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BERTRAND'S PARADOX: NEW PROBABILISTIC MODELS

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Abstract. In this paper two new generating procedure of a random chord are obtained and thereby new solutions of Bertrand's paradox are proposed.

1. Introduction

Paradox, on its own, is a puzzle that confronts some already established principles. Bertand's paradox was developed as a probability question that raised severe objections on the principle of indifference while dealing with geometrical probability. The question that defines this paradox: "What is the probability that a chord selected "at random" in a circle is larger than a side of the inscribed equilateral triangle?"

In [3], Bertrand obtained probabilities $1/3$, $1/2$ and $1/4$ by different random chord generation procedures: by choosing a chord with one end at a vertex of the inscribed equilateral triangle in a circle; by choosing a chord perpendicular to the diameter which is the right bisector of the equilateral triangle; and selecting a point inside a circle and denoting it as a chord midpoint, respectively. This puzzle has fascinated many since its discovery and a series of papers with outstanding solutions of this problem have been published, see e.g. [1, 2, 4–9]. Here, we provide two new models of random chord construction in a circle and obtain associated probabilities of Bertrand's paradox.

The paper is organized as follows. In Section 2, we propose two new procedures for generating a random chord in a circle and obtain probabilities of Bertrand's paradox for each case. Section 3 concludes this paper.

Key words and phrases. Bertrand's paradox, new solutions, Monte Carlo simulations.
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2. New Models and Solutions

In [7], an attempt was made to look at classical models of Bertrand's paradox as limits of a continuous family of planar probabilistic models. Such family is seen by fixing a point, say A, at a distance $h > 1$ from a unit circle and constructing lines that intersect the circle and point A. However, this family of chord constructing models undermines the randomness selection of distance h and, so, it yields inappropriate results with respect to Bertrand's paradox. Motivated by this issue, in [10] a chord generating procedure is presented that overcomes this obstacle. Here, we additionally provide two new methods of generating random chords in a circle with the same intention.

For both models, we will denote X as the distance from the center of the circle and the chord and L as the corresponding chord length.

2.1. **First model.** The first method is obtained as follows.

Step 1. Let a point A be such that its distance from the center of the circle OA is a random variable $Y \sim U(0,1)$ and is lying on the x axis.
Step 2. Using the circle invariance property we can obtain a point on a x axis, say P, so that the relation $OP \cdot OA = 1$ holds.
Step 3. Angle ϕ is determined by the circle tangent and the x axis, with P as its vertex;
Step 4. Select a line which is directed by an angle $\theta \in U(0,\phi)$, with P as its starting point. A chord is formed by its intersection with the circle (Figure 1.).

In this case, we have $X = \sqrt{1 - \frac{L^2}{4}}$, $\phi = \arcsin(Y)$ and $\theta = \arcsin\left(Y\sqrt{1 - \frac{L^2}{4}}\right)$.

Using transformation technique, the distribution function of L can be found as

$$F_I(l) = \int_0^1 \int_0^l \frac{xy}{4\arcsin(y)\sqrt{1 - \frac{x^2}{4}}\sqrt{1 - (1 - \frac{x^2}{4})y^2}}\, dx\, dy$$

$$= \int_0^1 \frac{\arcsin(y) - \arcsin\left(\frac{y\sqrt{4-l^2}}{2}\right)}{\arcsin(y)}\, dy, \quad 0 < l < 2. \tag{2.1}$$

Integral (2.1) cannot be obtained explicitly, so we can only provide numerical solutions. For the Bertrand's case $l = \sqrt{3}$ we have

$$P\left\{L_I \geq \sqrt{3}\right\} = 1 - F_I(\sqrt{3}) = 0.4694. \tag{2.2}$$

2.2. **Second model.** The second method is obtained as follows.

Step 1. Let a point A be determined by a random angle $\phi \sim U(0,\pi/2)$ on a circumference of a circle.
Step 2. Let a tangent t of a circle be determined by point A.
Step 3. Angle δ is determined by the circle tangent and the x axis, with P as its vertex.
Step 4. Select a line which is directed by an angle $\theta \in U(0,\delta)$, with P as its starting point. A chord is formed by its intersection with the circle (Figure 2).

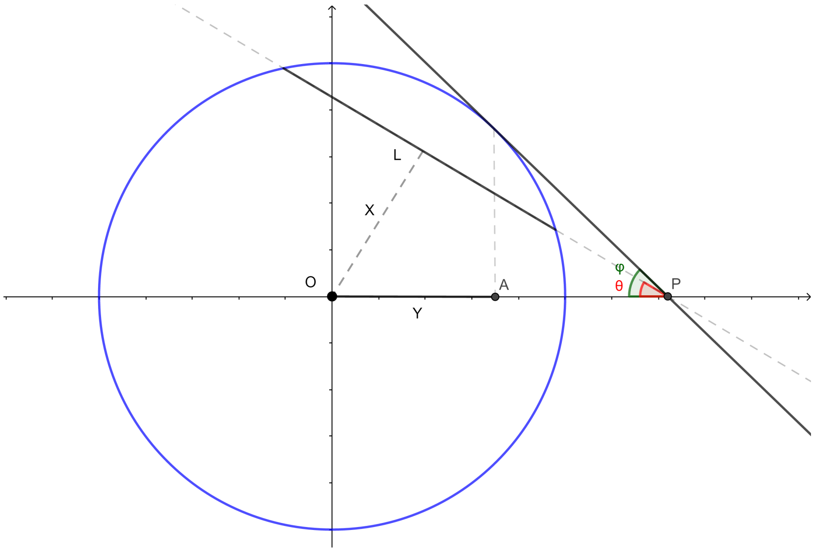

FIGURE 1. Solution I.

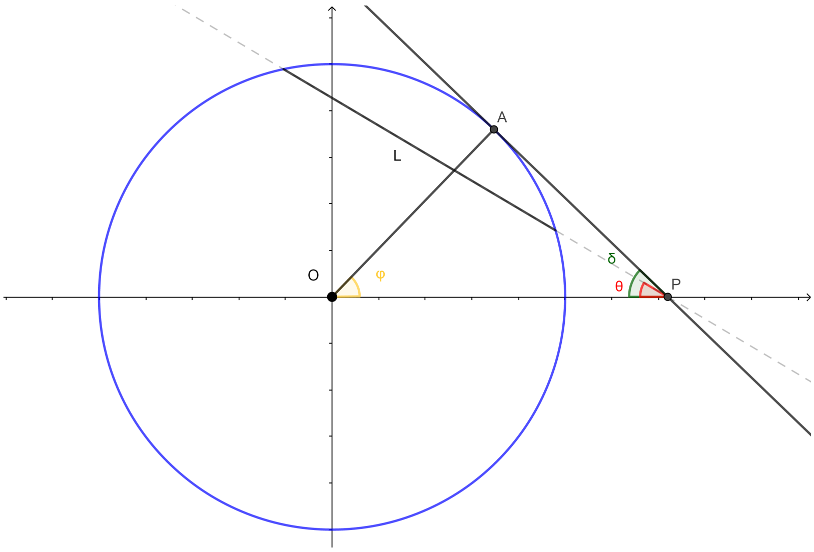

FIGURE 2. Solution II.

For this case, we have $X = \sqrt{1 - \frac{L^2}{4}}$ and $\sin\theta = \cos\phi\sqrt{1 - \frac{L^2}{4}}$. Further, the distribution function of L can be obtained as

$$F_{II}(l) = \frac{2}{\pi}\int_0^{\pi/2}\int_0^l \frac{x\cos y}{4(\frac{\pi}{2} - y)\sqrt{1 - \frac{x^2}{4}}\sqrt{1 - (1 - \frac{x^2}{4})\cos^2 y}}\,dx\,dy$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\operatorname{arcsec}\left(\frac{2\sec(y)}{\sqrt{4-l^2}}\right) - y}{\pi - 2y}\,dy, \quad 0 < l < 2. \tag{2.3}$$

As above, integral (2.3) cannot be obtained explicitly, so we can obtain numerical solutions. For the Bertrand's case $l = \sqrt{3}$ we have

$$P\left\{L_{II} \geq \sqrt{3}\right\} = 1 - F_{II}(\sqrt{3}) = 0.4454. \tag{2.4}$$

3. Conclusion

Overall, in this paper we presented two new generating procedures of random chords in a circle. The distribution function (2.3) is also obtained in [10] using a different method of constructing random chords. The results presented in this paper extend those can be found in [4, 9, 10] on Bertrand's paradox.

In [6], procedures of chord construction were classified by disjoint procedures: (i) inside the circle, (ii) on the circle circumference and (iii) outside of the circle. Proposed generating models connect procedures (i), (ii) and (iii), and confronts such classification. This may be a motivation to overlook Bertrand's paradox in a quite different manner.

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