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OPTIMIZING CHANCE CONSTRAINT MULTIPLE-OBJECTIVE FRACTIONAL MATHEMATICAL PROGRAMMING PROBLEM INVOLVING DEPENDENT RANDOM VARIABLE

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Abstract. This manuscript suggests a methodology to solve chance constraint multiple-objective linear fractional mathematical programming problem in which the parameters are dependent random variables to each other. The proposed problem is formulated by taking few of the parameters as continuous dependent random variables. The proposed model cannot be solved directly by using existing methodology. Thus in order to solve the proposed model, an equivalent deterministic model is derived. The procedure to solve the proposed model is accomplished in two main steps. Initially, the proposed multiple-objective chance constraint linear fractional mathematical problem is transformed to deterministic equivalent multiple-objective linear fractional mathematical programming by the help of chance constrained method. In the second step, multiple-objective functions, which consist of fractional functions is solved by using lexicographic programming approach. Finally, an example is mentioned to illustrate the methodology.

1. Introduction

Nowadays, in real world problems, many decision making problems have multiple and conflicting objectives. The mathematical programming problem involving more than one objective functions that are conflicting in nature is known as multiple-objective programming problem. If the objective functions are ratio of affine functions, the problem is called multiple-objective linear fractional mathematical programming problem.

Key words and phrases. Multiple-objective programming problem, chance constraint programming problem, fractional programming problem, lexicography method, dependent random variables.

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In a multiple-objective linear fractional programming problem, the optimal solution for one single objective need not be an optimal solution for the other single objective function. As a result, another solution which is called compromise solution must be needed to optimize all objective functions. A solution is said to be efficient solution, in the event that it cannot improve one objective function without degrading their performance in one of the other objective functions. There exist several methodologies to find efficient solution of multiple-objective fractional programming problem. Some of the methods seen in the literature are: J. S. Kornbluth and R. E. Steuer [11] proposed simplex based method to get weakly efficient solutions for multi-objective fractional programming problem. Luhandjula, [13] solved multi-objective fractional programming problem using a fuzzy programming approach. Dutta et al. [8] solved a special type of programming problem having identical denominators using variable transformation method. By applying techniques used in [6] for suitable transformation, M. Chakraborty and S. Gupta [3] solved multi-objective fractional programming problem based on set theoretic approach. Jain [9] proposed Gauss elimination method to solve multi-objective linear fractional programming problem. Porchelvi et al. [14] presented a method to find efficient solution of multi objective fractional programming problem with the help of complementary method proposed by Dheyab [7] by transforming fractional programming problem into equivalent programming problem. Tantawy [16] presented a feasible direction method for multi-objective fractional programming problem, where the denominators are identical functions.

In real world problems, the data of mathematical programming problem may not be known with certainty. If the uncertainty occurs due to randomness, then the programming problem is called stochastic programming problem. In this case, some or all of the data of the programming problem can be characterized with random variables following known distributions. There are two techniques that are used to solve stochastic mathematical programming problems. Namely, chance constraint and two stage mathematical programming. Our objective in this manuscript is to study the chance constraint mathematical programming problem. Chance constraint mathematical programming is one of the method used to solve mathematical programming problem at which the restrictions have fixed probability of violation. In this case the randomness can be shown either within the coefficient of objective functions, with in the constraint coefficients, within the right hand side parameters or in combination of constraint coefficients, objective function, and right hand side parameters.

In this manuscript, the randomness occurs only in the left side of the constraints. The difficulty of chance constraint programming problem is handling the chance constraints.

To handle these constraints some researchers obtained the deterministic equivalent of the problem with the concept of probability distribution function. Charnes and Cooper [5] presented the deterministic equivalence of chance constraint programming problem that includes independent normal random variables. Lingaraj and Wolfe [12] obtained the deterministic equivalence of chance constraint programming problem where the random variable follow gamma distribution. Knott [10] presented chance constraint mathematical programming by considering the parameters within the right side of limitations as uniform random variables. Biswal et al. [2] solved single objective probabilistic linear programming problem by considering few parameters as exponential random variables. Sahoo and Biswal [15] presented chance constraint programming problem where the random variables within the joint constraint have both normal and log normal distribution. Charles et al. [4] proposed chance constraint programming by considering the parameters within the right side of limitations having generalized continuous distribution. In spite of the fact that a few approaches are presented to obtain the deterministic equivalence of chance constraint programming problem including independent random variables, any method is not mentioned to find the deterministic equivalence of chance constraint programming problem including dependent random variables. i.e two random variables are called dependent random variable, if the probability of events associated with one random variable influence the distribution of probabilities of the other variable.

Chance constraint programming problem can be applied to the programming problem where the fractional objective functions are multiple, non commensurable and conflicting each other. In this case, there is no single solution that optimizes all fractional objective functions. The solutions of multiple-objective fractional programming problem are known as compromise solution or efficient solution. In multiple-objective fractional programming problem, decision makers need the satisfaction of criteria instead of optimizing the objective function. However, such type of problems are more complex when the parameters are uncertain. Recently Acharya et al. [1] solved multi-objective chance constraint fractional programming problems involving two parameters independent Cauchy random variables.

In this manuscript, an attempt has been made to get the lexicographic optimal solution of chance constraint multiple-objective linear fractional mathematical programming problem involving dependent normal random variable where the randomness occurs only in the constraint coefficient.

Multiple-objective chance constraint linear fractional programming is a special class of multiple-objective stochastic linear fractional programming problem.

This manuscript has been organized within six sections including the references. The first section states about the brief introduction of programming problem. The second section states the mathematical model of multiple-objective fractional programming problem. Section 3 presents the transformation strategy of multiple-objective chance constraint linear fractional programming problem into its deterministic equivalent. Section 4 states the solution procedure of multiple-objective chance constraint linear fractional programming problem. In Section 5 numerical example is given to demonstrate the proposed method. The final section presents the conclusion of the paper followed by references.

2. Mathematical Model

The general multiple-objective fractional programming problem can be stated as:

$$\max / \min : Z_k = \frac{N_k(X)}{D_k(X)} \tag{2.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, \geq, =) b_i, \quad i = 1, 2, \ldots, m, \tag{2.2}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.3}$$

where the functions $N_k(X)$ and $D_k(X)$ are continuous real valued functions defined from $\mathbb{R}^n \to \mathbb{R}$, the constraint functions can be linear or non linear functions, and the variable X is n-dimensional vector.

If Z_k are the objective functions which are defined on a compact set, then the point x^0 is compromise solution for the given problem, if and only if x^0 optimizes each objective function Z_k. The compromise solutions exist if the feasible space is non-empty and compact as well as the functions $N_k(X)$ and $D_k(X)$ are continuous functions and the denominator is different from zero.

If $N_k(X)$ and $D_k(X)$ are affine functions, the programming problem given by (2.1)–(2.3) is called multiple-objective linear fractional programming problem. If the parameters of multiple-objective linear fractional programming problem are uncertain due to randomness, then the given programming problem is called multiple-objective chance constraint linear fractional programming problem.

A multiple-objective chance constraint linear fractional programming problem is expressed as:

$$\max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{\sum_{j=1}^{n} c_{kj} x_j + c_{0k}}{\sum_{j=1}^{n} d_{kj} x_j + d_{0k}}, \quad k = 1, 2, \ldots, K, \tag{2.4}$$

subject to

$$P\left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right) \geq \alpha_i, \quad i = 1, 2, \ldots, m, \tag{2.5}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{2.6}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.7}$$

where

$$\sum_{j=1}^{n} c_{kj} x_j + c_{0k} \quad \text{and} \quad \sum_{j=1}^{n} d_{kj} x_j + d_{0k},$$

are linear functions of $x_j, c_{kj}, d_{kj} \in \mathbb{R}^n$, $a_{ij} \in \mathbb{R}^{m \times n}$, c_{0k} and d_{0k} are scalars, P indicates probability, α_i represents aspiration level for i-th constraint.

3. Transformation Technique

In order to understand the transformation of multiple-objective chance constraints linear fractional programming problem into its deterministic equivalent, we focus on the following two cases.

Case 1. Let's consider the following multiple-objective chance constraint linear fractional programming problem with two decision variables x_1 and x_2.

$$\min/\max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{c_{k1}x_1 + c_{k2}x_2 + c_{k0}}{d_{k1}x_1 + d_{k2}x_2 + d_{k0}}, \quad k = 1, 2, \ldots, K, \tag{3.1}$$

subject to

$$P(a_{i1}x_1 + a_{i2}x_2 \leq b_i) \geq \alpha_i, \quad i = 1, 2, \ldots, m, \tag{3.2}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{3.3}$$

$$x_1, x_2 \geq 0, \quad j = 1, 2, \ldots, n, \tag{3.4}$$

where $c_{k1}x_1 + c_{k2}x_2 + c_{k0}$ and $d_{k1}x_1 + d_{k2}x_2 + d_{k0}$ are linear functions of x_1 and x_2, $c_{k1}, \ldots, c_{k2}, \ldots, c_{k0}$, $d_{k1}, \ldots, d_{k2}, \ldots, d_{k0} \in \mathbb{R}$.

The mathematical programming problem (3.1)–(3.4) is equivalent to the mathematical programming problem given by:

$$\min/\max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{c_{k1}x_1 + c_{k2}x_2 + c_{k0}}{d_{k1}x_1 + d_{k2}x_2 + d_{k0}}, \quad k = 1, 2, \ldots, K, \tag{3.5}$$

subject to

$$\mathrm{E}(a_{i1}x_1 + a_{i2}x_2) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2)}, \quad i = 1, 2, \ldots, m, \tag{3.6}$$

$$0 \leq \alpha_i \leq 1, \quad i = 1, 2, \ldots, m, \tag{3.7}$$

$$x_1, x_2 \geq 0, \quad j = 1, 2, \ldots, n. \tag{3.8}$$

The equivalence of the two mathematical programming problems is proven by the existence of one to one function. In this case, the normal probability density function is used as a one to one function.

Let x be a normal random variable, then probability density function is expressed by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \sigma > 0, -\infty < \mu < \infty.$$

In (3.6) assume that the coefficients a_{i1} and a_{i2} are dependent random variables having normal distribution with variance σ^2 and mean μ. Let's assume the i-th constraint in the chance constraint given in (3.2)

$$P(a_{i1}x_1 + a_{i2}x_2 \leq b_i) \geq \alpha_i. \tag{3.9}$$

Let q be the random variable defined as $q = a_{i1}x_1 + a_{i2}x_2$, then (3.9) is expressed by

$$P(q \leq b_i) \geq \alpha_i. \tag{3.10}$$

Since q is a linear combination of normally distributed random variables, then it is a normal distributed random variable. Consequently the chance constraint given in (3.10) can be expressed as

$$P\left(\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}} \leq \frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \geq \alpha_i, \tag{3.11}$$

where $\mathrm{E}(q)$ and $\mathrm{Var}(q)$ are mean and variance of the random variable q and $\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}$ is a standard normal random variable.

The equation (3.11) can be written using cumulative distribution function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}} e^{-\frac{z^2}{2}}dz \geq \alpha_i, \quad \text{where } z=\frac{q-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}},$$

and

$$\varphi\left(\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \geq \alpha_i, \tag{3.12}$$

where $\varphi(\cdot)$ stands to standard normal random variable having $\mu=0$ and $\sigma=1$. Assume that k_{β_i} indicates the value of random variable with $\mu=0$ and $\sigma=1$ fulfilling $\varphi(k_{\beta_i})=\alpha_i$, at that point the constraint (3.12) is expressed as

$$\varphi\left(\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}}\right) \geq \varphi(k_{\beta_i}), \tag{3.13}$$

since φ continuous, the inequality (3.13) is satisfied only if

$$\frac{b_i-\mathrm{E}(q)}{\sqrt{\mathrm{Var}(q)}} \geq k_{\beta_i}$$

or

$$\mathrm{E}(q) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}(q)} \leq 0. \tag{3.14}$$

Substituting $q=a_{i1}x_1+a_{i2}x_2$ in (3.14), we have

$$\mathrm{E}(a_{i1}x_1+a_{i2}x_2) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}(a_{i1}x_1+a_{i2}x_2)}. \tag{3.15}$$

Substituting (3.15) in (3.2), the deterministic equivalent of the mathematical programming problem (3.1)–(3.4) is expressed as

$$\max: Z_k=\frac{N_k(X)}{D_k(X)}=\frac{c_{k1}x_1+c_{k2}x_2+c_{k0}}{d_{k1}x_1+d_{k2}x_2+d_{k0}}, \quad k=1,2,\ldots,K, \tag{3.16}$$

subject to

$$\text{E}(a_{i1}x_1 + a_{i2}x_2) \le b_i - k_{\beta_i}\sqrt{\text{Var}(a_{i1}x_1 + a_{i2}x_2)}, \quad i = 1, 2, \dots, m, \tag{3.17}$$

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \dots, m, \tag{3.18}$$

$$x_1, x_2 \ge 0, \quad j = 1, 2, \dots, n \tag{3.19}$$

because a_{i1} and a_{i2} are dependent random variables, then $\text{Var}(a_{i1}x_1 + a_{i2}x_2)$ is calculated as

$$\text{Var}(a_{i1}x_1 + a_{i2}x_2) = XH^tX, \tag{3.20}$$

where $X = (x_1, x_2)$ and H is 2×2 covariance matrix which is defined as:

$$H = \begin{pmatrix} \text{Var}(a_{i1}) & \text{Cov}(a_{i1}, a_{i2}) \\ \text{Cov}(a_{i2}, a_{i1}) & \text{Var}(a_{i2}) \end{pmatrix}.$$

Case 2. In this case, the multiple-objective chance constraint linear fractional programming with n decision variables is expressed as:

$$\min / \max : Z_k = \frac{N_k(X)}{D_k(X)} = \frac{\sum_{j=1}^{n} c_{kj}x_j + c_{0k}}{\sum_{j=1}^{n} d_{kj}x_j + d_{0k}}, \quad k = 1, 2, \dots, K, \tag{3.21}$$

subject to

$$P\left(\sum_{j=1}^{n} a_{ij}x_j \le b_i\right) \ge \alpha_i, \quad i = 1, 2, \dots, m, \tag{3.22}$$

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \dots, m, \tag{3.23}$$

$$x_j \ge 0, \quad j = 1, 2, \dots, n. \tag{3.24}$$

Assume that a_{ij} are dependent normal random variables having n decision variables, then the chance constraint (3.22) is given by

$$P\left(\sum_{i=1}^{n} a_{ij}x_j \le b_i\right) \ge \alpha_i. \tag{3.25}$$

Let q is a random variable defined as $q = \sum_{i=1}^{n} a_{ij}x_j - b_i$. Following the same procedure as case 1 above, the deterministic equivalent of the chance constraint programming problem is given by

$$\text{E}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \le b_i - k_{\beta_i}\sqrt{\text{Var}\left(\sum_{j=1}^{n} a_{ij}x_j\right)}, \tag{3.26}$$

substituting (3.26) in (3.22), the deterministic equivalent of the mathematical programming problem (3.21)–(3.24) is expressed by:

$$\text{(3.27)} \qquad \min/\max : Z_k = \frac{\sum_{j=1}^{n} c_{kj}x_j + c_{0k}}{\sum_{j=1}^{n} d_{kj}x_j + d_{0k}}, \quad k = 1, 2, \ldots, K,$$

subject to

$$\text{(3.28)} \qquad \mathrm{E}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \leq b_i - k_{\beta_i}\sqrt{\mathrm{Var}\left(\sum_{j=1}^{n} a_{ij}x_j\right)}, \quad i = 1, 2, \ldots, m,$$

$$\text{(3.29)} \qquad x_j \geq 0, \quad j = 1, 2, \ldots, n.$$

Since a_{ij} are dependent random variables then $\mathrm{Var}(q)$ is calculated as follows

$$\text{(3.30)} \qquad \mathrm{Var}(q) = \mathrm{Var}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n), \quad i = 1, 2, \ldots, m,$$

using the property of variance for the sum of dependent random variables we have $\mathrm{Var}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n) = XH^TX$, where H is $n \times n$ covariance matrix which is expressed by

$$H = \begin{pmatrix} \mathrm{Var}(a_{i1}) & \mathrm{Cov}(a_{i1}, a_{i2}) & \cdots & \mathrm{Cov}(a_{i1}, a_{in}) \\ \mathrm{Cov}(a_{i2}, a_{i1}) & \mathrm{Var}(a_{i2}) & \cdots & \mathrm{Cov}(a_{i2}, a_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(a_{in}, a_{i1}) & \mathrm{Cov}(a_{in}, a_{i2}) & \cdots & \mathrm{Var}(a_{in}) \end{pmatrix}.$$

4. Solution Procedure

Since the mathematical programming problem given in (2.4)–(2.7) involves uncertain parameters and several linear fractional objectives, it is difficult to discover the lexicographic optimal solution directly. To find the lexicographic optimal solution of the given multiple-objective chance constraint fractional programming problem, first convert the multiple-objective chance constraint linear fractional mathematical programming to deterministic equivalent multiple-objective linear fractional mathematical programming. Then lexicography approach is applied to get the lexicographic solution of the deterministic multiple-objective linear fractional programming problem.

We use the lexicographic ordering approach instead of general partial ordering since it is a special case of general partial ordering approach. In this case, when the lexicographic order has been imposed upon a set of objective functions, then all elements of the objective function will be comparable to one under the ordering where as partial orders are generated by a cone. In lexicography preferences are imposed by ordering the objective functions according to their importance rather than assigning weights. In this case, to solve single objective fractional programming problem, we used complementary method which is proposed by A. N. Dheyab [7]. The method is applied to change fractional mathematical model into equivalent mathematical model which is free from fractional functions. The idea is, to maximize fractional objective function, the numerator must be maximized and the denominator must be minimized. To do this, the fractional objective functions must be realized by subtracting the denominator function from the numerator function. The resulting objective function is maximized subject to given constraint. This shows that the single objective fractional mathematical programming is changed to single objective mathematical programming. Finally, the single objective programming problem is solved by a suitable strategy or existing software.

The basic steps of the methods of multiple-objective chance constraint linear fractional programming problem are given below.

Step 1. Transform the multiple-objective chance constraint linear fractional programming problem into deterministic equivalent multiple-objective linear fractional programming problem as mentioned in Section 3.

Step 2. From the objective function (minimization problem) take $k = 1$. The first objective function is expressed as $Z_1(x) = \frac{N_1(x)}{D_1(x)}$, then the value of Z_1 is taken as the minimum value of $N_1(x)$ and the maximum value of $D_1(x)$.

Step 3. Formulate a mathematical programming problem as $\min \overline{z_1}(x)$ together with the original constraints, where $\overline{z_1}(x) = N_1(x) - D_1(x)$. This is because to make the linear fractional programming problem minimum, the numerator must be as minimum as possible, while the denominator must be as greater as possible, i.e., let the numerator is denoted by $\min N_1(x)$ and the denominator is denoted by $\max D_1(x)$. Then the denominator $\max D_1(x)$ is converted to $\min D_1(x)$ by multiplying both sides by negative sign. Therefore, the new linear programming problem becomes $\min Z = \min N_1(x) - \min D_1(x)$ as stated in [7]. This can be written as $\min Z = N_1(x) - D_1(x)$. This is done by putting the variable of numerator linear on the opposite signal with code e_1, it is added to the simplex method table in the line $(m + 1)$ where as setting the variable of denominator linear to its opposite signal with code e_2, it is add to the simplex method table in the line $m + 2$, where the the bounds of the mathematical model for m is from numbers and the target linear problem is based on the following code,i.e., $Z = N_1 e_1 - D_1 e_2$, where N_1 is the value of the numerator after compensated the result of the value of x and D_1 is the value of the denominator after compensated the result of the value of x. Taking $e_1 = e_2$, we got $Z = N_1 - D_1$. Then the resulting problem is solved by methods of single objective programming or existing software.

Step 4. Apply the same procedure for the second objective function $Z_2(x) = \frac{N_2(x)}{D_2(x)}$. In this case, the minimization of earlier objective function $\min \overline{z_2}(x)$ is considered as an other constraint in addition to the original constraints.

Step 5. Once more the same strategy is applied for the third objective function and the resulting single objective programming problem is optimized subject to the previous objective function $\min \overline{z_3}(x)$ as constraint together with the original constraint.

Step 6. The method is continued until all the objective functions could be optimized.

The algorithm terminates once a unique optimum is determined. This means that if we have n objective functions, then we do have $n!$ sequence of objective functions. This shows that $n!$ possible lexicographic optimal solutions can be obtained from the given problem. Hence the algorithm terminates if all possible sequential ordered functions are optimized.

The values of the objective functions is obtained by substituting the lexicographic solution to the original objective functions.

5. Numerical Example

Consider the following multiple-objective chance constrained linear fractional mathematical programming where the constraint coefficient of the left hand restrictions follow dependent normal random variables.

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2}, \tag{5.1}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 + 3}, \tag{5.2}$$

subject to

$$P(a_{11}x_1 + a_{12}x_2 \leq 30) \geq 0.85, \tag{5.3}$$

$$P(a_{21}x_1 + a_{22}x_2 \leq 40) \geq 0.95, \tag{5.4}$$

$$x_j \geq 0, \quad j = 1, 2, \tag{5.5}$$

where a_{11}, a_{12}, a_{21}, a_{22} are random variables that follow dependent normal distribution with known parameters $E(a_{11}) = 2$, $E(a_{12}) = 4$, $E(a_{21}) = 1$, $E(a_{22}) = 2$, $\mathrm{Var}(a_{11}) = 16$, $\mathrm{Var}(a_{12}) = 25$, $\mathrm{Var}(a_{21}) = 49$, $\mathrm{Var}(a_{22}) = 36$, $\mathrm{Cov}(a_{11}, a_{12}) = 10$, $\mathrm{Cov}(a_{21}, a_{22}) = 14$.

Now, using equation the problem given in (3.27)–(3.29) the deterministic equivalent of the problem given in (5.1)–(5.5) is expressed as:

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2}, \tag{5.6}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 + 3}, \tag{5.7}$$

subject to

$$\mathrm{E}(a_{11}x_1 + a_{12}x_2) \leq b_1 - k_{\beta_1}y_1, \tag{5.8}$$

$$\mathrm{Var}(a_{11}x_1 + a_{12}x_2) - y_1^2 = 0, \tag{5.9}$$

$$\mathrm{E}(a_{21}x_1 + a_{22}x_2) \leq 40 - k_{\beta_2}y_2 \leq 0, \tag{5.10}$$

$$\mathrm{Var}(a_{21}x_1 + a_{22}x_2) - y_2^2 = 0, \tag{5.11}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.12}$$

Using the property of mean and variance of dependent random variables we have:

$$\mathrm{E}(a_{11}x_1 + a_{12}x_2) = \mathrm{E}(a_{11})x_1 + \mathrm{E}(a_{12})x_2 \tag{5.13}$$

and

$$\mathrm{Var}(a_{11}x_1 + a_{12}x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \mathrm{Var}(a_{11}) & \mathrm{Cov}(a_{11}, a_{12}) \\ \mathrm{Cov}(a_{12}, a_{11}) & \mathrm{Var}(a_{12}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

similarly, the variance of the second constraint is given by

$$\mathrm{Var}(a_{21}x_1 + a_{22}x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \mathrm{Var}(a_{21}) & \mathrm{Cov}(a_{21}, a_{22}) \\ \mathrm{Cov}(a_{22}, a_{21}) & \mathrm{Var}(a_{22}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Substituting all the values of the given data in the problem (5.6)–(5.12), we have the following deterministic multiple-objective linear fractional programming problems.

$$\max Z_1 = \frac{8x_1 + 14x_2}{2x_1 + 4x_2 + 2}, \tag{5.14}$$

$$\max Z_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 - 3}, \tag{5.15}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.16}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.17}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.18}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.19}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.20}$$

The deterministic programming problem given in (5.14)–(5.20) is multiple-objective nonlinear fractional programming problem. Using the above method, we can get the lexicographic solution of the given mathematical problem.

Now, consider the first objective function $\max Z_1 = \frac{-4x_1+3x_2}{2x_1+4x_2}$ and separate this function into two functions namely, numerator and denominator. Using the procedure in step 2, we have to formulate single objective programming problem together with the given constraints which is stated by (5.21)–(5.26) as follows

$$\max \bar{Z}_1 = (8x_1 + 14x_2) - (2x_1 + 4x_2 + 2) = 6x_1 + 10x_2 - 2, \tag{5.21}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.22}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.23}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.24}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.25}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.26}$$

Solving this nonlinear programming problem using LINGO software, we obtain the following optimal solutions: $x_1 = 1.113499$, $x_2 = 2.725417$, $y_1 = 16.31657$, $y_2 = 20.32563$, with maximum value $\bar{Z}_1 = 31.93516$.

Next, we consider the second objective function $Z_2 = \frac{-16x_1+9x_2}{-6x_1+4x_2-3}$. According to the above procedure given in step 4, we formulate the non linear programming problem as:

$$\max \bar{Z}_2 = \frac{-16x_1 + 9x_2}{-6x_1 + 5x_2 - 3} = -10x_1 + 4x_2 + 3, \tag{5.27}$$

subject to

$$2x_1 + 4x_2 \leq 30 - 1.034y_1, \tag{5.28}$$

$$(16x_1^2 + 20x_1x_2 + 25x_2^2) - y_1^2 = 0, \tag{5.29}$$

$$1x_1 + 2x_2 \leq 40 - 1.645y_2, \tag{5.30}$$

$$(49x_1^2 + 28x_1x_2 + 36x_2^2) - y_2^2 = 0, \tag{5.31}$$

$$6x_1 + 10x_2 = 33.93516, \tag{5.32}$$

$$x_1, x_2, y_1, y_2 \geq 0. \tag{5.33}$$

Here $6x_1 + 10x_2 \geq 33.93516$ is included in the constraint. Solving the nonlinear programming problem given in (5.27)–(5.33), we obtain the following lexicographic optimal solution: $x_1 = 1.113499$, $x_2 = 2.725417$, $y_1 = 16.31657$, $y_2 = 20.32563$, with maximum value $\bar{Z}_1 = 2.766681$.

Therefore, a lexicographic solution by above multiple-objective chance constraint fractional programming problem is $x_1 = 1.113499$, $x_2 = 2.725417$, with $\max Z_1 = \frac{47.06383}{17.854083}$, $\max Z_2 = \frac{6.712769}{3.946091}$.

In any multiple-objective programming problem, there exist a number of good lexicographic solutions. These lexicographic solutions are equally acceptable. Choosing the lexicographic solution depends on the situation that decision makers prefer. The preference of decision maker depends on different conditions like budget, row material, resource, time limit etc. Therefore, having more lexicographic solution to multiple-objective programming problem is necessary for decision makers to select the best solution among the given alternatives which satisfies their need and capacity. Hence, we need to search more lexicographic solution for the above programming problem. So, applying the above procedure given in section 4, first choose the second objective function and optimizing subject to the given constraints, we have an optimal solution $x_1 = 0.0000$, $x_2 = 3.271538$, $y_1 = 16.35769$, $y_2 = 19.62923$, with maximum value $\bar{Z}_2 = 16.08615$.

Next, optimizing the first objective function Z_1 subject to the original constraint including $\bar{Z}_2 = -10x_1 - 4x_2 \geq 14.19455$, obtain lexicographic optimal solution $x_1 = 0.2538795$, $x_2 = 3.156236$, $y_1 = 16.31267$, $y_2 = 19.60155$, with maximum value $\bar{Z}_1 = 31.08564$. Substituting these values to the original objective function gives to the lexicographic solution which is given by $x_1 = 0.2538795$, $x_2 = 3.156236$, with $\max Z_1 = \frac{46.2183}{18.288939}$, $\max Z_2 = \frac{24.2183}{11.257903}$.

Finally, the two lexicographic solutions are given in Table 1.

TABLE 1. Lexicographic solutions

x_1	x_2	Z_1	$Z_2(X)$
1.113499	2.725417	$\frac{47.06383}{17.854083}$	$\frac{6.712769}{3.946091}$
0.2538795	3.156236	$\frac{46.2183}{18.288939}$	$\frac{24.2183}{11.257903}$

6. Conclusion

Multiple-objective chance constraint linear fractional programming are solved by considering the coefficient of constraints as random variables following dependent normal distribution. We consider that other data of the model are deterministic. The formulated programming problem is converted to its deterministic equivalent programming problem using the concept of cumulative probability distribution for dependent random variables using the concepts of covariance. The resulting multiple-objective fractional programming is solved by using lexicography method which is prior method. Alternative lexicographic solutions are obtained using the proposed method. The problem can be extended to the same programming problems involving other dependent random variables.

References

[1] S. Acharya, B. Belay and R. Mishra, *Multi-objective probabilistic fractional programming problem involving two parameters cauchy distribution*, Math. Model. Anal. **2**4(3) (2019),385–403. https://doi.org/10.3846/mma.2019.024

[2] M. P. Biswal, N. Biswal and D. Li, *Probabilistic linear programming problems with exponential random variables*, European J. Oper. Res. **1**11(3) (1998), 589–597. https://doi.org/10.1016/S0377-2217(97)90319-2

[3] M. Chakraborty and S. Gupta, *Fuzzy mathematical programming for multi objective linear fractional programming problem*, Fuzzy Sets and Systems **1**25(3) (2002), 335–342. https://doi.org/10.1016/S0165-0114(01)00060-4

[4] V. Charles, S. Ansari and M. Khalid, *Multi-objective stochastic linear programming with general form of distributions*, International Journal of Operational Research & Optimization **2**(2) (2011), 261–278.

[5] A. Charnes and W. Cooper, *Chance constraints and normal deviates*, J. Amer. Statist. Assoc. **5**7(297) (1962), 134–148. https://doi.org/10.2307/2282444

[6] A. Charnes and W. W. Cooper, *Programming with linear fractional functionals*, Naval Research Logistics Quarterly **9**(3–4) (1962), 181–186. https://doi.org/10.1002/nav.3800090303

[7] A. N. Dheyab, *Finding the optimal solution for fractional linear programming problems with fuzzy numbers*, Journal of Kerbala University **1**0(3) (2012), 105–110.

[8] D. Dutta, J. Rao and R. Tiwari, *A restricted class of multi objective linear fractional programming problems*, European J. Oper. Res. **6**8(3) (1993), 352–355. https://doi.org/10.1016/0377-2217(93)90191-O

[9] S. Jain, *Modeling of Gauss elimination technique for multi-objective fractional programming problem*, South Asian Journal of Mathematics **4**(3) (2014), 148–153.

[10] M. Knott, *Randomized decisions in chance-constrained programming*, Journal of the Operational Research Society **36**(10) (1985), 959–962. https://doi.org/10.1057/jors.1985.167

[11] J. S. Kornbluth and R. E. Steuer, *Multiple objective linear fractional programming*, Management Science **27**(9) (1981), 1024–1039. https://doi.org/10.1287/mnsc.27.9.1024

[12] B. Lingaraj and H. Wolfe, *Certainty equivalent of a chance constraint if the random variable follows a gamma distribution*, The Indian Journal of Statistics **36**(2) (1974), 204–208.

[13] M. K. Luhandjula, *Fuzzy approaches for multiple objective linear fractional optimization*, Fuzzy Sets and Systems **13**(1) (1984), 11–23. https://doi.org/10.1016/0165-0114(84)90023-X

[14] R. Porchelvi, L. Vasanthi and R. Hepzibah, *On solving multi objective fractional linear programming problems*, International Journal of Current Research **6**(8) (2014), 8095–8102.

[15] N. Sahoo and M. Biswal, *Computation of probabilistic linear programming problems involving normal and log-normal random variables with a joint constraint*, Computer Mathematics **82**(11) (2005), 1323–1338. https://doi.org/10.1080/00207160500113058

[16] S. F. Tantawy, *Solving a special class of multiple objective linear fractional programming problems*, The ANZIAM Journal **56**(1) (2014), 91–103.

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