ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 49, Number 2, 2025

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Niš : Grafika Galeb). - 24 cm

Dvomesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2502

Published By: Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Email: krag_j_math@kg.ac.rs
Website: http://kjm.pmf.kg.ac.rs

Designed By: Thomas Lampert

Front Cover: Željko Mališić

Printed By: Grafika Galeb, Niš, Serbia
From 2021 the journal appears in one volume and six issues per annum.

Contents

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 167–180.

WEAVING g-FRAMES FOR OPERATORS

A. KHOSRAVI AND J. S. BANYARANI

Abstract. Bemrose et al. introduced weaving frames and later, Deepshikha et al. generalized them to weaving K-frames. In this note, as a generalization of these notions, we introduce approximate K-duals and investigate the properties of K-g-frames and weaving K-g-frames. We show that woven K-g-frames and weakly woven K-g-frames coincide. We also study perturbation and erasure of woven K-g-frames and we show that they are stable under small perturbations. Also we generalize some of the known results in frame theory to K-g-frames and weaving K-g-frames.

1. Introduction and Preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [7] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [5] and popularized from then on. Frames are generalizations of bases in Hilbert spaces. A frame as well as an orthonormal basis allows that each element in the underlying Hilbert space to be written as an unconditionally convergent series in linear combinations of the frame elements; however, in contrast to the situation for a basis, the coefficients might not be unique. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communications.

Sun in [14] introduced g-frames as another generalization of frames. He showed that frames, oblique frames, pseudo frames and fusion frames are special cases of g-frames see also [9] and [10]. Weaving frames were introduced in [1] and investigated in [2,3,12]. In [13] we have generalized weaving frames to the Banach spaces. This concept

Key words and phrases. K-frame, g-frame, weaving K-g-frame, perturbation.
2020 *Mathematics Subject Classification.* Primary: 42C15.
DOI 10.46793/KgJMat2502.167K
Received: May 13, 2021.
Accepted: January 13, 2022.

is motivated on distributed signal processing, see [1]. A potentional application of weaving frames together is dealing with wireless sensor networks which may be subjected to distributed processing under different frames. The theory can be used in the processing of signals using Gabor frames.

Frames for operators, which are also called K-frames are more general than ordinary frames, where K is a bounded linear operator in a separable Hilbert space H. K-frames were introduced by Găvruta [8] and investigated in [15]. Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames (for example, the corresponding synthesis operator for K-frames is not surjective). Deepshikha et .al in [6] generalized weaving frames to weaving K-frames.

Throughout this paper H denotes a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{I}$ is a finite or countable subset of $\mathbb{Z}$ and $\{H_i : i \in \mathcal{I}\}$ is a sequence of separable Hilbert spaces. Also, for every $i \in \mathcal{I}$, $L(H, H_i)$ is the set of all bounded linear operators from H to H_i, and $L(H, H)$ is denoted by $L(H)$. Also, $GL(H) = \{T \in L(H) : T$ is invertible $\}$. Also throughout this paper we let $K \in L(H)$, with closed range.

A family $\{\varphi_i\}_{i\in\mathcal{I}} \subseteq H$ is a *frame* for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A family $\{\varphi_i\}_{i\in\mathcal{I}} \subseteq H$ is a *K-frame* for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathcal{I}} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is called a *g-frame* for H with respect to $\{H_i : i \in \mathcal{I}\}$ if there exist $0 < A \leq B < \infty$ such that for every $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2,$$

A, B are called g-frame bounds. In this case we say that $\{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is an (A, B) g-frame. We call Λ a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. If only the right hand side inequality is required, Λ is called a *g-Bessel sequence* see [4].

For every sequence $\{H_i\}_{i\in\mathcal{I}}$, the space

$$\left(\sum_{i\in\mathcal{I}} \bigoplus H_i\right)_{\ell^2} = \left\{(f_i)_{i\in\mathcal{I}} : f_i \in H_i, i \in \mathcal{I}, \sum_{i\in\mathcal{I}} \|f_i\|^2 < \infty\right\},$$

with pointwise operations and the following inner product is a Hilbert space

$$\langle (f_i)_{i\in\mathcal{I}}, (g_i)_{i\in\mathcal{I}}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle.$$

If Λ is a g-Bessel sequence, then *the synthesis operator* for Λ is the linear operator

$$T_\Lambda : \left(\sum_{i\in\mathfrak{I}} \bigoplus H_i\right)_{\ell^2} \mapsto H, \quad T_\Lambda(f_i)_{i\in\mathfrak{I}} = \sum_{i\in\mathfrak{I}} \Lambda_i^* f_i.$$

The adjoint of the synthesis operator is called *the analysis operator* and is defined by

$$T_\Lambda^* : H \mapsto \left(\sum_{i\in\mathfrak{I}} \bigoplus H_i\right)_{\ell^2}, \quad T_\Lambda^* f = (\Lambda_i f)_{i\in\mathfrak{I}}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the *g-frame operator* of Λ and $S_\Lambda f = \sum_{i\in\mathfrak{I}} \Lambda_i^*\Lambda_i f$, $f \in H$.

If $\Lambda = (\Lambda_i)_{i\in\mathfrak{I}}$ is a g-frame with lower and upper g-frame bounds A, B, respectively, then the g-frame operator of Λ is a bounded, positive and invertible operator on H and

$$A\langle f, f\rangle \le \langle S_\Lambda f, f\rangle \le B\langle f, f\rangle, \quad f \in H,$$

so

$$A \cdot I \le S_\Lambda \le B \cdot I.$$

Let $K \in L(H)$. A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is called a *K-g-frame*, if there exist constants $0 < A \le B < \infty$ such that

$$A\|K^* f\|^2 \le \sum_{i\in\mathfrak{I}} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in H.$$

Remark 1.1. Plainly, every g-frame is a K-g-frame, $K \neq 0$, since

$$\frac{A}{\|K^*\|^2}\|K^* f\|^2 \le A\|f\|^2 \le \sum_{i\in\mathfrak{I}} \|\Lambda_i f\|^2 \le B\|f\|^2.$$

Conversly, if K^* is bounded from below (equivalently if K is surjective), then every K-g-frame is an ordinary g-frame.

Găvruta showed that every K-frame in H is a frame for $R(K)$ and so every element of $R(K)$ can be reconstructed see [8, 15]. We generalize this result to K-g-frames.

Lemma 1.1. *Let $K \in L(H)$ with closed range $R(K)$. Then*

(a) *$K\,|_{R(K^*)}: R(K^*) \to R(K)$ and $K^*\,|_{R(K)}: R(K) \to R(K^*)$ are isomorphisms.*
(b) *If $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is a K-g-frame with g-frame operator S, then $S\,|_{R(K)}: R(K) \to S(R(K))$ is an isomorphism, i.e., $\{\Lambda_i \in L(R(K), H_i) : i \in \mathfrak{I}\}$ is a g-frame.*

Proof. (a) Since $R(K)$ is closed, then $R(K^*)$ is also closed and $(\ker(K))^\perp = R(K^*)$, $(\ker(K^*))^\perp = R(K)$. Hence, $K\,|_{R(K^*)}: R(K^*) \to R(K)$ is a bounded bijective linear map. Now, by Banach isomorphism theorem $K\,|_{R(K^*)}$ is an isomorphism and similarly $K^*\,|_{R(K)}: R(K) \longrightarrow R(K^*)$ is an isomorphism. Therefore, there exist $A, B > 0$ such that for each $y \in R(K)$

$$A\|y\| \le \|K^* y\| \le B\|y\|.$$

(b) Since $\{\Lambda_i\}_{i\in\mathfrak{I}}$ is a K-g-frame, there exist $0 < A' < B' < \infty$ such that for each $x \in H$

$$A'\|K^*(x)\|^2 \leq \langle Sx, x\rangle \leq B'\|x\|^2,$$

specially for each $x \in H$, we have

$$A'A^2\|K(x)\|^2 \leq \langle SKx, Kx\rangle \leq \|S(K(x))\| \cdot \|Kx\|,$$

so by (2.1) for each $x \in H$, we have $A'A^2\|K(x)\| \leq \|S(K(x))\|$. Therefore, $S\mid_{R(K)}$ is one-to-one and $S(R(K))$ is closed. Now, again by Banach isomorphism theorem, we have the result. □

A small modification in [14] gains the following result.

Lemma 1.2. *Let for each $i \in \mathfrak{I}$, $\{e_{i,j} : j \in \mathfrak{I}_i\}$ be an orthonormal basis for H_i. Then $\{\Lambda_i\}_{i\in\mathfrak{I}}$ is a K-g-frame if and only if $\{\Lambda_i^*(e_{i,j})\}_{i\in\mathfrak{I},j\in\mathfrak{I}_i}$ is a K-frame.*

In [15] the authors defined the atomic system for K and by using this idea we introduce the following definition.

Definition 1.1. Let $K \in L(H)$. A sequence $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is called an *atomic g-system* for K, if the following conditions are satisfied:

(a) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ is a g-Bessel sequence;
(b) for any $x \in H$, there exists $\mathfrak{g}_x = (g_i)_i \in (\sum_{i\in\mathfrak{I}} \bigoplus H_i)_{\ell^2}$ such that $Kx = \sum_{i\in\mathfrak{I}} \Lambda_i^*(g_i)$, where $\|\mathfrak{g}_x\| \leq C\|x\|$, C is a positive constant.

We recall some definitions from [12].

Definition 1.2. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be two g-frames for H. We call $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *woven g-frames* if there exist $0 < A \leq B < \infty$ such that for every $\sigma \subset \mathfrak{I}$ and every $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i f\|^2 \leq B\|f\|^2.$$

In this case, for convenience we say that $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are an (A, B)-woven g-frame.

Proof of the following lemma is similar to [15, Theorem 3.5] which we reaffirm.

Lemma 1.3. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ be a g-Bessel sequence in H. Then $\{\Lambda_i\}_{i\in\mathfrak{I}}$ is a K-g-frame for H, if and only if there exists $A > 0$ such that $S \geq AKK^*$, where S is the g-frame operator for $\{\Lambda_i\}_{i\in\mathfrak{I}}$.*

Remark 1.2. Since $S^{\frac{1}{2}}S^{\frac{1}{2}} = S \geq AKK^*$, by Douglas theorem, there exists $C \in L(H)$ such that $K = S^{\frac{1}{2}}C$.

Definition 1.3. Let $K \in L(H)$ and $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be K-g-frames. We say that $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are *woven K-g-frames* if there exist constants $0 < A \leq B < \infty$ such that for every $\sigma \subset \mathfrak{I}$ and every $f \in H$

$$A\|K^*f\|^2 \leq \sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i f\|^2 \leq B\|f\|^2.$$

In this case we say that $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are (A, B) woven K-g-frames.

Example 1.1. Let H be a Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and let $\Lambda_n, \Gamma_n, K : H \to H$ be defined by

$$\Lambda_n(x) = \langle x, e_{5n}\rangle e_{5n} + \langle x, e_{5n-1}\rangle e_{5n-1},$$
$$\Gamma_n(x) = \langle x, e_{5n}\rangle e_{5n} + \langle x, e_{5n+1}\rangle e_{5n+1},$$

and $K(x) = \sum_{n\in\mathbb{N}}\langle x, e_{5n}\rangle e_{5n}$ for every $x \in H$.

Then $\{\Gamma_n : n \in \mathbb{N}\}$ and $\{\Lambda_n : n \in \mathbb{N}\}$ are woven K-g-frames.

Since K is the orthogonal projection of H onto M, the closed subspace of H generated by $\{e_{5n} : n \in \mathbb{N}\}$, then $K = K^*$. Now for every $x \in H$ and $\sigma \subseteq I$ we have

$$\begin{aligned}\|K^*(x)\|^2 &= \sum_{n\in\mathbb{N}} |\langle x, e_{5n}\rangle|^2 \le \sum_{n\in\sigma} \|\Lambda_n(x)\|^2 + \sum_{n\in\sigma^c} \|\Gamma_n(x)\|^2 \\ &= \sum_{n\in\sigma} |\langle x, e_{5n}\rangle|^2 + \sum_{n\in\sigma} |\langle x, e_{5n-1}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_{5n}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_{5n+1}\rangle|^2 \\ &\le 3\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 = 3\|x\|^2,\end{aligned}$$

and we have the result.

As we have in [12, Remark 3.2] if $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are g-Bessel sequences with bounds B and B' and g-frame operators S and S', respectively, then for every $\sigma \subset \mathfrak{I}$, $0 \le S_\sigma \le S \le B \cdot I$ and $0 \le S'_{\sigma^c} \le S' \le B' \cdot I$. Therefore, $0 \le S_\sigma + S'_{\sigma^c} \le (B + B') \cdot I$. Hence, $\{\Lambda_i\}_{i\in\sigma} \bigcup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-Bessel sequence with bound $B + B'$ and g-frame operator $S_\sigma + S'_{\sigma^c}$, where $S_\sigma f = \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f$ and $S'_{\sigma^c} f = \sum_{i\in\sigma^c} \Gamma_i^*\Gamma_i f$.

In this paper we try to generalize some of the known results in K-frames, weaving frames and weaving g-frames to K-g-frames.

2. Weaving K-g-frame

In [1], the authors introduced the concept of weaving frames. In this section we also study weaving K-g-frames.

Definition 2.1. Let $K \in L(H)$. The sequences $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are called a *woven atomic g-system* for K, if the following conditions are satisfied:

(a) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are g-Bessel sequences;
(b) there exist positive constants C_1, C_2 such that for any $x \in H$, and any $\sigma \subset \mathfrak{I}$ there exist $\mathfrak{g}_x = (g_i)_i, \mathfrak{g}'_x = (g'_i)_i \in (\sum_{i\in\mathfrak{I}} \bigoplus H_i)_{\ell^2}$ such that $Kx = \sum_{i\in\sigma} \Lambda_i^*(g_i) + \sum_{i\in\sigma^c} \Gamma_i^*(g'_i)$ with $\|\mathfrak{g}_x\| \le C_1\|x\|$ and $\|\mathfrak{g}'_x\| \le C_2\|x\|$.

Theorem 2.1. *Let $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be a woven atomic g-system for K. Then $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ are woven K-g-frames.*

Proof. Let $x \in H$. For every $y \in H$ with $\|y\| = 1$ and every $\sigma \subset \mathfrak{I}$, there exist $(g_i)_i, (g'_i)_i \in (\sum_{i\in\mathfrak{I}} \bigoplus H_i)_{\ell^2}$, such that $Ky = \sum_{i\in\sigma} \Lambda_i^* g_i + \sum_{i\in\sigma^c} \Gamma_i^* g'_i$, then

$$\|K^*x\| = \sup_{\|y\|=1} |\langle K^*x, y\rangle| = \sup_{\|y\|=1} \left|\left\langle x, \sum_{i\in\sigma} \Lambda_i^* g_i + \sum_{i\in\sigma^c} \Gamma_i^* g'_i \right\rangle\right|$$

$$\leq \sup_{\|y\|=1}\left|\left\langle x, \sum_{i\in\sigma}\Lambda_i^* g_i\right\rangle\right| + \sup_{\|y\|=1}\left|\left\langle x, \sum_{i\in\sigma^c}\Gamma_i^* g_i'\right\rangle\right|$$
$$= \sup_{\|y\|=1}\left|\sum_{i\in\sigma}\langle\Lambda_i x, g_i\rangle\right| + \sup_{\|y\|=1}\left|\sum_{i\in\sigma^c}\langle\Gamma_i x, g_i'\rangle\right|$$
$$\leq \sup_{\|y\|=1}\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in\sigma}\|g_i\|^2\right)^{\frac{1}{2}} + \sup_{\|y\|=1}\left(\sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in\sigma^c}\|g_i'\|^2\right)^{\frac{1}{2}}$$
$$\leq \sup_{\|y\|=1}\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}\left[\left(\sum_{i\in\mathcal{I}}\|g_i\|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in\mathcal{I}}\|g_i'\|^2\right)^{\frac{1}{2}}\right]$$
$$\leq (C_1+C_2)\sup_{\|y\|=1}\|y\|\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}.$$

Therefore, $\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2 \geq \frac{1}{(C_1+C_2)^2}\|K^*x\|^2$. □

Definition 2.2. We call $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ *weakly woven K-g-frames,* if for every $\sigma\subset\mathcal{I}$, $\{\Lambda_i\}_{i\in\sigma}\bigcup\{\Gamma_i\}_{i\in\sigma^c}$ is a K-g-frame.

Lemma 2.1. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ be K-g-frames. Suppose that for every $\epsilon>0$ and every two disjoint finite sets $I_1, J_1\subset\mathcal{I}$ there exists a subset $\sigma\subset\mathcal{I}\backslash(I_1\cup J_1)$ such that for $\delta=\mathcal{I}\backslash(I_1\cup J_1\cup\sigma)$ the lower K-g-frame bound of $\{\Lambda_i\}_{i\in I_1\cup\sigma}\bigcup\{\Gamma_i\}_{i\in J_1\cup\delta}$ is less than ϵ. Then there exists $\mathcal{Q}\subset\mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{Q}}\bigcup\{\Gamma_i\}_{i\in\mathcal{I}\backslash\mathcal{Q}}$ is not a K-g-frame, i.e, $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ are not weakly woven K-g-frames.*

Proof. Let $\epsilon>0$ and for each $p\in\mathbb{N}$, $A_p=[-p,p]\cap\mathcal{I}$ where $[-p,p]\cap\mathbb{Z}=\{-p,\ldots,0,1,\ldots,p\}$. We prove that there exist an increasing sequence $\{f_n\}_{n=1}^\infty\subset\mathbb{N}$, a sequence $\{h_n\}_{n=1}^\infty\subset H$ with $\|h_n\|=1$, and sequences $\{\sigma_n\}$, $\{\delta_n\}$ of subsets $\mathcal{I}$ with $\sigma_n\subset A_{n-1}^c=\mathcal{I}\backslash A_{n-1}$, $\delta_n=A_{n-1}^c\backslash\sigma_n$, such that $I_n=I_{n-1}\cup(\sigma_n\cap A_n)$, $J_n=J_{n-1}\cup(\delta_n\cap A_n)$ satisfy both

$$\sum_{i\in I_{n-1}\cup\sigma_n}\|\Lambda_i(h_n)\|^2 + \sum_{i\in J_{n-1}\cup\delta_n}\|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2,$$
$$\sum_{i\in\mathcal{I},|i|\geq f_n+1}\|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{I},|i|\geq f_n+1}\|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2.$$

We proceed by induction. By taking $I_0=J_0=\emptyset$, we can choose $\sigma_1\subset\mathcal{I}$ such that for $\delta_1=\sigma_1^c=\mathcal{I}\backslash\sigma_1$ the lower K-g-frame bound of $\{\Lambda_i\}_{i\in\sigma_1}\bigcup\{\Gamma_i\}_{i\in\delta_1}$ is less than ϵ. Therefore there is some $h_1\in H$ with $\|h_1\|=1$ such that

$$\sum_{i\in\sigma_1}\|\Lambda_i(h_1)\|^2 + \sum_{i\in\delta_1}\|\Gamma_i(h_1)\|^2 < \epsilon\|K^*\|^2.$$

Since

$$\sum_{i\in\mathcal{I}}\|\Lambda_i(h_1)\|^2 + \sum_{i\in\mathcal{I}}\|\Gamma_i(h_1)\|^2 < +\infty,$$

there is $f_1 \in \mathbb{N}$ such that

$$\sum_{i\in\mathcal{I},|i|\geq f_1+1} \|\Lambda_i(h_1)\|^2 + \sum_{i\in\mathcal{I},|i|\geq f_1+1} \|\Gamma_i(h_1)\|^2 < \epsilon\|K^*\|^2.$$

Let σ_i, δ_i, h_i and f_i for $i = 1, 2, \dots, n-1$ with the above conditions are given. Then $J_{n-1} \cap I_{n-1} = \emptyset$ and $I_{n-1} \cup J_{n-1} = A_{n-1}$. By the hypothesis there is $\sigma_n \subset \mathcal{I}\backslash A_{n-1}$ with $\delta_n = \mathcal{I}\backslash(A_{n-1} \cup \sigma_n)$ such that $\{\Lambda_i\}_{i\in I_{n-1}\cup\sigma_n} \bigcup \{\Gamma_i\}_{i\in J_{n-1}\cup\delta_n}$ has lower K-g-frame bound less than ϵ. Hence, there exist $h_n \in H$ with $\|h_n\| = 1$ such that

$$\sum_{i\in I_{n-1}\cup\sigma_n} \|\Lambda_i(h_n)\|^2 + \sum_{i\in J_{n-1}\cup\delta_n} \|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2.$$

Similar to the above argument there is $f_n > f_{n-1}$ such that

$$\sum_{i\in\mathcal{I},|i|\geq f_n+1} \|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{I},|i|\geq f_n+1} \|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2.$$

By taking $I_n = I_{n-1} \cup (\sigma_n \cap A_n)$, $J_n = J_{n-1} \cup (\delta_n \cap A_n)$ for each n, $J_n \cap I_n = \emptyset$ and $I_n \cup J_n = A_n$. Therefore,

$$\left(\bigcup_{i=1}^{\infty} I_i\right) \sqcup \left(\bigcup_{j=1}^{\infty} J_j\right) = \mathcal{I},$$

where $\sqcup$ denotes a disjoint union. For

$$\mathcal{Q} = \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \mathcal{Q}^c = \bigcup_{j=1}^{\infty} J_j,$$

we have

$$\begin{aligned}
\sum_{i\in\mathcal{Q}} \|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{I}\backslash\mathcal{Q}} \|\Gamma_i(h_n)\|^2 &= \left(\sum_{i\in I_n} \|\Lambda_i(h_n)\|^2 + \sum_{j\in J_n} \|\Gamma_i(h_n)\|^2\right) \\
&\quad + \left(\sum_{i\in\mathcal{Q}\cap A_n^c} \|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{Q}^c\cap A_n^c} \|\Gamma_i(h_n)\|^2\right) \\
&\leq \left(\sum_{i\in I_{n-1}\cup\sigma_n} \|\Lambda_i(h_n)\|^2 + \sum_{i\in J_{n-1}\cup\delta_n} \|\Gamma_i(h_n)\|^2\right) \\
&\quad + \left(\sum_{i\in\mathcal{I},|i|\geq f_n+1} \|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{I},|i|\geq f_n+1} \|\Gamma_i(h_n)\|^2\right) \\
&< \frac{\epsilon}{n}\|K^*\|^2 + \frac{\epsilon}{n}\|K^*\|^2.
\end{aligned}$$

So that the lower K-g-frame bound of $\{\Lambda_i\}_{i\in\mathcal{Q}} \bigcup \{\Gamma_i\}_{i\in\mathcal{I}\backslash\mathcal{Q}}$ is zero. Then, it is not a K-g-frame and the two original K-g-frames are not weakly woven. $\square$

Corollary 2.1. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ be K-g-frames. If they are weakly woven, then there exist $A > 0$ and finite disjoint subsets $J, Q \subset \mathcal{I}$ such that for each $\sigma \subset \mathcal{I} \setminus (J \cup Q)$*

and $\delta = \mathfrak{I}\backslash(J\cup Q\cup\sigma)$ *the sequence* $\{\Lambda_i\}_{i\in J\cup\sigma}\bigcup\{\Gamma_i\}_{i\in Q\cup\delta}$ *has lower* K*-g-frame bound* A*.*

In the proof of [2, Theorem 4.5], Casazza et al. dealt with frames, but their proof also works for K-g-frames and by a modification in their proof, we can get the following results.

Theorem 2.2. *Let* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *be* K*-g-frames. Then the following are equivalent:*

(a) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are woven* K*-g-frames;*
(b) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are weakly woven* K*-g-frames.*

Definition 2.3. Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be g-Bessel sequences, with bounds B, B', respectively. Then the *operator* $S_{\Gamma,\Lambda}: H\to H$ defined by

$$S_{\Gamma,\Lambda}(f) = T_\Gamma T_\Lambda^*(f) = \sum_{i\in\mathfrak{I}}\Gamma_i^*\Lambda_i(f), \quad f\in H,$$

is a bounded linear operator with $\|S_{\Gamma,\Lambda}\|\le\sqrt{BB'}$. Also, $S_{\Gamma,\Lambda}^* = S_{\Lambda,\Gamma}$ and $S_{\Gamma,\Gamma} = S_\Gamma$, see [11].

The proof of [11, Lemma 2.11] also works for K-g-frames and we have the following result.

Lemma 2.2. *Let* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *be g-Bessel sequences. If there exists* $\lambda>0$ *such that* $\|S_{\Lambda,\Gamma}(f)\|\ge\lambda\|K^*f\|$*, then* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are* K*-g-frames.*

Example 2.1. Let H be a Hilbert space with orthonormal basis $\{e_n : n\in\mathbb{N}\}$ and let Λ_n, Γ_n and $K: H\to H$ be defined by $\Lambda_n(x) = \langle x, e_{2n}\rangle e_{2n}$, $\Gamma_n(x) = \langle x, e_{2n}\rangle e_{2n} + \langle x, e_{2n+1}\rangle e_{2n+1}$, $K(x) = \sum_{n\in\mathbb{N}}\langle x, e_{2n}\rangle e_{2n}$, for every $x\in H$. Then $\{\Lambda_n : n\in\mathbb{N}\}$ and $\{\Gamma_n : n\in\mathbb{N}\}$ are woven K-g-frames for H with universal bounds 1 and 3. The reason is similar to Example 1.1.

Proposition 2.1. *Let* $\Lambda = \{\Lambda_i\in L(H,H_i) : i\in\mathfrak{I}\}$, $\Gamma = \{\Gamma_i\in L(H,H_i) : i\in\mathfrak{I}\}$, $\Lambda' = \{\Lambda'_i\in L(H,H'_i) : i\in\mathfrak{I}\}$ *and* $\Gamma' = \{\Gamma'_i\in L(H,H'_i) : i\in\mathfrak{I}\}$ *be g-Bessel sequences with bounds* D_1, D_2, D_3, D_4*, respectively. If there exists* $\lambda>0$ *such that* $\|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\|\ge\lambda\|K^*f\|$ *for each* $\sigma\subset\mathfrak{I}$ *and* $f\in H$*, then* $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ *are woven* K*-g-frames and also* $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are woven* K*-g-frames.*

Proof. As we saw before, they are woven g-Bessel sequences. Suppose that $\lambda>0$ such that for all $\sigma\subset\mathfrak{I}$ and $f\in H$

$$\lambda\|K^*f\|\le\|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\|,$$

then,

$$\begin{aligned}\|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\| &\le \|S^\sigma_{\Lambda,\Lambda'}f\| + \|S^{\sigma^c}_{\Gamma,\Gamma'}f\| = \|(T_\Lambda T^*_{\Lambda'})^\sigma(f)\| + \|(T_\Gamma T^*_{\Gamma'})^{\sigma^c}(f)\|\\ &\le \|T_\Lambda\|\left(\sum_{i\in\sigma}\|\Lambda'_i f\|^2\right)^{\frac12} + \|T_\Gamma\|\left(\sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\right)^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&\le \sqrt{D_1}\left(\sum_{i\in\sigma}\|\Lambda'_i f\|^2\right)^{\frac{1}{2}} + \sqrt{D_2}\left(\sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\right)^{\frac{1}{2}}\\
&\le \left(\sqrt{D_1}+\sqrt{D_2}\right)\left(\sum_{i\in\sigma}\|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$
\sum_{i\in\sigma}\|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma'_i f\|^2 \ge \frac{\lambda^2\|K^* f\|^2}{(\sqrt{D_1}+\sqrt{D_2})^2}.
$$

On the other hand, since $S^*_{\Gamma,\Lambda} = S_{\Lambda,\Gamma}$, then $(S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})^* = S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma',\Gamma}$ and we have the result. ☐

Theorem 2.3. *Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathfrak{I}\}$ and $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in \mathfrak{I}\}$ be (A,B) woven K-g-frames and $\Lambda' = \{\Lambda'_i \in L(H',H'_i) : i \in \mathfrak{I}\}$ and $\Gamma' = \{\Gamma'_i \in L(H',H'_i) : i \in \mathfrak{I}\}$ be (A',B') woven K-g-frames.*

(i) *Then $\{\Lambda_i \oplus \Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i \oplus \Gamma'_i\}_{i\in\mathfrak{I}}$ are $(\min\{A,A'\},\max\{B,B'\})$ woven K-g-frames.*

(ii) *If $H = H', H_i = H'_i$ for each $i \in \mathfrak{I}$, and for every $\sigma \subset \mathfrak{I}$*

$$
S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma,\Gamma'} + S^{\sigma^c}_{\Gamma',\Gamma} \ge 0,
$$

then $\{\Lambda_i + \Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i + \Gamma'_i\}_{i\in\mathfrak{I}}$ are woven K-g-frames, where $S^{\sigma^c}_{\Gamma,\Gamma'} = \sum_{i\in\sigma^c}\Gamma^*_i\Gamma'_i$.

Proof. (i) With a proof similar to the proof of [11, Proposition 2.16], $\{\Lambda_i \oplus \Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i \oplus \Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. For every $\sigma \subset \mathfrak{I}$ and every $(f,g) \in H \oplus H'$

$$
\begin{aligned}
&\sum_{i\in\sigma}\|(\Lambda_i \oplus \Lambda'_i)(f,g)\|^2 + \sum_{i\in\sigma^c}\|(\Gamma_i \oplus \Gamma'_i)(f,g)\|^2\\
=&\sum_{i\in\sigma}\|(\Lambda_i f, \Lambda'_i g)\|^2 + \sum_{i\in\sigma^c}\|(\Gamma_i f, \Gamma'_i g)\|^2\\
=&\sum_{i\in\sigma}\langle(\Lambda_i f, \Lambda'_i g), (\Lambda_i f, \Lambda'_i g)\rangle + \sum_{i\in\sigma^c}\langle(\Gamma_i f, \Gamma'_i g), (\Gamma_i f, \Gamma'_i g)\rangle\\
=&\sum_{i\in\sigma}(\|\Lambda_i f\|^2 + \|\Lambda'_i g\|^2) + \sum_{i\in\sigma^c}(\|\Gamma_i f\|^2 + \|\Gamma'_i g\|^2)\\
\le& B\|f\|^2 + B'\|g\|^2 \le \max\{B,B'\}\|(f,g)\|^2,
\end{aligned}
$$

similarly for the lower bound.

(ii) It is clear that $S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma,\Gamma'} + S^{\sigma^c}_{\Gamma',\Gamma}$ is a self-adjoint operator. For every $\sigma \subset \mathfrak{I}$ we have

$$
\begin{aligned}
S^{\sigma}_{\Lambda+\Lambda'} + S^{\sigma^c}_{\Gamma+\Gamma'} =& \sum_{i\in\sigma}(\Lambda_i + \Lambda'_i)^*(\Lambda_i + \Lambda'_i) + \sum_{i\in\sigma^c}(\Gamma_i + \Gamma'_i)^*(\Gamma_i + \Gamma'_i)\\
=& \sum_{i\in\sigma}{\Lambda_i}^*\Lambda_i + \sum_{i\in\sigma}{\Lambda'_i}^*\Lambda'_i + \sum_{i\in\sigma^c}{\Gamma_i}^*\Gamma_i + \sum_{i\in\sigma^c}{\Gamma'_i}^*\Gamma'_i\\
&+ \sum_{i\in\sigma}({\Lambda_i}^*\Lambda'_i + {\Lambda'_i}^*\Lambda_i) + \sum_{i\in\sigma^c}({\Gamma_i}^*\Gamma'_i + {\Gamma'_i}^*\Gamma_i)
\end{aligned}
$$

$$=S_{\Lambda}^{\sigma} + S_{\Gamma}^{\sigma^c} + S_{\Lambda'}^{\sigma} + S_{\Gamma'}^{\sigma^c} + S_{\Lambda,\Lambda'}^{\sigma} + S_{\Lambda',\Lambda}^{\sigma} + S_{\Gamma,\Gamma'}^{\sigma^c} + S_{\Gamma',\Gamma}^{\sigma^c}$$
$$\geq AKK^* + A'KK^* = (A + A')KK^*.$$

Also, plainly $\{\Lambda_i + \Lambda_i'\}_{i\in\sigma} \cup \{\Gamma_i + \Gamma_i'\}_{i\in\sigma^c}$ is a g-Bessel sequence. □

Definition 2.4. Let $\Lambda = \{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathfrak{I}}$ be g-Bessel sequences. Then

(a) Γ is a K-dual of Λ if for each $f \in H$, we have $Kf = S_{\Gamma,\Lambda}(f) = \sum_{i\in\mathfrak{I}} \Gamma_i^*\Lambda_i(f)$;

(b) Γ is an approximate K-dual of Λ if there exists $0 < r < 1$ such that for every $f \in H$,

$$\|K(f) - S_{\Gamma,\Lambda}(f)\| \leq r\|K(f)\|.$$

Plainly, every K-dual is an approximate K-dual, and for the converse we have the following result.

Proposition 2.2. *Let $\Gamma = \{\Gamma_i\}_{i\in\mathfrak{I}}$ be an approximate K-dual of Λ. Then Λ has a K-dual and every element $K(f)$ of $R(K)$ can be reconstructed from $\{\Gamma_i^* \circ \Lambda_i(f)\}_{i\in\mathfrak{I}}$.*

Proof. Since Γ is an approximate K-dual of Λ, there exists $0 < r < 1$ such that

$$\|K(f) - S_{\Gamma,\Lambda}(f)\| \leq r\|K(f)\|, \quad f \in H. \tag{2.1}$$

Now, from (2.1) it follows that $S_{\Gamma,\Lambda}(f) = 0$ if and only if $K(f) = 0$. Therefore, we can define $U : R(K) \to R(S_{\Gamma,\Lambda})$ by $U(K(f)) = S_{\Gamma,\Lambda}(f)$ for every $f \in H$. Hence U is an injective bounded linear map and by using (2.1) we have

$$\|Kf - U(Kf)\| \leq r\|Kf\|, \quad f \in H. \tag{2.2}$$

So, for every $f \in H$

$$(1 - r)\|Kf\| \leq \|U(Kf)\| \leq (1 + r)\|Kf\|.$$

Hence, U has a closed range, $R(U) = R(S_{\Gamma,\Lambda})$. Now by Banach isomorphism theorem $U^{-1} : R(S_{\Gamma,\Lambda}) \to R(K)$ is a bounded linear map, which can be extended to $V : H \to H$, by $V = U^{-1} \circ \pi_{R(K)}$, where $\pi_{R(K)}$ is the orthogonal projection of H onto $R(U)$. It is clear that

$$K(f) = V \circ S_{\Gamma,\Lambda}(f) = \sum_{i\in\mathfrak{I}} (V \circ \Gamma_i^*) \circ \Lambda_i(f), \quad f \in H.$$

Therefore, $\{\Gamma_i \circ V^*\}_{i\in\mathfrak{I}}$ is a K-dual of $\{\Lambda_i\}_{i\in\mathfrak{I}}$. □

Remark 2.1. If in the above Proposision $R(S_{\Gamma,\Lambda}) \subseteq R(K)$, then we can regard $U : R(K) \to R(K)$ and from (2.2) it follows that

$$\|g - U(g)\| \leq r\|g\|, \quad g \in R(K).$$

Then $\|I_{R(K)} - U\| \leq r < 1$ and consequently U is invertible and the above inequality is similar to the inequality for approximate K-dual.

A small modification in the proofs of [3, Proposition 15] and [12, Theorem 3.14] shows that these properties hold for K-g-frames.

3. Perturbation

In this section we study the behaviour of K-g-frames under some perturbations.

The following result shows that approximate K-duals are stable under small perturbation.

Theorem 3.1. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be a g-Bessel sequence and $\Psi = \{\psi_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be an approximate K-dual (resp. K-dual) of Λ with $0 < r < 1$ and upper bound C. If $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is a sequence such that*

$$\left(\sum_{i\in\mathfrak{I}} \| (\Lambda_i - \Gamma_i)(f) \|^2\right)^{\frac{1}{2}} \leq F\|K(f)\|, \quad f \in H,$$

and $\sqrt{CF} < 1 - r$ (resp. $CF < 1$), then Ψ is an approximate K-dual of Γ.

Proof. Let B be an upper bound for Λ. Then for any $f \in H$, we have

$$\left(\sum_{i\in\mathfrak{I}} \|\Gamma_i f\|^2\right)^{\frac{1}{2}} \leq \|\{\Lambda_i f\}_{i\in\mathfrak{I}}\|_2 + \|\{\Gamma_i f - \Lambda_i f\}_{i\in\mathfrak{I}}\|_2 \leq (\sqrt{B} + \sqrt{F}\|K\|)\|f\|,$$

so, Γ is a g-Bessel sequence. For any $f \in H$,

$$\|S_{\Psi,\Lambda} f - S_{\Psi,\Gamma} f\| \leq \sup_{\|g\|=1} \left\{ \left(\sum_{i\in\mathfrak{I}} \|(\Lambda_i - \Gamma_i) f\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in\mathfrak{I}} \|\psi_i g\|^2\right)^{\frac{1}{2}} \right\} \leq \sqrt{CF}\|Kf\|.$$

Hence, for every $f \in H$

$$\|Kf - S_{\Psi,\Gamma} f\| \leq \|Kf - S_{\Psi,\Lambda} f\| + \|S_{\Psi,\Lambda} f - S_{\Psi,\Gamma} f\| \leq (r + \sqrt{CF})\|Kf\|.$$

Since $r + \sqrt{CF} < 1$ we have the result. If Ψ is a K-dual of Λ, then $S_{\Psi,\Lambda} f = Kf$ and we have $\|K - S_{\Psi,\Lambda}\| \leq \sqrt{CF} < 1$. ☐

Theorem 3.2. *Let $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be (A, B) woven K-g-frames and let $T \in L(H)$ and $T_i, T_i' \in L(H_i)$ for each $i \in \mathfrak{I}$. If there exist $0 < m < M < \infty$ such that for each $i \in \mathfrak{I}$ and $f_i \in H_i$, $m\|f_i\| \leq \|T_i f\|, \|T_i' f_i\| \leq M\|f_i\|$, then $\{\Lambda_i' = T_i \Lambda_i T\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i' = T_i' \Gamma_i T\}_{i\in\mathfrak{I}}$ are woven T^*K-g-frames, with universal bounds $m^2 A$ and $M^2 B\|T\|^2$. Moreover if $TK^* = K^*T$, $m\|f\| \leq \|Tf\|$, then $\{\Lambda_i' \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i' \in L(H, H_i) : i \in \mathfrak{I}\}$ are woven K-g-frames, with universal bounds $m^4 A$ and $M^2 B\|T\|^2$.*

Proof. For every $\sigma \subset \mathfrak{I}$ and every $f \in H$

$$\begin{aligned}
\sum_{i\in\sigma} \|\Lambda_i' f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i' f\|^2 &= \sum_{i\in\sigma} \|T_i \Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|T_i' \Gamma_i T f\|^2 \\
&\leq \sum_{i\in\sigma} \|T_i\|^2 \|\Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|T_i'\|^2 \|\Gamma_i T f\|^2 \\
&\leq M^2 \left(\sum_{i\in\sigma} \|\Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i T f\|^2\right)
\end{aligned}$$

$$\leq M^2 B\|T\|^2\|f\|^2,$$

and similarly for every $\sigma \subset \mathfrak{I}$ and every $f \in H$ we have,

$$\sum_{i\in\sigma}\|\Lambda_i' f\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i' f\|^2 \geq m^2 A\|K^*Tf\|^2 = m^2 A\|(T^*K)^* f\|^2.$$

The rest of the proof is obvious. □

Corollary 3.1. *Let $\{\Lambda_i \in L(H, H_i)\}_{i\in\mathfrak{I}}$ be a K-g-frame for H and $T \in L(H)$ be invertible. Then*

(i) *$\{\Lambda_i T\}_{i\in\mathfrak{I}}$ is a K-g-frame, when $\Gamma K^* = K^*\Gamma$;*
(ii) *$\{T\Lambda_i\}_{i\in\mathfrak{I}}$ is a K-g-frame, when $H_i \subseteq H$ for each $i \in \mathfrak{I}$.*

Proof. Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ be a K-g-frame with bounds A and B.

(i) For every $x \in H$, we have

$$\begin{aligned}\frac{A}{\|T^{-1}\|^2}\|K^*x\|^2 &\leq A\|TK^*x\|^2 = A\|K^*(Tx)\|^2\\ &\leq \sum_{i\in I}\|\Lambda_i Tx\|^2 \leq B\|Tx\|^2 \leq B\|T\|^2\|x\|^2.\end{aligned}$$

For (ii),

$$\begin{aligned}\frac{A}{\|T^{-1}\|^2}\|K^*(x)\|^2 &\leq \frac{1}{\|T^{-1}\|^2}\sum\|\Lambda_i x\|^2\\ &\leq \sum\|T\Lambda_i x\|^2 \leq \|T\|^2\sum\|\Lambda_i x\|^2 \leq B\|T\|^2\|x\|^2.\end{aligned}$$ □

For the erasure of K-g-frames, the following result shows that it is possible to remove some elements of a woven K-g-frame and still have a woven K-g-frame.

Proposition 3.1. *Suppose that $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are (A, B) woven K-g-frames. If $\mathcal{J} \subset \mathfrak{I}$ and*

$$\sum_{i\in\mathcal{J}}\|\Lambda_i f\|^2 \leq D\|K^* f\|^2,$$

for some, $0 < D < A$, then $\{\Lambda_i\}_{i\in\mathfrak{I}\setminus\mathcal{J}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}\setminus\mathcal{J}}$ are $(A-D, B)$ woven K-g-frames.

Proof. The proof is similar to the proof of [3, Proposition 16]. □

Corollary 3.2. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ be a K-g-frame with lower frame bound A. If for some $\mathcal{J} \subset \mathfrak{I}$ and $0 < D < A$,*

$$\sum_{i\in\mathcal{J}}\|\Lambda_i f\|^2 \leq D\|K^* f\|^2, \quad f \in H,$$

then $\{\Lambda_i\}_{i\in\mathcal{J}^c}$ is a K-g-frame with lower bound $A - D$.

Definition 3.1. Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ be a K-g-frame and let $0 \leq \lambda_1, \lambda_2 < 1$. We say that the family $\{\Gamma_i\}_{i\in\mathfrak{I}}$ is a (λ_1, λ_2)-perturbation of $\{\Lambda_i\}_{i\in\mathfrak{I}}$ if we have

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda_1\|\Lambda_i f\| + \lambda_2\|\Gamma_i f\|, \quad \text{for all } f \in H.$$

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be woven K-g-frames and $\{\Lambda'_i\}_{i\in\mathfrak{I}}$, $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ be (λ_1,λ_2), (μ_1,μ_2)-perturbations of $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$, respectively. Then $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are woven K-g-frames.*

Proof. A simple calculation shows that $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. For each $f\in H$ we have

$$\|\Lambda'_i f\| - \|\Lambda_i f\| \le \|\Lambda_i f - \Lambda'_i f\| \le \lambda_1\|\Lambda_i f\| + \lambda_2\|\Lambda_i f\|,$$

hence

$$\frac{1-\lambda_1}{1+\lambda_2}\|\Lambda_i f\| \le \|\Lambda'_i f\| \le \frac{1+\lambda_1}{1-\lambda_2}\|\Lambda_i f\|.$$

Similarly, we have

$$\frac{1-\mu_1}{1+\mu_2}\|\Gamma_i f\| \le \|\Gamma'_i f\| \le \frac{1+\mu_1}{1-\mu_2}\|\Gamma_i f\|.$$

Now for every $\sigma\subset\mathfrak{I}$ and every $f\in H$

$$\begin{aligned}
&\min\left\{\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2,\left(\frac{1-\mu_1}{1+\mu_2}\right)^2\right\}\left(\sum_{i\in\sigma}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c}\|\Gamma_i f\|^2\right)\\
\le&\sum_{i\in\sigma}\|\Lambda'_i f\|^2+\sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\\
\le&\max\left\{\left(\frac{1+\lambda_1}{1-\lambda_2}\right)^2,\left(\frac{1+\mu_1}{1-\mu_2}\right)^2\right\}\left(\sum_{i\in\sigma}\|\Lambda_i f\|^2+\sum_{i\in\sigma^c}\|\Gamma_i f\|^2\right),
\end{aligned}$$

and we have the result. □

Corollary 3.3. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be woven K-g-frames and $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ be sequences and $0\le M_1, M_2$ such that for every $f\in H$, and every $i\in\mathfrak{I}$*

$$\|\Lambda_i f - \Lambda'_i f\| \le M_1\min\{\|\Lambda_i f\|,\|\Lambda'_i f\|\},$$
$$\|\Gamma_i f - \Gamma'_i f\| \le M_2\min\{\|\Gamma_i f\|,\|\Gamma'_i f\|\},$$

then $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are woven K-g-frames.

Proof. It is clear that $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. From the hypothesis it follows that for each $i\in\mathfrak{I}, f\in H$, we have

$$\frac{1}{M_1+1}\|\Lambda_i f\| \le \|\Lambda'_i f\| \le (M_1+1)\|\Lambda_i f\|,$$
$$\frac{1}{M_2+1}\|\Gamma_i f\| \le \|\Gamma'_i f\| \le (M_2+1)\|\Gamma_i f\|.$$

Now similar to the proof of the above theorem we have the result. □

Example 3.1. Let $\{\Lambda_n : n\in\mathbb{N}\}$, $\{\Gamma_n : n\in\mathbb{N}\}$, K and H be given as in Example 2.1 and $\Lambda'_n = \frac{1}{2}\Lambda_n$ and $\Gamma'_n = \frac{1}{3}\Gamma_n$. Then $\{\Lambda'_n : n\in\mathbb{N}\}$, $\{\Gamma'_n : n\in\mathbb{N}\}$ are a woven K-g-frame. It is enough to use Example 2.1 and Theorem 3.3.

Acknowledgements. The authors express their gratitude to the referee for valuable comments.

References

[1] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers and R. G. Lynch, *Weaving frames*, Oper. Matrices **10**(4) (2016), 1093–1116. https://doi.org/10.7153/oam-10-61
[2] P. G. Casazza, D. Freeman and R. G. Lynch, *Weaving Schauder frames*, J. Approx. Theory **211** (2016), 42–60. https://doi.org/10.1016/j.jat.2016.07.001
[3] P. G. Casazza and R. G. Lynch, *Weaving properties of Hilbert space frames*, 2015 International Conference on Sampling Theory and Applications (SampTA), 2015, 110–114. https://doi.org/10.1109/SAMPTA.2015.7148861
[4] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-25613-9
[5] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283. https://doi.org/10.1063/1.527388
[6] Deepshikha and L. K. Vashisht, *Weaving K-frames in Hilbert spaces,* Results Math. **73** (2018), Article ID 81. https://doi.org/10.1007/s00025-018-0843-4
[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. https://doi.org/10.1090/S0002-9947-1952-0047179-6
[8] L. Găvruta, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), 139–144. https://doi.org/10.1016/j.acha.2011.07.006
[9] A. Khosravi and B. Khosravi, *Fusion frames and g-frames in Banach spaces*, Proc. Indian Acad. Sci. Math. Sci. **121**(2) (2011), 155–164. https://doi.org/10.1007/s12044-011-0020-0
[10] A. Khosravi and M. M. Azandaryani, *Approximate duality of g-frames in Hilbert spaces*, Acta Math. Sci. **34B**(3) (2014), 639–652.
[11] A. Khosravi and K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl. **342 (2)** (2008), 1068–1083. https://doi.org/10.1016/j.jmaa.2008.01.002
[12] A. Khosravi and J. S. Banyarani, *Weaving g-frames and weaving fusion frames,* Bull. Malays. Math. Sci. Soc. **42** (2019), 3111–3129. https://doi.org/10.1007/s40840-018-0647-4
[13] A. Khosravi and J. S. Banyarani, *Some properties of g-frames in Banach spaces,* Int. J. Wavelets Multiresolut. Inf. Process. **16**(6) (2018), Article ID 1850051. https://doi.org/10.1142/S0219691318500510
[14] W. Sun, */g-Frames and g-Riesz bases*, J. Math. Anal. Appl. **322**(1) (2006), 437–452. https://doi.org/10.1016/j.jmaa.2005.09.039
[15] X. Xiao, Y. Zhu and L. Găvruta, *Some properties of K-frames in Hilbert spaces*, Results Math. **63** (2013), 1243–1255. https://doi.org/10.1007/s00025-012-0266-6

Faculty of Mathematical Sciences and Computer Scienes,
Kharazmi University, 599 Taleghani Ave.,
Tehran 15618, Iran
Email address: khosravi_amir@yahoo.com, khosravi@khu.ac.ir
Email address: jamaleh.sohrabi@yahoo.com

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 181–202.

A TOTALLY RELAXED SELF-ADAPTIVE SUBGRADIENT EXTRAGRADIENT SCHEME FOR EQUILIBRIUM AND FIXED POINT PROBLEMS IN A BANACH SPACE

OLAWALE KAZEEM OYEWOLE[1,2], HAMMED ANUOLUWAPO ABASS[1,2],
AND OLUWATOSIN TEMITOPE MEWOMO[1]

Abstract. The goal of this paper is to introduce a Totally Relaxed Self adaptive Subgradient Extragradient Method (TRSSEM) together with an Halpern iterative method for approximating a common solution of Fixed Point Problem (FPP) and Equilibrium Problem (EP) in 2-uniformly convex and uniformly smooth Banach space. We prove the strong convergence of the sequence generated by our proposed method. The proposed method does not require the computation of a projection onto a feasible set, it instead requires a projection onto a finite intersection of sub-level sets of convex functions. Our result generalizes, unifies and extends some related results in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^*. Let E be endowed with the duality pairing $\langle \cdot, \cdot \rangle$ of element from E and E^*, and also the corresponding norm $\| \cdot \|$. Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction such that $C \subset \operatorname{int}(\operatorname{dom}(f, \cdot))$, then for every $x \in C$, the Equilibrium Problem (EP) (see [3, 14]), is to find a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

We denote the EP and its solution set by $EP(C, f)$ and $Sol(C, f)$, respectively.

Key words and phrases. Equilibrium problem, strongly pseudomonotone, strong convergence, Banach space, quasi-ϕ-nonexpansive mapping, fixed point.
2020 *Mathematics Subject Classification.* Primary: 47H09. Secondary: 49J35, 90C47.
DOI 10.46793/KgJMat2502.181O
Received: September 21, 2021.
Accepted: February 02, 2022.

The EP is a generalization of many important optimization problems, such as Variational Inequality Problem (VIP), Fixed Point Problem (FPP) and so on (see [6,14] and the references therein). In particular, if $f(x,y) = \langle Ax, y-x\rangle$, where $A : C \to E^*$, is a nonlinear mapping, then $EP(C,f)$ (1.1) reduces to the classical VIP introduced by Stampacchia [47] (see also [36,38,41,52]), which is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

There are two important directions of research on EP: These are the existence of solution of EP and other related problems (see [14,29] for more details) and the development of iterative algorithms for approximating the solution of EP, its several generalizations and related optimization problems (see [1,12,13,33,34,42–44] and the references therein).

In 2018, Hieu [24] introduced some methods for solving strongly pseudomonotone and Lipschitz type bifunction EPs. We note that a bifunction f satisfies the Lipschitz type condition, if there exist positive constants $c_1, c_2 \in \mathbb{R}$ such that for all $x, y, z \in C$, the inequality

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2$$

holds.

In general EP, the Lipschitz type condition does not hold and when it does, finding the constants c_1 and c_2 is always not an easy task. This does have effect on the efficiency of the method involved. In addition, in the method of Hieu [24], there is the need to first solve at least one strongly convex programming problem. Also, if the bifunction and the feasible sets have complex structures, the computations could be expensive and time consuming.

Furthermore, the problem of finding a common point in the set of solutions of different generalizations of EP and the fixed point set of a nonlinear mapping in Hilbert, Banach and Hadamard spaces have been considered by several authors in literature (see [25,39,40,46,51,57]) and the references therein for further reading.

In 2013, Anh [9] introduced an extragradient algorithm for finding a common element of the fixed point set of a nonexpansive mapping and solution set of an EP involving pseudomonotone and Lipschitz type continuous bifunction in real Hilbert space. The author proved a strong convergence result of the sequence generated by his method under some standard conditions, see [8–10] for related results.

However, in Banach spaces, just like the extragradient method employed by Hieu [24], many existing methods for approximating a common solution FPP and EP involving a pseudomonotone bifunctions requires that a strongly convex programming is solved (see [26,27] and the references therein).

To avoid the assumptions of Lipschitz continuity on the bifunction and solving strongly convex progamming, Vinh and Gibali [53] introduced two gradient-type iterative algorithms involving a one-step projection method for solving $EP(C,f)$ (1.1) and proved strong convergence results for both algorithms with an adaptive step-size rule which does not require the Lipschitz condition of the associated method. The method proposed in [53] involves a projection onto a feasible set, and is known to be computationally expensive, time and memory consuming if the feasible set is not simple.

In an attempt to overcome this setback, Censor et al. [17] introduced the subgradient extragradient method which uses a projection onto a halfspace. Also, He et al. [23] introduced a TRSSEM for solving the VIP (1.2) in a real Hilbert space. Let $C^i := \{x \in H : h_i(x) \leq 0\}$, where $h_i : H \to \mathbb{R}$ for $i = 1, 2, \ldots, m$, are convex functions. In the TRSSEM, the feasible set is given as

$$C := \cap_{i=1}^{m} C^i.$$

On the other hand, for approximating a fixed point of a nonexpansive mapping T, Mainge [31] introdued an inertial Krasnoselskij-Mann Algorithm as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T w_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

and proved a weak convergence theorem under some mild assumptions on the sequences $\{\theta_n\}$ and $\{\alpha_n\}$. The term $\theta_n(x_n - x_{n-1})$ as given (1.3) is referred to as the inertial extrapolation term. It is known that the introduction of the inertial term helps to speed up the convergence rate of the algorithm. Due to its importance, lots of researchers have adopted the use of the inertial technique in their quest for approximating the solutions of fixed point and optimization problem (see [4, 5, 31] and the references therein).

In this paper, motivated by the works of He et al. [23], Vinh and Gibali [53] and other related results in literature, we introduce a TRSSEM for approximating a common solution of FPP and EP in 2-uniformly convex and uniformly smooth Banach space. We prove a strong convergence result for the sequence generated by the proposed method under some conditions. Finally, we give some applications of our main result. The rest of the section is organized as follows. In Section 2, we recall some important results and definitions that will be useful in establishing our main result. In Section 3, we state our proposed method and then discuss its convergence analysis. We give some theoretical application of our main result in Section 4 and give a concluding remark Section 5.

2. Preliminaries

We denote the weak and the strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let E be a real Banach space, given a function $g : E \to \mathbb{R}$.

• The function g is called Gâteaux differentiable at $x \in E$, if there exists an element E, denoted by $g'(x)$ or $\nabla g(x)$ such that

$$\lim_{t \to \infty} \frac{g(x + ty) - g(x)}{t} = \langle y, g'(x) \rangle, \quad y \in E,$$

where g' or $\nabla g(x)$ is called Gâteaux differential or gradient of g at x. We say g is Gâteaux on E if for each $x \in E$, g is Gâteaux differentiable at x.

• The function g is called weakly lower semicontinuous at $x \in E$, if $x_n \rightharpoonup x$ implies $g(x) \leq \liminf_{n\to\infty} g(x_n)$. We say that a function g is weakly lower semicontinuous on E, if for each $x \in E$, g is weakly lower semicontinuous at x.

• If g is a convex function, then it is said to be differentiable at a point $x \in E$ if the following set

$$\partial g(x) = \{f \in E : g(y) - g(x) \geq \langle f, y - x\rangle,\ y \in E\} \tag{2.1}$$

is nonempty. Each element $\partial g(x)$ is called a subgradient of g at x or the subdifferential of g and the inequality (2.1) is said to be the subdifferential inequality of g at x.

The function g is subdifferentiable at x, if g is subdifferntiable at every $x \in E$. It is well known that if g is Gâteaux differentiable at x, then g is subdifferentiable at x and $\partial g(x) = \{g'(x)\}$, that is, $\partial g(x)$ is just a singleton set. For more details on Gâteaux differentiable functions on Banach space, see [15].

Let C be a nonempty, closed and convex subset of a real Banach space with norm $\|\cdot\|$ and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2 \text{ for all } x \in E\},$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the duality pairing between the elements of E and E^*. Alber [7], introduced a generalized projection operator Π_C an analogue of the metric projection $P_C : H \to C$ in the Hilbert space H. He defines $\Pi_C : E \to C$ by

$$\Pi_C(x) = \inf_{y\in C}\{\phi(y, x) \text{ for all } x \in E\}.$$

In Hilbert spaces $P_C(x) \equiv \Pi_C(x)$.

Consider the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for all } x, y \in E.$$

In the real Hilbert space, we observe that $\phi(x, y) = \|x - y\|^2$. It is easy to see that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2.$$

The functional ϕ also satisfies the following important properties:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

and

$$\phi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\phi(x, y) + (1-\lambda)\phi(x, z), \tag{2.3}$$

for all $x, y, z \in E$ and $\lambda \in (0, 1)$.

Note. If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$, see [18, 48].

We are also concerned with the functional $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$. That is, $V(x,x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. It is well known that if E is a reflexive, strictly convex and smooth Banach space, then

$$V(x,x^*) \leq V(x,x^*+y^*) - 2\left\langle J^{-1}x^* - x, y^*\right\rangle,$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [50].

Let C be a closed and convex subset of E and $T : C \to C$ be a mapping, a point $x \in C$ is called a fixed point of T, if $x = Tx$. We denote the set of fixed points of T by $F(T)$. Let $T : C \to C$ be a mapping, a point $p \in C$ is called an asymptotic fixed point of T (see [45]) if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$ (see [16, 48]). T is said to be ϕ-nonexpansive if $\phi(Tx,Ty) \leq \phi(x,y)$ for all $x, y \in C$ and quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mapping which requires the strict condition $F(T) = \hat{F}(T)$ (see [16, 45, 48]).

Let E be a real Banach space. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}.$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$. E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$, with $\|x\| = \|y\| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0,2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+ty\| - \|x-ty\|) - 1 : \|x\| = \|y\| = 1\right\}.$$

E is said to be uniformly smooth if $\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $p^{-1} + q^{-1} = 1$. It is also known that every q-uniformly smooth Banach space is uniformly smooth.

It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. The following are some important and useful properties of J, for further details, see [2, 48].

Let C be a nonempty, closed and convex subset of a real Banach space E and $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. f is said to be

(i) strongly monotone on C, if there exists $\gamma \geq 0$ such that for any $x, y \in C$

$$f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2;$$

(ii) monotone on C, if

$$f(x,y) + f(y,x) \leq 0, \quad \text{for all } x, y \in C;$$

(iii) pseudomonotone on C, if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \text{for all } x, y \in C;$$

(iv) strongly γ-pseudomonotone on C, if there exists $\gamma > 0$ such that for any $x, y \in C$

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq -\gamma\|x-y\|^2.$$

From the above, it is clear (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). The converse is generally not true (see [53]).

We now give the following useful and important lemmas that are needed in establishing our main results.

Lemma 2.1 ([35])**.** *Let E be a 2-uniformly convex and smooth Banach space. Then for every $x, y \in E$*

$$\phi(x,y) \geq \nu\|x-y\|^2,$$

where $\nu > 0$ is the 2-uniformly convexity constant of E.

Lemma 2.2 ([28])**.** *Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.3 ([7])**.** *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq\rangle \leq 0, \quad \text{for all } y \in C, \tag{2.5}$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x), \quad \text{for all } y \in C, x \in X. \tag{2.6}$$

Lemma 2.4 ([55])**.** *Fix a number $s > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\psi(\|x-y\|),$$

for all $x, y \in X$, $\lambda \in [0,1]$, with $\|x\| < s$ and $\|y\| < s$.

Lemma 2.5 ([54])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where

(a) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} \sigma \leq 0$;
(c) $\gamma_n \geq 0$, $n \geq 1$, *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([32]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$: $a_{m_n} < a_{m_n+1}$ and $a_n < a_{m_n+1}$. In fact, $m_n = \max\{i \leq k : a_i < a_{i+1}\}$.*

3. Main Result

In this section, we give a concise and precise statement of our algorithm, discuss some of its elementary properties and its convergence analysis. The convergence analysis is given in the next section.

Statement 3.1. Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. For $i = 1, 2, \ldots, m$, let $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinous and Gâteaux differentiable functions. Let $S : E \to E$ be a quasi-ϕ-nonexpansive mapping and $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a strongly γ-pseudomonotone bifunction satisfying the following assumptions.

Assumption 3.2. We require the following assumptions for our operator and the solution set:

A1. $f(x, \cdot)$ is convex and lower semi-continuous for every $x \in E$;
A2. f is strongly γ-pseudomonotone on C,
A3. $Sol(C, f) \neq \emptyset$;
A4. if $\{x_n\}_{n=0}^{\infty} \subset E$ is bounded, then the sequence $\{g(x_n) \in \partial(f(x_n, \cdot))(x_n)_{n=0}^{\infty}\}$ is bounded.

Note. The assumption $A4$. is quite standard assumption and it holds for example when $f(x, \cdot)$ is bounded on bounded subsets (see [11]).

Assumption 3.3. To prove a strong convergence result using Algorithm 3.4, the following conditions are needed.

B1. The feasible set C is defined by $C := \cap_{i=1}^{m} C^i$, where $C^i := \{z \in E : h_i(z) \leq 0\}$;
B2. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
B3. $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
B4. $\sum\limits_{n=1}^{\infty} \phi(x_n, x_{n-1}) < \infty$.
B5. $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$.

Algorithm 3.4. $(TRSSEM)$ for $EP(C, f)$

Step 0. Choose the sequences $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\} \subset (0,1)$ satisfying Assumption 3.3, let $\mu \in (0,1)$ and $\beta_0 > 0$. For $u \in C$, select initial points x_0 and x_1 in C. Set $n = 1$.

Step 1. For $i = 1, 2 \ldots, m$, and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \leq 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$. Take $g(w_n) \in \partial(f(w_n, \cdot))(w_n)$, $n \geq 1$, and compute

$$z_n = \Pi_{C_n} J^{-1}(Jw_n - \beta_n g(w_n)), \tag{3.1}$$

where β_n is given by

$$\beta_{n+1} = \begin{cases} \min\left\{\beta_n, \frac{\mu\|w_n - z_n\|}{\|g(w_n) - g(z_n)\|}\right\}, & \text{if } g(w_n) \neq g(z_n), \\ \beta_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. If $w_n = z_n$ ($w_n \in Sol(C, f)$), then set $w_n = y_n$ and go to Step 3. Otherwise, compute the next iterate by

$$y_n = \Pi_{Q_n} J^{-1}(Jw_n - \beta_n g(z_n)), \tag{3.3}$$

where

$$Q_n = \{w \in E : \langle w - z_n, Jw_n - \beta_n g(w_n) - Jz_n\rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = J^{-1}((1 - \alpha_n)Ju + \alpha_n(1 - \gamma_n)Jy_n + \gamma_n JSy_n). \tag{3.4}$$

Step 4. Set $n := n + 1$ and go to Step 1.

Lemma 3.1. *If* $w_n = z_n$, *then* $w_n \in Sol(C, f)$.

Proof. Suppose $w_n = z_n$, then by (2.5) and (3.1), we have

$$\langle Jw_n - \beta_n g(w_n) - Jw_n, y - z_n\rangle \leq 0, \quad y \in C,$$

or equivalently

$$\langle g(w_n), y - w_n\rangle \geq 0, \quad \text{for all } y \in C. \tag{3.5}$$

Therefore, from (3.5) and the definition of the subdifferential f in the second argument, we obtain

$$f(w_n, y) = f(w_n, y) - f(w_n, w_n) \geq \langle g(w_n), y - w_n\rangle \geq 0.$$

Hence, $w_n \in Sol(C, f)$. □

Lemma 3.2 ([56]). *The sequence* $\{\beta_n\}$ *generated by* (3.2) *is a motonically decreasing sequence and*

$$\lim_{n\to\infty}\beta_n = \beta \geq \min\left\{\frac{\mu}{L}, \beta_0\right\}.$$

Remark 3.1. Note that if $w_n = z_n$ and $w_n = Sw_n$ we are at a common solution of the $EP(C,f)$ and fixed point of the mapping S. In our convergence analysis, we will assume implicitly that this does not occur after finitely many iterations so that our Algorithm 3.4 generates an infinite sequence satisfying, in particular $w_n \neq z_n$ and $w_n \neq Sw_n$ for all $n \in \mathbb{N}$.

We now prove some lemmas which are required components of the main result.

Lemma 3.3. *The sequence* $\{x_n\}$ *generated by Algorithm* 3.4 *is bounded.*

Proof. Let $x^* \in Sol(C,f)$, then we have from (2.6), that

$$\begin{aligned}
\phi(x^*, y_n) =& \phi(x^*, \Pi_{Q_n} J^{-1}(Jw_n - \beta_n g(w_n))) \\
\leq& \phi(x^*, J^{-1}(Jw_n - \beta_n g(z_n))) - \phi(y_n, J^{-1}(Jw_n - \beta_n g(w_n))) \\
=& \|x^*\|^2 - 2\langle x^*, Jw_n - \beta_n g(z_n)\rangle - \|y_n\|^2 + 2\langle y_n, Jw_n - \beta_n g(z_n)\rangle \\
=& \phi(x^*, w_n) - \phi(y_n, w_n) + 2\beta_n\langle x^* - y_n, g(z_n)\rangle \\
=& \phi(x^*, w_n) - (\phi(y_n, z_n) + \phi(z_n, w_n) \\
& + 2\langle y_n - z_n, Jz_n - Jw_n\rangle) + 2\beta_n\langle x^* - y_n, g(z_n)\rangle \\
=& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) \\
& + 2\langle y_n - z_n, Jw_n - Jz_n\rangle + 2\beta_n\langle x^* - y_n, g(z_n)\rangle. \qquad (3.6)
\end{aligned}$$

Now, we have from (3.6) that

$$\begin{aligned}
2\beta_n\langle x^* - y_n, g(z_n)\rangle =& 2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\beta_n\langle z_n - y_n, g(z_n)\rangle \\
=& 2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\langle y_n - z_n, -\beta_n g(z_n)\rangle. \qquad (3.7)
\end{aligned}$$

Substituting (3.7) into (3.6) and using the strongly pseudomonotonicity of f, we obtain

$$\begin{aligned}
\phi(x^*, y_n) =& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - Jz_n\rangle \\
& + 2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\langle y_n - z_n, -\beta_n g(z_n)\rangle \\
=& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \\
& + 2\beta_n\langle x^* - z_n, g(z_n)\rangle \\
\leq& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) \\
& + 2\beta_n\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle + 2\beta_n f(z_n, x^*) \\
\leq& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) \\
& - 2\beta_n\gamma\phi(x^*, z_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \\
\leq& \phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle. \qquad (3.8)
\end{aligned}$$

By the definition of Q_n and Cauchy-Schwartz inequality, we have

$$\begin{aligned}\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle =& 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \\ &+ 2\beta_n\langle y_n - z_n, g(w_n) - g(z_n)\rangle \\ \le& 2\beta_n\|y_n - z_n\|\|g(w_n) - g(z_n)\|. \end{aligned}\tag{3.9}$$

Using (3.2) and Lemma 2.1 in (3.9), we get

$$\begin{aligned}\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \le& 2\frac{\mu\beta_n}{\beta_{n+1}}\|y_n - z_n\|\|w_n - z_n\| \\ \le& 2\frac{\mu\beta_n}{\beta_{n+1}}\sqrt{\frac{\phi(y_n, z_n)}{\nu}}\sqrt{\frac{\phi(z_n, w_n)}{\nu}} \\ \le& \frac{\mu\beta_n}{\nu\beta_{n+1}}(\phi(y_n, z_n) + \phi(z_n, w_n)). \end{aligned}\tag{3.10}$$

Therefore, from (3.8) and (3.10), we have

$$\phi(x^*, y_n) \le \phi(x^*, w_n) - \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)).\tag{3.11}$$

From (2.3) and (3.4), we have

$$\begin{aligned}\phi(x^*, x_{n+1}) =& \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)(1-\gamma_n)Ju_n + \gamma_n JSy_n)) \\ =& \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)(1-\gamma_n)Jy_n + (1-\alpha_n)\gamma_n JSy_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)(1-\gamma_n)\phi(x^*, y_n) + (1-\alpha_n)\gamma_n\phi(x^*, Sy_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, y_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) \\ &- \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n). \end{aligned}\tag{3.12}$$

From Algorithm 3.4, we have

$$\begin{aligned}\phi(x^*, w_n) =& \phi(x^*, J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))) \\ \le& (1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}),\end{aligned}$$

hence

$$\begin{aligned}\phi(x^*, x_{n+1}) &\le \alpha_n\phi(x^*, u) + (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})) \\ &\le \alpha_n\phi(x^*, u) + (1-\alpha_n)(\phi(x^*, x_n) + \phi(x^*, x_{n-1})) \\ &\le \max\{\phi(x^*, u), (\phi(x^*, x_n) + \phi(x^*, x_{n-1}))\} \\ &\vdots \\ &\le \max\{\phi(x^*, u), (\phi(x^*, x_1) + \phi(x^*, x_0))\}, \quad n \ge 1. \end{aligned}\tag{3.13}$$

This implies that $\{\phi(x^*, x_n)\}$ is bounded. Therefore, $\{x_n\}$ is bounded. Consequently, $\{g(y_n)\}$ is bounded and by the nonexpansiveness of the projection operator and the mapping S, we have that $\{z_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{Sy_n\}$ are bounded. □

The boundedness of $\{x_n\}$ implies that there is at least one weak limit point. The next result provides a condition under which each of such weak limit is in the solution set of the equilibrium problem.

Lemma 3.4. *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging weakly to a point $p \in C$ and suppose that the conditions $\|w_{n_i} - z_{n_i}\| \to 0$ and $\|w_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$ hold on this subsequence. Then $p \in Sol(C, f)$.*

Proof. From Lemma 2.5 and the definition of subdifferential, we have

$$\begin{aligned} 0 &\leq \langle x - z_{n_i}, Jz_{n_i} - (Jw_{n_i} - \beta_{n_i} g(w_{n_i}))\rangle \\ &= \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle x - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle \\ &= \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle x - w_{n_i}, \beta_{n_i} g(w_{n_i})\rangle + \langle w_{n_i} - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle \\ &\leq \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle w_{n_i} - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle + f(w_{n_i}, x). \end{aligned} \tag{3.14}$$

Passing limit to the inequality in (3.14), we have

$$f(p, x) \geq 0, \quad \text{for all } x \in C.$$ □

In proving the strong convergence of our Algorithm 3.4, the underlying idea relies on certain estimate and other classical properties of the iterates which are given in the next lemmas below.

Lemma 3.5. *The sequence $\{x_n\}$ generated by Algorithm 3.4 satisfies the following estimates:*

(i) $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n$;
(ii) $-1 \leq \limsup_{n\to\infty} b_n < +\infty$,

where $a_n = \phi(x^*, x_n)$ and $b_n = \frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\langle Ju - Jx^*, x_{n+1} - x^*\rangle$.

Proof. Let $p_n = (1 - \gamma_n)Jy_n + \gamma_n JSy_n$, then from (2.4), we have

$$\begin{aligned} \phi(x^*, x_{n+1}) =& \phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jp_n)) \\ \leq& V(x^*, \alpha_n Ju + (1 - \alpha_n)Jp_n - \alpha_n(Ju - Jx^*)) \\ &- 2\langle -\alpha_n(Ju - Jx^*), J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jp_n)\rangle \\ \leq& V(x^*, \alpha_n Jx^* + (1 - \alpha_n)Jp_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\ \leq& \alpha_n V(x^*, Jx^*) + (1 - \alpha_n)V(x^*, Jp_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\ \leq& \alpha_n\phi(x^*, x^*) + (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\ \leq& (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\ \leq& (1 - \alpha_n)(1 - \gamma_n)\phi(x^*, y_n) + \gamma_n(1 - \alpha_n)\phi(x^*, Sy_n) \\ &+ 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\phi(x^*, y_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + \alpha_n\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\langle Ju - Jx^*, x_{n+1} - x^*\rangle\right).
\end{aligned}
$$

This established (i). Next we proof (ii). Since $\{x_n\}$ is bounded, then we have

$$
\sup_{n\geq 0} b_n \leq \sup\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\|Ju - Jx^*\|\|x_{n+1} - x^*\|\right) < \infty.
$$

This implies that $\limsup_{n\to\infty} b_n < \infty$. Next we show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, that is $\limsup_{n\to\infty} b_n \leq -1$. Then there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$ for all $n \geq n_0$. Then for all $n_0 \in \mathbb{N}$, we get from (i), that

$$
\begin{aligned}
a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n b_n \\
&< (1-\alpha_n)a_n - \alpha_n \\
&= a_n - \alpha_n(a_n + 1) \leq a_n - \alpha_n.
\end{aligned}
$$

Taking lim sup of both sides in the last inequality, we have

$$
\limsup_{n\to\infty} a_n \leq a_{n_0} - \lim_{n\to\infty}\sum_{i=n_0}^{n}\alpha_i = -\infty.
$$

This contradicts the definition of $\{a_n\}$ as a nonnegative integer.

Therefore, $\limsup_{n\to\infty} b_n \geq -1$. □

We now present our main theorem.

Theorem 3.5. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, for $i = 1, 2, \ldots, m$. Let $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying conditions A1-A4, let $S : C \to C$ be a quasi-ϕ-nonexpansive mapping such that $\Gamma = \{Sol(C, f) \cap F(S)\} \neq \emptyset$. Let $\{\theta_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying Assumption 3.3, then the sequence $\{x_n\}$ generated by Algorithm 3.4 converges strongly to $p = \Pi_\Gamma u$, where Π_Γ is the projection of C onto Γ.*

Proof. Let $p \in \Gamma$, we divide the proof into two cases.

Case I Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}$ is monotone non-increasing. Since $\{\phi(x^*, x_n)\}$ is bounded, then it is convergent and

$$
\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \quad \text{as } n \to \infty. \tag{3.15}
$$

Since $p_n = J^{-1}((1-\gamma_n)Jy_n + \gamma_n JSy)$, then from Lemma 2.4, we have

$$
\phi(x^*, p_n) = \phi(x^*, J^{-1}((1-\gamma_n)Jy_n + \gamma_n JSy))
$$

$$
\begin{aligned}
&= V(x^*, (1-\gamma_n)Jy_n + \gamma_n JSy) \\
&= \|x^*\|^2 - 2\langle x^*, (1-\gamma_n)Jy_n + \gamma_n JSy\rangle + \|(1-\gamma_n)Jy_n + \gamma_n JSy\|^2 \\
&= \|x^*\|^2 - 2(1-\gamma_n)\langle x^*, Jy_n\rangle - 2\gamma_n\langle x^*, JSy_n\rangle + (1-\gamma_n)\|y_n\|^2 + \gamma_n\|Sy_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&= \phi(x^*, y_n) + \phi(x^*, Sy_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \phi(x^*, y_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|). \qquad (3.16)
\end{aligned}
$$

Therefore, from (3.4), (3.11) and (3.16), we have

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jp_n)) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, p_n) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, y_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) \\
&\quad + \theta_n\phi(x^*, x_{n-1})) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) \\
&\quad - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|). \qquad (3.17)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \le &\alpha_n\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + \phi(x^*, u)\right) \\
&+ (1-\alpha_n)\phi(x^*, x_n) - \phi(x^*, x_{n-1}).
\end{aligned}
$$

By using $\alpha_n \to 0$, we obtain $\gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \to 0$ as $n \to \infty$. Therefore, by condition $B3$ and the property of ψ, we get

$$\lim_{n\to\infty} \|Jy_n - JSy_n\| = 0.$$

Since J^{-1} is norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \qquad (3.18)$$

Furthermore, from (3.12), we have

$$\phi(x^*, y_n) \le \phi(x^*, w_n) - \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)).$$

Therefore, it follows from (3.4) that

$$
\begin{aligned}
\phi(x^*, x_{n+1}) \le &\alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, p_n) \\
\le &\alpha_n\phi(x^*, u) + (1-\alpha)\phi(x^*, y_n) \\
\le &\alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) \\
&- (1-\alpha_n)\left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n))
\end{aligned}
$$

$$\leq \alpha_n \phi(x^*, u) + (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) + \theta_n \phi(x^*, x_{n-1})) - (1-\alpha_n)\left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)).$$

This implies that

$$(1-\alpha_n)\left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)) \leq \alpha_n \left(\phi(x^*, u) + \frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}\right) + (1-\alpha_n)\phi(x^*, x_n) - \phi(x^*, x_{n+1}).$$

By condition B2 and (3.15), we have $(\phi(y_n, z_n) + \phi(z_n, w_n)) \to 0,$ as $n \to \infty$, thus

$$\lim_{n\to\infty} \phi(y_n, z_n) = \lim_{n\to\infty} \phi(z_n, w_n) = 0.$$

Since the sequences $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded, we obtain by Lemma 2.2, that

$$\lim_{n\to\infty} \|y_n - z_n\| = \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.19}$$

From Algorithm 3.4 and condition B4, we obtain

$$\lim_{n\to\infty} \phi(w_n, x_n) = \lim_{n\to\infty} \theta_n \phi(x_n, x_{n-1}) = 0,$$

and by Lemma 2.2, we get

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.20}$$

It is easy to see from (3.19) and (3.20), that

$$\lim_{n\to\infty} \|x_n - z_n\| = \|x_n - y_n\| = 0. \tag{3.21}$$

Observe also that

$$\phi(y_n, p_n) = \phi(y_n, J^{-1}((1-\gamma_n)Jy_n + \gamma_n)JSy_n) \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

Hence, by Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|y_n - p_n\| = 0.$$

This and (3.21), imply

$$\lim_{n\to\infty} \|x_n - p_n\| = 0.$$

Furthermore,

$$\|Jx_{n+1} - Jp_n\| = \alpha_n \|Ju - Jp_n\| = \alpha_n \|Ju - Jp_n\| \to 0, \quad \text{as } n \to \infty.$$

Since J^{-1} is norm-to-norm continuous on bounded subsets of E, we have $\|x_{n+1} - p_n\| \to 0$, as $n \to \infty$. Hence,

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - p_n\| + \|p_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.23}$$

Now, since the sequence $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_n \rightharpoonup q \in E$. Then, by (3.19), (3.20) and Lemma 3.4, we obtain $q \in Sol(C, f)$. Also, since $\|y_n - Sy_n\| \to 0$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$, then we have $q \in \hat{F}(S) = F(S)$. Therefore, $q \in \Gamma$.

We now show that $\{x_n\}$ converges strongly to a point $x^* = \Pi_\Gamma u$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle Ju - Jx^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty}\langle Ju - Jx^*, x_{n_i+1} - x^*\rangle.$$

Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we have by (2.5), that

$$\begin{aligned}\limsup_{n\to\infty}\langle Ju - Jx^*, x_{n+1} - x^*\rangle &= \lim_{i\to\infty}\langle Ju - Jx^*, x_{n_i+1} - x^*\rangle \\ &= \langle Ju - Jx^*, q - x^*\rangle \leq 0. \end{aligned} \tag{3.24}$$

It follows from Lemma 2.5, Lemma 3.5 (i) and (3.24), that $\phi(p, x_n) \to$ as $n \to \infty$. Therefore, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

Case II Suppose there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\phi(x^*, x_{n_j+1}) > \phi(x^*, x_{n_j}), \quad \text{for all } n \in \mathbb{N}.$$

From Lemma 2.6, there exists a non-decreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following inequalities hold for all $n \in \mathbb{N}$:

$$\phi(x^*, x_{m_n}) \leq \phi(x^*, x_{m_n+1}) \quad \text{and} \quad \phi(p, x_n) \leq \phi(x^*, x_{m_n+1}). \tag{3.25}$$

We note from (3.11) and (3.12), that

$$\begin{aligned}\phi(x^*, x_{m_n}) \leq& \phi(x^*, x_{m_n+1}) \leq \alpha_{m_n}\phi(x^*, u) \\ &+ (1-\alpha_{m_n})\left[\phi(x^*, w_{m_n}) - \left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n}))\right] \\ \leq& \alpha_{m_n}\phi(x^*, u) + (1-\alpha_{m_n})\left(\left(\left(1-\theta_{m_n}\right)\phi(x^*, x_{m_n}) + \theta_{m_n}\phi(x^*, x_{m_n-1})\right)\right) \\ &- (1-\alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \\ \leq& \alpha_{m_n}\left(\phi(x^*, u) + \frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*, x_{m_n-1})\right) \\ &- (1-\alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})).\end{aligned}$$

Hence,

$$\begin{aligned}&(1-\alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right) \times (\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \\ \leq& \alpha_{m_n}\left(\phi(x^*, u) + \frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*, x_{m_n-1})\right) + (1-\alpha_{m_n})\phi(x^*, x_{m_n}) - \phi(x^*, x_{m_n}).\end{aligned}$$

Since $\alpha_{m_n} \to 0$ as $n \to \infty$, it follows that

$$\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \to 0, \quad \text{as } n \to \infty,$$

hence

$$\lim_{n\to\infty}\phi(y_{m_n},z_{m_n})=\lim_{n\to\infty}\phi(z_{m_n},w_{m_n})=0.$$

Since $\{x_{m_n}\}$, $\{y_{m_n}\}$ and $\{w_{m_n}\}$ are bounded, we have

$$\lim_{n\to\infty}\|y_{m_n}-z_{m_n}\|=\lim_{n\to\infty}\|z_{m_n}-w_{m_n}\|=0.$$

Following similar method as in Case I, we obtain

$$\lim_{n\to\infty}\|w_{m_n}-Sw_{m_n}\|=\lim_{n\to\infty}\|x_{m_n+1}-x_{m_n}\|=0. \tag{3.26}$$

By Lemma 3.4 and (3.26), we obtain a weak limit $q\in E$ of $\{x_{m_n}\}$ such that $q\in\Gamma$.

Again, since $\{x_{m_n}\}$ is bounded, we can choose a sequence $\{x_{m_n}\}$ of $\{x_{m_n}\}$, subsequencing if necessary such that $x_{m_n}\to q$ as $n\to\infty$ and

$$\limsup_{n\to\infty}\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle=\lim_{n\to\infty}\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle.$$

Hence, from (2.5), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle=&\lim_{n\to\infty}\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle\\ \le&\langle Ju-Jx^*,q-x^*\rangle\le 0.\end{aligned} \tag{3.27}$$

From (3.25), we have

$$\begin{aligned}0\le&\phi(x^*,x_{m_n+1})-\phi(x^*,x_{m_n})\\ \le&(1-\alpha_{m_n})\phi(x^*,x_{m_n})\\ &+\alpha_{m_n}\left(\frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*,x_{m_n-1})+2\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle\right)-\phi(x^*,x_{m_n}).\end{aligned}$$

That is

$$\phi(x^*,x_{m_n})\le\frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*,x_{m_n-1})+2\langle Ju-Jx^*,x_{m_n+1}-x^*\rangle. \tag{3.28}$$

Hence, by condition $(B5)$ and (3.27), we obtain $\phi(x^*,x_{m_n})\to 0$ as $n\to\infty$ and Lemma 2.2 implies $\|x_{m_n}-x^*\|\to 0$ as $n\to\infty$. Consequently, $\|x_n-x^*\|\to 0$ as $n\to\infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $x^*=\Pi_\Gamma u$. □

4. Applications

In this section, we present some theoretical applications of our main result.

4.1. **Variational Inequalities Problem.** Suppose we define the f in $EP(C,f)$ (1.1), by:

$$f(x,y):=\begin{cases}\langle Ax,y-x\rangle, & \text{if } x,y\in C,\\ +\infty, & \text{otherwise,}\end{cases} \tag{4.1}$$

where $A : C \to E^*$ is a strongly γ-pseudomonotone mapping. Then $EP(C, f)$ (1.1) reduces to $VIP(C, A)$, that is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in C. \tag{4.2}$$

We denote the set of solution of (4.2) by $Sol(C, A)$. Recall an operator A is said to be strongly γ-pseudomonotone, if there exists $\gamma > 0$ such that for any $x, y \in C$

$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq \gamma\phi(y, x).$$

In this situation, Algorithm 3.4 when modified provides a new method for solving variational inequality problems and fixed point problem for a quasi-ϕ-nonexpansive mapping. We give the new method as follows.

Algorithm 4.1. $(TRSSEM)$ *for* $VIP(C, A)$

Step 0. Choose the sequences $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\} \subset (0, 1)$ satisfying Assumption 3.3, take $\eta, \rho \in (0, 1)$ and $\beta_0 > 0$. For $u \in C$, select initial points x_0 and x_1 in C. Set $n = 1$.

Step 1. For $i = 1, 2, \dots, m$, and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n \rangle \leq 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$. Compute

$$z_n = \Pi_{C_n} J^{-1}(Jw_n - \beta_n Aw_n), \tag{4.3}$$

where β_n is given by

$$\beta_{n+1} = \begin{cases} \min\left\{\beta_n, \frac{\mu\|w_n - z_n\|}{\|g(w_n) - g(z_n)\|}\right\}, & \text{if } g(w_n) \neq g(z_n), \\ \beta_n, & \text{otherwise.} \end{cases} \tag{4.4}$$

Step 2. If $w_n = z_n$ ($w_n \in Sol(C, A)$), then set $w_n = y_n$ and go to Step 3. Otherwise, compute the next iterate by

$$y_n = \Pi_{Q_n} J^{-1}(Jw_n - \beta_n Az_n), \tag{4.5}$$

where

$$Q_n = \{w \in E : \langle w - z_n, Jw_n - \beta_n Aw_n - Jz_n \rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = J^{-1}((1 - \alpha_n)Ju + \alpha_n(1 - \gamma_n)Jy_n + \gamma_n JSy_n). \tag{4.6}$$

Step 4. Set $n := n + 1$ and go to Step 1.

A convergence result for solving VIP(C,A) (4.2) is given below without proof.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, for $i = 1, 2, \ldots, m$. Let $A : C \to E^*$ be a strongly γ-pseudomonotone operator that is bounded on bounded sets, let $S : E \to E$ be a quasi-ϕ-nonexpansive mapping such that $\Gamma = \{Sol(C, A) \cap F(S)\} \neq \emptyset$. Let $\{\theta_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying Assumption* 3.3*, then the sequence $\{x_n\}$ generated by Algorithm* 4.1 *converges strongly to $p = \Pi_\Gamma u$, where Π_Γ is the projection of C onto Γ.*

4.2. **Fixed Point Problem (FPP).** Given a closed set $C \subset E$, a fixed point of a mapping $T : C \to C$ is any point $x^* \in C$ such that $x^* = Tx^*$. Finding a fixed point amounts to solving $EP(C, f)$ with

$$f(x, y) = \langle x - Tx, y - x \rangle, \quad \text{for all } y \in C.$$

In this case, we define the operator $T = I - A$, where I is the identity mapping on C and A is the operator defined in Subsection 4.1. The method and result given in 4.1, thus apply.

5. Conclusion

We considered an iterative approximation of a common solution of EP and FPP. We introduced a totally relaxed self adaptive inertial subgradient extragradient method, Mann and Halpern iterative technique for solving this problem in 2-uniformly convex Banach space, which is also uniformly smooth. Our method uses a carefully selected adaptive stepsize which does not depend on any Lipschitz-type condition neither does it require the knowledge of the Lipschitz constant of the gradient of pseudomonotone operator.

Acknowledgements. The authors sincerely thank the anonymous referee for his careful reading, constructive comments and fruitful suggestions that improved the manuscript. The first and second authors acknowledge with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The third author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

References

[1] H. A. Abass, C. Izuchukwu, O. T. Mewomo and Q. L. Dong, *Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach space*, Fixed Point Theory **21**(2) (2020), 397–411. https://doi.org/10.24193/fpt-ro.2020.2.28

[2] R. P. Agarwal, D. O'Regan and D. R. Saha, *Fixed Point Theory for Lipschitizian-Type Mappings with Applications*, Springer, New York, 2009. https://link.springer.com/book/10.1007/978-0-387-75818-3
[3] T. O. Alakoya, L. O. Jolaoso and O. T. Mewomo, *Strong convergence theorems for finite families of pseudomonotone equilibrium and fixed point problems in Banach spaces*, Afr. Mat. **32** (2021), 897–923. https://doi.org/10.1007/s13370-020-00869-z
[4] T. O. Alakoya and O. T. Mewomo, *Viscosity S-iteration method with inertial technique and self-adaptive step size for split variational inclusion, equilibrium and fixed point problems*, Comput. Appl. Math. (2021), Article ID 39. https://doi.org/10.1007/s40314-021-01749-3
[5] T. O. Alakoya, A. O. E. Owolabi and O. T. Mewomo, *An inertial algorithm with a self-adaptive step size for a split equilibrium problem and a fixed point problem of an infinite family of strict pseudo-contractions*, J. Nonlinear Var. Anal. **5** (2021), 803–829. https://doi.org/10.1080/02331934.2021.1895154
[6] T. O. Alakoya, A. Taiwo and O. T. Mewomo, *On system of split generalised mixed equilibrium and fixed point problems for multivalued mappings with no prior knowledge of operator norm*, Fixed Point Theory **23**(1) (2022), 45–74. https://doi.org/10.24193/fpt-ro.2022.1.04
[7] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: A. G. Kartsatos (Ed.), *Theory and Applications of Nonlinear Operators and Accretive and Monotone Type*, Lecture Notes in Pure and Applied Mathematics **178**, Dekker, New York, 1996, 15–50.
[8] P. N. Anh, *A hybrid extragradient method extended to fixed point problems and equilibrium problems*, Optimization **62** (2013), 271–283. https://doi.org/10.1080/02331934.2011.607497
[9] P. N. Anh and H. A. Le Thi, *An Armijo-type method for pseudomotone equilibrium problems and its applications*, J. Glob. Optim. **57** (2013), 803–820. https://doi.org/10.1007/s10898-012-9970-8
[10] P. N. Anh, *Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities*, J. Optim. Theory Appl. **154** (2012), 303–320. https://doi.org/10.1007/s10957-012-0005-x
[11] H. H. Bauschke and J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev. **38**(3) (1996), 367–426. https://doi.org/10.1137/S0036144593251710
[12] J. Y. Bello Cruz, P. S. M. Santos and S. Scheimberg, *A two-phase algorithm for a variational inequality formulation of equilibrium problems*, J. Optim. Theory Appl. **159** (2013), 562–575. https://doi.org/10.1007/s10957-012-0181-8
[13] G. Bigi, M. Castellani, M. Pappalardo and M. Passacantando, *Existence and solution methods for equilibra*, Eur. J. Oper. Res. **227** (2013), 1–11. https://doi.org/10.1016/j.ejor.2012.11.037
[14] E. Blum and W. Oetlli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.
[15] Y. Censor and A. Lent, *An iterative row-action method for interval complex programming*, J. Optim. Theory Appl. **34** (1981), 321–353. https://doi.org/10.1007/BF00934676
[16] Y. Censor and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996), 323–339. https://doi.org/10.1080/02331939608844225
[17] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **148** (2011), 318–335. https://doi.org/10.1007/s10957-010-9757-3
[18] I. Cioranescu, *Geometry of Banach Spaces*, Duality Mappings and Nonlinear, Kluwer, Dordrecht, 1990.
[19] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[20] P. Debnath, N. Konwar and S. Radenović, *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Science*, Springer Verlag, Singapore, 2021. `https://doi.org/10.1007/978-981-16-4896-0`

[21] P. M. Duc, L. D. Muu and N. V. Quy, *Solution-existence and algorithms with their convergence rate for strongly pseudomonotone equilibrium problems*, Pacific J. Optim. **12** (2016), 833–845.

[22] M. Fukushima, *A relaxed projection method for variational inequalities*, Math. Program **35** (1986), 58–70. `https://doi.org/10.1007/BF01589441`

[23] S. He, T. Wu, A. Gibali and Q.-L. Dong, *Totally self relaxed self-adaptive algorithm for solving variational inequalities over the intersection of sub-level sets*, Optimization **67**(9) (2018), 1487–1504. `https://doi.org/10.1080/02331934.2018.1476515`

[24] D. V. Hieu, *New extragradient method for a class of equilibrium problems*, App. Anal. **97**(5) (2018), 811–824. `https://doi.org/10.1080/00036811.2017.1292350`

[25] C. Izuchukwu, K. O. Aremu, O. K. Oyewole, O. T. Mewomo and S. H. Khan, *On mixed equilibrium problems in Hadamard spaces*, J. Math. **2019** (2019), Article ID 3210649, 13 pages. `https://doi.org/10.1155/2019/3210649`

[26] Z. Jouymandi and F. Moradlou, *Extragradient and linesearch algorithms for solving equilibrium problems and fixed point problems in Banach space*, Fixed Point Theory **20**(2) (2019), 523–539. `https://doi.org/10.24193/fpt-ro.2019.2.34`

[27] Z. Jouymandi and F. Moradlou, *Retraction algorithms for solving variational inequalities, pseudomonotone equilibrium problems and fixed point problems in Banach spaces*, Numer. Algorithms **78**(2) (2018), 1153–1182. `https://doi.org/10.1007/s11075-017-0417-7`

[28] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945. `https://doi.org/10.1137/S105262340139611X`

[29] G. Kassay, M. Miholca and N. T. Vinh, *Vector quasi-equilibrium problems for the sum of two multivalued mappings*, J. Optim. Theory Appl. **169** (2016), 424–442. `https://doi.org/10.1007/s10957-016-0919-9`

[30] S. H. Khan, T. O. Alakoya and O. T. Mewomo, *Relaxed projection methods with self-adaptive step size for solving variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings in Banach spaces*, Math. Comput. Appl. **25** (2020), Article ID 54, 25 pages. `https://doi.org/10.3390/mca25030054`

[31] P. E. Maingé, *Convergence theorem for inertial KM-type algorithms*, J. Comput. Appl. Math. **219** (2008), 223–236. `https://doi.org/10.1016/j.cam.2007.07.021`

[32] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Var. Anal. **16** (2008), 899–912. `https://doi.org/10.1007/s11228-008-0102-z`

[33] O. T. Mewomo and F. U. Ogbuisi, *Convergence analysis of iterative method for multiple set split feasibility problems in certain Banach spaces*, Quaest. Math. **41**(1) (2018), 129–148. `https://doi.org/10.2989/16073606.2017.1375569`

[34] O. T. Mewomo and O. K. Oyewole, *An iterative approximation of common solutions of split generalized vector mixed equilibrium problem and some certain optimization problems*, Demonstr. Math. **54**(1) (2021), 335–358. `https://doi.org/10.1515/dema-2021-0019`

[35] K. Nakajo, *Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces*, Appl. Math. Comput. **271** (2015), 251–258. `https://doi.org/10.1016/j.amc.2015.08.096`

[36] G. N. Ogwo, T. O. Alakoya and O. T. Mewomo, *Iterative algorithm with self-adaptive step size for approximating the common solution of variational inequality and fixed point problems*, Optimization (2021). `https://doi.org/10.1080/02331934.2021.1981897`

[37] G. N. Ogwo, T. O. Alakoya and O. T. Mewomo, *Inertial iterative method with self-adaptive step size for finite family of split monotone variational inclusion and fixed point problems in Banach spaces*, Demonstr. Math. (2021). `https://doi.org/10.1515/dema-2020-0119`

[38] G. N. Ogwo, C. Izuchukwu, Y. Shehu and O. T. Mewomo, *Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems*, J. Sci. Comput. (2021). https://doi.org/10.1007/s10915-021-01670-1

[39] M. A. Olona, T. O. Alakoya, A. O.-E. Owolabi and O. T. Mewomo, *Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings*, Demonstr. Math. **54** (2021), 47–67. https://doi.org/10.1515/dema-2021-0006

[40] M. A. Olona, T. O. Alakoya, A. O.-E. Owolabi and O. T. Mewomo, *Inertial algorithm for solving equilibrium, variational inclusion and fixed point problems for an infinite family of strictly pseudocontractive mappings*, J. Nonlinear Funct. Anal. **2021** (2021), Article ID 10, 21 pages. https://doi.org/10.23952/jnfa.2020.10

[41] A. O.-E. Owolabi, T. O. Alakoya, A. Taiwo and O. T. Mewomo, *A new inertial-projection algorithm for approximating common solution of variational inequality and fixed point problems of multivalued mappings*, Numer. Algebra Control Optim. (2021). https://doi.org/10.3934/naco.2021004

[42] O. K. Oyewole, H. A. Abass and O. T. Mewomo, *Strong convergence algorithm for a fixed point constraint split null point problem*, Rend. Circ. Mat. Palermo (2) **70**(1) (2021), 389–408. https://doi.org/10.1007/s12215-020-00505-6

[43] O. K. Oyewole, K. O. Aremu and O. T. Mewomo, *A multi step inertial algorithm for approximating a common solution of split generalized mixed equilibrium and minimization problems*, Ric. Mat. (2021). https://doi.org/10.1007/s11587-021-00624-x

[44] O. K. Oyewole, O. T. Mewomo, L. O. Jolaoso and S. H. Khan, *An extragradient algorithm for split generalized equilibrium problem and the set of fixed points of quasi-ϕ-nonexpansive mappings in Banach spaces*, Turkish J. Math. **44**(4) (2020), 1146–1170. https://1146âĂŞ1170.doi:10.3906/mat-1911-83

[45] S. Reich, *A weak convergence theorem for alternating method with Bregman distance*, in: A. G. Kartsatos (Ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Marcel Dekker, New York, 1996, 313–318.

[46] S. Saewan and P. Kumam, *A generalized f-projection method for countable families of weak relatively nonexpansive mappings and the system of generalized Ky Fan inequalities*, J. Glob. Optim. **56** (2012), 1–23. https://doi.org/10.1007/s10898-012-9922-3

[47] G. Stampacchia, *Forms bilineaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris **258** (1964), 4414–4416.

[48] A. Taiwo, T. O. Alakoya and O. T. Mewomo, *Strong convergence theorem for solving equilibrium problem and fixed point of relatively nonexpansive multi-valued mappings in a Banach space with applications*, Asian-Eur. J. Math. **14**(8) (2021), Article ID 2150137, 31 pages. https://doi.org/10.1142/S1793557121501370

[49] V. Todorčević, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*, Springer Nature, Switzerland AG, 2019. https://doi.org/10.1007/978-3-030-22591-9

[50] R. T. Rockfellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1977), 877–808. https://doi.org/10.1137/0314056

[51] G. C. Ugwunnadi, C. Izuchukwu and O. T. Mewomo, *On nonspreading-type mappings in Hadamard spaces*, Bol. Soc. Parana. Mat. **(3)39**(5) (2021), 175–197. https://doi.org/10.5269/bspm.41768

[52] V. A. Uzor, T. O. Alakoya and O. T. Mewomo, *Strong convergence of a self-adaptive inertial Tseng's extragradient method for pseudomonotone variational inequalities and fixed point problems*, Open Math. (2022). https://doi.org/10.1515/math-2022-0429

[53] N. T. Vinh and A. Gibali, *Gradient projection-type algorithms for solving equilibrium problems and its applications*, Comput. Appl. Math. **38** (2019), Article ID 119. https://doi.org/10.1007/240314-019-0894-5

[54] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Aust. Math. Soc. **65** (2002), 109–113. https://doi.org/10.1017/S0004972700020116

[55] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138. https://doi.org/10.1016/0362-546X(91)90200-K

[56] J. Yang and H. W. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algorithms **80** (2019), 741–752. https://doi.org/10.1007/s11075-018-0504-4

[57] F. Yang, L. Zhao and J. K. Kim, *Hybrid projection method for generalized mixed equilibrium problem and fixed point problem of infinite family of asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces*, Appl. Math. Comput. **218**(10) (2012), 6072–6082. https://doi.org/10.1016/j.amc.2011.11.091

[1]School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Durban, South Africa

[2]DSI-NRF Center of Excellence in Mathematical and Statistical Sciences,
(CoE-MaSS) Johannesburg, South Africa
Email address: [1]217079141@stu.ukzn.ac.za
Email address: [2]216075727@stu.ukzn.ac.za
Email address: [3]mewomoo@ukzn.ac.za

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 203–221.

CONSTRUCTING SYMMETRIC EQUALITY ALGEBRAS

RAJAB ALI BORZOOEI[1], MONA AALY KOLOGANI[2], MOHAMMAD ALI HASHEMI[3], AND ELAHE MOHAMMADZADEH[3]

Abstract. In this paper, we introduce the notion of strong fuzzy filter on hyper equality algebras and investigate some equivalence definitions of it. Then by using this notion we constructed a symmetric equality algebra and define a special form of classes. By using these, we define the concept of a fuzzy hyper congruence relation on hyper equality algebra and we prove that the quotient is made by it is an equality algebra. Also, by using a fuzzy equivalence relation on hyper equality, we introduce a fuzzy hyper congruence relation and prove that this fuzzy hyper congruence is regular and finally we prove that the quotient structure that is made by it is a symmetric hyper equality algebra.

1. Introduction

The motivation for introducing equality algebras came from EQ-algebras which are defined by Novák in [18]. In EQ-algebras, compared to equality algebras, there is an additional operation $\otimes$, called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and intention was to define a structure similar to EQ-algebras but without the product. This new logical algebra, the equality algebra, has two connectives, a meet operation and an equivalence, and a constant. Equality algebra is introduced by Jeni [9], and since then many mathematicians have studied this algebraic structure and it in various fields. For instance, Novák et al. in [18] introduced a closure operator in the equality algebra class, and investigated that under what condition an equality algebra is a BCK-algebra. Zebardast et al. in [23] investigated the relation among

Key words and phrases. (Hyper) Equality algebra, symmetric equality algebra, strong fuzzy filter, congruence relation.
2020 *Mathematics Subject Classification.* Primary: 03G25. Secondary: 06D35, 06B10.
DOI 10.46793/KgJMat2502.203B
Received: November 03, 2021.
Accepted: March 02, 2022.

equality algebras and other logical algebra for instance, hoop, residuated lattice and etc. Also, Zebardast et al. in [23] studied commutative equality algebras and considered characterizations of commutative equality algebras. For more study we suggest [5, 7, 8, 16, 22, 23].

The hyper structure theory (called also multialgebra) was introduced in 1934 by F. Marty [14] at the 8th congress of Scandinavian Mathematicians. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer science. In [15] Mittas et al. applied the hyperstructures to lattices and introduced the concepts of hyperlattice and superlattice. Many authors studied different aspects of semihypergroups, Borzooei et al. exerted hyper structuers to logical algebras and introduced some hyper logical algebras (see [1–4]). Hyper equality algebras are introduced and studied in [6, 12, 19] and authors provided many basic properties of this class of hyper algebras. Fuzzy type theory was developed by Novák in [17] as a fuzzy counterpart of the classical higher-order logic. Filters have momentous role in the perusing logical deductive systems and logical algebraic systems. The notion of filters on equality algebras is introduced by Jeni in [10]. Then some different kinds of filters on equality algebras are defined and studied, see [5, 22], for more details. Also, Zadeh [21], the idea of the fuzzy sets have been used to other algebraic structures by many mathematicians that we refer to [11, 13]. Fuzzy filters on equality algebras are defined recently in [20], where they have defined fuzzy congruences on equality algebras and have showed that there is one-to-one correspondence between fuzzy filters and fuzzy congruences.

In Section 2, we give some notions and statements of hyper equality algebras from [6] and we recall some facts about fuzzy set theories. In Section 3, we defined the notion of strong fuzzy filters on hyper equality algebras and investigate some properties of strong fuzzy filters on these algebras. Section 4, we introduce the concept of fuzzy hyper congruence on hyper equality algebras and we give a relation between strong fuzzy filters and fuzzy hyper congruence on hyper equality algebras.

2. Preliminaries

In this section, we present some of the main definitions and results of equality algebras used in this paper.

Let $\emptyset \neq \mathcal{L}$. Then a fuzzy subset of $\mathcal{L}$ is $\varsigma : \mathcal{L} \to [0,1]$, where for $\mathfrak{t} \in [0,1]$, the set $\varsigma_{\mathfrak{t}} = \{\mathfrak{x} \in \mathcal{L} \mid \varsigma(\mathfrak{x}) \succcurlyeq \mathfrak{t}\}$ is said to be a level subset of ς. We say ς satisfies the sup-property if for every $\emptyset \neq \mathcal{S} \subseteq \mathcal{L}$ there exists $\mathfrak{i} \in \mathcal{S}$, where $\varsigma(\mathfrak{i}) = \sup_{\mathfrak{x}\in\mathcal{S}} \varsigma(\mathfrak{x})$. The set of all fuzzy subsets of $\mathcal{L}$, is shown by $\mathcal{FS}(\mathcal{L})$. A function $\varrho : \mathcal{L} \times \mathcal{L} \to [0,1]$ is said to be a fuzzy relation on $\mathcal{L}$. Also, ϱ on $\mathcal{L}$ is said to be a fuzzy equivalence relation if for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$:

(i) $\varrho(\mathfrak{x}, \mathfrak{x}) = \bigvee_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{y}, \mathfrak{z})$ (fuzzy reflexive);
(ii) $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{y}, \mathfrak{x})$ (fuzzy symmetric);
(iii) $\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \barwedge \varrho(\mathfrak{z}, \mathfrak{y}))$ (fuzzy transitive).

An equality algebra $\mathcal{E} = \langle \mathcal{E}, \backsim, \overline{\wedge}, 1\rangle$ is an algebra of type $(2,2,0)$ such that, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{E}$, the following axioms are fulfilled:

($\mathcal{E}1$) $\langle \mathcal{E}, \overline{\wedge}, 1\rangle$ is a $\overline{\wedge}$-semilattice with top element 1;

($\mathcal{E}2$) $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$;

($\mathcal{E}3$) $\mathfrak{x} \backsim \mathfrak{x} = 1$;

($\mathcal{E}4$) $\mathfrak{x} \backsim 1 = \mathfrak{x}$;

($\mathcal{E}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ implies $\mathfrak{x} \backsim \mathfrak{z} \preccurlyeq \mathfrak{y} \backsim \mathfrak{z}$ and $\mathfrak{x} \backsim \mathfrak{z} \preccurlyeq \mathfrak{x} \backsim \mathfrak{y}$;

($\mathcal{E}6$) $\mathfrak{x} \backsim \mathfrak{y} \preccurlyeq (\mathfrak{x} \overline{\wedge} \mathfrak{z}) \backsim (\mathfrak{y} \overline{\wedge} \mathfrak{z})$;

($\mathcal{E}7$) $\mathfrak{x} \backsim \mathfrak{y} \preccurlyeq (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$.

From now, $\langle \mathcal{E}, \backsim, \overline{\wedge}, 1\rangle$ or $\mathcal{E}$ is an equality algebra.

Now, define two operations $\curvearrowright$ (implication) and $\leftrightarrows$ (equivalence operation) on $\mathcal{E}$ by $\mathfrak{x} \curvearrowright \mathfrak{y} = \mathfrak{x} \backsim (\mathfrak{x} \overline{\wedge} \mathfrak{y})$ and $\mathfrak{x} \leftrightarrows \mathfrak{y} = (\mathfrak{x} \curvearrowright \mathfrak{y}) \overline{\wedge} (\mathfrak{y} \curvearrowright \mathfrak{x})$ (see [9]).

Let $\mathcal{L}$ be a non-empty set. A function $\circ : \mathcal{L} \times \mathcal{L} \to P(\mathcal{L})^* = P(\mathcal{L}) \setminus \{\emptyset\}$ is a hyper operation on $\mathcal{L}$.

A hyper equality algebra $\mathcal{L} = \langle \mathcal{L}; \backsim, \overline{\wedge}, 1\rangle$ is a non-empty set $\mathcal{L}$ endowed with a binary operation $\overline{\wedge}$, a hyper operation $\backsim$ and a top element 1 where for each $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$:

($\mathcal{HE}1$) $\langle \mathcal{L}, \overline{\wedge}, 1\rangle$ is a meet-semilattice with top element 1;

($\mathcal{HE}2$) $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{y} \backsim \mathfrak{x}$;

($\mathcal{HE}3$) $1 \in \mathfrak{x} \backsim \mathfrak{x}$;

($\mathcal{HE}4$) $\mathfrak{x} \in 1 \backsim \mathfrak{x}$;

($\mathcal{HE}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ implies $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}$ and $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{x} \backsim \mathfrak{y}$;

($\mathcal{HE}6$) $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \overline{\wedge} \mathfrak{z}) \backsim (\mathfrak{y} \overline{\wedge} \mathfrak{z})$;

($\mathcal{HE}7$) $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$, where $\mathfrak{x} \preccurlyeq \mathfrak{y}$ iff $\mathfrak{x} \overline{\wedge} \mathfrak{y} = \mathfrak{x}$ and $\mathcal{S} \lll \mathcal{R}$ is defined by, for all $\mathfrak{x} \in \mathcal{S}$, there is $\mathfrak{y} \in \mathcal{R}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$.

Notation. Throughout of this paper, we suppose $\mathcal{L} = \langle \mathcal{L}; \backsim, \overline{\wedge}, 1\rangle$ or $\mathcal{L}$ is a hyper equality algebra, unless otherwise stated (see [6]).

Define two operations, the implication and the equivalence on $\langle \mathcal{L}, \backsim, \overline{\wedge}, 1\rangle$ [6], such that for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we have

$$\mathfrak{x} \curvearrowright \mathfrak{y} = \mathfrak{x} \backsim (\mathfrak{x} \overline{\wedge} \mathfrak{y}) \quad \text{and} \quad \mathfrak{x} \leftrightarrows \mathfrak{y} = (\mathfrak{x} \curvearrowright \mathfrak{y}) \overline{\wedge} (\mathfrak{y} \curvearrowright \mathfrak{x}).$$

Proposition 2.1 ([6]). *For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, the next results are equivalent:*

($\mathcal{HE}5$) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ *implies* $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}$ *and* $\mathfrak{x} \backsim \mathfrak{z} \lll \mathfrak{x} \backsim \mathfrak{y}$*;*

($\mathcal{HE}5a$) $\mathfrak{x} \backsim (\mathfrak{x} \overline{\wedge} \mathfrak{y} \overline{\wedge} \mathfrak{z}) \lll \mathfrak{x} \backsim (\mathfrak{x} \overline{\wedge} \mathfrak{y})$*;*

($\mathcal{HE}5b$) $\mathfrak{x} \curvearrowright (\mathfrak{y} \overline{\wedge} \mathfrak{z}) \lll \mathfrak{x} \curvearrowright \mathfrak{y}$.

Proposition 2.2 ([6]). *For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$ and $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have:*

(P1) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *and* $\mathfrak{y} \preccurlyeq \mathfrak{x}$ *imply* $\mathfrak{x} = \mathfrak{y}$*;*

(P2) $1 \in \mathfrak{x} \curvearrowright \mathfrak{x}$, $1 \in \mathfrak{x} \curvearrowright 1$, $\mathfrak{x} \lll \mathfrak{x} \backsim 1$, $\mathfrak{x} \in 1 \curvearrowright \mathfrak{x}$ *and* $1 \in \mathfrak{x} \leftrightarrows \mathfrak{x}$*;*

(P3) $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{x} \curvearrowright \mathfrak{y}$ *and* $\mathfrak{x} \backsim \mathfrak{y} \lll \mathfrak{y} \curvearrowright \mathfrak{x}$*;*

(P4) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $1 \in \mathfrak{x} \curvearrowright \mathfrak{y}$*;*

(P5) $\mathfrak{x} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$ *implies* $\mathfrak{z} \backsim \mathfrak{x} \lll \mathfrak{z} \backsim \mathfrak{y}$ *and* $\mathfrak{z} \backsim \mathfrak{x} \lll \mathfrak{y} \backsim \mathfrak{x}$*;*

(P6) $\mathfrak{x} \lll \mathfrak{y} \curvearrowright \mathfrak{x}$ *and* $\mathcal{S} \lll \mathcal{R} \curvearrowright \mathcal{S}$*;*

(P7) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $\mathfrak{z} \curvearrowright \mathfrak{x} \lll \mathfrak{z} \curvearrowright \mathfrak{y}$ *and* $\mathfrak{y} \curvearrowright \mathfrak{z} \lll \mathfrak{x} \curvearrowright \mathfrak{z}$*;*

(P8) $\mathcal{S} \lll \mathcal{R}$ *implies* $\mathcal{T} \curvearrowright \mathcal{S} \lll \mathcal{T} \curvearrowright \mathcal{R}$ *and* $\mathcal{R} \curvearrowright \mathcal{T} \lll \mathcal{S} \curvearrowright \mathcal{T}$;
(P9) $\mathfrak{x} \preccurlyeq \mathfrak{y}$ *implies* $\mathfrak{x} \lll \mathfrak{y} \backsim \mathfrak{x}$;
(P10) $\mathfrak{y} \lll (\mathfrak{x} \curvearrowright \mathfrak{y}) \curvearrowright \mathfrak{y}$;
(P11) $\mathfrak{x} \curvearrowright \mathfrak{y} \lll (\mathfrak{y} \curvearrowright \mathfrak{z}) \curvearrowright (\mathfrak{x} \curvearrowright \mathfrak{z})$;
(P12) $\mathfrak{x} \curvearrowright (\mathfrak{y} \barwedge \mathfrak{z}) \lll (\mathfrak{x} \barwedge \mathfrak{z}) \curvearrowright \mathfrak{y}$.

Consider $\emptyset \neq \mathcal{G} \subseteq \mathcal{L}$ such that, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, if $\mathfrak{x} \in \mathcal{G}$ and $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\mathfrak{y} \in \mathcal{G}$. Thus, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{G}$, $\mathcal{G}$ is called a

(WF) weak filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $\mathfrak{x} \backsim \mathfrak{y} \subseteq \mathcal{G}$ imply $\mathfrak{y} \in \mathcal{G}$;
(F) filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $\mathcal{G} \lll \mathfrak{x} \backsim \mathfrak{y}$ imply $\mathfrak{y} \in \mathcal{G}$;
(SF) strong filter of $\mathcal{L}$ if $\mathfrak{x} \in \mathcal{G}$ and $(\mathfrak{x} \backsim \mathfrak{y}) \cap \mathcal{G} \neq \emptyset$ imply $\mathfrak{y} \in \mathcal{G}$.

Clearly, if $\mathcal{G}$ is one of the above stated cases, then $1 \in \mathcal{G}$. Also, any strong filter of $\mathcal{L}$ is a (weak) filter but the converse is not true (see [6, Remark 1, Examples 10, 11]). $\mathcal{L}$ is a symmetric if $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$, where $=$ denotes the equality between subsets of $\mathcal{L}$. $\mathcal{L}$ is separated if for every $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $1 \in \mathfrak{x} \backsim \mathfrak{y}$, then $\mathfrak{x} = \mathfrak{y}$. $\mathcal{L}$ is good if for each $\mathfrak{x} \in \mathcal{L}$, $\mathfrak{x} = 1 \backsim \mathfrak{x}$ (see [6]).

Assume θ is an equivalence relation on $\mathcal{L}$. For any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, $\mathcal{S}\overline{\theta}\mathcal{R}$ means, for any $\mathfrak{i} \in \mathcal{S}$, there exists $\mathfrak{h} \in \mathcal{R}$ such that $\mathfrak{i}\theta\mathfrak{h}$ and for any $\mathfrak{h} \in \mathcal{R}$ there exists $\mathfrak{i} \in \mathcal{S}$ such that $\mathfrak{i}\theta\mathfrak{h}$, and $\mathcal{S}\overline{\overline{\theta}}\mathcal{R}$ means, for any $\mathfrak{i} \in \mathcal{S}$ and any $\mathfrak{h} \in \mathcal{R}$, $\mathfrak{i}\theta\mathfrak{h}$. Moreover, θ is called a congruence relation on $\mathcal{L}$ if, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$, $\mathfrak{x}\theta\mathfrak{y}$ and $\mathfrak{u}\theta\mathfrak{v}$ imply $(\mathfrak{x} \backsim \mathfrak{u})\ \overline{\theta}\ (\mathfrak{y} \backsim \mathfrak{v})$ and $(\mathfrak{x} \barwedge \mathfrak{u})\ \theta\ (\mathfrak{y} \barwedge \mathfrak{v})$. Also, θ is called a strong congruence relation if for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$, $\mathfrak{x}\theta\mathfrak{y}$ and $\mathfrak{u}\theta\mathfrak{v}$ imply $(\mathfrak{x} \backsim \mathfrak{u})\ \overline{\overline{\theta}}\ (\mathfrak{y} \backsim \mathfrak{v})$ and $(\mathfrak{x} \barwedge \mathfrak{u})\ \theta\ (\mathfrak{y} \barwedge \mathfrak{v})$ (see [6, Definition 11]).

3. Strong Fuzzy Filters of Hyper Equality Algebras

In this section, we define the notion of strong fuzzy filter on hyper equality algebras and we characterize it. By using this notion we define a congruence relation on hyper equality algebra and prove that the quotient that is made by this is an equality algebra.

Definition 3.1. Let $\vartheta \in \mathcal{FS}(\mathcal{L})$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ if for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$
(FF1) $\vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i})\right) \preccurlyeq \vartheta(\mathfrak{y})$;
(FF2) if $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\vartheta(\mathfrak{x}) \preccurlyeq \vartheta(\mathfrak{y})$.
Note. For any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, the above relations are equivalent with the following statements:
(FF1′) $(\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i})) \barwedge (\sup_{\mathfrak{o} \in \mathcal{S} \backsim \mathcal{R}} \vartheta(\mathfrak{o})) \preccurlyeq \sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})$;
(FF2′) If $\mathcal{S} \lll \mathcal{R}$, then $\sup_{\mathfrak{i} \in \mathcal{S}} \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{h} \in \mathcal{R}} \vartheta(\mathfrak{h})$.

Example 3.1. (*i*) Assume $\mathcal{L} = [0, 1]$. For each $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we define the operations $\backsim$ and $\barwedge$ on $\mathcal{L}$ by $\mathfrak{x} \barwedge \mathfrak{y} = \min\{\mathfrak{x}, \mathfrak{y}\}$ and $\mathfrak{x} \backsim \mathfrak{y} = \{0, 1 - |\mathfrak{x} - \mathfrak{y}|\}$. Then $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1 \rangle$ is a hyper equality algebra. Define $\vartheta : \mathcal{L} \to [0, 1]$ by $\vartheta(1) = \beta$ and for any $\mathfrak{x} < 1$, $\vartheta(\mathfrak{x}) = \alpha$, where $0 < \alpha < \beta < 1$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$.

(*ii*) If $\mathcal{L} = \{0, \mathfrak{i}, 1\}$ such that $0 \preccurlyeq \mathfrak{i} \preccurlyeq 1$, then, for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, define

$$\mathfrak{x}\mathbin{\bar\wedge}\mathfrak{y}=\min\{\mathfrak{x},\mathfrak{y}\} \quad \text{and} \quad \begin{array}{c|ccc} \backsim & 0 & \mathfrak{i} & 1 \\ \hline 0 & \{1\} & \{0,\mathfrak{i}\} & \{0,\mathfrak{i}\} \\ \mathfrak{i} & \{0,\mathfrak{i}\} & \{1\} & \{\mathfrak{i}\} \\ 1 & \{0,\mathfrak{i}\} & \{0,\mathfrak{i}\} & \{1\} \end{array}.$$

Then $\mathcal{L}=\langle\mathcal{L};\backsim,\bar\wedge,1\rangle$ is a hyper equality algebra. Suppose $\vartheta:\mathcal{L}\to[0,1]$ is defined by $\vartheta(1)=\beta$ and $\vartheta(0)=\vartheta(\mathfrak{i})=\alpha$, where $0<\alpha<\beta<1$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$.

Theorem 3.1. *Consider $\vartheta\in\mathcal{FS}(\mathcal{L})$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ which satisfies the sup-property if and only if for all $\mathfrak{t}\in[0,1]$, $\vartheta_{\mathfrak{t}}\neq\emptyset$ is a strong filter of $\mathcal{L}$.*

Proof. Suppose ϑ is a strong fuzzy filter of $\mathcal{L}$ and $\mathfrak{t}\in[0,1]$ such that $\vartheta_{\mathfrak{t}}\neq\emptyset$. Then $\mathfrak{x}\in\vartheta_{\mathfrak{t}}$. Assume $\mathfrak{y}\in\mathcal{L}$ such that $\mathfrak{x}\preccurlyeq\mathfrak{y}$. Since ϑ is strong, by (FF2) we have $\mathfrak{t}\preccurlyeq\vartheta(\mathfrak{x})\preccurlyeq\vartheta(\mathfrak{y})$ and so $\mathfrak{y}\in\vartheta_{\mathfrak{t}}$. Now, suppose for any $\mathfrak{y}\in\mathcal{L}$, $(\mathfrak{x}\backsim\mathfrak{y})\cap\vartheta_{\mathfrak{t}}\neq\emptyset$. Thus, there is $\mathfrak{z}\in(\mathfrak{x}\backsim\mathfrak{y})\cap\vartheta_{\mathfrak{t}}$ where $\vartheta(\mathfrak{z})\succcurlyeq\mathfrak{t}$. By (FF1), since $\vartheta(\mathfrak{z})\preccurlyeq\sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{z})$, we have

$$\mathfrak{t}\preccurlyeq\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\vartheta(\mathfrak{z})\preccurlyeq\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\left(\sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{y}).$$

Hence, $\mathfrak{y}\in\vartheta_{\mathfrak{t}}$. Therefore, $\vartheta_{\mathfrak{t}}$ is a strong filter of $\mathcal{L}$.

Conversely, suppose $\mathfrak{x}\in\mathcal{L}$. Clearly, $\mathfrak{x}\in\vartheta_{\vartheta(\mathfrak{x})}$ and so $\vartheta_{\vartheta(\mathfrak{x})}\neq\emptyset$. If for $\mathfrak{y}\in\mathcal{L}$, $\mathfrak{x}\preccurlyeq\mathfrak{y}$, from $\vartheta_{\vartheta(\mathfrak{x})}$ is strong, then $\mathfrak{y}\in\vartheta_{\vartheta(\mathfrak{x})}$. Thus, $\vartheta(\mathfrak{x})\preccurlyeq\vartheta(\mathfrak{y})$ and so $(FF2)$ holds. Consider $\mathfrak{t}_1,\mathfrak{t}_2\in[0,1]$ and $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$ such that $\vartheta(\mathfrak{x})=\mathfrak{t}_1$ and $\mathfrak{t}_2=\sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{i})$. Suppose $\mathfrak{s}=\min\{\mathfrak{t}_1,\mathfrak{t}_2\}$. Since $\mathcal{L}$ satisfies the sup-property, there exists $\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}$ such that $\vartheta(\mathfrak{z})=\mathfrak{t}_2$ and so $\mathfrak{s}\preccurlyeq\vartheta(\mathfrak{z})$. Thus, $\mathfrak{z}\in(\mathfrak{x}\backsim\mathfrak{y})\cap\vartheta_{\mathfrak{s}}$. Since $\mathfrak{x}\in\vartheta_{\mathfrak{s}}$, $(\mathfrak{x}\backsim\mathfrak{y})\cap\vartheta_{\mathfrak{s}}\neq\emptyset$ and $\vartheta_{\mathfrak{s}}$ is a strong filter of $\mathcal{L}$, we get $\mathfrak{y}\in\vartheta_{\mathfrak{s}}$. Hence,

$$\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\left(\sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{i})\right)=\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\vartheta(\mathfrak{z})=\min\{\mathfrak{t}_1,\mathfrak{t}_2\}=\mathfrak{s}\preccurlyeq\vartheta(\mathfrak{y}).$$

Therefore, ϑ is a strong fuzzy filter of $\mathcal{L}$. □

Example 3.2. Consider Example 3.1 (*ii*) and let $\alpha=0.3$ and $\beta=0.7$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$ and $\vartheta_\alpha=\{0,\mathfrak{i},1\}$ and $\vartheta_\beta=\{1\}$. Obviously, ϑ_β and ϑ_α are strong filters of $\mathcal{L}$.

Theorem 3.2. *Assume $\vartheta\in\mathcal{FS}(\mathcal{L})$. Then, for each $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$ and $\mathcal{S},\mathcal{R},\mathcal{T}\subseteq\mathcal{L}$, the following statements are equivalent:*

(i) *ϑ is a strong fuzzy filter of $\mathcal{L}$;*
(ii) *$\vartheta(\mathfrak{x})\preccurlyeq\vartheta(1)$ and $\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\left(\sup_{\mathfrak{z}\in\mathfrak{x}\frown\mathfrak{y}}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{y})$;*
(iii) *if $\mathcal{S}\lll\mathcal{R}\curvearrowright\mathcal{T}$, then for all $\mathfrak{x}\in\mathcal{S}$, there exists $\mathfrak{y}\in\mathcal{R}$ and $\mathfrak{z}\in\mathcal{T}$ such that*

$$\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\vartheta(\mathfrak{y})\preccurlyeq\vartheta(\mathfrak{z}),\quad (\sup_{\mathfrak{i}\in\mathcal{S}}\vartheta(\mathfrak{i}))\mathbin{\bar\wedge}\left(\sup_{\mathfrak{h}\in\mathcal{R}}\vartheta(\mathfrak{h})\right)\preccurlyeq\sup_{\mathfrak{o}\in\mathcal{T}}\vartheta(\mathfrak{o}).$$

Proof. (i)⇒(ii) By (FF2) for any $\mathfrak{x}\in\mathcal{L}$ since $\mathfrak{x}\preccurlyeq 1$, we have $\vartheta(\mathfrak{x})\preccurlyeq\vartheta(1)$ and for any $\mathfrak{x}\in\mathcal{S}$ and $\mathfrak{y}\in\mathcal{R}$

$$\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\left(\sup_{\mathfrak{z}\in\mathfrak{x}\frown\mathfrak{y}}\vartheta(\mathfrak{z})\right)=\vartheta(\mathfrak{x})\mathbin{\bar\wedge}\left(\sup_{\mathfrak{z}\in\mathfrak{x}\backsim(\mathfrak{x}\bar\wedge\mathfrak{y})}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{x}\mathbin{\bar\wedge}\mathfrak{y})\preccurlyeq\vartheta(\mathfrak{y}).$$

(ii)⇒(i) Suppose $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$ such that $\mathfrak{x}\preccurlyeq\mathfrak{y}$. Then by Proposition 2.2 (P4), $1\in\mathfrak{x}\curvearrowright\mathfrak{y}$. Since, for any $\mathfrak{x}\in\mathcal{L}$, $\vartheta(\mathfrak{x})\preccurlyeq\vartheta(1)$, we have $\sup_{\mathfrak{z}\in\mathfrak{x}\curvearrowright\mathfrak{y}}\vartheta(\mathfrak{z})=\vartheta(1)$. Then by (ii) we have

$$\vartheta(\mathfrak{x})=\vartheta(\mathfrak{x})\barwedge\vartheta(1)=\vartheta(\mathfrak{x})\barwedge\left(\sup_{\mathfrak{z}\in\mathfrak{x}\curvearrowright\mathfrak{y}}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{y}).$$

By (ii) and Proposition 2.2 (P3), for any $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$, we have

$$\vartheta(\mathfrak{x})\barwedge\left(\sup_{\mathfrak{z}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{x})\barwedge\left(\sup_{\mathfrak{z}\in\mathfrak{x}\curvearrowright\mathfrak{y}}\vartheta(\mathfrak{z})\right)\preccurlyeq\vartheta(\mathfrak{y}).$$

Hence, ϑ is a strong fuzzy filter of $\mathcal{L}$.

(ii)⇒(iii) Consider $\mathcal{S}\lll\mathcal{R}\curvearrowright\mathcal{T}$. Then, for any $\mathfrak{x}\in\mathcal{S}$, there exists $\mathfrak{v}\in\mathcal{R}\curvearrowright\mathcal{T}$ such that $\mathfrak{x}\preccurlyeq\mathfrak{v}$. Since $\mathfrak{v}\in\mathcal{R}\curvearrowright\mathcal{T}$, there are $\mathfrak{y}\in\mathcal{R}$ and $\mathfrak{z}\in\mathcal{T}$ such that $\mathfrak{v}\in\mathfrak{y}\curvearrowright\mathfrak{z}$ and so $\mathfrak{x}\lll\mathfrak{y}\curvearrowright\mathfrak{z}$. From $\mathfrak{x}\lll\mathfrak{y}\curvearrowright\mathfrak{z}$, there exists $\mathfrak{w}\in\mathfrak{y}\curvearrowright\mathfrak{z}$ such that $\mathfrak{x}\preccurlyeq\mathfrak{w}$. Then $\vartheta(\mathfrak{x})\preccurlyeq\vartheta(\mathfrak{w})$, and so $\vartheta(\mathfrak{x})\preccurlyeq\sup_{\mathfrak{w}\in\mathfrak{y}\curvearrowright\mathfrak{z}}\vartheta(\mathfrak{w})$. Hence,

$$\vartheta(\mathfrak{x})\barwedge\vartheta(\mathfrak{y})\preccurlyeq\left(\sup_{\mathfrak{w}\in\mathfrak{y}\curvearrowright\mathfrak{z}}\vartheta(\mathfrak{w})\right)\barwedge\vartheta(\mathfrak{y})\preccurlyeq\vartheta(\mathfrak{z}).$$

Moreover,

$$\sup_{\mathfrak{i}\in\mathcal{S}}\ \vartheta(\mathfrak{i})\preccurlyeq\sup_{\mathfrak{w}\in\mathfrak{y}\curvearrowright\mathfrak{z}}\ \vartheta(\mathfrak{w})\preccurlyeq\sup_{\mathfrak{v}\in\mathcal{R}\curvearrowright\mathcal{T}}\ \vartheta(\mathfrak{v}).$$

Then, by (FF1′) and (FF2′), we get

$$\begin{aligned}(\sup_{\mathfrak{i}\in\mathcal{S}}\ \vartheta(\mathfrak{i}))\barwedge\left(\sup_{\mathfrak{h}\in\mathcal{R}}\ \vartheta(\mathfrak{h})\right)&\preccurlyeq\left(\sup_{\mathfrak{w}\in\mathfrak{y}\curvearrowright\mathfrak{z}}\ \vartheta(\mathfrak{w})\right)\barwedge\left(\sup_{\mathfrak{h}\in\mathcal{R}}\ \vartheta(\mathfrak{h})\right)\\&\preccurlyeq(\sup_{\mathfrak{v}\in\mathcal{R}\curvearrowright\mathcal{T}}\ \vartheta(\mathfrak{v}))\barwedge\left(\sup_{\mathfrak{h}\in\mathcal{R}}\ \vartheta(\mathfrak{h})\right)\\&=\left(\sup_{\mathfrak{v}\in\mathcal{R}\backsim(\mathcal{R}\barwedge\mathcal{T})}\ \vartheta(\mathfrak{v})\right)\barwedge\left(\sup_{\mathfrak{h}\in\mathcal{R}}\ \vartheta(\mathfrak{h})\right)\\&\preccurlyeq\left(\sup_{\mathfrak{u}\in\mathcal{R}\barwedge\mathcal{T}}\ \vartheta(\mathfrak{u})\right)\preccurlyeq\sup_{\mathfrak{o}\in\mathcal{T}}\ \vartheta(\mathfrak{o}).\end{aligned}\tag{3.1}$$

(iii)⇒(ii) Since for any $\mathfrak{x}\in\mathcal{L}$, $\mathfrak{x}\preccurlyeq 1$ by Proposition 2.2 (P9), $\mathfrak{x}\lll 1\backsim\mathfrak{x}$. Then by Proposition 2.2 (P3), $\mathfrak{x}\lll 1\backsim\mathfrak{x}\lll\mathfrak{x}\curvearrowright 1$. Now, by (iii) for all $\mathfrak{x}\in\mathcal{L}$, $\vartheta(\mathfrak{x})=\vartheta(\mathfrak{x})\barwedge\vartheta(\mathfrak{x})\preccurlyeq\vartheta(1)$. Also, since, for any $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$, $\mathfrak{x}\curvearrowright\mathfrak{y}\lll\mathfrak{x}\curvearrowright\mathfrak{y}$, by (iii) (indeed by (3.1)), we have $\left(\sup_{\mathfrak{z}\in\mathfrak{x}\curvearrowright\mathfrak{y}}\ \vartheta(\mathfrak{z})\right)\barwedge\vartheta(\mathfrak{x})\preccurlyeq\vartheta(\mathfrak{y})$. □

Example 3.3. Consider Example 3.1 (*ii*) and let $\alpha=0.3$ and $\beta=0.7$. Then ϑ is a strong fuzzy filter of $\mathcal{L}$. Clearly, Theorem 3.2 holds. For instance, $\vartheta(0)=\vartheta(\mathfrak{i})\preccurlyeq\vartheta(1)$ and

$$\vartheta(0)\barwedge\left(\sup_{\mathfrak{z}\in 0\curvearrowright\mathfrak{i}}\vartheta(\mathfrak{z})\right)=\vartheta(0)\barwedge\left(\sup_{\mathfrak{z}\in 0\backsim(0\barwedge\mathfrak{i})=0\backsim 0=\{1\}}\vartheta(\mathfrak{z})\right)=\vartheta(0)\barwedge\vartheta(1)\preccurlyeq\vartheta(\mathfrak{i}).$$

Assume $\vartheta\in\mathcal{FS}(\mathcal{L})$. For any $\mathcal{S}\subseteq\mathcal{L}$, we define a map $\varpi^{\mathcal{S}}:\mathcal{L}\to[0,1]$ by $\varpi^{\mathcal{S}}(\mathfrak{x})=\sup_{\mathfrak{z}\in\mathcal{S}\backsim\mathfrak{x}}\ \vartheta(\mathfrak{z})$, for any $\mathfrak{x}\in\mathcal{L}$. In particular, $\varpi^{\mathfrak{y}}(\mathfrak{x})=\sup_{\mathfrak{z}\in\mathfrak{y}\backsim\mathfrak{x}}\ \vartheta(\mathfrak{z})$ and for any $\mathcal{S},\mathcal{R}\subseteq\mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{R})=\sup_{\mathfrak{x}\in\mathcal{R}}\ \varpi^{\mathcal{S}}(\mathfrak{x})=\sup_{\mathfrak{z}\in\mathcal{S}\backsim\mathcal{R}}\ \vartheta(\mathfrak{z})$. If for all $\mathfrak{z}\in\mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{z})\preccurlyeq\varpi^{\mathfrak{y}}(\mathfrak{z})$, then we denote it by $\varpi^{\mathfrak{x}}\preccurlyeq\varpi^{\mathfrak{y}}$.

Proposition 3.1. *Suppose ϑ is a strong fuzzy filter of $\mathcal{L}$. If for $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$, $\varpi^{\mathfrak{x}}=\varpi^{\mathfrak{y}}$, then $\vartheta(\mathfrak{x})=\vartheta(\mathfrak{y})$.*

Proof. Let $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$, for $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$. Then by Proposition 3.2 (i), $\sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z}) = \varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$. From ϑ is strong, $\vartheta(\mathfrak{x}), \vartheta(\mathfrak{y}) \preccurlyeq \vartheta(1)$ and $\vartheta(\mathfrak{x}) \barwedge \left(\sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z})\right) \preccurlyeq \vartheta(\mathfrak{y})$. Thus, $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{x}) \barwedge \vartheta(1) \preccurlyeq \vartheta(\mathfrak{y})$. Similarly, $\vartheta(\mathfrak{y}) \preccurlyeq \vartheta(\mathfrak{x})$. Hence, $\vartheta(\mathfrak{x}) = \vartheta(\mathfrak{y})$. □

Example 3.4. Let $\mathcal{L} = \{0, \mathfrak{i}, \mathfrak{h}, 1\}$ be a set. Define the operation $\barwedge$ and $\backsim$ on $\mathcal{L}$ as follows:

$\backsim$	0	$\mathfrak{i}$	$\mathfrak{h}$	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{0, \mathfrak{i}\}$
$\mathfrak{i}$	$\{1\}$	$\{1\}$	$\{\mathfrak{i}, 1\}$	$\{\mathfrak{i}\}$
$\mathfrak{h}$	$\{\mathfrak{h}, 1\}$	$\{\mathfrak{i}, 1\}$	$\{1\}$	$\{\mathfrak{h}, 1\}$
1	$\{0, \mathfrak{i}\}$	$\{\mathfrak{i}\}$	$\{\mathfrak{h}\}$	$\{1\}$

,

$\barwedge$	0	$\mathfrak{i}$	$\mathfrak{h}$	1
0	0	0	0	0
$\mathfrak{i}$	0	$\mathfrak{i}$	0	$\mathfrak{i}$
$\mathfrak{h}$	0	0	$\mathfrak{h}$	$\mathfrak{h}$
1	0	$\mathfrak{i}$	$\mathfrak{h}$	1

.

Define ϑ on $\mathcal{L}$ by $\vartheta(0) = \vartheta(\alpha) = \vartheta(\mathfrak{h}) = \alpha$ and $\vartheta(1) = \beta$, where $0 \prec \alpha \prec \beta \prec 1$. Clearly, $\varpi^{\mathfrak{i}} = \varpi^{0}$, then $\vartheta(\mathfrak{i}) = \vartheta(0)$. But the converse may not be true, since $\vartheta(\mathfrak{i}) = \vartheta(\mathfrak{h})$ but

$$\alpha = \vartheta(\mathfrak{i}) = \varpi^{\mathfrak{i}}(1) \neq \varpi^{\mathfrak{h}}(1) = \sup\{\vartheta(\mathfrak{h}), \vartheta(1)\} = \vartheta(1) = \beta.$$

Proposition 3.2. *Consider ϑ is a strong fuzzy filter of $\mathcal{L}$. Then for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v} \in \mathcal{L}$ and $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$, we have:*

(i) *$\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ if and only if $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$;*
(ii) *$\varpi^{\mathcal{S}} = \varpi^{\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}(\mathcal{R}) = \vartheta(1)$;*
(iii) *if $\varpi^{\mathfrak{x}} = \vartheta$, then $\vartheta(\mathfrak{x}) = \vartheta(1)$;*
(iv) *if $\mathfrak{x} \preccurlyeq \mathfrak{y}$, then $\vartheta(\mathfrak{x}) \preccurlyeq \varpi^{\mathfrak{y}}(\mathfrak{x})$;*
(v) *if $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}} = \varpi^{\mathfrak{v}}$, then $\varpi^{\mathfrak{x} \backsim \mathfrak{u}} = \varpi^{\mathfrak{y} \backsim \mathfrak{u}}$, $\varpi^{\mathfrak{u} \backsim \mathfrak{y}} = \varpi^{\mathfrak{v} \backsim \mathfrak{y}}$ and $\varpi^{\mathfrak{u} \barwedge \mathfrak{y}} = \varpi^{\mathfrak{y} \barwedge \mathfrak{v}}$;*
(vi) *if $\mathfrak{y} \backsim \mathfrak{u} = \mathfrak{u} \backsim \mathfrak{y}$ and $\mathfrak{y} \backsim \mathfrak{v} = \mathfrak{v} \backsim \mathfrak{y}$, then $\varpi^{\mathfrak{x} \backsim \mathfrak{u}} = \varpi^{\mathfrak{y} \backsim \mathfrak{v}}$.*

Proof. (i) Suppose $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$. Then for all $\mathfrak{z} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{z}) = \varpi^{\mathfrak{y}}(\mathfrak{z})$. Consider $\mathfrak{y} = \mathfrak{z}$. Then $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \varpi^{\mathfrak{y}}(\mathfrak{y}) = \sup_{\mathfrak{z} \in \mathfrak{y} \backsim \mathfrak{y}} \vartheta(\mathfrak{z})$. By ($\mathcal{HE}3$), $1 \in \mathfrak{y} \backsim \mathfrak{y}$ and so $\sup_{\mathfrak{z} \in \mathfrak{y} \backsim \mathfrak{y}} \vartheta(\mathfrak{z}) = \vartheta(1)$. Hence, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$.

Conversely, assume that for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$. By ($\mathcal{HE}2$) and ($\mathcal{HE}7$), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z}) \lll (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})$. Then for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$, there exists $\mathfrak{h} \in (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})$ such that $\mathfrak{i} \preccurlyeq \mathfrak{h}$. Since ϑ is a strong fuzzy filter of $\mathcal{L}$, by (FF2), for any $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$, $\vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h})$ and so $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i}) \preccurlyeq \vartheta(\mathfrak{h}) \preccurlyeq \sup_{\mathfrak{h} \in (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})} \vartheta(\mathfrak{h})$. Then

$$\begin{aligned} \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \varpi^{\mathfrak{x}}(\mathfrak{y}) = \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \left(\sup_{\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{i})\right) &\preccurlyeq \left(\sup_{\mathfrak{u} \in \mathfrak{y} \backsim \mathfrak{z}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{h} \in (\mathfrak{y} \backsim \mathfrak{z}) \backsim (\mathfrak{x} \backsim \mathfrak{z})} \vartheta(\mathfrak{h})\right) \\ &\preccurlyeq \sup_{\mathfrak{v} \in \mathfrak{x} \backsim \mathfrak{z}} \vartheta(\mathfrak{v}) = \varpi^{\mathfrak{x}}(\mathfrak{z}). \end{aligned}$$

Since for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \vartheta(1)$, by above relation, we have

$$\varpi^{\mathfrak{y}}(\mathfrak{z}) = \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \vartheta(1) = \varpi^{\mathfrak{y}}(\mathfrak{z}) \barwedge \varpi^{\mathfrak{x}}(\mathfrak{y}) \preccurlyeq \varpi^{\mathfrak{x}}(\mathfrak{z}).$$

This shows that $\varpi^{\mathfrak{y}} \preccurlyeq \varpi^{\mathfrak{x}}$. By the similar way, we have $\varpi^{\mathfrak{x}} \preccurlyeq \varpi^{\mathfrak{y}}$. Hence, $\varpi^{\mathfrak{x}} = \varpi^{\mathfrak{y}}$.

(ii) Similar to (i).

(iii) Suppose $\varpi^{\mathfrak{x}} = \vartheta$. Then for any $\mathfrak{y} \in \mathcal{L}$, $\varpi^{\mathfrak{x}}(\mathfrak{y}) = \sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}} \vartheta(\mathfrak{z}) = \vartheta(\mathfrak{y})$. Let $\mathfrak{x} = \mathfrak{y}$. Since by ($\mathcal{HE}3$), $1 \in \mathfrak{x} \backsim \mathfrak{x}$, we get $\vartheta(\mathfrak{x}) = \varpi^{\mathfrak{x}}(\mathfrak{x}) = \sup_{\mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{x}} \vartheta(\mathfrak{z}) = \vartheta(1)$.

(iv) Assume $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$ where $\mathfrak{x}\preccurlyeq\mathfrak{y}$. By Proposition 2.2 (P9), $\mathfrak{x}\lll\mathfrak{y}\backsim\mathfrak{x}$. Then there is $\mathfrak{i}\in\mathfrak{y}\backsim\mathfrak{x}$ such that $\mathfrak{x}\preccurlyeq\mathfrak{i}$. Since ϑ is strong, by (FF2) we have

$$\vartheta(\mathfrak{x})\preccurlyeq\vartheta(\mathfrak{i})\preccurlyeq\sup_{\mathfrak{i}\in\mathfrak{y}\backsim\mathfrak{x}}\vartheta(\mathfrak{i})=\varpi^{\mathfrak{y}}(\mathfrak{x}).$$

(v) If $\varpi^{\mathfrak{x}}=\varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}}=\varpi^{\mathfrak{v}}$, then by (i), $\varpi^{\mathfrak{x}}(\mathfrak{y})=\vartheta(1)=\varpi^{\mathfrak{u}}(\mathfrak{v})$. By $(\mathcal{HE}7)$,

$$\mathfrak{x}\backsim\mathfrak{y}\lll(\mathfrak{x}\backsim\mathfrak{u})\backsim(\mathfrak{y}\backsim\mathfrak{u}),\quad \mathfrak{u}\backsim\mathfrak{v}\lll(\mathfrak{u}\backsim\mathfrak{y})\backsim(\mathfrak{v}\backsim\mathfrak{y}).$$

Thus, for any $\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}$, there exists $\mathfrak{h}\in(\mathfrak{x}\backsim\mathfrak{u})\backsim(\mathfrak{y}\backsim\mathfrak{u})$ such that $\mathfrak{i}\preccurlyeq\mathfrak{h}$. From ϑ is strong, by (FF2) we have $\vartheta(\mathfrak{i})\preccurlyeq\vartheta(\mathfrak{h})$ and so

$$\vartheta(1)=\varpi^{\mathfrak{x}}(\mathfrak{y})=\sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{i})\preccurlyeq\vartheta(\mathfrak{h})\preccurlyeq\sup_{\mathfrak{h}\in(\mathfrak{x}\backsim\mathfrak{u})\backsim(\mathfrak{y}\backsim\mathfrak{u})}\vartheta(\mathfrak{h}).$$

Hence, $\varpi^{\mathfrak{x}\backsim\mathfrak{u}}(\mathfrak{y}\backsim\mathfrak{u})=\vartheta(1)$. Now, by (i), we have $\varpi^{\mathfrak{x}\backsim\mathfrak{u}}=\varpi^{\mathfrak{y}\backsim\mathfrak{u}}$. Similarly, $\varpi^{\mathfrak{u}\backsim\mathfrak{y}}=\varpi^{\mathfrak{v}\backsim\mathfrak{y}}$. Moreover, by $(\mathcal{HE}6)$, $\mathfrak{x}\backsim\mathfrak{y}\lll(\mathfrak{x}\,\bar{\wedge}\,\mathfrak{u})\backsim(\mathfrak{y}\,\bar{\wedge}\,\mathfrak{u})$ and $\mathfrak{u}\backsim\mathfrak{v}\lll(\mathfrak{u}\,\bar{\wedge}\,\mathfrak{y})\backsim(\mathfrak{v}\,\bar{\wedge}\,\mathfrak{y})$. Then for any $\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}$ there is $\mathfrak{h}\in(\mathfrak{x}\,\bar{\wedge}\,\mathfrak{u})\backsim(\mathfrak{y}\,\bar{\wedge}\,\mathfrak{u})$ such that $\mathfrak{i}\preccurlyeq\mathfrak{h}$. Since ϑ is strong, by (FF2), we have $\vartheta(\mathfrak{i})\preccurlyeq\vartheta(\mathfrak{h})$ and so

$$\vartheta(1)=\varpi^{\mathfrak{x}}(\mathfrak{y})=\sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{y}}\vartheta(\mathfrak{i})\preccurlyeq\vartheta(\mathfrak{h})\preccurlyeq\sup_{\mathfrak{h}\in(\mathfrak{x}\bar{\wedge}\mathfrak{u})\backsim(\mathfrak{y}\bar{\wedge}\mathfrak{u})}\vartheta(\mathfrak{h}).$$

Hence, $\varpi^{\mathfrak{x}\bar{\wedge}\mathfrak{u}}(\mathfrak{y}\,\bar{\wedge}\,\mathfrak{u})=\vartheta(1)$. Now, by (i), we have $\varpi^{\mathfrak{x}\bar{\wedge}\mathfrak{u}}=\varpi^{\mathfrak{y}\bar{\wedge}\mathfrak{u}}$.

(vi) Similar to (v). □

Corollary 3.1. *Let $\mathcal{L}=\langle\mathcal{L};\backsim,\bar{\wedge},1\rangle$ be symmetric and ϑ be a strong fuzzy filter of $\mathcal{L}$. If for all $\mathfrak{x},\mathfrak{y},\mathfrak{u},\mathfrak{v}\in\mathcal{L}$, $\varpi^{\mathfrak{x}}=\varpi^{\mathfrak{y}}$ and $\varpi^{\mathfrak{u}}=\varpi^{\mathfrak{v}}$, then $\varpi^{\mathfrak{x}\backsim\mathfrak{u}}=\varpi^{\mathfrak{y}\backsim\mathfrak{v}}$ and $\varpi^{\mathfrak{u}\bar{\wedge}\mathfrak{y}}=\varpi^{\mathfrak{y}\bar{\wedge}\mathfrak{v}}$.*

Note. Consider ϑ is a strong fuzzy filter of $\mathcal{L}$ and

$$\mathcal{L}/\varpi=\{\varpi^{S}\mid\mathcal{S}\subseteq\mathcal{L}\}.$$

For any $\varpi^{\mathcal{S}},\varpi^{\mathcal{R}}\in\mathcal{L}/\varpi$, we consider the operations $\overline{\backsim}$ and $\overline{\bar{\wedge}}$ on $\mathcal{L}/\varpi$ as follow:

$$\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{R}}=\varpi^{\mathcal{S}\backsim\mathcal{R}}\quad\text{and}\quad\varpi^{\mathcal{S}}\overline{\bar{\wedge}}\varpi^{\mathcal{R}}=\varpi^{\mathcal{S}\bar{\wedge}\mathcal{R}},$$

where $\mathcal{S}\backsim\mathcal{R}=\bigcup_{\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}}\mathfrak{i}\backsim\mathfrak{h}$ and $\mathcal{S}\,\bar{\wedge}\,\mathcal{R}=\{\mathfrak{i}\,\bar{\wedge}\,\mathfrak{h}\mid\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}\}$. Also, we consider $\varpi^{1}=\varpi^{\mathcal{L}}$. Now, we prove that these operations are well-defined. Assume $\varpi^{\mathcal{S}},\varpi^{\mathcal{R}},\varpi^{\mathcal{T}}\in\mathcal{L}/\varpi$ such that $\varpi^{\mathcal{S}}=\varpi^{\mathcal{R}}$. Then by Proposition 3.2 (ii), $\varpi^{\mathcal{S}}(\mathcal{R})=\vartheta(1)$, and so $\sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}}\vartheta(\alpha)=\vartheta(1)$. At first we prove $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{T}}=\varpi^{\mathcal{R}}\overline{\backsim}\varpi^{\mathcal{T}}$. For this, by Proposition 3.2 (ii), we show $\varpi^{\mathcal{S}\backsim\mathcal{T}}(\mathcal{R}\backsim\mathcal{T})=\omega(1)$. By definition, for any $\alpha\in\mathcal{S}\backsim\mathcal{R}$, there exists $\mathfrak{i}\in\mathcal{S}$ and $\mathfrak{h}\in\mathcal{R}$ such that $\alpha\in\mathfrak{i}\backsim\mathfrak{h}$. By $(\mathcal{HE}7)$, for any $\mathfrak{o}\in\mathcal{T}$, we get $\mathfrak{i}\backsim\mathfrak{h}\lll(\mathfrak{i}\backsim\mathfrak{o})\backsim(\mathfrak{h}\backsim\mathfrak{o})$. So, for any $\alpha\in\mathfrak{i}\backsim\mathfrak{h}$, there exists $\beta\in(\mathfrak{i}\backsim\mathfrak{o})\backsim(\mathfrak{h}\backsim\mathfrak{o})$ such that $\alpha\preccurlyeq\beta$. Since ϑ is strong, we have $\vartheta(\alpha)\preccurlyeq\vartheta(\beta)$, and so $\sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}}\vartheta(\alpha)\preccurlyeq\vartheta(\beta)$. Then

$$\varpi^{\mathcal{S}\backsim\mathcal{T}}(\mathcal{R}\backsim\mathcal{T})=\sup_{\mathfrak{z}\in(\mathcal{S}\backsim\mathcal{T})\backsim(\mathcal{R}\backsim\mathcal{T})}\vartheta(\mathfrak{z})\succcurlyeq\vartheta(\beta)\succcurlyeq\sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}}\vartheta(\alpha)\succcurlyeq\vartheta(1).$$

Hence, $\varpi^{\mathcal{S}\backsim\mathcal{T}}(\mathcal{R}\backsim\mathcal{T})=\vartheta(1)$. By Proposition 3.2 (ii), $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{T}}=\varpi^{\mathcal{R}}\overline{\backsim}\varpi^{\mathcal{T}}$.

Now, we prove that $\varpi^{\mathcal{S}}\overline{\bar{\wedge}}\varpi^{\mathcal{T}}=\varpi^{\mathcal{R}}\overline{\bar{\wedge}}\varpi^{\mathcal{T}}$. For this, by Proposition 3.2 (ii), we show $\varpi^{\mathcal{S}\bar{\wedge}\mathcal{T}}(\mathcal{R}\,\bar{\wedge}\,\mathcal{T})=\omega(1)$. By definition, for any $\alpha\in\mathcal{S}\backsim\mathcal{R}$, there exists $\mathfrak{i}\in\mathcal{S}$ and $\mathfrak{h}\in\mathcal{R}$ such that $\alpha\in\mathfrak{i}\backsim\mathfrak{h}$. By $(\mathcal{HE}6)$, for any $\mathfrak{o}\in\mathcal{T}$, we have $\mathfrak{i}\backsim\mathfrak{h}\lll(\mathfrak{i}\,\bar{\wedge}\,\mathfrak{o})\backsim(\mathfrak{h}\,\bar{\wedge}\,\mathfrak{o})$.

So for any $\alpha \in \mathfrak{i} \backsim \mathfrak{h}$, there exists $\beta \in (\mathfrak{i} \barwedge \mathfrak{o}) \backsim (\mathfrak{h} \barwedge \mathfrak{o})$ such that $\alpha \preccurlyeq \beta$. Since ϑ is strong, we obtain $\vartheta(\alpha) \preccurlyeq \vartheta(\beta)$, and so $\sup_{\alpha\in \mathcal{S}\backsim\mathcal{R}} \vartheta(\alpha) \preccurlyeq \vartheta(\beta)$. Then

$$\varpi^{\mathcal{S}\barwedge\mathcal{T}}(\mathcal{R} \barwedge \mathcal{T}) = \sup_{\mathfrak{z}\in(\mathcal{S}\barwedge\mathcal{T})\backsim(\mathcal{R}\barwedge\mathcal{T})} \vartheta(\mathfrak{z}) \succcurlyeq \vartheta(\beta) \succcurlyeq \sup_{\alpha\in\mathcal{S}\backsim\mathcal{R}} \vartheta(\alpha) \succcurlyeq \vartheta(1).$$

Hence, $\varpi^{\mathcal{S}\barwedge\mathcal{T}}(\mathcal{R} \barwedge \mathcal{T}) = \vartheta(1)$. By Proposition 3.2 (ii), $\varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{T}} = \varpi^{\mathcal{R}}\overline{\barwedge}\varpi^{\mathcal{T}}$. Therefore, these operations are well-defined.

Now, suppose $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$. Then the relation $\preccurlyeq_{\varpi}$ on $\mathcal{L}/\varpi$ by $\varpi^{\mathcal{S}} \preccurlyeq_{\varpi} \varpi^{\mathcal{R}}$ if and only if for any $\mathcal{T} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{T}) \preccurlyeq \varpi^{\mathcal{R}}(\mathcal{T})$, is an order on $\mathcal{L}/\varpi$. By routine calculation, it is easy to see that $\varpi^{\mathcal{S}} \preccurlyeq_{\varpi} \varpi^{\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}}$ if and only if $\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}\barwedge\mathcal{R}}$ if and only if $\varpi^{\mathcal{S}}(\mathcal{S} \barwedge \mathcal{R}) = \vartheta(1)$ (by Proposition 3.2 (ii)) if and only if $\sup_{\mathfrak{h}\in\mathcal{S}\backsim(\mathcal{S}\barwedge\mathcal{R})} \vartheta(\mathfrak{h}) = \vartheta(1)$ if and only if $\sup_{\mathfrak{h}\in\mathcal{S}\frown\mathcal{R}} \vartheta(\mathfrak{h}) = \vartheta(1)$.

Theorem 3.3. *Let $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ be symmetric and ϑ be a strong fuzzy filter of $\mathcal{L}$. Then $\mathcal{L}/\varpi = \langle \mathcal{L}/\varpi; \overline{\backsim}, \overline{\barwedge}, \varpi^1\rangle$ is a symmetric equality algebra.*

Proof. We prove that $\overline{\backsim}$ and $\overline{\barwedge}$ are well-defined. Clearly, $(\mathcal{L}/\varpi, \preccurlyeq_{\varpi})$ is a poset. Now, we show that $\mathcal{L}/\varpi = \langle \mathcal{L}/\varpi; \overline{\backsim}, \overline{\barwedge}, \varpi^1\rangle$ is an equality algebra.

We have to prove that for any $\mathcal{S} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}} \preccurlyeq_{\varpi} \varpi^{\mathcal{L}}$. For this, suppose $\mathcal{R} \subseteq \mathcal{L}$. Then $\varpi^{\mathcal{L}}(\mathcal{R}) = \sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathcal{R}} \vartheta(\mathfrak{z})$. Since $\mathcal{L} \backsim \mathcal{R} = \bigcup_{\mathfrak{g}\in\mathcal{L},\mathfrak{h}\in\mathcal{R}} \mathfrak{g} \backsim \mathfrak{h}$ and $\mathcal{R} \subseteq \mathcal{L}$, we get

$$1 \in \mathfrak{h} \backsim \mathfrak{h} \in \bigcup_{\mathfrak{g}\in\mathcal{L},\mathfrak{h}\in\mathcal{R}} \mathfrak{g} \backsim \mathfrak{h} = \mathcal{L} \backsim \mathcal{R}.$$

Then $\varpi^{\mathcal{L}}(\mathcal{R}) = \vartheta(1)$. Moreover, since for any $\mathfrak{z} \in \mathcal{S} \backsim \mathcal{R}$, $\vartheta(\mathfrak{z}) \preccurlyeq \vartheta(1)$, we obtain $\varpi^{\mathcal{S}}(\mathcal{R}) = \sup_{\mathfrak{z}\in\mathcal{S}\backsim\mathcal{R}} \vartheta(\mathfrak{z}) \preccurlyeq \vartheta(1) = \varpi^{\mathcal{L}}(\mathcal{R})$. Hence for any $\mathcal{R} \subseteq \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathcal{R}) \preccurlyeq \varpi^{\mathcal{L}}(\mathcal{R})$, and so $\varpi^{\mathcal{S}} \preccurlyeq_{\varpi} \varpi^{\mathcal{L}}$.

($\mathcal{E}2$) Since $\mathcal{L}$ is symmetric, for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\mathfrak{x} \backsim \mathfrak{y} = \mathfrak{y} \backsim \mathfrak{x}$. Consider $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$. Then

$$\mathcal{S} \backsim \mathcal{R} = \bigcup_{\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}} \mathfrak{i} \backsim \mathfrak{h} = \bigcup_{\mathfrak{h}\in\mathcal{R},\mathfrak{i}\in\mathcal{S}} \mathfrak{h} \backsim \mathfrak{i} = \mathcal{R} \backsim \mathcal{S}.$$

Then $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} = \varpi^{\mathcal{R}\backsim\mathcal{S}} = \varpi^{\mathcal{R}}\overline{\backsim}\varpi^{\mathcal{S}}$.

($\mathcal{E}3$) Assume $\mathcal{S} \subseteq \mathcal{L}$. Then for any $\mathfrak{i} \in \mathcal{S}$ we have

$$1 \in 1 \backsim \mathfrak{i} \in (\mathfrak{i} \backsim \mathfrak{i}) \backsim \mathfrak{i} \in \bigcup_{\mathfrak{g}\in\mathcal{L}\ \ \mathfrak{i},\mathfrak{h}\in\mathcal{S}} (\mathfrak{i} \backsim \mathfrak{h}) \backsim \mathfrak{g} = (\mathcal{S} \backsim \mathcal{S}) \backsim \mathcal{L}.$$

Then $\varpi^{\mathcal{S}\backsim\mathcal{S}}(\mathcal{L}) = \sup_{\mathfrak{i}\in(\mathcal{S}\backsim\mathcal{S})\backsim\mathcal{L}} \vartheta(\mathfrak{i}) = \vartheta(1)$. Hence, by Proposition 3.2 (ii), $\varpi^{\mathcal{S}\backsim\mathcal{S}} = \varpi^{\mathcal{L}}$. Therefore, $\varpi^{\mathcal{S}}\overline{\backsim}\varpi^{\mathcal{S}} = \varpi^{\mathcal{L}}$.

($\mathcal{E}4$) Similar to ($\mathcal{E}3$), suppose $\mathcal{S} \subseteq \mathcal{L}$. Then for any $\mathfrak{i} \in \mathcal{S}$ we have

$$1 \in 1 \backsim \mathfrak{i} \in (\mathfrak{i} \backsim \mathfrak{i}) \backsim \mathfrak{i} \in \bigcup_{\mathfrak{g}\in\mathcal{L}\ \ \mathfrak{i},\mathfrak{h}\in\mathcal{S}} (\mathfrak{g} \backsim \mathfrak{i}) \backsim \mathfrak{h} = (\mathcal{L} \backsim \mathcal{S}) \backsim \mathcal{S}.$$

Then $\varpi^{\mathcal{L}\backsim\mathcal{S}}(\mathcal{S}) = \sup_{\mathfrak{i}\in(\mathcal{L}\backsim\mathcal{S})\backsim\mathcal{S}} \vartheta(\mathfrak{i}) = \vartheta(1)$. Hence, by Proposition 3.2 (ii), $\varpi^{\mathcal{L}\backsim\mathcal{S}} = \varpi^{\mathcal{S}}$. Therefore, $\varpi^{\mathcal{L}}\overline{\backsim}\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}}$.

($\mathcal{E}5$) Let $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$ such that $\varpi^{\mathcal{S}} \preccurlyeq_{\varpi} \varpi^{\mathcal{R}} \preccurlyeq_{\varpi} \varpi^{\mathcal{T}}$. Then

$$\varpi^{\mathcal{S}} = \varpi^{\mathcal{S}}\overline{\barwedge}\varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\barwedge\mathcal{R}}, \quad \varpi^{\mathcal{R}\barwedge\mathcal{T}} = \varpi^{\mathcal{R}}\overline{\barwedge}\varpi^{\mathcal{T}} = \varpi^{\mathcal{R}}.$$

Suppose $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$. Then there exist $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ where $\mathfrak{x} \in \mathfrak{i} \backsim \mathfrak{h}$. By Proposition 2.2 (P3), $\mathfrak{i} \backsim \mathfrak{h} \lll \mathfrak{i} \curvearrowright \mathfrak{h}$, then there exists $\mathfrak{y} \in \mathfrak{i} \curvearrowright \mathfrak{h}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Since

$$\mathfrak{i} \curvearrowright \mathfrak{h} \in \bigcup\nolimits_{\mathfrak{i}\in\mathcal{S},\mathfrak{h}\in\mathcal{R}} \mathfrak{i} \curvearrowright \mathfrak{h} = \mathcal{S} \curvearrowright \mathcal{R},$$

we get for any $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{y} \in \mathcal{S} \curvearrowright \mathcal{R}$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Hence, $\mathcal{S} \backsim \mathcal{R} \lll \mathcal{S} \curvearrowright \mathcal{R}$. By using this method, Proposition 3.2 (iv) and Proposition 2.2 (P12), we can prove that

$$\mathcal{T} \curvearrowright (\mathcal{S} \barwedge \mathcal{R}) \lll (\mathcal{T} \barwedge \mathcal{R}) \curvearrowright \mathcal{R} = (\mathcal{T} \barwedge \mathcal{R}) \backsim (\mathcal{S} \barwedge \mathcal{R} \barwedge \mathcal{T}).$$

Then

$$\begin{aligned}
\varpi^{\mathcal{S}\backsim\mathcal{T}} &= \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})\backsim\mathcal{T}} = \varpi^{\mathcal{T}\backsim(\mathcal{S}\barwedge\mathcal{R})} \\
&\lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright(\mathcal{S}\barwedge\mathcal{R})} \lll_{\varpi} \varpi^{(\mathcal{T}\barwedge\mathcal{R})\curvearrowright\mathcal{S}} = \varpi^{(\mathcal{T}\barwedge\mathcal{R})\backsim(\mathcal{S}\barwedge\mathcal{R}\barwedge\mathcal{T})} \\
&= \varpi^{(\mathcal{T}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{(\mathcal{S}\barwedge\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{S}} = \varpi^{\mathcal{R}\backsim\mathcal{S}} = \varpi^{\mathcal{S}\backsim\mathcal{R}}.
\end{aligned}$$

Hence, $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} \preccurlyeq_{\varpi} \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}}$. Again, since $\varpi^{\mathcal{S}\barwedge\mathcal{R}} = \varpi^{\mathcal{S}}$, similar to the above proof, by Propositions 3.2 (iv), 2.2 (P3) and 2.1 (HE5b), we have

$$\begin{aligned}
\varpi^{\mathcal{S}\backsim\mathcal{T}} &= \varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})} \overline{\backsim} \varpi^{\mathcal{T}} = \varpi^{(\mathcal{S}\barwedge\mathcal{R})\backsim\mathcal{T}} = \varpi^{\mathcal{T}\backsim(\mathcal{R}\barwedge\mathcal{S})} \\
&\lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright(\mathcal{R}\barwedge\mathcal{S})} \lll_{\varpi} \varpi^{\mathcal{T}\curvearrowright\mathcal{R}} \\
&= \varpi^{\mathcal{T}\backsim(\mathcal{T}\barwedge\mathcal{R})} = \varpi^{\mathcal{T}} \overline{\backsim} \varpi^{(\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{T}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{R}\backsim\mathcal{T}}.
\end{aligned}$$

Hence, $\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}} \preccurlyeq_{\varpi} \varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}$.

($\mathcal{E}6$) For all $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have $\mathcal{S} \backsim \mathcal{R} \lll (\mathcal{S} \barwedge \mathcal{T}) \backsim (\mathcal{R} \barwedge \mathcal{T})$. Because if $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, then there exist $\mathfrak{i} \in \mathcal{S}$ and $\mathfrak{h} \in \mathcal{R}$ such that $\mathfrak{x} \in \mathfrak{i} \backsim \mathfrak{h}$. Thus for any $\mathfrak{o} \in \mathcal{T}$, by ($\mathcal{HE}6$), $\mathfrak{y} \in (\mathfrak{i} \barwedge \mathfrak{o}) \backsim (\mathfrak{h} \barwedge \mathfrak{o})$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Hence for any $\mathfrak{x} \in \mathcal{S} \backsim \mathcal{R}$, there exists $\mathfrak{y} \in (\mathcal{S} \barwedge \mathcal{T}) \backsim (\mathcal{R} \barwedge \mathcal{T})$ such that $\mathfrak{x} \preccurlyeq \mathfrak{y}$. Then, for any $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have

$$\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} \preccurlyeq_{\varpi} \varpi^{(\mathcal{S}\barwedge\mathcal{T})\backsim(\mathcal{R}\barwedge\mathcal{T})} = \varpi^{\mathcal{S}\barwedge\mathcal{T}} \overline{\backsim} \varpi^{\mathcal{R}\barwedge\mathcal{T}} = \left(\varpi^{\mathcal{S}} \overline{\barwedge} \varpi^{\mathcal{T}}\right) \overline{\backsim} \left(\varpi^{\mathcal{R}} \overline{\barwedge} \varpi^{\mathcal{T}}\right).$$

($\mathcal{E}7$) Similar to the proof of ($\mathcal{E}6$), for any $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, by ($\mathcal{HE}7$),

$$\mathcal{S} \backsim \mathcal{R} \lll (\mathcal{S} \backsim \mathcal{T}) \backsim (\mathcal{R} \backsim \mathcal{T}).$$

Then

$$\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}} = \varpi^{\mathcal{S}\backsim\mathcal{R}} \preccurlyeq_{\varpi} \varpi^{(\mathcal{S}\backsim\mathcal{T})\backsim(\mathcal{R}\backsim\mathcal{T})} = \varpi^{(\mathcal{S}\backsim\mathcal{T})} \overline{\backsim} \varpi^{(\mathcal{R}\backsim\mathcal{T})} = \left(\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{T}}\right) \overline{\backsim} \left(\varpi^{\mathcal{R}} \overline{\backsim} \varpi^{\mathcal{T}}\right).$$

Therefore, $\mathcal{L}/\varpi$ is an equality algebra. □

In Theorem 3.3, the condition symmetric is essential, because we need it for ($\mathcal{E}5$) and in the absence of this assumption the axiom ($\mathcal{E}5$) does not hold.

Example 3.5. Let $\mathcal{L} = \{0, \mathfrak{o}, \mathfrak{i}, \mathfrak{h}, 1\}$ be a set with the following operations:

$\backsim$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1
0	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\mathfrak{o}$	$\{0\}$	$\{0,1\}$	$\{0,\mathfrak{h}\}$	$\{0,\mathfrak{i}\}$	$\{\mathfrak{o}\}$
$\mathfrak{i}$	$\{0\}$	$\{0,\mathfrak{h}\}$	$\{\mathfrak{i},1\}$	$\{0,\mathfrak{o}\}$	$\{\mathfrak{i}\}$
$\mathfrak{h}$	$\{0\}$	$\{0,\mathfrak{i}\}$	$\{0,\mathfrak{o}\}$	$\{1\}$	$\{\mathfrak{h}\}$
1	$\{0\}$	$\{\mathfrak{o}\}$	$\{\mathfrak{i}\}$	$\{\mathfrak{h}\}$	$\{1\}$

,

$\overline{\wedge}$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1
0	0	0	0	0	0
$\mathfrak{o}$	0	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{o}$
$\mathfrak{i}$	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{o}$	$\mathfrak{i}$
$\mathfrak{h}$	0	$\mathfrak{o}$	$\mathfrak{o}$	$\mathfrak{h}$	$\mathfrak{h}$
1	0	$\mathfrak{o}$	$\mathfrak{i}$	$\mathfrak{h}$	1

.

Define ϑ on $\mathcal{L}$ by $\vartheta(0) = \vartheta(\mathfrak{o}) \preccurlyeq \vartheta(\mathfrak{i}), \vartheta(\mathfrak{h}) \preccurlyeq \vartheta(1)$. By routine calculation, we have $\mathcal{L}/\varpi = \{\varpi^0, \varpi^{\mathfrak{o}}, \varpi^{\mathfrak{i}}, \varpi^{\mathfrak{h}}, \varpi^1\}$ which is a symmetric equality algebra.

Theorem 3.4. *Let $\mathcal{L} = \langle \mathcal{L}; \backsim, \overline{\wedge}, 1\rangle$ be symmetric such that for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, $(\mathfrak{x} \backsim \mathfrak{y}) \backsim \mathfrak{z} = (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$ and ϑ be a strong fuzzy filter of $\mathcal{L}$. Then there exists a strong fuzzy filter ϵ on $\mathcal{L}/\varpi$ such that $\epsilon \circ \pi \succcurlyeq \vartheta$, where π is the canonical epimorphism.*

Proof. Define $\epsilon : \mathcal{L}/\varpi \to [0,1]$, for any $\mathcal{S} \subseteq \mathcal{L}$, by $\epsilon(\varpi^{\mathcal{S}}) = \sup_{\mathfrak{z}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{z})$. First we prove that ϵ is well defined. Assume $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$ such that $\varpi^{\mathcal{S}} = \varpi^{\mathcal{R}}$. Then for any $\mathfrak{z} \in \mathcal{L}$, $\varpi^{\mathcal{S}}(\mathfrak{z}) = \varpi^{\mathcal{R}}(\mathfrak{z})$. Thus,

$$\epsilon(\varpi^{\mathcal{S}}) = \sup_{\mathfrak{z}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{z}) = \sup_{\mathfrak{z}\in\mathcal{L}} \varpi^{\mathcal{R}}(\mathfrak{z}) = \epsilon(\varpi^{\mathcal{R}}).$$

Since for any $\mathfrak{i} \in \mathcal{L}$, $1 \in \mathfrak{i} \backsim \mathfrak{i}$, we have $\varpi^{\mathcal{L}}(\mathfrak{i}) = \sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathfrak{i}} \vartheta(\mathfrak{z}) = \vartheta(1)$. Suppose $\mathcal{S} \subseteq \mathcal{L}$. Then

$$\epsilon(\varpi^{\mathcal{L}}) = \sup_{\mathfrak{i}\in\mathcal{L}} \varpi^{\mathcal{L}}(\mathfrak{i}) = \sup_{\mathfrak{i}\in\mathcal{L}} \sup_{\mathfrak{z}\in\mathcal{L}\backsim\mathfrak{i}} \vartheta(\mathfrak{z}) = \sup_{\mathfrak{i}\in\mathcal{L}} \vartheta(1) = \vartheta(1) \tag{3.2}$$

$$\succcurlyeq \sup_{\mathfrak{y}\in\mathcal{L}} \sup_{\mathfrak{h}\in\mathcal{S}\backsim\mathfrak{y}} \vartheta(\mathfrak{h}) = \sup_{\mathfrak{y}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{y}) = \epsilon(\varpi^{\mathcal{S}}). \tag{3.3}$$

Since for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, $(\mathfrak{x} \backsim \mathfrak{y}) \backsim \mathfrak{z} = (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$, obviously, for any $\mathcal{S}, \mathcal{R}, \mathcal{T} \subseteq \mathcal{L}$, we have $(\mathcal{S} \backsim \mathcal{R}) \backsim \mathcal{T} = (\mathcal{S} \backsim \mathcal{T}) \backsim (\mathcal{R} \backsim \mathcal{T})$. Moreover, from ϑ is a strong fuzzy filter of $\mathcal{L}$, by $(FF1')$ we have

$$\begin{aligned}
\epsilon(\varpi^{\mathcal{R}}) &= \sup_{\mathfrak{v}\in\mathcal{L}} \varpi^{\mathcal{R}}(\mathfrak{v}) = \sup_{\mathfrak{v}\in\mathcal{L}} \Big(\sup_{\mathfrak{h}\in\mathcal{R}\backsim\mathfrak{v}} \vartheta(\mathfrak{h})\Big) \\
&\succcurlyeq \sup_{\mathfrak{v}\in\mathcal{L}} \Big((\sup_{\mathfrak{i}\in\mathcal{S}\backsim\mathfrak{v}} \vartheta(\mathfrak{i})) \,\overline{\wedge}\, \Big(\sup_{\mathfrak{h}\in(\mathcal{S}\backsim\mathfrak{v})\backsim(\mathcal{R}\backsim\mathfrak{v})} \vartheta(\mathfrak{h})\Big)\Big) \\
&= \sup_{\mathfrak{v}\in\mathcal{L}} \Big((\sup_{\mathfrak{i}\in\mathcal{S}\backsim\mathfrak{v}} \vartheta(\mathfrak{i})) \,\overline{\wedge}\, \Big(\sup_{\mathfrak{h}\in(\mathcal{S}\backsim\mathcal{R})\backsim\mathfrak{v}} \vartheta(\mathfrak{h})\Big)\Big) \\
&\succcurlyeq \sup_{\mathfrak{v}\in\mathcal{L}} \Big(\varpi^{\mathcal{S}}(\mathfrak{v}) \,\overline{\wedge}\, \varpi^{\mathcal{S}\backsim\mathcal{R}}(\mathfrak{v})\Big) \\
&= \Big(\sup_{\mathfrak{v}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{v})\Big) \,\overline{\wedge}\, \Big(\sup_{\mathfrak{v}\in\mathcal{L}} \varpi^{\mathcal{S}\backsim\mathcal{R}}(\mathfrak{v})\Big) = \epsilon(\varpi^{\mathcal{S}}) \,\overline{\wedge}\, \epsilon(\varpi^{\mathcal{S}\backsim\mathcal{R}}) \\
&= \epsilon(\varpi^{\mathcal{S}}) \,\overline{\wedge}\, \epsilon(\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}}).
\end{aligned}$$

Hence, for any $\mathcal{S}, \mathcal{R} \subseteq \mathcal{L}$,

$$\epsilon(\varpi^{\mathcal{R}}) \succcurlyeq \epsilon(\varpi^{\mathcal{S}}) \,\overline{\wedge}\, \epsilon(\varpi^{\mathcal{S}} \overline{\backsim} \varpi^{\mathcal{R}}). \tag{3.4}$$

Thus, by (3.2), (3.4) and Theorem 3.2, we have ϵ is a strong fuzzy filter of $\mathcal{L}/\varpi$. Consider $\mathcal{S} \subseteq \mathcal{L}$. Define $\vartheta(\mathcal{S}) = \sup_{\mathfrak{z}\in\mathcal{S}} \vartheta(\mathfrak{z})$. Since $\mathcal{L}$ is symmetric, by Proposition 2.2

(P6), we get

$$\begin{aligned}\epsilon\circ\pi(\mathcal{S}) =&\epsilon(\varpi^{\mathcal{S}}) = \sup_{\mathfrak{z}\in\mathcal{L}} \varpi^{\mathcal{S}}(\mathfrak{z}) = \sup_{\mathfrak{z}\in\mathcal{L}} \sup_{\mathfrak{y}\in\mathcal{S}\backsim\mathfrak{z}} \vartheta(\mathfrak{y})\\ =&\sup_{\mathfrak{z}\in\mathcal{L}} \sup_{\mathfrak{y}\in\mathfrak{z}\backsim\mathcal{S}} \vartheta(\mathfrak{y}) \succcurlyeq \sup_{\mathfrak{i}\in\mathcal{S}} \vartheta(\mathfrak{i}) = \vartheta(\mathcal{S}).\end{aligned}$$

□

4. Fuzzy Hyper Congruence Relation

In this section, we introduce the notion of a fuzzy regular relation on hyper equality algebras and then we give some results related to quotient hyper equality algebras.

Definition 4.1. Let Ω be an equivalence relation on $\mathcal{L}$. Then Ω is called a regular relation on $\mathcal{L}$, if $(\mathfrak{x}\backsim\mathfrak{y})\Omega 1$ and $(\mathfrak{y}\backsim\mathfrak{x})\Omega 1$ imply $\mathfrak{x}\Omega\mathfrak{y}$.

Example 4.1. According to Example 3.1 (*ii*), define

$$\Omega = \{(0,0),(\mathfrak{i},\mathfrak{i}),(\mathfrak{i},0),(0,\mathfrak{i}),(1,1)\}.$$

Then Ω is a regular relation on $\mathcal{L}$.

Definition 4.2. Consider ϱ is a fuzzy equivalence relation on $\mathcal{L}$. Then we say ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$ if for all $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{y}) \preccurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}}\varrho(\mathfrak{x}\backsim\mathfrak{z},\mathfrak{y}\backsim\mathfrak{z}) \quad\text{and}\quad \varrho(\mathfrak{x},\mathfrak{y}) \preccurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}}\varrho(\mathfrak{x}\barwedge\mathfrak{z},\mathfrak{y}\barwedge\mathfrak{z}),$$

where

$$\varrho(\mathfrak{x}\backsim\mathfrak{z},\mathfrak{y}\backsim\mathfrak{z}) = \sup_{\mathfrak{i}\in\mathfrak{x}\backsim\mathfrak{z},\mathfrak{h}\in\mathfrak{y}\backsim\mathfrak{z}} \varrho(\mathfrak{i},\mathfrak{h}). \tag{4.1}$$

The fuzzy hyper congruence relation ϱ on $\mathcal{L}$ is called a fuzzy regular relation on $\mathcal{L}$, if for any $\mathfrak{x},\mathfrak{y}\in\mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{y}) \succcurlyeq \min\left\{\bigvee\varrho(\mathfrak{x}\backsim\mathfrak{y},1),\bigvee\varrho(\mathfrak{y}\backsim\mathfrak{x},1)\right\}. \tag{4.2}$$

Proposition 4.1. *Assume ϱ is a fuzzy regular relation on $\mathcal{L}$. Then for any $\mathfrak{t}\in[0,1]$, $\varrho_{\mathfrak{t}}\neq\emptyset$ is a regular relation on $\mathcal{L}$.*

Proof. First, we prove that for any $\mathfrak{t}\in[0,1]$, $\varrho_{\mathfrak{t}}$ is an equivalence relation on $\mathcal{L}$. Since $\varrho_{\mathfrak{t}}$ is a non-empty set, there exist $\mathfrak{y},\mathfrak{z}\in\mathcal{L}$ such that $\varrho(\mathfrak{y},\mathfrak{z})\succcurlyeq\mathfrak{t}$. Then for all $\mathfrak{x}\in\mathcal{L}$

$$\varrho(\mathfrak{x},\mathfrak{x}) = \bigvee_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}}\varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}.$$

Hence, $\varrho_{\mathfrak{t}}$ is a fuzzy reflexive on $\mathcal{L}$. Let $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{y}$. Then $\varrho(\mathfrak{x},\mathfrak{y})\succcurlyeq\mathfrak{t}$. From ϱ is regular, we have $\varrho(\mathfrak{y},\mathfrak{x})=\varrho(\mathfrak{x},\mathfrak{y})\succcurlyeq\mathfrak{t}$, and so $\mathfrak{y}\varrho_{\mathfrak{t}}\mathfrak{x}$. Hence, $\varrho_{\mathfrak{t}}$ is symmetric. Now, suppose $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{y}$ and $\mathfrak{y}\varrho_{\mathfrak{t}}\mathfrak{z}$. Then $\varrho(\mathfrak{x},\mathfrak{y})\succcurlyeq\mathfrak{t}$ and $\varrho(\mathfrak{y},\mathfrak{z})\succcurlyeq\mathfrak{t}$. Since ϱ is a fuzzy regular relation on $\mathcal{L}$, we have

$$\varrho(\mathfrak{x},\mathfrak{z}) \succcurlyeq \bigvee_{\mathfrak{y}\in\mathcal{L}}(\varrho(\mathfrak{x},\mathfrak{y})\barwedge\varrho(\mathfrak{y},\mathfrak{z})) \succcurlyeq \varrho(\mathfrak{x},\mathfrak{y})\barwedge\varrho(\mathfrak{y},\mathfrak{z}) \succcurlyeq \mathfrak{t}.$$

Hence, $\varrho(\mathfrak{x},\mathfrak{z})\succcurlyeq\mathfrak{t}$ and so $\mathfrak{x}\varrho_{\mathfrak{t}}\mathfrak{z}$. Thus, $\varrho_{\mathfrak{t}}$ is transitive. Therefore, $\varrho_{\mathfrak{t}}$ is an equivalence relation on $\mathcal{L}$.

Now, we prove that $\varrho_{\mathfrak{t}}$ is regular. For this, suppose $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ where $(\mathfrak{x} \backsim \mathfrak{y})\varrho_{\mathfrak{t}}1$ and $(\mathfrak{y} \backsim \mathfrak{x})\varrho_{\mathfrak{t}}1$. Then there exists $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$ and $\mathfrak{h} \in \mathfrak{y} \backsim \mathfrak{x}$ such that $\mathfrak{i}\varrho_{\mathfrak{t}}1$ and $\mathfrak{h}\varrho_{\mathfrak{t}}1$, respectively. Thus, $\varrho(\mathfrak{i}, 1) \succcurlyeq \mathfrak{t}$ and $\varrho(\mathfrak{h}, 1) \succcurlyeq \mathfrak{t}$. Hence,

$$\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\} \succcurlyeq \min\{\varrho(\mathfrak{i}, 1), \varrho(\mathfrak{h}, 1)\} \succcurlyeq \mathfrak{t}.$$

Therefore, $\varrho_{\mathfrak{t}}$ is a regular relation on $\mathcal{L}$. □

Proposition 4.2. *Consider ϱ is a fuzzy relation on $\mathcal{L}$ which satisfies the sup-property. If for each $\mathfrak{t} \in [0, 1]$, $\varrho_{\mathfrak{t}} \neq \emptyset$ is a regular relation on $\mathcal{L}$, then ϱ is a fuzzy regular relation on $\mathcal{L}$.*

Proof. Suppose $\mathfrak{t} = \bigvee_{(\mathfrak{u},\mathfrak{v})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v})$. By assumption ϱ is a fuzzy relation on $\mathcal{L}$ which satisfies the sup-property, then there exists $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{L} \times \mathcal{L}$ such that $\mathfrak{t} = \varrho(\mathfrak{u}, \mathfrak{v})$. Since $\mathfrak{t} \in [0, 1]$, we get $\mathfrak{u}\varrho_{\mathfrak{t}}\mathfrak{v}$, and so $\varrho_{\mathfrak{t}} \neq \emptyset$. Reflexivity of $\varrho_{\mathfrak{t}}$ implies that for all $\mathfrak{x} \in \mathcal{L}$, $(\mathfrak{x}, \mathfrak{x}) \in \varrho_{\mathfrak{t}}$. Thus for all $\mathfrak{x} \in \mathcal{L}$, $\varrho(\mathfrak{x}, \mathfrak{x}) \succcurlyeq \mathfrak{t}$. Then, for any $\mathfrak{x} \in \mathcal{L}$,

$$\varrho(\mathfrak{x}, \mathfrak{x}) \preccurlyeq \bigvee_{(\mathfrak{y},\mathfrak{z})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{y}, \mathfrak{z}) = \mathfrak{t} \preccurlyeq \varrho(\mathfrak{x}, \mathfrak{x}).$$

Hence, ϱ is a fuzzy reflexive relation on $\mathcal{L}$. Now, since $\varrho_{\mathfrak{t}} \neq \emptyset$, suppose $(\mathfrak{x}, \mathfrak{y}) \in \varrho_{\mathfrak{t}}$ and by symmetry property, we have $\varrho_{\mathfrak{t}}(\mathfrak{x}, \mathfrak{y}) = \varrho_{\mathfrak{t}}(\mathfrak{y}, \mathfrak{x})$ and so $\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \mathfrak{t}$ and $\varrho(\mathfrak{y}, \mathfrak{x}) \succcurlyeq \mathfrak{t}$. Then for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we get that

$$\varrho(\mathfrak{x}, \mathfrak{y}) \preccurlyeq \bigvee_{(\mathfrak{u},\mathfrak{v})\in\mathcal{L}\times\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{t} \preccurlyeq \varrho(\mathfrak{y}, \mathfrak{x}).$$

Similarly, $\varrho(\mathfrak{y}, \mathfrak{x}) \preccurlyeq \varrho(\mathfrak{x}, \mathfrak{y})$. Then ϱ is a fuzzy symmetric relation on $\mathcal{L}$. By a similar argument, it is easy to see that ϱ is a fuzzy transitive relation on $\mathcal{L}$. Thus ϱ is a fuzzy equivalent relation on $\mathcal{L}$. Now, we show that ϱ is a fuzzy regular relation on $\mathcal{L}$. Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ and

$$\mathfrak{t} = \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\}.$$

Since ϱ satisfies the sup-property, there exist $\mathfrak{i} \in \mathfrak{x} \backsim \mathfrak{y}$ and $\mathfrak{h} \in \mathfrak{y} \backsim \mathfrak{x}$ such that $\mathfrak{t} \preccurlyeq \bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1) = \varrho(\mathfrak{i}, 1)$ and $\mathfrak{t} \preccurlyeq \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1) = \varrho(\mathfrak{h}, 1)$. Thus, $(\mathfrak{i}, 1), (\mathfrak{h}, 1) \in \varrho_{\mathfrak{t}}$. Moreover, since ϱ satisfies the sup-property, by (4.1) we have

$$\varrho(\mathfrak{x} \backsim \mathfrak{y}, 1) = \sup_{m\in\mathfrak{x}\backsim\mathfrak{y}} \varrho(\mathfrak{m}, 1) = \varrho(\mathfrak{i}, 1) \succcurlyeq \mathfrak{t} \text{ and}$$
$$\varrho(\mathfrak{y} \backsim \mathfrak{x}, 1) = \sup_{\neg\in\mathfrak{y}\backsim\mathfrak{x}} \varrho(\neg, 1) = \varrho(\mathfrak{h}, 1) \succcurlyeq \mathfrak{t}.$$

Hence, $(\mathfrak{x} \backsim \mathfrak{y}, 1), (\mathfrak{y} \backsim \mathfrak{x}, 1) \in \varrho_{\mathfrak{t}}$. By our assumption, $\varrho_{\mathfrak{t}}$ is a regular relation on $\mathcal{L}$, then $(\mathfrak{x}, \mathfrak{y}) \in \varrho_{\mathfrak{t}}$. Thus for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, we get that

$$\varrho(\mathfrak{x}, \mathfrak{y}) \succcurlyeq \mathfrak{t} = \min\left\{\bigvee \varrho(\mathfrak{x} \backsim \mathfrak{y}, 1), \bigvee \varrho(\mathfrak{y} \backsim \mathfrak{x}, 1)\right\}.$$

Therefore, ϱ is a fuzzy regular relation on $\mathcal{L}$. □

Corollary 4.1. *Consider ϱ is a fuzzy relation on $\mathcal{L}$ such that satisfies the sup-property. Then ϱ is a fuzzy regular relation on $\mathcal{L}$ iff, for all $\mathfrak{t} \in [0, 1]$, $\varrho_{\mathfrak{t}} \neq \emptyset$ is a regular relation on $\mathcal{L}$.*

Proof. By Propositions 4.1 and 4.2, the proof is clear. □

By the following result, we show a relation between strong fuzzy filters on hyper equality algebras and fuzzy hyper congruence relation that is made by them.

Theorem 4.1. *Assume* ϑ *is a strong fuzzy filter on* $\mathcal{L}$*. Then for any* $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$*, relation* $\varrho : \mathcal{L} \times \mathcal{L} \to [0,1]$ *which is defined by*

$$\varrho(\mathfrak{x}, \mathfrak{y}) = \left(\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{v})\right)$$

is a fuzzy hyper congruence relation on $\mathcal{L}$*.*

Proof. Clearly, ϱ is reflexive and symmetry. We show that ϱ is a fuzzy transitive relation on $\mathcal{L}$. For this by Proposition 2.2 (P11), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, $\mathfrak{x} \curvearrowright \mathfrak{y} \lll (\mathfrak{y} \curvearrowright \mathfrak{z}) \curvearrowright (\mathfrak{x} \curvearrowright \mathfrak{z})$. Since ϑ is a strong fuzzy filter of $\mathcal{L}$, by Theorem 3.2 (iii), we have

$$\left(\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right) \preccurlyeq \sup_{\mathfrak{w} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{w}). \tag{4.3}$$

Then for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we get

$$\begin{aligned} &\bigvee_{\mathfrak{z} \in \mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \barwedge \varrho(\mathfrak{z}, \mathfrak{y})) \\ &= \bigvee_{\mathfrak{z} \in \mathcal{L}} \left[\left(\sup_{\mathfrak{i} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{i})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathfrak{z} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{h})\right) \barwedge \left(\sup_{\mathfrak{u} \in \mathfrak{z} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right) \barwedge \left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right)\right] \\ &= \bigvee_{\mathfrak{z} \in \mathcal{L}} \left[\left(\left(\sup_{\mathfrak{i} \in \mathfrak{x} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{i})\right) \barwedge \left(\sup_{\mathfrak{u} \in \mathfrak{z} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u})\right)\right) \barwedge \left(\left(\sup_{\mathfrak{v} \in \mathfrak{y} \curvearrowright \mathfrak{z}} \vartheta(\mathfrak{v})\right) \barwedge \left(\sup_{\mathfrak{h} \in \mathfrak{z} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{h})\right)\right)\right] \\ &\preccurlyeq \left(\sup_{\mathfrak{o} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{o})\right) \barwedge \left(\sup_{\mathfrak{w} \in \mathfrak{y} \curvearrowright \mathfrak{x}} \vartheta(\mathfrak{w})\right) \quad \text{(by (4.3))}, \\ &= \varrho(\mathfrak{x}, \mathfrak{y}). \end{aligned}$$

Thus ϱ is a fuzzy transitive relation on $\mathcal{L}$. Hence, ϱ is a fuzzy equivalence relation on $\mathcal{L}$. Now, we investigate the condition of Definition 3.1. By Proposition 2.2 (P9), for any $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$

$$\mathfrak{x} \barwedge \mathfrak{y} \preccurlyeq \mathfrak{y} \lll \mathfrak{y} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) = \mathfrak{y} \curvearrowright \mathfrak{x}.$$

Then there exists $\mathfrak{z} \in \mathfrak{y} \curvearrowright \mathfrak{x}$ such that $\mathfrak{x} \barwedge \mathfrak{y} \preccurlyeq \mathfrak{y} \preccurlyeq \mathfrak{z}$. By $(\mathcal{HE}5)$ we have

$$(\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z} \lll \mathfrak{y} \backsim \mathfrak{z}. \tag{4.4}$$

Thus,

$$\begin{aligned} \mathfrak{x} \curvearrowright \mathfrak{y} = \mathfrak{x} \backsim (\mathfrak{x} \barwedge \mathfrak{y}) \quad & \text{(by } (\mathcal{HE}7)) \\ \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim ((\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z}) \quad & \text{(by Proposition 2.2(P3))} \\ \lll (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright ((\mathfrak{x} \barwedge \mathfrak{y}) \backsim \mathfrak{z}) \quad & \text{(by (4.4) and Proposition 2.2(P8))} \\ \lll (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright (\mathfrak{y} \backsim \mathfrak{z}). & \end{aligned}$$

Hence, by (FF2) for any $\mathfrak{z} \in \mathfrak{y} \curvearrowright \mathfrak{x}$, we have

$$\sup_{\mathfrak{u} \in \mathfrak{x} \curvearrowright \mathfrak{y}} \vartheta(\mathfrak{u}) \preccurlyeq \sup_{\mathfrak{v} \in (\mathfrak{x} \backsim \mathfrak{z}) \curvearrowright (\mathfrak{y} \backsim \mathfrak{z})} \vartheta(\mathfrak{v}). \tag{4.5}$$

By the similar way, for any $\mathfrak{z} \in \mathfrak{x} \curvearrowright \mathfrak{y}$, we get

$$\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i}) \preccurlyeq \sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h}). \tag{4.6}$$

Then by (4.5) and (4.6), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\begin{aligned}
\varrho(\mathfrak{x},\mathfrak{y}) &= \Big(\sup_{\mathfrak{u}\in\mathfrak{x}\curvearrowright\mathfrak{y}} \vartheta(\mathfrak{u})\Big) \barwedge \Big(\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i})\Big)\\
&\preccurlyeq \Big(\sup_{\mathfrak{v}\in(\mathfrak{x}\backsim\mathfrak{z})\curvearrowright(\mathfrak{y}\backsim\mathfrak{z})} \vartheta(\mathfrak{v})\Big) \barwedge \Big(\sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})\Big)\\
&\preccurlyeq \bigvee_{\mathfrak{z}\in\mathcal{L}} \Big[\Big(\sup_{\mathfrak{v}\in(\mathfrak{x}\backsim\mathfrak{z})\curvearrowright(\mathfrak{y}\backsim\mathfrak{z})} \vartheta(\mathfrak{v})\Big) \barwedge \Big(\sup_{\mathfrak{h}\in(\mathfrak{y}\backsim\mathfrak{z})\curvearrowright(\mathfrak{x}\backsim\mathfrak{z})} \vartheta(\mathfrak{h})\Big)\Big]\\
&= \bigvee_{\mathfrak{z}\in\mathcal{L}} \varrho(\mathfrak{x}\backsim\mathfrak{z}, \mathfrak{y}\backsim\mathfrak{z}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathfrak{x}\curvearrowright\mathfrak{y} &= \mathfrak{x}\backsim(\mathfrak{x}\barwedge\mathfrak{y}) \quad (\text{by } (\mathcal{HE}6))\\
&\lll (\mathfrak{x}\barwedge\mathfrak{z})\backsim(\mathfrak{x}\barwedge\mathfrak{y}\barwedge\mathfrak{z}) \quad (\text{by Proposition 2.2 (P3)})\\
&\lll (\mathfrak{x}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{x}\barwedge\mathfrak{y}\barwedge\mathfrak{z}) \quad (\text{by Proposition 2.2 (P7)})\\
&\lll (\mathfrak{x}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{y}\barwedge\mathfrak{z}). \qquad (4.7)
\end{aligned}$$

Similarly, for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\mathfrak{y}\curvearrowright\mathfrak{x} \lll (\mathfrak{y}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{x}\barwedge\mathfrak{z}). \tag{4.8}$$

Then by (4.7), (4.8) and (FF2), for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we get that

$$\begin{aligned}
\varrho(\mathfrak{x},\mathfrak{y}) &= \Big(\sup_{\mathfrak{u}\in\mathfrak{x}\curvearrowright\mathfrak{y}} \vartheta(\mathfrak{u})\Big) \barwedge \Big(\sup_{\mathfrak{i}\in\mathfrak{y}\curvearrowright\mathfrak{x}} \vartheta(\mathfrak{i})\Big)\\
&\preccurlyeq \Big(\sup_{\mathfrak{v}\in(\mathfrak{x}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{y}\barwedge\mathfrak{z})} \vartheta(\mathfrak{v})\Big) \barwedge \Big(\sup_{\mathfrak{h}\in(\mathfrak{y}\barwedge\mathfrak{z})\curvearrowright(\mathfrak{x}\barwedge\mathfrak{z})} \vartheta(\mathfrak{h})\Big)\\
&= \varrho(\mathfrak{x}\barwedge\mathfrak{z}, \mathfrak{y}\barwedge\mathfrak{z}).
\end{aligned}$$

Hence, ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$. □

Let ϱ be a fuzzy hyper congruence relation on $\mathcal{L}$, we define the fuzzy subset $\vartheta^{\varrho}_{\mathfrak{x}} : \mathcal{L} \to [0,1]$ by $\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{y}) = \varrho(\mathfrak{y},\mathfrak{x})$ for all $\mathfrak{y} \in \mathcal{L}$.

Lemma 4.1. *Consider ϱ is a fuzzy hyper congruence relation on $\mathcal{L}$. Then for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$ iff $\varrho(\mathfrak{x},\mathfrak{y}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s},\mathfrak{t})$.*

Proof. Suppose $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$. Since ϱ is a fuzzy reflexive relation, we have

$$\vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{x}) = \vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{x}) = \varrho(\mathfrak{x},\mathfrak{x}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s},\mathfrak{t}).$$

Conversely, assume $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$ such that $\varrho(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})$. Then by fuzzy symmetric and fuzzy transitive relations defined on ϱ, for all $\mathfrak{z} \in \mathcal{L}$, we get

$$\begin{aligned}\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{z}) &= \varrho(\mathfrak{z}, \mathfrak{x}) = \varrho(\mathfrak{x}, \mathfrak{z}) \\ &\succcurlyeq \bigvee\nolimits_{\mathfrak{y}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{y}) \barwedge \varrho(\mathfrak{y}, \mathfrak{z})) \succcurlyeq \varrho(\mathfrak{x}, \mathfrak{y}) \barwedge \varrho(\mathfrak{y}, \mathfrak{z}) \\ &= \left(\bigvee\nolimits_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})\right) \barwedge \varrho(\mathfrak{y}, \mathfrak{z}) = \varrho(\mathfrak{y}, \mathfrak{z}) = \varrho(\mathfrak{z}, \mathfrak{y}) \\ &= \vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{z}).\end{aligned}$$

By replacing $\mathfrak{x}$ by $\mathfrak{y}$ throughout the above statements, we get $\vartheta^{\varrho}_{\mathfrak{x}}(\mathfrak{z}) \preccurlyeq \vartheta^{\varrho}_{\mathfrak{y}}(\mathfrak{z})$. Hence, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$. ☐

Theorem 4.2. *Consider $\mathcal{L} = \langle \mathcal{L}; \backsim, \barwedge, 1\rangle$ is symmetric and ϱ be a fuzzy hyper congruence relation on $\mathcal{L}$, satisfies the sup-property. Define $\frac{\mathcal{L}}{\varrho} = \{\vartheta^{\varrho}_{\mathfrak{x}} \mid \mathfrak{x} \in \mathcal{L}\}$. Then $\frac{\mathcal{L}}{\varrho} = \left\langle \frac{\mathcal{L}}{\varrho}; \backsim_{\varrho}, \barwedge_{\varrho}, \vartheta^{\varrho}_{1}\right\rangle$ is symmetric, where the operations $\backsim_{\varrho}$ and $\barwedge_{\varrho}$ are defined on $\frac{\mathcal{L}}{\varrho}$ as follows:*

$$\vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}\backsim\mathfrak{y}} = \{\vartheta^{\varrho}_{\mathfrak{z}} \mid \mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{y}\}, \quad \vartheta^{\varrho}_{\mathfrak{x}} \barwedge_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}\barwedge\mathfrak{y}},$$
$$\vartheta^{\varrho}_{\mathfrak{x}} \preccurlyeq_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} \Leftrightarrow \vartheta^{\varrho}_{\mathfrak{x}} \barwedge_{\varrho} \vartheta^{\varrho}_{\mathfrak{y}} = \vartheta^{\varrho}_{\mathfrak{x}},$$

and for any $\mathcal{S}, \mathcal{R} \subseteq \frac{\mathcal{L}}{\varrho}$, $\mathcal{S} \lll_{\varrho} \mathcal{R}$ if and only if for all $\mathfrak{i} \in \mathcal{S}$, there exists $\mathfrak{h} \in \mathcal{R}$ such that $\vartheta^{\varrho}_{\mathfrak{i}} \preccurlyeq_{\varrho} \vartheta^{\varrho}_{\mathfrak{h}}$.

Proof. Let $\mathfrak{x}, \mathfrak{y}, \mathfrak{s}, \mathfrak{t} \in \mathcal{L}$, $\vartheta^{\varrho}_{\mathfrak{x}} = \vartheta^{\varrho}_{\mathfrak{y}}$ and $\vartheta^{\varrho}_{\mathfrak{s}} = \vartheta^{\varrho}_{\mathfrak{t}}$. Then by Lemma 4.1, $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{s}, \mathfrak{t}) = \bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v})$. Set $\bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{m}$. Then by Proposition 4.1, $\varrho_{\mathfrak{m}}$ is a regular relation on $\mathcal{L}$ and since $\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{s}, \mathfrak{t}) = \mathfrak{m}$, we have $\mathfrak{x}\varrho_{\mathfrak{m}}\mathfrak{y}$ and $\mathfrak{s}\varrho_{\mathfrak{m}}\mathfrak{t}$. Moreover, since ϱ is a fuzzy regular relation on $\mathcal{L}$ such that satisfies the sup-property and for any $\mathfrak{z} \in \mathcal{L}$, $\mathfrak{z} \preccurlyeq 1$, we get $\varrho(\mathfrak{x}, \mathfrak{z}) \preccurlyeq \varrho(\mathfrak{x}, 1)$ and so

$$\varrho(\mathfrak{x}, \mathfrak{y}) = \varrho(\mathfrak{x}, 1) \barwedge \varrho(1, \mathfrak{y}) \succcurlyeq \bigvee\nolimits_{\mathfrak{z}\in\mathcal{L}} (\varrho(\mathfrak{x}, \mathfrak{z}) \barwedge \varrho(\mathfrak{z}, \mathfrak{y})).$$

Moreover, since ϱ is a fuzzy relation on $\mathcal{L}$, we have $\varrho(\mathfrak{t}, \mathfrak{t}) = \bigvee_{\mathfrak{u},\mathfrak{v}\in\mathcal{L}} \varrho(\mathfrak{u}, \mathfrak{v}) = \mathfrak{m}$. Also, from $\mathcal{L}$ is symmetric, by (4.2), Proposition 2.2(P9) and ($\mathcal{HE}7$), we have

$$\begin{aligned}\varrho(\mathfrak{x} \backsim \mathfrak{t}, \mathfrak{y} \backsim \mathfrak{t}) &\succcurlyeq \min\left\{\bigvee \varrho\left((\mathfrak{x} \backsim \mathfrak{t}) \backsim (\mathfrak{y} \backsim \mathfrak{t}), 1\right), \bigvee \varrho\left((\mathfrak{y} \backsim \mathfrak{t}) \backsim (\mathfrak{x} \backsim \mathfrak{t}), 1\right)\right\} \\ &\succcurlyeq \min\left\{\bigvee \varrho\left(\mathfrak{x} \backsim \mathfrak{y}, 1\right), \bigvee \varrho\left(\mathfrak{y} \backsim \mathfrak{x}, 1\right)\right\} = \bigvee \varrho\left(\mathfrak{x} \backsim \mathfrak{y}, 1\right) \\ &\succcurlyeq \varrho(\mathfrak{x}, 1) \barwedge \varrho(1, \mathfrak{y}) = \varrho(\mathfrak{x}, \mathfrak{y}) = m.\end{aligned}$$

Then $\mathfrak{x} \backsim \mathfrak{t}\varrho_{\mathfrak{m}}\mathfrak{y} \backsim \mathfrak{t}$. Similarly, $\mathfrak{y} \backsim \mathfrak{t}\varrho_{\mathfrak{m}}\mathfrak{y} \backsim \mathfrak{s}$. Since $\varrho_{\mathfrak{m}}$ is a regular relation on $\mathcal{L}$, we get $(\mathfrak{x} \backsim \mathfrak{s})\overline{\varrho_{\mathfrak{m}}}(\mathfrak{y} \backsim \mathfrak{t})$. Suppose $\vartheta^{\varrho}_{\mathfrak{z}} \in \vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}}$. Then there is $\mathfrak{x}' \in \mathfrak{x} \backsim \mathfrak{s}$ such that $\vartheta^{\varrho}_{\mathfrak{x}'} = \vartheta^{\varrho}_{\mathfrak{z}}$. Thus, $\mathfrak{z}\varrho_{\mathfrak{m}}\mathfrak{x}'$. On the other hand, $(\mathfrak{x} \backsim \mathfrak{s})\overline{\varrho_{\mathfrak{m}}}(\mathfrak{y} \backsim \mathfrak{t})$, so there exists $\mathfrak{y}' \in \mathfrak{y} \backsim \mathfrak{t}$ such that $\mathfrak{x}'\varrho_{\mathfrak{m}}\mathfrak{y}'$. Also, since $\varrho_{\mathfrak{m}}$ is transitive, and by Proposition 4.1 is an equivalence relation on $\mathcal{L}$, we have $\mathfrak{z}\varrho_{\mathfrak{m}}\mathfrak{y}'$. Then by Lemma 4.1, we get $\vartheta^{\varrho}_{\mathfrak{z}} = \vartheta^{\varrho}_{\mathfrak{y}'}$. This shows that $\vartheta^{\varrho}_{\mathfrak{z}} = \vartheta^{\varrho}_{\mathfrak{y}'} \in \vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}}$. Thus $\vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}} \subseteq \vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}}$. Similarly, we obtain $\vartheta^{\varrho}_{\mathfrak{y}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{t}} \subseteq \vartheta^{\varrho}_{\mathfrak{x}} \backsim_{\varrho} \vartheta^{\varrho}_{\mathfrak{s}}$. Hence, $\backsim_{\varrho}$ is well-defined. Easily, $\barwedge_{\varrho}$ is well-defined, too. Now, we show that $\frac{\mathcal{L}}{\varrho} = \left\langle \frac{\mathcal{L}}{\varrho}; \backsim_{\varrho}, \barwedge_{\varrho}, \vartheta^{\varrho}_{1}\right\rangle$ is a hyper equality algebra.

($\mathcal{HE}1$) Clearly, ϑ_1^ϱ is the top element and $\langle \frac{\mathcal{L}}{\varrho}; \bar{\wedge}_\varrho, \vartheta_1^\varrho \rangle$ is a meet-semilattice.

($\mathcal{HE}2$) For all $\mathfrak{x}, \mathfrak{y} \in \mathcal{L}$, since $\mathcal{L}$ is symmetric, we have

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}\backsim\mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}\backsim\mathfrak{x}}^\varrho = \vartheta_{\mathfrak{y}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{x}}^\varrho. \tag{4.9}$$

($\mathcal{HE}3$) Assume $\mathfrak{x} \in \mathcal{L}$. Since $1 \in \mathfrak{x} \backsim \mathfrak{x}$, we have

$$\vartheta_1^\varrho \in \{\vartheta_{\mathfrak{z}}^\varrho : \mathfrak{z} \in \mathfrak{x} \backsim \mathfrak{x}\} = \vartheta_{\mathfrak{x}\backsim\mathfrak{x}}^\varrho.$$

Thus, $\vartheta_1^\varrho \in \vartheta_{\mathfrak{x}\backsim\mathfrak{x}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{x}}^\varrho$.

($\mathcal{HE}4$) Simmilar to ($\mathcal{HE}3$), since for any $\mathfrak{x} \in 1 \backsim \mathfrak{x}$, we have $\vartheta_{\mathfrak{x}}^\varrho \in \vartheta_{1\backsim\mathfrak{x}} = \vartheta_1 \backsim_\varrho \vartheta_{\mathfrak{x}}$.

($\mathcal{HE}5$) Suppose $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$ such that $\vartheta_{\mathfrak{x}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{z}}^\varrho$. Since $\vartheta_{\mathfrak{x}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}}^\varrho$, we have $\vartheta_{\mathfrak{x}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho$ and so $\vartheta_{\mathfrak{x}\bar{\wedge}\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}}^\varrho$. By Lemma 4.1, $\varrho(\mathfrak{x} \bar{\wedge} \mathfrak{y}, \mathfrak{x}) = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t}) = \mathfrak{m}$. Thus $(\mathfrak{x} \bar{\wedge} \mathfrak{y})\varrho_{\mathfrak{m}}\mathfrak{x}$. Since $\mathcal{L}$ is symmetric, by Proposition 2.2(P9), we get $\mathfrak{x} \lll \mathfrak{x} \backsim \mathfrak{z}$ and $\mathfrak{x} \bar{\wedge} \mathfrak{y} \preccurlyeq \mathfrak{y} \lll \mathfrak{y} \backsim \mathfrak{z}$. Then $\mathfrak{m} = \varrho(\mathfrak{x}, \mathfrak{x} \bar{\wedge} \mathfrak{y}) \preccurlyeq \varrho(\mathfrak{x} \backsim \mathfrak{z}, \mathfrak{y} \backsim \mathfrak{z})$, and so $(\mathfrak{x} \backsim \mathfrak{z})\varrho_{\mathfrak{m}}(\mathfrak{y} \backsim \mathfrak{z})$. Consider $\vartheta_\gamma^\varrho \in \vartheta_{\mathfrak{x}\backsim\mathfrak{z}}^\varrho$. Then there exists $\alpha \in \mathfrak{x} \backsim \mathfrak{z}$ such that $\vartheta_\gamma^\varrho = \vartheta_\alpha^\varrho$. Thus $\gamma\varrho_{\mathfrak{m}}\alpha$. Since $(\mathfrak{x} \backsim \mathfrak{z})\varrho_{\mathfrak{m}}(\mathfrak{y} \backsim \mathfrak{z})$, there exists $\beta \in \mathfrak{y} \backsim \mathfrak{z}$ such that $\beta\varrho_{\mathfrak{m}}\alpha$. From $\varrho_{\mathfrak{m}}$ has transitivity, we get $\gamma\varrho_{\mathfrak{m}}\beta$. Thus, $\varrho(\beta, \gamma) \succcurlyeq \mathfrak{m} = \bigvee_{\mathfrak{s},\mathfrak{t}\in\mathcal{L}} \varrho(\mathfrak{s}, \mathfrak{t})$. Hence, $\vartheta_\gamma^\varrho = \vartheta_\beta^\varrho$ and so $\vartheta_\gamma^\varrho = \vartheta_\beta^\varrho \in \vartheta_{\mathfrak{y}\backsim\mathfrak{z}}^\varrho$. So $\vartheta_{\mathfrak{x}\backsim\mathfrak{z}}^\varrho \preccurlyeq_\varrho \vartheta_{\mathfrak{y}\backsim\mathfrak{z}}^\varrho$. The proof of other case is similar.

($\mathcal{HE}6$) For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \bar{\wedge} \mathfrak{z}) \backsim (\mathfrak{y} \bar{\wedge} \mathfrak{z})$. Then, for any $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we have

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}\backsim\mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{(\mathfrak{x}\bar{\wedge}\mathfrak{z})\backsim(\mathfrak{y}\bar{\wedge}\mathfrak{z})}^\varrho = \vartheta_{\mathfrak{x}\bar{\wedge}\mathfrak{z}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}\bar{\wedge}\mathfrak{z}}^\varrho = \left(\vartheta_{\mathfrak{x}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{z}}^\varrho\right) \backsim_\varrho \left(\vartheta_{\mathfrak{y}}^\varrho \bar{\wedge}_\varrho \vartheta_{\mathfrak{z}}^\varrho\right).$$

($\mathcal{HE}7$) For all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{L}$, we obtain $\mathfrak{x} \backsim \mathfrak{y} \lll (\mathfrak{x} \backsim \mathfrak{z}) \backsim (\mathfrak{y} \backsim \mathfrak{z})$. Then

$$\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}}^\varrho = \vartheta_{\mathfrak{x}\backsim\mathfrak{y}}^\varrho \preccurlyeq_\varrho \vartheta_{(\mathfrak{x}\backsim\mathfrak{z})\backsim(\mathfrak{y}\backsim\mathfrak{z})}^\varrho = \vartheta_{\mathfrak{x}\backsim\mathfrak{z}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{y}\backsim\mathfrak{z}}^\varrho = \left(\vartheta_{\mathfrak{x}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{z}}^\varrho\right) \backsim_\varrho \left(\vartheta_{\mathfrak{y}}^\varrho \backsim_\varrho \vartheta_{\mathfrak{z}}^\varrho\right).$$

Thus, the above facts and (4.9) show that $\frac{\mathcal{L}}{\varrho}$ is a symmetric hyper equality algebra. □

5. Conclusions and Future Works

In this paper, the notion of strong fuzzy filter on hyper equality algebras is introduced and some equivalence definitions of it are investigated. Then by using this notion, a symmetric equality algebra is constructed and defined a special form of classes. By using these, the concept of a fuzzy hyper congruence relation on hyper equality algebra is defined and prove that the quotient is made by it is an equality algebra. Also, by using a fuzzy equivalence relation on hyper equality, a fuzzy hyper congruence relation is introduced and proved that this fuzzy hyper congruence is regular. Finally, it is proved that the quotient structure that is made by it is a symmetric hyper equality algebra.

For future work, we can define different kinds of fuzzy ideal on hyper equality algebras and investigate properties of them and study about fuzzy hyper congruence relation on hyper equality algebra and the quotient structure that is made by it.

Acknowledgements. This research is supported by a grant of National Natural Science Foundation of China (11971384).

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the editor and referees.

References

[1] R. A. Borzooei, P. Corsini and M. M. Zahedi, *Some kinds of positive implicative hyper K-ideals*, J. Discrete Math. Sci. Cryptogr. **6**(1) (2003), 97–108. https://doi.org/10.1080/09720529.2003.10697966

[2] R. A. Borzooei, B. G. Saffar and R. Ameri, *On hyper EQ-algebras*, Italian Journal of Pure and Applied Mathematics **31** (2013), 77–96.

[3] R. A. Borzooei and S. Niazian, *Weak hyper residuated lattice*, Quasigroups Related Systems **21** (2013), 29–42.

[4] R. A. Borzooei, M. M. Zahedi, Y. B. Jun and A. Hasankhani, *On Hyper K-algebra*, Math. Jpn. **52**(1) (2000), 113–121.

[5] R. A. Borzooei, F. Zebardast and M. Aaly Kologani, *Some types of filters in equality algebras*, Categ. Gen. Algebr. Struct. Appl. **7** (2017), 33–55.

[6] X. Y. Cheng, X. L. Xin and Y. B. Jun, *Hyper equality algebras*, Quantitative Logic and Soft Computing 2016, Adv. Intell. Syst. Comput. **510** (2016), 415–428.

[7] L. C. Ciungu, *On pseudo-equality algebras*, Arch. Math. Log. **53** (2014), 561–570. https://doi.org/10.1007/s00153-014-0380-0

[8] L. C. Ciungu, *Internal states on equality algebras*, Soft Comput. **19** (2015), 939–953. https://doi.org/10.1007/s00500-014-1494-3

[9] S. Jenei, *Equality algebras*, Stud. Log. **56**(2) (2010), 183–186. https://doi.org/10.1109/CINTI.2010.5672249

[10] S. Jenei, *Equality algebras*, Proceedings of the CINTI2010 Conference (11th IEEE International Symposium on Computational Intelligence and Informatics), Budapest, 2010.

[11] Y. B. Jun and W. H. Shim, *Fuzzy strong implicative hyper BCK-ideals of hyper BCK-algebras*, Inf. Sci. **170** (2005), 351–361. https://doi.org/10.1016/j.ins.2004.03.009

[12] M. A. Hashemi and R. A. Borzooei, *Results on hyper equality algebras*, Journal of Mathematical Extension **14**(3) (2020), 169–193.

[13] Z. M. Ma and B. Q. Hu, *Fuzzy EQ-filters of EQ-algebras*, Quantitative Logic and Soft Computing **5** (2012), 528–535. https://doi.org/10.1142/9789814401531_0071

[14] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, 1934, 45–49.

[15] J. Mittas and M. Konstantinidou, *Sur une nouvelle generation de la notion detreillis, Les supertreillis et certaines de certaines de leurs proprites generales*, Ann. Sci. Univ. Blaise Pascal, Ser. Math. Fasc. **25** (1989), 61–83.

[16] M. Mohseni Takallo and M. Aaly Kologani, *MBJ-neutrosophic filters of equality algebras*, JAHLA **1**(2) (2020), 57–75. https://doi.org/10.29252/hatef.jahla.1.2.6

[17] V. Novák, *On fuzzy type theory*, Fuzzy Sets and Systems **149** (2005), 235–273. https://doi.org/10.1016/j.fss.2004.03.027

[18] V. Novák and B. De Baets, *EQ-algebras*, Fuzzy Sets and Systems **160** (2009), 2956–2978. https://doi.org/10.1016/j.fss.2009.04.010

[19] A. Paad and M. Bakhshi, *Hyper equality ideals: Basic properties*, JAHLA **1**(2) (2020), 45–56. https://doi.org/10.29252/HATEF.JAHLA.1.2.5

[20] X. Y. Yu, X. L. Xin and J. T. Wang, *Fuzzy filters on equality algebras with applications*, Journal of Intelligent & Fuzzy Systems **35** (2018), 3709–3719. https://doi.org/10.3233/JIFS-18509

[21] L. A. Zadeh, *Fuzzy sets*, Inf. Control. **8** (1965), 338–353. https://doi.org/10.1142/9789814261302_0021

[22] M. Zarean, R. A. Borzooei and O. Zahiri, *On state equality algebras*, Quasigroups Related Systems **25** (2017), 307–326.

[23] F. Zebardast, R. A. Borzooei and M. Aaly Kologani, *Results on equality algebras*, Inf. Sci. **381** (2017), 270–282. https://doi.org/10.1016/j.ins.2016.11.027

[1]Department of Mathematics,
Faculty of Mathematical Sciences,
Shahid Beheshti University,
Tehran, Iran
Email address: borzooei@sbu.ac.ir

[2]Hatef Higher Education Institute,
Zahedan,
Iran
Email address: mona4011@gmail.com

[3]Department of Mathematics,
Payame Noor University,
Tehran, Iran
Email address: ali1978hashemi@gmail.com

Email address: mohamadzadeh36@gmail.com

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 223–237.

ACENTRALIZERS OF SOME FINITE GROUPS

ZAHRA MOZAFAR[1] AND BIJAN TAERI[1]

Abstract. Let G be a finite group. The acentralizer of an automorphism α of G, is the subgroup of fixed points of α, i.e., $C_G(\alpha)=\{g\in G \mid \alpha(g)=g\}$. In this paper we determine the acentralizers of the dihedral group of order $2n$, the dicyclic group of order $4n$ and the symmetric group on n letters. As a result we see that if $n\geq 3$, then the number of acentralizers of the dihedral group and the dicyclic group of order $4n$ are equal. Also we determine the acentralizers of groups of orders pq and pqr, where p, q and r are distinct primes.

1. Introduction

Throughout this article, the usual notation will be used [17]. For example $\mathbb{Z}_n$ denotes the cyclic group of integers modulo n, $\mathbb{Z}_n^*$ denotes the group of invertible elements of $\mathbb{Z}_n$. The dihedral group of order $2n$ and the dicyclic group of order $4n$ are denoted by D_n, and Q_n, respectively. The symmetric group on a finite set of n symbols is denoted by S_n, or $\mathrm{Sym}(X)$, where $|X|=n$. The symbol $G=X\ltimes Y$ (or $G=Y\rtimes X$) indicates that G is a split extension (semidirect product) of a normal subgroup Y of G by a complement X.

Let G be a finite group. We write $\mathrm{Cent}(G)=\{C_G(g)\mid g\in G\}$, where $C_G(g)$ is the centralizer of the element g in G. The group G is called n-centralizer if $|\mathrm{Cent}(G)|=n$. There are some results on finite n-centralizers groups (see for instance [1–8, 12, 18]). Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. If $\alpha\in\mathrm{Aut}(G)$, then the acentralizer of α in G is defined as

$$C_G(\alpha)=\{g\in G\mid \alpha(g)=g\},$$

Key words and phrases. Automorphism, centralizer, acentralizer, finite groups.
2020 *Mathematics Subject Classification.* Primary: 20D45. Secondary: 20D25.
DOI 10.46793/KgJMat2502.223M
Received: August 16, 2021.
Accepted: March 09, 2022.

which is a subgroup of G. In particular if $\alpha = \tau_a$ is an inner automorphisms of G induced by $a \in G$, then $C_G(\tau_a) = C_G(a)$ is the centralizer of a in G. Let $\mathrm{Acent}(G)$ be the set of acentralizers of G, that is

$$\mathrm{Acent}(G) = \{C_G(\alpha) \mid \alpha \in \mathrm{Aut}(G)\}.$$

A group G is called n-acentralizer, if $|\mathrm{Acent}(G)| = n$. It is obvious that G is 1-acentralizer group if and only if G is a trivial group or $\mathbb{Z}_2$. Nasrabadi and Gholamian [14] proved that G is a 2-acentralizer group if and only if $G \cong \mathbb{Z}_4$, $\mathbb{Z}_p$ or $\mathbb{Z}_{2p}$, for some odd prime p. Furthermore, they characterized $3, 4, 5$-acentralizer groups. Seifizadeh et al. [16] characterized n-acentralizer groups, where $n \in \{6, 7, 8\}$, and obtained a lower bound on the number of acentralizer subgroups for p-groups, where p is a prime number. They showed that if $p \neq 2$, there is no n-acentralizer p-group for $n = 6, 7$. Moreover, if $p = 2$, then there is no 6-acentralizer p-group. In [13] we showed that if G is a finite abelian p-group of rank 2, where p is an odd prime, then the number of acentralizers of G is exactly the number of subgroups of G. Also we obtained acentralizers of infinite two-generator abelian groups.

Throughout the paper we use the presentations of the dihedral group of order $2n$, D_n, and the dicyclic group of order $4n$, Q_n, as follows

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle = \langle b\rangle \ltimes \langle a\rangle,$$
$$Q_n = \langle a, b \mid a^{2n} = 1,\ a^n = b^2,\ bab^{-1} = a^{-1}\rangle = \langle b\rangle \ltimes \langle a\rangle.$$

We note that if n is a power of 2, then Q_n is the generalized quaternion group. Computing the number of centralizers of finite group have been the object of some papers. For instance Ashrafi [2,3] showed that $|\mathrm{Cent}(Q_n)| = n + 2$ and

$$|\mathrm{Cent}(D_n)| = \begin{cases} n+2, & n \text{ is odd},\\ \frac{n}{2}+2, & n \text{ is even}.\end{cases}$$

In this paper we compute $|\mathrm{Acent}(D_n)|$, $|\mathrm{Acent}(Q_n)|$, $|\mathrm{Acent}(S_n)|$ and the number of acentralizers of groups of order pqr, where p, q and r are distinct primes.

2. Acentralizers of Dihedral and Dicyclic Groups

Recall that the dihedral group D_n have two type subgroups for $n > 3$, $\langle a^d\rangle$ and $\langle a^d, a^r b\rangle$, where $d \mid n$, $0 \le r < d$. The total number of these two type subgroups are $\tau(n) = \sum_{d|n} 1$, that is the number of positive divisors of n, and $\sigma(n) = \sum_{d|n} d$, that is the sum positive divisors of n, respectively. Recall that if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then $\tau(n) = \prod_{j=1}^{r}(k_j + 1)$ and $\sigma(n) = \prod_{j=1}^{r} \frac{p_j^{k_j+1}-1}{p_j-1}$.

For $n > 2$, the automorphism group of D_n is isomorphic to $\mathbb{Z}_n^* \ltimes \mathbb{Z}_n$, the semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_n^*$, with the canonical action of $\varepsilon : \mathbb{Z}_n^* \to \mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Explicitly,

$$\mathrm{Aut}(D_n) = \{\gamma_{s,t} \mid s \in \mathbb{Z}_n^*,\ t \in \mathbb{Z}_n\},$$

where $\gamma_{s,t}$ is defined by

$$\gamma_{s,t}(a^i) = a^{is} \quad \text{and} \quad \gamma_{s,t}(a^ib) = a^{is+t}b,$$

for all $0 \leq i \leq n-1$. Note that

$$\begin{aligned} a^i \in C_{D_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i) = a^i \\ &\Leftrightarrow a^{is} = a^i \\ &\Leftrightarrow is \equiv i \pmod n \\ &\Leftrightarrow i(s-1) \equiv 0 \pmod n \end{aligned}$$

and

$$\begin{aligned} a^ib \in C_{D_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^ib) = a^ib \\ &\Leftrightarrow a^{is+t}b = a^ib \\ &\Leftrightarrow is + t \equiv i \pmod n \\ &\Leftrightarrow i(s-1) + t \equiv 0 \pmod n. \end{aligned}$$

We use the following well-known theorem from elementary number theory.

Theorem 2.1. ([15, Page 102]) *Let a, b and m be integers such that $m > 0$ and let $c = \gcd(a, m)$. If c does not divide b, then the congruence $ax \equiv b \pmod m$ has no solutions. If $c \mid b$, then $ax \equiv b \pmod m$ has exactly c incongruent solutions modulo m.*

First we compute $\mathrm{Acent}(D_n)$. Clearly, $D_1 \cong \mathbb{Z}_2$ and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So $|\mathrm{Acent}(D_1)| = 1$ and $|\mathrm{Acent}(D_2)| = 5$.

Lemma 2.1. The identity subgroup is not an acentralizer for any automorphism of D_n. Also if n is even, the subgroups $\langle a^d \rangle$, $\langle a^d, a^rb \rangle$, where d is a divisor of n such that $d \nmid \frac{n}{2}$ and $0 \leq r < d$, are not acentralizers of D_n.

Proof. On the contrary, suppose that the identity subgroup $\langle a^n \rangle = \langle 1 \rangle$ is an acentralizer. Then there exists $\gamma_{s,t} \in \mathrm{Aut}(D_n)$ such that $\gamma_{s,t}$ fixes only the identity element. If $c := \gcd(n, s-1) \neq 1$, then

$$\gamma_{s,t}(a^{\frac{n}{c}}) = a^{\frac{n}{c}s} = a^{\frac{n}{c}} a^{\frac{s-1}{c}n} = a^{\frac{n}{c}},$$

which is a contradiction. Hence $\gcd(n, s-1) = 1$, and so by Theorem 2.1, there exists $0 < i < n-1$ such that $n \mid i(s-1) + t$. Since $\gamma_{s,t}(a^ib) = a^{is+t}b = a^{i(s-1)+t}a^ib \neq a^ib$, $n \nmid i(s-1) + t$, which is a contradiction. Thus the identity subgroup can not be an acentralizer.

Now suppose, for a contradiction, that $H := \langle a^d \rangle$, where d is a divisor of n and $d \nmid n/2$ is an acentralizer of D_n. Since $a^d \in C_{D_n}(\gamma_{s,t})$ we have $a^d = \gamma_{s,t}(a^d) = a^{sd}$. Thus $n \mid (s-1)d$ and so $s = \frac{n}{d}k + 1$, for some $0 \leq k < d$. Since $d \mid n$ and $d \nmid \frac{n}{2}$, d is even. Also k is even, as s is odd. Hence, $s = \frac{2n}{d}k_1 + 1$, for some non-negative integer k_1, and so $2n \mid (s-1)d$. Thus, $n \mid (s-1)\frac{d}{2}$ and

$$\gamma_{s,t}(a^{\frac{d}{2}}) = a^{s\frac{d}{2}} = a^{\frac{d}{2}}a^{(s-1)\frac{d}{2}} = a^{\frac{d}{2}},$$

which is a contradiction, as $a^{\frac{d}{2}} \notin H = C_{D_n}(\gamma_{s,t})$.

Similarly if $K := \langle a^d, a^r b\rangle$, where d is a divisor of n, $d \nmid n/2$, $0 \le r < d$, and $C_{D_n}(\gamma_{s,t}) = K$, for some $\gamma_{s,t} \in \mathrm{Aut}(D_n)$, we obtain a contradiction. □

Theorem 2.2. If n is an odd integer, then every non-identity subgroups of D_n is an acentralizer of D_n. If n is even, then $|\mathrm{Acent}(D_n)|$ is equal to the number of subgroups of $D_{\frac{n}{2}}$, that is

$$|\mathrm{Acent}(D_n)| = \begin{cases} \tau(n) + \sigma(n) - 1, & n \text{ is odd,} \\ \tau(\frac{n}{2}) + \sigma(\frac{n}{2}), & n \text{ is even.} \end{cases}$$

Proof. First suppose that n is odd. Let d be a divisor of n and put $d_1 := n/d$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and for $0 \le j \le n-1$, $\gamma_{1,1}(a^j b) = a^{j+1}b \ne a^j b$, we have $C_{D_n}(\gamma_{1,1}) = \langle a\rangle = \langle a^d\rangle$. If $d \ne 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(n, d_1) = d_1 \nmid 1$, by Theorem 2.1, for every $0 \le j \le n-1$, $n \nmid jd_1 + 1$, and so $\gamma_{1+d_1,1}(a^j b) = a^{j(1+d_1)+1}b = a^{jd_1+1}a^j b \ne a^j b$. It follows that $C_{D_n}(\gamma_{1+d_1,1}) = \langle a^d\rangle$.

Now consider the subgroup $H := \langle a^d, a^r b\rangle$ of D_n, where $0 \le r < d$. If $d = 1$, then $r = 0$ and $H = G = C_{D_n}(\gamma_{1,0})$. If $d = n$, then $\langle a^d, a^r b\rangle = \langle a^r b\rangle$. Note that $\gamma_{2,n-r}(a^i) = a^{2i} \ne a^i$, for all $1 \le i \le n-1$. On the other hand $\gamma_{2,n-r}(a^r b) = a^{2r+n-r}b = a^r b$ and hence $C_{D_n}(\gamma_{2,n-r}) = \langle a^r b\rangle = H$.

If $d \notin \{1, n\}$, then we put $s = 1 + d_1$ and $t = n - rd_1$. Since

$$\gamma_{s,t}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+n} = a^d,$$
$$\gamma_{s,t}(a^r b) = a^{rs+t}b = a^{r(1+d_1)+n-rd_1}b = a^r b,$$

it follows that $C_{D_n}(\gamma_{s,t}) = H$. Therefore $|\mathrm{Acent}(D_n)| = \tau(n) + \sigma(n) - 1$.

Now suppose that n is even. Let d be a divisor of $\frac{n}{2}$ and put $d_1 := n/d$. Let $H := \langle a^d\rangle$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and $\gamma_{1,1}(a^j b) = a^{j+1}b \ne a^j b$, for all $0 \le j \le n-1$, we have $C_{D_n}(\gamma_{1,1}) = \langle a\rangle = H$. If $d \ne 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(n, d_1) = d_1 \nmid 1$, by Theorem 2.1, for all $0 \le j \le n-1$, $n \nmid jd_1 + 1$, and so $\gamma_{1+d_1,1}(a^j b) = a^{j(1+d_1)+1}b = a^{jd_1+1}a^j b \ne a^j b$. It follows that $C_{D_n}(\gamma_{1+d_1,1}) = \langle a^d\rangle$.

Now we consider the subgroup $H := \langle a^d, a^r b\rangle$ of D_n, where $0 \le r < d$. If $d = 1$, then $H = G = C_{D_n}(\gamma_{1,0})$. If $d \ne 1$ and $r = 0$, then we have $\gamma_{s,0}(a^d) = a^{d(1+d_1)} = a^{d+n} = a^d$, $\gamma_{1+d_1,0}(b) = b$, and so $C_{D_n}(\gamma_{1+d_1,0}) = \langle a^d, b\rangle = H$. If $d \ne 1$ and $t \ne 0$, then we put $s = 1 + d_1$ and $t = n - rd_1$. Since

$$\gamma_{s,t}(a^d) = a^{d(1+d_1)} = a^{d+n} = a^d,$$
$$\gamma_{s,t}(a^r b) = a^{r(1+d_1)+n-rd_1}b = a^r b,$$

we have $C_{D_n}(\gamma_{s,t}) = H$. It follows that $|\mathrm{Acent}(D_n)| = \tau(\frac{n}{2}) + \sigma(\frac{n}{2})$. □

Now we compute $\mathrm{Acent}(Q_n)$. Recall that if $n > 2$, then the automorphism group of Q_n is isomorphic to $\mathbb{Z}_{2n}^* \ltimes \mathbb{Z}_{2n}$, with the canonical action of $\varepsilon : \mathbb{Z}_{2n}^* \to \mathrm{Aut}(\mathbb{Z}_{2n}) \cong \mathbb{Z}_{2n}^*$. In fact

$$\mathrm{Aut}(Q_n) = \{\gamma_{s,t} \mid s \in \mathbb{Z}_{2n}^*, t \in \mathbb{Z}_{2n}\},$$

where

$$\gamma_{s,t}(a^i) = a^{is} \quad \text{and} \quad \gamma_{s,t}(a^i b) = a^{is+t} b,$$

for all $0 \le i \le 2n - 1$. Hence $\mathrm{Aut}(Q_m) \cong \mathrm{Aut}(D_{2m})$, where $m > 2$. Note that $\mathrm{Aut}(Q_2) \cong S_4$ and $\mathrm{Aut}(D_4) \cong D_4$. We have

$$\begin{aligned} a^i \in C_{Q_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i) = a^i \\ &\Leftrightarrow a^{is} = a^i \\ &\Leftrightarrow is \equiv i \pmod{2n} \\ &\Leftrightarrow i(s-1) \equiv 0 \pmod{2n} \end{aligned}$$

and

$$\begin{aligned} a^i b \in C_{Q_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i b) = a^i b \\ &\Leftrightarrow a^{is+t} b = a^i b \\ &\Leftrightarrow is + t \equiv i \pmod{2n} \\ &\Leftrightarrow i(s-1) + t \equiv 0 \pmod{2n}. \end{aligned}$$

Lemma 2.2. (1) Every element, $x \in Q_n$ can be written uniquely as $x = a^i b^j$, where $0 \le i < 2n$ and $j = 0, 1$.

(2) $Z(Q_n) = \langle a^n \rangle \cong \mathbb{Z}_2$.

(3) $Q_n / Z(Q_n) \cong D_n$.

(4) $o(a^i) = 2n/i$ for $1 < i \leqslant 2n$ and $o(a^i b) = 4$ for all i.

(5) Every subgroup of Q_n is either cyclic or a dicyclic group.

Proof. (1)–(4) are straightforward.

Let H be a subgroup of Q_n. Suppose that $Z(Q_n) \le H$. Then $H/Z(Q_n)$ is a subgroup of D_n. Since every subgroup of D_n is either cyclic or dihedral, the same is true for $H/Z(Q_n)$. If $H/Z(Q_n)$ is cyclic, then H is cyclic (indeed H is a subgroup of $\langle a \rangle$ or $H = \langle a^i b \rangle$). Therefore, we may assume $H/Z(Q_n)$ is dihedral. Thus, $H/Z(Q_n)$ has a dihedral presentation $\langle x, y \mid x^m = y^2 = 1,\ yxy = x^{-1} \rangle$. Hence, H has the same presentation with $H/Z(Q_n)$ and so H is a dicyclic group.

Finally, if H does not contain $Z(Q_n)$ then H does not contain an element of the form $a^i b$. Therefore, $H \le \langle a \rangle$ and so it is cyclic. □

In what follows we compute acentralizers of Q_n.

Lemma 2.3. Let H be a subgroup of Q_n which does not contain $Z(Q_n)$. Then H is not an acentralizer of Q_n.

Proof. By Lemma 2.2, $H = \langle a^m \rangle$, where $m \mid 2n$, $m \nmid n$. Now suppose, for a contradiction that, H is an acentralizer of Q_n. Then there exists $\gamma_{s,t} \in \mathrm{Aut}(Q_n)$ such that $C_{Q_n}(\gamma_{s,t}) = H$. Thus, $a^m = \gamma_{s,t}(a^m) = a^{sm}$, and so $2n \mid (s-1)m$, i.e., $s = \frac{2n}{m}k + 1$, for some $0 \leq k < m$. Since $m \mid 2n$ and $m \nmid n$, m is even. Also k is even, as s is odd. Therefore, $s = \frac{4n}{m}k_1 + 1$, for some non-negative integer k_1, and hence $4n \mid (s-1)m$. Thus, $2n \mid (s-1)\frac{m}{2}$ and

$$\gamma_{s,t}(a^{\frac{m}{2}}) = a^{s\frac{m}{2}} = a^{\frac{m}{2}} a^{(s-1)\frac{m}{2}} = a^{\frac{m}{2}},$$

which is a contradiction, as $a^{\frac{m}{2}} \notin H = C_{Q_n}(\gamma_{s,t})$. □

Theorem 2.3. We have $|\mathrm{Acent}(Q_n)| = \tau(n) + \sigma(n)$.

Proof. Suppose d is a divisor of n such that $1 \leq d < n$, and $d_1 := 2n/d$. Let $H := \langle a^d \rangle$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and for $0 \leq j \leq 2n-1$, $\gamma_{1,1}(a^j b) = a^{j+1} b \neq a^j b$, we have $C_{Q_n}(\gamma_{1,1}) = \langle a \rangle$.

If $d \neq 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(2n, d_1) = d_1 \nmid 1$, by Theorem 2.1, $2n \nmid jd_1 + 1$, for all $0 \leq j \leq 2n-1$, and so $\gamma_{1+d_1,1}(a^j b) = a^{j(1+d_1)+1} b = a^{jd_1+1} a^j b \neq a^j b$. It follows that $C_{Q_n}(\gamma_{1+d_1,1}) = \langle a^d \rangle$.

Now consider the subgroup $H := \langle a^d, a^r b \rangle$ of Q_n, where $0 \leq r < d$. If $d = 1$, then $r = 0$ and $H = G = C_{Q_n}(\gamma_{1,0})$. If $d \neq 1$ and $r = 0$, then we put $s = 1 + d_1$ and $t = 0$, where $d_1 := \frac{2n}{d}$. We have $\gamma_{s,0}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+2n} = a^d$, $\gamma_{s,0}(b) = b$. Hence, $C_{Q_n}(\gamma_{1+d_1,0}) = \langle a^d, b \rangle = H$. If $d \neq 1$ and $r \neq 0$, then we put $s = 1 + d_1$ and $t = 2n - rd_1$, where $d_1 := \frac{2n}{d}$. We have

$$\gamma_{s,t}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+2n} = a^d,$$
$$\gamma_{s,t}(a^r b) = a^{rs+t} b = a^{r(1+d_1)+2n-rd_1} b = a^r b.$$

Hence $C_{Q_n}(\gamma_{s,t}) = H$. It follows that $|\mathrm{Acent}(Q_n)| = \tau(n) + \sigma(n) - 1$. □

Corollary 2.1. For all $n \geq 3$ we have $|\mathrm{Acent}(Q_n)| = |\mathrm{Acent}(D_{2n})|$.

3. Acentralizers of Groups of Order pq

It is well-known that the groups of order pq, where p and q are distinct primes, with $p > q$, are

$\mathbb{Z}_{pq}$,

$T_{p,q} = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^u \rangle$, where $o(u) = q$ in $\mathbb{Z}_p^*$ and $q \mid p-1$.

Using Theorem 3.1 below, we have $|\mathrm{Acent}(\mathbb{Z}_{pq})| = |\mathrm{Acent}(\mathbb{Z}_p)|\,|\mathrm{Acent}(\mathbb{Z}_q)| = 2 \times 2 = 4$.

Theorem 3.1. ([14, Lemma 2.1]) Let H and T be finite groups with $\gcd(|H|, |T|) = 1$. Then

$$|\mathrm{Acent}(H \times T)| = |\mathrm{Acent}(H)| \cdot |\mathrm{Acent}(T)|.$$

We compute $|\mathrm{Acent}(T_{p,q})|$. The proof of the following lemma is straightforward.

Lemma 3.1. Non-trivial subgroups of $T_{p,q}$ are $\langle a \rangle$, $\langle ba^j \rangle$, where $0 \leq j \leq p-1$.

A Frobenius group of order pq, where p is prime and $q \mid p-1$ is a group with the presentation $F_{p,q} = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^u \rangle$, where $o(u) = q$ in $\mathbb{Z}_p^*$. If q is a prime number, then $F_{p,q} \cong T_{p,q}$.

Theorem 3.2 ([10])**.** *Let p be a prime number and $q \mid p-1$. Then* $\mathrm{Aut}(F_{p,q}) \cong F_{p,p-1}$*, in fact*

$$\mathrm{Aut}(F_{p,q}) = \{\alpha_{i,j} \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\},$$

where

$$\alpha_{i,j}(a^m) = a^{im} \quad \text{and} \quad \alpha_{i,j}(b^n a^m) = b^n a^{(u^{n-1}+\cdots+u+1)j+im},$$

for all $0 \leq m \leq p-1$ and $1 \leq n \leq q-1$.

Note that if $G := F_{p,q}$, then

$$\begin{aligned} a^m \in C_G(\alpha_{i,j}) &\Leftrightarrow \alpha_{i,j}(a^m) = a^m \\ &\Leftrightarrow a^{im} = a^m \\ &\Leftrightarrow im \equiv m \pmod p \\ &\Leftrightarrow (i-1)m \equiv 0 \pmod p \end{aligned}$$

and

$$\begin{aligned} b^n a^m \in C_G(\alpha_{i,j}) &\Leftrightarrow \alpha_{i,j}(b^n a^m) = b^n a^m \\ &\Leftrightarrow b^n a^{(u^{n-1}+\cdots+u+1)j+im} = b^n a^m \\ &\Leftrightarrow im + (u^{n-1} + \cdots + u + 1)j \equiv m \pmod p \\ &\Leftrightarrow (i-1)m + (u^{n-1} + \cdots + u + 1)j \equiv 0 \pmod p. \end{aligned}$$

We note that if $p \mid u^{n-1} + \cdots + u + 1$, then $p \mid u^n - 1$ and $u^n \equiv 1 \pmod p$, which is a contradiction. Therefore, $p \nmid u^{n-1} + \cdots + u + 1$.

Lemma 3.2. The identity subgroup is not an acentralizer for any automorphism of $T_{p,q}$.

Proof. Suppose, contrary on our claim, that $\langle 1 \rangle$ is an acentralizer of $T_{p,q}$. Then there exists $\alpha_{i,j} \in \mathrm{Aut}(T_{p,q})$ such that $\alpha_{i,j}$ fixes only the identity element. If $i = 1$, then $\alpha_{1,j}(a^m) = a^m$, for all $1 \leq m \leq p-1$, which is a contradiction. Hence $\gcd(p, i-1) = 1$, and by Theorem 2.1, there exists $0 < m < p-1$, such that $p \mid (i-1)m + j$. But since $\alpha_{i,j}(ba^m) \neq ba^m$, we have $p \nmid (i-1)m + j$, which is a contradiction. Thus, the identity subgroup is not an acentralizer. □

Theorem 3.3. Every non-identity subgroup of $G := T_{p,q}$ is an acentralizer of an automorphism, and therefore $|\mathrm{Acent}(T_{p,q})| = p + 2$.

Proof. Let $H := \langle a \rangle$, which is a unique Sylow p-subgroup of G. Note that $\alpha_{1,1}(a^m) = a^m$. Since $p \nmid u^{n-1} + \cdots + u + 1$,

$$\alpha_{1,1}(b^n a^m) = b^n a^{(u^{n-1}+\cdots+u+1)+m} = b^n a^m a^{(u^{n-1}+\cdots+u+1)} \neq b^n a^m.$$

Hence, $C_G(\alpha_{1,1}) = H$.

Let $K := \langle ba^m \rangle$, where $0 \leq m \leq p-1$, which is a subgroup of G of order q. If $m = 0$, then $K = \langle b \rangle$, and since $\alpha_{2,0}(b) = b$, $\alpha_{2,0}(a) = a^2 \neq a$, it follows that $C_G(\alpha_{2,0}) = K$. If $1 \leq m \leq p-1$, then $\alpha_{2,p-m}(ba^m) = ba^{p-m+2m} = ba^m$. Also since $\alpha_{2,p-m}(a^m) = a^{2m} \neq a^m$, for all $1 \leq m \leq p-1$, we have $a^m \notin C_G(\alpha_{2,p-m})$. It follows that $C_G(\alpha_{2,p-m}) = K$. Hence, $|\mathrm{Acent}(T_{p,q})| = 1 + 1 + p = p + 2$. □

4. Acentralizers of Groups of Order pqr

In this section we compute acentralizers of groups of order pqr, where p, q, and r are distinct primes. The presentations of groups of order pqr, where p, q and r are primes such that $p > q > r$ are given in [11]. By [10] all groups of order pqr, $p > q > r$, are isomorphic to one of the following groups:

(1) $G_1 = \mathbb{Z}_{pqr}$;
(2) $G_2 = \mathbb{Z}_r \times T_{p,q}$, $q \mid p-1$;
(3) $G_3 = \mathbb{Z}_q \times T_{p,r}$, $r \mid p-1$;
(4) $G_4 = F_{p,qr}$, $qr \mid p-1)$;
(5) $G_5 = \mathbb{Z}_p \times T_{q,r}$, $r \mid q-1$;
(6) $G_{i+5} = \langle a, b, c \mid a^p = b^q = c^r = 1,\ ab = ba,\ c^{-1}bc = b^u,\ c^{-1}ac = a^{v^i} \rangle$, where $r \mid p-1$, $q-1$, $o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$, $1 \leq i \leq r-1$.

Using the above result, Theorem 3.3 and Theorem 3.1 it is suffices to compute the number of acentralizers of $F_{p,qr}$ and G_{i+5}. The proof of the following lemma is straightforward.

Lemma 4.1. Let $F_{p,qr} = \langle a, b \mid a^p = b^{qr} = 1, bab^{-1} = a^u \rangle = \langle b \rangle \ltimes \langle a \rangle$ and $o(u) = qr$ in $\mathbb{Z}_p^*$ where p, q, r are prime and $qr \mid p-1$. Then non-trivial subgroups of $F_{p,qr}$ are $A := \langle a \rangle$, $B_x := \langle ba^x \rangle$, $C_x := \langle b^q a^x \rangle$, $D_x := \langle b^r a^x \rangle$, where $0 \leq x \leq p-1$, $H := \langle b^r, a \rangle$ and $K := \langle b^q, a \rangle$.

Lemma 4.2. Non-trivial subgroups of G_{i+5} are $A := \langle a \rangle$, $B := \langle b \rangle$, AB, $H_{j,t} := \langle cb^t a^j \rangle$, $H_t := \langle a, cb^t \rangle$ and $K_j := \langle b, ca^j \rangle$, where $0 \leq j \leq p-1$, $0 \leq t \leq q-1$. In particular G_{i+5} have $pq + p + q + 5$ subgroups.

Proof. One can easily see that the order of elements of G_{i+5} is as in the Table 1,

Elements	a^j	b^t	$b^t a^j$	$c^k b^{i'} a^{j'}$
Orders	p	q	pq	r

Table 1. The order of elements G_{i+5}

where $1 \leq j \leq p-1$, $1 \leq t \leq q-1$, $0 \leq i' \leq q-1$, $0 \leq j' \leq p-1$, $1 \leq k \leq r-1$.

It is clear that $A = \langle a \rangle$ is a unique Sylow p-subgroup of G_{i+5} and $B = \langle b \rangle$ is a unique Sylow q-subgroup of G_{i+5}. Thus $AB = \langle a, b \rangle \trianglelefteq G_{i+5}$ is a unique subgroup of order pq of G_{i+5}. It is also clear that $H_{j,t} = \langle cb^t a^j \rangle$, where $0 \leq j \leq p-1$, $0 \leq t \leq q-1$, are subgroups of order r. Since A and B are normal in G_{i+5}, every subgroups of order pr should contain A and every subgroups of order qr should contain B. Thus $K_j = \langle b, ca^j \rangle$ and $H_t = \langle a, cb^t \rangle$, where $0 \le j \le p-1$, $0 \le t \le q-1$ are subgroups of order pr and qr of G_{i+5}, respectively. $\square$

Theorem 4.1 ([10])**.** Automorphism group of G_{i+5} is isomorphic to $F_{p,p-1} \times F_{q,q-1}$, in fact

$$\mathrm{Aut}(G_{i+5}) = \{\alpha_{j,t,j_1,i_1} \mid 1 \le j \le p-1, 1 \le t \le q-1, 0 \le j_1 \le p-1, 0 \le i_1 \le q-1\},$$

where

$$\begin{aligned}
&\alpha_{j,t,j_1,i_1}(a^m) = a^{jm},\\
&\alpha_{j,t,j_1,i_1}(b^n) = b^{tn},\\
&\alpha_{j,t,j_1,i_1}(c^k b^{n_1} a^{m_1}) = c^k b^{i_1(u^{k-1}+\cdots+u+1)+tn_1} a^{j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1},
\end{aligned}$$

for $1 \le m \le p-1$, $1 \le n \le q-1$, $0 \le m_1 \le p-1$, $0 \le n_1 \le q-1$ and $1 \le k \le r-1$.

Note that if $G := G_{i+5}$, then

$$\begin{aligned}
a^m \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(a^m) = a^m\\
&\Leftrightarrow a^{jm} = a^m\\
&\Leftrightarrow jm \equiv m \pmod p\\
&\Leftrightarrow m(j-1) \equiv 0 \pmod p
\end{aligned}$$

and

$$\begin{aligned}
b^n \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(b^n) = b^n\\
&\Leftrightarrow b^{tn} = b^n\\
&\Leftrightarrow tn \equiv n \pmod q\\
&\Leftrightarrow n(t-1) \equiv 0 \pmod q
\end{aligned}$$

and

$$\begin{aligned}
c^k b^{n_1} a^{m_1} \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(c^k b^{n_1} a^{m_1}) = c^k b^{n_1} a^{m_1}\\
&\Leftrightarrow c^k b^{i_1(u^{k-1}+\cdots+u+1)+tn_1} a^{j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1} = c^k b^{n_1} a^{m_1}\\
&\Leftrightarrow i_1(u^{k-1}+\cdots+u+1)+tn_1 \equiv n_1 \pmod q,\\
&\qquad j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1 \equiv m_1 \pmod p\\
&\Leftrightarrow i_1(u^{k-1}+\cdots+u+1)+(t-1)n_1 \equiv 0 \pmod q,\\
&\qquad j_1(v^{(k-1)i}+\cdots+v^i+1)+(j-1)m_1 \equiv 0 \pmod p.
\end{aligned}$$

Lemma 4.3. The identity subgroup and the subgroups C_x , D_x, where $0 \le x \le p-1$, H and K (defined in Lemma 4.1) are not acentralizers for any automorphism of $G := F_{p,qr}$.

Proof. As in the proof of Lemma 3.2 we can see that the identity subgroup is not an acentralizer.

Now suppose, for a contradiction that $C_x := \langle b^q a^x \rangle$, where $0 \leq x \leq p-1$ is an acentralizers of G. Then there exists $\alpha_{i,j} \in \mathrm{Aut}(G)$ such that $C_G(\alpha_{i,j}) = C_x$, where $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$. If $i = 1$, then $\alpha_{1,j}(a^m) = a^m$, for every $1 \leq m \leq p-1$, this contradicts $a^m \notin \langle b^q a^x \rangle$. Hence $\gcd(i-1, p) = 1$, by Theorem 2.1, there exists $0 < m < p-1$ such that $p \mid j + (i-1)m$. But since $ba^m \notin C_x = C_G(\alpha_{i,j})$, $\alpha_{i,j}(ba^m) = ba^{j+im} = ba^m a^{j+(i-1)m} \neq ba^m$, which implies that $p \nmid j + (i-1)m$, which is a contradiction.

Similarly we have H, D_x, and K are not acentralizers. □

Theorem 4.2. We have $|\mathrm{Acent}(F_{p,qr})| = p + 2$.

Proof. The proof is similar to that of Theorem 3.3. □

Lemma 4.4. The identity subgroup is not an acentralizer for any automorphism of G_{i+5}.

Proof. On the contrary, suppose that $\langle 1 \rangle$ is an acentralizer of G_{i+5}. Then there exists $\alpha_{j,t,j_1,i_1} \in \mathrm{Aut}(G_{i+5})$ such that α_{j,t,j_1,i_1} fixes only the identity element. If $j = 1$ or $t = 1$, then $\alpha_{1,t,j_1,i_1}(a^m) = a^m$ and $\alpha_{j,1,j_1,i_1}(b^n) = b^n$, for all $1 \leq m \leq p-1$ and $1 \leq n \leq q-1$, which is a contradiction. Hence $\gcd(j-1, p) = 1$ and $\gcd(t-1, q) = 1$. Hence, by Theorem 2.1, there exist $0 < m_1 < p-1$ and $0 < n_1 < q-1$ such that $p \mid j_1 + (j-1)m_1$ and $q \mid i_1 + (t-1)n_1$. But since

$$\alpha_{j,t,j_1,i_1}(cb^{n_1}a^{m_1}) = cb^{i_1+tn_1}a^{j_1+jm_1} = cb^{n_1}a^{m_1}b^{i_1+(t-1)n_1}a^{j_1+(j-1)m_1} \neq cb^{n_1}a^{m_1},$$

either $p \nmid j_1 + (j-1)m_1$ or $q \nmid i_1 + (t-1)n_1$, which is a contradiction. Thus, the identity subgroup is not an acentralizer. □

Theorem 4.3. Every non-identity subgroup of $G := G_{i+5}$ is an acentralizer of an automorphism, that is $|\mathrm{Acent}(G_{i+5})| = pq + p + q + 4$.

Proof. We use the notation of Theorem 4.1. Note that $\alpha_{1,1,0,0}$ is the identity automorphism of G and so $C_G(\alpha_{1,1,0,0}) = G$.

Now we show that $A = \langle a \rangle$ is an acentralizer. It is clear that $\alpha_{1,2,1,1}(a) = a$ and $\alpha_{1,2,1,1}(b^n) = b^{2n} = b^n b^n \neq b^n$, for all $1 \leq n \leq q-1$. Furthermore since $p \nmid (v^{(k-1)i} + \cdots + v^i + 1)$,

$$\begin{aligned}\alpha_{1,2,1,1}(c^k b^{n_1} a^{m_1}) &= c^k b^{(u^{k-1}+\cdots+u+1)+2n_1} a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1} \\ &= c^k b^{n_1} a^{m_1} b^{(u^{k-1}+\cdots+u+1)+n_1} a^{(v^{(k-1)i}+\cdots+v^i+1)} \neq c^k b^{n_1} a^{m_1}.\end{aligned}$$

It follows that $C_G(\alpha_{1,2,1,1}) = A$.

Let $B = \langle b \rangle$ be the unique Sylow q-subgroup of G. It is clear that $\alpha_{2,1,1,1}(b^n) = b^n$ and so $b^n \in C_G(\alpha_{2,1,1,1})$. Since $1 \leq m \leq p-1$, $\alpha_{2,1,1,1}(a^m) = a^{2m} = a^m a^m \neq a^m$. Also since $\gcd(u^{k-1}+\cdots+u+1,q)=1$, so $q\nmid(u^{k-1}+\cdots+u+1)$. Thus,

$$\begin{aligned}\alpha_{2,1,1,1}(c^kb^{n_1}a^{m_1}) &= c^kb^{(u^{k-1}+\cdots+u+1)+n_1}a^{(v^{(k-1)i}+\cdots+v^i+1)+2m_1}\\ &= c^kb^{n_1}a^{m_1}b^{(u^{k-1}+\cdots+u+1)}a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1}\neq c^kb^{n_1}a^{m_1}.\end{aligned}$$

Hence, $C_G(\alpha_{2,1,1,1})=B$.

Let $AB=\langle a,b\rangle$ be the unique subgroup of G of the order pq. It is clear that $\alpha_{1,1,1,1}(a^m)=a^m$ and $\alpha_{1,1,1,1}(b^n)=b^n$. Thus, $a^m,b^n\in C_G(\alpha_{1,1,1,1})$. Since $\gcd(u^{k-1}+\cdots+u+1,q)=1$ and $\gcd(v^{(k-1)i}+\cdots+v^i+1,p)=1$, so $q\nmid(u^{k-1}+\cdots+u+1)$ and $p\nmid(v^{(k-1)i}+\cdots+v^i+1)$. Thus,

$$\begin{aligned}\alpha_{1,1,1,1}(c^kb^{n_1}a^{m_1}) &= c^kb^{(u^{k-1}+\cdots+u+1)+n_1}a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1}\\ &= c^kb^{n_1}a^{m_1}b^{(u^{k-1}+\cdots+u+1)}a^{(v^{(k-1)i}+\cdots+v^i+1)}\neq c^kb^{n_1}a^{m_1}.\end{aligned}$$

Hence, $C_G(\alpha_{1,1,1,1})=AB$.

Let $H_{m_1,n_1}=\langle cb^{n_1}a^{m_1}\rangle$ where $0\leq m_1\leq p-1$ and $0\leq n_1\leq q-1$ be the unique subgroup of G of order pq. First suppose $m_1=n_1=0$. Then $\alpha_{2,2,0,0}(c)=c$. Since $1\leq m\leq p-1$, $1\leq n\leq q-1$, we have $\alpha_{2,2,0,0}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,0,0}(b^n)=b^{2n}\neq b^n$. Thus $C_G(\alpha_{2,2,0,0})=H_{0,0}=\langle c\rangle$. Now suppose $n_1=0$, $m_1\neq 0$. Then $\alpha_{2,2,p-m_1,0}(ca^{m_1})=ca^{p-m_1+2m_1}=ca^{m_1}$ and $\alpha_{2,2,p-m_1,0}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,p-m_1,0}(b^n)=b^{2n}\neq b^n$. So $C_G(\alpha_{2,2,p-m_1,0})=H_{m_1,0}=\langle ca^{m_1}\rangle$. Similarly, if $m_1=0$, $n_1\neq 0$, then $\alpha_{2,2,0,q-n_1}(cb^{n_1})=cb^{q-n_1+2n_1}=cb^{n_1}$, $\alpha_{2,2,0,n_1}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,p-m_1,0}(b^n)=b^{2n}\neq b^n$. Hence, $C_G(\alpha_{2,2,0,q-n_1})=H_{0,n_1}=\langle cb^{n_1}\rangle$. Finally suppose that $m_1\neq 0$ and $n_1\neq 0$. Then

$$\alpha_{2,2,p-m_1,q-n_1}(cb^{n_1}a^{m_1})=cb^{q-n_1+2n_1}a^{p-m_1+2m_1}=cb^{q+n_1}a^{p+m_1}=cb^{n_1}a^{m_1},$$

and so, $cb^{n_1}a^{m_1}\in C_G(\alpha_{2,2,p-m_1,q-n_1})$. Since $1\leq m\leq p-1$ and $1\leq n\leq q-1$, we have $\alpha_{2,2,p-m_1,q-n_1}(a^m)=a^{2m}=a^ma^m\neq a^m$ and $\alpha_{2,2,p-m_1,q-n_1}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{2,2,p-m_1,q-n_1})=H_{m_1,n_1}$.

Now we consider the unique subgroup $AH_{n_1}=\langle a,cb^{n_1}\rangle$, where $0\leq n_1\leq q-1$ of order rp. First suppose that $n_1=0$. Then $\alpha_{1,2,0,0}(a^m)=a^m$. Also $\alpha_{1,2,0,0}(c^k)=c^k$. So $a^m,c^k\in C_G(\alpha_{1,2,0,0})$. Since $1\leq n\leq q-1$ we have $\alpha_{1,2,0,0}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{1,2,0,0})=\langle a,c\rangle=AH_0$. Now suppose that $n_1\neq 0$. Then $\alpha_{1,2,0,q-n_1}(a^m)=a^m$. Also, $\alpha_{1,2,0,q-n_1}(cb^{n_1})=cb^{q-n_1+2n_1}=cb^{q+n_1}=cb^{n_1}$. So, $a^m,cb^{n_1}\in C_G(\alpha_{1,2,0,q-n_1})$. Since $1\leq n\leq q-1$, we have $\alpha_{1,2,0,q-n_1}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{1,2,0,q-n_1})=AH_{n_1}$.

Now consider the unique subgroup $BH_{m_1}=\langle b,ca^{m_1}\rangle$, where $0\leq m_1\leq p-1$, of order rq. First suppose that $m_1=0$. Then $\alpha_{2,1,0,0}(b^n)=b^n$. Also $\alpha_{2,1,0,0}(c^k)=c^k$. So $b^n,c^k\in C_G(\alpha_{2,1,0,0})$. Since $1\leq m\leq p-1$ we have $\alpha_{2,1,j_1,0}(a^m)=a^{2m}=a^ma^m\neq a^m$. Hence, $C_G(\alpha_{2,1,0,0})=\langle b,c\rangle=BH_0$. Now suppose that $m_1\neq 0$. Then $\alpha_{2,1,p-m_1,0}(b^n)=b^n$. Also, $\alpha_{2,1,p-m_1,0}(ca^{m_1})=ca^{p-m_1+2m_1}=ca^{p+m_1}=ca^{m_1}$. So, $b^n,ca^{m_1}\in C_G(\alpha_{2,1,p-m_1,0})$. Since $1\leq m\leq p-1$ we have $\alpha_{2,1,p-m_1,0}(a^m)=a^{2m}=a^ma^m\neq a^m$. Hence, $C_G(\alpha_{2,1,p-m_1,0})=BH_{m_1}$.

Therefore, $|\mathrm{Acent}(G_{i+5})| = 1+1+1+1+pq+q+p = pq+p+q+4$. □

5. Acentralizers of Finite Symmetric Groups

In this section we compute $|\mathrm{Acent}(S_n)|$. First we note that $S_2 \cong \mathbb{Z}_2$ and so $|\mathrm{Acent}(S_2)| = 1$. Also if $n = 6$, then $\mathrm{Aut}(S_6) = S_6 \rtimes \mathbb{Z}_2$ and by GAP [9] we see that $|\mathrm{Acent}(S_6)| = 443$. Now since for every $n \neq 6$, $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n) = S_n$, we have $\mathrm{Acent}(S_n) = \mathrm{Cent}(S_n)$. Hence in order to find $|\mathrm{Acent}(S_n)|$ we need to find $|\mathrm{Cent}(S_n)|$. Recall that the conjugacy class an element g of a group G, is the set of elements its conjugate, that is

$$x^G := \{xgx^{-1} \mid x \in G\}.$$

Let A and G be groups, and let G act on a set X. Let B be the group of all of functions from X into A. The product of two elements f and g of B $fg(x) = f(x)g(x)$. The group G acts on B via $f^g(x) = f(gxg^{-1})$. The semidirect product of B and G with respect to this action is called the general wreath product.

Theorem 5.1. ([17, Page 297]) Let α be an element of S_n of cycle type $(r_1^{\lambda_1}, \ldots, r_k^{\lambda_k})$, then the centralizer of α in S_n is a direct product of k groups of the form $\mathbb{Z}_{r_i} \wr S_{\lambda_i}$, the general wreath product. The order of $C_{S_n}(\alpha)$ is equal to $\prod \lambda_i! r_i^{\lambda_i}$.

Every permutation α in S_n can be written as the product of disjoint cycles $\alpha = \alpha_1 \cdots \alpha_k$, where $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$, $j = 1, \ldots k$, is a product λ_j disjoint cycles of length r_j such that $r_1 < r_2 < \cdots < r_k$. The cycle, type of α is

$$r = (\underbrace{r_1, \ldots, r_1}_{\lambda_1}, \ldots, \underbrace{r_k, \ldots, r_k}_{\lambda_k}) = (r_1^{\lambda_1}, \ldots, r_k^{\lambda_k}).$$

We will not omit those r_i which are 1, so we have $\lambda_1 r_1 + \cdots + \lambda_k r_k = n$. The r_j's are distinct and λ_j's describe their multiplicities in the partition r of n. For $j = 1, \ldots, k$ let Y_j be the of letters in $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$. In fact

$$Y_j = \left\{a_{j,1}^{(1)}, a_{j,1}^{(2)}, \ldots, a_{j,1}^{(r_j)}, \ldots, a_{j,\lambda_j}^{(1)}, a_{j,\lambda_j}^{(2)}, \ldots a_{j,\lambda_j}^{(r_j)}\right\},$$

where $\alpha_{j,1} = (a_{j,1}^{(1)}\ a_{j,1}^{(2)}\ \cdots a_{j,1}^{(r_j)})$, …, $\alpha_{j,\lambda_j} = (a_{j,\lambda_j}^{(1)}\ a_{j,\lambda_j}^{(2)}\ \cdots a_{j,\lambda_j}^{(r_j)})$. Clearly, Y_j is α-invariant and $C_G(\alpha)$-invariant; and the restriction of α to Y_j is α_j, A permutation θ commutes, with α if and only if $\alpha = \beta_1 \cdots \beta_k$, where $\beta_j = \beta_{j,1}\beta_{j,2}\cdots\beta_{j,\lambda_j}$, $\beta_{j,1} = (b_{j,1}^{(1)}\ b_{j,1}^{(2)}\ \cdots b_{j,1}^{(r_j)})$, …, $\beta_{j,\lambda_j} = (b_{j,\lambda_j}^{(1)}\ b_{j,\lambda_j}^{(2)}\ \cdots b_{j,\lambda_j}^{(r_j)})$, and $\theta(a_{j,\lambda_j}^{(r_j)}) = b_{j,\lambda_j}^{(r_j)}$. Now, θ commutes with α if and only if each Y_j is θ-invariant and if the restriction β_j of β on Y_j commutes with restriction of α_j of α on Y_j. Since $Y_i \cap Y_j = \emptyset$ for $i \neq j$, the permutation β is uniquely determined by giving its restrictions on Y_j. Hence we have $C_{S_n}(\alpha) = C_1 \times \cdots \times C_k$, where C_j is the centralizer of α_j in $\mathrm{Sym}(Y_j)$.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_\lambda$, where $\sigma_1 = (a_{1,0}\ a_{1,1}\ \cdots a_{1,r-1})$, $\sigma_2 = (a_{2,0}\ a_{2,1}\ \cdots a_{2,r-1})$, …, $\sigma_\lambda = (a_{\lambda,0}\ a_{\lambda,1}\ \cdots a_{\lambda,r-1})$ be the product of λ cycles of length r. Let Y be the set of all letters in σ, that is

$$Y = \{a_{1,0}\ a_{1,1}\ \cdots a_{1,r-1}, a_{2,0}\ a_{2,1}\ \cdots a_{2,r-1}, \ldots, a_{i,0}, a_{i,1}, \ldots a_{i,r-1}\}.$$

Let $M_r := \{m \in \mathbb{N} \mid m \leq r,\ \gcd(m,r) = 1\}$. Then we have $|M_r| = \phi(r)$, where ϕ is the Euler's totient function. For every $t \in M_r$, since $\gcd(r,t) = 1$ and the order of σ is r, we have $C_G(\sigma) = C_G(\sigma^t)$, where $G := \mathrm{Sym}(Y)$. It follows that the number of different centralizers of permeations which are product of λ cycles of the same length r with letters in Y is

$$\frac{|\sigma^{\mathrm{Sym}(Y)}|}{\phi(r)}.$$

Now suppose that $\alpha = \alpha_1 \cdots \alpha_k$, where $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$, $j = 1, \ldots k$, is a product λ_j disjoint cycles of length r_j such that $r_1 < r_2 < \cdots < r_k$. Let Y_j, $j = 1, \ldots, r$, be the set of letters in α_j. The cycle α_1 in the decomposition $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ in S_n can be chosen in $\binom{n}{|Y_1|} = \binom{n}{r_1\lambda_1}$ ways. The cycle α_2 can be chosen in $\binom{n-|Y_1|}{|Y_2|} = \binom{n-r_1\lambda_1}{r_2\lambda_2}$ ways. In general α_j can be chosen in

$$\binom{n - \sum_{i=1}^{j-1}|Y_i|}{|Y_j|} = \binom{n - \sum_{i=1}^{j-1}\lambda_i}{r_j\lambda_j} = \binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

ways. If $r_1 = 1$, $\lambda_1 = 2$, $r_2 = 2$, $\lambda_2 = 1$, and $\sum_{j=3}^{k}\lambda_j r_j = n-4$, then let $\widehat{\alpha_1}$ be two cycles of length 1 with letters in α_2 and $\widehat{\alpha_2}$ be a cycle of length 2 with letters in α_1. Then $\alpha_1\alpha_2\alpha_3\cdots\alpha_k$ and $\widehat{\alpha_1}\,\widehat{\alpha_2}\alpha_3\cdots\alpha_k$ have the same centralizers. Hence, in this case we have

$$\frac{1}{2}\prod_{j=1}^{k}\frac{|\alpha_j{}^{\mathrm{Sym}(Y_j)}|}{\phi(r_j)}\binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

different centralizers of permutations whose cycle types are the same with α. Otherwise there are

$$\prod_{j=1}^{k}\frac{|\alpha_j{}^{\mathrm{Sym}(Y_j)}|}{\phi(r_j)}\binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

different centralizers of permutations whose cycle types are the same with α in S_n.

In the following tables we denote the number of acentralizers of the same type as a permutation π by $\sharp C_{S_n}(\pi)$.

π	$()$	$(*,*)$	$(*,*,*)$
$\lvert\pi^{S_3}\rvert$	1	3	2
cycle type	(1^3)	$(1^1,2^1)$	(3^1)
$C_{S_3}(\pi) \cong$	$C_1 \wr S_3 \cong S_3$	$(C_2 \wr S_1) \times (C_1 \wr S_1) \cong C_2$	$C_3 \wr S_1 \cong C_3$
$\sharp C_{S_3}(\pi)$	1	3	1

So, $|\mathrm{Cent}(S_3)| = 5$.

π	$()$	$(*,*)$	$(*,*,*)$	$(*,*)(*,*)$	$(*,*,*,*)$
$\lvert\pi^{S_4}\rvert$	1	6	8	3	6
cycle type	(1^4)	$(1^2,2^1)$	$(1^1,3^1)$	(2^2)	(4^1)
$C_{S_4}(\pi) \cong$	S_4	$C_2 \times C_2$	C_3	D_4	C_4
$\sharp C_{S_4}(\pi)$	1	3	4	3	3

So, $|\mathrm{Cent}(S_4)| = 14$.

π	$()$	$(*,*)$	$(*,*,*)$	$(*,*)(*,*)$	$(*,*,*,*)$	$(*,*)(*,*,*)$	$(*,*,*,*,*)$
$\lvert\pi^{S_5}\rvert$	1	10	20	15	30	20	24
cycle type	(1^5)	$(1^3,2^1)$	$(1^2,3^1)$	$(1^1,2^2)$	$(1^1,4^1)$	$(2^1,3^1)$	(5^1)
$C_{S_5}(\pi)\cong$	S_5	$C_2\times S_3$	$C_3\times C_2$	D_8	C_4	$C_2\times C_3$	C_5
$\sharp C_{S_5}(\pi)$	1	10	10	15	15	10	6

So, $|\mathrm{Cent}(S_5)| = 67$.

6. Conclusion

The acentralizer of an automorphism of a group is defined to be the subgroup of its fixed points. In particular the acentralizer of an inner automorphism is just a centralizer. In this paper we computed the acentralizers of some classes of groups, namely dihedral, dicyclic and symmetric groups. As a result we see that if $n \geq 3$, then the numbers of acentralizers of the dihedral group and the dicyclic group of order $4n$ are equal. Also we determined the acentralizers of groups of orders pq and pqr, where p, q and r are distinct primes.

References

[1] A. Abdollahi, S. M. J. Amiri and A. M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math. **33**(1) (2007), 43–57. https://doi.org/10.1.1.640.5229

[2] A. R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq. **7**(1) (2000), 139–146. https://doi.org/10.1007/s10011-000-0139-5

[3] A. R. Ashrafi, *Counting the centralizers of some finite groups*, The Korean Journal of Computational & Applied Mathematics **7**(1) (2000), 115–124. https://doi.org/10.1007/BF03009931

[4] A. R. Ashrafi and B. Taeri, *On finite groups with a certain number of centralizers*, The Korean Journal of Computational & Applied Mathematics **17**(1–2) (2005), 217–227. https://doi.org/10.1007/BF02936050

[5] A. R. Ashrafi and B. Taeri, *On finite groups with exactly seven element centralizers*, The Korean Journal of Computational & Applied Mathematics **22**(1–2) (2006), 403–410. https://doi.org/10.1007/BF02896488

[6] S. M. Belcastro and G. J. Sherman, *Counting centralizers in finite groups*, Math. Mag. **67**(5) (1994), 366–374. https://doi.org/10.1080/0025570X.1994.11996252

[7] S. J. Baishya *Counting centralizers and z-classes of some F-groups*, Comm. Algebra (2021) (to appear). https://doi.org/10.1080/00927872.2021.2008413

[8] Z. Foruzanfar, Z. Mostaghim and M. Rezaei, *Classification of finite simple groups according to the number of centralizers*, Malays. J. Math. Sci. **12**(2) (2018), 211–222.

[9] The GAP Group. Groups, Algorithms and Programing, Version 4.4, 2005. http://www.gap-system.org

[10] M. Ghorbani and F. Nowroozi Larki, *Automorphism group of groups of order pqr*, Algebraic Structures and their Applications **1**(1) (2014), 49–56.

[11] H. Hölder, *Die Gruppen der Ordnungen* p^3, pq^2, pqr, p^4, Math. Ann. **43** (1893), 371–410. https://doi.org/10.1007/BF01443651

[12] M. A. Iranmanesh and M. H. Zareian, *On n-centralizer CA-groups*, Comm. Algebra **49**(10) (2021), 4186–4195. https://doi.org/10.1080/00927872.2021.1916513

[13] Z. Mozafar and B. Taeri, *Acentralizers of abelian groups of rank* 2, Hacet. J. Math. Stat. **49**(1) (2020), 273–281. https://doi.org/10.15672/hujms.546988

[14] M. M. Nasrabadi and A. Gholamian, *On finite n-Acentralizers groups*, Comm. Algebra **43**(2) (2015), 378–383. https://doi.org/10.1080/00927872.2013.842244
[15] K. H. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley, Reading, Massachusetts Menlo Park, California, 1983.
[16] P. Seifizadeh, A. Gholamian, A. Farokhianee and M. A. Nasrabadi, *On the autocentralizer subgroups of finite p-groups*, Turkish J. Math. **44** (2020), 1802–1812. https://doi.org/10.3906/mat-1911-89
[17] M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
[18] M. Zarrin, *On element-centralizers in finite groups*, Arch. Math. **93** (2009), 497–503. https://doi.org/10.1007/s00013-009-0060-1

[1]Department of Mathematical Sciences,
Isfahan University of Technology,
Isfahan 84156-83111,
Iran
Email address: z.mozafar@math.iut.ac.ir
Email address: b.taeri@iut.ac.ir

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 239–252.

OPTIMIZING CHANCE CONSTRAINT MULTIPLE-OBJECTIVE FRACTIONAL MATHEMATICAL PROGRAMMING PROBLEM INVOLVING DEPENDENT RANDOM VARIABLE

BERHANU BELAY[1] AND SRIKUMAR ACHARYA[2]

Abstract. This manuscript suggests a methodology to solve chance constraint multiple-objective linear fractional mathematical programming problem in which the parameters are dependent random variables to each other. The proposed problem is formulated by taking few of the parameters as continuous dependent random variables. The proposed model cannot be solved directly by using existing methodology. Thus in order to solve the proposed model, an equivalent deterministic model is derived. The procedure to solve the proposed model is accomplished in two main steps. Initially, the proposed multiple-objective chance constraint linear fractional mathematical problem is transformed to deterministic equivalent multiple-objective linear fractional mathematical programming by the help of chance constrained method. In the second step, multiple-objective functions, which consist of fractional functions is solved by using lexicographic programming approach. Finally, an example is mentioned to illustrate the methodology.

1. Introduction

Nowadays, in real world problems, many decision making problems have multiple and conflicting objectives. The mathematical programming problem involving more than one objective functions that are conflicting in nature is known as multiple-objective programming problem. If the objective functions are ratio of affine functions, the problem is called multiple-objective linear fractional mathematical programming problem.

Key words and phrases. Multiple-objective programming problem, chance constraint programming problem, fractional programming problem, lexicography method, dependent random variables.

2020 *Mathematics Subject Classification.* Primary: 90C05, 90C15. Secondary: 90C29, 90C32.

DOI 10.46793/KgJMat2502.239B

Received: May 16, 2021.

Accepted: March 11, 2022.

In a multiple-objective linear fractional programming problem, the optimal solution for one single objective need not be an optimal solution for the other single objective function. As a result, another solution which is called compromise solution must be needed to optimize all objective functions. A solution is said to be efficient solution, in the event that it cannot improve one objective function without degrading their performance in one of the other objective functions. There exist several methodologies to find efficient solution of multiple-objective fractional programming problem. Some of the methods seen in the literature are: J. S. Kornbluth and R. E. Steuer [11] proposed simplex based method to get weakly efficient solutions for multi-objective fractional programming problem. Luhandjula, [13] solved multi-objective fractional programming problem using a fuzzy programming approach. Dutta et al. [8] solved a special type of programming problem having identical denominators using variable transformation method. By applying techniques used in [6] for suitable transformation, M. Chakraborty and S. Gupta [3] solved multi-objective fractional programming problem based on set theoretic approach. Jain [9] proposed Gauss elimination method to solve multi-objective linear fractional programming problem. Porchelvi et al. [14] presented a method to find efficient solution of multi objective fractional programming problem with the help of complementary method proposed by Dheyab [7] by transforming fractional programming problem into equivalent programming problem. Tantawy [16] presented a feasible direction method for multi-objective fractional programming problem, where the denominators are identical functions.

In real world problems, the data of mathematical programming problem may not be known with certainty. If the uncertainty occurs due to randomness, then the programming problem is called stochastic programming problem. In this case, some or all of the data of the programming problem can be characterized with random variables following known distributions. There are two techniques that are used to solve stochastic mathematical programming problems. Namely, chance constraint and two stage mathematical programming. Our objective in this manuscript is to study the chance constraint mathematical programming problem. Chance constraint mathematical programming is one of the method used to solve mathematical programming problem at which the restrictions have fixed probability of violation. In this case the randomness can be shown either within the coefficient of objective functions, with in the constraint coefficients, within the right hand side parameters or in combination of constraint coefficients, objective function, and right hand side parameters.

In this manuscript, the randomness occurs only in the left side of the constraints. The difficulty of chance constraint programming problem is handling the chance constraints.

To handle these constraints some researchers obtained the deterministic equivalent of the problem with the concept of probability distribution function. Charnes and Cooper [5] presented the deterministic equivalence of chance constraint programming problem that includes independent normal random variables. Lingaraj and Wolfe [12] obtained the deterministic equivalence of chance constraint programming problem where the random variable follow gamma distribution. Knott [10] presented chance constraint mathematical programming by considering the parameters within the right side of limitations as uniform random variables. Biswal et al. [2] solved single objective probabilistic linear programming problem by considering few parameters as exponential random variables. Sahoo and Biswal [15] presented chance constraint programming problem where the random variables within the joint constraint have both normal and log normal distribution. Charles et al. [4] proposed chance constraint programming by considering the parameters within the right side of limitations having generalized continuous distribution. In spite of the fact that a few approaches are presented to obtain the deterministic equivalence of chance constraint programming problem including independent random variables, any method is not mentioned to find the deterministic equivalence of chance constraint programming problem including dependent random variables. i.e two random variables are called dependent random variable, if the probability of events associated with one random variable influence the distribution of probabilities of the other variable.

Chance constraint programming problem can be applied to the programming problem where the fractional objective functions are multiple, non commensurable and conflicting each other. In this case, there is no single solution that optimizes all fractional objective functions. The solutions of multiple-objective fractional programming problem are known as compromise solution or efficient solution. In multiple-objective fractional programming problem, decision makers need the satisfaction of criteria instead of optimizing the objective function. However, such type of problems are more complex when the parameters are uncertain. Recently Acharya et al. [1] solved multi-objective chance constraint fractional programming problems involving two parameters independent Cauchy random variables.

In this manuscript, an attempt has been made to get the lexicographic optimal solution of chance constraint multiple-objective linear fractional mathematical programming problem involving dependent normal random variable where the randomness occurs only in the constraint coefficient.

Multiple-objective chance constraint linear fractional programming is a special class of multiple-objective stochastic linear fractional programming problem.

This manuscript has been organized within six sections including the references. The first section states about the brief introduction of programming problem. The second section states the mathematical model of multiple-objective fractional programming problem. Section 3 presents the transformation strategy of multiple-objective chance constraint linear fractional programming problem into its deterministic equivalent. Section 4 states the solution procedure of multiple-objective chance constraint linear fractional programming problem. In Section 5 numerical example is given to demonstrate the proposed method. The final section presents the conclusion of the paper followed by references.

2. Mathematical Model

The general multiple-objective fractional programming problem can be stated as:

$$\max / \min : Z_k = \frac{N_k(X)}{D_k(X)} \tag{2.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, \geq, =) b_i, \quad i = 1, 2, \ldots, m, \tag{2.2}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.3}$$

where the functions $N_k(X)$ and $D_k(X)$ are continuous real valued functions defined from $\mathbb{R}^n \to \mathbb{R}$, the constraint functions can be linear or non linear functions, and the variable X is n-dimensional vector.

If Z_k are the objective functions which are defined on a compact set, then the point x^0 is compromise solution for the given problem, if and only if x^0 optimizes each objective function Z_k. The compromise solutions exist if the feasible space is non-empty and compact as well as the functions $N_k(X)$ and $D_k(X)$ are continuous functions and the denominator is different from zero.

If $N_k(X)$ and $D_k(X)$ are affine functions, the programming problem given by (2.1)–(2.3) is called multiple-objective linear fractional programming problem. If the parameters of multiple-objective linear fractional programming problem are uncertain due to randomness, then the given programming problem is called multiple-objective chance constraint linear fractional programming problem.

A multiple-objective chance constraint linear fractional programming problem is expressed as:

$$\max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{\sum\limits_{j=1}^{n} c_{kj} x_j + c_{0k}}{\sum\limits_{j=1}^{n} d_{kj} x_j + d_{0k}}, \quad k = 1, 2, \ldots, K, \tag{2.4}$$

subject to

$$P\left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right) \geq \alpha_i, \quad i = 1, 2, \ldots, m, \tag{2.5}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{2.6}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.7}$$

where

$$\sum_{j=1}^{n} c_{kj} x_j + c_{0k} \quad \text{and} \quad \sum_{j=1}^{n} d_{kj} x_j + d_{0k},$$

are linear functions of $x_j, c_{kj}, d_{kj} \in \mathbb{R}^n$, $a_{ij} \in \mathbb{R}^{m \times n}$, c_{0k} and d_{0k} are scalars, P indicates probability, α_i represents aspiration level for i-th constraint.

3. Transformation Technique

In order to understand the transformation of multiple-objective chance constraints linear fractional programming problem into its deterministic equivalent, we focus on the following two cases.

Case 1. Let's consider the following multiple-objective chance constraint linear fractional programming problem with two decision variables x_1 and x_2.

$$\min / \max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{c_{k1}x_1 + c_{k2}x_2 + c_{k0}}{d_{k1}x_1 + d_{k2}x_2 + d_{k0}}, \quad k = 1, 2, \dots, K, \tag{3.1}$$

subject to

$$P(a_{i1}x_1 + a_{i2}x_2 \le b_i) \ge \alpha_i, \quad i = 1, 2, \dots, m, \tag{3.2}$$

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \dots, m, \tag{3.3}$$

$$x_1, x_2 \ge 0, \quad j = 1, 2, \dots, n, \tag{3.4}$$

where $c_{k1}x_1 + c_{k2}x_2 + c_{k0}$ and $d_{k1}x_1 + d_{k2}x_2 + d_{k0}$ are linear functions of x_1 and x_2, $c_{k1}, \dots, c_{k2}, \dots, c_{k0}$, $d_{k1}, \dots, d_{k2}, \dots, d_{k0} \in \mathbb{R}$.

The mathematical programming problem (3.1)–(3.4) is equivalent to the mathematical programming problem given by:

$$\min / \max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{c_{k1}x_1 + c_{k2}x_2 + c_{k0}}{d_{k1}x_1 + d_{k2}x_2 + d_{k0}}, \quad k = 1, 2, \dots, K, \tag{3.5}$$

subject to

$$\mathrm{E}(a_{i1}x_1 + a_{i2}x_2) \le b_i - k_{\beta_i}\sqrt{\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2)}, \quad i = 1, 2, \dots, m, \tag{3.6}$$

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \dots, m, \tag{3.7}$$

$$x_1, x_2 \ge 0, \quad j = 1, 2, \dots, n. \tag{3.8}$$

The equivalence of the two mathematical programming problems is proven by the existence of one to one function. In this case, the normal probability density function is used as a one to one function.

Let x be a normal random variable, then probability density function is expressed by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \sigma > 0, -\infty < \mu < \infty.$$

In (3.6) assume that the coefficients a_{i1} and a_{i2} are dependent random variables having normal distribution with variance σ^2 and mean μ. Let's assume the i-th constraint in the chance constraint given in (3.2)

$$P(a_{i1}x_1 + a_{i2}x_2 \le b_i) \ge \alpha_i. \tag{3.9}$$

Let q be the random variable defined as $q = a_{i1}x_1 + a_{i2}x_2$, then (3.9) is expressed by

$$P(q \le b_i) \ge \alpha_i. \tag{3.10}$$

Since q is a linear combination of normally distributed random variables, then it is a normal distributed random variable. Consequently the chance constraint given in (3.10) can be expressed as

$$P\left(\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}} \le \frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \ge \alpha_i, \tag{3.11}$$

where $\mathrm{E}(q)$ and $\mathrm{Var}(q)$ are mean and variance of the random variable q and $\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}$ is a standard normal random variable.

The equation (3.11) can be written using cumulative distribution function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}} e^{-\frac{z^2}{2}}dz \ge \alpha_i, \quad \text{where } z=\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}},$$

and

$$\varphi\left(\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \ge \alpha_i, \tag{3.12}$$

where $\varphi(\cdot)$ stands to standard normal random variable having $\mu = 0$ and $\sigma = 1$. Assume that k_{β_i} indicates the value of random variable with $\mu = 0$ and $\sigma = 1$ fulfilling $\varphi(k_{\beta_i}) = \alpha_i$, at that point the constraint (3.12) is expressed as

$$\varphi\left(\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \ge \varphi(k_{\beta_i}), \tag{3.13}$$

since φ continuous, the inequality (3.13) is satisfied only if

$$\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}} \ge k_{\beta_i}$$

or

$$\mathrm{E}(q) \le b_i - k_{\beta_i}\sqrt{\mathrm{Var}(q)} \le 0. \tag{3.14}$$

Substituting $q = a_{i1}x_1 + a_{i2}x_2$ in (3.14), we have

$$\mathrm{E}(a_{i1}x_1 + a_{i2}x_2) \le b_i - k_{\beta_i}\sqrt{\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2)}. \tag{3.15}$$

Substituting (3.15) in (3.2), the deterministic equivalent of the mathematical programming problem (3.1)–(3.4) is expressed as

$$\max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{c_{k1}x_1 + c_{k2}x_2 + c_{k0}}{d_{k1}x_1 + d_{k2}x_2 + d_{k0}}, \quad k = 1, 2, \ldots, K, \tag{3.16}$$

subject to

$$\mathrm{E}(a_{i1}x_1 + a_{i2}x_2) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2)}, \quad i = 1, 2, \ldots, m, \tag{3.17}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{3.18}$$

$$x_1, x_2 \geq 0, \quad j = 1, 2, \ldots, n \tag{3.19}$$

because a_{i1} and a_{i2} are dependent random variables, then $\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2)$ is calculated as

$$\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2) = XH^tX, \tag{3.20}$$

where $X = (x_1, x_2)$ and H is 2×2 covariance matrix which is defined as:

$$H = \begin{pmatrix} \mathrm{Var}(a_{i1}) & \mathrm{Cov}(a_{i1}, a_{i2}) \\ \mathrm{Cov}(a_{i2}, a_{i1}) & \mathrm{Var}(a_{i2}) \end{pmatrix}.$$

Case 2. In this case, the multiple-objective chance constraint linear fractional programming with n decision variables is expressed as:

$$\min / \max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{\sum\limits_{j=1}^{n} c_{kj}x_j + c_{0k}}{\sum\limits_{j=1}^{n} d_{kj}x_j + d_{0k}}, \quad k = 1, 2, \ldots, K, \tag{3.21}$$

subject to

$$P\left(\sum_{j=1}^{n} a_{ij}x_j \leq b_i\right) \geq \alpha_i, \quad i = 1, 2, \ldots, m, \tag{3.22}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{3.23}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n. \tag{3.24}$$

Assume that a_{ij} are dependent normal random variables having n decision variables, then the chance constraint (3.22) is given by

$$P\left(\sum_{i=1}^{n} a_{ij}x_j \leq b_i\right) \geq \alpha_i. \tag{3.25}$$

Let q is a random variable defined as $q = \sum_{i=1}^{n} a_{ij}x_j - b_i$. Following the same procedure as case 1 above, the deterministic equivalent of the chance constraint programming problem is given by

$$\mathrm{E}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}\left(\sum_{j=1}^{n} a_{ij}x_j\right)}, \tag{3.26}$$

substituting (3.26) in (3.22), the deterministic equivalent of the mathematical programming problem (3.21)–(3.24) is expressed by:

$$\min / \max : Z_k = \frac{\sum_{j=1}^{n} c_{kj} x_j + c_{0k}}{\sum_{j=1}^{n} d_{kj} x_j + d_{0k}}, \quad k = 1, 2, \ldots, K, \tag{3.27}$$

subject to

$$\mathrm{E}\left(\sum_{j=1}^{n} a_{ij} x_j\right) \le b_i - k_{\beta_i} \sqrt{\mathrm{Var}\left(\sum_{j=1}^{n} a_{ij} x_j\right)}, \quad i = 1, 2, \ldots, m, \tag{3.28}$$

$$x_j \ge 0, \quad j = 1, 2, \ldots, n. \tag{3.29}$$

Since a_{ij} are dependent random variables then $\mathrm{Var}(q)$ is calculated as follows

$$\mathrm{Var}(q) = \mathrm{Var}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n), \quad i = 1, 2, \ldots, m, \tag{3.30}$$

using the property of variance for the sum of dependent random variables we have $\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n) = XH^TX$, where H is $n \times n$ covariance matrix which is expressed by

$$H = \begin{pmatrix} \mathrm{Var}(a_{i1}) & \mathrm{Cov}(a_{i1}, a_{i2}) & \cdots & \mathrm{Cov}(a_{i1}, a_{in}) \\ \mathrm{Cov}(a_{i2}, a_{i1}) & \mathrm{Var}(a_{i2}) & \cdots & \mathrm{Cov}(a_{i2}, a_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(a_{in}, a_{i1}) & \mathrm{Cov}(a_{in}, a_{i2}) & \cdots & \mathrm{Var}(a_{in}) \end{pmatrix}.$$

4. Solution Procedure

Since the mathematical programming problem given in (2.4)–(2.7) involves uncertain parameters and several linear fractional objectives, it is difficult to discover the lexicographic optimal solution directly. To find the lexicographic optimal solution of the given multiple-objective chance constraint fractional programming problem, first convert the multiple-objective chance constraint linear fractional mathematical programming to deterministic equivalent multiple-objective linear fractional mathematical programming. Then lexicography approach is applied to get the lexicographic solution of the deterministic multiple-objective linear fractional programming problem.

We use the lexicographic ordering approach instead of general partial ordering since it is a special case of general partial ordering approach. In this case, when the lexicographic order has been imposed upon a set of objective functions, then all elements of the objective function will be comparable to one under the ordering where as partial orders are generated by a cone. In lexicography preferences are imposed by ordering the objective functions according to their importance rather than assigning weights. In this case, to solve single objective fractional programming problem, we used complementary method which is proposed by A. N. Dheyab [7]. The method is applied to change fractional mathematical model into equivalent mathematical model which is free from fractional functions. The idea is, to maximize fractional objective function, the numerator must be maximized and the denominator must be minimized. To do this, the fractional objective functions must be realized by subtracting the denominator function from the numerator function. The resulting objective function is maximized subject to given constraint. This shows that the single objective fractional mathematical programming is changed to single objective mathematical programming. Finally, the single objective programming problem is solved by a suitable strategy or existing software.

The basic steps of the methods of multiple-objective chance constraint linear fractional programming problem are given below.

Step 1. Transform the multiple-objective chance constraint linear fractional programming problem into deterministic equivalent multiple-objective linear fractional programming problem as mentioned in Section 3.

Step 2. From the objective function (minimization problem) take $k = 1$. The first objective function is expressed as $Z_1(x) = \frac{N_1(x)}{D_1(x)}$, then the value of Z_1 is taken as the minimum value of $N_1(x)$ and the maximum value of $D_1(x)$.

Step 3. Formulate a mathematical programming problem as $\min \overline{z_1}(x)$ together with the original constraints, where $\overline{z_1}(x) = N_1(x) - D_1(x)$. This is because to make the linear fractional programming problem minimum, the numerator must be as minimum as possible, while the denominator must be as greater as possible, i.e., let the numerator is denoted by $\min N_1(x)$ and the denominator is denoted by $\max D_1(x)$. Then the denominator $\max D_1(x)$ is converted to $\min D_1(x)$ by multiplying both sides by negative sign. Therefore, the new linear programming problem becomes $\min Z = \min N_1(x) - \min D_1(x)$ as stated in [7]. This can be written as $\min Z = N_1(x) - D_1(x)$. This is done by putting the variable of numerator linear on the opposite signal with code e_1, it is added to the simplex method table in the line $(m + 1)$ where as setting the variable of denominator linear to its opposite signal with code e_2, it is add to the simplex method table in the line $m + 2$, where the the bounds of the mathematical model for m is from numbers and the target linear problem is based on the following code,i.e., $Z = N_1e_1 - D_1e_2$, where N_1 is the value of the numerator after compensated the result of the value of x and D_1 is the value of the denominator after compensated the result of the value of x. Taking $e_1 = e_2$, we got $Z = N_1 - D_1$. Then the resulting problem is solved by methods of single objective programming or existing software.

Step 4. Apply the same procedure for the second objective function $Z_2(x) = \frac{N_2(x)}{D_2(x)}$. In this case, the minimization of earlier objective function $\min \overline{z_2}(x)$ is considered as an other constraint in addition to the original constraints.

Step 5. Once more the same strategy is applied for the third objective function and the resulting single objective programming problem is optimized subject to the previous objective function $\min \overline{z_3}(x)$ as constraint together with the original constraint.

Step 6. The method is continued until all the objective functions could be optimized.

The algorithm terminates once a unique optimum is determined. This means that if we have n objective functions, then we do have $n!$ sequence of objective functions. This shows that $n!$ possible lexicographic optimal solutions can be obtained from the given problem. Hence the algorithm terminates if all possible sequential ordered functions are optimized.

The values of the objective functions is obtained by substituting the lexicographic solution to the original objective functions.

5. Numerical Example

Consider the following multiple-objective chance constrained linear fractional mathematical programming where the constraint coefficient of the left hand restrictions follow dependent normal random variables.

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2}, \tag{5.1}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 + 3}, \tag{5.2}$$

subject to

$$P(a_{11}x_1 + a_{12}x_2 \leq 30) \geq 0.85, \tag{5.3}$$

$$P(a_{21}x_1 + a_{22}x_2 \leq 40) \geq 0.95, \tag{5.4}$$

$$x_j \geq 0, \quad j = 1, 2, \tag{5.5}$$

where a_{11}, a_{12}, a_{21}, a_{22} are random variables that follow dependent normal distribution with known parameters $E(a_{11}) = 2$, $E(a_{12}) = 4$, $E(a_{21}) = 1$, $E(a_{22}) = 2$, $\mathrm{Var}(a_{11}) = 16$, $\mathrm{Var}(a_{12}) = 25$, $\mathrm{Var}(a_{21}) = 49$, $\mathrm{Var}(a_{22}) = 36$, $\mathrm{Cov}(a_{11}, a_{12}) = 10$, $\mathrm{Cov}(a_{21}, a_{22}) = 14$.

Now, using equation the problem given in (3.27)–(3.29) the deterministic equivalent of the problem given in (5.1)–(5.5) is expressed as:

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2}, \tag{5.6}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 + 3}, \tag{5.7}$$

subject to

$$\mathrm{E}(a_{11}x_1 + a_{12}x_2) \leq b_1 - k_{\beta_1} y_1, \tag{5.8}$$

$$\mathrm{Var}(a_{11}x_1 + a_{12}x_2) - y_1^2 = 0, \tag{5.9}$$

$$\mathrm{E}(a_{21}x_1 + a_{22}x_2) \leq 40 - k_{\beta_2} y_2 \leq 0, \tag{5.10}$$

$$\mathrm{Var}(a_{21}x_1 + a_{22}x_2) - y_2^2 = 0, \tag{5.11}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.12}$$

Using the property of mean and variance of dependent random variables we have:

$$\mathrm{E}(a_{11}x_1 + a_{12}x_2) = \mathrm{E}(a_{11})x_1 + \mathrm{E}(a_{12})x_2 \tag{5.13}$$

and

$$\mathrm{Var}(a_{11}x_1 + a_{12}x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \mathrm{Var}(a_{11}) & \mathrm{Cov}(a_{11}, a_{12}) \\ \mathrm{Cov}(a_{12}, a_{11}) & \mathrm{Var}(a_{12}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

similarly, the variance of the second constraint is given by

$$\mathrm{Var}(a_{21}x_1 + a_{22}x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \mathrm{Var}(a_{21}) & \mathrm{Cov}(a_{21}, a_{22}) \\ \mathrm{Cov}(a_{22}, a_{21}) & \mathrm{Var}(a_{22}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Substituting all the values of the given data in the problem (5.6)–(5.12), we have the following deterministic multiple-objective linear fractional programming problems.

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2 + 2}, \tag{5.14}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 - 3}, \tag{5.15}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.16}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.17}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.18}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.19}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.20}$$

The deterministic programming problem given in (5.14)–(5.20) is multiple-objective nonlinear fractional programming problem. Using the above method, we can get the lexicographic solution of the given mathematical problem.

Now, consider the first objective function $\max Z_1 = \frac{-4x_1+3x_2}{2x_1+4x_2}$ and separate this function into two functions namely, numerator and denominator. Using the procedure in step 2, we have to formulate single objective programming problem together with the given constraints which is stated by (5.21)–(5.26) as follows

$$\max \bar{Z}_1 = (8x_1 + 14x_2) - (2x_1 + 4x_2 + 2) = 6x_1 + 10x_2 - 2, \tag{5.21}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.22}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.23}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.24}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.25}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.26}$$

Solving this nonlinear programming problem using LINGO software, we obtain the following optimal solutions: $x_1 = 1.113499$, $x_2 = 2.725417$, $y_1 = 16.31657$, $y_2 = 20.32563$, with maximum value $\bar{Z}_1 = 31.93516$.

Next, we consider the second objective function $Z_2 = \frac{-16x_1+9x_2}{-6x_1+4x_2-3}$. According to the above procedure given in step 4, we formulate the non linear programming problem as:

$$\max \bar{Z}_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 - 3} = -10x_1 + 4x_2 + 3, \tag{5.27}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.28}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.29}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.30}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.31}$$

$$6x_1 + 10x_2 = 33.93516, \tag{5.32}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.33}$$

Here $6x_1 + 10x_2 \geq 33.93516$ is included in the constraint. Solving the nonlinear programming problem given in (5.27)–(5.33), we obtain the following lexicographic optimal solution: $x_1 = 1.113499$, $x_2 = 2.725417$, $y_1 = 16.31657$, $y_2 = 20.32563$, with maximum value $\bar{Z}_1 = 2.766681$.

Therefore, a lexicographic solution by above multiple-objective chance constraint fractional programming problem is $x_1 = 1.113499$, $x_2 = 2.725417$, with $\max Z_1 = \frac{47.06383}{17.854083}$, $\max Z_2 = \frac{6.712769}{3.946091}$.

In any multiple-objective programming problem, there exist a number of good lexicographic solutions. These lexicographic solutions are equally acceptable. Choosing the lexicographic solution depends on the situation that decision makers prefer. The preference of decision maker depends on different conditions like budget, row material, resource, time limit etc. Therefore, having more lexicographic solution to multiple-objective programming problem is necessary for decision makers to select the best solution among the given alternatives which satisfies their need and capacity. Hence, we need to search more lexicographic solution for the above programming problem. So, applying the above procedure given in section 4, first choose the second objective function and optimizing subject to the given constraints, we have an optimal solution $x_1 = 0.0000$, $x_2 = 3.271538$, $y_1 = 16.35769$, $y_2 = 19.62923$, with maximum value $\bar{Z}_2 = 16.08615$.

Next, optimizing the first objective function Z_1 subject to the original constraint including $\bar{Z}_2 = -10x_1 - 4x_2 \geq 14.19455$, obtain lexicographic optimal solution $x_1 = 0.2538795$, $x_2 = 3.156236$, $y_1 = 16.31267$, $y_2 = 19.60155$, with maximum value $\bar{Z}_1 = 31.08564$. Substituting these values to the original objective function gives to the lexicographic solution which is given by $x_1 = 0.2538795$, $x_2 = 3.156236$, with $\max Z_1 = \frac{46.2183}{18.288939}$, $\max Z_2 = \frac{24.2183}{11.257903}$.

Finally, the two lexicographic solutions are given in Table 1.

TABLE 1. Lexicographic solutions

x_1	x_2	Z_1	$Z_2(X)$
1.113499	2.725417	$\frac{47.06383}{17.854083}$	$\frac{6.712769}{3.946091}$
0.2538795	3.156236	$\frac{46.2183}{18.288939}$	$\frac{24.2183}{11.257903}$

6. Conclusion

Multiple-objective chance constraint linear fractional programming are solved by considering the coefficient of constraints as random variables following dependent normal distribution. We consider that other data of the model are deterministic. The formulated programming problem is converted to its deterministic equivalent programming problem using the concept of cumulative probability distribution for dependent random variables using the concepts of covariance. The resulting multiple-objective fractional programming is solved by using lexicography method which is prior method. Alternative lexicographic solutions are obtained using the proposed method. The problem can be extended to the same programming problems involving other dependent random variables.

References

[1] S. Acharya, B. Belay and R. Mishra, *Multi-objective probabilistic fractional programming problem involving two parameters cauchy distribution*, Math. Model. Anal. **2**4(3) (2019),385–403. https://doi.org/10.3846/mma.2019.024

[2] M. P. Biswal, N. Biswal and D. Li, *Probabilistic linear programming problems with exponential random variables*, European J. Oper. Res. **1**11(3) (1998), 589–597. https://doi.org/10.1016/S0377-2217(97)90319-2

[3] M. Chakraborty and S. Gupta, *Fuzzy mathematical programming for multi objective linear fractional programming problem*, Fuzzy Sets and Systems **1**25(3) (2002), 335–342. https://doi.org/10.1016/S0165-0114(01)00060-4

[4] V. Charles, S. Ansari and M. Khalid, *Multi-objective stochastic linear programming with general form of distributions*, International Journal of Operational Research & Optimization **2**(2) (2011), 261–278.

[5] A. Charnes and W. Cooper, *Chance constraints and normal deviates*, J. Amer. Statist. Assoc. **5**7(297) (1962), 134–148. https://doi.org/10.2307/2282444

[6] A. Charnes and W. W. Cooper, *Programming with linear fractional functionals*, Naval Research Logistics Quarterly **9**(3–4) (1962), 181–186. https://doi.org/10.1002/nav.3800090303

[7] A. N. Dheyab, *Finding the optimal solution for fractional linear programming problems with fuzzy numbers*, Journal of Kerbala University **1**0(3) (2012), 105–110.

[8] D. Dutta, J. Rao and R. Tiwari, *A restricted class of multi objective linear fractional programming problems*, European J. Oper. Res. **6**8(3) (1993), 352–355. https://doi.org/10.1016/0377-2217(93)90191-O

[9] S. Jain, *Modeling of Gauss elimination technique for multi-objective fractional programming problem*, South Asian Journal of Mathematics **4**(3) (2014), 148–153.

[10] M. Knott, *Randomized decisions in chance-constrained programming*, Journal of the Operational Research Society **36**(10) (1985), 959–962. https://doi.org/10.1057/jors.1985.167

[11] J. S. Kornbluth and R. E. Steuer, *Multiple objective linear fractional programming*, Management Science **27**(9) (1981), 1024–1039. https://doi.org/10.1287/mnsc.27.9.1024

[12] B. Lingaraj and H. Wolfe, *Certainty equivalent of a chance constraint if the random variable follows a gamma distribution*, The Indian Journal of Statistics **36**(2) (1974), 204–208.

[13] M. K. Luhandjula, *Fuzzy approaches for multiple objective linear fractional optimization*, Fuzzy Sets and Systems **13**(1) (1984), 11–23. https://doi.org/10.1016/0165-0114(84)90023-X

[14] R. Porchelvi, L. Vasanthi and R. Hepzibah, *On solving multi objective fractional linear programming problems*, International Journal of Current Research **6**(8) (2014), 8095–8102.

[15] N. Sahoo and M. Biswal, *Computation of probabilistic linear programming problems involving normal and log-normal random variables with a joint constraint*, Computer Mathematics **82**(11) (2005), 1323–1338. https://doi.org/10.1080/00207160500113058

[16] S. F. Tantawy, *Solving a special class of multiple objective linear fractional programming problems*, The ANZIAM Journal **56**(1) (2014), 91–103.

[1]Department of Mathematics,College of Natural and Computational Sciences
Debre Tabor University,
Debre Tabor, Ethiopia
Email address: berhanubelay2@gmail.com

[2]Department of Mathematics,School of Applied Sciences
KIIT Deemed to be University, Bhubanswar, India,
Bhubanswar, India
Email address: sacharyafma@kiit.ac.in

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 253–265.

POSITIVITY AND PERIODICITY IN NONLINEAR NEUTRAL MIXED TYPE LEVIN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

KARIMA BESSIOUD[1], ABDELOUAHEB ARDJOUNI[1], AND AHCENE DJOUDI[2]

Abstract. In this work, we give sufficient conditions for the existence of periodic and positive periodic solutions for a nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delays by using Krasnoselskii's fixed point theorem. Also, we obtain the existence of a unique periodic solution of the posed equation by means of the contraction mapping principle. As an application, we give an example to illustrate our results. Previous results are extended and generalized.

1. Introduction

Differential and integro-differential equations with delays have received great attention and have become an active area of research. This is due to the fact that several phenomena in life sciences, engineering, chemistry and physics can be described by means of delay equations. Indeed, problems concerning the positivity, periodicity and stability of solutions for differential and integro-differential equations with delays have received the considerable attention of many authors, see [1]–[24], [26, 27] and the references therein.

Key words and phrases. Fixed points, positivity, periodicity, Levin-Nohel integro-differential equations.
2020 *Mathematics Subject Classification.* Primary: 34K13. Secondary: 45J05, 45D05.
DOI 10.46793/KgJMat2502.253B
Received: September 08, 2021.
Accepted: April 01, 2022.

In this paper, we consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delays

$$\frac{d}{dt}x(t) = -\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,x(s)\,ds - \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,x(s)\,ds + \frac{d}{dt}g\left(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))\right), \tag{1.1}$$

where a_j, b_j, τ_j, σ_j and g are continuous functions with $\tau_j(t) > 0$, $\sigma_j(t) > 0$, $j = 1, \ldots, m$. In this work, we use the idea of integrating factor to convert the equation (1.1) into an integral equation. Then, we employ Krasnoselskii's fixed point theorem to show the existence of periodic and positive periodic solutions of (1.1). Also, we obtain the existence of a unique periodic solution by using the contraction mapping principle. An example is given to illustrate our main results.

In [9], we investigated the asymptotic stability of the zero solution for (1.1) by using the contraction mapping theorem. Also, in the special case $a_j(t,s) = 0$, $j = 2, \ldots, m$, $b_j(t,s) = 0$, $j = 1, \ldots, m$ and $g(t, x_1, x_2, \ldots, x_m) = g_1(t, x_1)$, in [10], we proved the existence and uniqueness of periodic solutions and the existence of positive solutions for (1.1) by appealing Krasnoselskii's fixed point theorem and the contraction mapping theorem. Then, the results presented in this paper extend and generalize the main results in [10].

2. Existence and Uniqueness of Periodic Solutions

For $T > 0$ let P_T be the set of all continuous scalar functions x periodic in t of period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Since we are searching for the existence of periodic solutions for (1.1), it is natural to suppose that

$$a_j(t+T, s+T) = a(t,s), \quad b_j(t+T, s+T) = b_j(t,s), \quad \tau_j(t+T) = \tau_j(t), \quad \sigma_j(t+T) = \sigma_j(t), \tag{2.1}$$

with τ_j and σ_j being scalar continuous functions, $\tau_j(t) \geq \tau_j^* > 0$ and $\sigma_j(t) \geq \sigma_j^* > 0$, $j = 1, \ldots, m$. Also, we suppose

$$\int_0^T A(z)\,dz > 0, \quad A(t) = \sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,ds + \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,ds. \tag{2.2}$$

The function $g(t, x_1, x_2, \ldots, x_m)$ is periodic in t of period T, it is also globally Lipschitz continuous in x_j, $j = 1, \ldots, m$. That is

$$g(t+T, x_1, x_2, \ldots, x_m) = g(t, x_1, x_2, \ldots, x_m), \tag{2.3}$$

and there are positive constants E_j, $j=1,\dots,m$, such that

$$|g(t,x_1,x_2,\dots,x_m)-g(t,y_1,y_2,\dots,y_m)|\leq\sum_{j=1}^{m}E_j|x_j-y_j|. \tag{2.4}$$

The next lemma is crucial to our results.

Lemma 2.1. *Suppose that* (2.1)–(2.3) *hold. Then,* $x\in P_T$ *is a solution of the equation* (1.1) *if and only if* x *satisfies*

$$\begin{aligned} x(t)=&G_x(t)-\left(1-e^{-\int_{t-T}^{t}A(z)dz}\right)^{-1}\\ &\times\int_{t-T}^{t}\left[L_x(s)+N_x(s)+A(s)G_x(s)\right]e^{-\int_s^t A(z)dz}ds, \end{aligned}\tag{2.5}$$

where

$$G_x(t)=g\left(t,x\left(t-\tau_1(t)\right),\dots,x\left(t-\tau_m(t)\right)\right), \tag{2.6}$$

and

$$\begin{aligned} L_x(t)=&\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\left(\int_s^t\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)x(\nu)d\nu\right.\right.\\ &\left.\left.+\sum_{k=1}^{m}\int_u^{u+\sigma_k(u)}b_k(u,\nu)x(\nu)d\nu\right)du+G_x(s)-G_x(t)\right)ds \end{aligned}\tag{2.7}$$

and

$$\begin{aligned} N_x(t)=&\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\left(\int_s^t\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)x(\nu)d\nu\right.\right.\\ &\left.\left.+\sum_{k=1}^{m}\int_u^{u+\sigma_k(u)}b_k(u,\nu)x(\nu)d\nu\right)du+G_x(s)-G_x(t)\right)ds. \end{aligned}\tag{2.8}$$

Proof. Obviously, we have

$$x(s)=x(t)-\int_s^t\frac{\partial}{\partial u}x(u)du.$$

Inserting this relation into (1.1), we obtain

$$\begin{aligned} &\frac{d}{dt}x(t)+\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\left(x(t)-\int_s^t\frac{\partial}{\partial u}x(u)du\right)ds\\ &+\sum_{j=1}^{m}\int_t^{t+\sigma_j(t)}b_j(t,s)\left(x(t)-\int_s^t\frac{\partial}{\partial u}x(u)du\right)ds-\frac{d}{dt}G_x(t)\ =0. \end{aligned}$$

So,

$$\begin{aligned}&\frac{d}{dt}x(t)+x(t)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\,ds+\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\,ds\right)\\&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\left(\int_s^t\frac{\partial}{\partial u}x(u)\,du\right)ds\\&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\left(\int_s^t\frac{\partial}{\partial u}x(u)\,du\right)ds-\frac{d}{dt}G_x(t)\;=\;0.\end{aligned}$$

Substituting $\frac{\partial x}{\partial u}$ from (1.1), we get

$$\begin{aligned}&\frac{d}{dt}x(t)+x(t)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\,ds+\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\,ds\right)\\&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k(u,\nu)\,x(\nu)\,d\nu+\frac{\partial}{\partial u}G_x(u)\right)du\right]ds\\&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j(t,s)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k(u,\nu)\,x(\nu)\,d\nu\right.\right.\\&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k(u,\nu)\,x(\nu)\,d\nu+\frac{\partial}{\partial u}G_x(u)\right)du\right]ds-\frac{d}{dt}G_x(t)\;=\;0.\end{aligned}\tag{2.9}$$

By performing the integration, we obtain

$$\int_s^t\frac{\partial}{\partial u}G_x(u)\,du=G_x(t)-G_x(s)\,.\tag{2.10}$$

Substituting (2.10) into (2.9), we get

$$\frac{d}{dt}x(t)+A(t)\,x(t)+L_x(t)+N_x(t)-\frac{d}{dt}G_x(t)=0,$$

where A and L_x and N_x are given by (2.2), (2.7) and (2.8), respectively. We rewrite this equation as

$$\frac{d}{dt}\left\{x(t)-G_x(t)\right\}=-A(t)\left(x(t)-G_x(t)\right)-A(t)\,G_x(t)-L_x(t)-N_x(t)\,.\tag{2.11}$$

Multiply both sides of (2.11) with $e^{\int_0^t A(z)dz}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t}\frac{\partial}{\partial s}\left[x(s)-G_x(s)\right]e^{\int_0^s A(z)dz}ds\\&=-\int_{t-T}^{t}\left[L_x(s)+N_x(s)+A(s)\,G_x(s)\right]e^{\int_0^s A(z)dz}ds.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&(x(t)-G_x(t))\,e^{\int_0^t A(z)dz}-(x(t-T)-G_x(t-T))\,e^{\int_0^{t-T}A(z)dz}\\&=-\int_{t-T}^{t}[L_x(s)+N_x(s)+A(s)\,G_x(s)]\,e^{\int_0^s A(z)dz}ds.\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t A(z)dz}$ and using the fact that $x(t)=x(t-T)$, we obtain

$$\begin{aligned}&x(t)-G_x(t)\\&=-\left(1-e^{-\int_{t-T}^t A(z)dz}\right)^{-1}\int_{t-T}^{t}[L_x(s)+N_x(s)+A(s)\,G_x(s)]\,e^{-\int_s^t A(z)dz}ds.\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

Define the mapping H by

$$\begin{aligned}(H\varphi)(t)=&G_\varphi(t)-\left(1-e^{-\int_{t-T}^t A(z)dz}\right)^{-1}\\&\times\int_{t-T}^{t}[L_\varphi(s)+N_\varphi(s)+A(s)\,G_\varphi(s)]\,e^{-\int_s^t A(z)dz}ds.\end{aligned}\tag{2.12}$$

It is clear form (2.12) that $H:P_T\to P_T$ by the way it was constructed in Lemma 2.1.

Next, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic and positive periodic solutions. For the proof of Krasnoselskii's fixed point theorem we refer the reader to [25].

Theorem 2.1 (Krasnoselskii)**.** *Let M be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B},\|\cdot\|)$. Suppose that C and B map M into $\mathbb{B}$ such that*
(i) $x,y\in M$ implies $Cx+By\in M$;
(ii) C is continuous and CM is contained in a compact set;
(iii) B is a contraction mapping.
Then there exists $z\in M$, with $z=Cz+Bz$.

We note that to apply the above theorem we need to construct two mappings; one is contraction and the other is continuous and compact. Therefore, we express (2.12) as

$$(H\varphi)(t)=(B\varphi)(t)+(C\varphi)(t),$$

where $C,B:P_T\to P_T$ are given by

$$(B\varphi)(t)=G_\varphi(t),\tag{2.13}$$

and

$$(C\varphi)(t)=-\left(1-e^{-\int_{t-T}^t A(z)dz}\right)^{-1}\int_{t-T}^{t}[L_\varphi(s)+N_\varphi(s)+A(s)\,G_\varphi(s)]\,e^{-\int_s^t A(z)dz}ds.\tag{2.14}$$

To simplify notations, we introduce the following constants

$$\eta=\left(1-e^{-\int_{t-T}^{t}A(z)dz}\right)^{-1},\quad \rho=\sup_{t\in[0,T]}\left(\sup_{s\in[t-T,t]}\sum_{j=1}^{m}\left(\int_{s-\tau_j(s)}^{s}\left|a_j\left(s,w\right)\right|dw\right)\right),$$
$$\varrho=\sup_{t\in[0,T]}\left(\sup_{s\in[t-T,t]}\sum_{j=1}^{m}\left(\int_{s}^{s+\sigma_j(s)}\left|b_j\left(s,w\right)\right|dw\right)\right),\quad \gamma=\sup_{t\in[0,T]}\left(\sup_{s\in[t-T,t]}e^{-\int_s^t A(z)dz}\right),$$
$$\delta=\sup_{t\in[0,T]}\left(\sup_{s\in[t-T,t]}\left(\sup_{w\in[t-T,t]}\int_w^s\left(\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}\left|a_k\left(u,\nu\right)\right|d\nu\right.\right.\right.$$
$$\left.\left.\left.+\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}\left|b_k\left(u,\nu\right)\right|d\nu\right)du\right)\right),\quad \alpha=\sup_{t\in[0,T]}\left|G_0\left(t\right)\right|. \tag{2.15}$$

Lemma 2.2. *Let C be given in* (2.14)*. Suppose that* (2.1)*–*(2.4) *hold. Then $C:P_T\to P_T$ is continuous and the image of C is contained in a compact set.*

Proof. To see that C is continuous, let $\varphi,\psi\in P_T$. Given $\epsilon>0$, take $\beta=\frac{\epsilon}{N}$ with $N=\eta\gamma T\left(\rho+\varrho\right)\left(\delta+3\sum_{j=1}^{m}E_j\right)$ where E_j, $j=1,\dots,m$, are given by (2.4). Now, for $\|\varphi-\psi\|<\beta$, we get

$$\begin{aligned}&\|C\varphi-C\psi\|\\ \leq&\eta\gamma\int_{t-T}^{t}\left[\rho\left(\delta+2\sum_{j=1}^{m}E_j\right)\|\varphi-\psi\|+\varrho\left(\delta+2\sum_{j=1}^{m}E_j\right)\|\varphi-\psi\|\right.\\ &\left.+\left(\rho+\varrho\right)\left(\sum_{j=1}^{m}E_j\right)\|\varphi-\psi\|\right]ds\\ \leq&\eta\gamma\int_{t-T}^{t}\left(\rho+\varrho\right)\left(\delta+3\sum_{j=1}^{m}E_j\right)\|\varphi-\psi\|\,ds\\ \leq&N\|\varphi-\psi\|<\epsilon.\end{aligned}$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D=\{\varphi\in P_T:\|\varphi\|\leq R\}$ where R is a fixed positive constant. Let $\varphi\in D$. Observe that in view of (2.4) we have

$$\left|G_\varphi\left(t\right)\right|=\left|G_\varphi\left(t\right)-G_0\left(t\right)+G_0\left(t\right)\right|\leq\left|G_\varphi\left(t\right)-G_0\left(t\right)\right|+\left|G_0\left(t\right)\right|\leq\sum_{j=1}^{m}E_j\|\varphi\|+\alpha.$$

Consequently,

$$\begin{aligned}\|C\varphi\| \leq & \eta\gamma \int_{t-T}^{t} \left[\rho \left(\delta R + 2\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right)\right. \\ & \left. + \varrho\left(\delta R + 2\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right) + (\rho+\varrho)\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right] ds \\ \leq & \eta\gamma T(\rho+\varrho)\left(\delta R + 3\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right) = L.\end{aligned}$$

So, $C(D)$ is uniformly bounded. Next, we calculate $(C\varphi)'(t)$ and prove that $C(D)$ is equicontinuous. By making use of (2.1)–(2.3) we get by taking the derivative in (2.14) that

$$(C\varphi)'(t) = -A(t)(C\varphi)(t) - L_{\varphi}(t) - N_{\varphi}(t) - A(t) G_{\varphi}(t).$$

Thus, the above expression yields $\left\|(C\varphi)'\right\| \leq F$, for some positive constant F. So, $C(D)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $C(D)$ is relatively compact. Then, $C(D)$ is contained in a compact set. □

Lemma 2.3. *Suppose that* (2.1), (2.3) *and* (2.4) *hold, and*

$$\sum_{j=1}^{m} E_j < 1, \tag{2.16}$$

where E_j, $j = 1, \ldots, m$, *are given by* (2.4). *If* B *is given by* (2.13), *then* B *is a contraction mapping.*

Proof. Let B be defined by (2.13). Then for $\varphi, \psi \in P_T$ we obtain

$$\begin{aligned}\|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)| \\ &\leq \sum_{j=1}^{m} E_j \sup_{t\in[0,T]} |\varphi(t-\tau_j(t)) - \psi(t-\tau_j(t))| \\ &\leq \left(\sum_{j=1}^{m} E_j\right) \|\varphi - \psi\|.\end{aligned}$$

Hence, B defines a contraction mapping. □

Theorem 2.2. *Assume that* (2.1)–(2.4) *and* (2.16) *hold. Let* J *be a positive constant satisfying the inequality*

$$J\sum_{j=1}^{m} E_j + \alpha + \eta\gamma T(\varrho+\rho)\left(\delta J + 3\left(J\sum_{j=1}^{m} E_j + \alpha\right)\right) \leq J. \tag{2.17}$$

Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. *Then the equation* (1.1) *has a solution in* M.

Proof. By Lemma 2.2, $C : M \to P_T$ is continuous and $C(M)$ is contained in a compact set. Also, by Lemma 2.3, the mapping B is a contraction and it is clear that $B : M \to P_T$. Next, we prove that if $\varphi, \psi \in M$, we have $\|C\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in M$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned}
&\|C\varphi + B\psi\| \\
\leq &\left(\sum_{j=1}^{m} E_j\right)\|\psi\| + \alpha + \eta\gamma\int_{t-T}^{t}\left[\rho\left(\delta\|\varphi\| + 2\left(\sum_{j=1}^{m} E_j\|\varphi\| + \alpha\right)\right)\right. \\
&\left.+\varrho\left(\delta\|\varphi\| + 2\left(\sum_{j=1}^{m} E_j\|\varphi\| + \alpha\right)\right) + (\varrho+\rho)\left(\left(\sum_{j=1}^{m} E_j\right)\|\varphi\| + \alpha\right)\right] ds \\
\leq &J\sum_{j=1}^{m} E_j + \alpha + \eta\gamma T(\varrho+\rho)\left(\delta J + 3\left(J\sum_{j=1}^{m} E_j + \alpha\right)\right) \\
\leq &J.
\end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in M such that $z = Cz + Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence, (1.1) has a T-periodic solution. ☐

Theorem 2.3. *Suppose that* (2.1)–(2.4) *hold. If*

$$\sum_{j=1}^{m} E_j + \eta\gamma T(\varrho+\rho)\left(\delta + 3\sum_{j=1}^{m} E_j\right) < 1, \tag{2.18}$$

then the equation (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (2.12). For $\varphi, \psi \in P_T$, in view of (2.12), we obtain

$$\|H\varphi - H\psi\| \leq \left(\sum_{j=1}^{m} E_j + \eta\gamma T(\varrho+\rho)\left(\delta + 3\sum_{j=1}^{m} E_j\right)\right)\|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. ☐

Corollary 2.1. *Suppose that* (2.1)–(2.3) *hold. Let J be a positive constant and define $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Suppose there are positive constants E_j^*, $j = 1, \ldots, m$, so that for $x, y \in M$ we have*

$$\begin{aligned}
&|g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))) - g(t, y(t-\tau_1(t)), \ldots, y(t-\tau_m(t)))| \\
\leq &\sum_{j=1}^{m} E_j^* |x(t-\tau_j(t)) - y(t-\tau_j(t))|.
\end{aligned}$$

If $\sum_{j=1}^{m} E_j^* < 1$ *and* $\|H\varphi\| \leq J$ *for* $\varphi \in M$*, then* (1.1) *has a* T*-periodic solution in* M*. Moreover, if*

$$\sum_{j=1}^{m} E_j^* + \eta\gamma T(\varrho + \rho)\left(\delta + 3\sum_{j=1}^{m} E_j^*\right) < 1,$$

then (1.1) *has a unique* T*-periodic solution in* M*.*

Proof. Let the mapping H be given by (2.12). Then, the results follow immediately from Theorem 2.2 and Theorem 2.3. □

Example 2.1. For small positive ϵ_1, ϵ_2 and ϵ_3, we consider the nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delay

$$\begin{aligned} &\frac{d}{dt}x(t) + \epsilon_1 \int_{t-\frac{2\pi}{\omega}}^{t} (1 + \sin\omega(t-s))\, x(s)\, ds \\ &+ \epsilon_2 \int_{t}^{t+\frac{\pi}{\omega}} (2 + \cos\omega(s-t))\, x(s)\, ds - \epsilon_3 \frac{d}{dt}\left(\sin(\omega t)\, x^2\left(t - \frac{2\pi}{\omega}\right)\right) = 0, \end{aligned} \tag{2.19}$$

where ω is a positive constant. So, we have

$$\begin{aligned} a_1(t,s) &= \epsilon_1(1 + \sin\omega(t-s)), \quad b_1(t,s) = \epsilon_2(2 + \cos\omega(s-t)), \\ a_j(t,s) &= b_j(t,s) = \tau_j(t) = \sigma_j(t) = 0, \quad j = 2, \ldots, m, \\ \tau_1(t) &= \frac{2\pi}{\omega}, \quad \sigma_1(t) = \frac{\pi}{\omega}, \end{aligned}$$

and

$$g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_m(t))) = \epsilon_3 \sin(\omega t)\, x^2\left(t - \frac{2\pi}{\omega}\right).$$

Proof. Define $M = \left\{\varphi \in P_{\frac{2\pi}{\omega}} : \|\varphi\| \leq J\right\}$, where J is a positive constant. For $\varphi \in M$, we have

$$\|H\varphi\| \leq \epsilon_3 J^2 + \left(1 - e^{-(\epsilon_1+\epsilon_2)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} (8\epsilon_1 + 6\epsilon_2)\frac{\pi^2}{\omega^2}\left[8\epsilon_1\frac{\pi^2}{\omega^2}J + 6\epsilon_2\frac{\pi^2}{\omega^2}J + 3\epsilon_3 J^2\right].$$

Thus, the inequality

$$\epsilon_3 J^2 + \left(1 - e^{-(\epsilon_1+\epsilon_2)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} (8\epsilon_1 + 6\epsilon_2)\frac{\pi^2}{\omega^2}\left[8\epsilon_1\frac{\pi^2}{\omega^2}J + 6\epsilon_2\frac{\pi^2}{\omega^2}J + 3\epsilon_3 J^2\right] \leq J, \tag{2.20}$$

which is satisfied for small ϵ_1, ϵ_2 and ϵ_3, implies $\|H\varphi\| \leq J$. Hence, (2.19) has a $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.1.

For the uniqueness of the periodic solution, we let $\varphi, \psi \in M$. From (2.19) we see that

$$\eta = \left(1 - e^{-(\varepsilon_1+\varepsilon_2)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1}, \quad \rho = \frac{2\pi}{\omega}\varepsilon_1, \quad \varrho = \frac{2\pi}{\omega}\varepsilon_2, \quad \gamma \leq 1.$$

Also $\alpha = 0$, $E = 2\varepsilon_3 J^2$, where J is given by (2.20). If

$$2\varepsilon_3 J + \left(1 - e^{-(\varepsilon_1+\varepsilon_2)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} (8\varepsilon_1 + 6\varepsilon_2)\frac{\pi^2}{\omega^2}\left[8\varepsilon_1\frac{\pi^2}{\omega^2} + 6\varepsilon_2\frac{\pi^2}{\omega^2} + 6\varepsilon_3 J\right] < 1,$$

is satisfied for small ε_1, ε_2 and ε_3, then (2.19) has a unique $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.1. □

3. Existence of Positive Periodic Solutions

For a non-negative constant L and a positive constant K, we define the set

$$\mathbb{M} = \{\varphi \in P_T : L \leq \varphi \leq K\},$$

which is a closed convex and bounded subset of the Banach space P_T. To simplify notation, we let

$$\theta = \max_{t\in[0,T]} \left(\max_{s\in[t-T,t]} e^{-\int_s^t A(z)dz} \right), \quad \lambda = \min_{t\in[0,T]} \left(\min_{s\in[t-T,t]} e^{-\int_s^t A(z)dz} \right).$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $G_x(t) \geq 0$ and $G_x(t) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{M}$.

In the case $G_x(t) \geq 0$, we assume that there exist non-negative constants k_{1j} and positive constants k_{2j}, $j = 1, \ldots, m$, such that

$$\sum_{j=1}^{m} k_{1j} x(t-\tau_j(t)) \leq G_x(t) \leq \sum_{j=1}^{m} k_{2j} x(t-\tau_j(t)), \tag{3.1}$$

$$\sum_{j=1}^{m} k_{2j} < 1, \tag{3.2}$$

and for all $t \in [0,T]$, $x \in \mathbb{M}$

$$\frac{L\left(1-\sum_{j=1}^{m} k_{1j}\right)}{\eta\lambda T} \leq F_x(t) \leq \frac{K\left(1-\sum_{j=1}^{m} k_{2j}\right)}{\eta\theta T}, \tag{3.3}$$

where $F_x(t) = -L_x(t) - N_x(t) - A(t) G_x(t)$.

Theorem 3.1. *Assume that* (2.1)–(2.4), (2.16) *and* (3.1)–(3.3) *hold. Then the equation* (1.1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$.

Proof. By Lemma 2.1 x is a solution of (1.1) if $x = Cx + Bx$, where C and B are given by (2.14) and (2.13), respectively. By Lemma 2.2, C is continuous and compact. Moreover, by Lemma 2.3, B is a contraction. We just need to prove that condition (i) of Theorem 2.1 is satisfied. Toward this, let $\varphi, \psi \in \mathbb{M}$, then

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
=&G_\psi(t) - \eta \int_{t-T}^{t} \left[L_\varphi(s) + N_\varphi(s) + A(s) G_\varphi(s)\right] e^{-\int_s^t A(z)dz} ds \\
\leq& K\sum_{j=1}^{m} k_{2j} + \eta\theta T \frac{K\left(1-\sum_{j=1}^{m} k_{2j}\right)}{\eta\theta T} = K.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&(B\psi)(t)+(C\varphi)(t)\\&=G_{\psi}(t)-\eta\int_{t-T}^{t}\left[L_{\varphi}(s)+N_{\varphi}(s)+A(s)G_{\varphi}(s)\right]e^{-\int_s^t A(z)dz}ds\\&\geq L\sum_{j=1}^{m}k_{1j}+\eta\lambda T\frac{L\left(1-\sum\limits_{j=1}^{m}k_{1j}\right)}{\eta\lambda T}=L.\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $x\in\mathbb{M}$ such that $x=Bx+Cx$. By Lemma 2.1 this fixed point is a solution of (1.1) and the proof is complete. □

In the case $G_x(t)\leq 0$, we substitute conditions (3.1)–(3.3) with the following conditions respectively. We suppose that there exist negative constants k_{3j} and non-positive constants k_{4j}, $j=1,\ldots,m$, such that

$$\sum_{j=1}^{m}k_{3j}x(t-\tau_j(t))\leq G_x(t)\leq\sum_{j=1}^{m}k_{4j}x(t-\tau_j(t)),\tag{3.4}$$

$$-\sum_{j=1}^{m}k_{3j}<1,\tag{3.5}$$

and for all $t\in[0,T]$, $x\in\mathbb{M}$

$$\frac{L-K\sum\limits_{j=1}^{m}k_{3j}}{\eta\lambda T}\leq F_x(t)\leq\frac{K-L\sum\limits_{j=1}^{m}k_{4j}}{\eta\theta T}.\tag{3.6}$$

Theorem 3.2. *Suppose that* (2.1)–(2.4)*,* (2.16) *and* (3.4)–(3.6) *hold. Then the equation* (1.1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$*.*

The proof follows along the lines of Theorem 3.1, and hence we omit it.

Acknowledgements. The authors gratefully acknowledge the reviewers for their helpful comments.

References

[1] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for two types of second order nonlinear neutral differential equations with variable delay*, Proyecciones **32**(4) (2013), 377–391. `https://doi.org/10.4067/S0716-09172013000400006`

[2] A. Ardjouni and A. Djoudi, *Existence of periodic solutions in totally nonlinear neutral dynamic equations with variable delay on a time scale*, Mathematics in Engineering, Science and Aerospace (MESA) **4**(3) (2013), 305–318.

[3] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for two kinds of nonlinear neutral differential equations with variable delay*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **20**(3) (2013), 357–366.

[4] A. Ardjouni and A. Djoudi, *Existence and positivity of solutions for a totally nonlinear neutral periodic differential equation*, Miskolc Math. Notes **14**(3) (2013), 757–768. https://doi.org/10.18514/MMN.2013.742

[5] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for a second-order nonlinear neutral differential equation with variable delay*, Adv. Nonlinear Anal. **2**(2) (2013), 151–161. https://doi.org/10.1515/anona-2012-0024

[6] A. Ardjouni and A. Djoudi, *Existence of periodic solutions for a second order nonlinear neutral differential equation with functional delay*, Electron. J. Qual. Theory Differ. Equ. **2012**(31) (2012), 1–9. https://doi.org/10.14232/ejqtde.2012.1.31

[7] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for a nonlinear neutral differential equation with variable delay*, Appl. Math. E-Notes **12** (2012), 94–101.

[8] L. C. Becker and T. A. Burton, *Stability, fixed points and inverse of delays*, Proc. Roy. Soc. Edinburgh Sect. A **136**(2) (2006), 245–275. https://doi.org/10.1017/S0308210500004546

[9] K. Bessioud, A. Ardjouni and A. Djoudi, *Stability of nonlinear neutral mixed type liven-nohel integro-differential equations*, Kragujevac J. Math. **46**(5) (2022), 721–732.

[10] K. Bessioud, A. Ardjouni and A. Djoudi, *Periodicity and positivity in neutral nonlinear Liven-Nohel integro-differential equation*, Honam Math. J. **42**(4) (2020), 667–680. https://doi.org/10.5831/HMJ.2020.42.4.667

[11] K. Bessioud, A. Ardjouni and A. Djoudi, *Asymptotic stability in nonlinear neutral Levin-Nohel integro-differential equations*, Journal of Nonlinear Functional Analysis **2017**(19) (2017), 1–12. https://doi.org/10.23952/jnfa.2017.19

[12] T. A. Burton, *Liapunov functionals, fixed points and stability by Krasnoselskii's theorem*, Nonlinear Stud. **9**(2) (2002), 181–190.

[13] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, New York, 2006.

[14] T. A. Burton, *A fixed point theorem of Krasnoselskii*, App. Math. Lett. **11**(1) (1998), 85–88. https://doi.org/10.1016/S0893-9659(97)00138-9

[15] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York, 1985.

[16] F. Chen, *Positive periodic solutions of neutral Lotka-Volterra system with feedback control*, Appl. Math. Comput. **162**(3) (2005), 1279–1302. https://doi.org/10.1016/j.amc.2004.03.009

[17] I. Derrardjia, A. Ardjouni and A. Djoudi, *Stability by Krasnoselskii's theorem in totally nonlinear neutral differential equation*, Opuscula Math. **33**(2) (2013), 255–272. http://dx.doi.org/10.7494/OpMath.2013.33.2.255

[18] N. T. Dung, *Asymptotic behavior of linear advanced differential equations*, Acta Math. Sci. Ser. B (Engl. Ed.) **35**(3) (2015), 610–618. https://doi.org/10.1016/S0252-9602(15)30007-2

[19] N. T. Dung, *New stability conditions for mixed linear Levin-Nohel integro-differential equations*, J. Math. Phys. **54**(8) (2013), 1–11. https://doi.org/10.1063/1.4819019

[20] M. Fan, K. Wang, P. J. Y. Wong and R. P. Agarwal, *Periodicity and stability in periodic n-species Lotka-Volterra competition system with feedback controls and deviating arguments*, Acta Math. Sin. (Engl. Ser.) **19**(4) (2003), 801–822. https://doi.org/10.1007/s10114-003-0311-1

[21] J. K. Hale, *Theory of Functional Differential Equations*, Second Edition, Applied Mathematical Sciences, Springer-Verlag, New York-Heidelberg, 1977. https://doi.org/10.1007/978-1-4612-9892-2

[22] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-4342-7

[23] M. B. Mesmouli, A. Ardjouni and A. Djoudi, *Study of the stability in nonlinear neutral differential equations with functional delay using Krasnoselskii-Burton's fixed-point*, Appl. Math. Comput. **243** (2014), 492–502. https://doi.org/10.1016/j.amc.2014.05.135

[24] M. B. Mesmouli, A. Ardjouni and A. Djoudi, *Stability in neutral nonlinear differential equations with functional delay using Krasnoselskii-Burton's fixed-point*, Nonlinear Stud. **21**(4) (2014), 601–617.

[25] D. R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, Cambridge University Press, London-New York, 1974.

[26] Y. Wang, H. Lian and W. Ge, *Periodic solutions for a second order nonlinear functional differential equation*, App. Math. Lett. **20**(1) (2007), 110–115. https://doi.org/10.1016/j.aml.2006.02.028

[27] E. Yankson, *Existence and positivity of solutions for a nonlinear periodic differential equation*, Arch. Math. (Brno) **48**(4) (2012), 261–270. https://doi.org/10.5817/AM2012-4-261

[1]Department of Mathematics and Informatics,
University of Souk Ahras,
P.O. Box 1553, Souk Ahras, Algeria
Email address: karima_bess@yahoo.fr
Email address: abd_ardjouni@yahoo.fr

[2]Department of Mathematics,
University of Annaba,
P.O. Box 12, Annaba, Algeria
Email address: adjoudi@yahoo.com

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 267–285.

ESSENTIAL APPROXIMATE PSEUDOSPECTRA OF MULTIVALUED LINEAR RELATIONS

AREF JERIBI[1] AND KAMEL MAHFOUDHI[1]

Abstract. One of the fundamental ideas investigated in A. Ammar, A. Jeribi and K. Mahfoudhi in [5] is that of providing conditions under which the essential approximate pseudospectrum of closed, densely defined linear operators have a relationship with Fredholm theory and perturbation theory. In this paper the approximate pseudospectrum and the essential approximate pseudospectrum of closed, densely defined multivalued linear relations are introduced and studied, and work done in the aforementioned papers are extended to general multivalued linear relations

1. Introduction

A vast number of the problems that have been investigated in the Banach algebra setting originated in the context of bounded linear operators or multivalued linear relations on a Banach space. Let X denote a linear vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. T multivalued linear operator on X is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T, into the collection of non empty subsets of X such that

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2),$$

for all non zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator, which is equivalent to $T(0) = \{0\}$.

Key words and phrases. Pseudospectrum, approximate pseudospectra, essential approximate pseudospectra, multivalued linear relations.

2020 *Mathematics Subject Classification.* Primary: 47A53. Secondary: 47A55, 54B35, 47A13.

DOI 10.46793/KgJMat2502.267J

Received: February 03, 2022.

Accepted: April 04, 2022.

We denote by $\mathrm{L}\mathcal{R}(X)$ the class of linear relations everywhere defined. $T \in \mathrm{L}\mathcal{R}(X)$ is uniquely determined by its graph $\mathcal{G}(T)$, which is defined by

$$\mathcal{G}(T) := \{(x,y) \in X \times X : x \in \mathcal{D}(T),\ y \in Tx\},$$

so that we can identify T with $\mathcal{G}(T)$. The closure and completion of T, denoted by $\overline{T}$ and $\widetilde{T}$, respectively, is the linear relation defined by

$$\mathcal{G}(\overline{T}) := \overline{\mathcal{G}(T)},$$
$$\mathcal{G}(\widetilde{T}) := \widetilde{\mathcal{G}(T)}.$$

We denote by $\mathrm{C}\mathcal{R}(X)$ the class of all closed linear relations from X into X. The inverse of T is a linear relation T^{-1} given by

$$\mathcal{G}(T^{-1}) := \{(y,x) : (x,y) \in \mathcal{G}(T)\}.$$

If $\mathcal{G}(T)$ is closed, then T is said to be closed,

$$\mathcal{N}(T) := \{x \in \mathcal{D}(T) : (x,0) \in \mathcal{G}(T)\} \quad \text{and} \quad \mathcal{R}(T) := T(\mathcal{D}(T))$$

denote kernel structure and the range of the relation T, respectively. The linear relation $T+S$ is defined by

$$\mathcal{G}(T+S) := \{(x,y) \in X \times X : y = u + v \text{ with } (x,u) \in \mathcal{G}(T), (x,v) \in \mathcal{G}(S)\}.$$

Let $T \in \mathrm{L}\mathcal{R}(X)$ and $S \in \mathrm{L}\mathcal{R}(X)$ where $\mathcal{R}(T) \cap \mathcal{D}(S) \neq \emptyset$. The product of ST is defined by

$$\mathcal{G}(ST) := \{(x,z) \in X \times X : (x,u) \in \mathcal{G}(T) \text{ and } (u,z) \in \mathcal{G}(S) \text{ for some } u \in X\}.$$

Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$. We shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly, $Q_T T$ is a single valued operator and the norm of T is defined by

$$\|T\| := \|Q_T T\|.$$

We say that T is continuous if for each neighborhood V in $\mathcal{R}(T)$, $T^{-1}(V)$ is a neighborhood in $\mathcal{D}(T)$ ($\|T\| < \infty$), bounded if it is continuous with $\mathcal{D}(T) = X$, open if T^{-1} is continuous, equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) := \sup\Big\{\lambda \geq 0 : \lambda d(x, \mathcal{N}(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T)\Big\},$$

where $d(x,\mathcal{N}(T))$ is the distance between x and $\mathcal{N}(T)$. If $\mathcal{D}(T)$ and if $\|T\| < \infty$, then we shall say that T is bounded.

The class of such relations is denoted by $\mathrm{L}\mathcal{R}(X)$ and we denote by $\mathcal{L}(X)$ the set of all bounded linear operators from X. For $T \in \mathrm{L}\mathcal{R}(X)$, we write

$$\alpha(T) := \dim \mathcal{N}(T), \quad \beta(T) := \dim X/\mathcal{R}(T), \quad \overline{\beta}(T) := \dim X/\overline{\mathcal{R}(T)},$$

and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite. We say T is upper semi-Fredholm, if there exists a finite codimensional subspace M of $\mathcal{D}(T)$ for which $T_{|M}$ is injective and open. If M and N are subspaces of X and of the dual space $X^{'}$ respectively, then

$$M^{\perp} := \Big\{x^{'} \in X^{'} : x^{'}(x) = 0 \text{ for all } x \in M\Big\}$$

and

$$N^{\top} := \Big\{x \in X : x^{'}(x) = 0 \text{ for all } x^{'} \in N\Big\}.$$

The conjugate of $T \in \mathrm{L}\mathcal{R}(X)$ is the linear relation $T^{'}$ defined by

$$\mathcal{G}(T^{'}) := \mathcal{G}(-T^{-1})^{\perp} \subset Y^{'} \times X^{'},$$

so that $(y^{'}, x^{'}) \in \mathcal{G}(T^{'})$ if, and only if, $y^{'}(y) = x^{'}(x)$ for all $(x, y) \in \mathcal{G}(T)$.

A closed linear relation T acts from X into X.

Definition 1.1. Let $T \in \mathrm{L}\mathcal{R}(X)$.

(i) T is said to be upper semi-Fredholm, if there exists a closed, finite, codimensional subspace M of X, such that the restriction $T|_M$ has a single valued continuous inverse.

(ii) T is said to be lower semi-Fredholm linear relation if its conjugate $T^{'}$ is uppersemi-Fredholm linear relation.

We denote by $\mathcal{F}_+(X)$, the set of upper semi-Fredholm linear relations and by $\mathcal{F}_-(X)$ the set of lower semi-Fredholm linear relations.

In the case when X is Banach space, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators, and note that the definitions of the classes $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are consistent, respectively, with

$$\Phi_+(X) = \Big\{T \in \mathrm{C}\mathcal{R}(X) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } X\Big\},$$

$$\Phi_-(X) = \Big\{T \in \mathrm{C}\mathcal{R}(X) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } X\Big\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm relations from X and $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ denotes the set of semi-Fredholm relations from X.

We say that T is strictly singular, if there is no infinite dimensional subspace M of $\mathcal{D}(T)$ for which the restriction $T|_M$ has a single valued continuous inverse.

The families of all compact and strictly singular linear relations will be denoted by $\mathrm{K}\mathcal{R}(X)$ and $SS\mathcal{R}(X)$, respectively.

Let $T \in \mathrm{L}\mathcal{R}(X)$, the set

$$\rho(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open with dense range on X}\Big\}.$$

Referring back to the closed theorem of linear relations (see [16, 17]), when T is closed and X is a Banach space, this coincides with the set

$$\Big\{\lambda \in \mathbb{C} : (\lambda - \widetilde{T})^{-1} \text{ is everywhere defined and single valued}\Big\}.$$

Therefore, our definition of a resolvent set coincides with the standard definition for bounded or closed operators in Banach spaces.

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$. The set $\rho(T)$ is open, whereas the spectrum $\sigma(T)$ of a closed linear relation T is closed. The approximate point spectrum of T is the set defined by

$$\sigma_{ap}(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below}\Big\}.$$

The defect spectrum of T is the set defined by

$$\sigma_{\delta}(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\Big\}.$$

Let $T \in \mathrm{L}\mathcal{R}(X)$ and $\varepsilon > 0$. We define the pseudospectra of a linear relation T by

$$\sigma_{\varepsilon}(T) := \sigma(T) \cup \Big\{\lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\Big\}.$$

The approximate pseudospectrum of a linear relation T by the set

$$\sigma_{ap,\varepsilon}(T) := \sigma_{ap}(T) \cup \Big\{\lambda \in \mathbb{C} : \inf_{\substack{x \in \mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| \leq \varepsilon\Big\},$$

and the defect pseudospectrum of a linear relation T by

$$\sigma_{\delta,\varepsilon}(T) = \sigma_{ap,\varepsilon}(T^{'}).$$

Our concern in this paper is mainly the following essential pseudospectra

$$\begin{aligned}
\sigma_{e1,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{+}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e2,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{-}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e3,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{\pm}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e4,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi(X) \text{ for all } S \in \mathcal{I}_T(X)\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_T(X) :=& \{S \in \mathrm{L}\mathcal{R}(X) \text{ is continuous} : \|S\| < \varepsilon, \mathcal{D}(S) \supset \mathcal{D}(T) \text{ and } S(0) \subset T(0)\},\\
\sigma_{w,\varepsilon}(T) =& \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon}(T + K),\\
\sigma_{eap,\varepsilon}(T) =& \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T + K),\\
\sigma_{e\delta,\varepsilon}(T) =& \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\delta,\varepsilon}(T + K)
\end{aligned}$$

and

$$\mathcal{K}_T(X) := \{K \in \mathrm{K}\mathcal{R}(X) : \mathcal{D}(K) \supset \mathcal{D}(T) \text{ and } K(0) \subset T(0)\}.$$

We turn our attention to the following inclusions

$$\begin{aligned}
\sigma_{e1,\varepsilon}(T) \cap \sigma_{e2,\varepsilon}(T) =& \sigma_{e3,\varepsilon}(T) \subset \sigma_{e4,\varepsilon}(T) \subset \sigma_{w,\varepsilon}(T) \subset \sigma_{\varepsilon}(T),\\
\sigma_{e1,\varepsilon}(T) \subset& \sigma_{eap,\varepsilon}(T) \text{ and } \sigma_{e2,\varepsilon}(T) \subset \sigma_{e\delta,\varepsilon}(T),\\
\sigma_{w,\varepsilon}(T) =& \sigma_{eap,\varepsilon}(T) \cup \sigma_{e\delta,\varepsilon}(T).
\end{aligned}$$

If ε tends to 0, we recover the usual definition of the essential spectra of a closed operator T. The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(\cdot)$ are the Gustafson and Weidmann essential spectra, $\sigma_{e3}(\cdot)$ is the Kato essential spectrum, $\sigma_{e4}(\cdot)$ is the Wolf essential spectrum, $\sigma_w(\cdot)$ is the Schechter essential spectrum, $\sigma_{eap}(\cdot)$ is the essential approximate point spectrum and $\sigma_{e\delta}(\cdot)$ is the essential defect spectrum.

Remark 1.1. Let $T \in L\mathcal{R}(T)$.

(i) If $\varepsilon_1 < \varepsilon_2$, then $\sigma_{j,\varepsilon_1}(T) \subset \sigma_{j,\varepsilon_2}(T)$ with $j = 1, 2, 3, 4, w, eap, \delta$.

(ii) It is clear that $\sigma_{j,\varepsilon}(T)$, with $j = w, eap, \delta$ has a remarkable stability, any compact perturbation $K \in \mathcal{K}_T(X)$ leaves the essential pseudospectrum invariant, then we have $\sigma_{j,\varepsilon}(T + K) = \sigma_{j,\varepsilon}(T)$, with $j = w, eap, \delta$.

This paper is a continuation of the research which was undertaken by A. Ammar and A. Jeribi in works [3,5,6,13] and was devoted to special subsets of the pseudospectrum and the essential pseudospectrum of closed, densely defined multivalued linear relations

$$\sigma_{w,\varepsilon}(T) := \bigcap_{K\in\mathcal{K}_T(X)} \sigma_\varepsilon(T + K) := \bigcup_{\|D\|<\varepsilon} \sigma_w(T + D),$$

where

$$\sigma_w(T) := \bigcap_{K\in\mathcal{K}_T(X)} \sigma(T + K).$$

Also, for the benefit of the reader we review an important result about pseudospectrum from [7–10] and [11,12].

After compressing or depressing them, certain parts of pseudospectrum of an linear relations acting between Banach space may be distinguished. Among these parts, we are interested in two: one is the approximate pseudospectrum and the other is the essential approximate pseudospectrum. Motivated by the approximate pseudospectrum versions introduced by M. P. H. Wolf [22] in the case of linear operator, it becomes possible to extend this definition to the case of multivalued linear relations of closed, densely defined multivalued linear relations. Recently, J. M. Varah [21], has introduced the first idea of pseudospectra. L. N. Trefethen [18,19], not only initiated the study of pseudospectrum for matrices and operators, but he also talked of approximate eigenvalues and pseudospectrum and used this notion to study interesting problems in mathematical physics. In the same vein, several authors have worked on this field. For example, we may refer to E. B. Davies [15].

The main aims of this work are the following: we introduce and study the approximate pseudospectrum and the essential approximate pseudospectrum of closed, densely defined multivalued linear relations. We begin by the definition and we investigate the characterization, the stability and some properties of these pseudospectrum.

We organize our paper in the following way. In Section 2 contains preliminary and auxiliary properties that will be necessary in order to prove the main results of the other sections. Some results concerning approximate pseudospectrum and essential approximate pseudospectrum are established in Sections 3 and 4. The main focus of this section are Theorems 3.4, 3.5 and 4.1. Subsequently, we apply the obtained results to study the invariance and the characterization of the essential approximate pseudospectrum of a closed multivalued linear operator.

2. Preliminary Results

In this section we collect some results of the theory of multivalued linear operators which will be needed in the following sections.

Definition 2.1. Let $S \in \mathrm{L}\mathcal{R}(X)$ be continuous where, X is normed spaces.

(i) S is called a Fredholm perturbation if $T + S \in \Phi(X)$, whenever $T \in \Phi(X)$.

(ii) S is called an upper semi-Fredholm perturbation if $T + S \in \Phi_+(X)$, whenever $T \in \Phi_+(X)$.

(iii) S is called a lower semi-Fredholm perturbation if $T + S \in \Phi_-(X)$, whenever $T \in \Phi_-(X)$.

The sets of Fredholm, upper and lower semi-Fredholm perturbations are denoted by $\mathcal{P}(\Phi(X))$, $\mathcal{P}(\Phi_+(X))$, and $\mathcal{P}(\Phi_-(X))$, respectively.

We denote also the set

$$\mathcal{P}_T(\Phi(X)) := \{S \in \mathcal{P}(\Phi(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\},$$

$$\mathcal{P}_T(\Phi_+(X)) := \{S \in \mathcal{P}(\Phi_+(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\}$$

and

$$\mathcal{P}_T(\Phi_-(X)) := \{S \in \mathcal{P}(\Phi_-(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\}.$$

In general by [2] we have

$$\mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_+(X)) \subset \mathcal{P}_T(\Phi(X)) \quad \text{and} \quad \mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_-(X)) \subset \mathcal{P}_T(\Phi(X)).$$

Lemma 2.1 ([2])**.** *Let $T \in C\mathcal{R}(X)$, where X is Banach spaces. Then the following hold.*

(i) *If $T \in \Phi_+(X)$ and $S \in \mathcal{P}_T(\Phi_+(X))$, then $T + S \in \Phi_+(X)$ and $i(T + S) = i(T)$.*

(ii) *If $T \in \Phi_-(X, Y)$ and $S \in \mathcal{P}_T(\Phi_-(X))$, then $T+S \in \Phi_-(X)$ and $i(T+S) = i(T)$.*

Lemma 2.2 ([16])**.** *Let $T \in L\mathcal{R}(X)$. Then for $x \in \mathcal{D}(T)$, we have the following equivalence:*

(i) $y \in Tx \Leftrightarrow Tx = y + T(0)$.

In particular,

(ii) $0 \in Tx \Leftrightarrow Tx = T(0)$.

Lemma 2.3 ([16, Corollary I.2.11])**.** *Let $T \in L\mathcal{R}(X)$. Then*

(i) $T^{-1}Tx = x + T^{-1}(0)$ *for all* $x \in \mathcal{D}(T)$;

(ii) $TT^{-1}y = y + T(0)$ *for all* $y \in R(T)$.

Lemma 2.4 ([16, Proposition II.1.4 and II.1.6])**.** *Let X is normed spaces and $T \in L\mathcal{R}(X)$. Then*

(i) $\|Tx\| = d(y, T(0))$ for any $y \in Tx$;

(ii) $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$ $(x \in \mathcal{D}(T))$;

(iii) $\|T\| = \sup_{x \in B_X} \|Tx\|$ with $B_X := \{x \in X : \|x\| \leq 1\}$.

Lemma 2.5 ([12, Proposition 3.1])**.** *Let S, $T \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$. If $T \in C\mathcal{R}(X)$ and S is continuous, then $S+T \in C\mathcal{R}(X)$.*

Theorem 2.1 ([16, Theorem III.4.2])**.** *Let $T \in C\mathcal{R}(X)$, then*
(i) *T is continuous if and only if $\mathcal{D}(T)$ is closed;*
(ii) *T is open if and only if $R(T)$ is closed.*

Lemma 2.6 ([16, Proposition II.3.20])**.** *Let T be open and injective and let $S \in L\mathcal{R}(X)$ be a linear operator satisfying $\|S\| < \gamma(T)$. Then $T+S$ is open and injective.*

Proposition 2.1 ([16, Proposition I.4.2], [14, Lemma 2.4])**.** *Let $R, S, T \in L\mathcal{R}(X)$. Then*
(i) *$(R+S)T \subset RT+ST$ with equality if T is single valued.*
(ii) *Let $T \in L\mathcal{R}(X)$ and $S, R \in L\mathcal{R}(Y,Z)$. If $T(0) \subset \mathcal{N}(S)$ or $T(0) \subset \mathcal{N}(R)$, then $(R+S)T = RT+ST$.*

Theorem 2.2 ([16, Theorem III.5.3])**.** *Let X, Y be Banach spaces and let $T \in C\mathcal{R}(X)$. Then T is open if and only if $R(T)$ is closed.*

Theorem 2.3 ([12, Theorem 2.2])**.** *Let $S, T \in L\mathcal{R}(X)$ be closed. We have the following.*
(i) *If $S, T \in \Phi_+(X)$, then $ST \in \Phi_+(X)$ and $TS \in \Phi_+(X)$.*
(ii) *If $S, T \in \Phi_-(X)$, with TS (resp. ST) closed, then $TS \in \Phi_-(X)$ (resp. $ST \in \Phi_-(X)$).*
(iii) *If $S, T \in \Phi(X)$, then $TS \in \Phi(X)$ and $i(TS) = i(T) + i(S) + \dim X\big/(R(S) + D(T)) - \dim[S(0) \cap N(T)]$.*
(iv) *If S and T are everywhere defined and $TS \in \Phi_+(X)$, then $S \in \Phi_+(X)$.*
(v) *If S and T are everywhere defined such that $TS \in \Phi(X)$ and $ST \in \Phi(X)$, then $S \in \Phi(X)$ and $T \in \Phi(X)$.*

3. The Approximate Pseudospectrum of Linear Relations

The goal of this section is to study the approximate of pseudospectrum of closed, densely defined multivalued linear relations.

Proposition 3.1. *Let $T \in L\mathcal{R}(X)$ where X is a normed space. Then*

$$\sigma_{ap,\varepsilon}(T) \subset \sigma_\varepsilon(T).$$

Proof. Let $\lambda \notin \sigma_\varepsilon(T)$, then $\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}$. Moreover,

$$\frac{1}{\inf\limits_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\|} = \sup_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \frac{\|x\|}{\|(\lambda-\tilde{T})x\|}$$

$$(3.1) \qquad = \sup_{0\neq x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)}} \frac{\|x\|}{\|(\lambda-\tilde{T})x\|}.$$

Putting $x := (\lambda - T)^{-1}y$ we have

$$\begin{aligned}(\lambda - T)x &= (\lambda - T)(\lambda - T)^{-1}y \quad \text{(by Lemma 2.3)}\\ &= y + (\lambda - T)(0).\end{aligned}$$

Using Lemma 2.2, we obtain that $y \in (\lambda - T)x$. On the other hand, $(\lambda - T)(0) = \lambda(0) - T(0) = 0 - T(0) = T(0)$. Also by Lemma 2.2, $0 \in T(0)$ and from Lemma 2.4 we have

$$\|(\lambda - T)x\| = d(y, (\lambda - T)(0)) = d(y, T(0)) \leq d(y, 0) = \|y\|. \tag{3.2}$$

Combining (3.1) and (3.2) that

$$\sup_{y \in X \setminus \{0\}} \frac{\|(\lambda - \tilde{T})^{-1}y\|}{\|y\|} = \|(\lambda - \tilde{T})^{-1}\| \leq \frac{1}{\varepsilon}.$$

Consequently,

$$\inf_{\substack{x \in \mathcal{D}(T) \setminus \overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| > \varepsilon.$$

Hence,

$$\lambda \notin \sigma_{ap,\varepsilon}(T).$$ □

Example 3.1. Let $X = \mathbb{C}^n$ and let T be the single-valued linear operator on $\mathcal{L}(\mathbb{C}^n)$ given for all $n \geq 2$ with the infinity norm by

$$\begin{cases} T : \mathbb{C}^n \to \mathbb{C}^n, \\ e_i \mapsto Te_i, \quad \text{where } Te_i = e_{(n+1)-i}. \end{cases}$$

It is easily checked that

$$\begin{cases} T = T^{-1}, \\ \|T\| = 1, \\ \sigma(T) \cup \{\infty\} = \{-1, 1\}. \end{cases}$$

Therefore, T is everywhere defined closed linear relation. We will check that if

$$\|(\lambda - T)e_i\| = \lambda e_i - e_{(n+1)-i},$$

then

$$\inf_{\substack{e_i \in \mathcal{D}(T) \setminus \overline{\mathcal{N}(T)},\\ \|e_i\|=1}} \|(\lambda - T)e_i\| = |\lambda| + 1,$$

and if

$$\|(\lambda - T)^{-1}e_i\| = \frac{\lambda e_i - e_{(n+1)-i}}{\lambda^2 - 1},$$

then

$$\|(\lambda - T)^{-1}\| = \frac{|\lambda| + 1}{|\lambda^2 - 1|}.$$

Moreover, for $\varepsilon > 1$ we obtain

$$\sigma_{ap,\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon - 1\},$$
$$\sigma_{\varepsilon}(T) = \left\{\lambda \in \mathbb{C} : \frac{|\lambda| + 1}{|\lambda^2 - 1|} > \frac{1}{\varepsilon}\right\}.$$

It is easy to verify that, for all λ with $0 \leq |\lambda| \leq 1$ we have

$$\sigma_{\varepsilon}(T) \neq \sigma_{ap,\varepsilon}(T).$$

Proposition 3.2. *Let $T \in L\mathcal{R}(X)$, where X is a normed space. Then*

$$\bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T).$$

Proof. It is clear that $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon}(T)$ for all $\varepsilon > 0$, then

$$\sigma_{ap}(T) \subset \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T).$$

Conversely, let $\lambda \notin \sigma_{ap}(T)$. Then $\lambda - T$ is bounded below, hence $\lambda - T$ is injective, open with dense range on X and $(\lambda - \tilde{T})^{-1}$ is a bounded linear operator, so there exists $\varepsilon > 0$ such that

$$\|(\lambda - \tilde{T})^{-1}\| \leq \frac{1}{\varepsilon}.$$

Therefore,

$$\lambda \notin \sigma_{\varepsilon}(T),$$

and we conclude from Proposition 3.1 that $\lambda \notin \sigma_{ap,\varepsilon}(T)$. So, $\lambda \notin \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$. ☐

Theorem 3.1. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, for any α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ we have the following.*
(i) If $\alpha \in \mathbb{C}$ and $\varepsilon > 0$, then $\sigma_{ap,\varepsilon}(T + \alpha I) = \alpha + \sigma_{ap,\varepsilon}(T)$.
(ii) If $\beta \in \mathbb{C}\backslash\{0\}$ and $\varepsilon > 0$, then $\sigma_{ap,|\beta|\varepsilon}(\beta T) = \beta\sigma_{ap,\varepsilon}(T)$.

Proof. (i) Let $\lambda \in \sigma_{ap,\varepsilon}(T + \alpha I)$, then

$$\lambda \in \sigma_{ap}(T + \alpha I) \quad \text{or} \quad \inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - \alpha I - T)x\| \leq \varepsilon.$$

Hence, $(\lambda - \alpha)I - T$ is not bounded below (not injective) or

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|((\lambda - \alpha)I - T)x\| \leq \varepsilon.$$

This yields to $\lambda \in \alpha + \sigma_{ap,\varepsilon}(T)$. For the second inclusion it is the same reasoning.
(ii) Let $\lambda \in \sigma_{ap,|\alpha|\varepsilon}(\alpha T)$, then $\lambda \in \sigma_{ap}(\alpha T)$ or

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - \alpha T)x\| \leq |\alpha|\varepsilon.$$

It follows that $\lambda - \alpha T$ is not bounded below (not injective) or

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\|\alpha\left(\frac{\lambda}{\alpha} - T\right)x\right\| \leq |\alpha|\varepsilon.$$

Hence, $\alpha(\frac{\lambda}{\alpha} - T)$ is not bounded below (not injective) or

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\|\alpha\left(\frac{\lambda}{\alpha} - T\right)x\right\| = |\alpha| \inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\|\left(\frac{\lambda}{\alpha} - T\right)x\right\| \leq |\alpha|\varepsilon.$$

Thus, $\frac{\lambda}{\alpha} \in \sigma_{ap}(T)$ or

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\|\left(\frac{\lambda}{\alpha} - T\right)x\right\| \leq \varepsilon.$$

So, $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) \subseteq \alpha\sigma_{ap,\varepsilon}(T)$. However, the reverse inclusion is similar. □

Corollary 3.1. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, for any α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ we have*

$$\sigma_{ap,\varepsilon}(\alpha I + \beta T) = \alpha + \beta\sigma_{ap,\varepsilon|\beta|}(T).$$

Definition 3.1. Given a polynomial $P(z) = \sum_{k=0}^{n} \alpha_k z^k$ with coefficients $\alpha_k \in \mathbb{C}$, we define the polynomial in T by $P(T) = \sum_{k=0}^{n} \alpha_k T^k$.

Theorem 3.2. *Let T be closed relation and assume that V is closed single valued bounded relation such that $0 \in \rho(V)$. Let $S = VTV^{-1}$. Then, for all polynomial $P(T)$ of degree n we have $P(S)$ is closed and*

$$P(S) = VP(T)V^{-1}.$$

Proof. We need to show that $P(S)$ is closed. Let T is closed relation, then from [1, Lemma 2.7] we obtain that $P(T)$ is closed. On the other hand, V is closed single valued bounded relation such that $0 \in \rho(V)$, then V has a closed range ($R(V) = X$). By the fact that V injective and open we have

$$\alpha(V) = 0 < \infty \quad \text{and} \quad \gamma(V) > 0.$$

By using [16, Proposition II.5.17], we deduce that $VP(T)$ is closed. Moreover, since V^{-1} is single valued and bounded, then from [16, Exercise II.5.18] we obtain $VP(T)V^{-1}$ is closed. Hence, $P(S)$ is closed. Now, let

$$P(S) = \sum_{k=0}^{n} \alpha_k S^k = \sum_{k=0}^{n} \alpha_k (VTV^{-1})^k.$$

Since, V is single valued injective we infer that

$$P(S) = \sum_{k=0}^{n} \alpha_k (VTV^{-1})^k = V\left(\sum_{k=0}^{n} \alpha_k T^k\right)V^{-1} = VP(T))V^{-1}.$$ □

Theorem 3.3. *Let $T \in C\mathcal{R}(X)$ where X is a complete space and assume that V as in Theorem 3.2. Let $k = \|V\|\|V^{-1}\|$ and $P(S) = VP(T)V^{-1}$. Then*

$$\sigma_{ap}(P(S)) = \sigma_{ap}(P(T)).$$

Proof. We have

$$\begin{aligned}\lambda - P(S) =& \lambda - VP(T)V^{-1},\\ (\lambda - P(S))V =& (\lambda - VP(T)V^{-1})V\\ =& (\lambda V - VP(T)V^{-1}V) \quad \text{(using [16, Proposition I.4.2])}\\ =& (\lambda V - VP(T)) \quad \text{(as V is injective)}\end{aligned}$$

and

$$\begin{aligned}V^{-1}(\lambda - P(S))V =& V^{-1}(\lambda V - VP(T)),\\ V^{-1}(\lambda - P(S))V =& (\lambda V^{-1}V - V^{-1}VP(T)) \quad \text{(using [16, Proposition I.4.2])}\\ V^{-1}(\lambda - P(S))V =& (\lambda - P(T)) \quad \text{(as V is injective)}\\ (\lambda - P(S)) =& V(\lambda - P(T))V^{-1} \quad \text{(as V is single valued).}\end{aligned}$$

Now, if $\lambda \notin \sigma_{ap}(P(T))$ then the closed relation $\lambda - P(T)$ is bounded below (injective, open). By [16, Proposition VI.5.2])

$$V(\lambda - P(T))V^{-1} = \lambda - P(S)$$

is closed, bounded below (injective, open). Hence, $\lambda \in\notin \sigma_{ap}(P(S))$.

Conversely, if $\lambda \notin \sigma_{ap}(P(S))$ then the closed relation $\lambda - P(S)$ is bounded below (injective, open). By [16, Proposition VI.5.2])

$$V^{-1}(\lambda - P(S))V = \lambda - P(T)$$

is also closed, bounded below (injective, open). Hence, $\lambda \notin \sigma_{ap}(P(T))$, which implies the result. ☐

Now, we are ready to give our first main result of this section.

Theorem 3.4. *Let $T \in C\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If $\lambda \in \sigma_{ap,\varepsilon}(T)$, then there is $S \in L\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T+S)$.*

Proof. Let $\lambda \in \sigma_{ap,\varepsilon}(T)$. We will discuss these two cases.

1. case. If $\lambda \in \sigma_{ap}(T)$, we may put $S = 0$.

2. case. If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$, $\|x_0\| = 1$ such that

$$\|(\lambda - T)x_0\| < \varepsilon,$$

and by the Hahn Banach Theorem (see [20]), there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$. We define the relation $S : X \to X$ by

$$S(x) := x'(x)(\lambda - T)x_0.$$

It is clear that S is everywhere defined and single valued (as $S(0) = 0$).

$$\|Sx\| = \|x'(x)(\lambda - T)x_0)\| \leq \|x'\|\|x\|\|(\lambda - T)x_0\|,$$

for $x \neq 0$, we have

$$\frac{\|Sx\|}{\|x\|} \leq \|(\lambda - T)x_0\|,$$

so,

$$\|S\| \leq \|(\lambda - T)x_0\| < \varepsilon.$$

We can rewrite

$$\begin{aligned}
\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T - S)x\| \leq & \|(\lambda - T - S)x_0\| \\
\leq & \|(\lambda - T)x_0 - Sx_0\| \\
\leq & \|(\lambda - T)x_0 - x'(x_0)(\lambda - T)x_0\| \\
\leq & \|(\lambda - T)(0)\| \\
\leq & \|\lambda(0) - T(0)\| \\
\leq & \|T(0)\| = d(T(0), T(0)) = 0.
\end{aligned}$$

Then, $\lambda \in \sigma_{ap}(T + S)$. ☐

Theorem 3.5. *Let $T \in C\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If there is $S \in L\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T + S)$. Then $\lambda \in \sigma_{ap,\varepsilon}(T)$.*

Proof. Suppose that there exists a continuous linear relation $D \in \mathrm{L}\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$ such that

$$\lambda \in \sigma_{ap}(T + S),$$

which means that

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T - S)x\| = 0.$$

In order to prove that

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| < \varepsilon,$$

we can write

$$\begin{aligned}
\|(\lambda - T)x_0\| = \|(\lambda - T - S + S)x_0\| \leq & \|(\lambda - T - S)x_0\| + \|Sx_0\| \\
\leq & \|T(0)\| + \varepsilon \\
\leq & \varepsilon.
\end{aligned}$$

Then

$$\inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| < \varepsilon.$$ ☐

Corollary 3.2. *In summary, at the present moment we have shown that from Theorems* 3.5 *and* 3.4*, that for* $T \in C\mathcal{R}(X)$ *and* $\varepsilon > 0$

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\mathcal{J}_T(X)} \sigma_{ap}(T+S),$$

where

$$\mathcal{J}_T(X) := \Big\{ S \in L\mathcal{R}(X) \text{ is continuous} : \|S\| < \varepsilon, \mathcal{D}(T) \subset \mathcal{D}(S) \text{ and } S(0) \subset T(0) \Big\}.$$

Theorem 3.6. *Let* $T \in C\mathcal{R}(X)$ *where* X *is a complete space, then for any* $\varepsilon > 0$ *and* $E \in L\mathcal{R}(X)$ *such that* $E(0) \subset T(0)$ *and* $\mathcal{D}(E) \supset \mathcal{D}(T)$

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T+E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T).$$

Proof. Let $\lambda \in \sigma_{ap,\varepsilon-\|E\|}(T)$. Then, by Theorem 3.4 there is $S \in \mathrm{L}\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon - \|E\|$ such that

$$\lambda \in \sigma_{ap}(T+S) = \sigma_{ap}\Big((T+E) + (S-E)\Big).$$

Using [16, Proposition II.1.7] we get

$$\begin{aligned} \|S-E\| \leq & \|S\| + \|-E\| \\ = & \|S\| + \|E\| < \varepsilon \quad \text{(using [16, Proposition II.1.7])}. \end{aligned}$$

Then, from Theorem 3.5, we deduce that $\lambda \in \sigma_{ap,\varepsilon}(T+E)$. Using a similar reasoning to the first inclusion, we deduce that $\lambda \in \sigma_{ap,\varepsilon+\|E\|}(T)$. ☐

4. Essential Approximate Pseudospectra of Linear Relations

We begin this section by showing that the essential approximate pseudospectra of linear relations are closed, and then illustrate some characteristic properties.

Theorem 4.1. *Let* $T \in C\mathcal{R}(X)$ *and* $\varepsilon > 0$. *Then the following statements are equivalent:*

(i) $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(ii) *For all continuous linear relations* $S \in L\mathcal{R}(X)$ *such that* $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ *and* $\|S\| < \varepsilon$*, we have*

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

(iii) *For all continuous single valued relations* $D \in L\mathcal{R}(X)$ *such that* $\mathcal{D}(T) \subset \mathcal{D}(D)$ *and* $\|D\| < \varepsilon$*, we have*

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

Proof. $(i) \Rightarrow (ii)$ Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. Then there exists $K \in \mathcal{K}_T(X)$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+K).$$

Using Theorems 3.5 and 3.4, for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that

$$\begin{aligned} \mathcal{D}(T+K) =& \mathcal{D}(T) \cap \mathcal{D}(K) \\ =& \mathcal{D}(T) \subset \mathcal{D}(S) \quad (\text{as } \mathcal{D}(\ T) \subset \mathcal{D}(K)) \\ (T+K)(0) =& T(0) \supset S(0) \quad (\text{as } K(0) \subset T(0)), \end{aligned}$$

and $\|S\| < \varepsilon$, we have $\lambda \notin \sigma_{ap}(T+S+K)$. Then, $\lambda - T - S - K$ is open, injective with dense range. On the other hand, T is closed and K is compact then K is continuous hence $\lambda - S - K$ is continuous, furthermore $(\lambda - S - K)(0) \subset T(0)$, then using Lemma 2.5, we obtain that $\lambda - T - S - K$ is closed. Hence, from Theorem 2.1, $R(\lambda - T - S - K)$ is closed. We conclude that, $R(\lambda - T - S - K) = X$. Therefore

$$\lambda - T - S - K \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - S - K) \leq 0,$$

for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$. It is obvious from [1, Lemma 2.3] that for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$ we have

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

$(ii) \Rightarrow (iii)$ Is trivial.

$(iii) \Rightarrow (i)$ We assume that for all $D \in \mathrm{L}\mathcal{R}(X)$ a continuous single valued relations such that $\mathcal{D}(T) \subset \mathcal{D}(D)$ and $\|D\| < \varepsilon$, then we have

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

By virtue of [2, Theorem 3.5 (i)], $\lambda - T - D$ can be expressed in the form

$$\lambda - T - D = S + K,$$

where $K \in K_{\lambda-T-D}(X) = \mathcal{K}_T(X)$ since

$$\begin{aligned} K(0) \subset& T(0) = (\lambda - T - D)(0), \\ \mathcal{D}(\lambda - T - D) =& \mathcal{D}(T) \subset \mathcal{D}(K), \end{aligned}$$

and S is a linear relation with closed range and S is injective linear relation (i.e., $\alpha(S) = 0$). So,

$$\lambda - T - D - K = S \quad \text{and} \quad \alpha(\lambda - T - D - K) = 0.$$

Since $\lambda - T - D - K$ is injective linear relation (bounded below), then there exists a constant $M > 0$ such that

$$\|(\lambda - T - D - K)x\| \geq M\|x\|, \quad \text{for all } x \in \mathcal{D}(T).$$

This proves that

$$\inf_{\substack{x \in \overline{\mathcal{D}(T) \backslash \mathcal{N}(T)}, \\ \|x\|=1}} \|(\lambda - T - D - K)x\| \geq M > 0.$$

This is equivalent to say that

$$\lambda \notin \sigma_{ap}(T + D + K),$$

and therefore, $\lambda \notin \sigma_{eap,\varepsilon}(T)$. □

Remark 4.1. In summary, we have shown that from Theorem 4.1, that for $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\substack{\|D\|<\varepsilon \\ \mathcal{D}(T)\subset\mathcal{D}(D)}} \sigma_{eap}(T+D) = \bigcup_{\substack{\|S\|<\varepsilon \\ S(0)\subset T(0) \\ \mathcal{D}(S)\supset\mathcal{D}(T)}} \sigma_{eap}(T+S).$$

Theorem 4.2. *Let $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$. Then $\sigma_{eap,\varepsilon}(T)$ is a closed set.*

Proof. Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$ and D be a single valued continuous linear relation such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$. Hence, by Theorem 4.1, we have

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

So, $R(\lambda - T - D)$ is closed and from Lemma 2.5, we have $\lambda - T - D$ is closed. Then by using Theorem 2.1, $\lambda - T - D$ is open and hence $\gamma(\lambda - T - D) > 0$. Let $r > 0$ such that $r < \gamma(\lambda - T - D)$, let $\mu \in B_f(\lambda, r)$ then $|\mu - \lambda| \leq r < \gamma(\lambda - T - D)$. According to Lemma 2.6, it is clear that

$$\mu - T - D = \lambda - T - D + \mu - \lambda$$

is open and injective. Since $\mu - T - D$ is closed and open, then from Theorem 2.1 we deduce $R(\mu - T - D)$ is closed. Then, $\mu - T - D \in \Phi_+(X)$. On the other hand, using [16, Corollary V.15.7], we have

$$i(\mu - T - D) = i(\lambda - T - D) \leq 0.$$

Consequently, $\mu \notin \sigma_{eap,\varepsilon}(T)$ and we infer that $\sigma_{eap,\varepsilon}(T)$ is a closed. □

Observe that as a direct consequence of Theorem 4.1, we infer the following result.

Proposition 4.1. *Let $T \in C\mathcal{R}(X)$.*
(i) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon_1}(T) \subset \sigma_{eap,\varepsilon_2}(T)$.*
(ii) *If $\varepsilon > 0$, then $\sigma_{eap,\varepsilon}(T) \subset \sigma_{ap,\varepsilon}(T)$.*
(iii) $\bigcap_{\varepsilon>0} \sigma_{eap,\varepsilon}(T) = \sigma_{eap}(T)$.

Theorem 4.3. *Let $T \in L\mathcal{R}(X)$ where X is a complete space, then the following hold.*
(i) *For any $\varepsilon > 0$ and $S \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \varepsilon$ we have*

$$\sigma_{eap,\varepsilon-\|S\|}(T) \subseteq \sigma_{eap,\varepsilon}(T+S) \subseteq \sigma_{eap,\varepsilon+\|S\|}(T).$$

(ii) *For every $\alpha, \beta \in \mathbb{C}$, with $\beta \neq 0$*

$$\sigma_{eap,\varepsilon}(\alpha I + \beta T) = \alpha + \beta\sigma_{eap,\varepsilon|\beta|}(T).$$

Proof. The proof of this theorem is inspired from the proof of Corollary 3.1 and Theorem 3.6 and [4, Propositions 4.2 and 4.4]. □

Theorem 4.4. *Let $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$. Then*

(*i*) $\sigma_{eap,\varepsilon}(T) = \bigcap\limits_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P)$;

(*ii*) $\sigma_{eap,\varepsilon}(T) = \bigcap\limits_{S \in SS\mathcal{R}(X)} \sigma_{ap,\varepsilon}(T+S)$.

Proof. (*i*) Because, $\mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_+(X))$, we have that

$$\bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P) \subset \sigma_{eap,\varepsilon}(T).$$

Conversely, let $\lambda \notin \bigcap\limits_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P)$ then there exists $P \in \mathcal{P}_T(\Phi_+(X))$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+P).$$

Since, P is continuous and the use of Lemma 2.5 we infer that $T+P$ is closed. Now, by using of Corollary 3.2 we see that $\lambda \notin \sigma_{ap}(T+S+P)$ for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $\|S\| < \varepsilon$ and

$$\begin{aligned} \mathcal{D}(T+P) =& \mathcal{D}(T) \cap \mathcal{D}(P) = \mathcal{D}(T) \subset \mathcal{D}(S), \\ (T+P)(0) =& T(0) \quad (\text{as } P(0) \subset T(0)) \\ \supset& S(0). \end{aligned}$$

On the other hand, $(\lambda - S - P)(0) \subset T(0)$, $\mathcal{D}(\lambda - S - P) = \mathcal{D}(S) \cap \mathcal{D}(P) \supset \mathcal{D}(T)$ and T is closed, by using of Lemma 2.5, $\lambda - T - S - P$ is closed, and $\lambda - T - S - P$ is open as $\lambda \notin \sigma_{ap}(T+P+S)$, then from Theorem 2.1, $R(\lambda - T - S - P)$ is closed. Hence $\lambda - T - S - P$ is injective and open. Therefore

$$\lambda - T - S - P \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S - P) \leq 0.$$

Since

$$\begin{aligned} P \in \mathcal{P}_T(\Phi_+(X)), \quad P(0) \subset& (\lambda - T - S - P)(0), \\ \mathcal{D}(P) \supset \mathcal{D}(\lambda - T - S - P) =& \mathcal{D}(T) \cap \mathcal{D}(S) \cap \mathcal{D}(P), \quad P \in \mathcal{P}_{\lambda - T - D - P}(\Phi_+(X)) \end{aligned}$$

Using Lemma 2.1, we obtain that for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $\|S\| < \varepsilon$

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

Finally, it follows from Theorem 4.1 that $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(*ii*) From [2, Theorem 3.3], we have the inclusion $\mathcal{K}_T(X) \subset SS\mathcal{R}(X) \subset \mathcal{P}_T(\Phi_+(X))$. Then

$$\begin{aligned} \sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+K) \subset& \bigcap_{S \in SS\mathcal{R}(X)} \sigma_{ap,\varepsilon}(T+S) \\ \subset& \bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P) = \sigma_{eap,\varepsilon}(T). \qquad \square \end{aligned}$$

We finally close this paper with the following theorem.

Theorem 4.5. *Let $T, S \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$ and $\varepsilon > 0$. If $D \in L\mathcal{R}(X)$ such that $D(0) \subset S(0) \subset \mathcal{N}(T)$, $\|D\| < \varepsilon$ and $T(S+D) \in \mathcal{F}_+(X)$, then*

$$\sigma_{eap,\varepsilon}(T+S) \subset \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T).$$

If, further, $(S+D)T \in \mathcal{F}_+(X)$ and $T(0) \subset \mathcal{N}(S)$ or $T(0) \subset \mathcal{N}(D)$ we have

$$\sigma_{eap,\varepsilon}(T+S) = \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T).$$

Proof. Since $\mathcal{D}(T) = X$, then using Proposition 2.1, we obtain that

$$(\lambda - T)(\lambda - S - D) = \lambda(\lambda - S - D) - T(\lambda - S - D).$$

Also, by Proposition 2.1, we have

$$\begin{aligned}(\lambda - T)(\lambda - S - D) =& \lambda^2 - \lambda S - \lambda D - \lambda T + TS + TD \\ =& \lambda(\lambda - S - D - T) + T(S + D)\end{aligned}$$

Let $\lambda \notin \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T)$, then $\lambda \notin \sigma_{eap,\varepsilon}(S)$ and $\lambda \notin \sigma_{eap}(T)$. Using [12, Corollary 4.1] we have

$$(\lambda - T) \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T) \leq 0.$$

According to Theorem 4.1 we obtain

$$\lambda - S - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - S - D) \leq 0,$$

for all continuous linear relations $D \in \mathrm{L}\mathcal{R}(X)$ such that $D(0) \subset S(0)$ and $\|D\| < \varepsilon$. We infer that

$$(\lambda - T)(\lambda - S - D) \in \Phi_+(X).$$

Since

$$T(S + D)(0) = TS(0) \subset TT^{-1}(0) = T(0) = (\lambda - S - D - T)(0) = T(0),$$

and $T(S + D) \in \mathcal{F}_+(X)$, then $\lambda - S - D - T \in \Phi_+(X)$ for all $D \in \mathrm{L}\mathcal{R}(X)$ such that $D(0) \subset S(0)$ and $\|D\| < \varepsilon$

$$\begin{aligned}i(\lambda - S - D - T) =& i(\lambda - T) + i(\lambda - S - D) - \dim\Big(T(0) \cap \mathcal{N}(\lambda - S - D)\Big) \\ =& i(\lambda - T) + i(\lambda - S - D) \leq 0.\end{aligned}$$

Hence, from Theorem 4.1, we have $\lambda \notin \sigma_{eap,\varepsilon}(T+S)$. The second inclusion is analogous to the previous one. □

Acknowledgements. The authors are very grateful for the comments and corrections of the referee, whose detailed attention allowed us to make very worthwhile improvements.

References

[1] F. Abdmouleh, T. Àlvarez, A. Ammar and A. Jeribi, *Spectral mapping theorem for Rakocević and Schmoeger essential spectra of a multivalued linear operator*, Mediterr. J. Math. **12**(3) (2015), 1019–1031. https://doi.org/10.1007/s00009-014-0437-7

[2] A. Ammar, *A characterization of some subsets of essential spectra of a multivalued linear operator*, Complex Anal. Oper. Theory **11**(1) (2017), 175–196. https://doi.org/10.1007/s11785-016-0591-y

[3] A. Ammar, H. Daoud and A. Jeribi, *Pseudospectra and essential pseudospectra of multivalued linear relations*, Mediterr. J. Math. **12**(4) (2015), 1377–1395. https://doi.org/10.1007/s00009-014-0469-z

[4] A. Ammar, H. Daoud and A. Jeribi, *Stability of pseudospectra and essential pseudospectra of linear relations*, J. Pseudo-Differ. Oper. Appl. **7**(4) (2015), 473–491. https://doi.org/10.1007/s11868-016-0150-3

[5] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the essential approximation pseudospectrum on a Banach space*, Filomat **31** (11) (2017), 3599–3610. https://doi.org/10.2298/FIL1711599A

[6] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the condition pseudospectrum on Banach space*, Funct. Anal. Approx. Comput. **10**(2) (2018), 13–21. http://www.pmf.ni.ac.rs/faac

[7] A. Ammar, A. Jeribi and K. Mahfoudhi, *The condition pseudospectrum of a operator pencil*, Asian-Eur. J. Math. **14**(4) (2021), 12 pages. https://doi.org/10.1142/S1793557121500571

[8] A. Ammar, A. Jeribi and K. Mahfoudhi, *Some description of essential structured approximate and defect pseudospectrum*, Korean J. Math. **28**(4) (2020), 673–697. http://dx.doi.org/10.11568/kjm.2020.28.4.673

[9] A. Ammar, A. Jeribi and K. Mahfoudhi, *The condition pseudospectrum subset and related results*, J. Pseudo-Differ. Oper. Appl. **11**(1) (2020), 491–504. https://doi.org/10.1007/s11868-018-0265-9

[10] A. Ammar, A. Jeribi and K. Mahfoudhi, *Browder essential approximate pseudospectrum and defect pseudospectrum on a Banach space*, Extracta Math. **34**(1) (2019), 29–40. https://doi.org/10.17398/2605-5686.34.1.29

[11] A. Ammar, A. Jeribi and K. Mahfoudhi, *Measure of noncompactness, essential approximation and defect pseudospectrum*, Methods Funct. Anal. Topology **25**(1) (2019), 1–11.

[12] T. Àlvarez, A. Ammar and A. Jeribi, *On the essential spectra of some matrix of lineair relations*, Math. Methods Appl. Sci. **37** (2014), 620–644. https://doi.org/10.1002/mma.2818

[13] T. Àlvarez, A. Ammar and A. Jeribi, *A characterization of some subsets of $\mathcal{S}$-essential spectra of a multivalued linear operator*, Colloq. Math. **135**(2) (2014), 171–186. https://doi.org/10.4064/cm135-2-2

[14] E. Chafai and M. Mnif, *Perturbation of normally solvable linear relations in paracomplete spaces*, Linear Algebra Appl. **439**(7) (2013), 1875–1885. https://doi.org/10.1016/j.laa.2013.05.019

[15] E. B. Davies, *Linear Operators and their Spectra,* Cambridge Studies in Advanced Mathematics **106**, Camb. Univ. Press, Cambridge, 2007.

[16] R. W. Cross, *Multivalued Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics **213**, Marcel Dekker, Inc., New York, 1998.

[17] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer-Verlag, New York, 2015.

[18] L. N. Trefethen,*Pseudospectra of Matrices*, reprinted from: D. F. Griffiths and G. A. Watson (Eds.), *Numerical Analysis*, Longman Science and Technology, Harlow, 1992, 234–266.

[19] L. N. Trefethen, *Pseudospectra of linear operators*, SAIM Review **39**(3) (1997), 383–406. https://doi.org/10.1137/s0036144595295284

[20] M. Schechter, *Principles of Functional Analysis*, Second Edition, Graduate Studies in Mathematics **36**, American Mathematical Society, Providence, RI, 2002.

[21] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator*, Ph.D. Thesis, Stanford University ProQuest LLC, (1967).

[22] M. P. H. Wolff, *Discrete approximation of unbounded operators and approximation of their spectra*, J. Approx. Theory **113** (2001), 229–244. https://doi.org10.1006/jath.2001.3588

[1]Department of Mathematics,
Faculty of Science,
University of Sfax
Email address: Aref.Jeribi@fss.rnu.tn
Email address: kamelmahfoudhi72@yahoo.com

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 287–303.

MULTIPLE SOLUTIONS FOR A NONLOCAL KIRCHHOFF PROBLEM IN FRACTIONAL ORLICZ-SOBOLEV SPACES

ELHOUSSINE AZROUL[1], ABDELMOUJIB BENKIRANE[1], MOHAMMED SRATI[1], AND MINGQI XIANG[2]

Abstract. In this paper, using the three critical points theorem we obtain the existence of three weak solutions for a Kirchhoff type problem driven by a nonlocal operator of the elliptic type in a fractional Orlicz-Sobolev space, with homogeneous Dirichlet boundary conditions.

1. Introduction

In the last decade, great attention has been devoted to the study of nonlinear problems involving non-local operators. These types of operator come up in a quite natural way in several applications such as phase transition phenomena, crystal dislocation, soft thin films, minimal surfaces and finance; see for instance [2, 18] and references therein. We also refer the interested reader to [33], where a more extensive bibliography and an introduction to the subject are given.

In this paper, we are concerned with a class of nonlocal problems in fractional Orlicz-Sobolev spaces of the form

$$(P_a)\begin{cases} M\left(\int_\Omega\int_\Omega A\left(\dfrac{|u(x)-u(y)|}{|x-y|^s}\right)\dfrac{dxdy}{|x-y|^N}\right)(-\Delta)^s_{a(\cdot)}u & \\ =\lambda f(x,u)+\beta g(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

Key words and phrases. Nonlocal Kirchhoff type problem, fractional $a(\cdot)$-Laplacian, fractional Orlicz-Sobolev spaces, three critical points theorem.
2020 *Mathematics Subject Classification.* Primary: 35R11. Secondary: 35J20, 58E05, 35J60.
DOI 10.46793/KgJMat2502.287A
Received: August 17, 2021.
Accepted: April 04, 2022.

where Ω is an open bounded subset in $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary $\partial\Omega$, $0 < s < 1$, A is an N-function, $M : [0, \infty) \to (0, \infty)$ is a nondecreasing continuous function, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, λ and β are two real parameters and $(-\Delta)^s_{a(\cdot)}$ is a nonlocal integro-differential operator of elliptic type defined as follows

$$(-\Delta)^s_{a(\cdot)} u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u(x) - u(y)}{|x - y|^s} \cdot \frac{dy}{|x - y|^{N+s}},$$

for all $x \in \mathbb{R}^N$, where $a : \mathbb{R} \to \mathbb{R}$ which will be specified later.

This problem (P_a) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = h(u, x), \tag{1.1}$$

presented by Kirchhoff [29] in 1883 which is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.1), L is the length of string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [23]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [35] for some related works. In [7], the authors obtained the existence of three weak solutions for a Kirchhoff type elliptic system involving nonlocal fractional p-Laplacian by using the three point critique theorem. In [10], by means of mountain pass theorem of Ambrosetti and Rabinowitz, direct variational approach and Ekeland's variational principle, the authors showed the existence of nontrivial weak solutions to a class of $p(x)$-Kirchhoff type problem. For the problems involving fractional Kirchhoff type, we refer the reader to the works [11, 13]. They use different methods to establish the existence of solutions.

Problems of this type have been intensively studied in the last few years, due to numerous and relevant applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids), calculus of variations, nonlinear potential theory, the theory of quasiconformalmappings, differential geometry, geometric function theory, probability theory and image processing (see, for instance [22]).

The problem (P_a) involves the fractional $a(\cdot)$-Laplacian operator, the most appropriate functional framework for dealing with this problem is the fractional Orlicz Sobolev space [8, 16], namely a fractional Sobolev space constructed from an Orlicz space at the place of $L^p(\Omega)$. As we know, the Orlicz spaces represent a generalization of classical Lebesgue spaces in which the role usually played by the convex function t^p is assumed by a more general convex function $A(t)$; they have been extensively studied in the monograph of Krasnoselśkii and Rutickii [28] as well as in Luxemburg's doctoral thesis [31]. If the role played by $L^p(\Omega)$ in the definition of fractional Sobolev spaces $W^{s,p}(\Omega)$ is assigned to an Orlicz $L_A(\Omega)$ space, the resulting space $W^sL_A(\Omega)$ is exactly a fractional Orlicz-Sobolev space . Many properties of fractional Sobolev spaces have been extended to fractional Orlicz-Sobolev spaces (see [4, 5, 8, 9, 12, 16, 17]). For this, many researchers have studied the existence of solutions for the eigenvalue problems involving nonhomogeneous operators in the divergence form through Orlicz-Sobolev spaces by using variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory (see for instance [14, 15, 20, 32]).

The problem (P_a) is motivated by the class of problems on the form

$$(P)\begin{cases} Au = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \partial\Omega,\end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$, $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions and λ, β are two real parameters. For $Au = -\Delta_p = -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$, the problem (P) has been studied in many papers, we refer to [35, 36], in which the authors have used different methods to get the existence of solutions for (P). In the case when $Au = -\Delta_{p(\cdot)} = -\mathrm{div}\left(|\nabla u|^{p(\cdot)-2}\nabla u\right)$, where $p(\cdot)$ is a continuous function, problem (P) has also been studied by many authors, see for examples [19, 24, 25]. On the other hand, Chung in [26], studied the problem (P) with $Au = -M\left(\int_\Omega \phi(|\nabla u|)dx\right)\mathrm{div}(a(|\nabla u|\nabla u))$. That is, the following problem in Orlicz-Soblev spaces:

$$(P_\phi)\quad \begin{cases} -M\left(\int_\Omega \phi(|\nabla u|)dx\right)\mathrm{div}(a(|\nabla u|)\nabla u) = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \partial\Omega,\end{cases}$$

where ϕ is an N-function, defend as

$$\phi(t) = \int_0^t a(\tau)\tau d\tau,$$

and $M : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous Kirchhoff function. Under some suitable conditions, the author obtained the existence of three weak solutions of (P_Φ), by using the three critical point theorem. For $M \equiv 1$ in the problem (P_Φ), Cammaroto and Vilasti in [20], by the same theorem, they showed the existence of three weak solutions.

In the fractional case, i.e., when we take $Au = M\left([u]^p_{s,p}\right)(-\Delta)^s_p u$. That is, we consider the following problem

$$(P_s)\begin{cases} M\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}}dxdy\right)(-\Delta)^s_p u = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases}$$

where Ω is an open bounded subset in $\mathbb{R}^N$ and $(-\Delta)_p^s$ is the fractional p-Laplace operator. In [6], by using the three critical point theorem, the authors obtained the existence of three weak solutions of (P_s).

To our knowledge, this is the first contribution to studying of non-local problems in this class of functional spaces. More precisely, using the ideas first presented in articles [6, 20, 26]. Our result in this article generalizes special cases, in which we will consider the problem (P_a) with $M(t) = 1$ or $M(t) \neq 1$ and $A(t) = \frac{1}{p}t^p$ (the problem (P_s)).

This paper is organized as follows. In the second section, we recall some properties of fractional Sobolev spaces. In the third section, using the three critical points theorem which introduced by Ricceri [34], we obtain the existence of a three weak solutions of problem (P_a). Finally, the fourth section is devoted to giving an example which illustrates the mains abstracts results.

2. Some Preliminaries Results

To deal with this situation we introduce the fractional Orlicz-Sobolev space to investigate problem (P_a). Let us recall the definitions and some elementary properties of this spaces. We refer the reader to [1, 3, 8, 16, 33] for further reference and for some of the proofs of the results in this section.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. We assume that $a : \mathbb{R} \to \mathbb{R}$ in (P_a) is such that : $\varphi : \mathbb{R} \to \mathbb{R}$ defined by:

$$\varphi(t) = \begin{cases} a(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$A(t) = \int_0^t \varphi(\tau)d\tau.$$

Then, A, is N-function, see [1], i.e., $A : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, convex, increasing function, with $\frac{A(t)}{t} \to 0$ as $t \to 0$ and $\frac{A(t)}{t} \to \infty$ as $t \to \infty$.

For the function A introduced above we define the Orlicz space:

$$L_A(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ mesurable } \int_\Omega A(\lambda|u(x)|)dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_A = \inf \left\{ \lambda > 0 : \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

The conjugate N-function of A is defined by $\overline{A}(t) = \int_0^t \overline{\varphi}(\tau)d\tau$, where $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ is given by $\overline{\varphi}(t) = \sup\{s : \varphi(s) \leq t\}$. Furthermore, it is possible to prove a Hölder type inequality, that is

$$\left| \int_\Omega uv dx \right| \leq 2\|u\|_A \|v\|_{\overline{A}}, \quad \text{for all } u \in L_A(\Omega) \text{ and } v \in L_{\overline{A}}(\Omega).$$

Throughout this paper, we assume that

$$1 < p^- := \inf_{t\geq 0} \frac{t\varphi(t)}{A(t)} \leq p^+ := \sup_{t\geq 0} \frac{t\varphi(t)}{A(t)} < +\infty. \tag{2.1}$$

The above relation implies that $A \in \Delta_2$, i.e., A satisfies the global Δ_2-condition (see [32]):

$$A(2t) \leq KA(t), \quad \text{for all } t \geq 0,$$

where K is a positive constant.

Furthermore, we assume that A satisfies the following condition

$$\text{the function } [0,\infty) \ni t \mapsto A(\sqrt{t}) \text{ is convex.} \tag{2.2}$$

The above relation assures that $L_A(\Omega)$ is an uniformly convex space (see [32]).

Lemma 2.1 ([16]). *Assume that $A \in \Delta_2$. Then we have*

$$\overline{A}(\varphi(t)) \leq cA(t), \quad \textit{for all } t \geq 0,$$

where $c > 0$.

Now, we defined the fractional Orlicz-Sobolev space $W^sL_A(\Omega)$ as follows

$$W^sL_A(\Omega) = \left\{ u \in L_A(\Omega) : \int_\Omega\int_\Omega A\left(\frac{\lambda|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N} < \infty \text{ for some } \lambda > 0 \right\}.$$

This space is equipped with the norm

$$\|u\|_{s,A} = \|u\|_A + [u]_{s,A},$$

where $[\cdot]_{s,A}$ is the Gagliardo seminorm, defined by

$$[u]_{s,A} = \inf\left\{ \lambda > 0 : \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dxdy}{|x-y|^N} \leq 1 \right\}.$$

We work in the closed linear subspace

$$W_0^sL_A(\Omega) = \left\{ u \in W^sL_A(\mathbb{R}^N) : u = 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega \right\},$$

which can be equivalently renormed by setting $\|\cdot\| := [\cdot]_{s,A}$. By [16], $W^sL_A(\Omega)$ and is Banach space, also separable (resp. reflexive) space if and only if $A \in \Delta_2$ (resp. $A \in \Delta_2$ and $\overline{A} \in \Delta_2$). Furthermore, if $A \in \Delta_2$ and $A(\sqrt{t})$ is convex, then the space $W^sL_A(\Omega)$ is uniformly convex.

To simplify the notation, we set

$$\Phi(u) = \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}, \quad D^su = \frac{u(x)-u(y)}{|x-y|^s}, \quad d\mu = \frac{dxdy}{|x-y|^s},$$

and the dual space of $(W^sL_A(\Omega), \|\cdot\|)$ is denoted by $((W^sL_A(\Omega))^*, \|\cdot\|_*)$. Note that $d\mu$ is a regular Borel measure on the set $\Omega \times \Omega$.

Theorem 2.1 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then*

$$C_0^2(\Omega) \subset W^sL_A(\Omega).$$

Remark 2.1. A trivial consequence of Theorem 2.1, $C_0^\infty(\Omega) \subset W^sL_A(\Omega)$ and $W^sL_A(\Omega)$ is non-empty.

Proposition 2.1 ([8]). *Let Ω be an open subset of $\mathbb{R}^N$ and let A be an N-function. Assume condition* (2.1) *is satisfied, then the following relations hold true*

$$[u]_{s,A}^{p^-} \leq \Phi(u) \leq [u]_{s,A}^{p^+}, \quad \text{for all } u \in W^sL_A(\Omega), \text{with } [u]_{s,A} > 1,$$

$$[u]_{s,A}^{p^+} \leq \Phi(u) \leq [u]_{s,A}^{p^-}, \quad \text{for all } u \in W^sL_A(\Omega), \text{with } [u]_{s,A} < 1.$$

Theorem 2.2 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$, with $C^{0,1}$-regularity and bounded boundary, let $0 < s' < s < 1$. Let A be an N-function, assume condition* (2.1) *is satisfied and we define*

$$p_{s'}^* = \begin{cases} \frac{Np^-}{N-s'p^-}, & \text{if } N > s'p^-, \\ \infty, & \text{if } N \leq s'p^-. \end{cases}$$

- *If $s'p^- < N$, then $W^sL_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, p_{s'}^*]$ and the embedding $W^sL_A(\Omega) \hookrightarrow L^q(\Omega)$, is compact for all $q \in [1, p_{s'}^*)$.*
- *If $s'p^- = N$, then $W^sL_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty]$ and the embedding $W^sL_A(\Omega) \hookrightarrow L^q(\Omega)$, is compact for all $q \in [1, \infty)$.*
- *If $sp^- > N$, then the embedding $W^sL_A(\Omega) \hookrightarrow L^\infty(\Omega)$, is compact.*

Definition 2.1. Let X be a real Banach space. We denote by $\mathcal{W}_A$ the class of all functionals $A : X \to \mathbb{R}$ possessing the following propositionerty: if $\{u_n\}$ is a sequence in X weakly converging to $u \in X$ and $\liminf_{n\to\infty} A(u_n) \leq A(u)$, then $\{u_n\}$ has a subsequence strongly converging to u.

Definition 2.2. Let $0 < s' < s < 1$, if $N > s'p^-$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1 + |t|^{q-1}} < \infty,$$

where $q \in [1, p_{s'}^*)$.

While when $N < s'p^-$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for each $C > 0$, the function $x \mapsto \sup_{|t|\leq C} |f(x,t)|$ belongs to $L^1(\Omega)$.

Theorem 2.3 ([34]). *Let X be a separable and reflexive real Banach space with norm $\|\cdot\|$, let $\Psi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_A$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and let $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that Ψ has a strict local minimum x_0, with $\Psi(x_0) = J(x_0) = 0$. Finally, assume that*

$$\max\left\{\limsup_{\|x\|\to+\infty} \frac{J(x)}{\Psi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Psi(x)}\right\} \leq 0$$

and that

$$\sup_{x\in X} \min\{\Psi(x), J(x)\} > 0.$$

Let

$$\theta^* := \inf\left\{\frac{\Psi(x)}{J(x)} : x \in X, \min\{\Psi(x), J(x)\} > 0\right\}.$$

Then, for each compact interval $\Lambda \subset (\theta^*, +\infty)$, there exists a number $\delta > 0$ with the following propositionerty: for every $\lambda \in \Lambda$ and every C^1 functional $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists $\beta^* > 0$ such that for each $\beta \in [0, \beta^*]$, the equation

$$\Psi'(x) = \lambda J'(x) + \beta\Gamma'(x),$$

has at least three solutions whose norms are less than δ.

3. Mains Results

In this section, we prove the existence of three weak solutions in fractional Orlicz-Sobolev spaces applying Theorem 2.3. For this, we suppose that the Kirchhoff function $M : [0, \infty) \to (0, \infty)$ is a continuous and nondecreasing function satisfying the following condition:

(M_0) there exists $m_0 > 0$ such that $M(t) \geq m_0$, for all $t \geq 0$.

For $f \in \mathcal{A}$, we assume that

(F_1) $$\sup_{u\in W_0^s L_A(\Omega)} \int_\Omega F(x, u)dx > 0,$$

(F_2) $$\limsup_{t\to 0} \frac{\sup_{x\in\Omega} F(x, t)}{|t|^{p^+}} \leq 0,$$

(F_3) $$\limsup_{|t|\to\infty} \frac{\sup_{x\in\Omega} F(x, t)}{|t|^{p^-}} \leq 0,$$

where $F(x, t) = \int_0^t f(x, \tau)d\tau$.

Under such hypothesis, we set

$$\theta^* = \inf\left\{\frac{\widehat{M}(\Phi(u))}{\displaystyle\int_\Omega F(x, u)dx} : u \in W_0^s L_A(\Omega), \int_\Omega F(x, u)dx > 0\right\}.$$

Definition 3.1. We say that $u \in W_0^s L_A(\Omega)$ is a weak solution of problem (P_a) if

$$M(\Phi(u)) \int_{\mathbb{R}^N\times\mathbb{R}^N} a(|D^s u|)D^s u D^s v d\mu = \lambda \int_\Omega f(x, u)v dx + \beta \int_\Omega g(x, u)v dx,$$

for all $v \in W_0^s L_A(\Omega)$.

Theorem 3.1. *Let A be an N-function. Suppose that M satisfy (M_1) and for $f \in \mathcal{A}$, we suppose that (F_1), (F_2) and (F_3) hold true. If $p^+ < p^*_{s'}$, then for each compact interval $\Lambda \subset (\theta^*, \infty)$, there exists a number $\delta > 0$ with the following propositionerty:*

for every $\lambda \in \Lambda$ and every $g \in \mathcal{A}$ there exists $\beta^* > 0$ such that, for each $\beta \in [0, \beta^*]$, problem (P_a) has at least three weak solutions whose norms are less than δ.

We first prove the following useful result, which helps us to apply Theorem 2.3. For this, we define the functionals $\Psi, J : W_0^s L_A(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega F(x,u)dx, \quad \Psi(u) = \widehat{M}\left(\int_\Omega \int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right),$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

Lemma 3.1. *Let $f \in \mathcal{A}$. Then the functional $J \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ with derivative given by*

$$\langle J'(u), v\rangle = \int_\Omega f(x,u)vdx,$$

for all $u, v \in W_0^s L_A(\Omega)$. Moreover $J' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is compact.

By using Theorem 2.2, the proof of this Lemma is seminary to Lemma 3.3 in [6].

Lemma 3.2. *Let (M_1) and (2.1) hold true. Then $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and*

$$\langle \Psi'(u), v\rangle = M(\varPhi(u)) \int_{\Omega\times\Omega} a(|D^s u|)D^s u D^s v d\mu,$$

for all $u, v \in W_0^s L_A(\Omega)$. Moreover, for each $u \in W_0^s L_A(\Omega)$, $\Psi'(u) \in (W_0^s L_A(\Omega))^*$.

Proof. First, it is easy to see that

$$\langle \Psi'(u), v\rangle = M(\varPhi(u)) \int_{\Omega\times\Omega} a(|D^s u|)D^s u D^s v d\mu, \tag{3.1}$$

for all $u, v \in W_0^s L_A(\Omega)$. It follows from (3.1) that $\Psi'(u) \in (W_0^s L_A(\Omega))^*$ for each $u \in W_0^s L_A(\Omega)$.

Next, we prove that $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$. Let $\{u_n\} \subset W_0^s L_A(\Omega)$ with $u_n \to u$ strongly in $W_0^s L_A(\Omega)$, then $D^s u_n \to D^s u$ in $L_A(\Omega \times \Omega, d\mu)$. So by dominated convergence theorem, there exist a subsequence $\{D^s u_{n_k}\}$ and a function h in $L_A(\Omega \times \Omega, d\mu)$ such that

$$a(|D^s u_{n_k}|)D^s u_{n_k} \to a(|D^s u|)D^s u$$

and

$$|a(|D^s u_{n_k}|)D^s u_{n_k}| \leq |a(|h|)h|,$$

for almost every (x, y) in $\Omega \times \Omega$, by Lemma 2.1, we have $|a(|h|)h| \in L_{\overline{A}}(\Omega \times \Omega, d\mu)$. So, for $v \in W_0^s L_A(\Omega)$, $D^s v \in L_A(\Omega \times \Omega, d\mu)$ and by Hölder's inequality

$$\begin{aligned}
&\left|\int_{\Omega\times\Omega} \left[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right] D^s v d\mu\right| \\
\leq& 2\left\|a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right\|_{L_{\overline{A}}} \left\|D^s v\right\|_{L_A} \\
\leq& 2\left\|a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right\|_{L_{\overline{A}}} \left\|v\right\|.
\end{aligned}$$

Then by dominated convergence theorem we obtain that

$$\sup_{\|v\|\le 1}\left|\int_{\Omega\times\Omega}[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u]\, D^s v d\mu\right| \to 0. \tag{3.2}$$

On the other hand, the continuity of M and Proposition 2.1, we have

$$M\left(\Phi(u_n)\right) \to M\left(\Phi(u)\right). \tag{3.3}$$

Combining (3.2)–(3.3) with the Hölder inequality, we have

$$\|\Psi'(u_n) - \Psi'(u)\|_* = \sup_{v\in W_0^s L_A(\Omega), \|v\|\le 1} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| \to 0.$$ □

Lemma 3.3. *The following properties hold true:*
(i) *the functional* Ψ *is sequentially weakly lower semi continuous;*
(ii) *the functional* Ψ *belongs to the class* $\mathcal{W}_{W_0^s L_A(\Omega)}$.

Proof. (i) First, note that the map

$$u \mapsto \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N},$$

is lower semi-continuous in the weak topology of $W_0^s L_A(\Omega)$. Indeed, similar to Lemma 3.1, we obtain $\Phi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_\Omega\int_\Omega a(|D^s u|)D^s u D^s v d\mu,$$

for all $u, v \in W_0^s L_A(\Omega)$. On the other hand, since A is a convex function so Φ is also convex.

Now, let $\{u_n\} \subset W_0^s L_A(\Omega)$ with $u_n \rightharpoonup u$ weakly in $W_0^s L_A(\Omega)$, then by convexity of Φ we have

$$\Phi(u_n) - \Phi(u) \ge \langle \Phi'(u), u_n - u\rangle,$$

and hence, we obtain $\Phi(u) \le \liminf \Phi(u_n)$, that is, the map

$$u \mapsto \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}$$

is lower semi-continuous. On the other hand by the continuity and monotonicity of the function $t \mapsto \widehat{M}(t)$, we get

$$\liminf_{n\to\infty} \Psi(u_n) = \liminf_{n\to\infty} \widehat{M}(\Phi(u_n)) \ge \widehat{M}(\liminf_{n\to\infty} \Phi(u_n)) \ge \widehat{M}(\Phi(u)).$$

Thus, the functional Ψ is sequentially weakly lower semicontinuous.

(ii) Since $\widehat{M}$ is continuous and strictly increasing, it suffices to show that $\Phi \in \mathcal{W}_{W_0^s L_A(\Omega)}$. Then, let $\{u_n\}$ be a sequence weakly converging to in $W_0^s L_A(\Omega)$ and let $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$. Since the functional Φ is sequentially weakly lower semicontinuous, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$\lim_{n\to\infty} \Phi(u_n) = \Phi(u).$$

On the other hand, since $\left\{\frac{u_n+u}{2}\right\}$ converges weakly to u in $W_0^s L_A(\Omega)$, from (i), we have

$$\liminf_{n\to\infty} \varPhi\left(\frac{u_n+u}{2}\right) \geq \varPhi(u). \tag{3.4}$$

We assume by contradiction that $\{u_n\}$ does not converge to u in $W_0^s L_A(\Omega)$. Hence, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and there exits $\varepsilon_0 > 0$ such that

$$\left\|\frac{u_n-u}{2}\right\| \geq \frac{\varepsilon_0}{2},$$

by Proposition 2.1, we have

$$\varPhi\left(\frac{u_n-u}{2}\right) \geq \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\}.$$

On the other hand, by the conditions (2.1) and (2.2), we can apply [30, Lemma 2.1] in order to obtain

$$\frac{1}{2}\varPhi(u_n) + \frac{1}{2}\varPhi(u) - \varPhi\left(\frac{u_n+u}{2}\right) \geq \varPhi\left(\frac{u_n-u}{2}\right) \geq \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\}. \tag{3.5}$$

It follows from (3.5) that

$$\varPhi(u) - \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\} \geq \limsup_{n\to\infty} \varPhi\left(\frac{u_n+u}{2}\right), \tag{3.6}$$

from (3.4) and (3.6) we obtain a contradiction. This shows that $\{u_n\}$ converges strongly to u and the functional Ψ belongs to the class $\mathcal{W}_{W_0^s L_A(\Omega)}$. □

Lemma 3.4. *Assume that the sequence $\{u_n\}$ converges weakly to u in $W_0^s L_A(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega \int_\Omega a(|D^s u_n|) D^s u_n \left(D^s u_n - D^s u\right) d\mu \leq 0. \tag{3.7}$$

Then the sequence $\{u_n\}$ converges strongly to u in $W_0^s L_A(\Omega)$.

Proof. Since u_n converges weakly to u in $W_0^s L_A(\Omega)$, then $\{\|u_n\|\}$ is a bounded sequence of real numbers, that fact and Proposition 2.1, implies that the $\{\varPhi(u_n)\}$ is bounded, then for a subsequence, we deduce that $\varPhi(u_n) \to c$. Or since $\varPhi$ is weak lower semi continuous, we get $\varPhi(u) \leq \liminf_{n\to\infty} \varPhi(u_n) = c$. On the other hand, by the convexity of $\varPhi$, we have

$$\varPhi(u) \geq \varPhi(u_n) + \langle \varPhi'(u_n), u_n - u\rangle.$$

Next, by the hypothesis (3.7), we conclude that $\varPhi(u) = c$. Since $\left\{\frac{u_n+u}{2}\right\}$ converges weakly to u in $W_0^s L_A(\Omega)$, so since $\varPhi$ is sequentially weakly lower semicontinuous:

$$c = \varPhi(u) \leq \liminf_{n\to\infty} \varPhi\left(\frac{u_n+u}{2}\right).$$

Seminary to proof of Lemma 3.3, we assume by contradiction that u_n converges strongly to u in $W_0^s L_A(\Omega)$. □

Lemma 3.5. *Let (M_1) hold, then the operator $\Psi' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is invertible and Ψ'^{-1} is continuous.*

Proof. First, we assume that the operator $\Psi' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is invertible on $W_0^s L_A(\Omega)$. By the Minty-Browder theorem (see [37]), it suffices to prove that Ψ' is strictly monotone, hemicontinuous and coercive in the sense of monotone operators.

So, let $u, v \in W_0^s L_A(\Omega)$, with $u \neq v$ and let $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$. Since $a(|t|)t$ is increasing, then

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle = \int_\Omega \int_\Omega (a(|D^s u|) D^s u - a(|D^s u|) D^s v) \, (D^s u - D^s v) \, d\mu > 0.$$

So, $\Psi' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is strictly monotone, so by [37, Proposition 25.10], Φ is strictly convex. Moreover, since M is nondecreasing the function $\widehat{M}$ is convex in $\mathbb{R}^+$. Thus,

$$\widehat{M}(\Phi(\lambda u + \mu v)) < \widehat{M}(\lambda \Phi(u) + \mu \Phi(v)) \leq \lambda \widehat{M}(\Phi(u)) + \mu \widehat{M}(\Phi(v)).$$

This shows that Ψ is strictly convex and already said, that Ψ' is strictly monotone.

Let $u \in W_0^s L_A(\Omega)$, with $\|u\| > 1$, by (M_1) and Proposition 2.1, we have

$$\frac{\langle \Psi'(u), u \rangle}{\|u\|} = \frac{M(\Phi(u)) \, \langle \Phi'(u), u \rangle}{\|u\|} \geq \frac{m_0 p^- \Phi(u)}{\|u\|} \geq m_0 p^- \|u\|^{p^- - 1}.$$

Thus,

$$\lim_{\|u\| \to \infty} \frac{\langle \Phi'(u), u \rangle}{\|u\|} = \infty,$$

that is, Ψ' is coercive.

Now, by Lemma 3.1, we have $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$, then Ψ is hemicontinuous. Thus, in view of the Minty-Browder theorem, there exists $\Psi'^{-1} : (W_0^s L_A(\Omega))^* \to W_0^s L_A(\Omega)$ and it is bounded.

Let us prove that Ψ'^{-1} is continuous by showing that its is sequentially continuous. Let $\{u_n\} \subset (W_0^s L_A(\Omega))^*$ be a sequence strongly is converging to $u \in (W_0^s L_A(\Omega))^*$ and let $v_n = \Psi'^{-1}(u_n)$ and $v = \Psi'^{-1}(u)$. Then, $\{v_n\}$ bounded in $W_0^s L_A(\Omega)$, then, we can assume that it converges weakly to a certain $v_0 \in W_0^s L_A(\Omega)$. Since u_n converges strongly to u, we have

$$\lim_{n \to \infty} \langle \Psi'(v_n), v_n - v_0 \rangle = \lim_{n \to \infty} \langle u_n, v_n - v_0 \rangle = 0,$$

i.e.,

$$\lim_{n \to \infty} M(\Phi(v_n)) \int_\Omega \int_\Omega a(|D^s v_n|) D^s v_n \, (D^s v_n - D^s v_0) \, d\mu = 0. \tag{3.8}$$

Since $\{v_n\}$ is bounded in $W_0^s L_A(\Omega)$, then by Proposition 2.1, $\Phi(v_n)$ is also bounded, then

$$\Phi(v_n) \to t_0 \geq 0, \quad \text{as } n \to \infty.$$

If $t_0 = 0$, then using Proposition 2.1, we get $\{v_n\}$ that strongly converges to v_0 in $W_0^s L_A(\Omega)$, by the continuity and injectivity of Ψ'^{-1} we obtain the desired result.

If $t_0 > 0$, it follows from the continuity of the function M that

$$M(\Phi(v_n)) \to M(t_0), \quad \text{as } n \to \infty.$$

Thus, by (M_1), for sufficiently large n, we get

$$M(\Phi(v_n)) \geq C_0 > 0. \tag{3.9}$$

By (3.8) and (3.9), we have

$$\int_\Omega \int_\Omega a(|D^s v_n|) D^s v_n \, (D^s v_n - D^s v_0) \, d\mu = 0. \tag{3.10}$$

From (3.10) and since v_n converges weakly to v_0 in $W_0^s L_A(\Omega)$, we can apply Lemma 3.4, in order to deduce that v_n converge strongly to v_0 in $W_0^s L_A(\Omega)$. □

Proof of Theorem 3.1. We wish to apply Theorem 2.3 taking $X = W_0^s L_A(\Omega)$, Ψ and J are as before, by Lemma 3.1 J is C^1-functional with compact derivative. Moreover by Lemma 3.3, Ψ is a sequentially weakly lower continuous and C^1-functional belongs to the class $\mathcal{W}_{W_0^s L_A(\Omega)}$, also by Lemma 3.5, the operator Ψ' admits a continuous inverse on $(W_0^s L_A(\Omega))^*$.

On the other hand, we show that Φ is coercive. In fact, if $\|u\| > 1$, by (M_1) and Proposition 2.1, we have

$$\Psi(u) = \widehat{M}(\Phi(u)) \geq m_0 \Phi(u) \geq m_0 \|u\|^{p^-},$$

from which we have the coercivety of Ψ.

It is evident that $u_0 = 0$ is the global minimum of Ψ and that $\Psi(u_0) = J(u_0) = 0$. Moreover, Ψ is bounded on each bounded subset of $W_0^s L_A(\Omega)$. Indeed, if $\|u\| \leq C$, then

$$\Psi(u) = \widehat{M}(\Phi(u)) \leq \begin{cases} \widehat{M}(C^{p^-}), & \text{if } \|u\| > 1, \\ \widehat{M}(1), & \text{if } \|u\| \leq 1. \end{cases}$$

So, $\Psi(u) \leq \max\left\{\widehat{M}(1), \widehat{M}(C^{p^-})\right\}$.

Now, by the assumption (F_2) for all $\varepsilon > 0$, there exits $\eta_1 > 0$ such that

$$|F(x,t)| \leq \varepsilon |t|^{p^+},$$

for each $x \in \Omega$ and $|t| \leq \eta_1$. Since $p^+ < p^*_{s'}$, so by Theorem 2.2, the embedding $W_0^s L_A(\Omega)$ in $L^{p^+}(\Omega)$ is compact. Then for some positive constant C_2, one has for all $u \in W_0^s L_A(\Omega)$ with $|u| \leq \eta_1$ and $\|u\| < 1$

$$J(u) \leq \varepsilon \|u\|^{p^+}_{L^{p^+}} \leq \varepsilon C_2 \|u\|^{p^+} \leq \varepsilon C_2 \Phi(u).$$

Or by (M_1), we have $\Phi(u) \leq \frac{1}{m_0}\Psi(u)$, then

$$J(u) \leq \varepsilon C_2 \frac{1}{m_0} \Psi(u).$$

Consequently, we have

$$\limsup_{u \to 0} \frac{J(u)}{\Psi(u)} \leq \varepsilon C_2 \frac{1}{m_0}. \tag{3.11}$$

By (F_3), for all $\varepsilon > 0$, there exists $\eta_2 > 0$ such that

$$|F(x,t)| \leq \varepsilon |t|^{p^-}, \tag{3.12}$$

for all $x \in \Omega$ and $|t| > \eta_2$.

For $\|u\| > 1$ large enough, from (3.12), Proposition 2.1 and Theorem 2.2, we have

$$
\begin{aligned}
\frac{J(u)}{\Psi(u)} &= \frac{J(u)}{\widehat{M}(\Phi(u))} \\
&\leq \frac{\int_{\{x\in\Omega:|u|\leq\eta_2\}} F(x,u)dx}{m_0\|u\|^{p^-}} + \frac{\int_{\{x\in\Omega:|u|>\eta_2\}} F(x,u)dx}{m_0\|u\|^{p^-}}, \\
&\leq \frac{|\Omega| \sup\limits_{\Omega\times[-\eta_2,\eta_2]} F}{m_0\|u\|^{p^-}} + \frac{\varepsilon\|u\|^{p^-}_{L^{p^-}(\Omega)}}{m_0\|u\|^{p^-}}, \\
&\leq \frac{|\Omega| \sup\limits_{\Omega\times[-\eta_2,\eta_2]} F}{m_0\|u\|^{p^-}} + C_3\varepsilon.
\end{aligned}
$$

So,

$$
\limsup_{\|u\|\to\infty} \frac{J(u)}{\Psi(u)} \leq \varepsilon C_3. \tag{3.13}
$$

Since $\varepsilon > 0$ is arbitrary, relations (3.11) and (3.13) imply that

$$
\max\left\{\limsup_{\|x\|\to+\infty} \frac{J(x)}{\Psi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Psi(x)}\right\} \leq 0.
$$

Hence, all assumptions of Theorem 2.3 are satisfied. So, for each compact interval $\Lambda \subset (\theta^*, +\infty)$, there exists a number $\delta > 0$ with the propositionerty described in the conclusion of Theorem 2.3. Fix $\lambda \in \Lambda$ and $g \in \mathcal{A}$. Put

$$
\Gamma(u) = \int_\Omega G(x,u)dx \text{ and } G(x,t) = \int_0^t g(x,s)ds,
$$

for all $u \in W_0^s L_A(\Omega)$. Then Γ is a C^1 functional on $W_0^s L_A(\Omega)$ with compact derivative. So, there exists $\beta^* > 0$ such that, for each $\beta \in [0, \beta^*]$, the equation

$$
\Psi'(x) = \lambda J'(x) + \beta\Gamma'(x),
$$

has at least three solutions whose norms are less than δ. But the solutions in $W_0^s L_A(\Omega)$ of the above equation are exactly the weak solutions of problem (P_a) and thus, the proof of Theorem 3.1 is completed. □

4. Example

We present in this section an example of functions that satisfies the conditions of Theorem 3.1. Let

$$
\varphi(t) = \log(1 + |t|)|t|^{p-2}t, \tag{4.1}
$$

where $p \in [2, N)$. Let $b > \max\{2, p^+\}$, $a > 0$, $b \geq 0$ and $\alpha \geq 1$ we consider

$$f(t) = b\cos(t)\sin(t)|\sin(t)|^{b-2}, \quad \text{for all } t \in \mathbb{R}, \tag{4.2}$$

$$M(t) = a + bt^{\alpha-1}, \quad \text{for all } t \geq 0. \tag{4.3}$$

So, from (4.1), (4.2) and (4.3), we have

$$A(t) = \frac{1}{p}\log(1+|t|)|t|^p - \frac{1}{p}\int_0^{|t|}\frac{t^p}{1+t}dt, \quad \widehat{M}(t) = at + \frac{b}{\alpha}t^\alpha, \tag{4.4}$$

$$F(x,t) = F(t) = |\sin(t)|^b. \tag{4.5}$$

We will next show that all the hypotheses of Theorem 3.1 are satisfied.

By Example 2 in [21, page 243], it follows that

$$p^+ = p + 1 \quad \text{and} \quad p^- = p.$$

On the other hand, we point out that trivial computations imply that

$$\frac{d^2A(\sqrt{t})}{dt^2} = \frac{1}{4}\left[\frac{1}{1+|\sqrt{t}|} + (p-2)\log(1+|\sqrt{t}|)\right] \geq 0,$$

for all $t \in \mathbb{R}$ and thus, relations (2.1)–(2.2) are satisfied.

• For each $t \in \mathbb{R}$, we claim that $f \in \mathcal{A}$. Actually, the inequality

$$\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{q-1}} < b < \infty,$$

holds for any $1 < q < p_s^*$ and on the other hand, we have

$$\lim_{|t|\to 0}\frac{|\sin(t)|^b}{|t|^{p^+}} = 0 \quad \text{and} \quad \lim_{|t|\to\infty}\frac{|\sin(t)|^b}{|t|^{p^-}} = 0.$$

Select a compact set $V \subset \Omega$ of positive measure and $v \in W_0^sL_A(\Omega)$ such that $v(x) = \frac{\pi}{2}$ in V and $0 \leq v(x) \leq \frac{\pi}{2}$ in $\Omega \setminus V$. We obtain

$$\int_\Omega |\sin(v(x)|^b dx = |V| + \int_{\Omega\setminus V}|\sin(v(x)|^b dx > 0,$$

which means that (F_1), (F_2) and (F_3) are verified. Also, for $m_0 = a$ the condition (M_1) is satisfied, we set

$$\theta^* = \inf\left\{\frac{a\Phi(u) + \frac{b}{\alpha}(\Phi(u))^\alpha}{\int_\Omega|\sin(u(x))|^b dx} : u \in W_0^sL_A(\Omega), \int_\Omega|\sin(u(x))|^b dx > 0\right\}.$$

Then, for a bounded domain Ω in $\mathbb{R}^N$ of class $C^{0,1}$, it follows from Theorem 3.1, that for each compact interval $\Lambda \subset (\theta^*, +\infty)$, there exist a number $\delta > 0$ and $\beta^* > 0$ such that, for every $\lambda \in \Lambda$ such that for all $\beta \in [0, \beta^*]$, and all $g \in \mathcal{A}$ the following problem

$$\begin{cases} (a + b(\Phi(u))^{\alpha-1})\,(-\Delta)^s_{\log}u = \lambda b\cos(u)\sin(u)|\sin(u)|^{b-2} + \beta g(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where

$$(-\Delta)^s_{\log} u = 2 \text{ p.v} \int_{\mathbb{R}^N} \log(1+|D^s u|)|D^s u|^{p-2} D^s u d\mu$$

has at least three weak solutions whose norms are less than δ.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] G. Alberti, G. Bouchitte and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Ration. Mech. Anal. **144**(1) (1998), 1–46.

[3] E. Azroul, A. Benkirane and M. Srati, *Mountain pass type solutions for a nonlacal fractional $a(\cdot)$-Kirchhoff type problems*, Journal of Nonlinear Functional Analysis **2021**(3) (2021), 1-18.

[4] E. Azroul, A. Benkirane, M. Srati and C. Torres, *Infinitely many solutions for a nonlocal type problem with sign-changing weight function*, Electron. J. Differential Equations **2021**(16) (2021), 1-15.

[5] E. Azroul, A. Benkirane and M. Srati, *Eigenvalue problem associated with nonhomogeneous integro-differential operators*, J. Elliptic Parabol. Equ. **7** (2021), 47–64. https://doi.org/10.1007/s41808-020-00092-8

[6] E. Azroul, A. Benkirane and M. Srati, *Three solutions for Kirchhoff problem involving the nonlocal fractional p-Laplacian*, Adv. Oper. Theory **4**(4) (2019), 821–835. https://doi.org/10.15352/AOT.1901-1464

[7] E. Azroul, A. Benkirane, A. Boumazourh and M. Srati, *Three solutions for a nonlocal fractional p-Kirchhoff type elliptic system*, Appl. Anal. **100**(9) (2021), 1871–1888. https://doi.org/10.1080/00036811.2019.1670347.

[8] E. Azroul, A. Benkirane and M. Srati, *Existence of solutions for a nonlocal type problem in fractional Orlicz Sobolev spaces*, Adv. Oper. Theory **5** (2020), 1350–1375. https://doi.org/10.1007/s43036-020-00042-0

[9] E. Azroul, A. Benkirane and M. Srati, *Nonlocal eigenvalue type problem in fractional Orlicz-Sobolev space*, Adv. Oper. Theory **5** (2020), 1599–1617. https://doi.org/10.1007/s43036-020-00067-5

[10] E. Azroul, A. Benkirane, M. Shimi and M. Srati, *On a class of fractional $p(x)$-Kirchhoff type problems*, Adv. Anal. **100**(2) (2019), 383–402. https://doi.org/10.1080/00036811.2019.1603372

[11] E. Azroul, A. Benkirane, M. Shimi and M. Srati, *Three solutions for fractional $p(x,\cdot)$-Laplacian Dirichlet problems with weight*, Journal of Nonlinear Functional Analysis **2020**(22) (2020), 1–18.

[12] E. Azroul, A. Benkirane, M. Shimi and M. Srati, *On a class of nonlocal problems in new fractional Musielak-Sobolev spaces*, Appl. Anal. (2020). https://doi.org/10.1080/00036811.2020.1789601

[13] E. Azroul, A. Benkirane and M. Srati, *Three solutions for a Schrödinger-Kirchhoff type equation involving nonlocal fractional integro-defferential operators*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 1915–1932. /https://doi.org/10.1007/s11868-020-00331-5

[14] G. Bonanno, G. Molica Bisci and V. Rádulescu, *Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, Comptes Rendus Mathematique **349**(5–6) (2011), 263–268. https://doi.org/10.1016/j.crma.2011.02.009

[15] G. Bonanno, G. Molica Bisci and V. Rádulescu, *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces*, Nonlinear Anal. **75** (2012), 4441–4456. https://doi.org/10.1016/j.na.2011.12.016

[16] J. F. Bonder and A. M. Salort, *Fractional order Orlicz-Soblev spaces*, Journal of Functional Analysis **277**(2) (2019), 333–367. https://doi.org/10.1016/j.jfa.2019.04.003

[17] A. Boumazourh and M. Srati, *Leray-Schauder's solution for a nonlocal problem in a fractional Orlicz-Sobolev space*, Moroccan Journal of Pure and Applied Analysis **6**(1) (2020), 42–52. https://doi.org/10.2478/mjpaa-2020-0004

[18] P. Biler, G. Karch and R. Monneau, *Nonlinear diffusion of dislocation density and self-similar solutions*, Comm. Math. Phys. **294**(1) (2010), 145–168. https://doi.org/10.1007/s00220-009-0855-8
[19] F. Cammaroto and L. Vilasi, *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal. **74**(5) (2011), 1841–1852. https://doi.org/10.1016/j.na.2010.10.057
[20] F. Cammaroto and L. Vilasi, *Multiple solutions for a non-homogeneous Dirichlet problem in Orlicz Sobolev spaces*, Appl. Math. Comput. **218**(23) (2012), 11518–11527. https://doi.org/10.1016/j.amc.2012.05.039
[21] Ph. Clément, B. de Pagter, G. Sweers and F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. **1** (2004), 241–267. https://doi.org/10.1007/s00009-004-0014-6
[22] Y. Chen, S. Levine and M. Rao, *Variable exponent linear growth functionals in image processing*, SIAM J. Appl. Math. **66**(4) (2006), 1383–1406. https://doi.org/10.1137/050624522
[23] M. Chipot and B. Lovat, *Some remarks on non local elliptic and parabolic problems*, Nonlinear Anal. **30**(7) (1997), 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7
[24] N. T. Chung, *Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Var. Elliptic Equ. **58**(12) (2012), 1–10. https://doi.org/10.1080/17476933.2012.701289
[25] N. T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, Electron. J. Qual. Theory Differ. Equ. **2012**(42) (2012), 1–13. https://doi.org/10.14232/ejqtde.2012.1.42
[26] N. T. Chung, *Three solutions for a class of nonlocal problems in Orlicz-Sobolev spaces*, J. Korean Math. Soc. **5**(6) (2013), 1257–1269. https://doi.org/10.4134/JKMS.2013.50.6.1257
[27] G. W. Dai and R. F. Hao, *Existence of solutions of a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), 275–284, https://doi.org/10.1016/j.jmaa.2009.05.031
[28] M. A. Krasnosel'skii and J. B. Rutickii, *Convex functions and Orlicz spaces*, translated from the first Russian edition by Leo F. Boron, P. Noordhoff Ltd, Groningen, **47**(361) (1961), 266–267. https://doi.org/10.2307/3613435
[29] G. Kirchhoff, *Mechanik. Teubner*, Leipzig, 1883.
[30] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8**(3) (1958), 459–466.
[31] W. Luxemburg, *Banach function spaces*, Ph.D. Thesis, Technische Hogeschool te Delft **17**, (1955).
[32] M. Mihäilescu and V. Rädulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Soboliv spaces*, Ann. Inst. Fourier **58**(6) (2008), 2087–2111. https://doi.org/10.5802/aif.2407
[33] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(5) (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
[34] B. Ricceri, *A further three critical points theorem*, Nonlinear Anal. **71**(9) (2009), 4151–4157. https://doi.org/10.1016/j.na.2009.02.074
[35] B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Global Optim. **46** (2010), 543–549. https://doi.org/10.1007/s10898-009-9438-7
[36] M. Xiang, B. Zhang and M. Ferrara, *Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian*, J. Math. Anal. Appl. **424**(2) (2015), 1021–1041. https://doi.org/10.1016/j.jmaa.2014.11.055
[37] E. Zeidler, *Nonlinear functional analysis and applications*, in: *Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990. https://doi.org/10.1007/978-1-4612-0985-0

References

[1]Faculty of Sciences Dhar Al Mahraz,
Sidi Mohamed Ben Abdellah University,
Fez, Morocco
Email address: elhoussine.azroul@gmail.com
Email address: abd.benkirane@gmail.com
Email address: srati93@gmail.com

[2]College of Science, Civil Aviation,
University of China,
300300, Tianjin, China,
Email address: mqxiang@cauc.edu.cn

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 305–311.

GROWTH ESTIMATE FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES AND RESTRICTED ZEROS

N. A. RATHER[1], M. SHAFI[2], AND ISHFAQ DAR[3*]

Abstract. Let $\mathcal{R}_n$ be the set of all rational functions of the type $r(z) = f(z)/w(z)$, where $f(z)$ is a polynomial of degree at most n and $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$ for $1 \leq j \leq n$. In this paper, we extend some famous results concerning to the growth of polynomials by T. J. Rivlin, A. Aziz and others to the rational functions with prescribed poles and thereby obtain the analogous results for such rational functions with restricted zeros.

1. Introduction

Let $\mathcal{P}_n$ be the set of all complex polynomials $f(z) = \sum_{j=1}^{n} a_j z^j$ of degree at most n and let $D_{k-} = \{z : |z| < k\}$, $D_{k+} = \{z : |z| > k\}$ and $T_k = \{z : |z| = k\}$.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, we set

$$w(z) = \prod_{j=1}^{n}(z - a_j), \quad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right)$$

and

$$\mathcal{R}_n = \mathcal{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{f(z)}{w(z)} : f \in \mathcal{P}_n\right\}.$$

Then clearly $\mathcal{R}_n$ is the space of all rational functions with at most n poles $a_1, a_2, \ldots, a_n$ with finite limit at infinity. We note that $B(z) \in \mathcal{R}_n$. Throughout this paper, we shall assume that all the poles $a_1, a_2, \ldots, a_n$ lie in D_{1+}.

Key words and phrases. Rational functions, polynomial inequalities, growth, zeros.
2020 *Mathematics Subject Classification.* Primary: 26D10. Secondary: 41A17, 30C15.
DOI 10.46793/KgJMat2502.305R
Received: December 21, 2020.
Accepted: April 04, 2022.

For a polynomial $f(z)$ of degree n having no zeros in D_{1-}, T. J. Rivlin [8] proved that, for $\rho < 1$ and $z \in T_1$,

$$|f(\rho z)| \geq \left(\frac{\rho+1}{2}\right)^n |f(z)|. \tag{1.1}$$

The result is best possible and equality holds for $f(z) = \alpha(z-\beta)^n$, $|\beta| = 1$.

A. Aziz [2] generalizes inequality (1.1) and proved that, if $f(z)$ is a polynomial of degree n having no zeros in D_{k-}, then for $z \in T_1$,

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad k \geq 1 \text{ and } \rho < 1, \tag{1.2}$$

and

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n |f(z)|, \quad k \leq 1 \text{ and } 0 \leq \rho \leq k^2. \tag{1.3}$$

The result is sharp and equality holds for $f(z) = (z+k)^n$.

Analogous to the above inequality, we have a result when $1 < R \leq k^2,\ k > 1$, which can be found in [7, page 432], which states that if $f(z)$ is a polynomial of degree n having all its zeros in $D_{k+} \cup T_k$, where $k > 1$, then for $z \in T_1$ and $1 < R \leq k^2$

$$|f(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n |f(z)|. \tag{1.4}$$

The result is sharp and equality holds if and only if $f(z) = c(z - ke^{i\gamma})^n$ for some $c \neq 0$ and $\gamma \in \mathbb{R}$.

In literature there exist various results in this direction related to the growth of polynomials for reference see [1, 3–6].

The main aim of this paper is to obtain certain growth estimates for rational functions $r(z) \in \mathcal{R}_n$ having no zero in D_{k-}. In this direction we first present an extension of inequality (1.2) to the rational functions. More precisely, we prove the following.

Theorem 1.1. *Let $r \in \mathcal{R}_n$ with no zero in D_{k-}, where $k \geq 1$, then for $\rho < 1$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|. \tag{1.5}$$

Remark 1.1. If we take $k = 1$ in Theorem 1.1, we get the following extension of inequality (1.1) to the rational functions.

Corollary 1.1. *Let $r \in \mathcal{R}_n$ with no zeros in D_{1-}, then for $\rho < 1$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+1}{2}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|.$$

Remark 1.2. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.1, then inequality (1.5) reduces to the following inequality

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.6}$$

Letting $|\alpha| \to \infty$ in inequality (1.6), we get inequality (1.2).

Theorem 1.2. *Let $r \in \mathcal{R}_n$ with no zeros in D_{k-}, where $k \leq 1$, then for $0 \leq \rho \leq k^2$ and $z \in T_1$,*

$$|r(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|-1}{|a_j|+\rho}\right) |r(z)|. \tag{1.7}$$

Remark 1.3. By taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.2, inequality (1.7) reduces to the following inequality

$$|f(\rho z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \left(\frac{|\alpha|-1}{|\alpha|+\rho}\right)^n \left|\frac{\rho z-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.8}$$

Letting $|\alpha| \to \infty$ in inequality (1.8), we get inequality (1.3).

Theorem 1.3. *Let $r \in \mathcal{R}_n$ with no zeros in D_{k-}, where $k > 1$, then for $1 < R \leq k^2$ and $z \in T_1$,*

$$|r(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j|+1}{\big||a_j|-R\big|}\right) |r(z)|. \tag{1.9}$$

Remark 1.4. Taking $w(z) = (z-\alpha)^n$, $|\alpha| > 1$, in Theorem 1.3, inequality (1.9) reduces to the following inequality

$$|f(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n \left(\frac{|\alpha|+1}{\big||\alpha|-R\big|}\right)^n \left|\frac{Rz-\alpha}{z-\alpha}\right|^n |f(z)|. \tag{1.10}$$

Letting $|\alpha| \to \infty$ in inequality (1.10), we obtain inequality (1.4).

2. Proofs of the Theorems

Proof of Theorem 1.1. By hypothesis $r \in \mathcal{R}_n$, therefore we have $r(z) = \frac{f(z)}{w(z)}$, where $w(z) = \prod_{j=1}^{n}(z-a_j)$, $|a_j| > 1$. Since all the zeros of $f(z)$ lie in $D_{k+} \cup T_k$, $k \geq 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \leq \theta < 2\pi$, $1 \leq j \leq n$, are the zeros of $f(z)$, then we write $f(z) = c\prod_{j=1}^{n}(z-\rho_j e^{i\theta_j})$, where $\rho_j \geq k \geq 1$, $j = 1, 2, \ldots, n$. Hence, for $\rho < 1$ and

$0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \left|\frac{f(\rho e^{i\theta})}{w(\rho e^{i\theta})}\right| \Big/ \left|\frac{f(e^{i\theta})}{w(e^{i\theta})}\right|$$
$$= \left|\frac{f(\rho e^{i\theta})}{f(e^{i\theta})}\right| \cdot \left|\frac{w(e^{i\theta})}{w(\rho e^{i\theta})}\right|$$
$$= \prod_{j=1}^{n} \left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n} \left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \tag{2.1}$$

Now,

$$\prod_{j=1}^{n} \left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| = \prod_{j=1}^{n} \left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right|$$
$$= \prod_{j=1}^{n} \left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j \cos(\theta - \theta_j)}{1 + \rho_j^2 - 2\rho_j \cos(\theta - \theta_j)}\right)^{1/2}$$
$$\geq \prod_{j=1}^{n} \frac{\rho + \rho_j}{1 + \rho_j} \quad (\text{as } \rho < 1)$$
$$\geq \prod_{j=1}^{n} \frac{\rho + k}{1 + k} \quad (\text{as } \rho_j \geq k)$$
$$= \left(\frac{\rho + k}{1 + k}\right)^n. \tag{2.2}$$

Also for $|a_j| > 1$, $j = 1, 2, \ldots, n$, we have

$$\prod_{j=1}^{n} \left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \geq \prod_{j=1}^{n} \frac{|a_j| - 1}{|a_j| + \rho}. \tag{2.3}$$

Using inequalities (2.2) and (2.3) in equation (2.1), we obtain for $0 \leq \theta < 2\pi$

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| \geq \left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right).$$

That is, for $z \in T_1$ and $\rho < 1$, we have

$$|r(\rho z)| \geq \left[\left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n} \left(\frac{|a_j| - 1}{|a_j| + \rho}\right)\right] |r(z)|.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By hypothesis $r \in \mathcal{R}_n$, therefore we have $r(z) = \frac{f(z)}{w(z)}$, where $w(z) = \prod_{j=1}^{n}(z - a_j)$, $|a_j| > 1$. Since all the zeros of $f(z)$ lie in $D_{k+} \cup T_k$, $k \leq 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $0 \leq \theta < 2\pi$, $1 \leq j \leq n$, are the zeros of $f(z)$, then we write

$f(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k$, $k \leq 1$, $j = 1, 2, \ldots, n$. Hence, for $0 \leq \rho \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\rho e^{i\theta})}{r(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right|. \tag{2.4}$$

Now,

$$\begin{aligned}
\prod_{j=1}^{n}\left|\frac{\rho e^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\rho e^{i(\theta-\theta_j)} - \rho_j}{e^{i(\theta-\theta_j)} - \rho_j}\right| \\
&= \prod_{j=1}^{n}\left(\frac{\rho^2 + \rho_j^2 - 2\rho\rho_j\cos(\theta - \theta_j)}{1 + \rho_j^2 - 2\rho_j\cos(\theta - \theta_j)}\right)^{1/2} \\
&\geq \prod_{j=1}^{n}\frac{\rho + \rho_j}{1 + \rho_j} \quad (\text{as } 0 \leq \rho \leq k^2) \\
&\geq \prod_{j=1}^{n}\frac{\rho + k}{1 + k} \quad (\text{as } \rho_j \geq k) \\
&= \left(\frac{\rho + k}{1 + k}\right)^n. \qquad (2.5)
\end{aligned}$$

Again as before, for $|a_j| > 1$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{\rho e^{i\theta} - a_j}\right| \geq \prod_{j=1}^{n}\frac{|a_j| - 1}{|a_j| + \rho}. \tag{2.6}$$

Using inequalities (2.5) and (2.6) in equation (2.4), we have for $z \in T_1$ and $0 \leq \rho \leq k^2$,

$$|r(\rho z)| \geq \left[\left(\frac{\rho + k}{1 + k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j| - 1}{|a_j| + \rho}\right)\right]|r(z)|,$$

which is the desired result. □

Proof of Theorem 1.3. Since all the zeros of $r(z)$ lie in $D_{k+} \cup T_k$, where $k > 1$, therefore it follows that all the zeros of polynomial $f(z)$ lie in $D_{k+} \cup T_k$, $k > 1$, therefore if $z_j = \rho_j e^{i\theta_j}$, $1 \leq j \leq n$, are the zeros of $f(z)$, then we write $f(z) = c\prod_{j=1}^{n}(z - \rho_j e^{i\theta_j})$, where $\rho_j \geq k > 1$, $j = 1, 2, \ldots, n$. Hence, for $1 < R \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(Re^{i\theta})}{r(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - \rho_j e^{i\theta_j}}{e^{i\theta} - \rho_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - a_j}{Re^{i\theta} - a_j}\right|. \tag{2.7}$$

Now,

$$
\begin{aligned}
\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-\rho_j e^{i\theta_j}}{e^{i\theta}-\rho_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i(\theta-\theta_j)}-\rho_j}{e^{i(\theta-\theta_j)}-\rho_j}\right| \\
&= \prod_{j=1}^{n}\left(\frac{R^2+\rho_j^2-2R\rho_j\cos(\theta-\theta_j)}{1+\rho_j^2-2\rho_j\cos(\theta-\theta_j)}\right)^{1/2} \\
&\leq \prod_{j=1}^{n}\frac{R+\rho_j}{1+\rho_j} \quad (\text{as } 1<R\leq k^2) \\
&\leq \prod_{j=1}^{n}\frac{R+k}{1+k} \quad (\text{as } \rho_j\geq k) \\
&= \left(\frac{R+k}{1+k}\right)^n. \qquad (2.8)
\end{aligned}
$$

Also for $|a_j|>1$, $j=1,2,\ldots,n$, we have

$$
\prod_{j=1}^{n}\left|\frac{e^{i\theta}-a_j}{Re^{i\theta}-a_j}\right| \leq \prod_{j=1}^{n}\frac{1+|a_j|}{|R-|a_j||}. \tag{2.9}
$$

Using inequalities (2.8) and (2.9) in equation (2.7), we obtain for $0\leq\theta<2\pi$,

$$
\left|\frac{r(Re^{i\theta})}{r(e^{i\theta})}\right| \leq \left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j|+1}{|R-|a_j||}\right).
$$

That is, for $z\in T_1$ and $1<R\leq k^2$, we have

$$
|r(Rz)| \leq \left[\left(\frac{R+k}{1+k}\right)^n \prod_{j=1}^{n}\left(\frac{|a_j|+1}{|R-|a_j||}\right)\right]|r(z)|.
$$

That completes the proof of Theorem 1.3. □

Acknowledgements. The authors wish to thank the referee, for the careful reading of the paper and for the helpful suggestions and comments.

References

[1] N. C. Ankeny and T. J. Rivlin, *On a Theorem of S. Bernstein*, Pacific J. Math. **5** (1955), 849–852.
[2] A. Aziz, *Growth of polynomials whose zeros are within or outside a circle*, Bull. Austral. Math. Soc. **35** (1987), 247–256.
[3] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory **54** (1998), 306–313.
[4] A. Aziz and N. A. Rather, *Growth of maximum modulus of rational functions with prescribed poles*, Mathematical Inequalities & Applications **2** (1999), 165–173. https://dx.doi.org/10.7153/mia-02-14
[5] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World Scientific Publishing Co., Singapore, 1994.

[6] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. I, Springer-Verlag, New York, 1972.
[7] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Clarendon Press Oxford, 2002.
[8] T. J. Rivlin, *On the maximum modulus of polynomials*, Amer. Math. Monthly **67** (1960), 251–253.

[1,2,3]Department of Mathematics,
University of Kashmir,
Hazratbal Srinagar–19006, J&K, India
Email address: dr.narather@gmail.com
Email address: wanishafi1933@gmail.com
Email address: ishfaq619@gmail.com

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 313–326.

RELATION BETWEEN CONVERGENCE AND ALMOST CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

BIROJIT DAS[1], BABY BHATTACHARYA[1], AND BINOD CHANDRA TRIPATHY[2]

Abstract. In this paper, we introduce a new type of almost convergent complex uncertain sequence with respect to uniformly almost surely. We characterize the notion of almost convergence of sequences of complex uncertain variables further. We establish the interconnection between convergent complex uncertain sequence, bounded complex uncertain sequence and almost convergent complex uncertain sequence in all five aspects of uncertainty.

1. Introduction and Preliminaries

In the real world, often we face various types of indeterminacy. Frequency generated by samples plays important role in the study to deal with those indeterminate situations. Probability theory is an efficient tool to study the frequency. However, sometimes it is difficult to collect observed data when some unexpected events occur. In this case, decision maker have to invite experts to estimate the belief degree of each events occurrence. For dealing with belief degree legitimately, an axiomatic system named uncertainty theory satisfied normality, duality, and subadditivity was proposed by Liu [9]. As a fundamental concept in uncertainty theory, the uncertain variable was presented by Liu [9]. In order to describe an uncertain variable, Liu [9] introduced the concepts of uncertain measure, uncertain distribution and expected value of uncertain variable. The uncertain measure follows the axioms of normality, duality, subadditivity and product. In the year 2007, the notion of uncertain sequences and their four

Key words and phrases. uncertainty space, complex uncertain sequence, almost convergence, convergence.

2020 *Mathematics Subject Classification.* Primary: 60B10. Secondary: 60E05, 40A05, 40A30, 40F05.

DOI 10.46793/KgJMat2502.313D

Received: October 02, 2021.

Accepted: April 05, 2022.

types of convergences, namely convergence in mean, measure, distribution, almost surely was introduced by Liu [9]. Then the same was extended by You [12] while he introduced a new type of convergent uncertain sequence with respect to uniformly almost surely. Thereafter, to describe the complex uncertain quantities, the notions of uncertain variable and uncertain distribution are presented by Peng [10] in that direction. Chen et al. [1] explored the work considering the sequence of complex uncertain variables due to Peng [10]. They reported five types of convergence concepts of uncertain sequences in complex environment by establishing interrelationships among them. Since its initiation, the study of complex uncertain sequences got the full attention of the researchers. These convergence concept of complex uncertain sequence has also been generalised by Datta and Tripathy [8], Das et al. [2–7]. Recently, Saha et al. [11] introduced the concept almost convergent complex uncertain sequence in a given uncertainty space. They have initiated almost convergence in four directions of uncertainty, namely almost convergence in mean, in measure, in distribution and in almost surely. Also, they established the interrelationships between each types of almost convergences upto some extent. In this article, at first we extend the study by introducing the fifth direction of uncertainty, i.e., almost convergent complex uncertain sequence with respect to uniformly almost surely. We show that every almost convergent complex uncertain sequence with respect to uniformly almost surely is almost convergent in almost surely. We further establish the interconnection between almost convergent, bounded and convergent sequences of complex uncertain variable.

We now present few concepts and related results in the following, which will be playing an important role in the whole study.

Definition 1.1 ([12])**.** Let us consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then a function ζ from Γ to the set of complex numbers which is measurable in the aspect of uncertainty is called a complex uncertain variable.

Definition 1.2 ([10])**.** Let us consider a sequence (ζ_n) of complex uncertain variables. Then (ζ_n) is said to be almost convergent to ζ in almost surely if there is such an uncertain event Λ with unit uncertain measure that

$$\lim_{m\to\infty} ||u_{n,m}(\gamma) - \zeta(\gamma)|| = 0,$$

uniformly in n and for all $\gamma \in \Lambda$, where $u_{n,m} = \frac{1}{m}\sum_{i=1}^{m} \zeta_{n+i-1}$.

Definition 1.3 ([10])**.** A sequence (ζ_n) of complex uncertain variables is called almost convergent in respect of measure to some finite limit ζ if the following condition is satisfied: for all positive integer n and a positive real ε

$$\lim_{m\to\infty} \mathcal{M}\{||u_{n,m} - \zeta|| \geq \varepsilon\} = 0, \quad \text{where } u_{n,m} = \frac{1}{m}\sum_{i=1}^{m} \zeta_{n+i-1}.$$

Definition 1.4 ([10])**.** Let us consider a sequence (ζ_n) of complex uncertain variables. The sequence is said to be almost convergent in respect of mean to a finite ζ if

$$\lim_{l\to\infty} E[||t_{n,l} - \zeta||] = 0, \quad \text{where } t_{n,l} = \frac{1}{l}\sum_{i=1}^{l} \zeta_{n+i-1}.$$

Here n runs uniformly over $\mathbb{N}$.

Definition 1.5 ([10])**.** Let us consider infinite numbers of complex uncertain variables given by $\xi, \xi_1, \xi_2, \ldots,$ and suppose $\Phi, \Phi_{1,m}, \Phi_{2,m}, \ldots,$ are the distribution functions in respect of the complex uncertain variables ξ, $\frac{\xi_1+\xi_2+\cdots+\xi_m}{m}$, $\frac{\xi_2+\xi_3+\cdots+\xi_{m+1}}{m}, \ldots,$ respectively. Then the sequence (ξ_n) is said to be almost convergent to ξ in respect of distribution if

$$\lim_{m\to\infty} \Phi_{n,m}(c) = \Phi(c),$$

for all $n \in \mathbb{N}$, c being the complex point of continuity of the function Φ.

Theorem 1.1 ([10])**.** *If the real and imaginary part (ξ_n) and (η_n) of a sequence (ζ_n) almost converges to the finite limits ξ and η respectively with respect to measure, then $(\zeta_n) = (\xi_n + i\eta_n)$ almost converges in distribution to $\xi + i\eta$.*

Theorem 1.2 ([10])**.** *If (ζ_n) is an almost convergent sequence of complex uncertain variables in mean to some finite limit ζ, then it almost converges in respect of measure by preserving the limit.*

2. Main Results

At first our intend is to define almost convergent sequence of complex uncertain variables with respect to uniformly almost surely. We show existence of such sequence and establish the interrelationship with the almost convergent complex uncertain sequence in almost surely. Then, we initiate boundedness property of sequences of complex uncertain variables and prove the interconnection between convergent, bounded and almost convergent sequences of complex uncertain variables in all five aspects of uncertainty.

Definition 2.1. A complex uncertain sequence (ζ_n) is called almost convergent to a finite limit ζ in uniformly almost surely if there exists events $\{E_x\}$ with $\mathcal{M}\{E_x\} \to 0$ such that (ζ_n) almost converges to the same ζ uniformly in the domain $\Gamma - E_x$, where $x \in \mathbb{N}$, i.e.,

$$\lim_{p\to\infty} \left\| \frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n+k}(\gamma) - \zeta(\gamma) \right\| = 0,$$

for all $\gamma \in \Gamma - E_x$ and uniformly for all n.

Example 2.1. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ be an infinite set of uncertain events and $\mathcal{L}$ be the power set of Γ. Then $\mathcal{L}$ becomes σ-algebra on Γ.

Let the measurable set function $\mathcal{M}$ be defined as follows

$$\mathcal{M}\{\beta\} = \sum_{\gamma_j \in \beta} \frac{1}{2^j}.$$

Obviously, $\sum_{\gamma_j \in \beta} \frac{1}{2^j}$ is unity and $\mathcal{M}$ holds the other axioms of uncertain measure. So $\mathcal{M}$ becomes uncertain measure and thus, $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Now, for a given $\varepsilon > 0$ (however small) exists $p \in \mathbb{N}$ such that $\frac{1}{2^p} < \varepsilon$.

Let (ζ_n) be a complex uncertain sequence, where the complex uncertain variable ζ_n is given by

$$\zeta_n(\gamma) = \begin{cases} \frac{1}{2}i, & \text{if } n \geq p, \\ 0, & \text{otherwise}, \end{cases}$$

for all $\gamma \in \Gamma$. Also, let ζ be the complex uncertain variable such that $\zeta(\gamma) = 0$ for all $\gamma \in \Gamma$. We have, $||\zeta_n(\gamma) - \zeta(\gamma)|| = \left|\left|\frac{1}{2}i\right|\right| = \frac{1}{2}$, whenever $n > N$ and $||\zeta_n(\gamma) - \zeta(\gamma)|| = 0$, for the remaining cases. Moreover, $\mathcal{M}\{\gamma_j\} \to 0$, as $j > p$. Then, from the above, one can see that $(\zeta_n(\gamma))$ almost converges uniformly to $\zeta(\gamma) = 0$, for all $\gamma \in \Gamma - \gamma_j,\ j > p$. Hence, (ζ_n) is almost convergent to ζ in uniformly almost surely.

The following theorem is due to Saha et al. [10].

Theorem 2.1 ([10]). *Suppose $(\zeta_n) = (\xi_n + i\eta_n)$ be a complex uncertain sequence. If the real uncertain sequences (ξ_n) and (η_n) almost converges to ξ and η respectively in respect of measure, then (ζ_n) is almost convergent to $\xi + i\eta$ in the same direction.*

We now establish the converse part of the same in the same context. This one result produces few more interrelationships between the other almost convergence concepts.

Theorem 2.2. *If a complex uncertain sequence (ζ_n), which is given by $\zeta_n = \xi_n + i\eta_n$, almost converges in measure to the finite limit $\xi + i\eta$, then the real part (ξ_n) and imaginary part (η_n) also almost converges to ξ and η in measure.*

Proof. Let (ζ_n), where $\zeta_n = \xi_n + i\eta_n$ is almost convergent to $\zeta = \xi + i\eta$ in measure. Then, for any $\delta > 0$, we have

$$\mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n+k} - \zeta\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1} (\xi_{n+k} + i\eta_{n+k}) - (\xi + i\eta)\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1} (\xi_{n+k} - \xi) + i\frac{1}{p}\sum_{k=0}^{p-1}(\eta_{n+k} - \eta)\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty.$$

This implies that there exists $0 < \delta' < \frac{\delta}{2}$ such that

$$\mathcal{M}\left\{\gamma : \left|\left|\frac{1}{p}\sum_{k=0}^{p-1} \xi_{n+k}(\gamma) - \xi(\gamma)\right|\right| \geq \delta'\right\} \to 0, \quad \text{as } n \to \infty,$$

and

$$\mathcal{M}\left\{\gamma : \left\|\frac{1}{p}\sum_{k=0}^{p-1}\eta_{n+k}(\gamma) - \eta(\gamma)\right\| \geq \delta^{'}\right\} \to 0, \quad \text{as } n \to \infty.$$

Evidently, the uncertain sequences (ξ_n) and (η_n) are almost convergent in measure to ξ and η. □

In view of the above Theorem 2.2 and Theorem 1.1, we can deduce the following result.

Corollary 2.1. *Almost convergence in measure implies almost convergence in distribution.*

From Theorem 1.2 and Corollary 2.1, we can give the following.

Corollary 2.2. *An almost convergent sequence in mean almost converges with respect of distribution therein.*

Remark 2.1. The notion of almost convergence in almost surely and almost convergence in measure are the concepts no way related.

In the following two examples, we demonstrates the validity of the statement.

Example 2.2. We consider the space $(\Gamma, \mathcal{L}, \mathcal{M})$, with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\mathcal{L} = P(\Gamma)$. Define $\mathcal{M}$ as follows:

$$\mathcal{M}\{\Delta\} = \begin{cases} 0, & \text{if } \Delta = \phi, \\ 1, & \text{if } \Delta = \Gamma, \\ 0.6, & \text{if } \gamma_1 \in \Delta, \\ 0.4, & \text{if } \gamma_1 \notin \Delta. \end{cases}$$

We define ζ_n and ζ as follows:

$$\zeta_n(\alpha) = \begin{cases} i, & \text{if } \alpha = \gamma_1, \\ 2i, & \text{if } \alpha = \gamma_2, \\ 3i, & \text{if } \alpha = \gamma_3, \\ 4i, & \text{if } \alpha = \gamma_4, \\ 0, & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{N}$ and $\zeta(\gamma) = 0$ for all $\gamma \in \Gamma$.

Observe that $\zeta_n \to \zeta$, except only for $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and so (ζ_n) is almost convergent to ζ in almost surely. However, for some $\delta > 0$, we have

$$\mathcal{M}\{||\zeta_n - \zeta|| \geq \delta\} = \mathcal{M}\{\gamma : ||\zeta_n(\gamma) - \zeta(\gamma)|| \geq \delta\} = \mathcal{M}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \mathcal{M}\{\Gamma\} = 1.$$

Consequently, the complex uncertain sequence (ζ_n) is not almost convergent in measure.

Example 2.3. Let us consider the space $(\Gamma, \mathcal{L}, \mathcal{M})$, with $\Gamma = [0,1]$, $\mathcal{L} = P[0,1]$. Here the uncertain measure is the Lebesgue measure.

Suppose ζ_n and ζ be given by

$$\zeta_n(\alpha) = \begin{cases} i, & \text{if } \frac{p}{2^t} \leq \alpha \leq \frac{1+p}{2^t}, \\ 0, & \text{elsewhere,} \end{cases}$$

and $\zeta(\alpha) = 0$ for all $\gamma \in \Gamma$, $n = 2^t + p \in \mathbb{N}$, where p, t are integers. Then,

$$\begin{aligned} \mathcal{M}\left\{ \left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k} - \zeta \right\| \geq \delta \right\} =& \mathcal{M}\left\{ \gamma : \left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} \\ =& \frac{1+p}{2^t} - \frac{p}{2^t} = \frac{1}{2^t} \end{aligned}$$

and hence,

$$\lim_{n\to\infty} \mathcal{M}\left\{ \gamma : \left\| \tfrac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} = \lim_{t\to\infty} \tfrac{1}{2^t} = 0.$$

Thus, the complex uncertain sequence (ζ_n) almost converges to ζ in measure.

On the other hand, let $\gamma \in [0,1]$. Then, there are intervals of the form $[\frac{p}{2^t}, \frac{p+1}{2^t}]$ containing γ, for different values of p. Therefore, (ζ_n) does not converges to ζ in almost surely and hence (ζ_n) is not almost convergent to ζ in almost surely.

Remark 2.2. An almost convergent complex uncertain sequence in almost surely may not be almost convergent in distribution. The following example satisfies the same.

Example 2.4. Consider the uncertainty space and sequence taken in example 2.2.
Let $\Phi_n(z)$ and $\Phi(z)$ be the uncertainty distribution functions of ζ_n and ζ, respectively. Then

$$\Phi_n(z) = \Phi_n(p+iq) = \begin{cases} 0, & \text{if } p < 0, q \in (-\infty, \infty), \\ 0, & \text{if } p \geq 0, q < 1, \\ 0.6, & \text{if } p \geq 0, 1 \leq q < 2, \\ 0.6, & \text{if } p \geq 0, 2 \leq q < 3, \\ 0.6, & \text{if } p \geq 0, 3 \leq q < 4, \\ 1, & \text{if } p \geq 0, q \geq 4, \end{cases}$$

and

$$\Phi(z = p+iq) = \begin{cases} 0, & \text{if } p < 0, q \in (-\infty, \infty), \\ 0, & \text{if } p \geq 0, q < 0, \\ 1, & \text{if } p \geq 0, q \geq 0. \end{cases}$$

Thus, (ζ_n) is not almost convergent in distribution to ζ.

Remark 2.3. An almost convergent complex uncertain sequence in mean may not be almost convergent in almost surely.

Example 2.5. From the Example 2.3,

$$E\left[\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma)-\zeta(\gamma)\right\|\right]=\frac{1}{2^t},$$

which tends to 0, as $n \to \infty$. Then, the sequence almost converges to ζ in mean also. But it was already proved that (ζ_n) is not almost convergent in almost surely.

Remark 2.4. A sequence (ζ_n) which is almost convergent in almost surely may not be almost convergent in mean. Explanation is provided in the following example.

Example 2.6. Consider the space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$, $\mathcal{L} = P(\Gamma)$ and

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_j \in \Lambda} \frac{2}{3} \cdot \frac{1}{3^{(j-1)}}.$$

Define ζ_n and ζ respectively by

$$\zeta_n(\alpha) = \begin{cases} 3^n i, & \text{if } \alpha = \gamma_n, \\ 0, & \text{elsewhere,} \end{cases}$$

for $n \in \mathbb{N}$ and $\zeta \equiv 0$.

One can easily observe that the sequence (ζ_n) almost converges to ζ in almost surely.

Now, for the uncertain variable $||\zeta_n||$, its uncertainty distribution function is given by

$$\Phi_n(p) = \begin{cases} 0, & \text{if } p < 0, \\ 1 - \frac{1}{3^n}, & \text{if } 0 \le p < 3^n, \\ 1, & \text{elsewhere,} \end{cases}$$

for $n \in \mathbb{N}$.

Now, integration to the above distribution function for expected value gives us

$$E\left[\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}-\zeta\right\|\right]=1.$$

Therefore, the complex uncertain sequence (ζ_n) is not almost convergent in mean to ζ.

Theorem 2.3. *The sequence (ζ_n) is almost convergent in almost surely to ζ if and only if for any $\varepsilon > 0$ exists $N \in \mathbb{N}$ in such a way that*

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1}\zeta_{n+x}(\gamma)-\zeta(\gamma)\right\| \ge \varepsilon\right\}\right\}=0.$$

Proof. The definition of almost convergence in almost surely leads us to the existence of such uncertain event Δ with $\mathcal{M}\{\Delta\} = 1$, such that

$$\lim_{n\to\infty}\left\|\frac{1}{p}\sum_{r=0}^{p-1}\zeta_{n+r}(\alpha)-\zeta(\alpha)\right\|=0, \quad \text{for all } \alpha \in \Delta.$$

Let ε be a preassigned positive number. Then, there exists $N \in \mathbb{N}$ such that for any $\alpha \in \Delta$, we have

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\alpha) - \zeta(\alpha)\right\| < \varepsilon\right\}\right\} = 1,$$

where $n > N$.

Applying the duality axiom of uncertain measure to the above, we get

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\alpha) - \zeta(\alpha)\right\| \geq \varepsilon\right\}\right\} = 0, \quad \text{for all } \alpha \in \Delta.$$

Hence, the theorem is proved. □

Theorem 2.4. *The necessary and sufficient condition for a complex uncertain sequence (ζ_n) to almost converges in uniformly almost surely to ζ is that for any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} < \varepsilon.$$

Proof. Let the sequence (ζ_n) of complex uncertain variable almost converges to ζ in uniformly almost surely. Then for $\varepsilon > 0$, there exists $\delta > 0$ and an event B with measure less than ν, $\nu \to 0^+$, such that the sequence (ζ_n) converges uniformly to ζ on $\Gamma - B$. That means, there exists $n_0 \in \mathbb{N}$ so that

$$\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma)\right\| < \varepsilon, \quad \text{for all } n \geq n_0 \text{ and all } \gamma \in \Gamma - B.$$

Also, $\nu < \varepsilon$. Thus, we have

$$\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} \subseteq B.$$

Applying the subadditivity axiom of uncertain measure, we get

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} \leq \mathcal{M}\{B\} < \nu < \varepsilon.$$

Conversely, let

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} < \varepsilon.$$

We take $\delta > 0$. Then for any $\nu > 0$, $a \geq 1$, there exists a positive integer a_s such that

$$\mathcal{M}\left\{\bigcup_{n=a_s}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \frac{1}{a}\right\}\right\} < \frac{\nu}{2^a}.$$

Consider $B = \bigcup_{a=1}^{\infty} \bigcup_{n=a_s}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \frac{1}{a} \right\}$. Then

$$\mathcal{M}\{B\} \leq \sum_{a=1}^{\infty} \mathcal{M}\left\{ \bigcup_{n=a_s}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \frac{1}{a} \right\} \right\} \leq \sum_{a=1}^{\infty} \frac{\nu}{2^a} = \nu.$$

Moreover, $\sup_{\gamma \in \Gamma - B} \frac{1}{p} \sum_{x=0}^{p-1} ||\zeta_{n+x}(\gamma) - \zeta(\gamma)|| < \frac{1}{n}$, where $m = 1, 2, 3, \ldots$, and $n > a_s$. Therefore, the result is established. □

Theorem 2.5. *Let the sequence (ζ_n) be almost convergent in uniformly almost surely to ζ. Then, the sequence (ζ_n) is almost convergent in almost surely to ζ.*

Proof. Taking Theorem 2.3 into consideration, we have if the complex uncertain sequence (ζ_n) almost converges to ζ in uniformly almost surely, then

$$\mathcal{M}\left\{ \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon.$$

Now, since

$$\mathcal{M}\left\{ \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} \leq \mathcal{M}\left\{ \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon,$$

hence, (ζ_n) almost converges in almost surely to ζ. □

Theorem 2.6. *A complex uncertain sequence (ζ_n), which almost converges with respect to uniformly almost surely to ζ is also almost convergent in measure therein.*

Proof. Let (ζ_n) be almost convergent in uniformly almost surely to ζ. Then, for $\varepsilon > 0$ and $\delta > 0$ there exists $n_0 \in \mathbb{N}$ so that

$$\mathcal{M}\left\{ \bigcup_{n=n_0}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon, \quad \text{for all } n \geq n_0.$$

Then

$$\begin{aligned} &\mathcal{M}\left\{ \gamma : \left\| \frac{1}{pq} \sum_{x=0}^{p-1} \sum_{y=0}^{q-1} \zeta_{m+x,n+y}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} \\ \leq &\mathcal{M}\left\{ \bigcup_{n=n_0}^{\infty} \left\{ \gamma : \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} \right\} < \varepsilon. \end{aligned}$$

Hence, the sequence (ζ_n) almost converges in measure to ζ. □

Theorem 2.7. *Almost convergence in uniformly almost surely of a complex uncertain sequence implies its almost convergence in distribution with preservation of limit.*

Proof. It is straightforward from the Theorem 2.6 and Corollary 2.1. □

Remark 2.5. From the above discussion, a more complete version of interrelationships between different almost convergence in an uncertainty space can be depicted in the Figure 1 given in the top of the following page.

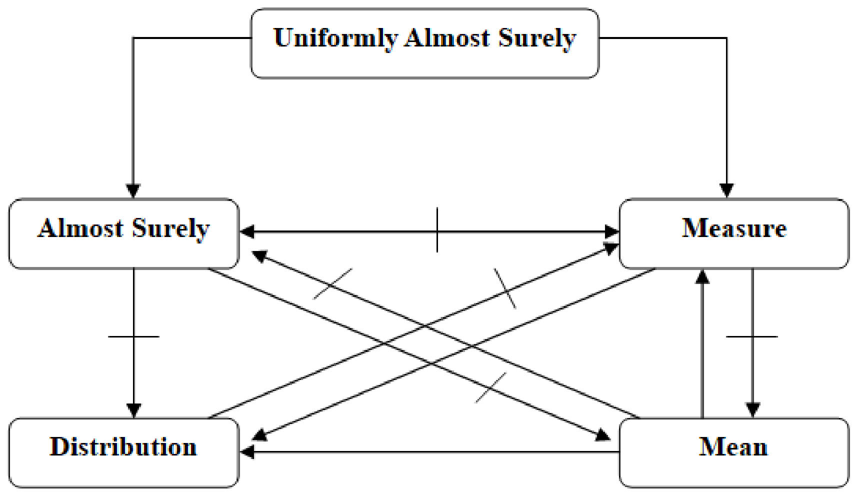

FIGURE 1. Interrelationships among five types of almost convergence

Saha et al. in [10] stated that in a given uncertainty space every convergent complex uncertain sequence is almost convergent to the same limit therein. The statement holds true for all the four aspects (in mean, measure, almost surely, distribution) introduced in [10] and in the fifth direction of uncertainty in uniformly almost surely, also. In this context, we give the detailed proof of the same below.

Theorem 2.8. *A convergent complex uncertain sequence which converges in uniformly almost surely to a finite limit, is also almost convergent to the same limit therein.*

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and (ζ_n) be a complex uncertain sequence which converges to ζ in uniformly almost surely. That means for any given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and a sequence (E_x) of uncertain events with uncertain measure of each of the events tending to zero such that

$$||\zeta_n(\gamma) - \zeta(\gamma)|| < \varepsilon, \quad \text{for all } n \geq n_0.$$

Now, for every positive integer p, $n \geq n_0$, $\gamma \in \Gamma - E_x$ and any $\varepsilon > 0$, we have

$$\begin{aligned} &\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma) - \zeta(\gamma)\right\| \\ =&\left\|\frac{\zeta_n(\gamma) + \zeta_{n+1}(\gamma) + \cdots + \zeta_{n+p-1}(\gamma)}{p} - \zeta(\gamma)\right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{\zeta_n(\gamma) + \zeta_{n+1}(\gamma) + \cdots + \zeta_{n+p-1}(\gamma) - p\zeta(\gamma)}{p} \right\| \\
&= \left\| \frac{\{\zeta_n(\gamma) - \zeta(\gamma)\} + \{\zeta_{n+1}(\gamma) - \zeta(\gamma)\} + \cdots + \{\zeta_{n+p-1}(\gamma) - \zeta(\gamma)\}}{p} \right\| \\
&\leq \left\{ \frac{||\zeta_n(\gamma) - \zeta(\gamma)||}{p} + \frac{||\zeta_{n+1}(\gamma) - \zeta(\gamma)||}{p} + \cdots + \frac{||\zeta_{n+p-1}(\gamma) - \zeta(\gamma)||}{p} \right\} \\
&< \frac{\varepsilon}{p} + \frac{\varepsilon}{p} + \cdots + \frac{\varepsilon}{p} = \frac{p\varepsilon}{p} = \varepsilon,
\end{aligned}
$$

uniformly for all n.

Since ε is chosen arbitrary, the obvious conclusion is that

$$\lim_{p\to\infty} \left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k}(\gamma) - \zeta(\gamma) \right\| = 0.$$

Hence, (ζ_n) is an almost convergent complex uncertain sequence in uniformly almost surely to ζ. □

Remark 2.6. In the above theorem if we replace the sub-collection $\Gamma - E_x$, by Λ, which is a subset of Γ with $\mathcal{M}\{\Lambda\} = 1$, then we can easily prove that every convergent almost surely complex uncertain sequence is almost convergent in the same direction.

Theorem 2.9. *A convergent complex uncertain sequence in mean almost converges in the same aspect. Also, limits of the both cases are identical.*

Proof. Suppose (ζ_n) converges to ζ in mean. Then $\lim_{n\to\infty} E[||\zeta_n - \zeta||] = 0$. This implies, for a preassigned $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$E[||\zeta_n - \zeta||] < \varepsilon, \quad \text{for all } n \geq n_0. \tag{2.1}$$

Suppose $p \in \mathbb{N}$ be given. Then

$$
\begin{aligned}
\left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k} - \zeta \right\| &= \left\| \frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1}}{p} - \zeta \right\| \\
&= \left\| \frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1} - p\zeta}{p} \right\| \\
&= \left\| \frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p} \right\| \\
&\leq \left\| \frac{\zeta_n - \zeta}{p} \right\| + \left\| \frac{\zeta_{n+1} - \zeta}{p} \right\| + \cdots + \left\| \frac{\zeta_{n+p-1} - \zeta}{p} \right\|.
\end{aligned}
$$

Applying the expected value operator to both sides, we get for any $n \geq n_0$

$$E\left[\left\| \frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p} \right\| \right]$$

$$\leq \frac{1}{p} E\left[||\zeta_n - \zeta|| + ||\zeta_{n+1} - \zeta|| + \cdots + ||\zeta_{n+p-1} - \zeta||\right]$$
$$= \frac{1}{p}\{E[||\zeta_n - \zeta||] + E[||\zeta_{n+1} - \zeta||] + \cdots + E[||\zeta_{n+p-1} - \zeta||]\}$$
$$< \frac{1}{p}(\varepsilon + \varepsilon + \cdots + \varepsilon) = \frac{p\varepsilon}{p} = \varepsilon.$$

Consequently, (ζ_n) is an almost convergent complex uncertain sequence in mean to ζ. □

Remark 2.7. Using the complex uncertainty distribution operator, instead of expected value operator in the above Theorem 2.9, one can verify that convergence in distribution of a complex uncertain sequence implies its almost convergence.

Theorem 2.10. *For a complex uncertain sequence*

convergence in measure $\Rightarrow$ almost convergence in measure.

Proof. Let (ζ_n) converges to ζ in measure. Then for any given $\varepsilon > 0$

$$\lim_{n\to\infty} \mathcal{M}\{||\zeta_n - \zeta|| > \varepsilon\} = 0.$$

Then for any $p \in \mathbb{N}$

$$\mathcal{M}\left\{\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k} - \zeta\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1}}{p} - \zeta\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1} - p\zeta}{p}\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p}\right\| > \varepsilon\right\}$$
$$\leq \mathcal{M}\left\{\left\|\frac{\zeta_n - \zeta}{p}\right\| > \varepsilon'\right\} + \mathcal{M}\left\{\left\|\frac{\zeta_{n+1} - \zeta}{p}\right\| > \varepsilon'\right\} + \cdots + \mathcal{M}\left\{\left\|\frac{\zeta_{n+p-1} - \zeta}{p}\right\| > \varepsilon'\right\}$$
$$= \mathcal{M}\{||\zeta_n - \zeta|| > p\varepsilon'\} + \mathcal{M}\{||\zeta_{n+1} - \zeta|| > p\varepsilon'\} + \cdots + \mathcal{M}\{\|\zeta_{n+p-1} - \zeta|| > p\varepsilon'\},$$

for some $\varepsilon' < \frac{\varepsilon}{p}$. Taking limiting case of $n \geq n_0$ to infinity, we get

$$\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k} - \zeta\right\| > \varepsilon\right\} = 0.$$

Consequently, (ζ_n) is an almost convergent complex uncertain sequence in measure to ζ. □

Theorem 2.11. *If a complex uncertain sequence (ζ_n) is almost convergent in mean then (ζ_n) is bounded in mean also.*

Proof. Since (ζ_n) converges to ζ in mean for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$E\left[\left|\left|\frac{1}{p}\sum_{x=0}^{p-1}\zeta_{n+x} - \zeta\right|\right|\right] < \varepsilon, \quad \text{for all } p > n_0 \text{ and uniformly for all } n \in \mathbb{N}$$

$$\Rightarrow E\left[\left|\left|\frac{1}{p}\sum_{k=n}^{n+p-1}\zeta_k - \zeta\right|\right|\right] < \varepsilon, \quad \text{for all } p > n_0 \text{ and uniformly for all } n \in \mathbb{N}.$$

This holds valid for $p = p + 1$ and so $E\left[\left|\left|\sum_{k=n}^{n+p-1}\zeta_k\right|\right|\right]$ is finite. Thus, there exists a finite real number M such that

$$E\left[\left|\left|\sum_{k=n}^{n+p-1}\zeta_k\right|\right|\right] \le \frac{M}{2}.$$

The above inequality can be established if we take $p = p + 1$ and $q = q + 1$. Now,

$$E[||\zeta_n||] = E\left[\left|\left|\sum_{k=n}^{n+p}\zeta_k - \sum_{k=n+1}^{n+p}\zeta_k\right|\right|\right] \le E\left[\left|\left|\sum_{k=n}^{n+p}\zeta_k\right|\right|\right] + E\left[\left|\left|\sum_{k=n+1}^{n+p}\zeta_k\right|\right|\right] \le \frac{M}{2} + \frac{M}{2} = M.$$

Therefore, $\sup_n E[||\zeta_n||] \le M$ and hence the complex uncertain sequence is bounded in mean. □

Remark 2.8. The above theorem holds good for the remaining cases of uncertainty. That is, almost convergence of complex uncertain sequences implies its boundedness in measure, distribution, almost surely and uniformly almost surely too.

References

[1] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Almost convergence of complex uncertain double sequences*, Filomat **35**(1) (2021), 61–78. https://doi.org/10.2298/FIL2101061D

[2] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Characterization of statistical convergence of complex uncertain double sequence*, Anal. Math. Phys. **10**(4) (2020), 1–20. https://doi.org/10.1007/s13324-020-00419-7

[3] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Statistical convergence of complex uncertain triple sequence*, Comm. Statist. Theory Methods (2021), 1–13. https://doi.org/10.1080/03610926.2020.1871016

[4] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Study of matrix transformation of uniformly almost surely convergent complex uncertain sequences*, Filomat **34**(14) (2020), 4907–4922. https://doi.org/10.2298/FIL2014907D

[5] B. Das, B. C. Tripathy, P. Debnath, J. Nath and B. Bhattacharya, *Almost convergence of complex uncertain triple sequences*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences **91**(2) (2021), 245–256. https://doi.org/10.1007/s40010-020-00721-w

[6] B. Das, P. Debnath and B. C. Tripathy, *Characterization of matrix classes transforming between almost sure convergent sequences of complex uncertain variables*, Journal of Uncertain Systems **14**(3) (2021), 1–12. https://doi.org/10.1142/S1752890921500197

[7] B. Liu, *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[8] C. You, *On the convergence of uncertain sequences*, Math. Comput. Model. **49** (2009), 482–487. https://doi.org/10.1016/j.mcm.2008.07.007

[9] D. Datta and B. C. Tripathy, *Convergence of complex uncertain double sequences*, New Mathematics and Natural Computation **16**(3) (2020), 447–459. https://doi.org/10.1142/S1793005720500271

[10] S. Saha, B. C. Tripathy and S. Roy, *On almost convergent of complex uncertain sequences*, New Mathematics and Natural Computation **16**(3) (2020), 573–580. https://doi.org/10.1142/S1793005720500349

[11] X. Chen, Y. Ning and X. Wang, *Convergence of complex uncertain sequences*, Journal of Intelligent & Fuzzy Systems **30** (2016), 3357–3366. https://doi.org/10.3233/IFS-152083

[12] Z. Peng, *Complex uncertain variable*, PhD Thesis, Tsinghua University, 2012.

[1]Department of Mathematics,
NIT Agartala,
Tripura, India, 799046
Email address: dasbirojit@gmail.com
Email address: babybhatt75@gmail.com

[2]Department of Mathematics,
Tripura University,
Tripura, India, 799022
Email address: tripathybc@rediffmail.com

KRAGUJEVAC JOURNAL OF MATHEMATICS

About this Journal

The *Kragujevac Journal of Mathematics* (KJM) is an international journal devoted to research concerning all aspects of mathematics. The journal's policy is to motivate authors to publish original research that represents a significant contribution and is of broad interest to the fields of pure and applied mathematics. All published papers are reviewed and final versions are freely available online upon receipt. Volumes are compiled and published and hard copies are available for purchase. From 2018 the journal appears in one volume and four issues per annum: in March, June, September and December. From 2021 the journal appears in one volume and six issues per annum: in February, April, June, August, October and December.

During the period 1980–1999 (volumes 1–21) the journal appeared under the name *Zbornik radova Prirodno–matematičkog fakulteta Kragujevac* (Collection of Scientific Papers from the Faculty of Science, Kragujevac), after which two separate journals—the *Kragujevac Journal of Mathematics* and the *Kragujevac Journal of Science*—were formed.

Instructions for Authors

The journal's acceptance criteria are originality, significance, and clarity of presentation. The submitted contributions must be written in English and be typeset in TEX or LATEX using the journal's defined style (please refer to the Information for Authors section of the journal's website `http://kjm.pmf.kg.ac.rs`). Papers should be submitted using the online system located on the journal's website by creating an account and following the submission instructions (the same account allows the paper's progress to be monitored). For additional information please contact the Editorial Board via e-mail (`krag_j_math@kg.ac.rs`).