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ACENTRALIZERS OF SOME FINITE GROUPS

ZAHRA MOZAFAR[1] AND BIJAN TAERI[1]

Abstract. Let G be a finite group. The acentralizer of an automorphism α of G, is the subgroup of fixed points of α, i.e., $C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$. In this paper we determine the acentralizers of the dihedral group of order $2n$, the dicyclic group of order $4n$ and the symmetric group on n letters. As a result we see that if $n \geq 3$, then the number of acentralizers of the dihedral group and the dicyclic group of order $4n$ are equal. Also we determine the acentralizers of groups of orders pq and pqr, where p, q and r are distinct primes.

1. Introduction

Throughout this article, the usual notation will be used [17]. For example $\mathbb{Z}_n$ denotes the cyclic group of integers modulo n, $\mathbb{Z}_n^*$ denotes the group of invertible elements of $\mathbb{Z}_n$. The dihedral group of order $2n$ and the dicyclic group of order $4n$ are denoted by D_n, and Q_n, respectively. The symmetric group on a finite set of n symbols is denoted by S_n, or $\mathrm{Sym}(X)$, where $|X| = n$. The symbol $G = X \ltimes Y$ (or $G = Y \rtimes X$) indicates that G is a split extension (semidirect product) of a normal subgroup Y of G by a complement X.

Let G be a finite group. We write $\mathrm{Cent}(G) = \{C_G(g) \mid g \in G\}$, where $C_G(g)$ is the centralizer of the element g in G. The group G is called n-centralizer if $|\mathrm{Cent}(G)| = n$. There are some results on finite n-centralizers groups (see for instance [1–8, 12, 18]). Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. If $\alpha \in \mathrm{Aut}(G)$, then the acentralizer of α in G is defined as

$$C_G(\alpha) = \{g \in G \mid \alpha(g) = g\},$$

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which is a subgroup of G. In particular if $\alpha = \tau_a$ is an inner automorphisms of G induced by $a \in G$, then $C_G(\tau_a) = C_G(a)$ is the centralizer of a in G. Let $\mathrm{Acent}(G)$ be the set of acentralizers of G, that is

$$\mathrm{Acent}(G) = \{C_G(\alpha) \mid \alpha \in \mathrm{Aut}(G)\}.$$

A group G is called n-acentralizer, if $|\mathrm{Acent}(G)| = n$. It is obvious that G is 1-acentralizer group if and only if G is a trivial group or $\mathbb{Z}_2$. Nasrabadi and Gholamian [14] proved that G is a 2-acentralizer group if and only if $G \cong \mathbb{Z}_4$, $\mathbb{Z}_p$ or $\mathbb{Z}_{2p}$, for some odd prime p. Furthermore, they characterized $3, 4, 5$-acentralizer groups. Seifizadeh et al. [16] characterized n-acentralizer groups, where $n \in \{6, 7, 8\}$, and obtained a lower bound on the number of acentralizer subgroups for p-groups, where p is a prime number. They showed that if $p \neq 2$, there is no n-acentralizer p-group for $n = 6, 7$. Moreover, if $p = 2$, then there is no 6-acentralizer p-group. In [13] we showed that if G is a finite abelian p-group of rank 2, where p is an odd prime, then the number of acentralizers of G is exactly the number of subgroups of G. Also we obtained acentralizers of infinite two-generator abelian groups.

Throughout the paper we use the presentations of the dihedral group of order $2n$, D_n, and the dicyclic group of order $4n$, Q_n, as follows

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1} \rangle = \langle b \rangle \ltimes \langle a \rangle,$$
$$Q_n = \langle a, b \mid a^{2n} = 1,\ a^n = b^2,\ bab^{-1} = a^{-1} \rangle = \langle b \rangle \ltimes \langle a \rangle.$$

We note that if n is a power of 2, then Q_n is the generalized quaternion group. Computing the number of centralizers of finite group have been the object of some papers. For instance Ashrafi [2, 3] showed that $|\mathrm{Cent}(Q_n)| = n + 2$ and

$$|\mathrm{Cent}(D_n)| = \begin{cases} n + 2, & n \text{ is odd,} \\ \frac{n}{2} + 2, & n \text{ is even.} \end{cases}$$

In this paper we compute $|\mathrm{Acent}(D_n)|$, $|\mathrm{Acent}(Q_n)|$, $|\mathrm{Acent}(S_n)|$ and the number of acentralizers of groups of order pqr, where p, q and r are distinct primes.

2. Acentralizers of Dihedral and Dicyclic Groups

Recall that the dihedral group D_n have two type subgroups for $n > 3$, $\langle a^d \rangle$ and $\langle a^d, a^r b \rangle$, where $d \mid n$, $0 \leq r < d$. The total number of these two type subgroups are $\tau(n) = \sum_{d|n} 1$, that is the number of positive divisors of n, and $\sigma(n) = \sum_{d|n} d$, that is the sum positive divisors of n, respectively. Recall that if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then $\tau(n) = \prod_{j=1}^{r} (k_j + 1)$ and $\sigma(n) = \prod_{j=1}^{r} \frac{p_j^{k_j+1} - 1}{p_j - 1}$.

For $n > 2$, the automorphism group of D_n is isomorphic to $\mathbb{Z}_n^* \ltimes \mathbb{Z}_n$, the semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_n^*$, with the canonical action of $\varepsilon : \mathbb{Z}_n^* \to \mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Explicitly,

$$\mathrm{Aut}(D_n) = \{\gamma_{s,t} \mid s \in \mathbb{Z}_n^*,\ t \in \mathbb{Z}_n\},$$

where $\gamma_{s,t}$ is defined by

$$\gamma_{s,t}(a^i) = a^{is} \quad \text{and} \quad \gamma_{s,t}(a^i b) = a^{is+t} b,$$

for all $0 \leq i \leq n-1$. Note that

$$\begin{aligned} a^i \in C_{D_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i) = a^i \\ &\Leftrightarrow a^{is} = a^i \\ &\Leftrightarrow is \equiv i \pmod{n} \\ &\Leftrightarrow i(s-1) \equiv 0 \pmod{n} \end{aligned}$$

and

$$\begin{aligned} a^i b \in C_{D_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i b) = a^i b \\ &\Leftrightarrow a^{is+t} b = a^i b \\ &\Leftrightarrow is + t \equiv i \pmod{n} \\ &\Leftrightarrow i(s-1) + t \equiv 0 \pmod{n}. \end{aligned}$$

We use the following well-known theorem from elementary number theory.

Theorem 2.1. ([15, Page 102]) *Let a, b and m be integers such that $m > 0$ and let $c = \gcd(a, m)$. If c does not divide b, then the congruence $ax \equiv b \pmod{m}$ has no solutions. If $c \mid b$, then $ax \equiv b \pmod{m}$ has exactly c incongruent solutions modulo m.*

First we compute $\mathrm{Acent}(D_n)$. Clearly, $D_1 \cong \mathbb{Z}_2$ and $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So $|\mathrm{Acent}(D_1)| = 1$ and $|\mathrm{Acent}(D_2)| = 5$.

Lemma 2.1. The identity subgroup is not an acentralizer for any automorphism of D_n. Also if n is even, the subgroups $\langle a^d \rangle$, $\langle a^d, a^r b \rangle$, where d is a divisor of n such that $d \nmid \frac{n}{2}$ and $0 \leq r < d$, are not acentralizers of D_n.

Proof. On the contrary, suppose that the identity subgroup $\langle a^n \rangle = \langle 1 \rangle$ is an acentralizer. Then there exists $\gamma_{s,t} \in \mathrm{Aut}(D_n)$ such that $\gamma_{s,t}$ fixes only the identity element. If $c := \gcd(n, s-1) \neq 1$, then

$$\gamma_{s,t}(a^{\frac{n}{c}}) = a^{\frac{n}{c}s} = a^{\frac{n}{c}} a^{\frac{s-1}{c}n} = a^{\frac{n}{c}},$$

which is a contradiction. Hence $\gcd(n, s-1) = 1$, and so by Theorem 2.1, there exists $0 < i < n-1$ such that $n \mid i(s-1) + t$. Since $\gamma_{s,t}(a^i b) = a^{is+t} b = a^{i(s-1)+t} a^i b \neq a^i b$, $n \nmid i(s-1) + t$, which is a contradiction. Thus the identity subgroup can not be an acentralizer.

Now suppose, for a contradiction, that $H := \langle a^d \rangle$, where d is a divisor of n and $d \nmid n/2$ is an acentralizer of D_n. Since $a^d \in C_{D_n}(\gamma_{s,t})$ we have $a^d = \gamma_{s,t}(a^d) = a^{sd}$. Thus $n \mid (s-1)d$ and so $s = \frac{n}{d}k + 1$, for some $0 \leq k < d$. Since $d \mid n$ and $d \nmid \frac{n}{2}$, d is even. Also k is even, as s is odd. Hence, $s = \frac{2n}{d}k_1 + 1$, for some non-negative integer k_1, and so $2n \mid (s-1)d$. Thus, $n \mid (s-1)\frac{d}{2}$ and

$$\gamma_{s,t}(a^{\frac{d}{2}}) = a^{s\frac{d}{2}} = a^{\frac{d}{2}}a^{(s-1)\frac{d}{2}} = a^{\frac{d}{2}},$$

which is a contradiction, as $a^{\frac{d}{2}} \notin H = C_{D_n}(\gamma_{s,t})$.

Similarly if $K := \langle a^d, a^r b\rangle$, where d is a divisor of n, $d \nmid n/2$, $0 \leq r < d$, and $C_{D_n}(\gamma_{s,t}) = K$, for some $\gamma_{s,t} \in \mathrm{Aut}(D_n)$, we obtain a contradiction. □

Theorem 2.2. If n is an odd integer, then every non-identity subgroups of D_n is an acentralizer of D_n. If n is even, then $|\mathrm{Acent}(D_n)|$ is equal to the number of subgroups of $D_{\frac{n}{2}}$, that is

$$|\mathrm{Acent}(D_n)| = \begin{cases} \tau(n) + \sigma(n) - 1, & n \text{ is odd,} \\ \tau(\frac{n}{2}) + \sigma(\frac{n}{2}), & n \text{ is even.} \end{cases}$$

Proof. First suppose that n is odd. Let d be a divisor of n and put $d_1 := n/d$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and for $0 \leq j \leq n-1$, $\gamma_{1,1}(a^j b) = a^{j+1}b \neq a^j b$, we have $C_{D_n}(\gamma_{1,1}) = \langle a\rangle = \langle a^d\rangle$. If $d \neq 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(n, d_1) = d_1 \nmid 1$, by Theorem 2.1, for every $0 \leq j \leq n-1$, $n \nmid jd_1 + 1$, and so $\gamma_{1+d_1,1}(a^j b) = a^{j(1+d_1)+1}b = a^{jd_1+1}a^j b \neq a^j b$. It follows that $C_{D_n}(\gamma_{1+d_1,1}) = \langle a^d\rangle$.

Now consider the subgroup $H := \langle a^d, a^r b\rangle$ of D_n, where $0 \leq r < d$. If $d = 1$, then $r = 0$ and $H = G = C_{D_n}(\gamma_{1,0})$. If $d = n$, then $\langle a^d, a^r b\rangle = \langle a^r b\rangle$. Note that $\gamma_{2,n-r}(a^i) = a^{2i} \neq a^i$, for all $1 \leq i \leq n-1$. On the other hand $\gamma_{2,n-r}(a^r b) = a^{2r+n-r}b = a^r b$ and hence $C_{D_n}(\gamma_{2,n-r}) = \langle a^r b\rangle = H$.

If $d \notin \{1, n\}$, then we put $s = 1 + d_1$ and $t = n - rd_1$. Since

$$\gamma_{s,t}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+n} = a^d,$$
$$\gamma_{s,t}(a^r b) = a^{rs+t}b = a^{r(1+d_1)+n-rd_1}b = a^r b,$$

it follows that $C_{D_n}(\gamma_{s,t}) = H$. Therefore $|\mathrm{Acent}(D_n)| = \tau(n) + \sigma(n) - 1$.

Now suppose that n is even. Let d be a divisor of $\frac{n}{2}$ and put $d_1 := n/d$. Let $H := \langle a^d\rangle$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and $\gamma_{1,1}(a^j b) = a^{j+1}b \neq a^j b$, for all $0 \leq j \leq n-1$, we have $C_{D_n}(\gamma_{1,1}) = \langle a\rangle = H$. If $d \neq 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(n, d_1) = d_1 \nmid 1$, by Theorem 2.1, for all $0 \leq j \leq n-1$, $n \nmid jd_1 + 1$, and so $\gamma_{1+d_1,1}(a^j b) = a^{j(1+d_1)+1}b = a^{jd_1+1}a^j b \neq a^j b$. It follows that $C_{D_n}(\gamma_{1+d_1,1}) = \langle a^d\rangle$.

Now we consider the subgroup $H := \langle a^d, a^r b\rangle$ of D_n, where $0 \leq r < d$. If $d = 1$, then $H = G = C_{D_n}(\gamma_{1,0})$. If $d \neq 1$ and $r = 0$, then we have $\gamma_{s,0}(a^d) = a^{d(1+d_1)} = a^{d+n} = a^d$, $\gamma_{1+d_1,0}(b) = b$, and so $C_{D_n}(\gamma_{1+d_1,0}) = \langle a^d, b\rangle = H$. If $d \neq 1$ and $t \neq 0$, then we put $s = 1 + d_1$ and $t = n - rd_1$. Since

$$\gamma_{s,t}(a^d) = a^{d(1+d_1)} = a^{d+n} = a^d,$$
$$\gamma_{s,t}(a^r b) = a^{r(1+d_1)+n-rd_1}b = a^r b,$$

we have $C_{D_n}(\gamma_{s,t}) = H$. It follows that $|\mathrm{Acent}(D_n)| = \tau(\frac{n}{2}) + \sigma(\frac{n}{2})$. □

Now we compute $\mathrm{Acent}(Q_n)$. Recall that if $n > 2$, then the automorphism group of Q_n is isomorphic to $\mathbb{Z}_{2n}^* \ltimes \mathbb{Z}_{2n}$, with the canonical action of $\varepsilon : \mathbb{Z}_{2n}^* \to \mathrm{Aut}(\mathbb{Z}_{2n}) \cong \mathbb{Z}_{2n}^*$. In fact

$$\mathrm{Aut}(Q_n) = \{\gamma_{s,t} \mid s \in \mathbb{Z}_{2n}^*, t \in \mathbb{Z}_{2n}\},$$

where

$$\gamma_{s,t}(a^i) = a^{is} \quad \text{and} \quad \gamma_{s,t}(a^i b) = a^{is+t} b,$$

for all $0 \leq i \leq 2n-1$. Hence $\mathrm{Aut}(Q_m) \cong \mathrm{Aut}(D_{2m})$, where $m > 2$. Note that $\mathrm{Aut}(Q_2) \cong S_4$ and $\mathrm{Aut}(D_4) \cong D_4$. We have

$$\begin{aligned} a^i \in C_{Q_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i) = a^i \\ &\Leftrightarrow a^{is} = a^i \\ &\Leftrightarrow is \equiv i \pmod{2n} \\ &\Leftrightarrow i(s-1) \equiv 0 \pmod{2n} \end{aligned}$$

and

$$\begin{aligned} a^i b \in C_{Q_n}(\gamma_{s,t}) &\Leftrightarrow \gamma_{s,t}(a^i b) = a^i b \\ &\Leftrightarrow a^{is+t} b = a^i b \\ &\Leftrightarrow is + t \equiv i \pmod{2n} \\ &\Leftrightarrow i(s-1) + t \equiv 0 \pmod{2n}. \end{aligned}$$

Lemma 2.2. (1) Every element, $x \in Q_n$ can be written uniquely as $x = a^i b^j$, where $0 \leq i < 2n$ and $j = 0, 1$.

(2) $Z(Q_n) = \langle a^n \rangle \cong \mathbb{Z}_2$.

(3) $Q_n / Z(Q_n) \cong D_n$.

(4) $o(a^i) = 2n/i$ for $1 < i \leqslant 2n$ and $o(a^i b) = 4$ for all i.

(5) Every subgroup of Q_n is either cyclic or a dicyclic group.

Proof. (1)–(4) are straightforward.

Let H be a subgroup of Q_n. Suppose that $Z(Q_n) \leq H$. Then $H/Z(Q_n)$ is a subgroup of D_n. Since every subgroup of D_n is either cyclic or dihedral, the same is true for $H/Z(Q_n)$. If $H/Z(Q_n)$ is cyclic, then H is cyclic (indeed H is a subgroup of $\langle a \rangle$ or $H = \langle a^i b \rangle$). Therefore, we may assume $H/Z(Q_n)$ is dihedral. Thus, $H/Z(Q_n)$ has a dihedral presentation $\langle x, y \mid x^m = y^2 = 1,\ yxy = x^{-1} \rangle$. Hence, H has the same presentation with $H/Z(Q_n)$ and so H is a dicyclic group.

Finally, if H does not contain $Z(Q_n)$ then H does not contain an element of the form $a^i b$. Therefore, $H \leq \langle a \rangle$ and so it is cyclic. □

In what follows we compute acentralizers of Q_n.

Lemma 2.3. Let H be a subgroup of Q_n which does not contain $Z(Q_n)$. Then H is not an acentralizer of Q_n.

Proof. By Lemma 2.2, $H = \langle a^m \rangle$, where $m \mid 2n$, $m \nmid n$. Now suppose, for a contradiction that, H is an acentralizer of Q_n. Then there exists $\gamma_{s,t} \in \mathrm{Aut}(Q_n)$ such that $C_{Q_n}(\gamma_{s,t}) = H$. Thus, $a^m = \gamma_{s,t}(a^m) = a^{sm}$, and so $2n \mid (s-1)m$, i.e., $s = \frac{2n}{m}k + 1$, for some $0 \le k < m$. Since $m \mid 2n$ and $m \nmid n$, m is even. Also k is even, as s is odd. Therefore, $s = \frac{4n}{m}k_1 + 1$, for some non-negative integer k_1, and hence $4n \mid (s-1)m$. Thus, $2n \mid (s-1)\frac{m}{2}$ and

$$\gamma_{s,t}(a^{\frac{m}{2}}) = a^{s\frac{m}{2}} = a^{\frac{m}{2}}a^{(s-1)\frac{m}{2}} = a^{\frac{m}{2}},$$

which is a contradiction, as $a^{\frac{m}{2}} \notin H = C_{Q_n}(\gamma_{s,t})$. □

Theorem 2.3. We have $|\mathrm{Acent}(Q_n)| = \tau(n) + \sigma(n)$.

Proof. Suppose d is a divisor of n such that $1 \le d < n$, and $d_1 := 2n/d$. Let $H := \langle a^d \rangle$. If $d = 1$, then since $\gamma_{1,1}(a) = a$ and for $0 \le j \le 2n-1$, $\gamma_{1,1}(a^jb) = a^{j+1}b \neq a^jb$, we have $C_{Q_n}(\gamma_{1,1}) = \langle a \rangle$.

If $d \neq 1$, then $\gamma_{1+d_1,1}(a^d) = a^{(1+d_1)d} = a^d$. Since $\gcd(2n, d_1) = d_1 \nmid 1$, by Theorem 2.1, $2n \nmid jd_1 + 1$, for all $0 \le j \le 2n-1$, and so $\gamma_{1+d_1,1}(a^jb) = a^{j(1+d_1)+1}b = a^{jd_1+1}a^jb \neq a^jb$. It follows that $C_{Q_n}(\gamma_{1+d_1,1}) = \langle a^d \rangle$.

Now consider the subgroup $H := \langle a^d, a^rb \rangle$ of Q_n, where $0 \le r < d$. If $d = 1$, then $r = 0$ and $H = G = C_{Q_n}(\gamma_{1,0})$. If $d \neq 1$ and $r = 0$, then we put $s = 1 + d_1$ and $t = 0$, where $d_1 := \frac{2n}{d}$. We have $\gamma_{s,0}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+2n} = a^d$, $\gamma_{s,0}(b) = b$. Hence, $C_{Q_n}(\gamma_{1+d_1,0}) = \langle a^d, b \rangle = H$. If $d \neq 1$ and $r \neq 0$, then we put $s = 1 + d_1$ and $t = 2n - rd_1$, where $d_1 := \frac{2n}{d}$. We have

$$\gamma_{s,t}(a^d) = a^{ds} = a^{d(1+d_1)} = a^{d+2n} = a^d,$$
$$\gamma_{s,t}(a^rb) = a^{rs+t}b = a^{r(1+d_1)+2n-rd_1}b = a^rb.$$

Hence $C_{Q_n}(\gamma_{s,t}) = H$. It follows that $|\mathrm{Acent}(Q_n)| = \tau(n) + \sigma(n) - 1$. □

Corollary 2.1. For all $n \ge 3$ we have $|\mathrm{Acent}(Q_n)| = |\mathrm{Acent}(D_{2n})|$.

3. Acentralizers of Groups of Order pq

It is well-known that the groups of order pq, where p and q are distinct primes, with $p > q$, are

$\mathbb{Z}_{pq}$,

$T_{p,q} = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^u \rangle$, where $o(u) = q$ in $\mathbb{Z}_p^*$ and $q \mid p - 1$.

Using Theorem 3.1 below, we have $|\mathrm{Acent}(\mathbb{Z}_{pq})| = |\mathrm{Acent}(\mathbb{Z}_p)|\ |\mathrm{Acent}(\mathbb{Z}_q)| = 2 \times 2 = 4$.

Theorem 3.1. ([14, Lemma 2.1]) Let H and T be finite groups with $\gcd(|H|, |T|) = 1$. Then

$$|\mathrm{Acent}(H \times T)| = |\mathrm{Acent}(H)| \cdot\ |\mathrm{Acent}(T)|.$$

We compute $|\mathrm{Acent}(T_{p,q})|$. The proof of the following lemma is straightforward.

Lemma 3.1. Non-trivial subgroups of $T_{p,q}$ are $\langle a \rangle$, $\langle ba^j \rangle$, where $0 \le j \le p-1$.

A Frobenius group of order pq, where p is prime and $q \mid p-1$ is a group with the presentation $F_{p,q} = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^u \rangle$, where $o(u) = q$ in $\mathbb{Z}_p^*$. If q is a prime number, then $F_{p,q} \cong T_{p,q}$.

Theorem 3.2 ([10])**.** *Let p be a prime number and $q \mid p-1$. Then $\mathrm{Aut}(F_{p,q}) \cong F_{p,p-1}$, in fact*

$$\mathrm{Aut}(F_{p,q}) = \{\alpha_{i,j} \mid 1 \leq i \leq p-1, 0 \leq j \leq p-1\},$$

where

$$\alpha_{i,j}(a^m) = a^{im} \quad \text{and} \quad \alpha_{i,j}(b^n a^m) = b^n a^{(u^{n-1}+\cdots+u+1)j+im},$$

for all $0 \leq m \leq p-1$ and $1 \leq n \leq q-1$.

Note that if $G := F_{p,q}$, then

$$\begin{aligned} a^m \in C_G(\alpha_{i,j}) &\Leftrightarrow \alpha_{i,j}(a^m) = a^m \\ &\Leftrightarrow a^{im} = a^m \\ &\Leftrightarrow im \equiv m \pmod p \\ &\Leftrightarrow (i-1)m \equiv 0 \pmod p \end{aligned}$$

and

$$\begin{aligned} b^n a^m \in C_G(\alpha_{i,j}) &\Leftrightarrow \alpha_{i,j}(b^n a^m) = b^n a^m \\ &\Leftrightarrow b^n a^{(u^{n-1}+\cdots+u+1)j+im} = b^n a^m \\ &\Leftrightarrow im + (u^{n-1} + \cdots + u + 1)j \equiv m \pmod p \\ &\Leftrightarrow (i-1)m + (u^{n-1} + \cdots + u + 1)j \equiv 0 \pmod p. \end{aligned}$$

We note that if $p \mid u^{n-1} + \cdots + u + 1$, then $p \mid u^n - 1$ and $u^n \equiv 1 \pmod p$, which is a contradiction. Therefore, $p \nmid u^{n-1} + \cdots + u + 1$.

Lemma 3.2. The identity subgroup is not an acentralizer for any automorphism of $T_{p,q}$.

Proof. Suppose, contrary on our claim, that $\langle 1 \rangle$ is an acentralizer of $T_{p,q}$. Then there exists $\alpha_{i,j} \in \mathrm{Aut}(T_{p,q})$ such that $\alpha_{i,j}$ fixes only the identity element. If $i = 1$, then $\alpha_{1,j}(a^m) = a^m$, for all $1 \leq m \leq p-1$, which is a contradiction. Hence $\gcd(p, i-1) = 1$, and by Theorem 2.1, there exists $0 < m < p-1$, such that $p \mid (i-1)m + j$. But since $\alpha_{i,j}(ba^m) \neq ba^m$, we have $p \nmid (i-1)m + j$, which is a contradiction. Thus, the identity subgroup is not an acentralizer. □

Theorem 3.3. Every non-identity subgroup of $G := T_{p,q}$ is an acentralizer of an automorphism, and therefore $|\mathrm{Acent}(T_{p,q})| = p + 2$.

Proof. Let $H := \langle a \rangle$, which is a unique Sylow p-subgroup of G. Note that $\alpha_{1,1}(a^m) = a^m$. Since $p \nmid u^{n-1} + \cdots + u + 1$,

$$\alpha_{1,1}(b^n a^m) = b^n a^{(u^{n-1}+\cdots+u+1)+m} = b^n a^m a^{(u^{n-1}+\cdots+u+1)} \neq b^n a^m.$$

Hence, $C_G(\alpha_{1,1}) = H$.

Let $K := \langle ba^m \rangle$, where $0 \leq m \leq p-1$, which is a subgroup of G of order q. If $m = 0$, then $K = \langle b \rangle$, and since $\alpha_{2,0}(b) = b$, $\alpha_{2,0}(a) = a^2 \neq a$, it follows that $C_G(\alpha_{2,0}) = K$. If $1 \leq m \leq p-1$, then $\alpha_{2,p-m}(ba^m) = ba^{p-m+2m} = ba^m$. Also since $\alpha_{2,p-m}(a^m) = a^{2m} \neq a^m$, for all $1 \leq m \leq p-1$, we have $a^m \notin C_G(\alpha_{2,p-m})$. It follows that $C_G(\alpha_{2,p-m}) = K$. Hence, $|\mathrm{Acent}(T_{p,q})| = 1 + 1 + p = p + 2$. $\square$

4. Acentralizers of Groups of Order pqr

In this section we compute acentralizers of groups of order pqr, where p, q, and r are distinct primes. The presentations of groups of order pqr, where p, q and r are primes such that $p > q > r$ are given in [11]. By [10] all groups of order pqr, $p > q > r$, are isomorphic to one of the following groups:

(1) $G_1 = \mathbb{Z}_{pqr}$;
(2) $G_2 = \mathbb{Z}_r \times T_{p,q}$, $q \mid p-1$;
(3) $G_3 = \mathbb{Z}_q \times T_{p,r}$, $r \mid p-1$;
(4) $G_4 = F_{p,qr}$, $qr \mid p-1)$;
(5) $G_5 = \mathbb{Z}_p \times T_{q,r}$, $r \mid q-1$;
(6) $G_{i+5} = \langle a, b, c \mid a^p = b^q = c^r = 1,\ ab = ba,\ c^{-1}bc = b^u,\ c^{-1}ac = a^{v^i} \rangle$, where $r \mid p-1$, $q-1$, $o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$, $1 \leq i \leq r-1$.

Using the above result, Theorem 3.3 and Theorem 3.1 it is suffices to compute the number of acentralizers of $F_{p,qr}$ and G_{i+5}. The proof of the following lemma is straightforward.

Lemma 4.1. Let $F_{p,qr} = \langle a, b \mid a^p = b^{qr} = 1, bab^{-1} = a^u \rangle = \langle b \rangle \ltimes \langle a \rangle$ and $o(u) = qr$ in $\mathbb{Z}_p^*$ where p, q, r are prime and $qr \mid p-1$. Then non-trivial subgroups of $F_{p,qr}$ are $A := \langle a \rangle$, $B_x := \langle ba^x \rangle$, $C_x := \langle b^q a^x \rangle$, $D_x := \langle b^r a^x \rangle$, where $0 \leq x \leq p-1$, $H := \langle b^r, a \rangle$ and $K := \langle b^q, a \rangle$.

Lemma 4.2. Non-trivial subgroups of G_{i+5} are $A := \langle a \rangle$, $B := \langle b \rangle$, AB, $H_{j,t} := \langle cb^t a^j \rangle$, $H_t := \langle a, cb^t \rangle$ and $K_j := \langle b, ca^j \rangle$, where $0 \leq j \leq p-1$, $0 \leq t \leq q-1$. In particular G_{i+5} have $pq + p + q + 5$ subgroups.

Proof. One can easily see that the order of elements of G_{i+5} is as in the Table 1,

Elements	a^j	b^t	$b^t a^j$	$c^k b^{i'} a^{j'}$
Orders	p	q	pq	r

Table 1. The order of elements G_{i+5}

where $1 \leq j \leq p-1$, $1 \leq t \leq q-1$, $0 \leq i' \leq q-1$, $0 \leq j' \leq p-1$, $1 \leq k \leq r-1$.

It is clear that $A = \langle a \rangle$ is a unique Sylow p-subgroup of G_{i+5} and $B = \langle b \rangle$ is a unique Sylow q-subgroup of G_{i+5}. Thus $AB = \langle a, b \rangle \trianglelefteq G_{i+5}$ is a unique subgroup of order pq of G_{i+5}. It is also clear that $H_{j,t} = \langle cb^t a^j \rangle$, where $0 \leq j \leq p-1$, $0 \leq t \leq q-1$, are subgroups of order r. Since A and B are normal in G_{i+5}, every subgroups of order pr should contain A and every subgroups of order qr should contain B. Thus $K_j = \langle b, ca^j \rangle$ and $H_t = \langle a, cb^t \rangle$, where $0 \le j \le p-1$, $0 \le t \le q-1$ are subgroups of order pr and qr of G_{i+5}, respectively. $\square$

Theorem 4.1 ([10]). Automorphism group of G_{i+5} is isomorphic to $F_{p,p-1} \times F_{q,q-1}$, in fact

$$\mathrm{Aut}(G_{i+5}) = \{\alpha_{j,t,j_1,i_1} \mid 1 \le j \le p-1, 1 \le t \le q-1, 0 \le j_1 \le p-1, 0 \le i_1 \le q-1\},$$

where

$$\begin{aligned}
&\alpha_{j,t,j_1,i_1}(a^m) = a^{jm},\\
&\alpha_{j,t,j_1,i_1}(b^n) = b^{tn},\\
&\alpha_{j,t,j_1,i_1}(c^k b^{n_1} a^{m_1}) = c^k b^{i_1(u^{k-1}+\cdots+u+1)+tn_1} a^{j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1},
\end{aligned}$$

for $1 \le m \le p-1$, $1 \le n \le q-1$, $0 \le m_1 \le p-1$, $0 \le n_1 \le q-1$ and $1 \le k \le r-1$.

Note that if $G := G_{i+5}$, then

$$\begin{aligned}
a^m \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(a^m) = a^m\\
&\Leftrightarrow a^{jm} = a^m\\
&\Leftrightarrow jm \equiv m \pmod p\\
&\Leftrightarrow m(j-1) \equiv 0 \pmod p
\end{aligned}$$

and

$$\begin{aligned}
b^n \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(b^n) = b^n\\
&\Leftrightarrow b^{tn} = b^n\\
&\Leftrightarrow tn \equiv n \pmod q\\
&\Leftrightarrow n(t-1) \equiv 0 \pmod q
\end{aligned}$$

and

$$\begin{aligned}
c^k b^{n_1} a^{m_1} \in C_G(\alpha_{j,t,j_1,i_1}) &\Leftrightarrow \alpha_{j,t,j_1,i_1}(c^k b^{n_1} a^{m_1}) = c^k b^{n_1} a^{m_1}\\
&\Leftrightarrow c^k b^{i_1(u^{k-1}+\cdots+u+1)+tn_1} a^{j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1} = c^k b^{n_1} a^{m_1}\\
&\Leftrightarrow i_1(u^{k-1}+\cdots+u+1)+tn_1 \equiv n_1 \pmod q,\\
&\qquad j_1(v^{(k-1)i}+\cdots+v^i+1)+jm_1 \equiv m_1 \pmod p\\
&\Leftrightarrow i_1(u^{k-1}+\cdots+u+1)+(t-1)n_1 \equiv 0 \pmod q,\\
&\qquad j_1(v^{(k-1)i}+\cdots+v^i+1)+(j-1)m_1 \equiv 0 \pmod p.
\end{aligned}$$

Lemma 4.3. The identity subgroup and the subgroups C_x , D_x, where $0 \le x \le p-1$, H and K (defined in Lemma 4.1) are not acentralizers for any automorphism of $G := F_{p,qr}$.

Proof. As in the proof of Lemma 3.2 we can see that the identity subgroup is not an acentralizer.

Now suppose, for a contradiction that $C_x := \langle b^q a^x \rangle$, where $0 \leq x \leq p-1$ is an acentralizers of G. Then there exists $\alpha_{i,j} \in \mathrm{Aut}(G)$ such that $C_G(\alpha_{i,j}) = C_x$, where $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$. If $i = 1$, then $\alpha_{1,j}(a^m) = a^m$, for every $1 \leq m \leq p-1$, this contradicts $a^m \notin \langle b^q a^x \rangle$. Hence $\gcd(i-1, p) = 1$, by Theorem 2.1, there exists $0 < m < p-1$ such that $p \mid j + (i-1)m$. But since $ba^m \notin C_x = C_G(\alpha_{i,j})$, $\alpha_{i,j}(ba^m) = ba^{j+im} = ba^m a^{j+(i-1)m} \neq ba^m$, which implies that $p \nmid j + (i-1)m$, which is a contradiction.

Similarly we have H, D_x, and K are not acentralizers. □

Theorem 4.2. We have $|\mathrm{Acent}(F_{p,qr})| = p + 2$.

Proof. The proof is similar to that of Theorem 3.3. □

Lemma 4.4. The identity subgroup is not an acentralizer for any automorphism of G_{i+5}.

Proof. On the contrary, suppose that $\langle 1 \rangle$ is an acentralizer of G_{i+5}. Then there exists $\alpha_{j,t,j_1,i_1} \in \mathrm{Aut}(G_{i+5})$ such that α_{j,t,j_1,i_1} fixes only the identity element. If $j = 1$ or $t = 1$, then $\alpha_{1,t,j_1,i_1}(a^m) = a^m$ and $\alpha_{j,1,j_1,i_1}(b^n) = b^n$, for all $1 \leq m \leq p-1$ and $1 \leq n \leq q-1$, which is a contradiction. Hence $\gcd(j-1, p) = 1$ and $\gcd(t-1, q) = 1$. Hence, by Theorem 2.1, there exist $0 < m_1 < p-1$ and $0 < n_1 < q-1$ such that $p \mid j_1 + (j-1)m_1$ and $q \mid i_1 + (t-1)n_1$. But since

$$\alpha_{j,t,j_1,i_1}(cb^{n_1}a^{m_1}) = cb^{i_1+tn_1}a^{j_1+jm_1} = cb^{n_1}a^{m_1}b^{i_1+(t-1)n_1}a^{j_1+(j-1)m_1} \neq cb^{n_1}a^{m_1},$$

either $p \nmid j_1 + (j-1)m_1$ or $q \nmid i_1 + (t-1)n_1$, which is a contradiction. Thus, the identity subgroup is not an acentralizer. □

Theorem 4.3. Every non-identity subgroup of $G := G_{i+5}$ is an acentralizer of an automorphism, that is $|\mathrm{Acent}(G_{i+5})| = pq + p + q + 4$.

Proof. We use the notation of Theorem 4.1. Note that $\alpha_{1,1,0,0}$ is the identity automorphism of G and so $C_G(\alpha_{1,1,0,0}) = G$.

Now we show that $A = \langle a \rangle$ is an acentralizer. It is clear that $\alpha_{1,2,1,1}(a) = a$ and $\alpha_{1,2,1,1}(b^n) = b^{2n} = b^n b^n \neq b^n$, for all $1 \leq n \leq q-1$. Furthermore since $p \nmid (v^{(k-1)i} + \cdots + v^i + 1)$,

$$\begin{aligned}\alpha_{1,2,1,1}(c^k b^{n_1} a^{m_1}) &= c^k b^{(u^{k-1}+\cdots+u+1)+2n_1} a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1} \\ &= c^k b^{n_1} a^{m_1} b^{(u^{k-1}+\cdots+u+1)+n_1} a^{(v^{(k-1)i}+\cdots+v^i+1)} \neq c^k b^{n_1} a^{m_1}.\end{aligned}$$

It follows that $C_G(\alpha_{1,2,1,1}) = A$.

Let $B = \langle b \rangle$ be the unique Sylow q-subgroup of G. It is clear that $\alpha_{2,1,1,1}(b^n) = b^n$ and so $b^n \in C_G(\alpha_{2,1,1,1})$. Since $1 \leq m \leq p-1$, $\alpha_{2,1,1,1}(a^m) = a^{2m} = a^m a^m \neq a^m$. Also since $\gcd(u^{k-1}+\cdots+u+1,q)=1$, so $q\nmid(u^{k-1}+\cdots+u+1)$. Thus,

$$\begin{aligned}\alpha_{2,1,1,1}(c^kb^{n_1}a^{m_1}) &= c^kb^{(u^{k-1}+\cdots+u+1)+n_1}a^{(v^{(k-1)i}+\cdots+v^i+1)+2m_1}\\ &= c^kb^{n_1}a^{m_1}b^{(u^{k-1}+\cdots+u+1)}a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1}\neq c^kb^{n_1}a^{m_1}.\end{aligned}$$

Hence, $C_G(\alpha_{2,1,1,1})=B$.

Let $AB=\langle a,b\rangle$ be the unique subgroup of G of the order pq. It is clear that $\alpha_{1,1,1,1}(a^m)=a^m$ and $\alpha_{1,1,1,1}(b^n)=b^n$. Thus, $a^m,b^n\in C_G(\alpha_{1,1,1,1})$. Since $\gcd(u^{k-1}+\cdots+u+1,q)=1$ and $\gcd(v^{(k-1)i}+\cdots+v^i+1,p)=1$, so $q\nmid(u^{k-1}+\cdots+u+1)$ and $p\nmid(v^{(k-1)i}+\cdots+v^i+1)$. Thus,

$$\begin{aligned}\alpha_{1,1,1,1}(c^kb^{n_1}a^{m_1}) &= c^kb^{(u^{k-1}+\cdots+u+1)+n_1}a^{(v^{(k-1)i}+\cdots+v^i+1)+m_1}\\ &= c^kb^{n_1}a^{m_1}b^{(u^{k-1}+\cdots+u+1)}a^{(v^{(k-1)i}+\cdots+v^i+1)}\neq c^kb^{n_1}a^{m_1}.\end{aligned}$$

Hence, $C_G(\alpha_{1,1,1,1})=AB$.

Let $H_{m_1,n_1}=\langle cb^{n_1}a^{m_1}\rangle$ where $0\le m_1\le p-1$ and $0\le n_1\le q-1$ be the unique subgroup of G of order pq. First suppose $m_1=n_1=0$. Then $\alpha_{2,2,0,0}(c)=c$. Since $1\le m\le p-1$, $1\le n\le q-1$, we have $\alpha_{2,2,0,0}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,0,0}(b^n)=b^{2n}\neq b^n$. Thus $C_G(\alpha_{2,2,0,0})=H_{0,0}=\langle c\rangle$. Now suppose $n_1=0$, $m_1\neq0$. Then $\alpha_{2,2,p-m_1,0}(ca^{m_1})=ca^{p-m_1+2m_1}=ca^{m_1}$ and $\alpha_{2,2,p-m_1,0}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,p-m_1,0}(b^n)=b^{2n}\neq b^n$. So $C_G(\alpha_{2,2,p-m_1,0})=H_{m_1,0}=\langle ca^{m_1}\rangle$. Similarly, if $m_1=0$, $n_1\neq0$, then $\alpha_{2,2,0,q-n_1}(cb^{n_1})=cb^{q-n_1+2n_1}=cb^{n_1}$, $\alpha_{2,2,0,n_1}(a^m)=a^{2m}\neq a^m$ and $\alpha_{2,2,p-m_1,0}(b^n)=b^{2n}\neq b^n$. Hence, $C_G(\alpha_{2,2,0,q-n_1})=H_{0,n_1}=\langle cb^{n_1}\rangle$. Finally suppose that $m_1\neq0$ and $n_1\neq0$. Then

$$\alpha_{2,2,p-m_1,q-n_1}(cb^{n_1}a^{m_1})=cb^{q-n_1+2n_1}a^{p-m_1+2m_1}=cb^{q+n_1}a^{p+m_1}=cb^{n_1}a^{m_1},$$

and so, $cb^{n_1}a^{m_1}\in C_G(\alpha_{2,2,p-m_1,q-n_1})$. Since $1\le m\le p-1$ and $1\le n\le q-1$, we have $\alpha_{2,2,p-m_1,q-n_1}(a^m)=a^{2m}=a^ma^m\neq a^m$ and $\alpha_{2,2,p-m_1,q-n_1}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{2,2,p-m_1,q-n_1})=H_{m_1,n_1}$.

Now we consider the unique subgroup $AH_{n_1}=\langle a,cb^{n_1}\rangle$, where $0\le n_1\le q-1$ of order rp. First suppose that $n_1=0$. Then $\alpha_{1,2,0,0}(a^m)=a^m$. Also $\alpha_{1,2,0,0}(c^k)=c^k$. So $a^m,c^k\in C_G(\alpha_{1,2,0,0})$. Since $1\le n\le q-1$ we have $\alpha_{1,2,0,0}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{1,2,0,0})=\langle a,c\rangle=AH_0$. Now suppose that $n_1\neq0$. Then $\alpha_{1,2,0,q-n_1}(a^m)=a^m$. Also, $\alpha_{1,2,0,q-n_1}(cb^{n_1})=cb^{q-n_1+2n_1}=cb^{q+n_1}=cb^{n_1}$. So, $a^m,cb^{n_1}\in C_G(\alpha_{1,2,0,q-n_1})$. Since $1\le n\le q-1$, we have $\alpha_{1,2,0,q-n_1}(b^n)=b^{2n}=b^nb^n\neq b^n$. Hence, $C_G(\alpha_{1,2,0,q-n_1})=AH_{n_1}$.

Now consider the unique subgroup $BH_{m_1}=\langle b,ca^{m_1}\rangle$, where $0\le m_1\le p-1$, of order rq. First suppose that $m_1=0$. Then $\alpha_{2,1,0,0}(b^n)=b^n$. Also $\alpha_{2,1,0,0}(c^k)=c^k$. So $b^n,c^k\in C_G(\alpha_{2,1,0,0})$. Since $1\le m\le p-1$ we have $\alpha_{2,1,j_1,0}(a^m)=a^{2m}=a^ma^m\neq a^m$. Hence, $C_G(\alpha_{2,1,0,0})=\langle b,c\rangle=BH_0$. Now suppose that $m_1\neq0$. Then $\alpha_{2,1,p-m_1,0}(b^n)=b^n$. Also, $\alpha_{2,1,p-m_1,0}(ca^{m_1})=ca^{p-m_1+2m_1}=ca^{p+m_1}=ca^{m_1}$. So, $b^n,ca^{m_1}\in C_G(\alpha_{2,1,p-m_1,0})$. Since $1\le m\le p-1$ we have $\alpha_{2,1,p-m_1,0}(a^m)=a^{2m}=a^ma^m\neq a^m$. Hence, $C_G(\alpha_{2,1,p-m_1,0})=BH_{m_1}$.

Therefore, $|\mathrm{Acent}(G_{i+5})| = 1+1+1+1+pq+q+p = pq+p+q+4$. □

5. Acentralizers of Finite Symmetric Groups

In this section we compute $|\mathrm{Acent}(S_n)|$. First we note that $S_2 \cong \mathbb{Z}_2$ and so $|\mathrm{Acent}(S_2)| = 1$. Also if $n = 6$, then $\mathrm{Aut}(S_6) = S_6 \rtimes \mathbb{Z}_2$ and by GAP [9] we see that $|\mathrm{Acent}(S_6)| = 443$. Now since for every $n \neq 6$, $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n) = S_n$, we have $\mathrm{Acent}(S_n) = \mathrm{Cent}(S_n)$. Hence in order to find $|\mathrm{Acent}(S_n)|$ we need to find $|\mathrm{Cent}(S_n)|$. Recall that the conjugacy class an element g of a group G, is the set of elements its conjugate, that is

$$x^G := \{xgx^{-1} \mid x \in G\}.$$

Let A and G be groups, and let G act on a set X. Let B be the group of all of functions from X into A. The product of two elements f and g of B $fg(x) = f(x)g(x)$. The group G acts on B via $f^g(x) = f(gxg^{-1})$. The semidirect product of B and G with respect to this action is called the general wreath product.

Theorem 5.1. ([17, Page 297]) Let α be an element of S_n of cycle type $(r_1^{\lambda_1}, \ldots, r_k^{\lambda_k})$, then the centralizer of α in S_n is a direct product of k groups of the form $\mathbb{Z}_{r_i} \wr S_{\lambda_i}$, the general wreath product. The order of $C_{S_n}(\alpha)$ is equal to $\prod \lambda_i! r_i^{\lambda_i}$.

Every permutation α in S_n can be written as the product of disjoint cycles $\alpha = \alpha_1 \cdots \alpha_k$, where $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$, $j = 1, \ldots k$, is a product λ_j disjoint cycles of length r_j such that $r_1 < r_2 < \cdots < r_k$. The cycle, type of α is

$$r = (\underbrace{r_1, \ldots, r_1}_{\lambda_1}, \ldots, \underbrace{r_k, \ldots, r_k}_{\lambda_k}) = (r_1^{\lambda_1}, \ldots, r_k^{\lambda_k}).$$

We will not omit those r_i which are 1, so we have $\lambda_1 r_1 + \cdots + \lambda_k r_k = n$. The r_j's are distinct and λ_j's describe their multiplicities in the partition r of n. For $j = 1, \ldots, k$ let Y_j be the of letters in $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$. In fact

$$Y_j = \left\{a_{j,1}^{(1)}, a_{j,1}^{(2)}, \ldots, a_{j,1}^{(r_j)}, \ldots, a_{j,\lambda_j}^{(1)}, a_{j,\lambda_j}^{(2)}, \ldots a_{j,\lambda_j}^{(r_j)}\right\},$$

where $\alpha_{j,1} = (a_{j,1}^{(1)}\ a_{j,1}^{(2)}\ \cdots a_{j,1}^{(r_j)})$, …, $\alpha_{j,\lambda_j} = (a_{j,\lambda_j}^{(1)}\ a_{j,\lambda_j}^{(2)}\ \cdots a_{j,\lambda_j}^{(r_j)})$. Clearly, Y_j is α-invariant and $C_G(\alpha)$-invariant; and the restriction of α to Y_j is α_j, A permutation θ commutes, with α if and only if $\alpha = \beta_1 \cdots \beta_k$, where $\beta_j = \beta_{j,1}\beta_{j,2}\cdots\beta_{j,\lambda_j}$, $\beta_{j,1} = (b_{j,1}^{(1)}\ b_{j,1}^{(2)}\ \cdots b_{j,1}^{(r_j)})$, …, $\beta_{j,\lambda_j} = (b_{j,\lambda_j}^{(1)}\ b_{j,\lambda_j}^{(2)}\ \cdots b_{j,\lambda_j}^{(r_j)})$, and $\theta(a_{j,\lambda_j}^{(r_j)}) = b_{j,\lambda_j}^{(r_j)}$. Now, θ commutes with α if and only if each Y_j is θ-invariant and if the restriction β_j of β on Y_j commutes with restriction of α_j of α on Y_j. Since $Y_i \cap Y_j = \emptyset$ for $i \neq j$, the permutation β is uniquely determined by giving its restrictions on Y_j. Hence we have $C_{S_n}(\alpha) = C_1 \times \cdots \times C_k$, where C_j is the centralizer of α_j in $\mathrm{Sym}(Y_j)$.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_\lambda$, where $\sigma_1 = (a_{1,0}\ a_{1,1}\ \cdots a_{1,r-1})$, $\sigma_2 = (a_{2,0}\ a_{2,1}\ \cdots a_{2,r-1})$, …, $\sigma_\lambda = (a_{\lambda,0}\ a_{\lambda,1}\ \cdots a_{\lambda,r-1})$ be the product of λ cycles of length r. Let Y be the set of all letters in σ, that is

$$Y = \{a_{1,0}\ a_{1,1}\ \cdots a_{1,r-1}, a_{2,0}\ a_{2,1}\ \cdots a_{2,r-1}, \ldots, a_{i,0}, a_{i,1}, \ldots a_{i,r-1}\}.$$

Let $M_r := \{m \in \mathbb{N} \mid m \leq r,\ \gcd(m,r) = 1\}$. Then we have $|M_r| = \phi(r)$, where ϕ is the Euler's totient function. For every $t \in M_r$, since $\gcd(r,t) = 1$ and the order of σ is r, we have $C_G(\sigma) = C_G(\sigma^t)$, where $G := \mathrm{Sym}(Y)$. It follows that the number of different centralizers of permeations which are product of λ cycles of the same length r with letters in Y is

$$\frac{|\sigma^{\mathrm{Sym}(Y)}|}{\phi(r)}.$$

Now suppose that $\alpha = \alpha_1 \cdots \alpha_k$, where $\alpha_j = \alpha_{j,1}\alpha_{j,2}\cdots\alpha_{j,\lambda_j}$, $j = 1, \ldots k$, is a product λ_j disjoint cycles of length r_j such that $r_1 < r_2 < \cdots < r_k$. Let Y_j, $j = 1, \ldots, r$, be the set of letters in α_j. The cycle α_1 in the decomposition $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ in S_n can be chosen in $\binom{n}{|Y_1|} = \binom{n}{r_1\lambda_1}$ ways. The cycle α_2 can be chosen in $\binom{n-|Y_1|}{|Y_2|} = \binom{n-r_1\lambda_1}{r_2\lambda_2}$ ways. In general α_j can be chosen in

$$\binom{n - \sum_{i=1}^{j-1}|Y_i|}{|Y_j|} = \binom{n - \sum_{i=1}^{j-1}\lambda_i}{r_j\lambda_j} = \binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

ways. If $r_1 = 1$, $\lambda_1 = 2$, $r_2 = 2$, $\lambda_2 = 1$, and $\sum_{j=3}^{k}\lambda_j r_j = n - 4$, then let $\widehat{\alpha_1}$ be two cycles of length 1 with letters in α_2 and $\widehat{\alpha_2}$ be a cycle of length 2 with letters in α_1. Then $\alpha_1\alpha_2\alpha_3\cdots\alpha_k$ and $\widehat{\alpha_1}\,\widehat{\alpha_2}\alpha_3\cdots\alpha_k$ have the same centralizers. Hence, in this case we have

$$\frac{1}{2}\prod_{j=1}^{k}\frac{|{\alpha_j}^{\mathrm{Sym}(Y_j)}|}{\phi(r_j)}\binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

different centralizers of permutations whose cycle types are the same with α. Otherwise there are

$$\prod_{j=1}^{k}\frac{|{\alpha_j}^{\mathrm{Sym}(Y_j)}|}{\phi(r_j)}\binom{\sum_{i=j}^{k} r_i\lambda_i}{r_j\lambda_j}$$

different centralizers of permutations whose cycle types are the same with α in S_n.

In the following tables we denote the number of acentralizers of the same type as a permutation π by $\sharp C_{S_n}(\pi)$.

π	$()$	$(*,*)$	$(*,*,*)$
$\lvert\pi^{S_3}\rvert$	1	3	2
cycle type	(1^3)	$(1^1, 2^1)$	(3^1)
$C_{S_3}(\pi) \cong$	$C_1 \wr S_3 \cong S_3$	$(C_2 \wr S_1) \times (C_1 \wr S_1) \cong C_2$	$C_3 \wr S_1 \cong C_3$
$\sharp C_{S_3}(\pi)$	1	3	1

So, $|\mathrm{Cent}(S_3)| = 5$.

π	$()$	$(*,*)$	$(*,*,*)$	$(*,*)(*,*)$	$(*,*,*,*)$
$\lvert\pi^{S_4}\rvert$	1	6	8	3	6
cycle type	(1^4)	$(1^2, 2^1)$	$(1^1, 3^1)$	(2^2)	(4^1)
$C_{S_4}(\pi) \cong$	S_4	$C_2 \times C_2$	C_3	D_4	C_4
$\sharp C_{S_4}(\pi)$	1	3	4	3	3

So, $|\mathrm{Cent}(S_4)| = 14$.

π	$()$	$(*,*)$	$(*,*,*)$	$(*,*)(*,*)$	$(*,*,*,*)$	$(*,*)(*,*,*)$	$(*,*,*,*,*)$
$\lvert\pi^{S_5}\rvert$	1	10	20	15	30	20	24
cycle type	(1^5)	$(1^3,2^1)$	$(1^2,3^1)$	$(1^1,2^2)$	$(1^1,4^1)$	$(2^1,3^1)$	(5^1)
$C_{S_5}(\pi)\cong$	S_5	$C_2\times S_3$	$C_3\times C_2$	D_8	C_4	$C_2\times C_3$	C_5
$\sharp C_{S_5}(\pi)$	1	10	10	15	15	10	6

So, $|\mathrm{Cent}(S_5)| = 67$.

6. Conclusion

The acentralizer of an automorphism of a group is defined to be the subgroup of its fixed points. In particular the acentralizer of an inner automorphism is just a centralizer. In this paper we computed the acentralizers of some classes of groups, namely dihedral, dicyclic and symmetric groups. As a result we see that if $n \geq 3$, then the numbers of acentralizers of the dihedral group and the dicyclic group of order $4n$ are equal. Also we determined the acentralizers of groups of orders pq and pqr, where p, q and r are distinct primes.

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[1]Department of Mathematical Sciences,
Isfahan University of Technology,
Isfahan 84156-83111,
Iran
Email address: z.mozafar@math.iut.ac.ir
Email address: b.taeri@iut.ac.ir