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A TOTALLY RELAXED SELF-ADAPTIVE SUBGRADIENT EXTRAGRADIENT SCHEME FOR EQUILIBRIUM AND FIXED POINT PROBLEMS IN A BANACH SPACE

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Abstract. The goal of this paper is to introduce a Totally Relaxed Self adaptive Subgradient Extragradient Method (TRSSEM) together with an Halpern iterative method for approximating a common solution of Fixed Point Problem (FPP) and Equilibrium Problem (EP) in 2-uniformly convex and uniformly smooth Banach space. We prove the strong convergence of the sequence generated by our proposed method. The proposed method does not require the computation of a projection onto a feasible set, it instead requires a projection onto a finite intersection of sub-level sets of convex functions. Our result generalizes, unifies and extends some related results in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^*. Let E be endowed with the duality pairing $\langle \cdot, \cdot \rangle$ of element from E and E^*, and also the corresponding norm $\| \cdot \|$. Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction such that $C \subset \operatorname{int}(\operatorname{dom}(f, \cdot))$, then for every $x \in C$, the Equilibrium Problem (EP) (see [3, 14]), is to find a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

We denote the EP and its solution set by $EP(C, f)$ and $Sol(C, f)$, respectively.

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The EP is a generalization of many important optimization problems, such as Variational Inequality Problem (VIP), Fixed Point Problem (FPP) and so on (see [6,14] and the references therein). In particular, if $f(x,y) = \langle Ax, y-x\rangle$, where $A : C \to E^*$, is a nonlinear mapping, then $EP(C,f)$ (1.1) reduces to the classical VIP introduced by Stampacchia [47] (see also [36,38,41,52]), which is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

There are two important directions of research on EP: These are the existence of solution of EP and other related problems (see [14,29] for more details) and the development of iterative algorithms for approximating the solution of EP, its several generalizations and related optimization problems (see [1,12,13,33,34,42–44] and the references therein).

In 2018, Hieu [24] introduced some methods for solving strongly pseudomonotone and Lipschitz type bifunction EPs. We note that a bifunction f satisfies the Lipschitz type condition, if there exist positive constants $c_1, c_2 \in \mathbb{R}$ such that for all $x, y, z \in C$, the inequality

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2$$

holds.

In general EP, the Lipschitz type condition does not hold and when it does, finding the constants c_1 and c_2 is always not an easy task. This does have effect on the efficiency of the method involved. In addition, in the method of Hieu [24], there is the need to first solve at least one strongly convex programming problem. Also, if the bifunction and the feasible sets have complex structures, the computations could be expensive and time consuming.

Furthermore, the problem of finding a common point in the set of solutions of different generalizations of EP and the fixed point set of a nonlinear mapping in Hilbert, Banach and Hadamard spaces have been considered by several authors in literature (see [25,39,40,46,51,57]) and the references therein for further reading.

In 2013, Anh [9] introduced an extragradient algorithm for finding a common element of the fixed point set of a nonexpansive mapping and solution set of an EP involving pseudomonotone and Lipschitz type continuous bifunction in real Hilbert space. The author proved a strong convergence result of the sequence generated by his method under some standard conditions, see [8–10] for related results.

However, in Banach spaces, just like the extragradient method employed by Hieu [24], many existing methods for approximating a common solution FPP and EP involving a pseudomonotone bifunctions requires that a strongly convex programming is solved (see [26,27] and the references therein).

To avoid the assumptions of Lipschitz continuity on the bifunction and solving strongly convex progamming, Vinh and Gibali [53] introduced two gradient-type iterative algorithms involving a one-step projection method for solving $EP(C,f)$ (1.1) and proved strong convergence results for both algorithms with an adaptive step-size rule which does not require the Lipschitz condition of the associated method. The method proposed in [53] involves a projection onto a feasible set, and is known to be computationally expensive, time and memory consuming if the feasible set is not simple.

In an attempt to overcome this setback, Censor et al. [17] introduced the subgradient extragradient method which uses a projection onto a halfspace. Also, He et al. [23] introduced a TRSSEM for solving the VIP (1.2) in a real Hilbert space. Let $C^i := \{x \in H : h_i(x) \leq 0\}$, where $h_i : H \to \mathbb{R}$ for $i = 1, 2, \ldots, m$, are convex functions. In the TRSSEM, the feasible set is given as

$$C := \cap_{i=1}^m C^i.$$

On the other hand, for approximating a fixed point of a nonexpansive mapping T, Mainge [31] introdued an inertial Krasnoselskij-Mann Algorithm as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T w_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

and proved a weak convergence theorem under some mild assumptions on the sequences $\{\theta_n\}$ and $\{\alpha_n\}$. The term $\theta_n(x_n - x_{n-1})$ as given (1.3) is referred to as the inertial extrapolation term. It is known that the introduction of the inertial term helps to speed up the convergence rate of the algorithm. Due to its importance, lots of researchers have adopted the use of the inertial technique in their quest for approximating the solutions of fixed point and optimization problem (see [4, 5, 31] and the references therein).

In this paper, motivated by the works of He et al. [23], Vinh and Gibali [53] and other related results in literature, we introduce a TRSSEM for approximating a common solution of FPP and EP in 2-uniformly convex and uniformly smooth Banach space. We prove a strong convergence result for the sequence generated by the proposed method under some conditions. Finally, we give some applications of our main result. The rest of the section is organized as follows. In Section 2, we recall some important results and definitions that will be useful in establishing our main result. In Section 3, we state our proposed method and then discuss its convergence analysis. We give some theoretical application of our main result in Section 4 and give a concluding remark Section 5.

2. Preliminaries

We denote the weak and the strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let E be a real Banach space, given a function $g : E \to \mathbb{R}$.

• The function g is called Gâteaux differentiable at $x \in E$, if there exists an element E, denoted by $g'(x)$ or $\nabla g(x)$ such that

$$\lim_{t\to\infty} \frac{g(x+ty) - g(x)}{t} = \langle y, g'(x)\rangle, \quad y \in E,$$

where g' or $\nabla g(x)$ is called Gâteaux differential or gradient of g at x. We say g is Gâteaux on E if for each $x \in E$, g is Gâteaux differentiable at x.

• The function g is called weakly lower semicontinuous at $x \in E$, if $x_n \rightharpoonup x$ implies $g(x) \leq \liminf_{n\to\infty} g(x_n)$. We say that a function g is weakly lower semicontinuous on E, if for each $x \in E$, g is weakly lower semicontinuous at x.

• If g is a convex function, then it is said to be differentiable at a point $x \in E$ if the following set

$$\partial g(x) = \{f \in E : g(y) - g(x) \geq \langle f, y - x\rangle,\ y \in E\} \tag{2.1}$$

is nonempty. Each element $\partial g(x)$ is called a subgradient of g at x or the subdifferential of g and the inequality (2.1) is said to be the subdifferential inequality of g at x.

The function g is subdifferentiable at x, if g is subdifferntiable at every $x \in E$. It is well known that if g is Gâteaux differentiable at x, then g is subdifferentiable at x and $\partial g(x) = \{g'(x)\}$, that is, $\partial g(x)$ is just a singleton set. For more details on Gâteaux differentiable functions on Banach space, see [15].

Let C be a nonempty, closed and convex subset of a real Banach space with norm $\|\cdot\|$ and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2 \text{ for all } x \in E\},$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the duality pairing between the elements of E and E^*. Alber [7], introduced a generalized projection operator Π_C an analogue of the metric projection $P_C : H \to C$ in the Hilbert space H. He defines $\Pi_C : E \to C$ by

$$\Pi_C(x) = \inf_{y\in C}\{\phi(y, x) \text{ for all } x \in E\}.$$

In Hilbert spaces $P_C(x) \equiv \Pi_C(x)$.

Consider the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for all } x, y \in E.$$

In the real Hilbert space, we observe that $\phi(x, y) = \|x - y\|^2$. It is easy to see that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2.$$

The functional ϕ also satisfies the following important properties:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

and

$$\phi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\phi(x, y) + (1-\lambda)\phi(x, z), \tag{2.3}$$

for all $x, y, z \in E$ and $\lambda \in (0, 1)$.

Note. If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$, see [18, 48].

We are also concerned with the functional $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{2.4}$$

for all $x \in E$ and $x^* \in E^*$. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. It is well known that if E is a reflexive, strictly convex and smooth Banach space, then

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\left\langle J^{-1}x^* - x, y^* \right\rangle,$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [50].

Let C be a closed and convex subset of E and $T : C \to C$ be a mapping, a point $x \in C$ is called a fixed point of T, if $x = Tx$. We denote the set of fixed points of T by $F(T)$. Let $T : C \to C$ be a mapping, a point $p \in C$ is called an asymptotic fixed point of T (see [45]) if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$ (see [16, 48]). T is said to be ϕ-nonexpansive if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$ and quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mapping which requires the strict condition $F(T) = \hat{F}(T)$ (see [16, 45, 48]).

Let E be a real Banach space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}.$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$, with $\|x\| = \|y\| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + ty\| - \|x - ty\|) - 1 : \|x\| = \|y\| = 1\right\}.$$

E is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $p^{-1} + q^{-1} = 1$. It is also known that every q-uniformly smooth Banach space is uniformly smooth.

It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. The following are some important and useful properties of J, for further details, see [2, 48].

Let C be a nonempty, closed and convex subset of a real Banach space E and $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. f is said to be

(i) strongly monotone on C, if there exists $\gamma \geq 0$ such that for any $x, y \in C$

$$f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2;$$

(ii) monotone on C, if

$$f(x,y) + f(y,x) \leq 0, \quad \text{for all } x, y \in C;$$

(iii) pseudomonotone on C, if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \text{for all } x, y \in C;$$

(iv) strongly γ-pseudomonotone on C, if there exists $\gamma > 0$ such that for any $x, y \in C$

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq -\gamma\|x-y\|^2.$$

From the above, it is clear (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). The converse is generally not true (see [53]).

We now give the following useful and important lemmas that are needed in establishing our main results.

Lemma 2.1 ([35])**.** *Let E be a 2-uniformly convex and smooth Banach space. Then for every $x, y \in E$*

$$\phi(x,y) \geq \nu\|x-y\|^2,$$

where $\nu > 0$ is the 2-uniformly convexity constant of E.

Lemma 2.2 ([28])**.** *Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.3 ([7])**.** *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq \rangle \leq 0, \quad \text{for all } y \in C, \tag{2.5}$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x), \quad \text{for all } y \in C, x \in X. \tag{2.6}$$

Lemma 2.4 ([55])**.** *Fix a number $s > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\psi(\|x-y\|),$$

for all $x, y \in X$, $\lambda \in [0,1]$, with $\|x\| < s$ and $\|y\| < s$.

Lemma 2.5 ([54])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where

(a) $\{\alpha_n\} \subset [0,1]$, $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup\limits_{n\to\infty} \sigma \leq 0$;
(c) $\gamma_n \geq 0$, $n \geq 1$, *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.6 ([32]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$: $a_{m_n} < a_{m_n+1}$ and $a_n < a_{m_n+1}$. In fact, $m_n = \max\{i \leq k : a_i < a_{i+1}\}$.*

3. Main Result

In this section, we give a concise and precise statement of our algorithm, discuss some of its elementary properties and its convergence analysis. The convergence analysis is given in the next section.

Statement 3.1. Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. For $i = 1, 2, \ldots, m$, let $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinous and Gâteaux differentiable functions. Let $S : E \to E$ be a quasi-ϕ-nonexpansive mapping and $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a strongly γ-pseudomonotone bifunction satisfying the following assumptions.

Assumption 3.2. We require the following assumptions for our operator and the solution set:

A1. $f(x, \cdot)$ is convex and lower semi-continuous for every $x \in E$;
A2. f is strongly γ-pseudomonotone on C,
A3. $Sol(C, f) \neq \emptyset$;
A4. if $\{x_n\}_{n=0}^{\infty} \subset E$ is bounded, then the sequence $\{g(x_n) \in \partial(f(x_n, \cdot))(x_n)_{n=0}^{\infty}\}$ is bounded.

Note. The assumption $A4$. is quite standard assumption and it holds for example when $f(x, \cdot)$ is bounded on bounded subsets (see [11]).

Assumption 3.3. To prove a strong convergence result using Algorithm 3.4, the following conditions are needed.

B1. The feasible set C is defined by $C := \cap_{i=1}^{m} C^i$, where $C^i := \{z \in E : h_i(z) \leq 0\}$;
B2. $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
B3. $0 < \liminf\limits_{n\to\infty} \gamma_n \leq \limsup\limits_{n\to\infty} \gamma_n < 1$;
B4. $\sum\limits_{n=1}^{\infty} \phi(x_n, x_{n-1}) < \infty$.
B5. $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$.

Algorithm 3.4. $(TRSSEM)$ for $EP(C,f)$

Step 0. Choose the sequences $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\} \subset (0,1)$ satisfying Assumption 3.3, let $\mu \in (0,1)$ and $\beta_0 > 0$. For $u \in C$, select initial points x_0 and x_1 in C. Set $n = 1$.

Step 1. For $i = 1, 2 \dots, m$, and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \leq 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$. Take $g(w_n) \in \partial(f(w_n, \cdot))(w_n)$, $n \geq 1$, and compute

$$z_n = \Pi_{C_n} J^{-1}(Jw_n - \beta_n g(w_n)), \tag{3.1}$$

where β_n is given by

$$\beta_{n+1} = \begin{cases} \min\left\{\beta_n, \frac{\mu\|w_n - z_n\|}{\|g(w_n) - g(z_n)\|}\right\}, & \text{if } g(w_n) \neq g(z_n), \\ \beta_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. If $w_n = z_n$ ($w_n \in Sol(C,f)$), then set $w_n = y_n$ and go to Step 3. Otherwise, compute the next iterate by

$$y_n = \Pi_{Q_n} J^{-1}(Jw_n - \beta_n g(z_n)), \tag{3.3}$$

where

$$Q_n = \{w \in E : \langle w - z_n, Jw_n - \beta_n g(w_n) - Jz_n\rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = J^{-1}((1-\alpha_n)Ju + \alpha_n(1-\gamma_n)Jy_n + \gamma_n JSy_n). \tag{3.4}$$

Step 4. Set $n := n+1$ and go to Step 1.

Lemma 3.1. *If $w_n = z_n$, then $w_n \in Sol(C,f)$.*

Proof. Suppose $w_n = z_n$, then by (2.5) and (3.1), we have

$$\langle Jw_n - \beta_n g(w_n) - Jw_n, y - z_n\rangle \leq 0, \quad y \in C,$$

or equivalently

$$\langle g(w_n), y - w_n\rangle \geq 0, \quad \text{for all } y \in C. \tag{3.5}$$

Therefore, from (3.5) and the definition of the subdifferential f in the second argument, we obtain

$$f(w_n, y) = f(w_n, y) - f(w_n, w_n) \geq \langle g(w_n), y - w_n\rangle \geq 0.$$

Hence, $w_n \in Sol(C,f)$. ☐

Lemma 3.2 ([56]). *The sequence $\{\beta_n\}$ generated by* (3.2) *is a motonically decreasing sequence and*

$$\lim_{n\to\infty} \beta_n = \beta \geq \min\left\{\frac{\mu}{L}, \beta_0\right\}.$$

Remark 3.1. Note that if $w_n = z_n$ and $w_n = Sw_n$ we are at a common solution of the $EP(C, f)$ and fixed point of the mapping S. In our convergence analysis, we will assume implicitly that this does not occur after finitely many iterations so that our Algorithm 3.4 generates an infinite sequence satisfying, in particular $w_n \neq z_n$ and $w_n \neq Sw_n$ for all $n \in \mathbb{N}$.

We now prove some lemmas which are required components of the main result.

Lemma 3.3. *The sequence $\{x_n\}$ generated by Algorithm* 3.4 *is bounded.*

Proof. Let $x^* \in Sol(C, f)$, then we have from (2.6), that

$$\begin{aligned}
\phi(x^*, y_n) =&\phi(x^*, \Pi_{Q_n} J^{-1}(Jw_n - \beta_n g(w_n)))\\
\leq&\phi(x^*, J^{-1}(Jw_n - \beta_n g(z_n))) - \phi(y_n, J^{-1}(Jw_n - \beta_n g(w_n)))\\
=&\|x^*\|^2 - 2\langle x^*, Jw_n - \beta_n g(z_n)\rangle - \|y_n\|^2 + 2\langle y_n, Jw_n - \beta_n g(z_n)\rangle\\
=&\phi(x^*, w_n) - \phi(y_n, w_n) + 2\beta_n\langle x^* - y_n, g(z_n)\rangle\\
=&\phi(x^*, w_n) - (\phi(y_n, z_n) + \phi(z_n, w_n)\\
&+ 2\langle y_n - z_n, Jz_n - Jw_n\rangle) + 2\beta_n\langle x^* - y_n, g(z_n)\rangle\\
=&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n)\\
&+ 2\langle y_n - z_n, Jw_n - Jz_n\rangle + 2\beta_n\langle x^* - y_n, g(z_n)\rangle.
\end{aligned} \tag{3.6}$$

Now, we have from (3.6) that

$$\begin{aligned}
2\beta_n\langle x^* - y_n, g(z_n)\rangle =&2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\beta_n\langle z_n - y_n, g(z_n)\rangle\\
=&2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\langle y_n - z_n, -\beta_n g(z_n)\rangle.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) and using the strongly pseudomonotonicity of f, we obtain

$$\begin{aligned}
\phi(x^*, y_n) =&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - Jz_n\rangle\\
&+ 2\beta_n\langle x^* - z_n, g(z_n)\rangle + 2\langle y_n - z_n, -\beta_n g(z_n)\rangle\\
=&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle\\
&+ 2\beta_n\langle x^* - z_n, g(z_n)\rangle\\
\leq&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n)\\
&+ 2\beta_n\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle + 2\beta_n f(z_n, x^*)\\
\leq&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n)\\
&- 2\beta_n\gamma\phi(x^*, z_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle\\
\leq&\phi(x^*, w_n) - \phi(y_n, z_n) - \phi(z_n, w_n) + 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle.
\end{aligned} \tag{3.8}$$

By the definition of Q_n and Cauchy-Schwartz inequality, we have

$$\begin{aligned}\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle =& 2\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \\ &+ 2\beta_n\langle y_n - z_n, g(w_n) - g(z_n)\rangle \\ \le& 2\beta_n\|y_n - z_n\|\|g(w_n) - g(z_n)\|. \end{aligned} \tag{3.9}$$

Using (3.2) and Lemma 2.1 in (3.9), we get

$$\begin{aligned}\langle y_n - z_n, Jw_n - \beta_n g(z_n) - Jz_n\rangle \le& 2\frac{\mu\beta_n}{\beta_{n+1}}\|y_n - z_n\|\|w_n - z_n\| \\ \le& 2\frac{\mu\beta_n}{\beta_{n+1}}\sqrt{\frac{\phi(y_n, z_n)}{\nu}}\sqrt{\frac{\phi(z_n, w_n)}{\nu}} \\ \le& \frac{\mu\beta_n}{\nu\beta_{n+1}}(\phi(y_n, z_n) + \phi(z_n, w_n)). \end{aligned} \tag{3.10}$$

Therefore, from (3.8) and (3.10), we have

$$\phi(x^*, y_n) \le \phi(x^*, w_n) - \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)). \tag{3.11}$$

From (2.3) and (3.4), we have

$$\begin{aligned}\phi(x^*, x_{n+1}) =& \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)(1-\gamma_n)Ju_n + \gamma_n JSy_n)) \\ =& \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)(1-\gamma_n)Jy_n + (1-\alpha_n)\gamma_n JSy_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)(1-\gamma_n)\phi(x^*, y_n) + (1-\alpha_n)\gamma_n\phi(x^*, Sy_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, y_n) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) \\ &- \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)) \\ \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n). \end{aligned} \tag{3.12}$$

From Algorithm 3.4, we have

$$\begin{aligned}\phi(x^*, w_n) =& \phi(x^*, J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))) \\ \le& (1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}), \end{aligned}$$

hence

$$\begin{aligned}\phi(x^*, x_{n+1}) &\le \alpha_n\phi(x^*, u) + (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1})) \\ &\le \alpha_n\phi(x^*, u) + (1-\alpha_n)(\phi(x^*, x_n) + \phi(x^*, x_{n-1})) \\ &\le \max\{\phi(x^*, u), (\phi(x^*, x_n) + \phi(x^*, x_{n-1}))\} \\ &\vdots \\ &\le \max\{\phi(x^*, u), (\phi(x^*, x_1) + \phi(x^*, x_0))\}, \quad n \ge 1. \end{aligned} \tag{3.13}$$

This implies that $\{\phi(x^*, x_n)\}$ is bounded. Therefore, $\{x_n\}$ is bounded. Consequently, $\{g(y_n)\}$ is bounded and by the nonexpansiveness of the projection operator and the mapping S, we have that $\{z_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{Sy_n\}$ are bounded. $\square$

The boundedness of $\{x_n\}$ implies that there is at least one weak limit point. The next result provides a condition under which each of such weak limit is in the solution set of the equilibrium problem.

Lemma 3.4. *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging weakly to a point $p \in C$ and suppose that the conditions $\|w_{n_i} - z_{n_i}\| \to 0$ and $\|w_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$ hold on this subsequence. Then $p \in Sol(C, f)$.*

Proof. From Lemma 2.5 and the definition of subdifferential, we have

$$\begin{aligned}
0 &\le \langle x - z_{n_i}, Jz_{n_i} - (Jw_{n_i} - \beta_{n_i} g(w_{n_i}))\rangle \\
&= \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle x - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle \\
&= \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle x - w_{n_i}, \beta_{n_i} g(w_{n_i})\rangle + \langle w_{n_i} - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle \\
&\le \langle x - z_{n_i}, Jz_{n_i} - Jw_{n_i}\rangle + \langle w_{n_i} - z_{n_i}, \beta_{n_i} g(w_{n_i})\rangle + f(w_{n_i}, x). \qquad (3.14)
\end{aligned}$$

Passing limit to the inequality in (3.14), we have

$$f(p, x) \ge 0, \quad \text{for all } x \in C. \qquad \square$$

In proving the strong convergence of our Algorithm 3.4, the underlying idea relies on certain estimate and other classical properties of the iterates which are given in the next lemmas below.

Lemma 3.5. *The sequence $\{x_n\}$ generated by Algorithm 3.4 satisfies the following estimates:*

(i) $a_{n+1} \le (1 - \alpha_n) a_n + \alpha_n b_n$;
(ii) $-1 \le \limsup_{n\to\infty} b_n < +\infty$,

where $a_n = \phi(x^*, x_n)$ and $b_n = \frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\langle Ju - Jx^*, x_{n+1} - x^*\rangle$.

Proof. Let $p_n = (1 - \gamma_n) Jy_n + \gamma_n JSy_n$, then from (2.4), we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) =&\phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n) Jp_n)) \\
\le& V(x^*, \alpha_n Ju + (1 - \alpha_n) Jp_n - \alpha_n (Ju - Jx^*)) \\
&- 2\langle -\alpha_n (Ju - Jx^*), J^{-1}(\alpha_n Ju + (1 - \alpha_n) Jp_n)\rangle \\
\le& V(x^*, \alpha_n Jx^* + (1 - \alpha_n) Jp_n) + 2\alpha_n \langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
\le& \alpha_n V(x^*, Jx^*) + (1 - \alpha_n) V(x^*, Jp_n) + 2\alpha_n \langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
\le& \alpha_n \phi(x^*, x^*) + (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
\le& (1 - \alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
\le& (1 - \alpha_n)(1 - \gamma_n)\phi(x^*, y_n) + \gamma_n (1 - \alpha_n)\phi(x^*, Sy_n) \\
&+ 2\alpha_n \langle Ju - Jx^*, x_{n+1} - x^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\phi(x^*, y_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) + 2\alpha_n\langle Ju - Jx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + \alpha_n\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\langle Ju - Jx^*, x_{n+1} - x^*\rangle\right).
\end{aligned}
$$

This established (i). Next we proof (ii). Since $\{x_n\}$ is bounded, then we have

$$
\sup_{n\geq 0} b_n \leq \sup\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + 2\|Ju - Jx^*\|\|x_{n+1} - x^*\|\right) < \infty.
$$

This implies that $\limsup_{n\to\infty} b_n < \infty$. Next we show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary, that is $\limsup_{n\to\infty} b_n \leq -1$. Then there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$ for all $n \geq n_0$. Then for all $n_0 \in \mathbb{N}$, we get from (i), that

$$
\begin{aligned}
a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n b_n \\
&< (1-\alpha_n)a_n - \alpha_n \\
&= a_n - \alpha_n(a_n + 1) \leq a_n - \alpha_n.
\end{aligned}
$$

Taking lim sup of both sides in the last inequality, we have

$$
\limsup_{n\to\infty} a_n \leq a_{n_0} - \lim_{n\to\infty}\sum_{i=n_0}^{n}\alpha_i = -\infty.
$$

This contradicts the definition of $\{a_n\}$ as a nonnegative integer.

Therefore, $\limsup_{n\to\infty} b_n \geq -1$. □

We now present our main theorem.

Theorem 3.5. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, for $i = 1, 2, \dots, m$. Let $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying conditions A1-A4, let $S : C \to C$ be a quasi-ϕ-nonexpansive mapping such that $\Gamma = \{Sol(C, f) \cap F(S)\} \neq \emptyset$. Let $\{\theta_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying Assumption 3.3, then the sequence $\{x_n\}$ generated by Algorithm 3.4 converges strongly to $p = \Pi_\Gamma u$, where Π_Γ is the projection of C onto Γ.*

Proof. Let $p \in \Gamma$, we divide the proof into two cases.

Case I Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}$ is monotone non-increasing. Since $\{\phi(x^*, x_n)\}$ is bounded, then it is convergent and

$$
\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \quad \text{as } n \to \infty. \tag{3.15}
$$

Since $p_n = J^{-1}((1-\gamma_n)Jy_n + \gamma_n JSy)$, then from Lemma 2.4, we have

$$
\phi(x^*, p_n) = \phi(x^*, J^{-1}((1-\gamma_n)Jy_n + \gamma_n JSy))
$$

$$
\begin{aligned}
&= V(x^*, (1-\gamma_n)Jy_n + \gamma_n JSy) \\
&= \|x^*\|^2 - 2\langle x^*, (1-\gamma_n)Jy_n + \gamma_n JSy\rangle + \|(1-\gamma_n)Jy_n + \gamma_n JSy\|^2 \\
&= \|x^*\|^2 - 2(1-\gamma_n)\langle x^*, Jy_n\rangle - 2\gamma_n\langle x^*, JSy_n\rangle + (1-\gamma_n)\|y_n\|^2 + \gamma_n\|Sy_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&= \phi(x^*, y_n) + \phi(x^*, Sy_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \phi(x^*, y_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|). \qquad (3.16)
\end{aligned}
$$

Therefore, from (3.4), (3.11) and (3.16), we have

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jp_n)) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, p_n) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, y_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)((1-\theta_n)\phi(x^*, x_n) \\
&\quad + \theta_n\phi(x^*, x_{n-1})) - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \\
&\le \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n) + \theta_n\phi(x^*, x_{n-1}) \\
&\quad - \gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|). \qquad (3.17)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \le& \alpha_n\left(\frac{\theta_n}{\alpha_n}\phi(x^*, x_{n-1}) + \phi(x^*, u)\right) \\
&+ (1-\alpha_n)\phi(x^*, x_n) - \phi(x^*, x_{n-1}).
\end{aligned}
$$

By using $\alpha_n \to 0$, we obtain $\gamma_n(1-\gamma_n)\psi(\|Jy_n - JSy_n\|) \to 0$ as $n \to \infty$. Therefore, by condition $B3$ and the property of ψ, we get

$$\lim_{n\to\infty}\|Jy_n - JSy_n\| = 0.$$

Since J^{-1} is norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\|y_n - Sy_n\| = 0. \qquad (3.18)$$

Furthermore, from (3.12), we have

$$\phi(x^*, y_n) \le \phi(x^*, w_n) - \left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n)).$$

Therefore, it follows from (3.4) that

$$
\begin{aligned}
\phi(x^*, x_{n+1}) \le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, p_n) \\
\le& \alpha_n\phi(x^*, u) + (1-\alpha)\phi(x^*, y_n) \\
\le& \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, w_n) \\
&- (1-\alpha_n)\left(1 - \frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n, z_n) + \phi(z_n, w_n))
\end{aligned}
$$

$$\begin{aligned}&\leq\alpha_n\phi(x^*,u)+(1-\alpha_n)((1-\theta_n)\phi(x^*,x_n)+\theta_n\phi(x^*,x_{n-1}))\\&\quad-(1-\alpha_n)\left(1-\frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n,z_n)+\phi(z_n,w_n)).\end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_n)\left(1-\frac{\mu\beta_n}{\nu\beta_{n+1}}\right)(\phi(y_n,z_n)+\phi(z_n,w_n))\leq&\alpha_n\left(\phi(x^*,u)+\frac{\theta_n}{\alpha_n}\phi(x^*,x_{n-1}\right)\\&+(1-\alpha_n)\phi(x^*,x_n)-\phi(x^*,x_{n+1}).\end{aligned}$$

By condition B2 and (3.15), we have $(\phi(y_n,z_n)+\phi(z_n,w_n))\to 0,$ as $n\to\infty$, thus

$$\lim_{n\to\infty}\phi(y_n,z_n)=\lim_{n\to\infty}\phi(z_n,w_n)=0.$$

Since the sequences $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded, we obtain by Lemma 2.2, that

$$\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|z_n-w_n\|=0. \tag{3.19}$$

From Algorithm 3.4 and condition B4, we obtain

$$\lim_{n\to\infty}\phi(w_n,x_n)=\lim_{n\to\infty}\theta_n\phi(x_n,x_{n-1})=0,$$

and by Lemma 2.2, we get

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.20}$$

It is easy to see from (3.19) and (3.20), that

$$\lim_{n\to\infty}\|x_n-z_n\|=\|x_n-y_n\|=0. \tag{3.21}$$

Observe also that

$$\phi(y_n,p_n)=\phi(y_n,J^{-1}((1-\gamma_n)Jy_n+\gamma_n)JSy_n)\to 0,\quad \text{as } n\to\infty. \tag{3.22}$$

Hence, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|y_n-p_n\|=0.$$

This and (3.21), imply

$$\lim_{n\to\infty}\|x_n-p_n\|=0.$$

Furthermore,

$$\|Jx_{n+1}-Jp_n\|=\alpha_n\|Ju-Jp_n\|=\alpha_n\|Ju-Jp_n\|\to 0,\quad \text{as } n\to\infty.$$

Since J^{-1} is norm-to-norm continuous on bounded subsets of E, we have $\|x_{n+1}-p_n\|\to 0$, as $n\to\infty$. Hence,

$$\|x_{n+1}-x_n\|\leq\|x_{n+1}-p_n\|+\|p_n-x_n\|\to 0,\quad \text{as } n\to\infty. \tag{3.23}$$

Now, since the sequence $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_n\rightharpoonup q\in E$. Then, by (3.19), (3.20) and Lemma 3.4, we obtain $q\in Sol(C,f)$. Also, since $\|y_n-Sy_n\|\to 0$ and $\|x_n-y_n\|\to 0$ as $n\to\infty$, then we have $q\in\hat{F}(S)=F(S)$. Therefore, $q\in\Gamma$.

We now show that $\{x_n\}$ converges strongly to a point $x^* = \Pi_\Gamma u$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle Ju - Jx^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty}\langle Ju - Jx^*, x_{n_i+1} - x^*\rangle.$$

Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we have by (2.5), that

$$\begin{aligned}\limsup_{n\to\infty}\langle Ju - Jx^*, x_{n+1} - x^*\rangle &= \lim_{i\to\infty}\langle Ju - Jx^*, x_{n_i+1} - x^*\rangle \\ &= \langle Ju - Jx^*, q - x^*\rangle \le 0. \end{aligned} \tag{3.24}$$

It follows from Lemma 2.5, Lemma 3.5 (i) and (3.24), that $\phi(p, x_n) \to$ as $n \to \infty$. Therefore, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

Case II Suppose there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\phi(x^*, x_{n_j+1}) > \phi(x^*, x_{n_j}), \quad \text{for all } n \in \mathbb{N}.$$

From Lemma 2.6, there exists a non-decreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following inequalities hold for all $n \in \mathbb{N}$:

$$\phi(x^*, x_{m_n}) \le \phi(x^*, x_{m_n+1}) \quad \text{and} \quad \phi(p, x_n) \le \phi(x^*, x_{m_n+1}). \tag{3.25}$$

We note from (3.11) and (3.12), that

$$\begin{aligned}\phi(x^*, x_{m_n}) \le& \phi(x^*, x_{m_n+1}) \le \alpha_{m_n}\phi(x^*, u) \\ &+ (1 - \alpha_{m_n})\left[\phi(x^*, w_{m_n}) - \left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n}))\right] \\ \le& \alpha_{m_n}\phi(x^*, u) + (1 - \alpha_{m_n})\left(((1 - \theta_{m_n})\phi(x^*, x_{m_n}) + \theta_{m_n}\phi(x^*, x_{m_n-1}))\right) \\ &- (1 - \alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \\ \le& \alpha_{m_n}\left(\phi(x^*, u) + \frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*, x_{m_n-1})\right) \\ &- (1 - \alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})).\end{aligned}$$

Hence,

$$\begin{aligned}&(1 - \alpha_{m_n})\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right) \times (\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \\ \le& \alpha_{m_n}\left(\phi(x^*, u) + \frac{\theta_{m_n}}{\alpha_{m_n}}\phi(x^*, x_{m_n-1})\right) + (1 - \alpha_{m_n})\phi(x^*, x_{m_n}) - \phi(x^*, x_{m_n}).\end{aligned}$$

Since $\alpha_{m_n} \to 0$ as $n \to \infty$, it follows that

$$\left(1 - \frac{\mu\beta_{m_n}}{\nu\beta_{m_n+1}}\right)(\phi(y_{m_n}, z_{m_n}) + \phi(z_{m_n}, w_{m_n})) \to 0, \quad \text{as } n \to \infty,$$

hence

$$\lim_{n\to\infty} \phi(y_{m_n}, z_{m_n}) = \lim_{n\to\infty} \phi(z_{m_n}, w_{m_n}) = 0.$$

Since $\{x_{m_n}\}$, $\{y_{m_n}\}$ and $\{w_{m_n}\}$ are bounded, we have

$$\lim_{n\to\infty} \|y_{m_n} - z_{m_n}\| = \lim_{n\to\infty} \|z_{m_n} - w_{m_n}\| = 0.$$

Following similar method as in Case I, we obtain

$$\lim_{n\to\infty} \|w_{m_n} - Sw_{m_n}\| = \lim_{n\to\infty} \|x_{m_n+1} - x_{m_n}\| = 0. \tag{3.26}$$

By Lemma 3.4 and (3.26), we obtain a weak limit $q \in E$ of $\{x_{m_n}\}$ such that $q \in \Gamma$.

Again, since $\{x_{m_n}\}$ is bounded, we can choose a sequence $\{x_{m_n}\}$ of $\{x_{m_n}\}$, subsequencing if necessary such that $x_{m_n} \to q$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle Ju - Jx^*, x_{m_n+1} - x^*\rangle = \lim_{n\to\infty} \langle Ju - Jx^*, x_{m_n+1} - x^*\rangle.$$

Hence, from (2.5), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle Ju - Jx^*, x_{m_n+1} - x^*\rangle &= \lim_{n\to\infty} \langle Ju - Jx^*, x_{m_n+1} - x^*\rangle \\ &\leq \langle Ju - Jx^*, q - x^*\rangle \leq 0. \end{aligned} \tag{3.27}$$

From (3.25), we have

$$\begin{aligned} 0 \leq& \phi(x^*, x_{m_n+1}) - \phi(x^*, x_{m_n}) \\ \leq& (1 - \alpha_{m_n})\phi(x^*, x_{m_n}) \\ &+ \alpha_{m_n} \left(\frac{\theta_{m_n}}{\alpha_{m_n}} \phi(x^*, x_{m_n-1}) + 2\langle Ju - Jx^*, x_{m_n+1} - x^*\rangle \right) - \phi(x^*, x_{m_n}). \end{aligned}$$

That is

$$\phi(x^*, x_{m_n}) \leq \frac{\theta_{m_n}}{\alpha_{m_n}} \phi(x^*, x_{m_n-1}) + 2\langle Ju - Jx^*, x_{m_n+1} - x^*\rangle. \tag{3.28}$$

Hence, by condition $(B5)$ and (3.27), we obtain $\phi(x^*, x_{m_n}) \to 0$ as $n \to \infty$ and Lemma 2.2 implies $\|x_{m_n} - x^*\| \to 0$ as $n \to \infty$. Consequently, $\|x_n - x^*\| \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $x^* = \Pi_\Gamma u$. □

4. Applications

In this section, we present some theoretical applications of our main result.

4.1. **Variational Inequalities Problem.** Suppose we define the f in $EP(C, f)$ (1.1), by:

$$f(x, y) := \begin{cases} \langle Ax, y - x\rangle, & \text{if } x, y \in C, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $A : C \to E^*$ is a strongly γ-pseudomonotone mapping. Then $EP(C, f)$ (1.1) reduces to $VIP(C, A)$, that is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \text{for all } y \in C. \tag{4.2}$$

We denote the set of solution of (4.2) by $Sol(C, A)$. Recall an operator A is said to be strongly γ-pseudomonotone, if there exists $\gamma > 0$ such that for any $x, y \in C$

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq \gamma\phi(y, x).$$

In this situation, Algorithm 3.4 when modified provides a new method for solving variational inequality problems and fixed point problem for a quasi-ϕ-nonexpansive mapping. We give the new method as follows.

Algorithm 4.1. $(TRSSEM)$ *for* $VIP(C, A)$

Step 0. Choose the sequences $\{\theta_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\} \subset (0, 1)$ satisfying Assumption 3.3, take $\eta, \rho \in (0, 1)$ and $\beta_0 > 0$. For $u \in C$, select initial points x_0 and x_1 in C. Set $n = 1$.

Step 1. For $i = 1, 2, \ldots, m$, and given the current iterate w_n, construct the family of half spaces

$$C_n^i := \{z \in E : h_i(w_n) + \langle h_i'(w_n), z - w_n\rangle \leq 0\}$$

and set

$$C_n = \cap_{i=1}^m C_n^i.$$

Let $w_n := J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$. Compute

$$z_n = \Pi_{C_n} J^{-1}(Jw_n - \beta_n Aw_n), \tag{4.3}$$

where β_n is given by

$$\beta_{n+1} = \begin{cases} \min\left\{\beta_n, \frac{\mu\|w_n - z_n\|}{\|g(w_n) - g(z_n)\|}\right\}, & \text{if } g(w_n) \neq g(z_n), \\ \beta_n, & \text{otherwise.} \end{cases} \tag{4.4}$$

Step 2. If $w_n = z_n$ ($w_n \in Sol(C, A)$), then set $w_n = y_n$ and go to Step 3. Otherwise, compute the next iterate by

$$y_n = \Pi_{Q_n} J^{-1}(Jw_n - \beta_n Az_n), \tag{4.5}$$

where

$$Q_n = \{w \in E : \langle w - z_n, Jw_n - \beta_n Aw_n - Jz_n\rangle \leq 0\}.$$

Step 3. Compute

$$x_{n+1} = J^{-1}((1 - \alpha_n)Ju + \alpha_n(1 - \gamma_n)Jy_n + \gamma_n JSy_n). \tag{4.6}$$

Step 4. Set $n := n + 1$ and go to Step 1.

A convergence result for solving VIP(C,A) (4.2) is given below without proof.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E and $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions, for $i = 1, 2, \dots, m$. Let $A : C \to E^*$ be a strongly γ-pseudomonotone operator that is bounded on bounded sets, let $S : E \to E$ be a quasi-ϕ-nonexpansive mapping such that $\Gamma = \{Sol(C, A) \cap F(S)\} \neq \emptyset$. Let $\{\theta_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying Assumption* 3.3*, then the sequence $\{x_n\}$ generated by Algorithm* 4.1 *converges strongly to $p = \Pi_\Gamma u$, where Π_Γ is the projection of C onto Γ.*

4.2. **Fixed Point Problem (FPP).** Given a closed set $C \subset E$, a fixed point of a mapping $T : C \to C$ is any point $x^* \in C$ such that $x^* = Tx^*$. Finding a fixed point amounts to solving $EP(C, f)$ with

$$f(x, y) = \langle x - Tx, y - x\rangle, \quad \text{for all } y \in C.$$

In this case, we define the operator $T = I - A$, where I is the identity mapping on C and A is the operator defined in Subsection 4.1. The method and result given in 4.1, thus apply.

5. Conclusion

We considered an iterative approximation of a common solution of EP and FPP. We introduced a totally relaxed self adaptive inertial subgradient extragradient method, Mann and Halpern iterative technique for solving this problem in 2-uniformly convex Banach space, which is also uniformly smooth. Our method uses a carefully selected adaptive stepsize which does not depend on any Lipschitz-type condition neither does it require the knowledge of the Lipschitz constant of the gradient of pseudomonotone operator.

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