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# UNIFORM ULTIMATE BOUNDEDNESS RESULTS FOR SOME SYSTEM OF THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. The paper is concerned with the study of the uniform ultimate boundedness of solutions of the third-order system of nonlinear delay differential equation

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t-r)) = P(t, X, \dot{X}, \ddot{X}),$$

where $A, B$ are real $n \times n$ constant symmetric matrices, $r$ is a positive real constant and $X \in \mathbb{R}^n$, using the Lyapunov-Krasovskii functional method and following the arguments used in [1] and [10], we obtained results which give an $n-$dimensional analogue of an earlier result of [13] and extend other earlier results for the case in which we do not necessarily require that $H(X(t-r))$ be differentiable.

## 1. Introduction

Let $\mathbb{R}$ denote the real line, $-\infty < t < \infty$ and $\mathbb{R}^n$ denote the real $n$-dimensional Euclidean space $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (in $n$ places) with the usual norm which will be represented throughout by $\| \cdot \|$.

Consider the delay differential equation of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t-r)) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

where $X \in \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $A$ and $B$ are real $n \times n$ constant symmetric matrices, $r$ is a positive real constant and the dots indicate differentiation with respect to $t$. We shall assume that $H$ and $P$ are continuous in their respective arguments.

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Equation (1.1) is the vector version for systems of real third-order nonlinear delay differential equations of the form

$$\begin{aligned}&\dddot{x}_i+\sum_{k=1}^{n}a_{ik}\ddot{x}_k+\sum_{k=1}^{n}b_{ik}\dot{x}_k+h_i(x_1(t-r),x_2(t-r),\dots,x_n(t-r))\\=&p_i(t,x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n,\ddot{x}_1,\dots,\ddot{x}_n),\end{aligned}$$

$i=1,2,\dots,n$, in which $a_{ik}$ and $b_{ik}$ are real constants, $r$ is a positive real constant and $h_i$, $p_i$ are continuous in their respective arguments. The case when $n=1$ and $r=0$ which give rise to the nonlinear differential equations of the form

$$\dddot{x}+a\ddot{x}+b\dot{x}+h(x)=p(t,x,\dot{x},\ddot{x}) \tag{1.2}$$

have been greatly studied by several authors for stability, boundedness, convergence and periodicity of solutions (see [5, 8, 14]). Similarly, equations of the form (1.2) for which $a$, $b$ are not necessarily constants have been studied by several authors in the literature (see [14]). For the case $n=1$ and $r>0$, delay differential equations of the form

$$\dddot{x}+a\ddot{x}+b\dot{x}+h(x(t-r))=p(t,x,\dot{x},\ddot{x}) \tag{1.3}$$

have been studied for stability, boundedness and periodicity of solutions by several authors in the literature. In [18], sufficient conditions which ensure the stability (for $p(t,x,\dot{x},\ddot{x})=0$) and boundedness (for $p(t,x,\dot{x},\ddot{x})\neq 0$) of solutions of equation (1.3) were obtained. In [13], equation (1.3) (in which $h$ is not necessarily differentiable) was studied, and the author obtained conditions which ensure that solutions are bounded. Similarly, equations of the form (1.3) for which $a, b$ are not necessarily constants have been studied by several authors in the literature. It is worth mentioning that equation (1.1), when $r=0$, gives rise to the nonlinear vector differential equations of the form

$$\dddot{X}+A\ddot{X}+B\dot{X}+H(X)=P(t,X,\dot{X},\ddot{X}), \tag{1.4}$$

where $A, B, H$ and $P$ are as defined above. Equations of the form (1.4) have been studied by several authors for boundedness and periodicity of solutions ([1,6,10]). In [6] the authors studied equation (1.4) when $H(X)$ is continuous and differentiable, while in [1] and [10] the authors studied (1.4) when $H(X)$ is not necessarily differentiable. Similarly, qualitative properties of solutions of (1.4) for which $A, B$ are not necessarily constants have been investigated by several authors (see [2,7,9,15]). However, there are few papers in connection with the qualitative properties of solutions of systems of third order nonlinear delay differential equations in literature. Recently, in [12], equation (1.1) in which $r=r(t)$, $H\in\mathcal{C}'(\mathbb{R}^n)$ and $P(t,X,\dot{X},\ddot{X})=P(t)$ was investigated for the boundedness of solutions, while in [17], the author studied the stability and boundedness of solutions of the equation

$$\dddot{X}+H(\dot{X})\ddot{X}+G(\dot{X}(t-r))+cX(t-r)=P(t,X,\dot{X},\ddot{X}),$$

where $H$, $G$ are continuous and differentiable in their arguments and $P$ is continuous in its arguments. To the best of our knowledge the extension of the results in [1] and [13] to equation (1.1) does not exist in literature. Throughout all the foregoing papers, the Lyapunov's second (or direct) method has been used as a main tool to carry out the proofs of the main results for scalar and vector ordinary differential equations, while the Lyapunov-Krasovskii functional method has been used for scalar and vector delay differential equations ([1] - [19]). In the present paper, we shall use the Lyapunov-Krasovskii functional method as a basic tool in our proofs. In the present paper, we also used the same method as a basic tool in our proofs. The motivation for the present work is derived from the papers mentioned above, and the object of this paper is to prove the uniform boundedness results under specified conditions on $H$ and $P$. Specifically, unlike in [12], we shall only assume that $H$ is not necessarily differentiable, and that for any $X, Y \in \mathbb{R}^n$ (following [1] and [10]), there exists an $n \times n$ operator $C(X, Y)$ such that

$$H(X) = H(Y) + C(X, Y)(X - Y) \tag{1.5}$$

for which the eigenvalues $\lambda_i(C(X, Y))$, $i = 1, 2, \ldots, n$, are continuous and satisfy

$$0 < \delta_h \leq \lambda_i(C(X, Y)) \leq \Delta_h$$

for fixed constants $\delta_h$ and $\Delta_h$. Moreover, we shall assume that

$$\Delta_h \leq k\delta_a\delta_b, \quad k < 1,$$

where

$$k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}, \frac{\alpha(1-\beta)\delta_a}{(\delta_a+2\alpha)^2}\right\} \tag{1.6}$$

and

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a, \tag{1.7}$$

$$0 < \delta_b \leq \lambda_i(B) \leq \Delta_b, \tag{1.8}$$

with $\lambda_i(A)$ and $\lambda_i(B)$ as the eigenvalues of $A$ and $B$, respectively.

The result in this paper is the $n$-dimensional analog of a result in [13]. Moreover, we shall improve on the results in [12] when $H(X(t-r))$ is not necessarily differentiable and $r(t) = r > 0$.

1.1. **Notation and definitions.** Given any $X, Y$ in $\mathbb{R}^n$ the symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$, that is $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$. Thus $\|X\|^2 = \langle X, X\rangle$. The matrix $A$ is said to be positive definite when $\langle AX, X\rangle > 0$ for all nonzero $X$ in $\mathbb{R}^n$.

The following notations (see [12, 13]) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of $x$. For a given $r > 0$, $t_1 \in \mathbb{R}$, $C(t_1) = \{\phi : [t_1 - r, t_1] \to \mathbb{R}^n / \phi$ is continuous$\}$. In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = \sup_{-r\leq\theta\leq 0} |\phi(0)|$. $C_{\mathbf{H}}$ will denote the set of $\phi$ such that $\|\phi\| \leq \mathbf{H}$. For any continuous function $x(u)$ defined on $-h \leq u < A$, $A > 0$, and any fixed $t$, $0 \leq t < A$, the symbol $x_t$ will denote the restriction of $x(u)$ to the interval $[t-r,t]$, that is, $x_t$ is an element of $C$ defined by $x_t(\theta)=x(t+\theta)$, $-r\leq\theta\leq 0$.

## 2. Some Preliminary Results

In this section, we shall state the algebraic results required in the proofs of our main results. The proofs are not given since they are found in [1,2,6,7,9–11,15,16].

**Lemma 2.1** ([1,2,6,7,9–11,15,16]). *Let $D$ be a real symmetric positive definite $n\times n$ matrix, then for any $X$ in $\mathbb{R}^n$, we have*

$$\delta_d\|X\|^2\leq\langle DX,X\rangle\leq\Delta_d\|X\|^2,$$

*where $\delta_d$, $\Delta_d$ are the least and the greatest eigenvalues of $D$, respectively.*

**Lemma 2.2** ([1, 2, 6, 7, 9–11, 15, 16]). *Let $Q,D$ be any two real $n\times n$ commuting symmetric matrices. Then*

(i) *the eigenvalues $\lambda_i(QD)$, $i=1,2,\dots,n$, of the product matrix $QD$ are all real and satisfy*

$$\min_{1\leq j,k\leq n}\lambda_j(Q)\lambda_k(D)\leq\lambda_i(QD)\leq\max_{1\leq j,k\leq n}\lambda_j(Q)\lambda_k(D);$$

(ii) *the eigenvalues $\lambda_i(Q+D)$, $i=1,2,\dots,n$, of the sum of matrices $Q$ and $D$ are real and satisfy*

$$\left\{\min_{1\leq j\leq n}\lambda_j(Q)+\min_{1\leq k\leq n}\lambda_k(D)\right\}\leq\lambda_i(Q+D)\leq\left\{\max_{1\leq j\leq n}\lambda_j(Q)+\max_{1\leq k\leq n}\lambda_k(D)\right\}.$$

**Lemma 2.3.** *Let $H\in\mathcal{C}(\mathbb{R}^n)$ be a continuous vector function and that $H(0)=0$. Then*

$$H(U)=C(U,0)X(t)-C(U,0)\int_{t-r}^{t}Y(s)ds,$$

*where $U=X(t-r)$.*

*Proof of Lemma* 2.3. From (1.5), we have that

$$H(X(t-r))=H(Y(t-r))+C(X(t-r),Y(t-r))(X(t-r)-Y(t-r)).\tag{2.1}$$

If we set $Y(t-r)=0$ in (2.1), we obtain

$$H(X(t-r))=C(X(t-r),0)X(t-r).\tag{2.2}$$

Since

$$X(t-r)=X(t)-\int_{t-r}^{t}Y(s)ds,$$

where

$$\dot{X}(t)=\frac{dX(t)}{dt}=Y(t),$$

it follows from (2.2) that

$$H(X(t-r))=C(X(t-r),0)X(t)-C(X(t-r),0)\int_{t-r}^{t}Y(s)ds.$$

Let $U=X(t-r)$, hence the result follows. □

**Corollary 2.1.** *If* $r = 0$, *then* (2.2) *reduces to* $H(X) = C(X, 0)X$.

## 3. Boundedness

First, consider a system of delay differential equations

$$\dot{x} = F(t, x_t), \quad x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0, \tag{3.1}$$

where $F : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets. The following lemma is a well-known result obtained in [4].

**Lemma 3.1** ([4]). *Let* $V(t, \phi) : \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}$ *be continuous and locally Lipschitz in* $\phi$. *If*

(i) $W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\right)$, *and*

(ii) $\dot{V}_{(3.1)} \le -W_3(|x(s)|) + M$,

*for some* $M > 0$, *where* $W(r)$, $W_i$, $i = 1, 2, 3$, *are wedges, then the solutions of* (3.1) *are uniformly bounded and uniformly ultimately bounded for bound* $\mathbf{B}$.

To study the boundedness of solutions of (1.1) for which $P(t, X, \dot{X}, \ddot{X}) \neq 0$, we would need to write (1.1) in the form

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -AZ - BY - H(X(t - r)) + P(t, X, Y, Z). \end{aligned} \tag{3.2}$$

Our main theorem in this paper stated with respect to (3.2), which is an $n$-dimensional analogue of a result in [13] is the following.

**Theorem 3.1.** *Consider* (3.2), *let* $H(0) = 0$ *and suppose that*

(i) *there exists an* $n \times n$ *real continuous operator* $C(X, Y)$ *for any vectors* $X, Y \in \mathbb{R}^n$ *such that*

$$H(X) = H(Y) + C(X, Y)(X - Y)$$

*whose eigenvalues* $\lambda_i(C(X, Y))$, $i = 1, 2, \ldots, n$, *satisfy*

$$0 < \delta_h \le \lambda_i(C(X, Y)) \le \Delta_h; \tag{3.3}$$

(ii) *the constant symmetric matrices* $A$ *and* $B$ *have positive eigenvalues, commute with themselves as well with the operator* $C(X, Y)$ *for any* $X, Y \in \mathbb{R}^n$ *and that*

$$\Delta_h \le k\delta_a\delta_b,$$

*where* $k$ $(< 1)$ *is the constant defined in* (1.6);

(iii) *there exist finite constants* $\Delta_0 \ge 0$, $\Delta_1 \ge 0$, *such that the vector* $P$ *satisfies*

$$\|P(t, X, Y, Z)\| \le \Delta_0 + \Delta_1(\|X\| + \|Y\| + \|Z\|) \tag{3.4}$$

*uniformly in* $t$, *for all arbitrary* $X, Y, Z \in \mathbb{R}^n$. *Then, if* $\Delta_1$ *is sufficiently small, the solutions to the system* (3.2) *are uniformly bounded and uniformly ultimately bounded provided*

$$r < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h}, \frac{2\beta\delta_a\delta_b}{\Delta_h[1+(1-\beta)\Delta_b+2(\Delta_a+\alpha+\alpha\delta_a^{-1})]}, \frac{\alpha}{\Delta_h(1+2\alpha\delta_a^{-1})}\right\}.$$

*Proof.* The main tool in the proof of Theorem 3.1 is the Lyapunov functional

$$\begin{aligned} 2V(X_t,Y_t,Z_t) =&\beta(1-\beta)\langle BX,BX\rangle + \beta\langle BY,Y\rangle + 2\alpha\langle BY, A^{-1}Y\rangle \\ &+ \alpha\langle A^{-1}Z,Z\rangle + \alpha\langle A^{-1}(AY+Z), AY+Z\rangle \\ &+ \langle Z+AY+(1-\beta)BX, Z+AY+(1-\beta)BX\rangle \\ &+ \lambda\int_{-r}^{0}\int_{t+s}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta ds, \end{aligned} \tag{3.5}$$

where $0<\beta<1$ and $\alpha,\lambda>0$ are constants.

Obviously, the function $V(X_t,Y_t,Z_t)$ is positive definite since each term of (3.5) is positive. Hence the condition (i) of Lemma 3.1 is satisfied. Now, let us compute the time derivative of the functional $V(X_t,Y_t,Z_t)$ for the solution $(X_t,Y_t,Z_t)$ of system (3.2). By $\dot V$, we denote the time derivative of the function $V=V(X_t,Y_t,Z_t)$ for the solution $(X_t,Y_t,Z_t)$ of the system (3.2). Then

$$\begin{aligned} \frac{dV}{dt} =& -\langle(1-\beta)BX, H(X(t-r))\rangle - \langle\alpha BY,Y\rangle - \langle\beta AY,BY\rangle \\ &- \langle(I+2\alpha A^{-1})Z, H(X(t-r))\rangle - \langle(\alpha I+A)Y, H(X(t-r))\rangle \\ &- \langle\alpha Z,Z\rangle + \langle\lambda rY,Y\rangle - \lambda\int_{t-r}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta \\ &+ \langle(1-\beta)BX+(\alpha I+A)Y+(I+2\alpha A^{-1})Z, P(t,X,Y,Z)\rangle. \end{aligned}$$

Upon using (2.2), we obtain

$$\begin{aligned} \frac{dV}{dt} =& -\langle(1-\beta)BX, C(U,0))X\rangle - \langle\alpha BY,Y\rangle - \langle\beta AY,BY\rangle \\ &- \langle\alpha Z,Z\rangle - \langle(I+2\alpha A^{-1})Z, C(U,0)X\rangle - \langle(\alpha I+A)Y, C(U,0)X\rangle \\ &+ \int_{t-r}^{t}\langle(1-\beta)BX(s)+(\alpha I+A)Y(s) \\ &+ (I+2\alpha A^{-1})Z(s), C(U,0)Y(s)\rangle ds \\ &+ \langle\lambda rY,Y\rangle - \lambda\int_{t-r}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta \\ &+ \langle(1-\beta)BX+(\alpha I+A)Y+(I+2\alpha A^{-1})Z, P(t,X,Y,Z)\rangle \\ =& -U_1-U_2-U_3+U_4+U_5, \end{aligned}$$

where

$$
\begin{aligned}
U_1 =& \frac{1}{2}\langle X, (1-\beta)BC(U,0)X\rangle + \langle Y, \beta ABY\rangle + \frac{1}{2}\langle \alpha Z, Z\rangle,\\
U_2 =& \frac{1}{4}\langle X, (1-\beta)BC(U,0)X\rangle + \langle (\alpha I + A)Y, C(U,0)X\rangle + \langle \alpha BY, Y\rangle,\\
U_3 =& \frac{1}{4}\langle X, (1-\beta)BC(U,0)X\rangle + \langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle + \frac{1}{2}\langle \alpha Z, Z\rangle,\\
U_4 =& \int_{t-r}^{t} \langle (1-\beta)BX(s) + (\alpha I + A)Y(s)\\
& + (I + 2\alpha A^{-1})Z(s), C(U,0)Y(s)\rangle ds + \langle \lambda r Y, Y\rangle - \lambda \int_{t-r}^{t} \langle Y(\theta), Y(\theta)\rangle d\theta
\end{aligned}
$$

and

$$U_5 = \langle (1-\beta)BX + (\alpha I + A)Y + (I + 2\alpha A^{-1})Z, P(t,X,Y,Z)\rangle.$$

From (1.7), (1.8) and (3.3), we have

$$
\begin{aligned}
U_1 \geq& \frac{1-\beta}{2}\delta_b\delta_h\|X\|^2 + \beta\delta_a\delta_b\|Y\|^2 + \frac{\alpha}{2}\|Z\|^2\\
\geq& \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2),
\end{aligned}
\tag{3.6}
$$

where $\delta_1 = \min\left\{\frac{(1-\beta)}{2}\delta_b\delta_h, \beta\delta_a\delta_b, \frac{\alpha}{2}\right\}$.

Next, we give estimates for $\langle (\alpha I + A)Y, C(U,0)X\rangle$ and $\langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle$. For some $k_1 > 0$, $k_2 > 0$, conveniently chosen later, we obtain

$$
\begin{aligned}
\langle (\alpha I + A)Y, C(U,0)X\rangle =& \left\| k_1(\alpha I + A)^{\frac{1}{2}}Y + \frac{1}{2}k_1^{-1}(\alpha I + A)^{\frac{1}{2}}C(U,0)X\right\|^2\\
& - \langle k_1^2(\alpha I + A)Y, Y\rangle\\
& - \frac{1}{4}k_1^{-2}\langle (\alpha I + A)C(U,0)X, C(U,0)X\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\langle (I + 2\alpha A^{-1})Z, C(U,0)X\rangle =& \left\| k_2(I + 2\alpha A^{-1})^{\frac{1}{2}}Z + \frac{1}{2}k_2^{-1}(I + 2\alpha A^{-1})^{\frac{1}{2}}C(U,0)X\right\|^2\\
& - \langle k_2^2(I + 2\alpha A^{-1})Z, Z\rangle\\
& - \frac{1}{4}k_2^{-2}\langle (I + 2\alpha A^{-1})C(U,0)X, C(U,0)X\rangle,
\end{aligned}
$$

thus

$$
\begin{aligned}
U_2 =& \left\| k_1(\alpha I + A)^{\frac{1}{2}}Y + \frac{1}{2}k_1^{-1}(\alpha I + A)^{\frac{1}{2}}C(U,0)X\right\|^2\\
& + \left\langle \left\{\alpha B - k_1^2(\alpha I + A)\right\} Y, Y\right\rangle\\
& + \left\langle \frac{1}{4}\left\{(1-\beta)B - k_1^{-2}(\alpha I + A)C(U,0)\right\} C(U,0)X, X\right\rangle
\end{aligned}
$$

and

$$U_3 = \left\| k_2(I+2\alpha A^{-1})^{\frac{1}{2}}Z + \frac{1}{2}k_2^{-1}(I+2\alpha A^{-1})^{\frac{1}{2}}C(U,0)X \right\|^2$$
$$+ \left\langle \left\{ \alpha I - k_2^2(I+2\alpha A^{-1}) \right\} Z, Z \right\rangle$$
$$+ \left\langle \frac{1}{4}\left\{ (1-\beta)B - k_2^{-2}(I+2\alpha A^{-1})C(U,0) \right\} C(U,0)X, X \right\rangle.$$

By Lemma 2.1 and Lemma 2.2, we have

$$U_2 \geq \left\{ \alpha\delta_b - k_1^2(\alpha+\Delta_a) \right\} \|Y\|^2 + \frac{1}{4}\delta_h \left\{ (1-\beta)\delta_b - \frac{1}{k_1^2}(\alpha+\Delta_a)\Delta_h \right\} \|X\|^2 \geq 0, \tag{3.7}$$

provided

$$\frac{(\alpha+\Delta_a)\Delta_h}{(1-\beta)\delta_b} \leq k_1^2 \leq \frac{\alpha\delta_b}{\alpha+\Delta_a}$$

and

$$\Delta_h \leq \frac{\alpha\delta_b^2(1-\beta)}{(\alpha+\Delta_a)^2}. \tag{3.8}$$

In a similar manner,

$$U_3 \geq 0, \tag{3.9}$$

provided

$$\frac{(2\alpha+\delta_a)\Delta_h}{(1-\beta)\delta_a\delta_b} \leq k_2^2 \leq \frac{\alpha\delta_a}{2\alpha+\delta_a}$$

and

$$\Delta_h \leq \frac{\alpha\delta_b\delta_a^2(1-\beta)}{(2\alpha+\delta_a)^2}. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\Delta_h \leq k\delta_a\delta_b,$$

where

$$k = \min\left\{ \frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}, \frac{\alpha(1-\beta)\delta_a}{(\delta_a+2\alpha)^2} \right\} < 1.$$

For $U_4$, using the identity $2|\langle u,v\rangle| \leq \|u\|^2 + \|v\|^2$, we obtain

$$\begin{aligned} |U_4| \leq & \frac{1}{2}(1-\beta)\Delta_b\Delta_h r\|X\|^2 + \frac{1}{2}(\alpha+\Delta_a)\Delta_h r\|Y\|^2 \\ & + \frac{1}{2}(1+2\alpha\delta_a^{-1})\Delta_h r\|Z\|^2 + \left\{ \frac{1}{2}(1-\beta)\Delta_b\Delta_h \right. \\ & \left. + \frac{1}{2}(\alpha+\Delta_a)\Delta_h + \frac{1}{2}(1+2\alpha\delta_a^{-1})\Delta_h \right\} \int_{t-r}^{t} \langle Y(s), Y(s)\rangle ds \\ & + \langle \lambda r Y, Y\rangle - \lambda \int_{t-r}^{t} \langle Y(\theta), Y(\theta)\rangle d\theta. \end{aligned} \tag{3.11}$$

If we choose

$$\lambda = \frac{1}{2}\Delta_h\left[(1-\beta)\Delta_b + (\alpha+\Delta_a) + (1+2\alpha\delta_a^{-1})\right]$$

in (3.11), we obtain

$$|U_4| \leq \frac{1}{2}(1-\beta)\Delta_b\Delta_h r\|X\|^2 + \frac{1}{2}(1+2\alpha\delta_a^{-1})\Delta_h r\|Z\|^2 + \frac{1}{2}\Delta_h r\left[1+(1-\beta)\Delta_b + 2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]\|Y\|^2. \tag{3.12}$$

Finally, we are left with $U_5$. Since $P(t,X,Y,Z)$ satisfies (3.4), by Schwarz's inequality we obtain

$$\begin{aligned} |U_5| &\leq \left[(1-\beta)\Delta_b\|X\| + (\alpha+\Delta_a)\|Y\| + (1+2\alpha\delta_a^{-1})\|Z\|\right]\|P(t,X,Y,Z)\| \\ &\leq \delta_2(\|X\|+\|Y\|+\|Z\|)\left[\Delta_0 + \Delta_1(\|X\|+\|Y\|+\|Z\|)\right], \end{aligned} \tag{3.13}$$

where $\delta_2 = \max\{(1-\beta)\Delta_b, (\alpha+\Delta_a), (1+2\alpha\delta_a^{-1})\}$.

Combining inequalities (3.6), (3.7), (3.9), (3.12) and (3.13), we obtain

$$\begin{aligned} \frac{dV}{dt} \leq &-\frac{1}{2}(1-\beta)[\delta_b\delta_h - r\Delta_b\Delta_h]\|X\|^2 \\ &-\left(\beta\delta_a\delta_b - \frac{1}{2}\Delta_h r\left[1+(1-\beta)\Delta_b + 2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]\right)\|Y\|^2 \\ &-\frac{1}{2}\left[\alpha - \Delta_h r(1+2\alpha\delta_a^{-1})\right]\|Z\|^2 \\ &+\delta_2(\|X\|+\|Y\|+\|Z\|)\left[\Delta_0+\Delta_1(\|X\|+\|Y\|+\|Z\|)\right]. \end{aligned}$$

Now if we choose

$$r < \min\left\{\frac{\delta_b\delta_h}{\Delta_b\Delta_h}, \frac{2\beta\delta_a\delta_b}{\Delta_h\left[1+(1-\beta)\Delta_b+2(\Delta_a+\alpha+\alpha\delta_a^{-1})\right]}, \frac{\alpha}{\Delta_h(1+2\alpha\delta_a^{-1})}\right\},$$

we get

$$\begin{aligned} \frac{dV}{dt} \leq &-\gamma(\|X\|^2+\|Y\|^2+\|Z\|^2) + 3\delta_2\Delta_1(\|X\|^2+\|Y\|^2+\|Z\|^2) \\ &+\delta_2\Delta_0(\|X\|+\|Y\|+\|Z\|) \\ = &-(\gamma-3\delta_2\Delta_1)(\|X\|^2+\|Y\|^2+\|Z\|^2) + \delta_2\Delta_0(\|X\|+\|Y\|+\|Z\|). \end{aligned}$$

If we choose $\Delta_1 < \frac{\gamma}{3\delta_2}$, then there is some $\theta > 0$, such that

$$\begin{aligned} \frac{d}{dt}V(X_t,Y_t,Z_t) \leq &-\theta(\|X\|^2+\|Y\|^2+\|Z\|^2) + n\theta(\|X\|+\|Y\|+\|Z\|) \\ = &-\frac{\theta}{2}(\|X\|^2+\|Y\|^2+\|Z\|^2) \\ &-\frac{\theta}{2}\left\{(\|X\|-n)^2+(\|Y\|-n)^2+(\|Z\|-n)^2\right\} + \frac{3\theta}{2}n^2 \\ \leq &-\frac{\theta}{2}\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right) + \frac{3\theta}{2}n^2, \end{aligned}$$

for some $n, \theta > 0$.

This completes the proof. □

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