Kragujevac Journal of Mathematics
Volume 49(1) (2025), Pages 81–92.

A STUDY ON THE BLOW-UP OF SOLUTIONS FOR A LAMÉ SYSTEM OF INVERSE PROBLEM

MOHAMMAD SHAHROUZI[1]

Abstract. We consider the Lamé system of inverse problem in a bounded domain with nonlinear boundary condition. When $2 < m \leq \frac{p}{4}$, we obtain the blow-up result for the weak solution with positive initial energy and sufficiently large initial data.

1. Introduction

We study the following Lamé system of inverse problem of determining a pair of functions $\{u(x,t), f(t)\}$ that satisfy:

(1.1)
$$u_{tt} - \Delta_e u - \operatorname{div}(|\nabla u|^{m-2}\nabla u) + h(x,t,u,\nabla u) = |u|^{p-2}u + f(t)\omega(x), \quad x \in \Omega, t > 0,$$

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \mu\frac{\partial u}{\partial \nu}(x,t) + |\frac{\partial u}{\partial \nu}|^{m-2}\frac{\partial u}{\partial \nu} + (\lambda+\mu)\operatorname{div} u = 0, & x \in \Gamma_1, t > 0, \end{cases} \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$\int_\Omega u(x,t)\omega(x)dx = 1, \quad t > 0, \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and ν is the unit outward normal to $\partial\Omega$. Let $u = (u^1, \dots, u^n)$ be a vector function, $\operatorname{div} u = u^1_{x_1} + u^2_{x_2} + \cdots + u^n_{x_n}$ be the divergence of u, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. We write

$$\Delta u = \left(\sum_{i=1}^n u^1_{x_i x_i}, \sum_{i=1}^n u^2_{x_i x_i}, \dots, \sum_{i=1}^n u^n_{x_i x_i}\right)^T.$$

Key words and phrases. Blow-up, Lamé system, Inverse problem.
2020 *Mathematics Subject Classification.* Primary: 35B44. Secondary: 35R30, 74B05.
DOI 10.46793/KgJMat2501.081S
Received: July 19, 2020.
Accepted: December 01, 2021.

Here Δ_e denotes the elasticity operator, which is the $n \times n$ matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\operatorname{div} u),$$

μ and λ are the Lamé constants which satisfy the following conditions

$$\mu > 0, \quad \lambda + \mu \geq 0.$$

Also, m and p are constants such that $p, m > 2$. In addition, $h(x,t,u,\nabla u)$ and $\omega(x)$ are real functions that satisfy specific conditions that will be enunciated later (see **(A1)**-**(A3)**).

Elasticity systems with constants Lamé coefficients in direct problems ($\omega(x) \equiv 0$) has attracted considerable attention in recent years, where diverse type of dissipative mechanisms have been introduced and several results have been obtained. In [1] Bchatnia and Daolati studied behavior of the energy for solutions to a Lamé system on a bounded domain with localized nonlinear damping and external force. Later, Bchatnia and Guesmia [2] considered the Lamé system in 3-dimension bounded domain with infinite memories and proved that system is well-possed and stable. Moreover, they established solutions converge to zero at infinity in terms of the growth of the infinite memories. Li and Bao [19] investigated the following memory-type elasticity problem

$$\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega \times (0,\infty),\\
&u = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
&\mu\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + (\mu+\lambda)(\operatorname{div} u)\nu + h(u_t) = 0, \quad \text{on } \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \text{in } \Omega.
\end{aligned}$$

The authors obtained global existence and the general energy decay of solutions by using perturbed energy method.

Boulaaras [6] proved asymptotic stability result of global solution for a coupled Lamé system with a viscoelastic term and the logarithmic nonlinearity. He obtained this result taking into account that the kernel is not necessarily decreasing. Recently, Bocanegra-Rodríguez et al. [5] investigated the longtime dynamics of the following semilinear Lamé systems

$$\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\partial_t u + f(u) = b,$$

defined in bounded domains of $\mathbb{R}^3$ with Dirichlet boundary condition. They proved the existence of finite dimensional global attractors subjected to a critical forcing $f(u)$. Moreover, they showed the upper-semicontinuity of attractors with respect to the parameter when $(\lambda + \mu) \to 0$ (see also [3,4,9,10]).

Inverse problems are the problems that consist of finding an unknown property of an object, or medium, to a probing signal (see [21]). In contrast with the extensive literature on global behaviour of solutions in direct problems, we know little about the inverse problems. For instance, Eden and Kalantarov in [8] studied the following inverse source problem:

$$\begin{aligned}
&u_t - \Delta u - |u|^p u + b(x,t,u,\nabla u) = F(t)\omega(x), \quad x \in \Omega, t > 0,\\
&u(x,t) = 0, \quad x \in \partial\Omega, t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega,\\
&\int_\Omega u(x,t)\omega(x)dx = 1, \quad t > 0,
\end{aligned}$$

and by using the modified concavity method established global nonexistence results as well as stability results depending on the sign of nonlinearity. For more information about the concavity argument, we refer the readers to [16–18]. In [26] Shahrouzi and Tahamtani by using the same method found conditions on data that guaranteeing the global nonexistence and asymptotic stability results for a class of Petrovsky inverse source problems (see also [22–24, 27]). Bukhgeǐm et al. [7] considered an inverse problem for the stationary elasticity system with constant Lamé coefficients and variable matrix coefficient depending on the spatial variables and frequency. They proved uniqueness theorem by reduction of the inverse problem to a family of equations with the M. Riesz potential. For more results on the Lamé system of inverse problems, we refer the reader to [11–15, 25] and references therein.

The paper is organized as follows. In Section 2, we present some notations, assumptions and known results needed for our work and state our main result: Theorem 2.1. Section 3 is devoted to the proof of the blow-up result.

2. Preliminaries and Main Result

We begin this section by introducing some hypotheses and our main result. We shall assume that the functions $\omega(x), h(x,t,u,\nabla u)$ and the functions appearing in the data satisfy the following conditions:

(A1) $u_0 \in H_0^1(\Omega) \cap L^{p+2}(\Omega)$, $u_1 \in L^2(\Omega)$, $\int_\Omega u_0(x)\omega(x)dx = 1$;
(A2) $\omega \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p+2}(\Omega)$, $\int_\Omega \omega^2(x)dx = 1$;
(A3) for some positive M_1, M_2 we have $|h(x,t,u,\nabla u)| \leq M_1|u|^{\frac{p}{2}} + M_2|\nabla u|^{\frac{m}{2}}$.

Throughout this paper all the functions considered are real-valued. We denote by $\|\cdot\|_q$ the L^q-norm over Ω . In particular, the L^2-norm is denoted $\|\cdot\|$ in Ω and $\|\cdot\|_{\Gamma_i}$ in Γ_i. Also we use familiar function spaces H_0^1, H^2.

We recall the trace Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1), \quad \text{for } 2 \leq q < \frac{2(n-1)}{n-2},$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \leq B_q\|\nabla u\|_2,$$

where B_q is the optimal constant.

We sometimes use the Young's inequality

$$ab \leq \beta a^q + C(\beta, q) b^{q'}, \quad a, b \geq 0, \beta > 0, \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.1}$$

where $C(\beta, q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$ are constants.

The following lemma was introduced in [16]. It will be used in the next section in order to prove the blow-up result.

Lemma 2.1. *Let $\alpha > 0$, $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that*

$$\psi''\psi - (1+\alpha)\left[\psi'\right]^2 \geq -2c_1\psi\psi' - c_2[\psi]^2,$$

for all $t \geq 0$. If

$$\psi(0) > 0 \quad \text{and} \quad \psi'(0) + \gamma_2\alpha^{-1}\psi(0) > 0,$$

then

$$\psi(t) \to +\infty, \quad \text{as } t \to t_1 \leq t_2 = \frac{1}{2\sqrt{c_1^2 + \alpha c_2}} \log \frac{\gamma_1\psi(0) + \alpha\psi'(0)}{\gamma_2\psi(0) + \alpha\psi'(0)}.$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \alpha c_2} \quad \text{and} \quad \gamma_2 = -c_1 - \sqrt{c_1^2 + \alpha c_2}.$$

We consider the following problem that is obtained from (1.1)–(1.4) by substituting $u(x,t) = e^{\xi t} v(x,t)$:

$$v_{tt} + 2\xi v_t + \xi^2 v - \Delta_e v - e^{\xi(m-2)t}\text{div}\left(|\nabla v|^{m-2}\nabla v\right) + e^{-\xi t}\hat{h}(t,v)$$
$$= e^{\xi(p-2)t}|v|^{p-2}v + e^{-\xi t} f(t)\omega(t), \quad x \in \Omega, t > 0, \tag{2.2}$$

$$\begin{cases} v(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \mu\frac{\partial v}{\partial \nu}(x,t) + e^{\xi(m-2)t}|\frac{\partial v}{\partial \nu}|^{m-2}\frac{\partial v}{\partial \nu} + (\lambda+\mu)\text{div}\, v = 0, & x \in \Gamma_1, t > 0, \end{cases} \tag{2.3}$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) - \xi u_0(x), \quad x \in \Omega, \tag{2.4}$$

$$\int_\Omega v(x,t)\omega(x)dx = e^{-\xi t}, \quad t > 0, \tag{2.5}$$

where

$$\hat{h}(t,v) := h(x,t,e^{\xi t}v, e^{\xi t}\nabla v),$$

and the value of the parameter ξ will be prescribed later.

By using the idea of Prilepko et al. [20] and **(A2)**, one can easily see that the problem (2.2)–(2.5) is equivalent to (2.2)–(2.4) in which the unknown function $f(t)$ is replaced by

$$\begin{aligned} e^{-\xi t}f(t) =&\mu \int_\Omega \nabla v \nabla \omega dx + (\lambda+\mu)\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))dx \\ &+ e^{\xi(m-2)t}\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega dx + e^{-\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx \\ &- e^{\xi(p-2)t}\int_\Omega |v|^{p-2}v\omega(x)dx. \end{aligned} \tag{2.6}$$

Define the total energy functional associated with problem (2.2)–(2.4) as follows

$$E_\xi(t) = \frac{1}{p}e^{\xi(p-2)t}\|v\|_p^p - \frac{1}{2}I(t), \tag{2.7}$$

where

$$I(t) = \|v_t\|^2 + \xi^2\|v\|^2 + \mu\|\nabla v\|^2 + (\lambda+\mu)\int_\Omega (\operatorname{div} v)^2 dx + \frac{2}{m}e^{\xi(m-2)t}\|\nabla v\|_m^m.$$

Now, we are in a position to state blow-up result.

Theorem 2.1. *Let the conditions* **(A1)-(A3)** *be satisfied. Assume that* $2 < m \leq \frac{p}{4}$ *and for sufficiently large initial data and* $\xi > 0$

$$\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}} \leq \xi < \frac{(m-1)\int_\Omega u_0u_1dx}{(m+1)\|u_0\|^2}, \tag{2.8}$$

$$E_\xi(0) \geq \frac{2D_1}{\xi(p+2)} + \frac{D_2}{2m},$$

where

$$\begin{aligned} D_1 =&\frac{\mu\xi}{p-2}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)\xi}{p-2}\int_\Omega (\operatorname{div}\omega(x))^2 dx + \frac{\xi^2(pM_1^2+2mM_2^2)}{4p+8m}\|\omega\|^2 \\ &+ \frac{\xi\|\nabla\omega\|_m^m}{m\left[\frac{p-2}{12(m-1)}\right]^{m-1}} + \frac{\xi\|\omega\|_p^p}{p\left[\frac{p-2}{6(p-1)}\right]^{p-1}}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} D_2 =&\frac{\mu}{2m}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)}{2m}\int_\Omega (\operatorname{div}\omega(x))^2 dx + \frac{\xi^2(3pM_1^2+6mM_2^2)}{8m}\|\omega\|^2 \\ &+ \frac{\xi^m\|\nabla\omega\|_m^m}{m^m\left[\frac{1}{3(m-1)}\right]^{m-1}} + \frac{\xi^p\|\omega\|_p^p}{p\left[\frac{2m}{3(p-1)}\right]^{p-1}}. \end{aligned} \tag{2.10}$$

Then there exists a finite time t_1 *such that the solution of the problem* (1.1)–(1.4) *blows up in* t_1*, that is*

$$\|u(t)\| \to +\infty, \quad \textit{as } t \to t_1.$$

3. Blow-up

In this section we are going to prove that for sufficiently large initial data some of the solutions blow up in a finite time. To prove the blow-up result (Theorem 2.1) for certain solutions with positive initial energy, we need the following lemma for the problem (2.2)–(2.5).

Lemma 3.1. *Under the conditions of Theorem* 2.1, *the energy functional* $E_\xi(t)$, *defined by* (2.7), *satisfies*

$$E_\xi(t) \geq E_\xi(0) - \frac{2D_1}{\xi(p+2)}.$$

Proof. A multiplication of equation (2.2) by v_t and integrating over Ω gives

$$\begin{aligned} E'_\xi(t) =& 2\xi\|v_t\|^2 - \frac{\xi(m-2)}{m}e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi(p-2)}{p}e^{\xi(p-2)t}\|v\|_p^p \\ &+ e^{-\xi t}\int_\Omega v_t\hat{h}(t,v)dx + \xi e^{-2\xi t}f(t). \end{aligned} \tag{3.1}$$

Plugging definition of $f(t)$, (2.6) into (3.1), we obtain

$$\begin{aligned} E'_\xi(t) =& 2\xi\|v_t\|^2 - \frac{\xi(m-2)}{m}e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi(p-2)}{p}e^{\xi(p-2)t}\|v\|_p^p + e^{-\xi t}\int_\Omega v_t\hat{h}(t,v)dx \\ &+ \xi\mu e^{-\xi t}\int_\Omega \nabla v\nabla\omega dx + \xi(\lambda+\mu)e^{-\xi t}\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))\,dx \\ &+ \xi e^{\xi(m-3)t}\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega(x)dx + \xi e^{-2\xi t}\int_\Omega \hat{h}(t,v)\omega(x)dx \\ &- \xi e^{\xi(p-3)t}\int_\Omega |v|^p v\omega(x)dx. \end{aligned} \tag{3.2}$$

Next, we estimate the terms on the right-hand side of (3.2). Using **(A3)**, Cauchy-Schwartz and Young's inequality (2.1), we obtain

$$\begin{aligned} e^{-\xi t}\left|\int_\Omega v_t\hat{h}(t,v)dx\right| \leq& M_1\int_\Omega v_t e^{\xi(\frac{p}{2}-1)t}|v|^{\frac{p}{2}}dx + M_2\int_\Omega v_t e^{\xi(\frac{m}{2}-1)t}|\nabla v|^{\frac{m}{2}}dx \\ \leq& M_1\|v_t\|e^{\xi(\frac{p}{2}-1)t}\|v\|_p^{\frac{p}{2}} + M_2\|v_t\|e^{\xi(\frac{m}{2}-1)t}\|\nabla v\|_m^{\frac{m}{2}} \\ \leq& \beta_1 e^{\xi(p-2)t}\|v\|_p^p + \beta_2 e^{\xi(m-2)t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\|v_t\|^2, \end{aligned} \tag{3.3}$$

where β_1 and β_2 are arbitrary positive constants

$$\mu\xi e^{-\xi t}\left|\int_\Omega \nabla v\nabla\omega dx\right| \leq \frac{\mu\xi(p-2)}{4}\|\nabla v\|^2 + \frac{\mu\xi}{p-2}e^{-2\xi t}\|\nabla\omega\|^2, \tag{3.4}$$

$$\begin{aligned} &\xi(\lambda+\mu)e^{-\xi t}\left|\int_\Omega (\operatorname{div} v)(\operatorname{div}\omega(x))\,dx\right| \\ \leq& \frac{\xi(\lambda+\mu)(p-2)}{4}\int_\Omega (\operatorname{div} v)^2\,dx + \frac{\xi(\lambda+\mu)}{p-2}e^{-2\xi t}\int_\Omega (\operatorname{div}\omega(x))^2\,dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\xi e^{\xi(m-3)t}\left|\int_\Omega |\nabla v|^{m-2}\nabla v\nabla\omega(x)dx\right| \leq \xi e^{\xi(m-2)t}\|\nabla v\|_m^{m-1}e^{-\xi t}\|\nabla\omega\|_m \\ \leq& \beta_3 e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi^m e^{-2\xi t}}{m\left[\frac{\beta_3 m}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m, \end{aligned} \tag{3.6}$$

where β_3 is an arbitrary positive constant,

$$
\begin{aligned}
&\xi e^{-2\xi t}\left|\int_\Omega \hat{h}(t,v)\omega(x)dx\right| \\
\leq & M_1\int_\Omega e^{\xi(\frac{p}{2}-1)t}|v|^{\frac{p}{2}}\xi e^{-\xi t}\omega(x)dx + M_2\int_\Omega e^{\xi(\frac{m}{2}-1)t}|\nabla v|^{\frac{m}{2}}\xi e^{-\xi t}\omega(x)dx \\
\leq & e^{\xi(\frac{p}{2}-1)t}\|v\|_p^{\frac{p}{2}}.M_1\xi e^{-\xi t}\|\omega\| + e^{\xi(\frac{m}{2}-1)t}\|\nabla v\|_m^{\frac{m}{2}}.M_2\xi e^{-\xi t}\|\omega\| \\
\leq & \beta_4 e^{\xi(p-2)t}\|v\|_p^p + \beta_5 e^{\xi(m-2)t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\beta_4}+\frac{M_2^2}{4\beta_5}\right)\xi^2 e^{-2\xi t}\|\omega\|^2,
\end{aligned}
\tag{3.7}
$$

where β_4 and β_5 are arbitrary positive constants.

Finally, we have for any positive β_6:

$$
\begin{aligned}
\xi e^{\xi(p-3)t}\left|\int_\Omega |v|^p v\omega(x)dx\right| \leq & \xi E^{\xi(p-2)t}\|v\|_p^{p-1}e^{-\xi t}\|\omega\|_p \\
\leq & \beta_6 e^{\xi(p-2)t}\|v\|_p^p + \frac{\xi^p e^{-2\xi t}}{p\left[\frac{\beta_6 p}{p-1}\right]^{p-1}}\|\omega\|_p^p.
\end{aligned}
\tag{3.8}
$$

Combining (3.3)–(3.8) with (3.2), we deduce

$$
\begin{aligned}
E_\xi'(t) \geq & \left[2\xi - \left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\right]\|v_t\|^2 - \left(\frac{\xi(m-2)}{m}+\beta_2+\beta_3+\beta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\
& + \left(\frac{\xi(p-2)}{p}-\beta_1-\beta_4-\beta_6\right)e^{\xi(p-2)t}\|v\|_p^p - \frac{\mu\xi(p-2)}{4}\|\nabla v\|^2 \\
& - \frac{\xi(\lambda+\mu)(p-2)}{4}\int_\Omega (\operatorname{div} v)^2\,dx - e^{-2\xi t}D_1,
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
D_1 = & \frac{\mu\xi}{p-2}\|\nabla\omega\|^2 + \frac{(\lambda+\mu)\xi}{p-2}\int_\Omega (\operatorname{div}\omega(x))^2\,dx + \frac{\xi^2(\beta_5 M_1^2+\beta_4 M_2^2)}{4(\beta_4+\beta_5)}\|\omega\|^2 \\
& + \frac{\xi^m\|\nabla\omega\|_m^m}{m\left[\frac{\beta_3 m}{m-1}\right]^{m-1}} + \frac{\xi^p\|\omega\|_p^p}{p\left[\frac{\beta_6 p}{p-1}\right]^{p-1}}.
\end{aligned}
$$

By virtue of (3.9), we obtain from (2.7) the following inequality

$$
\begin{aligned}
E_\xi'(t) - \frac{\xi(p-2)}{2}E_\xi(t) \geq & \left(\frac{\xi(p-2m+2)}{2m}-\beta_2-\beta_3-\beta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\
& + \left(\frac{\xi(p-2)}{2p}-\beta_1-\beta_4-\beta_6\right)e^{\xi(p-2)t}\|v\|_p^p \\
& + \left[\frac{\xi(p+6)}{4}-\left(\frac{M_1^2}{4\beta_1}+\frac{M_2^2}{4\beta_2}\right)\right]\|v_t\|^2 - e^{-2\xi t}D_1.
\end{aligned}
$$

At this point if we choose $\beta_1 = \beta_4 = \beta_6 = \frac{\xi(p-2)}{6p}$ and $\beta_2 = \beta_3 - \beta_5 = \frac{\xi(p-2)}{12m}$, then we gain

$$E'_\xi(t) - \frac{\xi(p-2)}{2} E_\xi(t) \geq \frac{\xi(p-4m+6)}{4m} \|\nabla v\|_m^m + \left[\frac{\xi(p+6)}{4} - \frac{3pM_1^2 + 6mM_2^2}{2\xi(p-2)}\right] \|v_t\|^2 - e^{-2\xi t} D_1.$$

Hence, by choosing $m \leq \frac{p+6}{4}$ and

$$\xi \geq \sqrt{\frac{6(pM_1^2 + 2mM_2^2)}{p^2 + 4p - 12}},$$

we get

$$E'_\xi(t) - \frac{\xi(p-2)}{2} E_\xi(t) \geq -e^{-2\xi t} D_1. \tag{3.10}$$

Integrating the differential inequality (3.10) between 0 and t gives that

$$E_\xi(t) \geq E_\xi(0) - \frac{D_1}{\xi(p+2)},$$

where D_1 satisfies (2.9) and proof of Lemma 3.1 is completed. □

Proof of Theorem 2.1**.** For obtain the blow-up result, the choice of the following functional is standard (see [17, 18])

$$\psi(t) = \|v(t)\|^2, \tag{3.11}$$

then

$$\psi'(t) = 2\int_\Omega v v_t dx, \quad \psi''(t) = 2\int_\Omega v v_{tt} dx + 2\|v_t\|^2. \tag{3.12}$$

A multiplication of equation (2.2) by v and integrating over Ω gives

$$\begin{aligned}\int_\Omega v v_{tt} dx = &- 2\xi \int_\Omega v v_t dx - \xi^2 \|v\|^2 - \mu \|\nabla v\|^2 - (\lambda + \mu) \int_\Omega (\operatorname{div} v)^2 dx \\ &- e^{\xi(m-2)t} \|\nabla v\|_m^m - e^{-\xi t} \int_\Omega v \hat{h}(t,v) dx + e^{\xi(p-2)t} \|v\|_p^p \\ &+ \mu e^{-\xi t} \int_\Omega \nabla v \nabla \omega dx + (\lambda+\mu) e^{-\xi t} \int_\Omega (\operatorname{div} v)(\operatorname{div} \omega(x)) dx \\ &+ e^{\xi(m-3)t} \int_\Omega |\nabla v|^{m-2} \nabla v \nabla \omega dx + e^{-2\xi t} \int_\Omega \hat{h}(t,v) \omega(x) dx \\ &- e^{\xi(p-3)t} \int_\Omega |v|^{p-2} v \omega(x) dx,\end{aligned} \tag{3.13}$$

where the definition of unknown function (2.6) has been used.

By combining (2.7) with (3.13), one can easily verify that

$$\begin{aligned}\int_\Omega vv_{tt}dx =&\eta E_\xi(t) - 2\xi\int_\Omega vv_tdx + \frac{\eta}{2}\|v_t\|^2 + \xi^2\left(\frac{\eta}{2}-1\right)\|v\|^2 + \mu\left(\frac{\eta}{2}-1\right)\|\nabla v\|^2\\ &+ (\lambda+\mu)\left(\frac{\eta}{2}-1\right)\int_\Omega(\operatorname{div} v)^2dx + \left(\frac{\eta}{m}-1\right)e^{\xi(m-2)t}\|\nabla v\|_m^m\\ &+\left(1-\frac{\eta}{p}\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-\xi t}\int_\Omega v\hat{h}(t,v)dx + \mu e^{-\xi t}\int_\Omega \nabla v\nabla\omega dx\\ &+(\lambda+\mu)e^{-\xi t}\int_\Omega(\operatorname{div} v)(\operatorname{div}\omega(x))dx + e^{\xi(m-3)t}\int_\Omega|\nabla v|^{m-2}\nabla v\nabla\omega dx\\ &+e^{-2\xi t}\int_\Omega\hat{h}(t,v)\omega(x)dx - e^{\xi(p-3)t}\int_\Omega|v|^{p-2}v\omega(x)dx. \end{aligned}\tag{3.14}$$

Applying **(A3)**, Cauchy-Schwartz inequality and the Young's inequality (2.1) to estimate the terms on the right-hand side of (3.14)

$$\begin{aligned}e^{-\xi t}\left|\int_\Omega v\hat{h}(t,v)dx\right| \le& M_1\|v\|e^{(\frac{p}{2}-1)\xi t}\|v\|_p^{\frac{p}{2}} + M_2\|v\|e^{(\frac{m}{2}-1)\xi t}\|\nabla v\|_m^{\frac{m}{2}}\\ \le&\theta_1 e^{(p-2)\xi t}\|v\|_p^p + \theta_2 e^{(m-2)\xi t}\|\nabla v\|_m^m + \left(\frac{M_1^2}{4\theta_1}+\frac{M_2^2}{4\theta_2}\right)\|v\|^2,\end{aligned}\tag{3.15}$$

where θ_1 and θ_2 are positive constants,

$$\mu e^{-\xi t}\left|\int_\Omega\nabla v\nabla\omega dx\right| \le \frac{\mu\eta}{4}\|\nabla v\|^2 + \frac{\mu e^{-2\xi t}}{\eta}\|\nabla\omega\|^2,\tag{3.16}$$

$$\begin{aligned}&(\lambda+\mu)e^{-\xi t}\left|\int_\Omega(\operatorname{div} v)(\operatorname{div}\omega(x))dx\right|\\ \le&\frac{\eta(\lambda+\mu)}{4}\int_\Omega(\operatorname{div} v)^2dx + \frac{(\lambda+\mu)e^{-2\xi t}}{\eta}\int_\Omega(\operatorname{div}\omega(x))^2dx,\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}e^{\xi(m-3)t}\left|\int_\Omega|\nabla v|^{m-2}\nabla v\nabla\omega dx\right| \le&\xi e^{\xi(m-2)t}\|\nabla v\|_m^{m-1}e^{-\xi t}\|\nabla\omega\|_m\\ \le&\theta_3 e^{\xi(m-2)t}\|\nabla v\|_m^m + \frac{\xi^m e^{-2\xi t}}{m\left[\frac{m\theta_3}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m,\end{aligned}\tag{3.18}$$

where θ_3 is an arbitrary positive constant. Also, similar to (3.7), we have

$$\begin{aligned}e^{-2\xi t}\left|\int_\Omega\hat{h}(t,v)\omega(x)dx\right| \le&\theta_4 e^{\xi(p-2)t}\|v\|_p^p + \theta_5 e^{\xi(m-2)t}\|\nabla v\|_m^m\\ &+\left(\frac{M_1^2}{4\theta_4}+\frac{M_2^2}{4\theta_5}\right)\xi^2 e^{-2\xi t}\|\omega\|^2,\end{aligned}\tag{3.19}$$

where θ_4 and θ_5 are positive constants. Furthermore, for $\theta_6 > 0$ we derive

$$e^{\xi(p-3)t}\int_\Omega|v|^{p-2}v\omega(x)dx \le \theta_6 e^{\xi(p-2)t}\|v\|_p^p + \frac{\xi^p e^{-2\xi t}}{p\left[\frac{p\theta_6}{p-1}\right]^{p-1}}\|\omega\|_p^p.\tag{3.20}$$

Utilizing (3.15)–(3.20) with (3.14), we get

$$\begin{aligned}\int_\Omega vv_{tt}dx \geq & \eta E_\xi(t) - 2\xi\int_\Omega vv_tdx + \left[\xi^2(\frac{\eta}{2}-1) - \left(\frac{M_1^2}{4\theta_1}+\frac{M_2^2}{4\theta_2}\right)\right]\|v\|^2 \\ & + \mu\left(\frac{\eta}{4}-1\right)\|\nabla v\|^2 + \frac{\eta}{2}\|v_t\|^2 + (\lambda+\mu)\left(\frac{\eta}{4}-1\right)\int_\Omega(\operatorname{div} v)^2dx \\ & + \left(\frac{\eta}{m}-1-\theta_2-\theta_3-\theta_5\right)e^{\xi(m-2)t}\|\nabla v\|_m^m \\ & + \left(1-\frac{\eta}{p}-\theta_1-\theta_4-\theta_6\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-2\xi t}\left(\frac{\mu}{\eta}\|\nabla\omega\|^2\right. \\ & + \frac{\lambda+\mu}{\eta}\int_\Omega(\operatorname{div}\omega(x))^2dx + \frac{\xi^m}{m\left[\frac{m\theta_3}{m-1}\right]^{m-1}}\|\nabla\omega\|_m^m \\ & \left. + \left(\frac{M_1^2}{4\theta_4}+\frac{M_2^2}{4\theta_5}\right)\xi^2\|\omega\|^2 + \frac{\xi^p}{p\left[\frac{p\theta_6}{p-1}\right]^{p-1}}\|\omega\|_p^p\right).\end{aligned}$$

Now by choosing $\eta = 2m$, $\theta_2=\theta_3=\theta_5=\frac{1}{3}$ and $\theta_1=\theta_4=\theta_6=\frac{2m}{3p}$, we conclude that

$$\begin{aligned}\int_\Omega vv_{tt}dx \geq & 2mE_\xi(t) - 2\xi\int_\Omega vv_tdx + \frac{\eta}{2}\|v_t\|^2 + \left[\xi^2(m-1) - \left(\frac{3pM_1^2}{8m}+\frac{3M_2^2}{4}\right)\right]\|v\|^2 \\ & + \mu\left(\frac{m}{2}-1\right)\|\nabla v\|^2 + (\lambda+\mu)\left(\frac{m}{2}-1\right)\int_\Omega(\operatorname{div} v)^2dx \\ & + \left(1-\frac{4m}{p}\right)e^{\xi(p-2)t}\|v\|_p^p - e^{-2\xi t}D_2,\end{aligned}$$

where D_2 satisfies (2.10). Let $2 < m \leq \frac{p}{4}$ and

$$\xi \geq \sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}},$$

it holds that

$$\int_\Omega vv_{tt}dx \geq 2mE_\xi(t) - 2\xi\int_\Omega vv_tdx + m\|v_t\|^2 - e^{-2\xi t}D_2.$$

According to Lemma 3.1 and hypothesis of Theorem 2.1, we obtain

$$\int_\Omega vv_{tt}dx \geq -2\xi\int_\Omega vv_tdx + m\|v_t\|^2. \tag{3.21}$$

To this end, by substituting (3.11) and (3.12) in (3.21), we arrive at

$$\psi''(t) \geq -2\xi\psi'(t) + 2(m+1)\|v_t\|^2,$$

finally we get

$$\psi(t)\psi''(t) \geq \frac{(m+1)}{2}[\psi'(t)]^2 - 2\xi\psi(t)\psi'(t).$$

Considering $2 < m \leq \frac{p}{4}$, it is obvious that

$$\max\left\{\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}},\sqrt{\frac{6(pM_1^2+2mM_2^2)}{p^2+4p-12}}\right\}=\sqrt{\frac{3(pM_1^2+2mM_2^2)}{8m(m-1)}}.$$

Hence by attention to (2.8) we see that the hypotheses of Lemma 2.1 are fulfilled with $\alpha = \frac{m-1}{2}$, $c_1 = \xi$, $c_2 = 0$ and

$$\psi'(0) - \frac{4\xi}{m-1}\psi(0) > 0,$$

thus conclusion of Lemma 2.1 gives us that some solutions of problem (2.2)–(2.5) blow up in a finite time and since this system is equivalent to (1.1)–(1.4), the proof is completed.

References

[1] A. Bchatnia and M. Daolati, *Behaviour of the energy for Lamé system in bounded domains with nonlinear damping and internal force*, Electron. J. Differ. Equ. **1** (2013), 1–17.

[2] A. Bchatnia and A. Guesmia, *Well-posedness and asymptotic stability for the Lamé system with infinite memories in a bounded domain*, Math. Control Relat. Fields **4**(4) (2014), 451–463.

[3] M. I. Belishev and I. Lasiecka, *The dynamical Lamé system: regularity of solutions, boundary controllability and boundary data continuation*, ESAIM Control Optim. Calc. Var. **8** (2002), 143–167.

[4] A. Beniani, N. Taouaf and A. Benaissa, *Well-posedness and exponential stability for coupled Lamé system with viscoelastic term and strong damping*, Comput. Math. Appl. **75**(12) (2018), 4397–4404.

[5] L. E. Bocanegra-Rodríguez, M. A. J. Silva, T. F. Ma and P. N. Seminario-Huertas, *Longtime dynamics of a semilinear Lamé system*, Dynam. Differential Equations (2021), 1–22. `https://doi.org/10.1007/s10884-021-09955-7`

[6] S. Boulaaras, *A well-posedness and exponential decay of solutions for a coupled Lamé system with viscoelastic term and logarithmic source term*, Appl. Anal. **100**(7) (2021), 1514–1532.

[7] A. L. Bukhgeĭm, G. V. Dyatlov, V. B. Kardakov and E. V. Tantserev, *Uniqueness in one inverse problem for the elasticity system*, Sib. Math. J. **45**(4) (2004), 618–627.

[8] A. Eden and V. K. Kalantarov, *On global behavior of solutions to an inverse problem for nonlinear parabolic equations*, J. Math. Anal. Appl. **228** (1998), 181–205.

[9] B. Feng, Z. Hajjej and M. Balegh, *Existence and general decay rate estimates of a coupled Lamé system only with viscoelastic dampings*, Math. Methods Appl. Sci. (2020), 1–18. `https://doi.org/10.1002/mma.6586`

[10] B. Feng and H. Li, *Decay rates for a coupled viscoelastic Lamé system with strong damping*, Math. Model. Anal. **25**(2) (2020), 226–240.

[11] M. Grasselli, M. Ikehata and M. Yamamoto, *Direct and inverse inequalities for the isotropic Lamé system with variable coefficients and applications to an inverse source problem*, Appl. Anal. **84**(4) (2005), 357–375.

[12] M. Ikehata, G. Nakamura and M. Yamamoto, *Uniqueness in inverse problems for the isotropic Lamé system*, J. Math. Sci. Univ. Tokyo **5** (1998), 627–692.

[13] O. Yu. Imanuvilov, V. Isakov and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 1366–1382.

[14] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2nd Edition, Springer-Verlag, New York, 2006. `https://10.1007/0-387-32183-7`

[15] V. Isakov, G. Nakamura and J. N. Wang, *Uniqueness and stability in the Cauchy problem for the elasticity system with residual stress*, Contemp. Math. 333 (2003), 99–113.

[16] V. K. Kalantarov and O. A. Ladyzhenskaya, *Formation of collapses in quasilinear equations of parabolic and hyperbolic types*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **69** (1977), 77–102.

[17] H. A. Levine and S. Ro. Park, *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation*, J. Math. Anal. Appl. **228** (1998), 181–205.

[18] H. A. Levine and J. Serrin, *Global nonexistence theorems for quasilinear evolution equation with dissipation*, Arch. Ration. Mech. Anal. **137** (1997), 341–361.

[19] F. Li and G. Y. Bao, *Uniform stability of the solution for a memory-type elasticity system with nonhomogeneous boundary control condition*, J. Dyn. Control Syst. **23** (2017), 301–315.

[20] A. I. Prilepko, D. G. Orlovskii and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, Inc., New York, 2000.

[21] A. G. Ramm, *Inverse Problems: Mathematical and Analytical Techniques with Applications to Engineering*, Springer Science Publishing House, 2005. `https://doi:10.1007/b100958`

[22] M. Shahrouzi, *Asymptotic stability and blow up of solutions for a class of viscoelastic inverse problem with boundary feedback*, Math. Methods Appl. Sci. **39** (2016), 2368–2379.

[23] M. Shahrouzi, *On behavior of solutions to a class of nonlinear hyperbolic inverse source problem*, Acta Math. Sin. (Engl. Ser.) **32** (2016), 683–698.

[24] M. Shahrouzi, *Blow up of solutions to a class of damped viscoelastic inverse source problem*, Differ. Equ. Dyn. Syst. **28** (2020), 889–899.

[25] M. Shahrouzi, *General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities*, Math. Methods Appl. Sci. (2021), 1–15. `https://doi.org/10.1002/mma.7891`

[26] M. Shahrouzi and F. Tahamtani, *Global nonexistence and stability of the solutions of inverse problems for a class of Petrovsky systems*, Georgian Math. J. **19** (2012), 575–586.

[27] F. Tahamtani and M. Shahrouzi, *Asymptotic stability and blow up of solutions for a Petrovsky inverse source problem with dissipative boundary condition*, Math. Methods Appl. Sci. **36** (2013), 829–839.

[1]Department of Mathematics,
Jahrom University,
Jahrom, Iran, P.O.Box: 74137-66171
Email address: `mshahrouzi@jahromu.ac.ir`