underline{\dim}(\mu) &= \text{ess inf}_{\mu} \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}, \\ \underline{\text{Dim}}(\mu) &= \text{ess inf}_{\mu} \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}(\mu) &= \text{ess sup}_{\mu} \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}, \\ \overline{\text{Dim}}(\mu) &= \text{ess sup}_{\mu} \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)},\end{aligned}$$

where  $B(t, r)$  stands for the closed ball of radius  $r$  centered at  $t$ . If  $\underline{\dim}(\mu) = \overline{\dim}(\mu)$  (resp.  $\underline{\text{Dim}}(\mu) = \overline{\text{Dim}}(\mu)$ ), this common value is denoted  $\dim \mu$  (resp.  $\text{Dim}(\mu)$ ), and if  $\dim \mu = \text{Dim} \mu$ , one says that  $\mu$  is exact dimensional.

Recall the mass distribution principle.

**Theorem A.1.** ([13, Theorem 4.2]). *Let  $\nu$  be a positive and finite Borel probability measure on a compact metric space  $(X, d)$ . Assume that  $M \subseteq X$  is a Borel set such that  $\nu(M) > 0$  and*

$$M \subseteq \left\{ t \in X : \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \right\}.$$

*Then the Hausdorff dimension of  $M$  is bounded from below by  $\delta$ .*

## APPENDIX B. CAUCHY FORMULA IN SEVERAL VARIABLES

Let us recall the Cauchy formula for holomorphic functions in several variables.

**Definition B.1.** Let  $d \geq 1$ , a subset  $D$  of  $\mathbb{C}^d$  is an open polydisc if there exist open discs  $D_1, \dots, D_d$  of  $\mathbb{C}$  such that  $D = D_1 \times \dots \times D_d$ . If we denote by  $\zeta_j$  the centre of  $D_j$ , then  $\zeta = (\zeta_1, \dots, \zeta_d)$  is the centre of  $D$  and if  $r_j$  is the radius of  $D_j$  then  $r = (r_1, \dots, r_d)$  is the multiradius of  $D$ . The set  $\partial D = \partial D_1 \times \dots \times \partial D_d$  is the distinguished boundary of  $D$ . We denote by  $D(\zeta, r)$  the polydisc with center  $\zeta$  and radius  $r$ .



Let  $D = D(\zeta, r)$  be a polydisc of  $\mathbb{C}^d$  and  $g \in C(\partial D)$  a continuous function on  $\partial D$ . We define the integral of  $g$  on  $\partial D$  as

$$\int_{\partial D} g(\zeta) d\zeta_1 \cdots d\zeta_d = (2i\pi)^d r_1 \cdots r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1} \cdots e^{i2\pi\theta_d} d\theta_1 \cdots d\theta_d,$$

where  $\zeta(\theta) = (\zeta_1(\theta), \dots, \zeta_d(\theta))$  and  $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$  for  $j = 1, \dots, d$ .

**Theorem B.1.** *Let  $D = D(a, r)$  be polydisc in  $\mathbb{C}^d$  with a multiradius whose components are positive, and  $f$  be a holomorphic function in a neighborhood of  $D$ . Then, for all  $z \in D$*

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_d}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| d\theta_1 \cdots d\theta_d.$$

**Acknowledgements.** The first author would like to thank the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia.

## REFERENCES

- [1] N. Attia, *On the exact dimension of Mandelbrot measure*, Probab. Math. Statist. **39**(2) (2019), 299–314. <https://doi.org/10.19195/0208-4147.39.2.4>
- [2] N. Attia, *Hausdorff and packing dimensions of Mandelbrot measure*, Internat. J. Math. **31**(9) (2020), Article ID 2050068. <https://doi.org/10.1142/S0129167X20500688>
- [3] N. Attia, *On the multifractal analysis of branching random walk on Galton-Watson tree with random metric*, J. Theoret. Probab. **34**(1) (2020), 90–102. <https://doi.org/10.1007/s10959-019-00984-z>
- [4] N. Attia and J. Barral, *Hausdorff and packing spectra, large deviations and free energy for branching random walks in  $\mathbb{R}^d$* , Comm. Math. Phys. **331** (2014), 139–187. <https://doi.org/10.1007/s00220-014-2087-9>
- [5] N. Attia, *On the multifractal analysis of covering number on the Galton Watson tree*, J. Appl. Probab. **56**(1) (2019), 265–281. <https://doi.org/10.1017/jpr.2019.17>
- [6] N. Attia, *Comportement asymptotique de marches aléatoires de branchement dans  $\mathbb{R}^d$  et dimension de Hausdorff*, SISYPHE - Signals and Systems in Physiology & Engineering - Thèse de doctorat, tel-00841496, (2012).
- [7] N. Attia, *On the Multifractal Analysis of the Branching Random Walk in  $\mathbb{R}^d$* , J. Theoret. Probab. **27** (2014), 1329–1349. <https://doi.org/10.1007/s10959-013-0488-x>
- [8] J. Barral, *Continuity of the multifractal spectrum of a statistically self-similar measure*, J. Theoret. Probab. **13** (2000), 1027–1060. <https://doi.org/10.1023/A:1007866024819>
- [9] J. D. Biggins, *Martingale convergence in the branching random walk*, J. Appl. Probab. **14** (1977), 25–37. <https://doi.org/10.2307/3213258>
- [10] J. D. Biggins, *Uniform convergence of martingales in the branching random walk*, Ann. Probab. **20** (1992), 137–151. <https://doi.org/10.1214/aop/1176989921>
- [11] J. D. Biggins, B. M. Hambly and O. D. Jones, *Multifractal spectra for random self-similar measures via branching processes*, Adv. in Appl. Probab. **43**(1) (2011), 1–39. <https://doi.org/10.1239/aap/1300198510>

- [12] C. D. Cutler, *Connecting ergodicity and dimension in dynamical systems*, Ergodic Theory Dynam. Systems **10** (1990), 451–462. <https://doi.org/10.1017/S014338570000568X>
- [13] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2<sup>nd</sup> Edition, Wiley, Chichester, 2003. <https://doi.org/10.1002/0470013850>
- [14] K. J. Falconer, *The multifractal spectrum of statistically self-similar measures*, J. Theoret. Probab. **7**(3) (1994), 681–702. <https://doi.org/10.1007/BF02213576>
- [15] A. H. Fan and J. P. Kahane, *How many intervals cover a point in random dyadic covering?* Port. Math. **58**(1) (2001), 59–75.
- [16] R. Holley and E. C. Waymire, *Multifractal dimensions and scaling exponents for strongly bounded random fractals*, Ann. Appl. Probab. **2** (1992), 819–845. <https://doi.org/10.1214/aop/1177005577>
- [17] R. Lyons, *Random walks and percolation on trees*, Ann. Probab. **18** (1990), 931–958. <https://doi.org/10.1214/aop/1176990730>
- [18] R. Lyons and R. Pemantle, *Random walks in a random environment and first passage percolation on trees*, Ann. Probab. **20** (1992), 125–136. <https://doi.org/10.1214/aop/1176989920>
- [19] Q. Liu and A. Rouault, *On two measures defined on the boundary of a branching tree*, IMA Vol. Math. Appl. **84** (1997), 187–201. [https://doi.org/10.1007/978-1-4612-1862-3\\_15](https://doi.org/10.1007/978-1-4612-1862-3_15)
- [20] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995. <https://doi.org/10.1017/cbo9780511623813>
- [21] G. M. Molchan, *Scaling exponents and multifractal dimensions for independent random cascades*, Comm. Math. Phys. **179** (1996), 681–702.

<sup>1</sup>ANALYSIS, PROBABILITY AND FRACTALS LABORATORY LR18ES17,  
FACULTY OF SCIENCES OF MONASTIR

<sup>2</sup>DEPARTMENT OF MATHEMATICS AND STATISTICS, COLLEGE OF SCIENCE,  
KING FAISAL UNIVERSITY, PO. BOX : 400 AL-AHSA 31982, SAUDI ARABIA  
*Email address:* najmeddine.attia@gmail.com  
*Email address:* nattia@kfu.edu.sa

<sup>3</sup>ESPRIT SCHOOL OF ENGINEERING, TUNIS, TUNISIA  
*Email address:* meriem.benhadjkhalifa@esprit.tn