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ON Z-SYMMETRIC MANIFOLD WITH CONHARMONIC CURVATURE TENSOR IN SPECIAL CONDITIONS

AYŞE YAVUZ TAŞCI[1] AND FÜSUN ÖZEN ZENGIN[2]

Abstract. The object of the present paper is to study the Z-symmetric manifold with conharmonic curvature tensor in special conditions. In this paper, we prove some theorems about these manifolds by using the properties of the Z-tensor.

1. Introduction

Conformal geometry has deep importance in pure mathematics, such as complex analysis, Riemann surface theory, differential geometry and algebraic topology, [2, 21, 22]. Computational conformal geometry is important in digital geometry processing. Discrete conformal geometry has been presented to compute conformal mapping which has been broadly applied in numerous practical fields, including computer vision and graphics, visualization, medical imaging, etc. In medical imaging, conformal geometry has been applied to surface parametrization and extract intrinsic features for natural objects like brain, colon, spleen and other human organs.

Historically, conformal mappings have been considered in many monographs, surveys and papers. Also, the theory of conformal mappings has very important applications in general relativity.

Let (M, g) and $(\bar{M}, \bar{g})$ be two n-dimensional Riemannian manifolds with metric tensors g_{ij} and $\bar{g}_{ij}$, respectively. Both metrics are defined in a common coordinate system (x^i). The correspondence between (M, g) and $(\bar{M}, \bar{g})$ is conformal, if the

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fundamental tensors g_{ij} and $\bar{g}_{ij}$ of two manifolds M and $\bar{M}$ are in the relation

$$\bar{g}_{ij}(x) = e^{2\sigma(X)} g_{ij}(x), \tag{1.1}$$

where $\sigma(x)$ is a scalar function of the x's.

By the transformation (1.1), it also follows that the relation between the Christoffel symbols Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ compatible with the metrics g_{ij} and $\bar{g}_{ij}$, respectively is given by

$$\bar{\Gamma}^h_{ij} = \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{1.2}$$

where $\sigma_i = \frac{\partial \sigma}{\partial x^i}$, $\sigma^h = g^{hi}\sigma_i$, g^{ij} are the components of the inverse matrix to g_{ij}, and δ^h_i is the Kronecker delta.

A conformal mapping is called homothetic if the function σ is a constant, that is, $\bar{g}_{ij}(x) = c g_{ij}(x)$. The condition is equivalent to $\sigma_i = 0$, hence, the mapping is also an affine one.

Denoting R^h_{ijk} and $\bar{R}^h_{ijk}$ are the Riemann tensors of the manifolds M and $\bar{M}$, respectively, then we have [11, 20]

$$\begin{aligned}
\bar{R}^h_{ijk} =& R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{hl}(\sigma_{lk} g_{ij} - \sigma_{li} g_{jk}) + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1 \sigma, \\
\bar{S_{ij}} =& S_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij}, \\
\bar{r} =& e^{-2\sigma}(r + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma),
\end{aligned} \tag{1.3}$$

where $\sigma_i = \partial_i \sigma$, $\Delta_l \sigma = g^{ij}\sigma_i \sigma_j$, $\Delta_2 \sigma = g^{ij}\sigma_{i,j}$, $\sigma_{ij} = \sigma_{i,j} - \sigma_i \sigma_j$. We denote that $S_{ij} = R^h_{ijh}$ and $\bar{S}_{ij} = \bar{R}^h_{ijh}$ are their Ricci tensors and $r = S_{ij} g^{ij}$ and $\bar{r} = \bar{S}_{ij}\bar{g}^{ij}$ are their scalar curvatures.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In generally, the harmonic function is not invariant under the conformal transformation. In [14], Ishii obtained the conditions which a harmonic function remains invariant and he introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition [14]

$$\sigma^h_{,h} + \sigma^h_{,h}\sigma^h_{,} = 0, \tag{1.4}$$

where comma denotes the covariant differentiation with respect to the metric g.

Thus, we can say that the conharmonic transformation which is a special type of conformal transformations preserves the harmonicity of smooth functions. It is well known that such transformations have an invariant tensor, so-called the conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor, that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

A rank-four tensor L that remains invariant under conharmonic transformation of a Riemannian manifold (M, g) is given by

$$\begin{aligned}
L(X,Y,Z,U) =& R(X,Y,Z,U) - \frac{1}{n-2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\
& + g(X,U)S(Y,Z) - g(Y,U)S(X,Z)],
\end{aligned} \tag{1.5}$$

where R and S denote the Riemannian curvature tensor of type $(0,4)$ defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type $(0,2)$, respectively. The curvature tensor defined by (1.5) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness.

Q denotes the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type $(0,2)$, that is

$$g(QX,Y) = S(X,Y). \tag{1.6}$$

Let $\Big\{e_i, i = 1,2,\dots,n\Big\}$ be an orthonormal basis of the tangent space at each point of the manifold. From (1.5), we have

$$\overline{L}(X,Y) = \sum_{i=1}^{n} L(X,e_i,e_i,Y) = \sum_{i=1}^{n} L(e_i,X,Y,e_i) = -\frac{r}{n-2} g(X,Y) \tag{1.7}$$

and

$$\sum_{i=1}^{n} L(e_i,e_i,X,Y) = \sum_{i=1}^{n} L(X,Y,e_i,e_i) = 0, \tag{1.8}$$

where r is the scalar curvature of the manifold. Also, from (1.5) it follows that [26]

$$\begin{aligned} L(X,Y,Z,U) &= -L(Y,X,Z,U),\\ L(X,Y,Z,U) &= -L(X,Y,U,Z),\\ L(X,Y,Z,U) &= L(Z,U,X,Y),\\ L(X,Y,Z,U) &+ L(X,Z,U,Y) + L(X,U,Y,Z) = 0. \end{aligned} \tag{1.9}$$

In [26], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetries and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. This tensor has valuable applications in general relativity. In [1], Abdussatter investigated its physical significance in the theory of general relativity. The conharmonic transformation has also been studied by Siddique and Ahsan [27], Ghosh, De and Taleshian [12], and many others.

A non-flat Riemannian manifold which is called a recurrent manifold [25] if the curvature tensor of this manifold satisfies the relation

$$(\nabla_W R)(X,Y,Z,U) = A(W)R(X,Y,Z,U), \tag{1.10}$$

where A is a non-zero 1-form. A non-flat Riemannian manifold which is called a Ricci-recurrent manifold if the Ricci tensor of this manifold satisfies the relation [5,23,28]

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z), \tag{1.11}$$

where A is a non-zero 1-form.

A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition $\nabla_X \xi = \alpha X + \lambda(X)\xi$, where $X \in TM$, $\lambda(X)$ is a linear form and α is a function, [4, 19, 29].

In the local transcription, this reads

$$\xi^h_{,i} = \alpha \delta^h_i + \xi^h \lambda_i, \tag{1.12}$$

where ξ^h and λ_i are the components of ξ and λ respectively, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called, [19, 29],

$i)$ recurrent if $\alpha = 0$, i.e.,

$$\xi^h_{,i} = \xi^h \lambda_i; \tag{1.13}$$

$ii)$ concircular if the form λ_i is gradient covector (i.e., $\lambda_i = \lambda_{,i}$), i.e.,

$$\xi^h_{,i} = \alpha \delta^h_i; \tag{1.14}$$

$iii)$ convergent if it is concircular and $\alpha = const.exp(\lambda)$.

A $\varphi(Ric)$-vector field is a vector field on an n dimensional Riemannian manifold (M, g) with metric g and Levi-Civita connection ∇, which satisfies the condition [13]

$$\nabla \varphi = \mu Ric, \tag{1.15}$$

where μ is some constant and Ric is the Ricci tensor. Obviously, when (M, g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu = 0$, the vector field φ is covariantly constant. In the following we suppose that $\mu \neq 0$ and (M, g) is neither an Einstein space nor a vacuum solution of the Einstein equations. In a locally coordinate neighbourhood $U(x)$, the equation (1.15) is written as

$$\varphi^h_{,i} = \mu S^h_i, \tag{1.16}$$

where φ^i and S^h_i are components of φ and Ric, respectively. After lowering indices, (1.16) has the form

$$\varphi_{i,j} = \mu S_{ij}, \tag{1.17}$$

where $\varphi_i = \varphi^\alpha g_{i\alpha}$ and $S_{ij} = g_{i\alpha} S^\alpha_j$.

2. Z-Tensor on a Riemannian Manifold

In 2012, Mantica and Molinari defined a generalized symmetric tensor of type $(0, 2)$ which is called Z-tensor and given by [15]

$$Z_{kl} = S_{kl} + \phi g_{kl}, \tag{2.1}$$

where ϕ is an arbitrary scalar function. The scalar $\bar{Z}$ is the trace of Z-tensor and from (2.1), it can be written as

$$\bar{Z} = g^{kl} Z_{kl} = r + n\phi. \tag{2.2}$$

The classical Z-tensor is obtained with the choice $\phi = -\frac{1}{n} r$. Shortly, the generalized Z-tensor is called as the Z-tensor. In some cases, the Z-tensor gives the several well known structures on Riemannian manifolds. For example, i) if $Z_{kl} = 0$ (i.e, Z-flat), then this manifold reduces to an Einstein manifold [3]; ii) if $\nabla_j Z_{kl} = \lambda_j Z_{kl}$ (Z-recurrent), then this manifold reduces to a generalized Ricci recurrent manifold [6]; iii) if $\nabla_j Z_{kl} = \nabla_k Z_{jl}$ (Codazzi tensor), then we find $\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)r$ [10]. This result gives us that this manifold is a nearly conformal symmetric manifold $(NCS)_n$ [24]. iv) The relation between the Z-tensor and the energy-stress tensor of Einstein' s equations [9], with cosmological constant Λ is $Z_{jl} = kT_{jl}$, where $\phi = -\frac{1}{2}r + \Lambda$ and k is the gravitational constant. In this case, the Z-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution ($Z = 0$) determines an Einstein space $\Lambda = \left(\frac{n-2}{2n}\right)r$; the conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla_j Z_{kl} = 0$ then this spacetime gives the conserved enery-momentum density.

This manifold has received a great deal of attention and is studied in considerable detail by many authors [7, 8, 15–18, 30, 31]), etc. Motivated by the above studies, in the present, we examine the properties of a Z-symmetric manifold with conharmonic curvature tensor.

The present paper is organized as follows. In Section 1 and Section 2, after reviewing the basics about symmetric spaces and Z-tensor, respectively. In Section 3 we will discuss Z-symmetric manifolds with conharmonic curvature tensor and mention some properties of these manifolds. We will concentrate on this paper that will be of relevance in our forthcoming paper.

3. Z-Symmetric Manifold with Conharmonic Curvature Tensor

In this section, we consider a Z-symmetric manifold with conharmonic curvature tensor. In the local coordinates, consider the equations (1.5) and (2.1), the relation between the Z-tensor and the conharmonic curvature tensor is found as

$$L_{hijk} = R_{hijk} - \frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{3.1}$$

By taking the covariant derivative of (3.1), we can find

$$\begin{aligned} L_{hijk,l} = &R_{hijk,l} - \frac{1}{n-2}[g_{ij}Z_{hk,l} - g_{ik}Z_{hj,l} + g_{hk}Z_{ij,l} - g_{hj}Z_{ik,l}] \\ &+ \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \end{aligned} \tag{3.2}$$

Suppose now that our manifold is Z-recurrent. Considering the equation (1.11) for Z-tensor, we can write $Z_{ij,l} = \lambda_l Z_{ij}$. Hence, we see from (3.2) that

$$L_{hijk,l} = R_{hijk,l} - \frac{\lambda_l}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] + \frac{2\phi_l}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}]. \tag{3.3}$$

It is obtained by (3.1)

(3.4)
$$\frac{1}{n-2}[g_{ij}Z_{hk} - g_{ik}Z_{hj} + g_{hk}Z_{ij} - g_{hj}Z_{ik}] = R_{hijk} - L_{hijk} + \frac{2\phi}{n-2}[g_{ij}g_{hk} - g_{ik}g_{hj}].$$

By the aid of (3.4), the expression (3.3) can be written as

$$L_{hijk,l} - \lambda_l L_{hijk} = R_{hijk,l} - \lambda_l R_{hijk} + \frac{2}{n-2}(g_{ij}g_{hk} - g_{ik}g_{hj})(\phi_l - \lambda_l \phi). \tag{3.5}$$

In the following theorems, a Riemannian manifold admitting covariantly constant conharmonic curvature tensor and recurrent Z-tensor with the recurrence vector field λ_l will be shown by (M, g).

Theorem 3.1. *The vector field ϕ_l and the recurrence vector field λ_l of (M, g) must be parallel and they satisfy the relation*

$$\phi_l = \left(\frac{r}{n} + \phi\right)\lambda_l.$$

Proof. Differentiating covariantly of (1.5) and assuming that the conharmonic curvature tensor is covariantly constant, it is not hard to see that the scalar curvature must be constant. If the Z-tensor is recurrent tensor admitting λ_l recurrence vector field then we have from (1.11) and (2.1)

$$\lambda_l Z_{ij} = S_{ij,l} + \phi_l g_{ij}. \tag{3.6}$$

Multiplying (3.6) by g^{ij}, we get

$$\lambda_l \bar{Z} = r_{,l} + n\phi_l. \tag{3.7}$$

Since r must be constant, from (2.2), the equation (3.7) takes the following form

$$\lambda_l(r + n\phi) = n\phi_l. \tag{3.8}$$

Arraying the equation (3.8), finally we obtain

$$\phi_l = \left(\frac{r}{n} + \phi\right)\lambda_l. \tag{3.9}$$

Hence, the proof is completed. □

Theorem 3.2. *On (M, g), $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector δ defined by $\lambda(X) = g(X, \delta)$.*

Proof. Suppose that the Z-tensor is recurrent tensor. As we already know from the equation (1.11), we have

$$Z_{ij,l} = \lambda_l Z_{ij}. \tag{3.10}$$

Multiplying (3.10) by g^{il}, we get

$$Z^l_{j,l} = \lambda^l Z_{jl}. \tag{3.11}$$

We remark that in a Riemannian manifold with covariantly constant conharmonic curvature tensor, the scalar curvature is constant. From this result and the Ricci Identity, we have $S^l_{j,l} = 0$. Thus, we see that

$$Z^l_{j,l} = \phi_j. \tag{3.12}$$

From (2.1), (3.11) and (3.12), one can show that

$$\phi_j = \lambda^l (S_{jl} + \phi g_{jl}). \tag{3.13}$$

On the other hand, if we use the equation (3.9), (3.13) takes the form

$$\left(\frac{r}{n} + \phi\right) \lambda_j = \lambda^l S_{jl} + \phi \lambda_j. \tag{3.14}$$

Finally, the equation (3.14) shows that

$$\lambda^l S_{jl} = \frac{r}{n} \lambda_j. \tag{3.15}$$

Hence, the proof is completed. □

Theorem 3.3. *A necessary and sufficient condition for the vector field ϕ^l generated by the scalar function ϕ of (M, g) to be divergence-free is that the divergence of the vector field λ^l be of negative value in the form*

$$\lambda^l_{,l} = -||\lambda||^2.$$

Proof. From Theorem 3.1, we know that the relation between ϕ_l and λ_l vector fields is in the form

$$\phi_l = \left(\frac{r}{n} + \phi\right) \lambda_l. \tag{3.16}$$

Taking the covariant derivative of (3.16), we get

$$\phi_{l,m} = \phi_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.17}$$

Substituting the equation (3.16) in (3.17), one can prove the relation

$$\phi_{l,m} = \left(\frac{r}{n} + \phi\right) \lambda_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.18}$$

Multiplying (3.18) by g^{lm}, we find

$$\phi^l_{,l} = \left(\frac{r}{n} + \phi\right) \left(||\lambda||^2 + \lambda^l_{,l}\right). \tag{3.19}$$

Now, suppose that the vector field ϕ_l is divergence-free. Of course, $r \neq -n\phi$ from (3.16), then by using (3.19), we obtain

$$\lambda^l_{,l} = -||\lambda||^2. \tag{3.20}$$

Conversely, if the equation (3.20) is satisfied then from (3.19), we can find $\phi^l_{,l} = 0$. Hence, the proof is completed. □

Theorem 3.4. *If the vector field λ_l on (M, g) is divergence-free then the divergence of the vector field ϕ_l is in the form*

$$\phi^l_{,l} = \frac{n}{r+n\phi}||\phi||^2.$$

Proof. From Theorem 3.3, we know that the relation (3.17) holds. In this case, if we use (3.16) and (3.17) then we get

$$\phi_{l,m} = \left(\frac{n}{r+n\phi}\right)\phi_m\phi_l + \left(\frac{r+n\phi}{n}\right)\lambda_{l,m}. \tag{3.21}$$

Multiplying (3.21) by g^{lm}, we find

$$\phi^l_{,l} = \left(\frac{n}{r+n\phi}\right)||\phi||^2 + \left(\frac{r+n\phi}{n}\right)\lambda^l_{,l}. \tag{3.22}$$

Now, suppose that the vector field λ_l is divergence-free. Finally, the divergence of the vector field ϕ_l is found in the following form

$$\phi^l_{,l} = \frac{n}{r+n\phi}||\phi||^2. \tag{3.23}$$

Thus, the proof is completed. □

Theorem 3.5. *If (M, g) admits a torse-forming vector field associated by the 1-form ϕ_l in the relation $\phi_{l,m} = \rho g_{lm} + \alpha_m\phi_l$, then the vector field λ_l is also torse-forming vector field satisfying the equation*

$$\lambda_{l,m} = \gamma g_{lm} + \beta_m\lambda_l,$$

where $\gamma = \frac{n\rho}{r+n\phi}$ and $\beta_m = \alpha_m - \lambda_m$.

Proof. Assume that the vector field ϕ_l is a torse-forming vector field with a scalar function ρ and a vector field α_m. As we know from (1.12) that

$$\phi_{l,m} = \rho g_{lm} + \alpha_m\phi_l. \tag{3.24}$$

Substituting the equation (3.24) in (3.17), thus we see that

$$\rho g_{lm} + \alpha_m\phi_l = \phi_m\lambda_l + \left(\frac{r}{n} + \phi\right)\lambda_{l,m}. \tag{3.25}$$

Also, we can use the equation (3.16) in (3.25). Then

$$\lambda_{l,m} = \frac{n\rho}{r+n\phi}g_{lm} + (\alpha_m - \lambda_m)\lambda_l. \tag{3.26}$$

Defining $\gamma = \frac{n\rho}{r+n\phi}$ and $\beta_m = \alpha_m - \lambda_m$, (3.26) takes the form

$$\lambda_{l,m} = \gamma g_{lm} + \beta_m\lambda_l. \tag{3.27}$$

Thus, the vector field λ_l is a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.6. *If* (M, g) *admits a torse-forming vector field associated by the* 1*-form* ϕ_l *in the relation* $\phi_{l,m} = \rho g_{lm} + \lambda_m \phi_l$*, then the vector field* λ_l *forms a concircular vector field in the form* $\lambda_{l,m} = \gamma g_{lm}$*, where* $\gamma = \frac{n\rho}{r+n\phi}$*.*

Proof. Assume that the vector field ϕ_l is a torse-forming vector field with a scalar function ρ and a vector field λ_l. If we take $\lambda_m = \alpha_m$ in (3.26), we get

$$\lambda_{l,m} = \frac{n\rho}{r + n\phi} g_{lm}. \tag{3.28}$$

Taking $\gamma = \frac{n\rho}{r+n\phi}$, we obtain

$$\lambda_{l,m} = \gamma g_{lm}. \tag{3.29}$$

Thus, the vector field λ_l forms a concircular vector field. Hence, the proof is completed. □

Theorem 3.7. *If the vector field* ϕ_l *of* (M, g) *is a concircular vector field, then the vector field* λ_l *forms a torse-forming vector field in the relation*

$$\lambda_{l,m} = \frac{n\rho}{r + n\phi} g_{lm} - \lambda_l \lambda_m.$$

Proof. Assume that the vector field ϕ_l is a concircular vector field with a scalar function ρ, i.e.,

$$\phi_{l,m} = \rho g_{lm}. \tag{3.30}$$

Using the equation (3.30) in (3.18), we get

$$\rho g_{lm} = \left(\frac{r}{n} + \phi\right) \lambda_l \lambda_m + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.31}$$

Finally, from (3.31), we obtain

$$\lambda_{l,m} = \frac{n\rho}{r + n\phi} g_{lm} - \lambda_l \lambda_m. \tag{3.32}$$

Thus, the vector field λ_l forms a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.8. *If the vector field* λ_l *of* (M, g) *has constant length and the vector field* ϕ_l *is a concircular vector field, then the equation* $\rho = c^2(\frac{r}{n} + \phi)$ *holds, where* $||\lambda|| = c$*.*

Proof. Assume that the vector field λ_l is of constant length, i.e., $\lambda_l \lambda^l = c^2$. Multiplying (3.32) by λ^l, we find

$$\lambda^l \lambda_{l,m} = \left(\frac{n\rho}{r + n\phi} - \lambda_l \lambda^l\right) \lambda_m. \tag{3.33}$$

Since λ_l is of constant length, then we have $\lambda^l \lambda_{l,m} = 0$. By substituting the last relation and $\lambda_l \lambda^l = c^2$ in (3.33), finally we obtain

$$\rho = c^2 \left(\frac{r}{n} + \phi\right). \tag{3.34}$$

Thus, the proof is completed. □

Theorem 3.9. *If the vector field λ_l of (M, g) is a concircular vector field in the form $\lambda_{l,m} = \rho g_{lm}$, then the vector field ϕ_l is a torse-forming vector field satisfying the equation*

$$\phi_{l,m} = \frac{n}{r+n\phi}\phi_m\phi_l + \frac{\rho(r+n\phi)}{n}g_{lm}.$$

Proof. Suppose that the vector field λ_l of (M, g) is a concircular vector field, i.e., the equation

$$\lambda_{l,m} = \rho g_{lm} \tag{3.35}$$

holds. Using the equations (3.21) and (3.35), we see that

$$\phi_{l,m} = \left(\frac{n}{r+n\phi}\right)\phi_m\phi_l + \frac{\rho(r+n\phi)}{n}g_{lm}. \tag{3.36}$$

This result shows that ϕ_l is a torse-forming vector field. Hence, the proof is completed. □

Theorem 3.10. *If the vector field λ_l of (M, g) is a concircular vector field in the form $\lambda_{l,m} = \rho g_{lm}$ and the vector field ϕ_l has constant length, then the scalar function ρ generating the vector field λ_l has negative value and it satisfies the equation $\rho = -(\frac{nc}{r+n\phi})^2$.*

Proof. Let the vector field λ_l be a concircular vector field and the vector field ϕ_l be of constant length. Multiplying (3.36) by ϕ^l, we get

$$\phi^l\phi_{l,m} = \left(\frac{n}{r+n\phi}\right)\phi_m\phi_l\phi^l + \frac{\rho(r+n\phi)}{n}\phi_m. \tag{3.37}$$

Since the vector field ϕ_l is of constant length, then we have $\phi^l\phi_{l,m} = 0$. If we take $||\phi|| = c$, the equation (3.37) reduces to

$$\left(\frac{nc^2}{r+n\phi}\right)\phi_m + \frac{\rho(r+n\phi)}{n}\phi_m = 0. \tag{3.38}$$

Finally, from (3.38), we obtain

$$\rho = -\left(\frac{nc}{r+n\phi}\right)^2.$$

Thus, the proof is completed. □

Theorem 3.11. *If the vector field ϕ_l of (M, g) is a recurrent vector field in the form $\phi_{l,m} = \alpha_m\phi_l$, then the vector field λ_l is also recurrent vector field in the form $\lambda_{l,m} = (\alpha_m - \lambda_m)\lambda_l$.*

Proof. Suppose that the vector field ϕ_l is recurrent vector field. Thus, we have

$$\phi_{l,m} = \alpha_m\phi_l. \tag{3.39}$$

Substituting the equation (3.39) in (3.17), we get

$$\alpha_m \phi_l = \phi_m \lambda_l + \left(\frac{r}{n} + \phi\right) \lambda_{l,m}. \tag{3.40}$$

Finally, from (3.40), it can be obtained that

$$\lambda_{l,m} = (\alpha_m - \lambda_m)\lambda_l. \tag{3.41}$$

Thus, the proof is completed. □

Theorem 3.12. *A recurrent vector field ϕ_m, with the recurrence vector field α_m of (M, g) admits the relation $\alpha_m = \lambda_m$ if and only if the vector field λ_m is covariantly constant or is of constant length.*

Proof. If we take $\alpha_m = \lambda_m$ in Theorem 3.11 then from (3.41), we get

$$\lambda_{l,m} = 0. \tag{3.42}$$

Thus, we can say that the vector field λ_l is covariantly constant. Conversely, if the relation (3.42) is satisfied, from (3.41) we have $\alpha_m = \lambda_m$. Similarly, suppose that the vector field λ_l has constant length. If we multiply the equation (3.41) by λ^l, then we have $\alpha_m = \lambda_m$. The converse is also true. Hence, the proof is completed. □

Theorem 3.13. *Let the vector field λ_l of (M, g) be a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$. A necessary and sufficient condition the vector field ϕ_l to be divergence-free is that the scalar function μ to be in the form*

$$\mu = -\left(\frac{n}{r + n\phi}\right)^2 \frac{||\phi||^2}{r}.$$

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field, from (1.17),

$$\lambda_{l,m} = \mu S_{lm}, \tag{3.43}$$

where μ is a scalar function. Putting the equation (3.43) in (3.21), one can easily obtain that

$$\phi_{l,m} = \left(\frac{n}{r + n\phi}\right) \phi_l \phi_m + \mu \left(\frac{r + n\phi}{n}\right) S_{lm}. \tag{3.44}$$

Multiplying the equation (3.44) by g^{lm}, it is found that

$$\phi^l_{,l} = \left(\frac{n}{r + n\phi}\right) ||\phi||^2 + \mu \left(\frac{r + n\phi}{n}\right) r. \tag{3.45}$$

Now, assume that the vector field ϕ_l is divergence-free. In this case, the equation (3.45) reduces to

$$\mu = -\left(\frac{n}{r + n\phi}\right)^2 \frac{||\phi||^2}{r}. \tag{3.46}$$

Conversely, if the scalar function μ satisfies the relation (3.46), from (3.45), it can be obtained that ϕ_l is divergence-free. Thus, the proof is completed. □

Theorem 3.14. *If the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) has constant length, then the value $-\frac{1}{\mu}(\frac{n||\phi||}{r+n\phi})^2$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector δ defined by $\phi(X) = g(X, \delta)$.*

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l is of constant length. Multiplying (3.44) by ϕ^l then we get

$$\phi^l \phi_{l,m} = \left(\frac{n}{r+n\phi}\right) ||\phi||^2 \phi_m + \mu \left(\frac{r+n\phi}{n}\right) \phi^l S_{lm}. \tag{3.47}$$

Because ϕ_l is of constant length, we have $\phi^l \phi_{l,m} = 0$. In this case, from (3.47), we obtain

$$\phi^l S_{lm} = -\frac{1}{\mu} \left(\frac{n||\phi||}{r+n\phi}\right)^2 \phi_m. \tag{3.48}$$

Thus, the proof is completed. □

Theorem 3.15. *If the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) is a concircular vector field, then the Ricci tensor is in the following form*

$$S_{lm} = a g_{lm} + b \phi_m \phi_l,$$

which is a quasi-Einstein manifold where $a = \frac{n\rho}{\mu(r+n\phi)}$, $b = -\frac{1}{\mu}(\frac{n}{r+n\phi})^2$.

Proof. Assume that the vector field λ_l is a $\lambda(Ric)$ vector field in the form $\lambda_{l,m} = \mu S_{lm}$ and the vector field ϕ_l of (M, g) is a concircular vector field. From the equation (3.44), one can obtain that

$$\rho g_{lm} = \left(\frac{n}{r+n\phi}\right) \phi_l \phi_m + \left(\frac{r+n\phi}{n}\right) \mu S_{lm}. \tag{3.49}$$

We easily find from (3.49) that

$$S_{lm} = \frac{n\rho}{\mu(r+n\phi)} g_{lm} - \frac{1}{\mu} \left(\frac{n}{r+n\phi}\right)^2 \phi_l \phi_m. \tag{3.50}$$

Finally, the Ricci tensor can be written in the form

$$S_{lm} = a g_{lm} + b \phi_m \phi_l, \tag{3.51}$$

where $a = \frac{n\rho}{\mu(r+n\phi)}$, $b = -\frac{1}{\mu}(\frac{n}{r+n\phi})^2$. Therefore, this manifold is a quasi-Einstein manifold. In this case, the proof is completed. □

Theorem 3.16. *If the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields in the forms $\lambda_{l,m} = \mu S_{lm}$ and $\phi_{l,m} = \alpha S_{lm}$, respectively, then the Ricci tensor is in the following form*

$$S_{lm} = \gamma \phi_l \phi_m,$$

where $\gamma = \frac{n^2}{(r+n\phi)(n\alpha-\mu(r+n\phi))}$, $r + n\phi \neq 0$, $\alpha \neq \mu(\frac{r+n\phi}{n})$ and α, μ, γ are scalar functions. Thus, this manifold reduces to a quasi-Einstein manifold.

Proof. Assume that the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields, respectively. In this case, we have from (1.17)

$$\lambda_{l,m} = \mu S_{lm} \quad \text{and} \quad \phi_{l,m} = \alpha S_{lm}. \tag{3.52}$$

Substituting the relations (3.52) in (3.21), we get

$$\alpha S_{lm} = \frac{n}{r + n\phi}\phi_l\phi_m + \mu\left(\frac{r + n\phi}{n}\right) S_{lm}. \tag{3.53}$$

Finally, Ricci tensor takes the form

$$S_{lm} = \gamma\phi_l\phi_m, \tag{3.54}$$

where $\gamma = \frac{n^2}{(r+n\phi)(n\alpha-\mu(r+n\phi))}$ and $r + n\phi \neq 0$, $\alpha \neq \mu(\frac{r+n\phi}{n})$. This means that this manifold reduces to a quasi-Einstein manifold. Hence, the proof is completed. ☐

Theorem 3.17. *The vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields in the forms $\lambda_{l,m} = \mu S_{lm}$ and $\phi_{l,m} = \alpha S_{lm}$, respectively. If the eigenvalue determined by the vector field α_k is r, then the eigenvalue determined by the vector field μ_k is also r.*

Proof. Assume that the vector fields λ_l and ϕ_l of (M, g) are $\lambda(Ric)$ and $\phi(Ric)$ vector fields, respectively. As we already know the relations in (3.52), if we use the equations (3.18) and (3.52), then we get

$$\alpha S_{lm} = \left(\frac{r}{n} + \phi\right)(\mu S_{lm} + \lambda_l\lambda_m). \tag{3.55}$$

Arraying the equation (3.55), we find

$$\left(\alpha - \left(\frac{r}{n} + \phi\right)\mu\right) S_{lm} = \left(\frac{r}{n} + \phi\right)\lambda_l\lambda_m. \tag{3.56}$$

Now, let's find the covariant derivative of (3.56) and use the equation (3.16), one can easily see that

$$\begin{aligned} &\left[\alpha_k - \left(\frac{r}{n} + \phi\right)(\lambda_k\mu + \mu_k)\right] S_{lm} + \left(\alpha - \left(\frac{r}{n} + \phi\right)\mu\right) S_{lm,k} \\ &= \left(\frac{r}{n} + \phi\right)[\mu(\lambda_l S_{mk} + \lambda_m S_{lk}) + \lambda_l\lambda_m\lambda_k]. \end{aligned} \tag{3.57}$$

We arrive at the following relation multiplying (3.57) by g^{lm}

$$\left[\alpha_k - \left(\frac{r}{n} + \phi\right)\mu_k\right] r = \left(\frac{r}{n} + \phi\right)[2\mu\lambda^l S_{lk} + ||\lambda||^2\lambda_k + \lambda_k\mu r]. \tag{3.58}$$

Again, multiplying (3.57) by g^{lk}, we get

$$\alpha^l S_{lk} - \left(\frac{r}{n} + \phi\right)\mu^l S_{lk} = \left(\frac{r}{n} + \phi\right)[2\mu(\lambda^l S_{lk} + ||\lambda||^2\lambda_k + \lambda_k\mu r]. \tag{3.59}$$

At the end, substracting the equations (3.58) and (3.59), we obtain

$$\alpha^l S_{lk} - \alpha_k r = \left(\frac{r}{n} + \phi\right)(\mu^l S_{lk} - \mu_k r). \tag{3.60}$$

So, we can say that if the eigenvalue determined by the vector field α_k is r, then the eigenvalue determined by the vector field μ_k is also r. Thus, the proof is completed. □

Theorem 3.18. *If the vector field ϕ_l of (M, g) is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$, then the Laplacian of the trace function of the Z-tensor is*

$$\Delta \bar{Z} = n\alpha r.$$

Proof. As we know that in a Riemannian manifold with covariantly constant conharmonic curvature tensor, the scalar curvature must be constant. Thus, going back to the relation (2.1), we get

$$\bar{Z}_{,k} = n\phi_k. \tag{3.61}$$

By taking the covariant derivative of (3.61), it can be found

$$\bar{Z}_{,kl} = n\phi_{k,l}. \tag{3.62}$$

Now, let us asuume that the vector field ϕ_l is a $\phi(Ric)$ vector field. In this case, the equation (3.62) takes the form

$$\bar{Z}_{,kl} = n\alpha S_{kl}. \tag{3.63}$$

Multiplying the equation (3.63) by g^{kl}, we obtain

$$g^{kl}\bar{Z}_{,kl} = \Delta \bar{Z} = n\alpha r. \tag{3.64}$$

Hence, the proof is completed. □

Theorem 3.19. *If the vector field ϕ_l of (M, g) is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$, then the scalar curvature satisfies the relation*

$$r = \frac{n\phi\delta}{n\alpha - \delta},$$

where $\delta \neq n\alpha$.

Proof. Assume that the vector field ϕ_l is a $\phi(Ric)$ vector field in the form $\phi_{l,m} = \alpha S_{lm}$. Hence, from (3.18), one can easily find that

$$\alpha S_{lm} = \left(\frac{r + n\phi}{n}\right)(\lambda_l \lambda_m + \lambda_{l,m}). \tag{3.65}$$

Let's multiply (3.65) by g^{lm}. Thus, it takes the form

$$\alpha r = \left(\frac{r + n\phi}{n}\right)\left(||\lambda||^2 + \lambda^l_{,l}\right). \tag{3.66}$$

Now, let's take $\delta = ||\lambda||^2 + \lambda^l_{,l}$ and $\delta \neq n\alpha$. Finally, it is obtained that

$$r = \frac{n\phi\delta}{n\alpha - \delta}. \tag{3.67}$$

Hence, this completes the proof. □

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[1]Department of Mechanical Engineering,
Piri Reis University,
Istanbul, Turkey
Email address: aytasci@pirireis.edu.tr

[2]Department of Mathematics Engineering,
Istanbul Technical University,
Istanbul,Turkey
Email address: fozen@itu.edu.tr