

NOTE ON THE MULTIFRACTAL FORMALISM OF COVERING NUMBER ON THE GALTON-WATSON TREE

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ABSTRACT. We consider, for t in the boundary of Galton-Watson tree (∂T), the covering number $N_n(t)$ by cylinder of generation n . For a suitable set I and a sequence $(s_{n,\gamma})$, we establish almost surely, and uniformly on γ , the Hausdorff and packing dimensions of the set $\{t \in \partial T : N_n(t) - nb \sim s_{n,\gamma}\}$ for $b \in I$.

1. INTRODUCTION AND MAIN RESULTS

Let (N, X) be a random vector with independent components taking values in \mathbb{N}^2 , where \mathbb{N} denotes the set of non-negative integers. Then let $\{(N_u, X_u)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ be a family of independent copies of the vector (N, X) indexed by the set of finite words over the alphabet \mathbb{N}_+ : the set of positive integers ($n = 0$ corresponds to the empty sequence denoted \emptyset). Let T be the Galton-Watson tree with defining elements $\{N_u\}$: we have $\emptyset \in T$, if $u \in T$ and $i \in \mathbb{N}_+$ then ui , the concatenation of u and i , belongs to T if and only if $1 \leq i \leq N_u$ and if $ui \in T$, then $u \in T$. Similarly, for each $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, denote by $T(u)$ the Galton-Watson tree rooted at u and defined by the $\{N_{uv}\}$, $v \in \bigcup_{n \geq 0} \mathbb{N}_+^n$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton-Watson tree is supercritical. We also assume that the probability of extinction is equal to 0, so that $\mathbb{P}(N \geq 1) = 1$.

For each infinite word $t = t_1 t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}}$ and $n \geq 0$, we set $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$ ($t_{|0} = \emptyset$). If $u \in \mathbb{N}_+^n$ for some $n \geq 0$, then n is the length of u and it is denoted by $|u|$. We denote by $[u]$ the set of infinite words $t \in \mathbb{N}_+^{\mathbb{N}}$ such that $t_{|u} = u$.

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The set $\mathbb{N}_+^{\mathbb{N}}$ is endowed with the standard ultrametric distance

$$d : (u, v) \mapsto e^{-\sup\{|w| : u \in [w], v \in [w]\}},$$

with the convention $\exp(-\infty) = 0$. The boundary of the Galton-Watson tree \mathbb{T} is defined as the compact set

$$\partial\mathbb{T} = \bigcap_{n \geq 1} \bigcup_{u \in \mathbb{T}_n} [u],$$

where $\mathbb{T}_n = \mathbb{T} \cap \mathbb{N}_+^n$.

We consider X_u as the covering number of the cylinder $[u]$, that is to say, the cylinder $[u]$ is cut off with probability $p_0 = \mathbb{P}(X = 0)$ and is covered m times with probability $p_m = \mathbb{P}(X = m)$, $m = 1, 2, \dots$

For $t \in \partial\mathbb{T}$, set

$$\mathbf{N}_n(t) = \sum_{k=1}^n X_{t_1 \dots t_k}.$$

Since this quantity depends on $t_1 \dots t_n$ only, we also denote by $\mathbf{N}_n(u)$ the constant value of $\mathbf{N}_n(\cdot)$ over $[u]$ whenever $u \in \mathbb{T}_n$. The quantity $\mathbf{N}_n(t)$ is called the covered number (or more precisely the n -covered number) of the point t by cylinder of generation k , $k = 1, 2, \dots, n$.

Consider an individual infinite branch $t_1 \dots t_n \dots$ in $\partial\mathbb{T}$. When $\mathbb{E}(X)$ is defined, the strong law of large number yields $\lim_{n \rightarrow \infty} n^{-1} \mathbf{N}_n(t) = \mathbb{E}(X)$. It is also well known, in the theory of the birth process, (see [15]) that almost surely (a.s.) $\lim_{n \rightarrow \infty} \mathbf{N}_n(t) = +\infty$ for every $t \in \mathcal{D} = \{0, 1\}^{\mathbb{N}}$ if and only if

$$p_0 = \mathbb{P}(X = 0) < \frac{1}{2}.$$

If this condition is satisfied, then a.s. every point is infinitely covered.

We consider, for $b \in \mathbb{R}$, the set

$$E_b = \left\{ t \in \partial\mathbb{T} : \lim_{n \rightarrow \infty} \frac{\mathbf{N}_n(t)}{n} = b \right\}.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [3, 8, 11, 14, 16, 21] and [4, 7] for a general case. All these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [1, 2, 19] for the study of Mandelbrot measures dimension).

We will assume that the free energy of X defined as

$$\tau(q) = \log \mathbb{E} \left(\sum_{i=1}^N e^{qX_i} \right)$$

is finite over \mathbb{R} . We will assume, without loss of generality, that X is not constant so that the function τ is strictly convex. Let τ^* stand for the Legendre transform of the function τ , defined as

$$\tau^*(b) := \inf_{q \in \mathbb{R}} \left(\tau(q) - qb \right), \quad b \in \mathbb{R}.$$

We say that the multifractal formalism holds at $b \in \mathbb{R}$ if

$$\dim E_b = \text{Dim } E_b = \tau^*(b),$$

where $\dim E_b$ is the Hausdorff dimension of E_b and $\text{Dim } E_b$ is the packing dimension of E_b (see Section A for the definition). In the following, we define the sets

$$\begin{aligned} J &= \left\{ q \in \mathbb{R}; \tau(q) - q\tau'(q) > 0 \right\}, \\ \Omega_\alpha^1 &= \text{int} \left\{ q : \mathbb{E} \left[\left| \sum_{i=1}^N e^{qX_i} \right|^\alpha \right] < \infty \right\}, \\ \Omega^1 &= \bigcup_{\alpha \in (1,2]} \Omega_\alpha^1, \\ \mathcal{J} &= J \cap \Omega^1 \quad \text{and} \quad I = \left\{ \tau'(q); q \in \mathcal{J} \right\}. \end{aligned}$$

Remark 1.1. It is well known, see [6, Proposition 3.1], that $L = \{\alpha \in \mathbb{R}, \tau^*(\alpha) \geq 0\}$, is a convex, compact and non-empty set. In addition, if we assume that $J = \mathcal{J}$ then $I = \text{int}(L)$, where $\text{int}(L)$ is the interior of L (see also [6, Proposition 3.1.]) In particular, I is an interval.

Next, we define for $b, \gamma \in \mathbb{R}$ and for any positive sequence $s^\gamma = \{s_{n,\gamma}\}_n$ such that $s_{n,\gamma} = o(n)$ and $\gamma \mapsto s_{n,\gamma}$ is analytic function, the set

$$E_{b,s^\gamma} = \left\{ t \in \partial T : \mathbf{N}_n(t) - nb \sim s_{n,\gamma} \text{ as } n \rightarrow +\infty \right\},$$

where $\mathbf{N}_n(t) - nb \sim s_{n,\gamma}$ means that $(\mathbf{N}_n(t) - nb)_n$ and $(s_{n,\gamma})_n$ are two equivalent sequences. It is clear that $E_{b,s^\gamma} \subset E_b$. So, we can get with a simple covering argument, with probability 1, for all $b \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$(1.1) \quad \dim E_{b,s^\gamma} \leq \dim E_b \leq \text{Dim } E_b \leq \tau^*(b),$$

(see Proposition 1 in [5] and Proposition 2.7 in [4]). Let us mention that the methods used to compute Hausdorff dimension of the sets E_b in, for example, [4, 7, 17, 18]) do not give results on $\dim E_{b,s^\gamma}$. These sets were considered by Kahane and Fan in [15]. The authors considered the space $\{0, 1\}^\mathbb{N}$ and they compute, for each b , almost surely (a.s.), the Hausdorff dimension of E_{b,s^γ} under the hypothesis :

$$s_{n,\gamma} = o(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = o(1) \quad \text{and} \quad \sqrt{n \ln \ln n} = o(s_{n,\gamma}).$$

A special case of a sequence satisfying the above hypothesis is $s_{n,\gamma} = n^\gamma$ with $\gamma \in (1/2, 1)$. Later, Attia in [5], gives a stronger result in the sense that, a.s. for all $b \in I$, he computed the Hausdorff dimensions of the sets E_{b,s^γ} under the hypothesis

$$(1.2) \quad s_{n,\gamma} = o(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = o(1)$$

and there exists $\epsilon_n \rightarrow 0$ such that

$$(1.3) \quad \sum_{n \geq 1} \exp \left(- \epsilon \sum_{k=1}^n \epsilon_k \eta_k(\gamma)^2 \right) < +\infty, \quad \text{for all } \epsilon > 0.$$

In particular, we can choose

$$s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma} \quad \text{with } \gamma \in (0, 1/2).$$

Theorem 1.1 ([5]). *Let s^γ be a positive sequence satisfying (1.2) and (1.3). Then, a.s. for all $b \in I$*

$$\dim E_{b,s^\gamma} = \dim E_b = \tau^*(b).$$

This requires, for a given sequence s^γ , a simultaneous building of an inhomogeneous Mandelbrot measure and a computing of their dimensions. In particular, for

$$s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma},$$

we have for all $\gamma \in (0, 1/2)$, a.s. $\dim E_{b,s^\gamma} = \tau^*(b)$. To state our main result, let $s^\gamma = (s_{n,\gamma})_n$ be a positive sequence and we define the set Λ_s to be any set of \mathbb{R} such that

$$(1.4) \quad \Lambda_s \subseteq \left\{ \gamma \in \mathbb{R}, \text{ such that } (s_{n,\gamma}) \text{ satisfies (1.2) and (1.3)} \right\}$$

and, for $k \geq 1$

$$(1.5) \quad \tilde{\eta}_k = \inf_{\gamma \in \Lambda_s} \eta_k(\gamma) > 0.$$

We suppose the following hypothesis.

Hypothesis 1.2. There exists a sequence $\epsilon_n \rightarrow 0$ such that

$$\sum_{n \geq 1} \exp \left(-\epsilon \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right) < +\infty, \quad \text{for all } \epsilon > 0.$$

Clearly this hypothesis is satisfied, for $s_{n,\gamma} = \sum_{k=1}^n \frac{1}{k^\gamma}$, with $\Lambda_s = [\epsilon, 1/2)$, $\epsilon > 0$. Applying the previous theorem we get the conclusion for each $\gamma \in \Lambda_s$ a.s. The goal of this note is to give a uniform result on γ . In addition, we determine the packing dimensions of the sets E_{b,s^γ} . More precisely we have the following result.

Theorem 1.3. *Let $s^\gamma = (s_{n,\gamma})_{n \geq 1}$ be a positive sequence and consider a set Λ_s satisfying (1.4) and (1.5). Under Hypothesis 1.2, we have, a.s.. for all $b \in I$ and for all $\gamma \in \Lambda_s$*

$$\dim E_{b,s^\gamma} = \dim E_b = \text{Dim } E_b = \text{Dim } E_{b,s^\gamma} = \tau^*(b).$$

2. CONSTRUCTION OF INHOMOGENEOUS MANDELBROT MEASURES

We define, for $(q, p) \in \mathcal{J} \times [1, \infty)$, the function

$$\varphi(p, q) = \exp \left(\tau(pq) - p\tau(q) \right).$$

From [5], for all nontrivial compact sets $K \subset \mathcal{J}$ there exist $1 < p_K < 2$ and $\tilde{p}_K > 1$ such that we have

$$(2.1) \quad \sup_{q \in K} \varphi(p_K, q) < 1, \quad \text{for all } 1 < p \leq p_K,$$

and

$$(2.2) \quad \sup_{q \in K} \mathbb{E} \left(\left(\sum_{i=1}^N e^{qX_i} \right)^{\tilde{p}_K} \right) < \infty.$$

Now, we will construct the inhomogeneous Mandelbrot measure. For $q \in \mathcal{J}$ and $k \geq 1$, we define $\psi_k(q, \gamma)$ as the unique t , such that

$$\tau'(t) = \tau'(q) + \eta_k(\gamma).$$

For $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ and $q \in \mathcal{J}$ we define, for $1 \leq i \leq N_u$

$$V(ui, q) = \frac{\exp(qX_{ui})}{\mathbb{E} \left(\sum_{i=1}^N \exp(qX_i) \right)} = \exp(qX_{ui} - \tau(q))$$

and, for all $n \geq 0$

$$Y_n(q, \gamma, u) = \sum_{v_1 \cdots v_n \in \mathbb{T}_n(u)} \prod_{k=1}^n V(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(q, \gamma)).$$

When $u = \emptyset$, this quantity will be denoted by $Y_n(q, \gamma)$ and when $n = 0$, their values equals 1.

The sequence $(Y_n(q, \gamma, u))_{n \geq 1}$ is a positive martingale with expectation 1, which converges almost surely and in L^1 norm to a positive random variable $Y(q, \gamma, u)$ (see [9] or [10, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits.

Proposition 2.1. (a) *Let $\mathbf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. There exists $p_{\mathbf{K}} \in (1, 2]$ such that for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ the continuous functions $(q, \gamma) \in \mathbf{K} \mapsto Y_n(q, \gamma, u)$ converge uniformly, almost surely and in $L_{p_{\mathbf{K}}}$ norm, to a limit $(q, \gamma) \in \mathbf{K} \mapsto Y(q, \gamma, u)$. In particular, $\mathbb{E}(\sup_{(q, \gamma) \in \mathbf{K}} Y(q, \gamma, u)^{p_{\mathbf{K}}}) < \infty$. Moreover, $Y(\cdot, \cdot, u)$ is positive almost surely.*

In addition, for all $n \geq 0$, $\sigma(\{(X_{u_1}, \dots, X_{u_{N_u}}), u \in \mathbb{T}_n\})$ and $\sigma(\{Y(\cdot, \cdot, u), u \in \mathbb{T}_{n+1}\})$ are independent, and the random functions $Y(\cdot, \cdot, u), u \in \mathbb{T}_{n+1}$, are independent copies of $Y(\cdot, \cdot) := Y(\cdot, \cdot, \emptyset)$.

(b) *With probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$, the weights*

$$\mu_q^\gamma([u]) = \left[\prod_{k=1}^n \exp(\psi_k(q, \gamma) X_{u_1 \dots u_k} - \tau(\psi_k(q, \gamma))) \right] Y(q, \gamma, u)$$

define a measure on $\partial \mathbb{T}$, where $n = |u|$.

The measure μ_q^γ will be used to approximate from below the Hausdorff dimension of the set E_{b,s^γ} .

Proof. (a) Fix a compact $K \subset \mathcal{J}$ and a compact $K_\delta \subset \Lambda_s$. Since $\eta_k(\gamma) = o(1)$, we can fix, without loss of generality, a compact neighborhood $K' \subset \mathcal{J}$ of K and suppose that,

$$\forall (q, \gamma) \in \mathbf{K} = K \times K_\gamma, \quad \text{for all } k \geq 1, \psi_k(q, \gamma) \in K'.$$

Fix a compact neighborhood $\mathbf{K}'' = K'' \times K_\gamma''$ of $K' \times K_\gamma$. By (2.2), we can find $\tilde{p}_{\mathbf{K}''} > 1$, such that

$$\sup_{q \in \mathbf{K}''} \mathbb{E} \left(\left(\sum_{i=1}^N e^{qX_i} \right)^{\tilde{p}_{\mathbf{K}''}} \right) < \infty.$$

By (2.1), we can fix $1 < p_{\mathbf{K}} \leq \min(2, \tilde{p}_{\mathbf{K}''})$ such that $\sup_{q \in \mathbf{K}''} \varphi(p_{\mathbf{K}}, q) < 1$. Then for each $(q, \gamma) \in K' \times K$, there exists a neighborhood $V_q \times V_\gamma \subset \mathbb{C}^2$ of (q, γ) , whose projection to \mathbb{R}^2 is contained in \mathbf{K}'' , and such that for all $u \in \mathbb{T}$, $(z, z') \in V_q \times V_\gamma$ and $k \geq 1$, the random variable

$$V(u, z) = \frac{\exp(zX_u)}{\mathbb{E} \left(\sum_{i=1}^N \exp(zX_i) \right)}, \quad \Gamma(z) = \frac{\mathbb{E} \left(\sum_{i=1}^N X_i \exp(zX_i) \right)}{\mathbb{E} \left(\sum_{i=1}^N \exp(zX_i) \right)}$$

and the analytic extension of η_k , denoted also by η_k , are well defined. For $(z, z') \in V_q \times V_\gamma$ and $k \geq 1$, we define $\psi_k(z, z')$ as the unique t such that

$$\Gamma(t) = \Gamma(z) + |\eta_k(z')|.$$

Moreover, we have

$$\sup_{z \in V_q} \varphi(p_{\mathbf{K}}, z) < 1, \quad \text{where } \varphi(p_{\mathbf{K}}, z) = \frac{\mathbb{E} \left(\sum_{i=1}^N |e^{zX_i}|^{p_{\mathbf{K}}} \right)}{\left| \mathbb{E} \left(\sum_{i=1}^N e^{zX_i} \right) \right|^{p_{\mathbf{K}}}}.$$

By extracting a finite covering of $K' \times K_\gamma$ from $\bigcup_{q, \gamma} V_q \times V_\gamma$, we find a neighborhood $\mathbf{V} = V_K \times V_{K_\gamma} \subset \mathbb{C}^2$ of $K' \times K_\gamma$ such that

$$\sup_{z \in V_K} \varphi(p_{\mathbf{K}}, z) < 1$$

and for all $(z, z') \in \mathbf{V}$, $\psi_k(z, z')$ is defined and belongs to V_K . Since the projection of V_K to \mathbb{R} is included in K'' and the mapping $z \mapsto \mathbb{E} \left(\sum_{i=1}^N e^{zX_i} \right)$ is continuous and does not vanish on V_K , by considering a smaller neighborhood of K' included in V_K if necessary, we can assume that

$$C_{V_K} = \sup_{z \in V_K} \mathbb{E} \left(\left| \sum_{i=1}^N e^{zX_i} \right|^{p_{\mathbf{K}}} \right) \left| \mathbb{E} \left(\sum_{i=1}^N e^{zX_i} \right) \right|^{-p_{\mathbf{K}}} < \infty.$$

Now, for $u \in \mathbb{T}$, we define the analytic extension to \mathbb{V} of $Y_n(q, \gamma, u)$ given by

$$\begin{aligned} Y_n(z, z', u) &= \sum_{v \in \mathbb{T}_n(u)} \prod_{k=1}^n V(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(z, z')) \\ &= \left[\prod_{k=1}^n \mathbb{E} \left(\sum_{i=1}^N e^{\psi_k(z, z') X_i} \right) \right]^{-1} \sum_{v \in \mathbb{T}_n(u)} \prod_{k=1}^n e^{\psi_{|u|+k}(z, z') X(uv_k)}. \end{aligned}$$

We denote also $Y_n(z, z', \emptyset)$ by $Y_n(z, z')$. By Lemma 3 in [5], there exists a constant C_{p_K} such that for all $(z, z') \in \mathbb{V}$

$$\begin{aligned} &\mathbb{E} \left(|Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) \\ &\leq C_{p_K} \mathbb{E} \left(\left| \sum_{i=1}^N V(i, \psi_n(z, z')) \right|^{p_K} \right) \prod_{k=1}^{n-1} \mathbb{E} \left(\sum_{i=1}^N |V(i, \psi_k(z, z'))|^{p_K} \right). \end{aligned}$$

Notice that $\mathbb{E} \left(\sum_{i=1}^N |V(i, \psi_k(z, z'))|^{p_K} \right) = \varphi(p_K, \psi_k(z, z'))$. Then

$$\begin{aligned} \mathbb{E} \left(|Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) &\leq C_{p_K} \mathbb{E} \left(\left| \sum_{i=1}^N V(i, \psi_n(z, z')) \right|^{p_K} \right) \prod_{k=1}^{n-1} \varphi(p_K, \psi_k(z, z')). \\ &\leq C_{p_K} C_{V_K} \prod_{k=1}^{n-1} \sup_{z \in V_K} \varphi(p_K, z), \end{aligned}$$

where we have used the fact that $\psi_k(z, z') \in V_K$ for all $k \geq 1$. With probability 1, the functions $(z, z') \in \mathbb{V} \mapsto Y_n(z, z')$, $n \geq 0$, are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset \mathbb{V}$ with $z_0 = (z_1, z'_1)$ and $\rho = (\rho_1, \rho_2)$. Theorem B.1 gives

$$\sup_{(z, z') \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')| \leq 4 \int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))| dt,$$

where, for $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned} \mathbb{E} \left(\sup_{z \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')|^{p_K} \right) &\leq \mathbb{E} \left(\left(4 \int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))| dt \right)^{p_K} \right) \\ &\leq 4^{p_K} \mathbb{E} \left(\int_{[0,1]^2} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))|^{p_K} dt \right) \\ &= 4^{p_K} \int_{[0,1]^2} \mathbb{E} |Y_n(\zeta(t)) - Y_{n-1}(\zeta(t))|^{p_K} dt \\ &\leq 4^{p_K} C_{V_K} C_{p_K} \prod_{k=1}^{n-1} \sup_{z \in V_K} \varphi(p_K, z). \end{aligned}$$

Since $\sup_{z \in V_K} \varphi(p_K, z) < 1$, it follows that

$$\sum_{n \geq 1} \left\| \sup_{(z, z') \in D(z_0, \rho)} |Y_n(z, z') - Y_{n-1}(z, z')| \right\|_{p_K} < \infty.$$

This implies, $(z, z') \mapsto Y_n(z, z')$ converges uniformly, almost surely and in L^{p_K} norm over the compact $D(z_0, \rho)$ to a limit $(z, z') \mapsto Y(z, z')$. This also implies that

$$\left\| \sup_{z \in D(z_0, \rho)} Y(z, z') \right\|_{p_K} < \infty.$$

Since K can be covered by finitely many such discs $D(z_0, \rho)$ we get the uniform convergence, almost surely and in L^{p_K} norm, of the sequence $((q, \gamma) \in K \mapsto Y_n(q, \gamma))_{n \geq 1}$ to $(q, \gamma) \in K \mapsto Y(q, \gamma)$. Moreover, since $\mathcal{J} \times \Lambda_s$ can be covered by a countable union of such compact K we get the simultaneous convergence for all $(q, \gamma) \in \mathcal{J} \times \Lambda_s$. The same holds simultaneously for all the functions $(q, \gamma) \in \mathcal{J} \times \Lambda_s \mapsto Y_n(q, \gamma, u)$, $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, because $\bigcup_{n \geq 0} \mathbb{N}_+^n$ is countable.

To finish the proof of Proposition 2.1 (1), we must show that with probability 1, $(q, \gamma) \in K \mapsto Y(q, \gamma)$ does not vanish. Without loss of generality we can suppose that $K = [0, 1]^2$. If I is a dyadic closed subcube of $[0, 1]^2$, we denote by E_I the event $\{\exists (q, \gamma) \in I : Y(q, \gamma) = 0\}$. Let I_0, I_1, I_2, I_3 stand for the 2^2 dyadic intervals of I in the next generation. The event E_I being a tail event of probability 0 or 1. If we suppose that $\mathbb{P}(E_I) = 1$, then there exists $j \in \{0, 1, 2, 3\}$ such that $\mathbb{P}(E_{I_j}) = 1$. Suppose now that $\mathbb{P}(E_K) = 1$. The previous remark allows to construct a decreasing sequence $(I(n))_{n \geq 0}$ of dyadic subcubes of K such that $\mathbb{P}(E_{I(n)}) = 1$. Let (q_0, γ_0) be the unique element of $\bigcap_{n \geq 0} I(n)$. Since $(q, \gamma) \mapsto Y(q, \gamma)$ is continuous we have $\mathbb{P}(Y(q_0, \gamma_0) = 0) = 1$, which contradicts the fact that $(Y_n(q_0, \gamma_0))_{n \geq 1}$ converges to $Y(q_0, \gamma_0)$ in L^1 .

(b) It is a consequence of the branching property

$$Y_{n+1}(q, \gamma, u) = \sum_{i=1}^N \exp\left(\psi_{n+1}(q, \gamma) X_{ui} - \tau(\psi_{n+1}(q, \gamma))\right) Y_n(q, \gamma, ui). \quad \square$$

3. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 can be deduced from the two following propositions. Their proof are developed in the next section.

Proposition 3.1. *Suppose Hypothesis 1.2, with probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$,*

$$\mathbf{N}_n(t) - nb \sim s_{n, \gamma}, \quad \text{for } \mu_q^\gamma\text{-almost every } t \in \partial T,$$

where $b = \tau'(q)$.

Proposition 3.2. *With probability 1, for all $(q, \gamma) \in \mathcal{J} \times \Lambda_s$, for μ_q^γ -almost every $t \in \partial T$*

$$\lim_{n \rightarrow \infty} \frac{\log Y(q, \gamma, t|_n)}{n} = 0.$$

From Proposition 3.1, we have with probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$, that $\mu_q^\gamma(E_{b,s\gamma}) = 1$, ($b = \tau'(q)$). In addition, with probability 1, for all $(q, \gamma) \in \mathcal{J} \times \Lambda_s$, for μ_q^γ -almost every $t \in E_{b,s\gamma}$, from the same Proposition and proposition 3.2, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{\log(\mu_q^\gamma[t_{|n}|])}{\log(\text{diam}([t_{|n}|]))} \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \log \left(\prod_{k=1}^n \exp(\psi_k(q, \gamma) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma))) Y(q, \gamma, t_{|n|}) \right) \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{k=1}^n \psi_k(q, \gamma) X_{t_1 \dots t_k} + \frac{1}{n} \sum_{k=1}^n \tau(\psi_k(q, \gamma)) - \frac{\log Y(q, \gamma, t_{|n|})}{n} \\ &= \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{k=1}^n \psi_k(q, \gamma) X_{t_1 \dots t_k} + \frac{1}{n} \sum_{k=1}^n \tau(\psi_k(q, \gamma)). \end{aligned}$$

Since $\eta_k(\gamma) = o(1)$ and then $\psi_k(q, \gamma) \rightarrow q$, we get

$$\lim_{n \rightarrow \infty} \frac{\log(\mu_q^\gamma[t_{|n}|])}{\log(\text{diam}([t_{|n}|]))} = -q\tau'(q) + \tau(q) = \tau^*(\tau'(q)).$$

We deduce the result from the mass distribution principle (Theorem A.1) and (1.1).

4. PROOF OF PROPOSITIONS 3.1 AND 3.2

4.1. Proof of Proposition 3.1. Let $K = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $b = \tau'(q)$, $q \in \mathcal{J}$, $\gamma \in \Lambda_s$, $n \geq 1$, $\epsilon > 0$ and $s^\gamma = (s_{n,\gamma})_{n \geq 1}$ we set

$$\begin{aligned} E_{b,n,\gamma,\epsilon}^1 &= \left\{ t \in \partial T : \sum_{k=1}^n \left(X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) \right) \geq \epsilon \sum_{k=1}^n \eta_k(\gamma) \right\}, \\ E_{b,n,\gamma,\epsilon}^{-1} &= \left\{ t \in \partial T : \sum_{k=1}^n \left(X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) \right) \leq -\epsilon \sum_{k=1}^n \eta_k(\gamma) \right\}. \end{aligned}$$

Suppose that we have shown that for, $\lambda \in \{-1, 1\}$, we have:

$$(4.1) \quad \mathbb{E} \left(\sup_{(q,\gamma) \in K} \sum_{n \geq 1} \mu_q^\gamma(E_{b,n,\gamma,\epsilon}^\lambda) \right) < \infty.$$

Then, with probability 1, for all $(q, \gamma) \in \mathcal{J} \times \Lambda_s$, $\lambda \in \{-1, 1\}$, and $\epsilon \in \mathbb{Q}_+^*$,

$$\sum_{n \geq 1} \mu_q^\gamma(E_{b,n,\gamma,\epsilon}^\lambda) < \infty,$$

consequently, by the Borel-Cantelli lemma, for μ_q^γ -almost every t , we have

$$\sum_{k=1}^n X_{t_1 \dots t_k}(t) - b - \eta_k(\gamma) = o \left(\sum_{k=1}^n \eta_k(\gamma) \right), \quad \text{so } N_n(t) - nb \sim s_{n,\gamma},$$

which yields the desired result.

Let us prove (4.1) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\theta = (\theta_n)$ be a positive sequence and $(q, \gamma) \in \mathbf{K}$. One has

$$\sup_{(q, \gamma) \in \mathbf{K}} \mu_q^\gamma \left(E_{b, n, \gamma, \epsilon}^1 \right) \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \mu_q^\gamma([u]) \mathbf{1}_{\left\{ E_{b, n, \gamma, \epsilon}^1 \right\}}(t_u),$$

where t_u is any point in $[u]$. Denote t_u simply by t , then

$$\begin{aligned} & \sup_{(q, \gamma) \in \mathbf{K}} \mu_q^\gamma \left(E_{b, n, \gamma, \epsilon}^1 \right) \\ & \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \mu_q^\gamma([u]) \prod_{k=1}^n \exp \left(\theta_k X_{t_1 \dots t_k} - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) \\ & \leq \sup_{(q, \gamma) \in \mathbf{K}} \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left((\psi_k(q, \gamma) + \theta_k) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) \\ & \quad \times Y(q, \gamma, u). \end{aligned}$$

For $(q, \gamma) \in \mathbf{K}$, $\theta = (\theta_n)$ and $n \geq 1$, we set

$$\begin{aligned} & H_n(q, \gamma, \theta) \\ & = \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left((\psi_k(q, \gamma) + \theta_k) X_{t_1 \dots t_k} - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) M(u), \end{aligned}$$

where

$$M(u) = \sup_{(q, \gamma) \in \mathbf{K}} Y(q, \gamma, u).$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathbf{V} = V_K \times V_{K_\gamma} \subset \mathbb{C}^2$ of $\mathbf{K} = K \times K_\gamma$ such that

$$\Gamma(z) = \frac{\mathbb{E} \left(\sum_{i=1}^N X_i \exp(z X_i) \right)}{\mathbb{E} \left(\sum_{i=1}^N \exp(z X_i) \right)}$$

is well defined for $z \in V_K$, for $k \geq 1$, $\eta_k(z')$ is defined for $z' \in V_{K_\gamma}$ and $\forall (z, z') \in \mathbf{V}$, $\psi_k(z, z')$ is defined and belongs to V_K .

For $\epsilon > 0$, $(z, z') \in \mathbf{V}$ and $n \geq 1$, we define

$$\begin{aligned} H_n(z, z', \theta) & = \sum_{u \in \mathbb{T}_n} \prod_{k=1}^n \exp \left((\psi_k(z, z') + \theta_k) X_{u|_k} - \theta_k \Gamma(z) - \theta_k \eta_k(z')(1 + \epsilon) \right) \\ & \quad \times \mathbb{E} \left(\sum_{i=1}^N \exp \left(\psi_k(z, z') X_i \right) \right)^{-1} M(u). \end{aligned}$$

Proposition 4.1. *There exist a neighborhood $\mathbf{V}' \subset \mathbf{V}$ of \mathbf{K} , a positive constant $\mathcal{C}_\mathbf{K}$ and a positive sequence θ such that for all $(z, z') \in \mathbf{V}'$, for all $n \in \mathbb{N}^*$*

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathcal{C}_\mathbf{K} \exp \left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right),$$

where the sequences $(\epsilon_n)_n$ and $(\tilde{\eta}_n)_n$ are the sequences used in Hypothesis 1.2.

Lemma 4.1. *There exist a positive sequence $\theta = (\theta_n)$ and a positive constant \mathcal{C}_K such that for all $(q, \gamma) \in K$ we have*

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}_K \exp \left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right).$$

Proof of Lemma 4.1. Let $\theta = (\theta_n)$ be a positive sequence, clearly we have

$$\begin{aligned} \mathbb{E}(H_n(q, \gamma, \theta)) &= \prod_{k=1}^n \mathbb{E} \left(\sum_{i=1}^N \exp \left((\psi_k(q, \gamma) + \theta_k) X_i \right) \right. \\ &\quad \times \exp \left(-\tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right) \mathbb{E}(M(u)) \\ &\leq \mathcal{C}'_K \prod_{k=1}^n \exp \left(\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right), \end{aligned}$$

where, by Proposition 2.1, $\mathcal{C}'_K = \mathbb{E}(M(u)) = \mathbb{E}(M(\emptyset)) < \infty$ for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$.

Since $\eta_k(\gamma) = o(1)$, we can fix a compact neighborhood K' of K and suppose that for all $k \geq 1$ and $(q, \gamma) \in K$, we have $\psi_k(q, \gamma) \in K'$. For $(q, \gamma) \in K$ and $k \geq 1$, writing the Taylor expansion with integral rest of order 2 of the function $g : \theta \mapsto \tau(\psi_k(q, \gamma) + \theta)$ at 0, we get

$$g(\theta) = g(0) + \theta g'(0) + \theta^2 \int_0^1 (1-t) g''(t\theta) dt,$$

with $g''(t\theta) \leq m_K = \sup_{t \in [0,1]} \sup_{q \in K'} \sup_{\gamma \in K_\gamma} g''(t\theta)$. It follows that for all $k \geq 1$

$$\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k \tau'(\psi_k(q, \gamma)) \leq \theta_k^2 m_K.$$

Recall that $\tau'(\psi_k(q, \gamma)) = \tau'(q) + \eta_k(\gamma)$. Then

$$\begin{aligned} \mathbb{E}(H_n(q, \gamma, \theta)) &\leq \mathcal{C}'_K \prod_{k=1}^n \exp \left(\tau(\psi_k(q, \gamma) + \theta_k) - \tau(\psi_k(q, \gamma)) - \theta_k b - \theta_k \eta_k(\gamma)(1 + \epsilon) \right), \\ &\leq \mathcal{C}'_K \prod_{k=1}^n \exp \left(-\theta_k \eta_k(\gamma) \epsilon + \theta_k^2 m_K \right). \end{aligned}$$

Choose the sequence θ such that $\theta_k = \epsilon_k \tilde{\eta}_k$. Then

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}'_K \prod_{k=1}^n \exp \left(-\epsilon_k \tilde{\eta}_k^2 (\epsilon - \epsilon_k m_K) \right).$$

Since $\epsilon_k \rightarrow 0$ then for k large enough we have $\epsilon - \epsilon_k m_K > \frac{\epsilon}{2}$. Then, there exists a constant \mathcal{C}_K such that

$$\mathbb{E}(H_n(q, \gamma, \theta)) \leq \mathcal{C}_K \exp \left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2 \right). \quad \square$$

Proof of Proposition 4.1. Since $\mathbb{E}(|H_n(q, \gamma, \theta)|) \leq \mathcal{C}_K \exp\left(-\frac{\epsilon}{2} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right)$ for $q \in K$, there exists a neighborhood $V_{q,\gamma} \subset V$ of (q, γ) such that for all $(z, z') \in V_{q,\gamma}$ we have

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathcal{C}_K \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right).$$

By extracting a finite covering of K from $\bigcup_{(q,\gamma) \in K} V_{q,\gamma}$, we find a neighborhood $V' \subset V$ of K such that

$$\mathbb{E}(|H_n(z, z', \theta)|) \leq \mathcal{C}_K \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right). \quad \square$$

With probability 1, the functions $(z, z') \in V' \mapsto H_n(z, z', \theta)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$, with $z_0 = (z_1, z'_1)$ and $\rho = (\rho_1, \rho_2)$. Theorem B.1 gives

$$\sup_{(z, z') \in D(z_0, \rho)} |H_n(z, z', \theta)| \leq 2 \int_{[0,1]^2} |H_n(\zeta(t), \theta)| dt,$$

where for $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned} \mathbb{E}\left(\sup_{z \in D(z_0, \rho)} |H_n^s(z, z', \theta)|\right) &\leq \mathbb{E}\left(2 \int_{[0,1]^2} |H_n(\zeta(t), \theta)| dt\right) \\ &\leq 4 \int_{[0,1]^2} \mathbb{E} |H_n(\zeta(t), \theta)| dt \\ &\leq 4 \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right). \end{aligned}$$

Finally, we get

$$\mathbb{E}\left(\sup_{(q,\gamma) \in K} \mu_q^\gamma(E_{b,n,\gamma,\epsilon}^1)\right) \leq 4 \exp\left(-\frac{\epsilon}{4} \sum_{k=1}^n \epsilon_k \tilde{\eta}_k^2\right)$$

and, then, under Hypothesis 1.2, we get (4.1), which finish the proof of Proposition 3.1.

4.2. Proof of Proposition 3.2. Let $K = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $a > 1$, $(q, \gamma) \in K$ and $n \geq 1$, we set

$$E_{n,a}^+ = \{t \in \partial T : Y(q, \gamma, t|_n) > a^n\}$$

and

$$E_{n,a}^- = \{t \in \partial T : Y(q, \gamma, t|_n) < a^{-n}\}.$$

It is sufficient to show that for $E \in \{E_{n,a}^+, E_{n,a}^-\}$

$$(4.2) \quad \mathbb{E}\left(\sup_{(q,\gamma) \in K} \sum_{n \geq 1} \mu_q^\gamma(E)\right) < \infty.$$

Indeed, if this holds, then with probability 1, for each $(q, \gamma) \in \mathbf{K}$ and $E \in \{E_{n,a}^+, E_{n,a}^-\}$, $\sum_{n \geq 1} \mu_q^\gamma(E) < \infty$, hence by the Borel-Cantelli lemma, for μ_q^γ -almost every $t \in \partial \mathbf{T}$, if n is big enough we have

$$-\log a \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log Y(q, \gamma, t|_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log Y(q, \gamma, t|_n) \leq \log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (4.2) for $E = E_{n,a}^+$ (the case $E = E_{n,a}^-$ is similar). At first we have,

$$\begin{aligned} \sup_{(q,\gamma) \in \mathbf{K}} \mu_q^\gamma(E_{n,a}^+) &= \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbf{T}_n} \mu_q^\gamma([u]) \mathbf{1}_{\{Y(q,\gamma,u) > a^n\}} \\ &= \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbf{T}_n} Y(q, \gamma, u) \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X(u) - \tau(\psi_k(q, \gamma))\right) \mathbf{1}_{\{Y(q,\gamma,u) > a^n\}} \\ &\leq \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbf{T}_n} (Y(q, \gamma, u))^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}, \\ &\leq \sup_{(q,\gamma) \in \mathbf{K}} \sum_{u \in \mathbf{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}, \end{aligned}$$

where $M(u) = \sup_{(q,\gamma) \in \mathbf{K}} Y(q, \gamma, u)$ and $\nu > 0$ is an arbitrary parameter. For $q \in K$, $\gamma \in K_\gamma$ and $\nu > 0$ we set

$$L_n(q, \gamma, \nu) = \sum_{u \in \mathbf{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(q, \gamma) X_u - \tau(\psi_k(q, \gamma))\right) a^{-\nu}.$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathbf{V} \subset \mathbb{C}^2$ of \mathbf{K} such that for all $(z, z') \in \mathbf{V}$ and $k \geq 1$ $\psi_k(z, z')$ is well defined and $\mathbb{E}\left(\sum_{i=1}^N e^{\psi_k(z, z') X_i}\right) \neq 0$.

Lemma 4.2. *Fix $a > 1$. For $(z, z') \in \mathbf{V}$ and $\nu > 0$, let*

$$\begin{aligned} L_n(z, z', \nu) &= \left[\prod_{k=1}^n \mathbb{E}\left(\sum_{i=1}^N \exp\left(\psi_k(z, z') X_i\right)\right) \right]^{-1} \\ &\quad \times \sum_{u \in \mathbf{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp\left(\psi_k(z, z') X_{u|_k}\right) a^{-\nu}. \end{aligned}$$

There exist a neighborhood $\mathbf{V}' \subset \mathbb{C}^2$ of \mathbf{K} and a positive constant C_K such that, for all $(z, z') \in \mathbf{V}'$, for all integer $n \geq 1$

$$(4.3) \quad \mathbb{E}\left(\left|L_n(z, z', p_K - 1)\right|\right) \leq C_K a^{-n(p_K - 1)/4},$$

where p_K provided by Proposition 2.1.

Proof. Write $V = V_K \times V_{K_\gamma}$. For $z \in V_K$ and $\nu > 0$, let

$$\tilde{L}_1(z, \nu) = \left| \mathbb{E}\left(\sum_{i=1}^N \exp(z X_i)\right) \right|^{-1} \mathbb{E}\left(\sum_{i=1}^N \left| \exp(z X_i) \right| \right) a^{-\nu}.$$

Let $q \in K$. Since $\mathbb{E}(\tilde{L}_1(q, \nu)) = a^{-\nu}$, there exists a neighborhood $V_q \subset V_K$ of q such that for all $z \in V_q$ we have $\mathbb{E}(|\tilde{L}_1(z, \nu)|) \leq a^{-\nu/2}$. Let $\gamma \in K_\gamma$. Recall the proof of Proposition 2.1 and since $\eta_k(\gamma) = o(1)$, we can find a neighborhood $V_\gamma \subset V_{K_\gamma}$ of K_γ such that, for all $k \geq 1$, $(z, z') \in V_q \times V_\gamma$, we have

$$\mathbb{E}(|\tilde{L}_1(\psi_k(z, z'), \nu)|) \leq a^{-\nu/3}.$$

By extracting a finite covering of K from $\bigcup_{(q, \gamma)} V_q \times V_\gamma$, we find a neighborhood $V' \subset V$ of K such that for all $(z, z') \in V'$ and $k \geq 1$

$$\mathbb{E}(|\tilde{L}_1(\psi_k(z, z'), \nu)|) \leq a^{-\nu/4}.$$

Therefore,

$$\begin{aligned} & \mathbb{E}(|L_n(z, z', \nu)|) \\ &= \left[\prod_{k=1}^n \mathbb{E} \left(\left| \sum_{i=1}^N \exp(\psi_k(z, z') X_i) \right| \right)^{-1} \right] \mathbb{E} \left(\left| \sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \exp(\psi_k(z, z') X_u) \right| \right) a^{-n\nu} \\ &\leq \left[\prod_{k=1}^n \mathbb{E} \left(\left| \sum_{i=1}^N \exp(\psi_k(z, z') X_i) \right| \right)^{-1} \right] \mathbb{E} \left(\sum_{u \in \mathbb{T}_n} M(u)^{1+\nu} \prod_{k=1}^n \left| \exp(\psi_k(z, z') X_u) \right| \right) a^{-n\nu}. \end{aligned}$$

By Proposition 2.1, there exists $p_K \in (1, 2]$ such that for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$,

$$\mathbb{E}(M(u)^{p_K}) = \mathbb{E}(M(\emptyset)^{p_K}) = C_K < \infty.$$

Now take $\nu = p_K - 1$ in the last calculation, it follows, from the independence of $\sigma(\{Y(\cdot, \cdot, u), u \in \mathbb{T}_n\})$ and $\sigma(\{(X_{u1}, \dots, X_{uN_u}), u \in \mathbb{T}_{n-1}\})$ for all $n \geq 1$, that

$$\begin{aligned} & \mathbb{E}(|L_n(z, z', p_K - 1)|) \\ &\leq \left[\prod_{k=1}^n \mathbb{E} \left(\left| \sum_{i=1}^N \exp(\psi_k(z, z') X_i) \right| \right)^{-1} \right] \prod_{k=1}^n \mathbb{E} \left(\sum_{i=1}^N \left| \exp(\psi_k(z, z') X_i) \right| \right)^n C_K a^{-n(p_K - 1)} \\ &= C_K \prod_{k=1}^n \mathbb{E}(|\tilde{L}_1(\psi_k(z, z'), p_K - 1)|) \\ &\leq C_K a^{-n(p_K - 1)/4}, \end{aligned}$$

then lemma is now proved. \square

With probability 1, the functions $(z, z') \in V' \mapsto L_n(z, z', \nu)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V'$, with $z_0 = (z_1, z'_1)$ and $\rho = (\rho_1, \rho_2)$. Theorem B.1 gives

$$\sup_{z \in D(z_0, \rho)} |L_n(z, p_K - 1)| \leq 4 \int_{[0,1]^2} |L_n(\zeta(t), p_K - 1)| dt,$$

where, for $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z'_1 + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned} \mathbb{E} \left(\sup_{z \in D(z_0, \rho)} |L_n(z, p_K - 1)| \right) &\leq \mathbb{E} \left(4 \int_{[0, 1]^2} |L_n(\zeta(t), p_K - 1)| dt \right) \\ &\leq 4 \int_{[0, 1]^2} \mathbb{E} |L_n(\zeta(t), p_K - 1)| dt \\ &\leq 4C_K a^{-n(p_K - 1)/4}. \end{aligned}$$

Since $a > 1$ and $p_K - 1 > 0$, we get (4.2).

APPENDIX A. HAUSDORFF AND PACKING DIMENSIONS

Given a subset K of $\mathbb{N}_+^{\mathbb{N}}$ endowed with a metric d making it σ -compact, $s > 0$ and E a subset of K , the s -dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} (\text{diam}(U_i))^s \right\},$$

the infimum being taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ . Then, the Hausdorff dimension of E is defined as

$$\dim E = \sup \{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf \{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given $s > 0$ and $E \subset K$ as above, one first defines

$$\overline{P}^s(E) = \lim_{\delta \rightarrow 0^+} \sup \left\{ \sum_{i \in \mathbb{N}} (\text{diam}(B_i))^s \right\},$$

the supremum being taken over all the packings $\{B_i\}_{i \in \mathbb{N}}$ of E by balls centered on E and with diameter smaller than or equal to δ . Then, the s -dimensional packing measure of E is defined as

$$P^s(E) = \lim_{\delta \rightarrow 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} \overline{P}^s(E_i) \right\},$$

the infimum being taken over all the countable coverings $(E_i)_{i \in \mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ . Then, the packing dimension of E is defined as

$$\text{Dim } E = \sup \{s > 0 : P^s(E) = \infty\} = \inf \{s > 0 : P^s(E) = 0\},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. For more details the reader is referred to [13, 20].

If μ is a positive and finite Borel measure supported on K , then its lower Hausdorff and packing dimensions is defined as

$$\begin{aligned}\underline{\dim}(\mu) &= \inf \left\{ \dim F : F \text{ Borel}, \mu(F) > 0 \right\} \\ \underline{\text{Dim}}(\mu) &= \inf \left\{ \text{Dim } F : F \text{ Borel}, \mu(F) > 0 \right\}\end{aligned}$$

and its upper Hausdorff and packing dimensions are defined as

$$\begin{aligned}\overline{\dim}(\mu) &= \inf \left\{ \dim F : F \text{ Borel}, \mu(F) = \|\mu\| \right\} \\ \overline{\text{Dim}}(\mu) &= \inf \left\{ \text{Dim } F : F \text{ Borel}, \mu(F) = \|\mu\| \right\}.\end{aligned}$$

We have (see [12])

$$\begin{aligned}\underline{\dim}(\mu) &= \text{ess inf}_{\mu} \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}, \\ \underline{\text{Dim}}(\mu) &= \text{ess inf}_{\mu} \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}(\mu) &= \text{ess sup}_{\mu} \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)}, \\ \overline{\text{Dim}}(\mu) &= \text{ess sup}_{\mu} \limsup_{r \rightarrow 0^+} \frac{\log \mu(B(t, r))}{\log(r)},\end{aligned}$$

where $B(t, r)$ stands for the closed ball of radius r centered at t . If $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{\text{Dim}}(\mu) = \overline{\text{Dim}}(\mu)$), this common value is denoted $\dim \mu$ (resp. $\text{Dim}(\mu)$), and if $\dim \mu = \text{Dim} \mu$, one says that μ is exact dimensional.

Recall the mass distribution principle.

Theorem A.1. ([13, Theorem 4.2]). *Let ν be a positive and finite Borel probability measure on a compact metric space (X, d) . Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and*

$$M \subseteq \left\{ t \in X : \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ .

APPENDIX B. CAUCHY FORMULA IN SEVERAL VARIABLES

Let us recall the Cauchy formula for holomorphic functions in several variables.

Definition B.1. Let $d \geq 1$, a subset D of \mathbb{C}^d is an open polydisc if there exist open discs D_1, \dots, D_d of \mathbb{C} such that $D = D_1 \times \dots \times D_d$. If we denote by ζ_j the centre of D_j , then $\zeta = (\zeta_1, \dots, \zeta_d)$ is the centre of D and if r_j is the radius of D_j then $r = (r_1, \dots, r_d)$ is the multiradius of D . The set $\partial D = \partial D_1 \times \dots \times \partial D_d$ is the distinguished boundary of D . We denote by $D(\zeta, r)$ the polydisc with center ζ and radius r .

Let $D = D(\zeta, r)$ be a polydisc of \mathbb{C}^d and $g \in C(\partial D)$ a continuous function on ∂D . We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta_1 \cdots d\zeta_d = (2i\pi)^d r_1 \cdots r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1} \cdots e^{i2\pi\theta_d} d\theta_1 \cdots d\theta_d,$$

where $\zeta(\theta) = (\zeta_1(\theta), \dots, \zeta_d(\theta))$ and $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$ for $j = 1, \dots, d$.

Theorem B.1. *Let $D = D(a, r)$ be polydisc in \mathbb{C}^d with a multiradius whose components are positive, and f be a holomorphic function in a neighborhood of D . Then, for all $z \in D$*

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_d}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| d\theta_1 \cdots d\theta_d.$$

REFERENCES

- [1] N. Attia, *On the exact dimension of Mandelbrot measure*, Probab. Math. Statist. **39**(2) (2019), 299–314. <https://doi.org/10.19195/0208-4147.39.2.4>
- [2] N. Attia, *Hausdorff and packing dimensions of Mandelbrot measure*, Internat. J. Math. **31**(9) (2020), Article ID 2050068. <https://doi.org/10.1142/S0129167X20500688>
- [3] N. Attia, *On the multifractal analysis of branching random walk on Galton-Watson tree with random metric*, J. Theoret. Probab. **34**(1) (2020), 90–102. <https://doi.org/10.1007/s10959-019-00984-z>
- [4] N. Attia and J. Barral, *Hausdorff and packing spectra, large deviations and free energy for branching random walks in \mathbb{R}^d* , Comm. Math. Phys. **331** (2014), 139–187. <https://doi.org/10.1007/s00220-014-2087-9>
- [5] N. Attia, *On the multifractal analysis of covering number on the Galton Watson tree*, J. Appl. Probab. **56**(1) (2019), 265–281. <https://doi.org/10.1017/jpr.2019.17>
- [6] N. Attia, *Comportement asymptotique de marches aléatoires de branchement dans \mathbb{R}^d et dimension de Hausdorff*, SISYPHE - Signals and Systems in Physiology & Engineering - Thèse de doctorat, tel-00841496, (2012).
- [7] N. Attia, *On the Multifractal Analysis of the Branching Random Walk in \mathbb{R}^d* , J. Theoret. Probab. **27** (2014), 1329–1349. <https://doi.org/10.1007/s10959-013-0488-x>
- [8] J. Barral, *Continuity of the multifractal spectrum of a statistically self-similar measure*, J. Theoret. Probab. **13** (2000), 1027–1060. <https://doi.org/10.1023/A:1007866024819>
- [9] J. D. Biggins, *Martingale convergence in the branching random walk*, J. Appl. Probab. **14** (1977), 25–37. <https://doi.org/10.2307/3213258>
- [10] J. D. Biggins, *Uniform convergence of martingales in the branching random walk*, Ann. Probab. **20** (1992), 137–151. <https://doi.org/10.1214/aop/1176989921>
- [11] J. D. Biggins, B. M. Hambly and O. D. Jones, *Multifractal spectra for random self-similar measures via branching processes*, Adv. in Appl. Probab. **43**(1) (2011), 1–39. <https://doi.org/10.1239/aap/1300198510>
- [12] C. D. Cutler, *Connecting ergodicity and dimension in dynamical systems*, Ergodic Theory Dynam. Systems **10** (1990), 451–462. <https://doi.org/10.1017/S014338570000568X>
- [13] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd Edition, Wiley, Chichester, 2003. <https://doi.org/10.1002/0470013850>
- [14] K. J. Falconer, *The multifractal spectrum of statistically self-similar measures*, J. Theoret. Probab. **7**(3) (1994), 681–702. <https://doi.org/10.1007/BF02213576>

- [15] A. H. Fan and J. P. Kahane, *How many intervals cover a point in random dyadic covering?* Port. Math. **58**(1) (2001), 59–75.
- [16] R. Holley and E. C. Waymire, *Multifractal dimensions and scaling exponents for strongly bounded random fractals*, Ann. Appl. Probab. **2** (1992), 819–845. <https://doi.org/10.1214/aoap/1177005577>
- [17] R. Lyons, *Random walks and percolation on trees*, Ann. Probab. **18** (1990), 931–958. <https://doi.org/10.1214/aop/1176990730>
- [18] R. Lyons and R. Pemantle, *Random walks in a random environment and first passage percolation on trees*, Ann. Probab. **20** (1992), 125–136. <https://doi.org/10.1214/aop/1176989920>
- [19] Q. Liu and A. Rouault, *On two measures defined on the boundary of a branching tree*, IMA Vol. Math. Appl. **84** (1997), 187–201. https://doi.org/10.1007/978-1-4612-1862-3_15
- [20] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability*, Cambridges Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995. <https://doi.org/10.1017/cbo9780511623813>
- [21] G. M. Molchan, *Scaling exponents and multifractal dimensions for independent random cascades*, Comm. Math. Phys. **179** (1996), 681–702.

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